DIFFERENTIATION AND INTEGRATION IN HIGHER DIMENSION

Introduction. For our discussion of differentiation and integration in higher dimension, for notational simplicity we first confine ourselves to the case of two real variables. For a real-valued function $f(x, y)$ of many real variables x, y, to perform differentiation we can always keep one variable fixed and regard the function as a function of only the remaining variable. In that case the derivatives we obtain are partial derivatives

$$\begin{aligned}\frac{\partial f}{\partial x}(a,b) &= D_1 f(a,b) = \lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a},\\ \frac{\partial f}{\partial y}(a,b) &= D_2 f(a,b) = \lim_{y\to b}\frac{f(a,y)-f(a,b)}{y-b}\end{aligned}$$

without introducing any new theory. The geometric interpretation of the derivative in the one variable case as using a polynomial of degree at most 1 to approximate the function to an order higher than the first can be extended to the case of many variables. We are going to define the differentiation in the many variable case by using this geometric interpretation of derivative. After we introduce this definition of differentiation, we will do two results about it. One is to relate this differentiation to partial differentiation. Another is to discuss the question of commutativity of partial differentiation.

For differentiation in many variables, in the special case of a complex-valued function $w = f(z)$ of a complex variable $z = x + iy$ the definition of a derivative as the limit of the difference quotient

$$\lim_{z\to c}\frac{f(z)-f(c)}{z-c}$$

can be used to define the complex derivative $f'(c)$ of $w = f(z)$ at $z = c$. The existence of $f'(c)$ will be shown to be equivalent to the total differentiability of the real part $u(x, y)$ and the imaginary part $v(x, y)$ of f as functions of two real variables and a system of two equations involving the partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to x and y evaluated at $z = c$ which are known as the Cauchy-Riemann equations.

For integration involving functions of many variables, we can also fix all the variables except one and consider integration with respect to the remaining variable without introducing any new theory. To handle integration in all the variables altogether, we will go through again the route of the Riemann sum, the upper sum, the lower sum, the upper integral, and the lower integral with the modification that the domain of integration will be approximated by sandwiching between two unions of closed parallelepiped defined by partitions of intervals on coordinate axes. This definition of integral for all the variables at the same time will be related to the result obtained by integrating with respect to one variable at a time by Fubini's theorem.

Finally the higher-dimensional analogue of the Fundamental Theorem of Calculus will be Stokes's theorem.

Differentiability in Many Variables. Let $a < b$ and $c < b$ be real numbers and let $f(x, y)$ be a function on $(a, b) \times (c, d)$. Let $(\xi, \eta) \in (a, b) \times (c, d)$.

Definition of Total Differentiability. The function $f(x, y)$ is said to be *totally differentiable* at (ξ, η) if there exists a polynomial $Ax + By + C$ of degree ≤ 1 with real coefficients in the variables x, y such that for any $\varepsilon > 0$ there exists some $\delta > 0$ with the property that

$$\left|\frac{f(x,y) - (A(x-\xi) + B(y-\eta) + f(\xi,\eta))}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}\right| < \varepsilon$$

for

$$0 < \sqrt{(x-\xi)^2 + (y-\eta)^2} < \delta.$$

In other words, total differentiability of $f(x, y)$ at (ξ, η) means approximability of $f(x, y)$ at (ξ, η) by a real polynomial of degree ≤ 1 in x, y to an order higher than the first. Total differentiability of $f(x, y)$ is also known simply as differentiability. The qualifier "total" is added to distinguish it from partial differentiability when it is necessary to avoid ambiguity.

Specialization to Yield Partial Derivatives. When we specialize to the case of $y = b$, we conclude from

$$\left|\frac{f(x,y) - (A(x-\xi) + B(y-\eta) + f(\xi,\eta))}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}\right| < \varepsilon$$

for

$$0 < \sqrt{(x-\xi)^2 + (y-\eta)^2} < \delta$$

that

$$\left| \frac{f\left(x, b\right) - \left(A(x-\xi) + f(\xi,\eta)\right)}{x-\xi} \right| < \varepsilon$$

for $0 < |x-\xi| < \delta$. This implies that $\frac{\partial f}{\partial x}(\xi,\eta) = A$. Likewise the specialization to the case of $x = a$ yields $\frac{\partial f}{\partial y}(\xi,\eta) = B$. Thus the differentiability of $f(x,y)$ at (ξ,η) as a function of two real variables implies the existence of the two partial derivatives $\frac{\partial f}{\partial x}(\xi,\eta)$ and $\frac{\partial f}{\partial y}(\xi,\eta)$. The converse is not true. The existence of both partial derivatives $\frac{\partial f}{\partial x}(\xi,\eta)$ and $\frac{\partial f}{\partial y}(\xi,\eta)$ even for all points $(\xi,\eta) \in (a,b)\times(c,d)$ does not even imply the continuity of the function at a points of $(a,b)\times(c,d)$. An example for this statement is given in the homework assignment.

Total Differentiability from Continuous Partial Derivatives. On the other hand, when both partial derivatives $\frac{\partial f}{\partial x}(\xi,\eta)$ and $\frac{\partial f}{\partial y}(\xi,\eta)$ are continuous for all points $(\xi,\eta) \in (a,b)\times(c,d)$, the function $f(x,y)$ is totally differentiable at every point of $(a,b)\times(c,d)$. The continuity of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at (ξ,η) means that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$(*) \qquad |D_1 f(x,y) - D_1 f(\xi,\eta)| < \varepsilon \quad \text{for} \quad 0 < \sqrt{(x-\xi)^2 + (y-\eta)^2} < \delta.$$

Now choose arbitrarily x, y such that

$$0 < \sqrt{(x-\xi)^2 + (y-\eta)^2} < \delta.$$

First consider the case $x > \xi$ and $y > \eta$. We write

$$f(x,y) - f(\xi,\eta) = f(x,y) - f(x,\eta) + f(x,\eta) - f(\xi,\eta).$$

By applying the Mean-Value Theorem to the function $f(x,\eta) - f(\xi,\eta)$ of the variable x on the interval $[\xi, x]$, we conclude that there exists some $\sigma \in (\xi, x)$ such that

$$f(x,\eta) - f(\xi,\eta) = D_1 f(\sigma,\eta)(x-\xi).$$

By applying the Mean-Value Theorem to the function $f(x,y) - f(x,\eta)$ of the variable y on the interval $[\eta, y]$, we conclude that there exists some $\tau \in (\eta, y)$ such that

$$f(x,y) - f(x,\eta) = D_2 f(x,\tau)(y-\eta).$$

We obtain

$$\begin{aligned} f(x,y)-f(\xi,\eta) &= D_1f(\sigma,\eta)(x-\xi)+D_2f(x,\tau)(y-\eta) \\ &= D_1f(\xi,\eta)(x-\xi)+D_2f(\sigma,\eta)(y-\eta) \\ &\quad + \left(D_1f(\sigma,\eta)-D_1f(\xi,\eta)\right)(x-\xi)+\left(D_2f(x,\tau)-D_2f(\sigma,\eta)\right)(y-\eta) \end{aligned}$$

or

$$\begin{aligned} &f(x,y)-\left(f(\xi,\eta)+D_1f(\sigma,\eta)(x-\xi)+D_2f(x,\tau)(y-\eta)\right) \\ &\quad = \left(D_1f(\sigma,\eta)-D_1f(\xi,\eta)\right)(x-\xi)+\left(D_2f(x,\tau)-D_2f(\sigma,\eta)\right)(y-\eta). \end{aligned}$$

By (*)

$$\left|D_1f(\sigma,\eta)-D_1f(\xi,\eta)\right|<\varepsilon \quad \text{and} \quad \left|D_1f(x,\tau)-D_1f(\xi,\eta)\right|<\varepsilon.$$

Thus

$$\left|f(x,y)-\left(f(\xi,\eta)+D_1f(\sigma,\eta)(x-\xi)+D_2f(x,\tau)(y-\eta)\right)\right|<\varepsilon\left|x-\xi\right|+\varepsilon\left|y-\eta\right|,$$

which implies that

$$\left|\frac{f(x,y)-\left(f(\xi,\eta)+D_1f(\sigma,\eta)(x-\xi)+D_2f(x,\tau)(y-\eta)\right)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\right|<2\varepsilon.$$

The other three cases of $x<\xi, y>\eta$, $x>\xi, y<\eta$, $x<\xi, y<\eta$ and the cases when $x=\xi$ or $y=\eta$ can be handled analogously to enable us to conclude that

$$\left|\frac{f(x,y)-\left(f(\xi,\eta)+D_1f(\sigma,\eta)(x-\xi)+D_2f(x,\tau)(y-\eta)\right)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\right|<2\varepsilon$$

for $0<\sqrt{(x-\xi)^2+(y-\eta)^2}<\delta$. We can now conclude that $f(x,y)$ is differentiable at (ξ,η) as a function of two real variables x,y.

Complex Derivative. Let c be a complex number and $R>0$. Let $w=f(z)$ be a complex-valued function of a complex variable z on $|z-c|<R$. The function $f(z)$ is said to be *complex differentiable* with complex derivative $f'(c)$ at $z=c$ if foe very $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|\frac{f(z)-f(c)}{z-c}-f'(c)\right|<\varepsilon$$

for $0 < |z-c| < \delta$. Let $c = a + ib$ and $z = x + iy$. We write $f(z)$ also as $f(x,y)$ and let $u(x,y)$ be the real part of $f(x,y)$ and let $v(x,y)$ be the imaginary part of $f(x,y)$. Specializing to the special case $y = b$, we obtain

$$\left|\frac{f(x,b)-f(a,b)}{x-a} - f'(c)\right| < \varepsilon$$

for $0 < |x-a| < \delta$, because $z - c = x - a$ when $y = b$. This means that $\frac{\partial f}{\partial x}(c) = f'(c)$. Here $\frac{\partial f}{\partial x}(c)$ means $\frac{\partial u}{\partial x}(a,b) + i\frac{\partial u}{\partial x}(a,b)$. Specializing to the special case $x = a$, we obtain

$$\left|\frac{f(x,b)-f(a,b)}{i(y-b)} - f'(c)\right| < \varepsilon$$

for $0 < |y-b| < \delta$, because $z - c = i(y-b)$ when $x = a$. This means that $\frac{1}{i}\frac{\partial f}{\partial y}(c) = f'(c)$. Here $\frac{\partial f}{\partial y}(c)$ means $\frac{\partial u}{\partial y}(a,b) + i\frac{\partial u}{\partial y}(a,b)$. Since both $\frac{\partial f}{\partial x}(c)$ and $\frac{1}{i}\frac{\partial f}{\partial x}(c)$ are both equal to $f'(c)$, we have the equation

$$\frac{\partial f}{\partial x}(c) = \frac{1}{i}\frac{\partial f}{\partial y}(c), \tag{$\sharp$}$$

which is known as the Cauchy-Riemann equation. Taking the real and imaginary parts of both sides, we obtain

$$\begin{cases} \frac{\partial u}{\partial x}(a,b) = \frac{\partial v}{\partial y}(a,b) \\ \\ \frac{\partial v}{\partial x}(a,b) = -\frac{\partial u}{\partial y}(a,b), \end{cases} \tag{$\natural$}$$

which is another form of the Cauchy-Riemann equation.

Let

$$E_1(z) = \frac{f(z)-f(c)}{z-c} - f'(c).$$

We can rewrite it as

$$f(z) = f(c) + f'(c)(z-c) + E(z),$$

where $E(z) = E_1(z)(z-c)$. The statement that

$$\left|\frac{f(z)-f(c)}{z-c} - f'(c)\right| < \varepsilon$$

for $0 < |z - c| < \delta$ can be reformulated as $|E_1(z)| < \varepsilon$ for $0 < |z - c| < \delta$. This means that

$$f(z) = f(c) + f'(c)(z - c) + E(z),$$

with $\left|\frac{E(z)}{z-c}\right| < \varepsilon$ for $0 < |z - c| < \delta$. Let A be the real part of $f'(c)$ and B be the imaginary part of $f'(c)$. Taking the real parts of both sides of

$$f(z) = f(c) + f'(c)(z - c) + E(z),$$

we get

$$u(x, y) = u(a, b) + A(x - a) - B(y - b) + \operatorname{Re} E(z)$$

with $\left|\frac{\operatorname{Re} E(z)}{|z-c|}\right| < \varepsilon$ for $0 < |z - c| < \delta$. This means that $u(x, y)$ is differentiable at $(x, y) = (a, b)$ as a function of two real variables x, y. Likewise, taking the real parts of both sides of

$$f(z) = f(c) + f'(c)(z - c) + E(z),$$

we get

$$v(x, y) = u(a, b) + B(x - a) + C(y - b) + \operatorname{Im} E(z)$$

with $\left|\frac{\operatorname{Im} E(z)}{|z-c|}\right| < \varepsilon$ for $0 < |z - c| < \delta$. This means that $v(x, y)$ is differentiable at $(x, y) = (a, b)$ as a function of two real variables x, y.

To summarize, we have the following statement.

If the complex derivative $f'(c)$ of the complex-valued function $w = f(z)$ of a complex variable z exists at the point $z = c$, then (i) the real part $u(x, y)$ and the imaginary part $v(x, y)$ of $f(z)$ are differentiable at $(x, y) = (a, b)$ as functions of two real variables x, y and (ii) the Cauchy-Riemann equation ($\natural$) or its equivalent form ($\sharp$) is satisfied.

The converse of this statement is also true. Suppose (i) the real part $u(x, y)$ and the imaginary part $v(x, y)$ of $f(z)$ are differentiable at $(x, y) = (a, b)$ as functions of two real variables x, y and (ii) the Cauchy-Riemann equation ($\natural$) or its equivalent form ($\sharp$) is satisfied. We are going to prove that $f'(c)$ exists. Since the real part $u(x, y)$ and the imaginary part $v(x, y)$ of $f(z)$ are differentiable at $(x, y) = (a, b)$ as functions of two real variables x, y, it follows that

$$\begin{aligned} u(x,y) &= u(a,b) + \frac{\partial u}{\partial x}(a,b)(x-a) + \frac{\partial u}{\partial y}(a,b)(y-b) + \hat{E}(x,y), \\ v(x,y) &= v(a,b) + \frac{\partial v}{\partial x}(a,b)(x-a) + \frac{\partial v}{\partial y}(a,b)(y-b) + \hat{\hat{E}}(x,y), \end{aligned}$$

with

$$\lim_{z\to c} \frac{\hat{E}(x,y)}{z-c} = 0 \quad \text{and} \quad \lim_{z\to c} \frac{\hat{\hat{E}}(x,y)}{z-c} = 0.$$

Let $A = \frac{\partial u}{\partial x}(a,b)$ and $B = \frac{\partial v}{\partial x}(a,b)$. From the Cauchy-Riemann equation $(\natural)$ it follows that $A = \frac{\partial v}{\partial y}(a,b)$ and $B = -\frac{\partial u}{\partial y}(a,b)$. After we multiply the second equation of

$$\begin{aligned} u(x,y) &= u(a,b) + A(x-a) - B(y-b) + \hat{E}(x,y), \\ v(x,y) &= v(a,b) + B(x-a) + A(y-b) + \hat{\hat{E}}(x,y) \end{aligned}$$

by i and adding it to the first equation, we obtain

$$f(z) = f(c) + (A+iB)(x-a) + (-B+iA)(y-b) + \hat{E}(x,y) + i\hat{\hat{E}}(x,y),$$

which is the same as

$$f(z) = f(c) + (A+iB)(z-c) + \hat{E}(x,y) + i\hat{\hat{E}}(x,y).$$

From

$$\lim_{z\to c} \frac{\hat{E}(x,y) + i\hat{\hat{E}}(x,y)}{z-c} = 0,$$

it follows that

$$\lim_{z\to c} \left(\frac{f(z)-f(c)}{z-c} - (A+iB) \right) = \lim_{z\to c} \frac{\hat{E}(x,y) + i\hat{\hat{E}}(x,y)}{z-c} = 0,$$

which means that $f'(c)$ exists.

We would like to give two interpretations of the Cauchy-Riemann equation. The first one concerns the definition of differentiability as approximability by polynomial of degree ≤ 1. For complex differentiability the approximation is by a polynomial $f(c) + f'(c)(z-c)$ of degree ≤ 1 in a complex variable z. The differentiability of $u(x,y)$ and $v(x,y)$ at $(x,y)=(a,b)$ means by polynomials $u(a,b)+\frac{\partial u}{\partial x}(a,b)(x-a)+\frac{\partial u}{\partial y}(a,b)(y-b)$ and $v(a,b)+\frac{\partial v}{\partial x}(a,b)(x-a)+\frac{\partial v}{\partial y}(a,b)(y-b)$ of degree ≤ 1 in the two real variables respectively. Putting together these two approximation we have the approximation of $u(x,y)+iv(x,y)$ by the polynomial

$$(u(a,b)+iv(a,b))+\left(\frac{\partial u}{\partial x}(a,b)+i\frac{\partial v}{\partial x}(a,b)\right)(x-a)+\left(\frac{\partial u}{\partial y}(a,b)+i\frac{\partial v}{\partial y}(a,b)\right)(y-b)$$

of degree ≤ 1 in the two real variables x, y with complex coefficients. To go from the differentiability of $u(x,y)$ and $v(x,y)$ at $(x,y)=(a,b)$ to the complex differentiability of $f(z)$ at $z=c$, what is needed to say that the polynomial

$$(u(a,b)+iv(a,b))+\left(\frac{\partial u}{\partial x}(a,b)+i\frac{\partial v}{\partial x}(a,b)\right)(x-a)+\left(\frac{\partial u}{\partial y}(a,b)+i\frac{\partial v}{\partial y}(a,b)\right)(y-b)$$

of degree ≤ 1 in the two real variables x, y with complex coefficients is a polynomial of degree ≤ 1 in the complex variable $z=x+iy$. This means that we need the identidy

$$\frac{\partial u}{\partial x}(a,b)+i\frac{\partial v}{\partial x}(a,b)=\frac{1}{i}\left(\frac{\partial u}{\partial y}(a,b)+i\frac{\partial v}{\partial y}(a,b)\right).$$

Its real and imaginary parts give precisely the Cauchy-Riemann equation ($\natural$).

Another interpretation of the Cauchy-Riemann equation is the condition for a $\mathbb{R}$-linear transformation T of the 2-dimensional $\mathbb{R}$-vector space $\mathbb{R}^2$ to itself to be a $\mathbb{C}$-linear transformation of the 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}$ to itself. Let

$$J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

which as the $\mathbb{R}$-linear transformation of the 2-dimensional $\mathbb{R}$-vector space $\mathbb{R}^2$ to itself satisfies $J^2=I_2$ and can serve as the scalar multiplication by i to make $\mathbb{R}^2$ a vector space over $\mathbb{C}$. Let

$$T=\begin{pmatrix}A & B\\ C & D\end{pmatrix}.$$

Then T commutes with J if and only if $A=D$ and $C=-B$, because

$$TJ=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}B & -A\\ D & -C\end{pmatrix}$$

and

$$JT = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}.$$

When

$$T = \begin{pmatrix} \frac{\partial u}{\partial x}(a,b) & \frac{\partial u}{\partial y}(a,b) \\ & \\ \frac{\partial v}{\partial x}(a,b) & \frac{\partial v}{\partial y}(a,b) \end{pmatrix},$$

the Cauchy-Riemann equation ($\natural$) is precisely the condition for the $\mathbb{R}$-linear transformation T of the 2-dimensional $\mathbb{R}$-vector space $\mathbb{R}^2$ to itself to be a $\mathbb{C}$-linear transformation of the 1-dimensional $\mathbb{C}$-vector space $\mathbb{C}$ to itself. The reason for this interpretation is that for a polynomial of degree ≤ 1 in the two real variables x, y with complex coefficients to be a polynomial of degree ≤ 1 in the complex variable $z = x + iy$ the obstruction is the homogeneous linear part which can be interpreted as defining a linear transformation of vector spaces.

TO BE CONTINUED ...