Math212a1419

The discrete and the essential spectrum,
Compact and relatively compact operators,
The Schrödinger operator.

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The main results of today’s lecture are about the Schrödinger operator $H = H_0 + V$. They are:

- If $V$ is bounded and $V \to 0$ as $x \to \infty$ then
  \[ \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0). \]

So this is a generalization of the results of the lecture on the square well. I will review some definitions and facts about the discrete and the essential spectrum.

- If $V \to \infty$ as $x \to \infty$ then $\sigma_{\text{ess}}(H)$ is empty.

Both of these results will require a jazzed up version of Weyl’s theorem on the stability of the essential spectrum as worked out in lecture on the square well. This in turn will require some results on compact operators that we did not do in Lecture 3.
1. The discrete and the essential spectrum.

2. Finite rank operators.
   - A normal form for norm limits of finite rank operators.

3. Compact operators.

4. Hilbert Schmidt operators
   - Hilbert Schmidt integral operators.
   - Hilbert Schmidt operators in general.

5. Weyl’s theorem on the essential spectrum.
   - Applications to Schrödinger operators.
The discrete spectrum and the essential spectrum.

Let $H$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $\sigma = \sigma(H) \subset \mathbb{R}$ denote is spectrum.

Recall that the **discrete spectrum** of $H$ is defined to be those eigenvalues $\lambda$ of $H$ which are of finite multiplicity and are also isolated points of the spectrum. This latter condition says that there is some $\epsilon > 0$ such that the intersection of the interval $(\lambda - \epsilon, \lambda + \epsilon)$ with $\sigma$ consists of the single point $\{\lambda\}$. The discrete spectrum of $H$ will be denoted by $\sigma_d(H)$ or simply by $\sigma_d$ when $H$ is fixed in the discussion.
The discrete spectrum and the essential spectrum.

Let $H$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let
\[ \sigma = \sigma(H) \subset \mathbb{R} \] denote its spectrum. Recall that the **discrete spectrum** of $H$ is defined to be those eigenvalues $\lambda$ of $H$ which are of finite multiplicity and are also isolated points of the spectrum. This latter condition says that there is some $\epsilon > 0$ such that the intersection of the interval $(\lambda - \epsilon, \lambda + \epsilon)$ with $\sigma$ consists of the single point $\{\lambda\}$. The discrete spectrum of $H$ will be denoted by $\sigma_d(H)$ or simply by $\sigma_d$ when $H$ is fixed in the discussion. The complement in $\sigma_d(H)$ in $\sigma(H)$ is called the **essential spectrum** of $H$ and is denoted by $\sigma_{\text{ess}}(H)$ or simply by $\sigma_{\text{ess}}$ when $H$ is fixed in the discussion.
Characterizing the discrete spectrum.

If $\lambda \in \sigma_d(H)$ then for sufficiently small $\epsilon > 0$ the spectral projection $P = P((\lambda - \epsilon, \lambda + \epsilon))$ has the property that it commutes with $H$ and the restriction of $H$ to the image of $P$ has only $\lambda$ in its spectrum and hence $P(\mathcal{H})$ is finite dimensional, since the multiplicity of $\lambda$ is finite by assumption.
Conversely, suppose that $\lambda \in \sigma(H)$ and that $P(\lambda - \epsilon, \lambda + \epsilon)$ is finite dimensional. This means that in a multiplication spectral representation of $H$, the subset $S_{(\lambda-\epsilon,\lambda+\epsilon)}$ of

$$S = \sigma \times \mathbb{N}$$

consisting of all $\{(s, n) | \lambda - \epsilon < s < \lambda + \epsilon\}$ has the property that

$$L_2(S_{(\lambda-\epsilon,\lambda+\epsilon)}, \, d\mu)$$

is finite dimensional.
$L_2(S_{(\lambda-\epsilon, \lambda+\epsilon)}, d\mu)$

is finite dimensional. If we write

$$S_{(\lambda-\epsilon, \lambda+\epsilon)} = \bigcup_{n \in \mathbb{N}} (\lambda - \epsilon, \lambda + \epsilon) \times \{n\}$$

then since

$$L_2(S_{(\lambda-\epsilon, \lambda+\epsilon)}) = \bigoplus_{n} L_2((\lambda - \epsilon, \lambda + \epsilon), \{n\})$$

we conclude that all but finitely many of the summands on the right are zero, which implies that for all but finitely many $n$ we have

$$\mu((\lambda - \epsilon, \lambda + \epsilon) \times \{n\}) = 0.$$
For all but finitely many \( n \) we have

\[
\mu \left( (\lambda - \epsilon, \lambda + \epsilon) \times \{n\} \right) = 0.
\]

For each of the finite non-zero summands, we can apply the case \( N = 1 \) of the following lemma:

**Lemma**

Let \( \nu \) be a measure on \( \mathbb{R}^N \) such that \( L^2(\mathbb{R}^N, \nu) \) is finite dimensional. Then \( \nu \) is supported on a finite set in the sense that there is some finite set of \( m \) distinct points \( x_1, \ldots, x_m \) each of positive measure and such that the complement of the union of these points has \( \nu \) measure zero.
Proof.

Partition $\mathbb{R}^N$ into cubes whose vertices have all coordinates of the form $t/2^r$ for a vector $t$ with integer coordinates and with integer $r$ and so that this is a disjoint union. The corresponding decomposition of the $L_2$ spaces shows that only finitely many of these cubes have positive measure, and as we increase $r$ the cubes with positive measure are nested downward, and can not increase in number beyond $n = \dim L_2(\mathbb{R}^N, \nu)$. Hence they converge in measure to at most $n$ distinct points each of positive $\nu$ measure and the complement of their union has measure zero.
We conclude from this lemma that there are at most finitely many points \((s_r, k)\) with \(s_r \in (\lambda - \epsilon, \lambda + \epsilon)\) which have finite measure in the multiplicative spectral representation of \(H\), each giving rise to an eigenvector of \(H\) with eigenvalue \(s_r\), and the complement of these points has measure zero. This shows that \(\lambda \in \sigma_d(H)\). We have proved

**Proposition.**

\[
\lambda \in \sigma(H) \text{ belongs to } \sigma_d(H) \text{ if and only if there is some } \epsilon > 0 \text{ such that } P((\lambda - \epsilon, \lambda + \epsilon))(\mathcal{H}) \text{ is finite dimensional.}
\]
Characterizing the essential spectrum

This is simply the contrapositive of the preceding proposition:

Proposition.

\[ \lambda \in \sigma(H) \text{ belongs to } \sigma_{\text{ess}}(H) \text{ if and only if for every } \epsilon > 0 \text{ the space } \]

\[ P((\lambda - \epsilon, \lambda + \epsilon))(\mathcal{H}) \]

is infinite dimensional.
Finite rank operators.

An operator $K \in \mathcal{L}(\mathcal{H})$ is said to be of finite rank if its range, $\text{Ran}(K)$, is finite dimensional, in which case the dimension of its range is called the rank of $K$. 
Finite rank operators.

An operator $K \in \mathcal{L}(\mathcal{H})$ is said to be of finite rank if its range, $\text{Ran}(K)$, is finite dimensional, in which case the dimension of its range is called the rank of $K$. If $\psi_1, \ldots, \psi_n$ is an orthonormal basis of $\text{Ran}(K)$ and $\psi$ is any element of $\mathcal{H}$ then

$$K\psi = \sum_{j=1}^{n} (K\psi, \psi_j) \psi_j.$$ 

If we set $\phi_j := K^* \psi_j$ we can write this last equation as

$$K\psi = \sum_{j=1}^{n} (\psi, \phi_j) \psi_j.$$
The elements $\phi_j = K^* \psi_j$ are linearly independent, for if
\[
\sum a_j \phi_j = 0
\]
then
\[
\left( \sum \bar{a}_j \psi_j, Kf \right) = \sum_{j,k} (\bar{a}_j \psi_j, (f, \phi_k) \psi_k) = (f, \sum_j a_j \phi_j) = 0
\]
for all $f \in \mathcal{H}$ which can not happen unless all the $a_j = 0$ since $\psi_1, \ldots, \psi_n$ is an orthonormal basis of $\text{Ran}(K)$. So
\[
K \psi = \sum_{j=1}^n (\psi, \phi_j) \psi_j
\]
is the most general expression of a finite rank operator.
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is the most general expression of a finite rank operator. Then \( K^* \) is given by

\[ K^* f = \sum (f, \psi_j) \phi_j \]

as can immediately be checked, and so the adjoint of a finite rank operator is of finite rank.
The most general expression of a finite rank operator is

\[ K\psi = \sum_{j=1}^{n} (\psi, \phi_j) \psi_j \]

is the most general expression of a finite rank operator. Then \( K^* \) is given by

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If \( K \) is an operator of finite rank and \( B \) is a bounded operator then clearly \( BK \) and \( KB \) are of finite rank.
A compact operator is the norm limit of finite rank operators.

**Proof.** Let $K$ be a compact operator and $\{\phi_j\}$ an orthonormal basis of $H$. Let $K_n := \sum_{j=1}^{n} (\cdot, \phi_j)K\phi_j$. 
A compact operator is the norm limit of finite rank operators.

**Proof.** Let $K$ be a compact operator and $\{\phi_j\}$ an orthonormal basis of $\mathcal{H}$. Let $K_n := \sum_{j=1}^{n} (\cdot, \phi_j) K \phi_j$. So $K = K_n$ on the space spanned by the first $n$ $\phi_j$’s. Let $\mathcal{H}_n$ be the orthocomplement of this space, i.e. the space spanned by $\phi_{n+1}, \phi_{n+2}, \ldots$. So

$$\|K - K_n\| = \sup_{\psi \in \mathcal{H}_n, \|\psi\|=1} \|K\psi\|.$$
\[ \| K - K_n \| = \sup_{\psi \in \mathcal{H}_n, \| \psi \| = 1} \| K \psi \|. \]

The numbers \( \| K - K_n \| \) are decreasing and hence tend to some limit, \( \ell \), and so we can find a sequence \( \psi_n \) with \( \psi_n \in \mathcal{H}_n, \| \psi_n \| = 1 \) and \( \lim \| K \psi_n \| \geq \ell \). The \( \psi_n \) converge weakly to 0, and hence so do the \( K \psi_n \). We can choose a (strongly) convergent subsequence which must therefore converge to 0, and so \( \ell = 0 \). \qed
The converse.

Over the next few slides we will assume that $K$ is the norm limit of finite rank operators.

**Theorem**

*Suppose that $K$ is the norm limit of finite rank operators and is self-adjoint. Then $\sigma_{\text{ess}}(K) \subset \{0\}$ with equality if and only if $\mathcal{H}$ is infinite dimensional.*

We must show that if $\lambda \neq 0$ then for $0 < \epsilon < |\lambda|$ the image of the spectral projection $P_K(\lambda - \epsilon, \lambda + \epsilon)$ is finite dimensional. Without loss of generality we may assume that $\lambda > 0$. 
Proof.

Let $K_n$ be a sequence of finite rank operators such that $\|K - K_n\| \leq 1/n$. If the image of the spectral projection $P_K(\lambda - \epsilon, \lambda + \epsilon)$ is infinite dimensional, we can find $\psi_n$ in this image with $\|\psi_n\| = 1$ and $K_n\psi_n = 0$. Then

$$\frac{1}{n} \geq |((K - K_n)\psi_n, \psi_n)| = |(K\psi_n, \psi_n)| \geq \lambda - \epsilon,$$

a contradiction for large enough $n$.  

\[\square\]
Singular values

**Theorem**

Suppose that $K$ is the norm limit of finite rank operators. There exist orthonormal sets $\{\psi_j\}$ and $\{\phi_j\}$ and positive numbers $s_j = s_j(K)$ (called the **singular values** of $K$) such that

$$K = \sum_j s_j(\cdot, \phi_j) \psi_j, \quad K^* = \sum_j s_j(\cdot, \psi_j) \phi_j.$$ 

There are either finitely many $s_j$ (if $K$ is of finite rank) or they converge to 0.
We know that $K^*K$ is the norm limit of finite rank operators and is self-adjoint so we can apply the preceding theorem. We also know that $K^*K\psi = 0 \Rightarrow (\psi, K^*K\psi) = \|K\psi\|^2 = 0$ so $\ker(K^*K) = \ker K$.

Let $s_j^2$ be the non-zero eigenvalues of $K^*K$ (arranged in decreasing order) and $\phi_j$ a corresponding orthonormal set.
We know that $K^*K$ is the norm limit of finite rank operators and is self-adjoint so we can apply the preceding theorem. We also know that $K^*K\psi = 0 \Rightarrow (\psi, K^*K\psi) = \|K\psi\|^2 = 0$ so $\ker(K^*K) = \ker K$.

Let $s_j^2$ be the non-zero eigenvalues of $K^*K$ (arranged in decreasing order) and $\phi_j$ a corresponding orthonormal set.

If $\zeta$ is orthogonal to all the $\phi_j$ then we know by the spectral theorem that $\|K^*K\zeta\| \leq \epsilon\|\zeta\|$ for any $\epsilon > 0$ and hence $K^*K\zeta = 0$ which implies that $K\zeta = 0$.

We can now proceed to the proof of the theorem:
Proof.

For any \( \psi \in \mathcal{H} \) we have

\[
\psi = \sum_j (\psi, \phi_j) \phi_j + \zeta, \quad \zeta \in \ker(K^*K) = \ker(K).
\]

Let \( \psi_j := s_j^{-1} K \phi_j \). Then \( K \psi = \sum_j s_j (\psi, \phi_j) \psi_j \). To see that the \( \psi_j \) are orthonormal, observe that

\[
(\psi_j, \psi_k) = s_j^{-1} s_k^{-1} (K \phi_j, K \phi_k) = s_j^{-1} s_k^{-1} (K^* K \phi_j, \phi_k)
\]

\[
= s_j s_k^{-1} (\phi_j, \phi_k).
\]

The formula for \( K^* \) follows by taking adjoints. \( \square \)
From \( K = \sum_j s_j(\cdot, \phi_j)\psi_j \) we see that \( K\phi_j = s_j\psi_j \) and from the formula for \( K^* \) we see that \( K^*\psi_j = s_j\phi_j \). Hence

\[
K^*K\phi_j = s_j^2\phi_j \quad \text{and} \quad KK^*\psi_j = s_j^2\psi_j.
\]
Characterizations of compact operators.

Let $K$ be a bounded operator on $\mathcal{H}$.

**Theorem**

The following statements are equivalent:

1. $K$ is the norm limit of finite rank operators.
2. If $A_n \in \mathcal{L}(\mathcal{H})$ with $A_n \to A$ strongly then $A_n K \to AK$ in norm.
3. $\psi_n \to \psi$ weakly implies that $K \psi_n \to K \psi$ in norm.
4. $K$ is compact.

We have already proved that 4. implies 1.
1. ⇒ 2.

Replacing $A_n$ by $A_n - A$ we may assume that $A = 0$. By the uniform boundedness principle, there is some $M$ such that $\|A_n\| \leq M$. If $K_j$ is a sequence of finite rank operators with $K_j \to K$ in norm, then

$$\|A_n K\| \leq \|A_n (K - K_j)\| + \|A_n K_j\| \leq M \|K - K_j\| + \|A_n K_j\|$$

so we may assume in proving 1. ⇒ 2. that $K$ is of finite rank.
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so we may assume in proving 1. ⇒ 2. that $K$ is of finite rank.

Write $K$ as the finite sum

$$K = \sum_{j=1}^{N} s_j (\cdot, \phi_j) \psi_j.$$

Then

$$\|A_nK\|^2 \leq \sup_{\|\psi\|=1} \sum_{1}^{N} s_j |(\psi, \phi_j)|^2 \|A_n\psi_j\|^2 \leq \sum_{1}^{N} s_j \|A_n\psi_j\|^2 \to 0.$$
2. ⇒ 3.

Again replacing ψₙ by ψₙ − ψ we may assume that ψₙ → 0 weakly. Choose any φ with ∥φ∥ = 1 and consider the rank one operators  

\[ Aₙ := (\cdot, ψₙ)φ. \]

Then \( Aₙ \to 0 \) strongly. Hence, by 2.,

\[
\| AₙK^* \| = \sup_{\|ζ\|=1} |(K^*ζ, ψₙ)| \|φ\| \to 0.
\]

But

\[
\sup_{\|ζ\|=1} |(K^*ζ, ψₙ)| = \|Kψₙ\|
\]

so \( \|Kψₙ\| \to 0. \)
3. $\Rightarrow$ 4.

If $\psi_n$ is bounded, it has a weakly convergent subsequence. (Just choose an orthonormal basis $\phi_j$ and then a subsequence that all the $(\psi_{n_k}, \phi_j)$ converge.) Then apply 3. to this subsequence. This completes the proof of the theorem.
Hilbert-Schmidt operators aka operators given by $L_2$ integral kernels.

Let $(M, \mu)$ be a measure space and $K = K(x, y) \in L_2(M \times M, \mu \otimes \mu)$. For $\psi \in L_2(M, \mu)$ define $K\psi$ by

$$K\psi(x) := \int_M K(x, y)\psi(y)d\mu(y).$$

So by Cauchy-Schwarz,

$$\int_M |K\psi(x)|^2d\mu(x) = \int_M \left| \int_M K(x, y)\psi(y)d\mu(y) \right|^2 d\mu(x) \leq \int_M \left( \int_M |K(x, y)|^2d\mu(y) \right) \left( \int_M |\psi(y)|^2d\mu(y) \right) d\mu(x) = \left( \int_{M \times M} |K(x, y)|^2d\mu(x)d\mu(y) \right) \left( \int_M |\psi(y)|^2d\mu(y) \right).$$
Hilbert Schmidt integral operators.

\[
\int_M |K\psi(x)|^2 \leq \left( \int_{M \times M} |K(x, y)|^2 d\mu(x) d\mu(y) \right) \left( \int_M |\psi(y)|^2 d\mu(y) \right).
\]

We see that $K$ is a bounded operator on $L_2(M, \mu)$.
Hilbert Schmidt integral operators.

Hilbert Schmidt operators are compact.

Let $\phi_j$ be an orthonormal basis of $L_2(M, \mu)$ so that $\phi_i(x)\phi_j(y)$ is an orthonormal basis of $L_2(M \times M, \mu \otimes \mu)$. 

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Hilbert Schmidt integral operators.

Hilbert Schmidt operators are compact.

Let $\phi_j$ be an orthonormal basis of $L_2(M, \mu)$ so that $\phi_i(x)\phi_j(y)$ is an orthonormal basis of $L_2(M \times M, \mu \otimes \mu)$. We can expand $K$ in terms of this orthonormal basis:

$$K(x, y) = \sum_{i,j} c_{i,j} \phi_i(x) \phi_j(y), \quad c_{i,j} = \int K(x, y) \overline{\phi_i(x)} \overline{\phi_j(y)} d\mu(x) d\mu(y)$$

so $c_{i,j} = (K \overline{\phi_j}, \phi_i)$, and

$$K\psi(x) = \sum_{i,j} c_{i,j} (\psi, \overline{\phi_j}) \phi_i(x).$$
\[ K\psi(x) = \sum_{i,j} c_{i,j}(\psi, \bar{\phi}_j)\phi_i(x). \]

Since \( \sum_{i,j} |c_{i,j}|^2 < \infty \) we see that \( K \) can be approximated in norm by finite rank operators and hence is compact.
When is a compact operator Hilbert-Schmidt?

Let $\mathcal{H} = L_2(M, \mu)$ and $K$ a compact operator on $\mathcal{H}$ with $\{s_j\}$ its singular values.

**Proposition.**

$K$ is Hilbert Schmidt if and only if

$$\sum_j s_j^2 < \infty$$

in which case $K$ has an integral kernel with

$$\sum_j s_j^2 = \int_M |K(x, y)|^2 d\mu(x) d\mu(y).$$
Proof.

We know that \( K = \sum_j s_j(\cdot, \phi_j)\psi_j \), for orthonormal sets \( \phi_j \) and \( \psi_j \). Replacing this (possibly infinite) sum by a finite sum gives

\[
K_N := \sum_{j=1}^{N} s_j(\cdot, \phi_j)\psi_j
\]

as an approximating finite rank operator to \( K \). Now \( K_N \) has the integral kernel

\[
K_N(x, y) = \sum_{j=1}^{N} s_j \overline{\phi_j(y)}\psi_j(x).
\]

Furthermore,

\[
\int_{M \times M} |K_N(x, y)|^2 d\mu(x) d\mu(y) = \sum_{j=1}^{N} s_j^2.
\]

If one side of

\[
\sum_j s_j^2 = \int_M |K(x, y)|^2 d\mu(x) d\mu(y)
\]

converges, so does the other.
A more general definition of Hilbert Schmidt operator.

Let us call a compact operator on a Hilbert space $\mathcal{H}$ **Hilbert Schmidt** if $\sum_j s_j(K)^2 < \infty$. In case $\mathcal{H} = L_2(M, \mu)$ we know that this coincides with our earlier definition.
Hilbert Schmidt operators in general.

### A more general definition of Hilbert Schmidt operator.

Let us call a compact operator on a Hilbert space $\mathcal{H}$ **Hilbert Schmidt** if $\sum_j s_j(K)^2 < \infty$. In case $\mathcal{H} = L_2(M, \mu)$ we know that this coincides with our earlier definition.

Since every Hilbert space is isomorphic to some $L_2(M, \mu)$ we see that the Hilbert Schmidt operators together with the norm

$$
\|K\|_2 = \left(\sum_j s_j(K)^2\right)^{\frac{1}{2}}
$$

form a Hilbert space isomorphic to $L_2(M \times M, \mu \otimes \mu)$. 

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Another characterization of Hilbert-Schmidt operators.

Proposition.

A compact operator $K$ is Hilbert Schmidt if and only if there is an orthonormal basis $\{\zeta_n\}$ with

$$\sum_{n} \|K\zeta_n\|^2 < \infty$$

in which case this holds for any orthonormal basis. Furthermore

$$\|K\|^2_2 = \sum_{n} \|K\zeta_n\|^2.$$
Hilbert Schmidt operators in general.

Proof.

We have

\[ \sum_n \| K \zeta_n \|^2 = \sum_{n,j} |(K \zeta_n, \psi_j)|^2 = \sum_{n,j} |\zeta_n, K^* \psi_j|^2 = \]

\[ = \sum_n \| K^* \psi_n \|^2 = \sum_j s_j(K)^2. \]

All terms in the sums are positive so they converge or diverge together and this holds for any orthonormal basis.
An important example.

Let $\mathcal{H} = L_2(\mathbb{R}^n)$ and let $\mathcal{F}$ denote the Fourier transform (extended to $\mathcal{H}$ via Plancherel). For any function $g$ we will let $g(x)$ (bad notation) denote the operator of multiplication by $g$ on $\mathcal{H}$ (where defined).
An important example.

Let $\mathcal{H} = L_2(\mathbb{R}^n)$ and let $\mathcal{F}$ denote the Fourier transform (extended to $\mathcal{H}$ via Plancherel). For any function $g$ we will let $g(x)$ (bad notation) denote the operator of multiplication by $g$ on $\mathcal{H}$ (where defined).

For a function $f$ we will let $f(p)$ denote the operator

$$\psi \mapsto \mathcal{F}^{-1} (f(p)(\mathcal{F}\psi)(p)).$$

In other words, we take the Fourier transform $\hat{\psi} = \mathcal{F}(\psi)$, then multiply by $f(p)$ and then inverse Fourier transform back.
So, for example, the operator $g(x)f(p)$ is given by

$$(g(x)f(p)\psi)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(x)\tilde{f}(x-y)\psi(y)dy$$

where $\tilde{f}$ denotes the inverse Fourier transform of $f$. 
So, for example, the operator $g(x)f(p)$ is given by

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where $\tilde{f}$ denotes the inverse Fourier transform of $f$. If $f$ and $g$ are in $L_2(\mathbb{R}^n)$ then this kernel belongs to $L_2(\mathbb{R}^n \times \mathbb{R}^n)$ and so is Hilbert Schmidt.
So, for example, the operator $g(x)f(p)$ is given by

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where $\tilde{f}$ denotes the inverse Fourier transform of $f$. If $f$ and $g$ are in $L_2(\mathbb{R}^n)$ then this kernel belongs to $L_2(\mathbb{R}^n \times \mathbb{R}^n)$ and so is Hilbert Schmidt.

By symmetry, the operator $f(p)g(x)$ is then also Hilbert Schmidt.
Theorem

Let $f$ and $g$ be bounded (Borel measurable) functions which tend to 0 at $\infty$. Then $f(p)g(x)$ and $g(x)f(p)$ are compact.
Proof.

By symmetry it is enough to consider \( g(x)f(p) \). Let \( f_R \) be the function equal to \( f \) on the ball of radius \( R \) centered at the origin and zero outside, with a similar definition of \( g_R \). Then \( g_R \) and \( f_R \) belong to \( L_2 \) and hence \( g_R(x)f_R(p) \) is Hilbert Schmidt and hence compact.
Proof.

By symmetry it is enough to consider $g(x)f(p)$. Let $f_R$ be the function equal to $f$ on the ball of radius $R$ centered at the origin and zero outside, with a similar definition of $g_R$. Then $g_R$ and $f_R$ belong to $L_2$ and hence $g_R(x)f_R(p)$ is Hilbert Schmidt and hence compact.

We have

$$
\|g(x)f(p) - g_R(x)f_R(p)\| \leq \|g\|_{\infty} \|f - f_R\|_{\infty} + \|g - g_R\|_{\infty} \|f_R\|_{\infty},
$$

where the norm on the left is the operator norm.

So $g_R(x)f_R(p)$ approaches $g(x)f(p)$ in norm.
For example, suppose that $H_0$ is the free Hamiltonian and we consider its resolvent at some negative real number, for example at $-1$. This operator is of the form $f(p)$ where

$$f(p) = -\frac{1}{1 + |p|^2}.$$ 

Let $V$ be a function which vanishes at $\infty$. Then the operator $V(x)f(p)$ is compact.
For example, suppose that $H_0$ is the free Hamiltonian and we consider its resolvent at some negative real number, for example at $-1$. This operator is of the form $f(p)$ where

$$f(p) = -\frac{1}{1 + |p|^2}.$$ 

Let $V$ be a function which vanishes at $\infty$. Then the operator $V(x)f(p)$ is compact.

This result will be of importance to us in conjunction with Weyl’s theorem on the essential spectrum.
Weyl’s characterization of the essential spectrum.

Recall the following from the lecture on the square well:

**Proposition.**

A $\lambda \in \mathbb{R}$ belongs to the essential spectrum, $\sigma_{\text{ess}}(A)$, of a self adjoint operator $A$, if and only if there exists a sequence $\psi_n \in D(A)$ such that

1. $\|\psi_n\| = 1$,
2. $\psi_n$ converges weakly to 0, and
3. $(\lambda I - A)\psi_n \to 0$.

Such a sequence is called a **singular Weyl sequence**.

I want to quickly remind you how we proved this Proposition:
Characterizing the spectrum as “approximate eigenvalues”.

Theorem

Let $A$ be a self-adjoint operator on a Hilbert space $H$. A real number $\lambda$ belongs to the spectrum of $A$ if and only if there exists a sequence of vectors $u_n \in D(A)$ such that

$$\|u_n\| = 1 \text{ and } \| (\lambda I - A) u_n \| \to 0.$$ 

If $\lambda \notin \sigma(A)$ then $(\lambda I - A)^{-1}$ exists and is bounded, so there is a $\mu > 0$ such that $\| (\lambda I - A) u \| \geq \mu \| u \| \quad \forall \ u \in D(A)$. So if $\|u_n\| = 1$ then $\| (\lambda I - A) u_n \| \geq \mu$ and can not approach 0.

So if a sequence as in the theorem exists, then $\lambda$ belongs to the spectrum of $A$. This is the easy direction; it is practically the definition of the spectrum. We now turn to the other direction:
Suppose that no such sequence exists. I claim that there exists a constant $c > 0$ such that

$$\|u\| \leq c\|(\lambda I - A)u\| \quad \forall \ u \in D(A). \tag{1}$$

For if not, there would exist a sequence $v_n$ of non-zero elements of $D(A)$ with

$$\|(\lambda I - A)v_n\|/\|v_n\| \to 0.$$

Replacing $v_n$ by $u_n := v_n/\|v_n\|$ gives a sequence of unit vector in $D(A)$ with $\|(\lambda I - A)u_n\| \to 0$, proving (1).

From the inequality (1) we conclude that the map $\lambda I - A$ is injective. We can also conclude that its image is closed. For if $w_n = (\lambda I - A)u_n, \ u_n \in D(A)$ with $w_n \to w$, then (1) implies that the sequence $u_n$ is Cauchy and so converges to some $u \in \mathcal{H}$. We want to show that $u \in D(A)$. For any $v \in D(A)$ we have:
\[(u, (\lambda I - A)v) = \lim_{n \to \infty} (u_n, (\lambda I - A)v) = \lim ((\lambda I - A)u_n, v) = \lim_{n \to \infty} (w_n, v) = (w, v).\]

So \(u \in D((\lambda I - A)^*) = D(A^*) = D(A)\) since \(A\) is self-adjoint. To show that \(\lambda\) is in the resolvent set of \(A\) we must show that the image of \((\lambda I - A)\) is all of \(\mathcal{H}\):
Let $f \in \mathcal{H}$ and consider the linear function on the image of $\lambda I - A$ given by

$$w \mapsto (v, f) \quad \text{where} \quad w = (\lambda I - A)v.$$  

Now $|(v, f)| \leq \|f\|\|v\|$ and by (1), this is $\leq c\|f\|\|w\|$. So this linear function is bounded on the image of $\lambda I - A$. But this image, being a closed subspace of $\mathcal{H}$ as we have just proved, is a Hilbert space in its own right, and so we may apply the Riesz representation theorem to conclude that there is a $u$ in the image of $\lambda I - A$ such that

$$((\lambda I - A)v, u) = (v, f) \quad \forall \ v \in D(A).$$

Since $A$, and hence $\lambda I - A$ is self-adjoint, this implies that $u \in D(A)$ and $(\lambda I - A)u = f$.

We now review Weyl’s theorem characterizing the essential spectrum:
Suppose that $\lambda$ belongs to the discrete spectrum of $A$. Let $\mathcal{H}_\lambda$ denote the eigenspace with eigenvalue $\lambda$ so $\mathcal{H}_\lambda$ is finite dimensional by assumption. Decompose $\mathcal{H}$ into the direct sum

$$\mathcal{H} = \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda^\perp.$$ 

The entire interval $(\lambda - \epsilon, \lambda + \epsilon)$ lies in the resolvent set of the restriction of $A$ to $\mathcal{H}_\lambda^\perp$. So the restriction of $\lambda I - A$ to $\mathcal{H}_\lambda^\perp$ has a bounded inverse. So if $\psi_n \in D(A)$ is a sequence of elements of $\mathcal{H}$ with
\[ \| \psi_n \| = 1 \quad \text{and} \quad (\lambda I - A)\psi_n \to 0 \]

then the \( H_\lambda \) components of the \( \psi_n \) must tend to 0. The \( H_\lambda \) components then form a bounded sequence in a finite dimensional space, and hence we can extract a convergent subsequence. So we have proved one half of the following theorem of Weyl:

**Theorem**

A point \( \lambda \) belongs to the essential spectrum of a self-adjoint operator \( A \) if and only if there exists a sequence \( \psi_n \in D(A) \) such that

- \( \| \psi_n \| = 1 \),
- \( \psi_n \) has no convergent subsequence, and
- \( (\lambda I - A)\psi_n \to 0 \).
Conversely, suppose that $\lambda$ lies in the essential spectrum. We want to construct a sequence as in the theorem. If $\lambda$ is an eigenvalue of infinite multiplicity, we can construct an orthonormal sequence of eigenvectors $\psi_n$ so $(\lambda I - A)\psi_n = 0$ and the $\psi_n$ have no convergent subsequence.

Otherwise we can find a sequence of $\lambda_n$ lying in the spectrum of $A$ with $\lambda_n \neq \lambda$ and $\lambda_n \to \lambda$. So $\lambda - \lambda_n \neq 0$ and $\lambda - \lambda_n$ lies in the spectrum of $A - \lambda_n I$ and hence by our theorem characterizing the spectrum as “approximate eigenvalues”, we can find $\psi_n$ with $\|\psi_n\| = 1$ and

$$\|(\lambda_n I - A)\psi_n\| \leq \frac{1}{n}|\lambda_n - \lambda|,$$

so $(\lambda I - A)\psi_n \to 0$. We wish to prove that $\psi_n$ has no convergent subsequence. If it did, then passing to the subsequence (and relabeling) we would have $\psi_n \to \psi$ with $\|\psi\| = 1$ and $\psi$ an eigenvector of $A$ with eigenvalue $\lambda$. 
Then

$$(\lambda - \lambda_n)(\psi_n, \psi) = (\psi_n, A\psi) - \lambda_n(\psi_n, \psi) = ((A - \lambda I)\psi_n, \psi),$$

so

$$|\lambda - \lambda_n||(\psi_n, \psi)| \leq \frac{1}{n}|\lambda_n - \lambda|,$$

which is impossible since $(\psi_n, \psi) \to 1$. $\square$
We now turn to the proof of the proposition: A sequence $\psi_n$ is said to converge **weakly** to $\psi$ if for every $v \in \mathcal{H}$

$$ (\psi_n, v) \to (\psi, v). $$

It is easy to check that every bounded sequence in a separable Hilbert space has a weakly convergent subsequence. On the other hand, suppose that $\psi_n \in \mathcal{H}$ satisfies

$$ \|\psi_n\| = 1 \quad \text{and} \quad \psi_n \text{ converges weakly to } 0. $$

Then $\psi_n$ can have no (strongly) convergent subsequence because the strong limit of any subsequence would have to equal the weak limit and hence $= 0$ contradicting the hypothesis $\|\psi_n\| = 1$.

I repeat the statement of the proposition we want to prove:
Proposition

$\lambda \in \mathbb{R}$ belongs to the essential spectrum of a self adjoint operator $A$ if and only if there exists a sequence $\psi_n \in D(A)$ such that

1. $\|\psi_n\| = 1$,

2. $\psi_n$ converges weakly to 0, and

3. $(\lambda I - A)\psi_n \to 0$.

If the first two conditions are satisfied then the $\psi_n$ can not have a convergent subsequence, as we have just seen. So the conditions of Weyl’s theorem on the essential spectrum that we have just proved are satisfied, and hence $\lambda$ belongs to the essential spectrum of $A$. 

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Conversely, if \( \lambda \) belongs to the essential spectrum of \( A \) then we can find a sequence \( \psi_n \) satisfying the three conditions of the above theorem. The first condition implies that the \( \psi_n \) are uniformly bounded so we can choose a subsequence which converges weakly to some \( \psi \in \mathcal{F} \). Let us pass to this subsequence (and relabel). Since our \( \psi_n \) has no convergent subsequence, there is an \( \epsilon > 0 \) such that

\[
\| \psi_n - \psi \| \geq \epsilon
\]

for all \( n \). By the third condition in the theorem,

\[
(\psi_n, (A - \lambda I)\phi) = ((A - \lambda I)\psi_n, \phi) \rightarrow 0 \quad \forall \phi \in D(A)
\]

so

\[
(\psi, (A - \lambda I)\phi) = 0 \quad \forall \phi \in D(A).
\]

Since \( A \) is self-adjoint, this implies that \( \psi \in D(A) \) and \( A\psi = \lambda \psi \).
Consider the sequence

$$\tilde{\psi}_n := \frac{1}{\|\psi_n - \psi\|}(\psi_n - \psi).$$

We have $\|\tilde{\psi}_n\| = 1$ and $(\tilde{\psi}_n, u) \to 0$ for all $u \in \mathcal{H}$. So the first two conditions of the Proposition are satisfied. So is the third because

$$(A - \lambda I)\tilde{\psi}_n = \frac{1}{\|\psi_n - \psi\|}(A - \lambda I)\psi_n \to 0$$

since $\|\psi_n - \psi\| \geq \epsilon > 0$. 

□
Weyl’s theorem.

Let $A$ and $B$ be self-adjoint operators. Let $z$ be a point in the resolvent set of $A$ and of $B$ and let $R_A = R_A(z)$ and $R_B = R_B(z)$ be the corresponding resolvents.

**Theorem**

If $R_A - R_B$ is compact, then

$$\sigma_{ess}(A) = \sigma_{ess}(B).$$
Before proving the theorem observe the following identity involving the resolvent:

\[
\frac{1}{z - \lambda} R_A(z)(\lambda I - A) = \frac{1}{z - \lambda} R_A(z)(zI - A + (\lambda - z)I) = \\
= \frac{1}{z - \lambda} I - R_A(z).
\]

So if \( \psi_n \) is a singular Weyl sequence at \( \lambda \) then

\[
\left\| \left( R_A(z) - \frac{1}{z - \lambda} I \right) \psi_n \right\| \to 0.
\]
Proof.

Since the \( \psi_n \) converge weakly to 0 and \( R_A - R_B \) is compact, we know that \((R_A - R_B)\psi_n\) converges strongly to zero and hence

\[
\left\| \left( R_B(z) - \frac{1}{z-\lambda} I \right) \psi_n \right\| \to 0.
\]

In particular \( \| R_B(z)\psi_n \| \to \frac{1}{|z-\lambda|} \neq 0 \). But

\[
R_B(z) - \frac{1}{z-\lambda} I = \frac{1}{\lambda - z} (\lambda I - B)R_B(z)
\]

so (after an innocuous renormalizing) the \( R_B(z)\psi_n \) are a singular Weyl sequence for \( B \). So \( \sigma_{\text{ess}}(A) \subset \sigma_{\text{ess}}(B) \). Interchanging the roles of \( A \) and \( B \) then proves the theorem.
Recall: The second resolvent identity.

Let $a$ and $b$ be operators whose range is the whole space and with bounded inverses. Then

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$$

assuming that the right hand side is defined. For example, if $A$ and $B$ are closed operators with $D(B - A) \supset D(A)$ we get

$$R_A(z) - R_B(z) = R_A(z)(B - A)R_B(z).$$

This is known as the second resolvent identity.
Let us now say that an operator $K$ is **relatively compact with respect to** an operator $A$ if $KR_A(z)$ is compact for some $z$ in the resolvent set of $A$. Notice that from the first resolvent identity,

$$R_A(z) - R_A(w) = (z - w)R_A(z)R_A(w),$$

we see that if $KR_A(z)$ is compact for some $z$ then $KR_A(z)$ is compact for all $z$ in the resolvent set. So the condition is independent of $z$. 
Let us now say that an operator $K$ is \textbf{relatively compact with respect to} an operator $A$ if $KR_A(z)$ is compact for some $z$ in the resolvent set of $A$. Notice that from the first resolvent identity,

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we see that if $KR_A(z)$ is compact for \textit{some} $z$ then $KR_A(z)$ is compact for \textit{all} $z$ in the resolvent set. So the condition is independent of $z$.

From the second resolvent identity we see that if $K$ is relatively compact with respect to a self adjoint operator $A$ then $R_{A+K}(z) - R_A(z)$ is compact. Therefore:
Weyl’s theorem

**Theorem.** [Hermann Weyl.]

If $K$ is a self-adjoint operator which is relatively compact with respect to to a self adjoint operator $A$ then $\sigma_{\text{ess}}(A + K) = \sigma_{\text{ess}}(A)$. 
Applications to Schrödinger operators.

The resolvent of $H_0$ at $-1$ is $\frac{-1}{1+\|p\|^2}$ which tends to $0$ as $\|p\| \to \infty$. If $V$ is a bounded potential which tends to zero at infinity then $VR(-1, H_0)$ is

$$-V(x)\frac{1}{1 + \|p\|^2}.$$ 

In view of what we have proved above, we conclude that

$$\sigma_{\text{ess}}(H_0 + V) = \sigma_{\text{ess}}(H_0).$$
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The purpose of the next few slides is to study the opposite situation - where the potential tends to infinity at infinity. We will show that under this circumstance $H_0 + V$ has no essential spectrum.
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The purpose of the next few slides is to study the opposite situation - where the potential tends to infinity at infinity. We will show that under this circumstance $H_0 + V$ has no essential spectrum. For this we will need a lemma.
First observe that for all potentials $W$ of compact support, the operator $W(I + H_0^{1/2})^{-1}$ is compact. This is again because it is of the form $g(x)f(p)$ where $f$ and $g$ are functions going to zero at infinity.
First observe that for all potentials \( W \) of compact support, the operator \( W(I + H_0^\frac{1}{2})^{-1} \) is compact. This is again because it is of the form \( g(x)f(p) \) where \( f \) and \( g \) are functions going to zero at infinity. Let \( V \geq 0 \) and set \( H = H_0 + V \). Notice that \( H \geq 0 \) as an operator, and in fact \( H \geq H_0 \). So for any \( u \in \text{Dom}(H_1^\frac{1}{2}) \)

\[
\| H_0^\frac{1}{2} u \|^2 = (u, H_0 u) \leq (u, Hu) = \| H^\frac{1}{2} u \|
\]

so \( H_0^\frac{1}{2} \) is \( H^\frac{1}{2} \) bounded and since \( (u, Hu) \leq (u, (H + I)u) \) we see that

\[
(H_0^\frac{1}{2} + I)(H + I)^{-\frac{1}{2}}
\]

is bounded. More directly, this is obvious in a multiplicative spectral representation.
Lemma

If $V \geq 0$ is a non-negative potential and $W$ is multiplication by a bounded function of compact support then $W$ is $H = H_0 + V$ compact, i.e. $W(I + H)^{-1}$ is compact.
Lemma

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Proof.

$$W(I + H)^{-\frac{1}{2}} = W(I + H_0^{\frac{1}{2}})^{-1}(I + H_0^{\frac{1}{2}})(I + H)^{-\frac{1}{2}}$$

and we know that the product of the first two factors is compact and the product of the second two factors is bounded. So $W(I + H)^{-\frac{1}{2}}$ is compact. Multiply by the bounded operator $(I + H)^{-\frac{1}{2}}$ on the right to conclude that $W(I + H)^{-1}$ is compact.
A theorem of Friedrichs, 1932.

**Theorem**

If \( V(x) \to \infty \) as \( x \to \infty \) then \( \sigma_{\text{ess}}(H_0 + V) \) is empty.

**Proof.**

For any \( E \) write \( V - E = f - g \) where \( f \geq 0 \) and \( g \) has compact support. See the next slide.

By the lemma, \( g \) is \( H_0 + f \) compact, so \( \sigma_{\text{ess}}(H_0 + f) = \sigma_{\text{ess}}(H - E) \). Since \( f \geq 0 \), we know that \( \sigma_{\text{ess}}(H - E) \subset [0, \infty) \) so \( \sigma_{\text{ess}}(H) \subset [E, \infty) \). Since this is true for all \( E \) we conclude that \( \sigma_{\text{ess}}(H_0 + V) \) is empty.
Applications to Schrödinger operators.

\[ V = f + E - g \]

\[ E - g \rightarrow E \]

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