LEIBNIZ'S NOTATIONS, DIFFERENTIALS AND VECTORS AS DIFFERENTIAL OPERATORS

Differential. In the case a real-valued function $f(x)$ of a single real variable x, differentiability at $x = a$ can be defined as approximability at a by a polynomial $f(a)+A(x-a)$ of degree ≤ 1 to an order higher than the first. The derivative is the slope A of the graph of the polynomial. Leibniz introduced the notation $\frac{df}{dx}$ as suggested by being the limit of the difference quotient

$$\frac{f(x)-f(a)}{x-a} = \frac{\Delta f}{\Delta x}$$

with the two difference Δf and Δx in the notation replaced by df and dx.

In the case a real-valued function $f(x, y)$ of two real variables x, y, differentiability at $(x, y) = (a, b)$ can likewise be defined as approximability at (a, b) by a polynomial $f(a, b) + A(x-a) + B(y-b)$ of degree ≤ 1 to an order higher than the first. The number A is the partial derivative $\frac{\partial f}{\partial x}$ at (a, b) and the number A is the partial derivative $\frac{\partial f}{\partial x}$ at (a, b). In this case what is to take the place of the derivative of the case of one single real variable. For one real variable the derivative is the slope A of the graph of the polynomial which is from the homogeneous linear part $A(x-a) = A\Delta x$ of the polynomial, homogeneous in Δx. For two real variables x, y the counterpart of $A(x-a) = A\Delta x$ is the homogeneous linear part $A(x-a) + B(y-b) = A\Delta x + B\Delta y$ of the polynomial, homogeneous in Δx and Δy. The total differential df of f is defined as

$$df = Adx + Bdy = \frac{\partial f}{\partial x}(a, b)dx + \frac{\partial f}{\partial y}(a, b)dy,$$

suggested by the limiting case of the approximation

$$\Delta f \approx A\Delta x + B\Delta y.$$

The meaning of $df = Adx + Bdy$ means the following: when we restrict the function $f(x, y)$ to the line $x = a + \alpha t$ and $y = b + \beta t$ with parameter t to construct the function $f(a + \alpha t, b + \beta t)$ of a single variable t, we have $\frac{dx}{dt} = \alpha$ and $\frac{dy}{dt} = \beta$ and from $df = Adx + Bdy$ we can formally divide dt to get

$$\frac{df}{dt} = A\frac{dx}{dt} + B\frac{dy}{dt} = A\alpha + B\beta.$$

This means that, when we are given a vector (α, β), then we end up with the number $A\alpha + B\beta$ by formally dividing $df = Adx + Bdy$ by dt. This means that the total differential is an $\mathbb{R}$-linear transformation from the vector space $\mathbb{R}^2$ to the vector space $\mathbb{R}$ defined by $(\alpha, \beta) \mapsto A\alpha + B\beta$. In the special case $f \equiv x$, we have $A = \frac{\partial f}{\partial x} = 1$ and $B = \frac{\partial f}{\partial y} = 0$ so that $df = dx$. This means that the dx on the right-hand side of $df = Adx + Bdy$ actually carries the meaning of the differential of x. Likewise the dy on the right-hand side of $df = Adx + Bdy$ actually carries the meaning of the differential of y.

Identifying Vector with Differential Operator. Let us look once more at the question of finding $\frac{df}{dt}$ when we restrict the function $f(x, y)$ to the line $x = a + \alpha t$ and $y = b + \beta t$ with parameter t to construct the function $f(a + \alpha t, b + \beta t)$ of a single variable t. We have the formula

$$\frac{df}{dt} = \alpha A + \beta B = \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}.$$

This works for any f and we rewrite the equation as applying the operator

$$\frac{d}{dt} = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$$

to f. This operator is in one-one correspondence with the vector (α, β). We identify the differential operator

$$\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$$

with the vector (α, β). The advantage of such an identification is that the representation of a vector by a pair of numbers (α, β) depends on the coordinate system chosen while a differential operator acting on a function to give a number is independent of the coordinate system. When we have chosen a coordinate system (x, y), the vector which is represented by a differential operator gives us its representation as a pair of numbers by its values at the functions x and y.

Evaluation of Differential at Differential Operator as Vector. We now combine the two definitions together so that when we are given a differential $Adx + Bdy$ and a vector $\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$ in the form of a differential operator, we can evaluate the differential $Adx + Bdy$ at the vector $\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$ to form the number $\alpha A+\beta B$. In particular, the differential dx evaluated at the vector $\frac{\partial}{\partial x}$ gives the value 1 and the the differential dx evaluated at the vector $\frac{\partial}{\partial y}$ gives the value 0. Likewise, the differential dy evaluated at the vector $\frac{\partial}{\partial x}$ gives the value 0 and the the differential dx evaluated at the vector $\frac{\partial}{\partial x}$ gives the value 1.