Math 55a Homework Assigned February 10, 2012
due February 17, 2012

Problem 1 *(Continuity of Integral in Parameter Variable and Fubini's Theorem).* Let $a < b$ be real numbers and $g(x), h(x)$ be continuous functions on $[a, b]$ such that $g(x) < h(x)$ on $[a, b]$. Let E be the subset

$$\left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, a \leq x \leq b,\ g(x) \leq y \leq h(x) \right\}$$

of $\mathbb{R}^2$. Let $f(x, y)$ be a continuous function on E (in the sense that for any $(x_0, y_0) \in E$ and any $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(x_0, y_0) - f(x, y)| < \varepsilon$ for $(x, y) \in E$ with $\left(|x - x_0|^2 + |y - y_0|^2\right)^{\frac{1}{2}} < \delta$). Let

$$F(x) = \int_{g(x)}^{h(x)} f(x, y) dy$$

for $a \leq x \leq b$. Let $\tilde{a} < a < b < \tilde{b}$ and $\tilde{c} < \tilde{d}$ be real numbers such that $\tilde{c} < \inf_{[a,b]} g$ and $\sup_{[a,b]} h < \tilde{d}$.

(a) Prove that $F(x)$ is continuous in x for $a \leq x \leq b$.

(b) Assume that $f \geq 0$ on E. Let $\varepsilon > 0$. Prove

$$\int_E f = \int_a^b F(x) dx$$

by showing that there exist a partition P

$$\tilde{a} = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = \tilde{b}$$

of $\left[\tilde{a}, \tilde{b}\right]$ and a partition Q

$$\tilde{c} = y_0 \leq y_1 \leq \cdots \leq y_{m-1} \leq y_m = \tilde{d}$$

of $\left[\tilde{c}, \tilde{d}\right]$ such that

$$\sum_{j \in \bar{J}} \left(\sup_{R_{jk} \cap E} f \right) |R_{jk}| < \varepsilon + \int_a^b F(x) dx$$

and

$$\sum_{j \in \underline{J}} \left(\inf_{R_{jk}} f \right) |R_{jk}| > -\varepsilon + \int_a^b F(x) dx,$$

where R_{jk} is the closed rectangle $[x_j, x_{j+1}] \times [y_k, y_{k+1}]$ for $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$ and $|R_{jk}|$ is the area of R_{jk} and

$$\bar{J} = \left\{ (j,k) \,\middle|\, 0 \leq j \leq n-1 \text{ and } 0 \leq k \leq m-1 \text{ with } R_{jk} \cap E \neq \emptyset \right\}$$

and

$$\underline{J} = \left\{ (j,k) \,\middle|\, 0 \leq j \leq n-1 \text{ and } 0 \leq k \leq m-1 \text{ with } R_{jk} \subset E \right\}.$$

Problem 2 *(Approximability by Polynomial of Degree $\leq k$ to Order $> k$).* Let $a < \xi < b$ be real numbers. Let k be a positive integer ≥ 2 and $f(x)$ be a function on (a,b). Assume that $f(x)$ is $(k-1)$-times differentiable at every point of (a,b).

(a) Show that if $f(x)$ is k-times differentiable at ξ, then there exists some polynomial $P(x) = \sum_{j=0}^{k} A_k \, (x-\xi)^j$ of degree $\leq k$ such that

$$\lim_{x \to \xi} \frac{|f(x) - P(x)|}{|x-\xi|^k} = 0.$$

(b) Let $a < \sigma < \tau < b$. If the derivative $f^{(k)}(\xi)$ of f at ξ of order k exists for every $\xi \in (a,b)$ and is continuous as a function of $\xi \in (a,b)$, show that for every $\xi \in (a,b)$ there there exists some polynomial

$$P_\xi(x) = \sum_{j=0}^{k} A_{\xi,k} \, (x-\xi)^j$$

of degree $\leq k$ in x with the property that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{|f(x) - P_\xi(x)|}{|x-\xi|^k} < \varepsilon$$

for $|x - \xi| < \delta$ and $\xi \in [\sigma, \tau]$.

Problem 3 *(Vector as Derivation).* Let $R > 0$ and let G be the set of points $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$ whose distance $\|x\| = \sqrt{\sum_{j=1}^n |x_j|^2}$ to the origin is $< R$. A real-valued function f on G is said to be *infinitely differentiable if partial derivatives*

$$\frac{\partial^{k_1+\cdots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

of f are continuous for all nonnegative integers $k_1, \cdots, k_n$. (Note that in this case all partial differentiations applied to f commute.)

(a) Let ℓ be a positive integer and $a = (a_1, \cdots, a_n)$ be a point of G. Let D be the set of points $x = (x_1, \cdots, x_n)$ of $\mathbb{R}^n$ such that the distance $\|x - a\|$ between x and a is $< R - \|a\|$ so that D is contained in G. Show that for any infinitely differentiable function f on G there exist some infinitely differentiable functions $h_{j_1 \cdots j_n}(x)$ on D (for $j_1, \cdots, j_n \in \mathbb{N} \cup \{0\}$ with $j_1 + \cdots + j_n = \ell$) such that

$$f(x) - \left(\sum_{\substack{k_1+\cdots+k_n \leq \ell-1 \\ k_1,\cdots,k_n \in \mathbb{N}\cup\{0\}}} \frac{(x_1 - a_1)^{k_1}}{k_1!} \cdots \frac{(x_n - a_n)^{k_n}}{k_n!} \frac{\partial^{k_1+\cdots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}(a) \right)$$

is equal to

$$\sum_{\substack{j_1+\cdots+j_n = \ell \\ j_1,\cdots,j_n \in \mathbb{N}\cup\{0\}}} (x_1 - a_1)^{j_1} \cdots (x_n - a_n)^{j_n} \, h_{j_1 \cdots j_n}(x)$$

for $x \in D$. Here $\mathbb{N}$ is the set of all positive integers.

Hint: For the case of $\ell = 1$, apply the fundamental theorem of calculus to the integral, over $0 \leq t \leq R - \|a\|$, of the derivative of the function

$$t \to f\left(a_1 + t\left(x_1 - a_1\right), \cdots, a_n + t\left(x_n - a_n\right)\right)$$

with respect to t. Use induction for a general ℓ.

(b) Let $\mathcal{F}$ be the set of all infinitely differentiable functions on G. Let $a \in G$. By a *derivation* of $\mathcal{F}$ at a is meant a function $\varphi : \mathcal{F} \to \mathbb{R}$ satisfying the following two properties.

(i) $\varphi(\lambda f + \mu g) = \lambda \varphi(f) + \mu \varphi(g)$ for $f, g \in \mathcal{F}$ and $\lambda, \mu \in \mathbb{R}$.

(ii) $\varphi(fg) = f(a)\varphi(g) + g(a)\varphi(f)$ for $f, g \in \mathcal{F}$.

Prove that every derivation φ of $\mathcal{F}$ at a is equal to the map

$$f \mapsto \sum_{j=1}^{n} A_j \frac{\partial f}{\partial x_j}(a),$$

where A_j is the real number $\varphi(x_j)$ for $1 \leq j \leq n$. Verify that conversely for any $A_1, \cdots, A_n \in \mathbb{R}$ the map

$$f \mapsto \sum_{j=1}^{n} A_j \frac{\partial f}{\partial x_j}(a)$$

from $\mathcal{F}$ to $\mathbb{R}$ is a derivation of $\mathcal{F}$ at a.

Hint: First show that any derivation maps a constant function to 0 and then apply the case $\ell = 2$ of Part(a).

Remark. Part (b) of this problem identifies (tangent) vectors of $\mathbb{R}^n$ at the point a with derivations of $\mathcal{F}$ at a when (tangent) vectors of $\mathbb{R}^n$ are defined by the coordinate-free formulation of directional derivatives. The significance is that a derivation of $\mathcal{F}$ at a is *algebraically* defined without the use of any differentiation when the set $\mathcal{F}$, endowed with the processes of addition, scalar multiplication, and multiplication of two elements, is given.

Problem 4 *(Lebesgue Measurable Subsets of* $\mathbb{R}$*).* Recall that a subset E of $\mathbb{R}$ is said to be *closed* if whenever all the terms of a convergent sequence are in E, its limit must be also in E. A subset G of $\mathbb{R}$ is said to be *open* if its complement $\mathbb{R} - G$ in $\mathbb{R}$ is closed.

(a) Verify that a subset G of $\mathbb{R}$ is open if and only if for every point a of E the open interval $(a - \varepsilon, a + \varepsilon)$ of length 2ε centered at a is contained in E for some $\varepsilon > 0$.

(b) Let G be an open subset of $\mathbb{R}$. Define a relation $\sim$ for elements x, y of G as follows.

$x \sim y$ if and only if there exist a finite number of bounded open intervals $I_1, \cdots, I_k$ inside G such that $x \in I_1$, $I_j \cap I_{j+1} \neq \emptyset$ for $1 \leq j \leq k-1$, and $y \in I_k$.

Verify that the relation $\sim$ is an *equivalence relation.* That is, (i) $x \sim x$ (reflexivity), (ii) $x \sim y$ implies $y \sim x$ (symmetry), and (iii) $x \sim y$ and $y \sim z$ imply $x \sim z$ (transitivity).

By using the equivalence classes of the equivalence relation $\sim$, show that G is a union of disjoint open intervals (with at most a countable number of such open intervals in the disjoint union).

Define the *Lebesque measure* $m(G)$ of an open subset G of $\mathbb{R}$ as $\sum_{j\in J}(b_j - a_j)$ when G is the disjoint union of open intervals (a_j, b_j) for $j \in J$, if such a sum is finite.

Define the *Lebesque measure* $m(F)$ of a closed set F of $\mathbb{R}$ as

$$\sup_{a<b \text{ in } \mathbb{R}} \left((b-a) - m\left((a,b) - F\right)\right)$$

if the supremum is finite. Note that the subset $(a,b) - F$ of $\mathbb{R}$ is open. In particular, if $F \subset (a,b)$ for some $a < b$ in $\mathbb{R}$, then

$$m(F) = (b-a) - m\left((a,b) - F\right)$$

so that the sum of the Lebesgue measure of the closed subset F of $\mathbb{R}$ and the Lebesgue measure of the open subset $(a,b) - F$ of $\mathbb{R}$ is equal to the Lebesque measure of the open subset (a,b) of $\mathbb{R}$.

(c) A subset S of $\mathbb{R}$ is said to be *Lebesgue measurable* (with finite Lebesgue measure) if for any $\varepsilon > 0$ there exist a closed subset F_ε of $\mathbb{R}$ and an open subset G_ε of $\mathbb{R}$ with $m(G_\varepsilon) < \infty$ such that $F_\varepsilon \subset S \subset G_\varepsilon$ and $m(G_\varepsilon) - m(F_\varepsilon) < \varepsilon$. In such a case the Lebesgue measure $m(S)$ of S is defined as

$$m(S) = \sup_{\substack{F \subset S \\ F \text{ closed}}} m(F) = \inf_{\substack{S \subset G \\ G \text{ open}}} m(G).$$

Prove that if S_1 and S_2 are subsets of $\mathbb{R}$ which are both Lebesgue measurable with finite Lebesgue measure , then $S_1 \cup S_2$ and $S_1 - S_2$ are also Lebesgue measurable with finite Lebesgue measure.

Remark. The purpose of this problem is show how the use of closed rectangles inside a subset E of $\mathbb{R}^2$ and the use of partitions for $\left[\tilde{a}, \tilde{b}\right]$ and $\left[\tilde{c}, \tilde{d}\right]$ with $E \subset \left(\tilde{a}, \tilde{b}\right) \times \left(\tilde{c}, \tilde{d}\right)$ in the definition of a double integral over E correspond to the use of a closed subset F of S and the use of an open superset G of S in the definition of the Lebesgue measure of a subset S of $\mathbb{R}$.