

**Math 55a Homework Assigned February 3, 2012
due February 10, 2012**

Problem 1 (Space of Compact Subsets of Real Numbers with Hausdorff Metric). Let \mathcal{C} be the collection of all compact subsets of \mathbb{R} . For $A, B \in \mathcal{C}$ define the *distance* between A and B as the maximum of the two nonnegative numbers

$$\sup_{x \in A} \inf_{y \in B} |x - y| \quad \text{and} \quad \sup_{y \in B} \inf_{x \in A} |x - y|,$$

which is denoted by $d_H(A, B)$ (known as the *Hausdorff metric*). Verify that the following properties for the Hausdorff metric $d_H(\cdot, \cdot)$.

- (i) (*Nonnegativity and Separation*) $d_H(A, B) \geq 0$ for $A, B \in \mathcal{C}$ with $d_H(A, B) = 0$ if and only if $A = B$.
- (ii) (*Symmetry*) $d_H(A, B) = d_H(B, A)$ for $A, B \in \mathcal{C}$.
- (iii) (*Triangle Inequality*) $d_H(A, C) \leq d_H(A, B) + d_H(B, C)$ for $A, B, C \in \mathcal{C}$.

Remark. Any set X with a function $d(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ satisfying the three axioms of Nonnegativity and Separation, Symmetry, and Triangle Inequality is known as a *metric space*. The set \mathcal{C} of all compact subsets of \mathbb{R} with the Hausdorff metric $d_H(\cdot, \cdot)$ is a metric space.

Problem 2 (Interpretation of Riemann Integral as Limit in the Space of Compact Subsets of Real Numbers with Hausdorff Metric). Let \mathcal{C} be the collection of all compact subsets of \mathbb{R} . Let $a < b$ be real numbers and let $f(x)$ be a continuous function on $[a, b]$. For a partition P

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$$

of $[a, b]$ denote by $\text{mesh}(P)$ the maximum of $|x_{j+1} - x_j|$ for $0 \leq j \leq n - 1$. By a Riemann sum $R(f, P, \xi)$ is meant the sum

$$\sum_{j=0}^{n-1} f(\xi_j)(x_{j+1} - x_j)$$

for some choice of $\xi_j \in [x_j, x_{j+1}]$ (for $0 \leq j \leq n - 1$), where $\xi = (\xi_0, \dots, \xi_{n-1})$. For $\eta > 0$ let $\Phi(\eta)$ be the compact subset of \mathbb{R} which is the closure of all

Riemann sums $R(f, P, \xi)$ with $\text{mesh}(P) < \eta$. Show that the Riemann integral $\int_a^b f$ can be alternatively defined as the number I such that the singleton $\{I\}$ as an element of \mathcal{C} is the limit of the map $\eta \mapsto \Phi(\eta)$ from \mathbb{R}_+ to \mathcal{C} as $\eta \rightarrow 0$, where \mathbb{R}_+ is the set of all positive real numbers. More precisely, given any $\varepsilon > 0$ there exists some $\delta > 0$ such that the Hausdorff distance $d_H(\Phi(\eta), \{I\})$ between $\Phi(\eta)$ and $\{I\}$ is $< \varepsilon$ when $0 < \eta < \delta$.

Problem 3 (Total Differentiability and Continuous Partial Differentiability).

Let $a_j < b_j$ be real numbers for $1 \leq j \leq n$. Let $f(x_1, \dots, x_n)$ be a real-valued functions of n real variables x_1, \dots, x_n on the open subset $\prod_{j=1}^n (a_j, b_j)$ of \mathbb{R}^n . For $(\xi_1, \dots, \xi_n) \in \prod_{j=1}^n (a_j, b_j)$ the notions of limit, continuity, total differentiability, and partial differentiability are defined as follows.

Definition of Limit. A real number L is said to be the *limit* of the function $f(x_1, \dots, x_n)$ at (ξ_1, \dots, ξ_n) if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x_1, \dots, x_n) - L| < \varepsilon$$

for

$$0 < \sqrt{(x_1 - \xi_1)^2 + \dots + (x_n - \xi_n)^2} < \delta.$$

The notation for the statement that L is the limit of $f(x_1, \dots, x_n)$ at (ξ_1, \dots, ξ_n) is

$$\lim_{(x_1, \dots, x_n) \rightarrow (\xi_1, \dots, \xi_n)} f(x_1, \dots, x_n) = L.$$

Definition of Continuity. The function $f(x_1, \dots, x_n)$ is said to be *continuous* at (ξ_1, \dots, ξ_n) if

$$\lim_{(x_1, \dots, x_n) \rightarrow (\xi_1, \dots, \xi_n)} f(x_1, \dots, x_n) = f(\xi_1, \dots, \xi_n).$$

Definition of Partial Differentiability. For $1 \leq \nu \leq n$ the ν -th *partial derivative* $D_\nu f(\xi_1, \dots, \xi_n)$ of the function $f(x_1, \dots, x_n)$ at (ξ_1, \dots, ξ_n) is defined as

$$\lim_{x_\nu \rightarrow \xi_\nu} \frac{f(x_1, \dots, x_n) - f(\xi_1, \dots, \xi_n)}{x_\nu - \xi_\nu}$$

if the limit exists. The ν -th partial derivative $D_\nu f(\xi_1, \dots, \xi_n)$ of the function $f(x_1, \dots, x_n)$ at (ξ_1, \dots, ξ_n) is also known as the *partial derivative* of $f(x_1, \dots, x_n)$ at (ξ_1, \dots, ξ_n) with respect to x_ν and is denoted by $\frac{\partial f}{\partial x_\nu}(\xi_1, \dots, \xi_n)$.

The notation $D_{\nu_k} D_{\nu_{k-1}} \cdots D_{\nu_2} D_{\nu_1} f$ means the result which is obtained by applying D_{ν_1} first to f and then applying D_{ν_2} first to $D_{\nu_1} f$, et cetera, and then finally applying D_{ν_k} to $D_{\nu_{k-1}} \cdots D_{\nu_2} D_{\nu_1} f$.

Definition of Total Differentiability. The function $f(x_1, \dots, x_n)$ is said to be *totally differentiable* at (ξ_1, \dots, ξ_n) if there exists a polynomial

$$A_1(x_1 - \xi_1) + \cdots + A_n(x_n - \xi_n) + f(\xi_1, \dots, \xi_n)$$

of degree ≤ 1 with real coefficients in the variables x_1, \dots, x_n such that for any $\varepsilon > 0$ there exists some $\delta > 0$ with the property that

$$\left| \frac{f(x_1, \dots, x_n) - (A_1(x_1 - \xi_1) + \cdots + A_n(x_n - \xi_n) + f(\xi_1, \dots, \xi_n))}{\sqrt{(x_1 - \xi_1)^2 + \cdots + (x_n - \xi_n)^2}} \right| < \varepsilon$$

for

$$0 < \sqrt{(x_1 - \xi_1)^2 + \cdots + (x_n - \xi_n)^2} < \delta.$$

In other words, total differentiability of $f(x_1, \dots, x_n)$ at (ξ_1, \dots, ξ_n) means approximability of $f(x_1, \dots, x_n)$ at (ξ_1, \dots, ξ_n) by a real polynomial of degree ≤ 1 in x_1, \dots, x_n to an order higher than the first. Total differentiability of $f(x_1, \dots, x_n)$ is also known simply as differentiability. The qualifier “total” is added to distinguish it from partial differentiability when it is necessary to avoid ambiguity.

(1) In the above definition of (total) differentiability, verify that

$$A_\nu = D_\nu f(\xi_1, \dots, \xi_n) \quad \text{for } 1 \leq \nu \leq n.$$

Moreover, show that the function $f(x_1, \dots, x_n)$ is continuous at (ξ_1, \dots, ξ_n) if $f(x_1, \dots, x_n)$ is (total) differentiable at (ξ_1, \dots, ξ_n) .

(2) Prove that the following two statements (i) and (ii) are equivalent.

- (i) $D_{\nu_1} \cdots D_{\nu_k} f$ exists and is continuous at every point (ξ_1, \dots, ξ_n) of $\prod_{j=1}^n (a_j, b_j)$ for any $k \in \mathbb{N}$ and any $1 \leq \nu_1, \dots, \nu_k \leq n$.
- (ii) $D_{\nu_1} \cdots D_{\nu_k} f$ is totally differentiable at every point (ξ_1, \dots, ξ_n) of $\prod_{j=1}^n (a_j, b_j)$ for any $k \in \mathbb{N}$ and any $1 \leq \nu_1, \dots, \nu_k \leq n$.

Problem 4 (Quaternionic Differentiability). When the standard basis of the vector space \mathbb{R}^4 over \mathbb{R} is denoted by $1, i, j, k$ with the multiplication table

\times	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

(where the (μ, ν) -entry of the table is the product of the μ -th element on the left-column as the first factor and the ν -th element on the top row as the second factor, for example, $ij = -ji = k$), the \mathbb{R} -vector space \mathbb{R}^4 endowed with such an additional multiplication operation is known as the set of all *quaternions* which is a 4-dimensional associative *non-commutative* algebra over \mathbb{R} and is denoted by \mathbb{H} . (The use of the letter \mathbb{H} for the set of all quaternions is to honor William Rowan Hamilton (1805 - 1865) who discovered the quaternions in 1843.) When a quaternion q is written as $q = t + ix + jy + kz$ with $t, x, y, z \in \mathbb{R}$,

- (i) its quaternion-conjugate \bar{q} is defined as $\bar{q} = t - ix - jy - kz$ with $\overline{qq'} = \bar{q}\bar{q}'$ for any other quaternion q' ;
- (ii) its absolute-value $|q|$ is defined as $|q| = \sqrt{q\bar{q}} = \sqrt{t^2 + x^2 + y^2 + z^2}$ with $|qq'| = |q| |q'|$ for any other quaternion q' ;
- (iii) its real part, denoted by $\text{Re } q$, is t ;
- (iv) its purely imaginary part, denoted by $\text{Pu } q$ is $ix + jy + kz$;
- (iv) its inverse, denoted by q^{-1} is $\frac{\bar{q}}{|q|^2}$;
- (v) its inner product with another quaternion $q' = t' + ix' + jy' + kz'$, denoted by $\langle q, q' \rangle$, is $\text{Re}(qq') = tt' + xx' + yy' + zz'$.

Let $a_j < b_j$ be real numbers with $0 \leq j \leq 3$. Let D be the set of all quaternions $q = t + ix + jy + kz$ with $(t, x, y, z) \in \prod_{j=0}^3 (a_j, b_j)$. Let $F(q) = f(q) + ig(q) + ju(q) + kv(q)$ be a quaternion-valued function on D (so that $F(q) \in \mathbb{H}$ and $f(q), g(q), u(q), v(q) \in \mathbb{R}$ for $q \in \mathbb{H}$). Let $q_0 = t_0 + ix_0 + jy_0 + kz_0 \in D$. The quaternion-valued function $F(q)$ is said to be *quaternionic-differentiable at $q = q_0$ on the left* with the quaternion

$D_{\text{left}} F(q_0)$ at its derivative (on the left) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|(q - q_0)^{-1} (F(q) - F(q_0)) - D_{\text{left}} F(q_0)| < \varepsilon$$

for $0 < |q - q_0| < \delta$.

Prove that $F(q)$ is quaternionic-differentiable at $q = q_0$ on the left if and only the following two conditions (a) and (b) are satisfied.

(a) Each of the four real-valued functions $f(q)$, $g(q)$, $u(q)$, $v(q)$ as functions of the four real variables t, x, y, z are totally differentiable at (t_0, x_0, y_0, z_0) .

(b) The 4×4 matrix

$$\begin{pmatrix} \left(\frac{\partial f}{\partial t}\right)_{q_0} & \left(\frac{\partial f}{\partial x}\right)_{q_0} & \left(\frac{\partial f}{\partial y}\right)_{q_0} & \left(\frac{\partial f}{\partial z}\right)_{q_0} \\ \left(\frac{\partial g}{\partial t}\right)_{q_0} & \left(\frac{\partial g}{\partial x}\right)_{q_0} & \left(\frac{\partial g}{\partial y}\right)_{q_0} & \left(\frac{\partial g}{\partial z}\right)_{q_0} \\ \left(\frac{\partial u}{\partial t}\right)_{q_0} & \left(\frac{\partial u}{\partial x}\right)_{q_0} & \left(\frac{\partial u}{\partial y}\right)_{q_0} & \left(\frac{\partial u}{\partial z}\right)_{q_0} \\ \left(\frac{\partial v}{\partial t}\right)_{q_0} & \left(\frac{\partial v}{\partial x}\right)_{q_0} & \left(\frac{\partial v}{\partial y}\right)_{q_0} & \left(\frac{\partial v}{\partial z}\right)_{q_0} \end{pmatrix}$$

of partial derivatives of $f(t, x, y, z)$, $g(t, x, y, z)$, $u(t, x, y, z)$, $v(t, x, y, z)$ with respect to t, x, y, z evaluated at (t_0, x_0, y_0, z_0) is of the form

$$\begin{pmatrix} T & -A & -B & -C \\ A & T & C & -B \\ B & -C & T & A \\ C & B & -A & T \end{pmatrix}$$

for some real numbers T, A, B, C .

In such a case show that $D_{\text{left}} F(q)$ is equal to the quaternion $T + iA + jB + kC$.

Problem 5 (Partially Differentiable Non-Continuous Function). Let $f(x, y)$ be the real-valued function defined on all of \mathbb{R}^2 as follows.

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

Show that $f(x, y)$ is not continuous at $(x, y) = (0, 0)$, yet both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point of \mathbb{R}^2 .