Math 55a Homework Assigned January 27, 2012
due February 3, 2012

Problem 1 (Open Cover of Bounded Closed Interval Admits Finite Subcover). Let $a < b$ be in $\mathbb{R}$. Let (α_j, β_j) be a collection of bounded open intervals of $\mathbb{R}$ indexed by $j \in J$ with the index set J being infinite. Assume that $[a, b]$ is contained in $\bigcup_{j \in J} (\alpha_j, \beta_j)$. Prove that there exists some *finite* subset $F = \{j_1, \cdots, j_n\}$ of J such that $[a, b] \subset \bigcup_{\nu=1}^{n} (\alpha_{j_\nu}, \beta_{j_\nu})$.

Hint. First reduce the general case to the case where $J = \mathbb{N}$ by replacing the collection $\{(\alpha_j, \beta_j)\}_{j \in J}$ of bounded open intervals by the collection

$$\left\{ (\sigma, \tau) \;\middle|\; \sigma, \tau \in \mathbb{Q} \text{ and } (\sigma, \tau) \subset (\alpha_j, \beta_j) \text{ for some } j \in J \right\}$$

of bounded open intervals. For the special case $J = \mathbb{N}$, if the contrary is assumed, then for $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that x_n is not contained in $\bigcup_{j=1}^{n} (\alpha_j, \beta_j)$. Some subsequence x_{n_k} of x_n converges to a point x^* of $[a, b]$ as $k \to \infty$. Derive a contradiction from the fact that x^* belongs to $(\alpha_\ell, \beta_\ell)$ for some $\ell \in \mathbb{N}$.

Problem 2 (Continuity of Uniform Limit of Continuous Functions). Let $a < b$ in $\mathbb{R}$ and for every $n \in \mathbb{N}$ let $f_n(x)$ be a real-valued function on (a, b). Suppose for every fixed $x \in (a, b)$ the sequence of numbers $f_n(x)$ converges to some real number $f(x)$ as $n \to \infty$ so that the convergence is uniform in $x \in (a, b)$. That is, for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for $n \in N$ and for $x \in (a, b)$. If $\xi \in (a, b)$ and f_n is continuous at ξ for every $n \in \mathbb{N}$, prove that $f_x(x)$ is continuous at ξ.

Hint. Use the existence and equality of both sides of

$$\lim_{m \to \infty} \lim_{n \to \infty} x_n^{(m)} = \lim_{n \to \infty} \lim_{m \to \infty} x_n^{(m)}$$

when (i) the sequence $\left(x_n^{(m)}\right)_{n \in \mathbb{N}}$ is Cauchy as $n \to \infty$ uniformly in m and (ii) for each fixed n the sequence $\left(x_n^{(m)}\right)_{m \in \mathbb{N}}$ is Cauchy.

Problem 3 (Uniformity of Convergence for Monotone Sequence of Continuous Functions on Bounded Closed Interval with Continuous Limit). Let $a < b$ in $\mathbb{R}$ and for $n \in \mathbb{N}$ let $f_n(x)$ be a continuous real-valued function on $[a, b]$

such that $f_n(x) \leq f_{n+1}(x)$ for $x \in [a,b]$ and $n \in \mathbb{N}$. Assume that for each $x \in [a,b]$ the sequence $f_n(x)$ converges to some $f(x)$ as $n \to \infty$. If $f(x)$ is continuous on $[a,b]$, prove that the convergence of $f_n(x)$ to $f(x)$ as $n \to \infty$ is uniform in $x \in [a,b]$. That is, for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N$ and $x \in [a,b]$.

Hint. Take $\varepsilon > 0$. For every point $x_0 \in [a,b]$ there exists N_{x_0} such that $|f_n(x_0) - f(x_0)| < \varepsilon$ for $n \geq N_{x_0}$. By the continuity of f and f_n at x_0 there exists $\delta_{x_0} > 0$ such that $|f_n(x) - f_n(x_0)| < \varepsilon$ and $|f(x) - f(x_0)| < \varepsilon$ for $|x - x_0| < \delta_{x_0}$. By the monotonicity $f_n(x) \leq f_{n+1}(x)$ for $n \in \mathbb{N}$ and $x \in [a,b]$, verify by the triangle inequality that $|f_n(x) - f(x)| < 3\varepsilon$ for $|x - x_0| < \delta_{x_0}$ and $n \geq N_{x_0}$. Apply the result of Problem 1 to the collection

$$\left\{ (x_0 - \delta_{x_0}, x_0 + \delta_{x_0}) \;\middle|\; x_0 \in [a,b] \right\}$$

of bounded open intervals of $\mathbb{R}$.

Remark. When Problem 2 is regarded as uniform convergence implying continuity of limit, Problem 3 which gives uniform convergence from the continuity of limit can be regarded as some version of converse of Problem 2 under the additional assumption of monotonicity of the convergent sequence of functions.

Problem 4 (Example of a Continuous Nowhere Differentiable Function on $\mathbb{R}$ *due to Bartel Leendert van der Waerden).* For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ let $f_n(x)$ be the distance between x and the nearest number of the form $\frac{m}{10^n}$ with $m \in \mathbb{Z}$. Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for $x \in \mathbb{R}$. By following the steps given below, prove that $f(x)$ is a continuous function on $\mathbb{R}$ but $f'(x)$ does not exist for any $x \in \mathbb{R}$.

Step 1. Use $|f_n(x)| < \frac{1}{10^n}$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$ to show that the convergence of $\sum_{k=1}^{n} f_k(x)$ to $f(x)$ as $n \to \infty$ is uniform in $x \in \mathbb{R}$. Use Problem 1 to show that $f(x)$ is continuous on $\mathbb{R}$.

Step 2. Check non-differentiability of $f(x)$ for $x \in (0,1)$ as follows. Write

$$x = \sum_{q=1}^{\infty} \frac{a_q}{10^q}$$

with $a_q \in \{0, 1, \cdots, 9\}$ for $q \in \mathbb{N}$. For $p \in \mathbb{N}$ let

$$y_{x,p} = \sum_{q=1}^{\infty} \frac{b_{q,p}}{10^q},$$

where $b_{q,p} = a_q$ if $q \neq p$ and

$$b_{p,p} = \begin{cases} a_p - 1 & \text{if} \quad a_p \in \{4, 9\} \\ \\ a_p + 1 & \text{if} \quad a_p \notin \{4, 9\}. \end{cases}$$

We have $|y_{x,p} - x| = \frac{1}{10^p}$ and

$$|f_n(x) - f_n(y_{x,p})| = \begin{cases} |y_{x,p} - x| & \text{for} \quad n < p \\ \\ 0 & \text{for} \quad n \geq p. \end{cases}$$

Verify that $f(y_{x,p}) - f(x) = m_p (y_{x,p} - x)$ for some integer m_p which is odd or even according as $p - 1$ is odd or even. Hence the difference quotient

$$\frac{f(y_{x,p}) - f(x)}{y_{x,p} - x}$$

cannot have a limit as $p \to \infty$.

Problem 5 (Nonpositivity of Second Derivative at Maximum Point). Let $a < b$ in $\mathbb{R}$ and let $f_n(x)$ be a real-valued function on (a, b). Assume that $f'(x)$ exists at every point of (a, b) and also assume that the derivative $f''(x)$ of the function $f'(x)$ exists at every point of (a, b). Assume that the supremum $\sup_{a<x<b} f$ of f on (a, b) is achieved at some point ξ of (a, b). By using the Mean-Value Theorem, show that $f''(\xi) \leq 0$.

Hint. A real-valued function which is differentiable at every point of (a, b) is automatically continuous at every point of (a, b).

Problem 6 (Generalized Mean-Value Theorem). Let $a < b$ in $\mathbb{R}$ and let $f(x)$ and $g(x)$ be continuous real-valued functions on $[a, b]$ which are differentiable at every point of (a, b). Prove that there exists some $\xi \in (a, b)$ such that

$$(f(b) - f(a))\, g'(\xi) = (g(b) - g(a))\, f'(\xi).$$

Note that the usual Mean-Value Theorem is the special case of $g(x) = x$.

Hint. Consider the function

$$x \mapsto (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x)$$

for $x \in [a, b]$.