212a1415
Radon Nikodym.
Dualities and Riesz representation theorems.
The spectral theorem, improved.
Riesz redux

Math 212a

November 4 - 6, 2014
## Contents

1. Review
2. A Riesz representation theorem for measures
   - Integration on locally compact Hausdorff spaces.
3. The spectral theorem
   - Resolutions of the identity.
4. Radon Nikodym
5. The dual space of $L^p$.
   - Duality of $L^p$ and $L^q$ when $\mu(S) < \infty$.
   - The case where $\mu(S) = \infty$.
   - Fubini’s theorem.
6. The Riesz representation theorem redux.
   - Propositions in topology.
   - Proof of the uniqueness and regularity of the measure.
Integration according to Daniel

Daniel’s theory starts with a space of bounded real valued functions on a set $S$ which was closed under the operations $\wedge$ and $\vee$ and a real valued function $I$ on $L$ which satisfies

1. $I$ is linear: $I(af + bg) = aI(f) + bI(g)$

2. $I$ is non-negative: $f \geq 0 \implies I(f) \geq 0$ or equivalently $f \geq g \implies I(f) \geq I(g)$.

3. $f_n \downarrow 0 \implies I(f_n) \downarrow 0$.

By using monotone limits the space $L$ and the function $I$ on it is enlarged to a space $L^1$, and if $\mathcal{B}$ denotes the smallest monotone class containing $L$ then

Theorem

[12J] If $f \in \mathcal{B}$ then $f \in L^1 \iff \exists g \in L^1$ with $|f| \leq g$. 
Integrable functions

A function $f \in B^+$ is called **integrable** if either $f^+$ or $f^-$ is summable. Then

$$I(f) := I(f^+) - I(f^-)$$

is unambiguously defined, but may $= +\infty$ or $-\infty$. So a function $f$ is summable iff it is integrable and $|I(f)| < \infty$.

To summarize:

**Theorem**

**[12K]** If $f$ and $g$ are integrable then $f + g$ is integrable with $I(f + g) = I(f) + I(g)$ provided that $I(f)$ and $I(g)$ are not oppositely infinite. If $f_n$ is integrable, $I(f_1) > -\infty$, and $f_n \nearrow f$ then $f$ is integrable and $I(f_n) \to I(f)$.
Integrable sets.

Loomis calls a set $A$ **integrable** if $1_A \in \mathcal{B}$. The monotone class properties of $\mathcal{B}$ imply that the integrable sets form a $\sigma$-field. Then define

$$\mu(A) := I(1_A)$$

and the monotone convergence theorem guarantees that $\mu$ is a measure. Loomis calls a set $A$ **summable** if it is integrable and $\mu(A) < \infty$.

Notice that if $S$ (the entire space) is summable then $I$ is bounded with respect to the uniform norm.
Using Stone’s axiom

If we add Stone’s axiom which says that

\[ f \in L \implies f \land 1 \in L \]

then the Lebesgue integral of \( f \in L^1 \) with respect to the measure \( \mu \) coming from \( I \) is the same as \( I \):

\[ \int_S f d\mu = I(f). \]
Integration on locally compact Hausdorff spaces.

We now suppose that $S$ is a locally compact Hausdorff space. As in the case of $\mathbb{R}^n$, we can (and will) take $L$ to be the space of continuous functions of compact support. Dini’s lemma then says that if $f_n \in L$, $f_n \downarrow 0$ then $f_n \to 0$ in the uniform topology. If $A$ is any subset of $S$ we will denote the set of $f \in L$ whose support is contained in $A$ by $L_A$. 

Integration on locally compact Hausdorff spaces.

**Non-negative linear functions**

**Lemma**

A non-negative linear function $I$ is bounded in the uniform norm on $L^C$ whenever $C$ is compact.

**Proof.**

Choose $g \geq 0 \in L$ so that $g(x) \geq 1$ for $x \in C$. If $f \in L^C$ then

$$|f| \leq \|f\|_{\infty}g$$

so

$$|I(f)| \leq I(|f|) \leq I(g) \cdot \|f\|_{\infty}.$$
Theorem

Every non-negative linear functional $I$ on $L$ is an integral.

Proof.

This is Dini's lemma together with the preceding lemma. Indeed, by Dini we know that $f_n \in L \downarrow 0$ implies that $\|f_n\|_\infty \downarrow 0$. Since $f_1$ has compact support, let $C$ be its support, a compact set. All the succeeding $f_n$ are then also supported in $C$ and so by the preceding lemma $I(f_n) \downarrow 0$.

I now want to replace the "non negative" assumption in the theorem by a boundedness assumption. I want to prove
A better Riesz representation theorem

**Theorem**

Let $F$ be a bounded linear function on $L$ (with respect to the uniform norm). Then there are two integrals $I^+$ and $I^−$ such that

$$F(f) = I^+(f) − I^−(f).$$

Furthermore, if $A$ is any integrable set with respect to $I^+$ then

$$\mu_{I^+}(A) \leq \|F\|_\infty$$

and similarly for $I^−$.

For this I need some facts about the “variations” of bounded linear functions.
Variations of a bounded linear function

The “positive part” of a bounded linear functional.

Suppose we start with an arbitrary $L$ and $I$. For each $1 \leq p \leq \infty$ we have the norm $\| \cdot \|_p$ on $L$ which makes $L$ into a real normed linear space. Let $F$ be a linear function on $L$ which is bounded with respect to this norm, so that

$$|F(f)| \leq C \|f\|_p$$

for all $f \in L$ where $C$ is some non-negative constant. The least upper bound of the set of $C$ which is called $\|F\|_p$ as usual. If $f \geq 0 \in L$, define

$$F^+(f) := \text{lub}\{F(g) : 0 \leq g \leq f, \ g \in L\}.$$
Variations of a bounded linear function
The “positive part” of a bounded linear functional, continued.

Then

\[ F^+(f) \geq 0 \]

and

\[ F^+(f) \leq \|F\|_p \|f\|_p \]

since \( F(g) \leq |F(g)| \leq \|F\|_p \|g\|_p \leq \|F\|_p \|f\|_p \) for all \( 0 \leq g \leq f \), \( g \in L \), since \( 0 \leq g \leq f \) implies \( |g|^p \leq |f|^p \) for \( 1 \leq p < \infty \) and also implies \( \|g\|_\infty \leq \|f\|_\infty \). Also

\[ F^+(cf) = cF^+(f) \quad \forall \ c \geq 0 \]

as follows directly from the definition.
Variations of a bounded linear function
The “positive part” of a bounded linear functional, continued.

Suppose that $f_1$ and $f_2$ are both non-negative elements of $L$. If $g_1, g_2 \in L$ with

$$0 \leq g_1 \leq f_1 \text{ and } 0 \leq g_2 \leq f_2$$

then

$$F^+(f_1+f_2) \geq \text{lub } F(g_1+g_2) = \text{lub } F(g_1) + \text{lub } F(g_2) = F^+(f_1) + F^+(f_2).$$
Variations of a bounded linear function

The “positive part” of a bounded linear functional, continued.

On the other hand, if $g \in L$ satisfies $0 \leq g \leq (f_1 + f_2)$ then $0 \leq g \land f_1 \leq f_1$, and $g \land f_1 \in L$. Also $g - g \land f_1 \in L$ and vanishes at points $x$ where $g(x) \leq f_1(x)$ while at points where $g(x) > f_1(x)$ we have $g(x) - g \land f_1(x) = g(x) - f_1(x) \leq f_2(x)$. So

$$g - g \land f_1 \leq f_2$$

and so

$$F^+(f_1 + f_2) = \text{lub } F(g) \leq \text{lub } F(g \land f_1) + \text{lub } F(g - g \land f_1) \leq F^+(f_1) + F^+(f_2)$$

So

$$F^+(f_1 + f_2) = F^+(f_1) + F^+(f_2)$$

if both $f_1$ and $f_2$ are non-negative elements of $L$. 
Variations of a bounded linear function

The “positive part” of a bounded linear functional, continued.

Now write any \( f \in L \) as \( f = f_1 - g_1 \) where \( f_1 \) and \( g_1 \) are non-negative. (For example we could take \( f_1 = f^+ \) and \( g_1 = f^- \).)

Define: \( F^+(f) = F^+(f_1) - F^+(g_1) \).

This is well defined, for if we also had \( f = f_2 - g_2 \) then \( f_1 + g_2 = f_2 + g_1 \) so

\[
F^+(f_1) + F^+(g_2) = F^+(f_1 + g_2) = F^+(f_2 + g_1) = F^+(f_2) + F^+(g_1)
\]

so

\[
F^+(f_1) - F^+(g_1) = F^+(f_2) - F^+(g_2).
\]

From this it follows that \( F^+ \) so extended is linear, and

\[
|F^+(f)| \leq F^+(|f|) \leq \|F\|_p f_p
\]

so \( F^+ \) is bounded.
Variations of a bounded linear function

The “negative part” of a bounded linear functional.

Define $F^-$ by

$$F^-(f) := F^+(f) - F(f).$$

As $F^-$ is the difference of two linear functions it is linear. Since by its definition, $F^+(f) \geq F(f)$ if $f \geq 0$, we see that $F^-(f) \geq 0$ if $f \geq 0$. Clearly $\|F^-(f)\|_p \leq \|F^+(f)\|_p + \|F\| \leq 2\|F\|_p$. We have proved:
Variations of a bounded linear function

Proposition

Every linear function on \( L \) which is bounded with respect to the \( \| \cdot \|_p \) norm can be written as the difference \( F = F^+ - F^- \) of two linear functions which are bounded with respect to the \( \| \cdot \|_p \) norm and take non-negative values on non-negative functions.

In fact, we could formulate this proposition more abstractly as dealing with a normed vector space which has an order relation consistent with its metric but we shall refrain from this more abstract formulation. Together with the theorem about non-negative linear functions, the proposition allows us to conclude the desired
The Riesz representation theorem for measures

**Theorem**

Let \( F \) be a bounded linear function on \( L \) (with respect to the uniform norm). Then there are two integrals \( I^+ \) and \( I^- \) such that

\[
F(f) = I^+(f) - I^-(f).
\]

Furthermore, if \( A \) is any integrable set with respect to \( I^+ \) then

\[
\mu_{I^+}(A) \leq \|F\|_{\infty}
\]

and similarly for \( I^- \).
The complex case

So far, we have been dealing with vector spaces and linear functions over the real numbers, because we used monotonicity to develop the theory. But any vector space over the complex numbers can be considered as a vector space over the real numbers, and any (complex) linear function $G$ can be written as $G = F_1 + iF_2$ where $F_1$ and $F_2$ are real linear functions. We can similarly consider complex valued measures. So, if $L$ now denotes the space of complex valued functions of compact support on a locally compact Hausdorff space $S$ we have:
The Riesz representation theorem for measures

**Theorem**

*Let $F$ be a bounded linear function on $L$ (with respect to the uniform norm). Then there is a complex valued measure $\mu$ on $S$ such that*

$$F(f) = \int_S f d\mu.$$  

I will later need an improved version of this theorem which gives more details about $\mu$. I will present this at the end of these slides. But I will now pause in the treatment of measure and integration to give a remarkable application of this Riesz representation theorem, together with the Riesz representation theorem on Hilbert spaces to get an improved version of the spectral theorem.
Notational reminder

I am now going to use these Riesz representation theorems to extend the functional calculus version of the spectral theorem that we have already proved from continuous functions vanishing at infinity to bounded Borel functions. I remind you of our notation and of the theorem:

\( C_0(\mathbb{R}) \) denotes the space of continuous functions vanishing at infinity, and for \( f \in C_0(\mathbb{R}) \), \( \| f \|_\infty \) denotes its sup norm, i.e.

\[
\| f \|_\infty = \sup_{t \in \mathbb{R}} |f(t)|.
\]

\( \mathcal{H} \) denotes a (separable) Hilbert space and \( \mathcal{L}(\mathcal{H}) \) denotes the space of bounded linear operators on \( \mathcal{H} \).
That part of the spectral theorem that we have already proved.

**Theorem**

*Let $A$ be a self-adjoint operator on a Hilbert space, $\mathcal{H}$. There exists a unique linear map $f \mapsto f(A)$ from $C_0(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$ such that*

- *The map is multiplicative, i.e. $(f_1f_2)(A) = f_1(A)f_2(A)$,*
That part of the spectral theorem that we have already proved.

**Theorem**

Let $A$ be a self-adjoint operator on a Hilbert space, $\mathcal{H}$. There exists a unique linear map $f \mapsto f(A)$ from $C_0(\mathbb{R})$ to $L(\mathcal{H})$ such that

- The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,
- $(\overline{f})(A) = f(A)^*$,
That part of the spectral theorem that we have already proved.

**Theorem**

Let $A$ be a self-adjoint operator on a Hilbert space, $\mathcal{H}$. There exists a unique linear map $f \mapsto f(A)$ from $C_0(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$ such that

- The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A)f_2(A)$,
- $(\tilde{f})(A) = f(A)^*$,
- $\|f(A)\| \leq \|f\|_\infty$. 

Math 212a
212a1415 Radon Nikodym. Dualities and Riesz representation theorems. The spectral theorem, improved. Riesz redux
That part of the spectral theorem that we have already proved.

**Theorem**

Let $A$ be a self-adjoint operator on a Hilbert space, $\mathcal{H}$. There exists a unique linear map $f \mapsto f(A)$ from $C_0(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$ such that

- The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,
- $(\bar{f})(A) = f(A)^*$,
- $\|f(A)\| \leq \|f\|_\infty$,
- if $w \not\in \mathbb{R}$ and $r_w$ is the function $r_w(x) = 1/(w - x)$ then

$$r_w(A) = R(w, A),$$
That part of the spectral theorem that we have already proved.

**Theorem**

Let $A$ be a self-adjoint operator on a Hilbert space, $\mathcal{H}$. There exists a unique linear map $f \mapsto f(A)$ from $C_0(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$ such that

- The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A) f_2(A)$,
- $(\bar{f})(A) = f(A)^*$,
- $\|f(A)\| \leq \|f\|_{\infty}$,
- if $w \notin \mathbb{R}$ and $r_w$ is the function $r_w(x) = 1/(w - x)$ then
  $$r_w(A) = R(w, A),$$
- if the support of $f$ is disjoint from $\text{Spec}(A)$ then $f(A) = 0$. 

Math 212a

212a1415 Radon Nikodym. Dualities and Riesz representation theorems. The spectral theorem, improved. Riesz redux
That part of the spectral theorem that we have already proved.

**Theorem**

Let $A$ be a self-adjoint operator on a Hilbert space, $\mathcal{H}$. There exists a unique linear map $f \mapsto f(A)$ from $C_0(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$ such that

- The map is multiplicative, i.e. $(f_1 f_2)(A) = f_1(A)f_2(A)$,
- $(\overline{f})(A) = f(A)^*$,
- $\|f(A)\| \leq \|f\|_{\infty}$,
- if $w \notin \mathbb{R}$ and $r_w$ is the function $r_w(x) = 1/(w - x)$ then
  \[ r_w(A) = R(w, A), \]
- if the support of $f$ is disjoint from $\text{Spec}(A)$ then $f(A) = 0$. 
Formulas for $f(A)$

If $f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t)e^{ixt} dt$ and $U(t)$ is the one parameter group generated by $iA$ (guaranteed to exist by Stone’s theorem) then

$$f(A) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t)U(t)dt.$$ 

If $f \in C_0^\infty(\mathbb{R})$ has compact support and $\tilde{f}$ is a almost holomorphic extension of $f$ then

$$f(A) = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial \tilde{f}}(z)R(z, A)dz.$$ 

I now embark on extending the class of functions for which $f(A)$ is defined using our Riesz representation theorems:
I will temporarily use the notation $O(f)$ (the operator corresponding to $f$) instead of $f(A)$.

Fix vectors $x, y$ in our Hilbert space $\mathcal{H}$. If $f \in C_0(\mathbb{R})$ then

$$| (O(f)x, y) | \leq \| O(f) \| \| x \| \| y \| \leq \| f \|_\infty \| x \| \| y \| .$$

So the map

$$f \mapsto (O(f)x, y)$$

is a continuous linear function on $C_0(\mathbb{R})$ bounded in norm by

$$\| f \|_\infty \| x \| \| y \| .$$

By the Riesz representation theorem for measures, this means that there is a unique complex valued bounded measure

$$\mu_{x,y}$$

on $\mathbb{R}$ such that

$$(O(f)x, y) = \int_{\mathbb{R}} f d\mu_{x,y}.$$
Now

\[(O(f)x, y) = (x, O(f)^*y) = (x, O(f)y) = (O(f)y, x)\]

\[\int_{\mathbb{R}} \overline{f} \, d\mu_{y,x} = \int_{\mathbb{R}} f \, d\overline{\mu_{y,x}}.\]

The uniqueness of the measure implies that

\[\mu_{y,x} = \overline{\mu_{x,y}}.\]

Thus, for each fixed Borel set \(U \subset \mathbb{R}\) its measure \(\mu_{x,y}(U)\) depends linearly on \(x\) and anti-linearly on \(y\).
We can choose a sequence of $f_n \in C_0(\mathbb{R})$ which satisfy $0 \leq f_n \leq 1$ and which are monotone increasing to the constant function $1_{\mathbb{R}}$. It then follows from the monotone convergence theorem and $|(O(f)x, x)| \leq \|f\|_\infty \|x\|^2$ that $\mu_{x,x}(\mathbb{R}) \leq \|x\|^2$ and hence that $\mu_{x,y}$ defines a bounded measure on $\mathbb{R}$.

So if $f$ is now any bounded Borel function on $\mathbb{R}$, the integral

$$\int_{\mathbb{R}} f d\mu_{x,y}$$

is well defined, and is bounded in absolute value by some constant times $\|f\|_\infty \|x\| \|y\|$. If we hold $f$ and $x$ fixed, this integral is a bounded anti-linear function of $y$, and hence by the Riesz representation theorem (for Hilbert space) there exists a $w \in \mathcal{H}$ such that this integral is given by $(w, y)$. The $w$ in question depends linearly on $f$ and on $x$ because the integral does, and so:
We have defined a linear map $O$ from bounded Borel functions on $\mathbb{R}$ to bounded operators on $\mathcal{H}$ such that

$$(O(f)x, y) = \int_{\mathbb{R}} f d\mu_{x,y}.$$ 

We now want to verify that this map $f \mapsto O(f)$ satisfies the properties listed in the theorem above, and so gives an extension of that theorem from continuous functions vanishing at infinity to bounded Borel functions.
Verifying that \( O(\overline{f}) = O(f)^* \).

Recall that

\[
(O(f)x, y) = \int_{\mathbb{R}} f d\mu_{x,y}.
\]

So
\[
(O(\overline{f})x, y) = \overline{\int_{\mathbb{R}} \overline{f} d\mu_{x,y}} = \int_{\mathbb{R}} fd\mu_{y,x}
\]

\[
= (O(f)y, x) = (x, O(f)y) = (O(f)^*x, y)
\]

We now turn to the proof of the multiplicativity which we know to be true for \( f, g \in C_0(\mathbb{R}) \).

for \( f, g \in C_0(\mathbb{R}) \) we have:
\[ \int_{\mathbb{R}} f g d \mu_{x,y} = (O(fg)x, y) = (O(f)O(g)x, y) = \int_{\mathbb{R}} f d \mu_{O(g)x,y}. \]

Since this holds for all \( f \in C_0(\mathbb{R}) \) (for fixed \( g, x, y \)) we conclude by the uniqueness of the measure that

\[ \mu_{O(g)x,y} = g \mu_{x,y}. \]

Therefore, for any bounded Borel function \( f \) we have

\[ (O(g)x, O(f)^* y) = (O(f)O(g)x, y) \]

\[ = \int_{\mathbb{R}} f d \mu_{O(g)x,y} = \int_{\mathbb{R}} f g d \mu_{x,y}. \]
\((O(g)x, O(f)^*y) = \int_{\mathbb{R}} fgd\mu_{x,y}\). 

This holds for all \(g \in C_0(\mathbb{R})\) and so by the uniqueness of the measure again, we conclude that \(\mu_{x, O(f)^*y} = f\mu_{x,y}\) and hence
\[
(O(fg)x, y) = \int_{\mathbb{R}} gfd\mu_{x,y}
\]
\[
= \int_{\mathbb{R}} gd\mu_{x, O(f)^*y} = (O(g)x, O(f)^*y) = (O(f)O(g)x, y)
\]
or
\[
O(fg) = O(f)O(g)
\]
as desired. \(\square\)

We have extended the homomorphism from \(C_0(\mathbb{R})\) to a homomorphism from the bounded Borel functions on \(\mathbb{R}\) to bounded operators on \(\mathcal{H}\).
Partitions of unity aka resolutions of the identity.

Define: \( P(U) := O(\mathbf{1}_U) \) for any Borel set \( U \). The following facts are immediate:

1. \( P(\emptyset) = 0 \)

Such a \( P \) is called a resolution of the identity. It follows from the last item that for any fixed \( x \in \mathcal{H} \), the map \( U \mapsto P(U)x \) is an \( \mathcal{H} \) valued measure.
Partitions of unity aka resolutions of the identity.

Define: \( P(U) := O(1_U) \) for any Borel set \( U \). The following facts are immediate:

1. \( P(\emptyset) = 0 \)
2. \( P(\mathbb{R}) = I \), the identity

Such a \( P \) is called a **resolution of the identity**. It follows from the last item that for any fixed \( x \in \mathcal{H} \), the map \( U \mapsto P(U)x \) is an \( \mathcal{H} \) valued measure.
Resolutions of the identity.

Partitions of unity aka resolutions of the identity.

Define: \( P(U) := O(1_U) \) for any Borel set \( U \). The following facts are immediate:

1. \( P(\emptyset) = 0 \)
2. \( P(\mathbb{R}) = I \), the identity
3. \( P(U \cap V) = P(U)P(V) \) and \( P(U)^* = P(U) \). In particular, \( P(U) \) is a self-adjoint projection operator.

Such a \( P \) is called a resolution of the identity. It follows from the last item that for any fixed \( x \in \mathcal{H} \), the map \( U \mapsto P(U)x \) is an \( \mathcal{H} \) valued measure.
Partitions of unity aka resolutions of the identity.

Define: \( P(U) := O(\mathbf{1}_U) \) for any Borel set \( U \). The following facts are immediate:

1. \( P(\emptyset) = 0 \)
2. \( P(\mathbb{R}) = I \), the identity
3. \( P(U \cap V) = P(U)P(V) \) and \( P(U)^* = P(U) \). In particular, \( P(U) \) is a self-adjoint projection operator.
4. If \( U \cap V = \emptyset \) then \( P(U \cup V) = P(U) + P(V) \).

Such a \( P \) is called a **resolution of the identity**. It follows from the last item that for any fixed \( x \in \mathcal{H} \), the map \( U \mapsto P(U)x \) is an \( \mathcal{H} \) valued measure.
Partitions of unity aka resolutions of the identity.

Define: \( P(U) := O(1_U) \) for any Borel set \( U \). The following facts are immediate:

1. \( P(\emptyset) = 0 \)
2. \( P(\mathbb{R}) = I \), the identity
3. \( P(U \cap V) = P(U)P(V) \) and \( P(U)^* = P(U) \). In particular, \( P(U) \) is a self-adjoint projection operator.
4. If \( U \cap V = \emptyset \) then \( P(U \cup V) = P(U) + P(V) \).
5. For each fixed \( x, y \in \mathcal{H} \) the set function \( P_{x,y} : U \mapsto (P(U)x, y) \) is a complex valued measure.

Such a \( P \) is called a resolution of the identity. It follows from the last item that for any fixed \( x \in \mathcal{H} \), the map \( U \mapsto P(U)x \) is an \( \mathcal{H} \) valued measure.
Resolutions of the identity.

**Partitions of unity aka resolutions of the identity.**

Define: \( P(U) := O(1_U) \) for any Borel set \( U \). The following facts are immediate:

1. \( P(\emptyset) = 0 \)
2. \( P(\mathbb{R}) = I \), the identity
3. \( P(U \cap V) = P(U)P(V) \) and \( P(U)^* = P(U) \). In particular, \( P(U) \) is a self-adjoint projection operator.
4. If \( U \cap V = \emptyset \) then \( P(U \cup V) = P(U) + P(V) \).
5. For each fixed \( x, y \in \mathcal{H} \) the set function \( P_{x,y} : U \mapsto (P(U)x, y) \) is a complex valued measure.
6. Such a \( P \) is called a **resolution of the identity.** It follows from the last item that for any fixed \( x \in \mathcal{H} \), the map \( U \mapsto P(U)x \) is an \( \mathcal{H} \) valued measure.
Actually, given any resolution of the identity we can give a meaning to the integral
\[ \int_{\mathbb{R}} f dP \]
for any bounded Borel function \( f \) in the strong sense as follows: If \( s = \sum \alpha_i 1_{U_i} \) is a simple function where
\[ \mathbb{R} = U_1 \cup \cdots \cup U_n, \quad U_i \cap U_j = \emptyset, \quad i \neq j \]
and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), define
\[ O(s) := \sum \alpha_i P(U_i) =: \int_{\mathbb{C}} s dP. \]
Actually, given any resolution of the identity we can give a meaning to the integral

\[ \int_{\mathbb{R}} fdP \]

for any bounded Borel function \( f \) in the strong sense as follows: If \( s = \sum \alpha_i 1_{U_i} \) is a simple function where

\[ \mathbb{R} = U_1 \cup \cdots \cup U_n, \quad U_i \cap U_j = \emptyset, \quad i \neq j \]

and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), define

\[ O(s) := \sum \alpha_i P(U_i) =: \int_{\mathbb{C}} sdP. \]

This is well defined on simple functions (is independent of the expression) and is multiplicative

\[ O(st) = O(s)O(t). \]
Also, since the $P(U)$ are self adjoint,

$$O(\bar{s}) = O(s)^*.$$ 

It is also clear that $O$ is linear and

$$(O(s)x, y) = \int_{\mathbb{R}} s dP_{x,y}.$$
Resolutions of the identity.

\[(O(s)x, y) = \int_{\mathbb{C}} s dP_{x, y}.\]

As a consequence, we get

\[\|O(s)x\|^2 = (O(s)^* O(s)x, x) = \int_{\mathbb{C}} |s|^2 dP_{x, x}\]

so

\[\|O(s)x\|^2 \leq \|s\|_{\infty} \|x\|^2.\]
Resolutions of the identity.

\[(O(s)x, y) = \int_{\mathbb{C}} sdP_{x,y}.\]

As a consequence, we get

\[\|O(s)x\|^2 = (O(s)^*O(s)x, x) = \int_{\mathbb{C}} |s|^2 dP_{x,x}\]

so

\[\|O(s)x\|^2 \leq \|s\|_\infty \|x\|^2.\]

If we choose \(i\) such that \(|\alpha_i| = \|s\|_\infty\) and take \(x = P(U_i)y \neq 0\), then we see that

\[\|O(s)\| = \|s\|_\infty\]

provided we now take \(\|f\|_\infty\) to denote the essential supremum which (I recall) means the following:
Recall: the definition of the essential supremum

It follows from the properties of a resolution of the identity that if $U_n$ is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if $f$ is any complex valued Borel function on $\mathbb{R}$, there will exist a largest open subset $V \subset \mathbb{R}$ such that $P(f^{-1}(V)) = 0$. 
Resolutions of the identity.

Recall: the definition of the essential supremum

It follows from the properties of a resolution of the identity that if $U_n$ is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if $f$ is any complex valued Borel function on $\mathbb{R}$, there will exist a largest open subset $V \subset \mathbb{R}$ such that $P(f^{-1}(V)) = 0$. We define the essential range of $f$ to be the complement of $V$, 

\[ |\lambda| \]
Resolutions of the identity.

Recall: the definition of the essential supremum

It follows from the properties of a resolution of the identity that if $U_n$ is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if $f$ is any complex valued Borel function on $\mathbb{R}$, there will exist a largest open subset $V \subset \mathbb{R}$ such that $P(f^{-1}(V)) = 0$. We define the **essential range** of $f$ to be the complement of $V$, say that $f$ is **essentially bounded** if its essential range is compact,
Resolutions of the identity.

Recall: the definition of the essential supremum

It follows from the properties of a resolution of the identity that if $U_n$ is a sequence of Borel sets such that $P(U_n) = 0$, then $P(U) = 0$ if $U = \bigcup U_n$. So if $f$ is any complex valued Borel function on $\mathbb{R}$, there will exist a largest open subset $V \subset \mathbb{R}$ such that $P(f^{-1}(V)) = 0$. We define the **essential range** of $f$ to be the complement of $V$, say that $f$ is **essentially bounded** if its essential range is compact, and then define its **essential supremum** $\|f\|_\infty$ to be the supremum of $|\lambda|$ for $\lambda$ in the essential range of $f$. Furthermore we identify two essentially bounded functions $f$ and $g$ if $\|f - g\|_\infty = 0$ and call the corresponding space $L^\infty(P)$. 

Math 212a

212a1415 Radon Nikodym. Dualities and Riesz representation theorems. The spectral theorem, improved. Riesz redux
Resolutions of the identity.

Every element of $L^\infty(P)$ can be approximated in the $\| \cdot \|_\infty$ norm by simple functions, and hence the integral

$$O(f) = \int \limits_\mathbb{R} f dP$$

is defined as the strong limit of the integrals of the corresponding simple functions. The map $f \mapsto O(f)$ is linear, multiplicative, and satisfies

$$O(\overline{f}) = O(f)^*$$

and

$$\|O(f)\| \leq \|f\|_\infty$$

as before.

I now return to the general study of measures and integration.
Absolute continuity.

Suppose we are given two integrals, $I$ and $J$ on the same space $L$. That is, both $I$ and $J$ satisfy the three conditions of linearity, positivity, and the monotone limit property that went into our definition of the term “integral”. We say that $J$ is absolutely continuous with respect to $I$ if every set which is $I$ null (i.e. has measure zero with respect to the measure associated to $I$) is $J$ null.
Bounded integrals or measures.

The integral $I$ is said to be **bounded** if

$$I(1) < \infty,$$

or, what amounts to the same thing, that

$$\mu_I(S) < \infty$$

where $\mu_I$ is the measure associated to $I$. 
The Radon-Nikodym theorem.

We will first formulate the Radon-Nikodym theorem for the case of bounded integrals, where there is a very clever proof due to von-Neumann which reduces it to the Riesz representation theorem in Hilbert space theory.

**Theorem**

[Radon-Nikodym] Let $I$ and $J$ be bounded integrals, and suppose that $J$ is absolutely continuous with respect to $I$. Then there exists an element $f_0 \in \mathcal{L}^1(I)$ such that

$$J(f) = I(ff_0) \quad \forall \ f \in \mathcal{L}^1(J).$$

The element $f_0$ is unique up to equality almost everywhere (with respect to $\mu_I$).
Proof, after von-Neumann.

Consider the linear function

\[ K := I + J \]

on \( L \). Then \( K \) satisfies all three conditions in our definition of an integral, and in addition is bounded. We know from the case \( p = 2 \) of the completeness of \( L^p \) that \( L^2(K) \) is a (real) Hilbert space. (Assume for this argument that we have passed to the quotient space so an element of \( L^2(K) \) is an equivalence class of functions.) The fact that \( K \) is bounded, says that \( 1 := 1_s \in L^2(K) \). If \( f \in L^2(K) \) then the Cauchy-Schwartz inequality says that

\[ K(|f|) = K(|f| \cdot 1) = (|f|, 1)_{2,K} \leq \|f\|_{2,K} \|1\|_{2,K} < \infty \]

so \(|f|\) and hence \( f \) are elements of \( L^1(K) \).
Furthermore, if \( f \in L^2(K) \),

\[
|J(f)| \leq J(|f|) \leq K(|f|) \leq \|f\|_{2,K} \|1\|_{2,K}.
\]

So \( f \mapsto J(f) \) is a bounded linear function on \( L^2(K) \). We conclude from the real version of the Riesz representation theorem, that there exists a unique \( g \in L^2(K) \) such that

\[
J(f) = (f, g)_{2,K} = K(fg).
\]

If \( A \) is any \( K \)-measurable subset of \( S \), then \( 0 \leq J(1_A) = K(1_A g) \)
so \( g \) is non-negative. (More precisely, \( g \) is equivalent \( K \)-almost everywhere to a function which is non-negative.)
We obtain inductively

\[ J(f) = K(fg) = \]
\[ I(fg) + J(fg) = I(fg) + I(fg^2) + J(fg^2) = \]
\[ \vdots \]
\[ = I \left( f \cdot \sum_{i=1}^{n} g^i \right) + J(fg^n). \]
Let $N$ be the set of all $x$ where $g(x) \geq 1$. Taking $f = 1_N$ in the preceding string of equalities shows that

$$J(1_N) \geq nI(1_N).$$

Since $n$ is arbitrary, we have proved

**Lemma**

*The set where $g \geq 1$ has $I$ measure zero.*
\[ J(f) = K(fg) = \]
\[ I(fg) + J(fg) = I(fg) + I(fg^2) + J(fg^2) = \]
\[ \vdots \]
\[ = I \left( f \cdot \sum_{i=1}^{n} g^i \right) + J(fg^n). \]

We have not yet used the assumption that \( J \) is absolutely continuous with respect to \( I \). We now use this assumption to conclude that \( N \) is also \( J \)-null. This implies that if \( f \geq 0 \) and \( f \in L^1(J) \) then \( fg^n \downarrow 0 \) almost everywhere \((J)\), and hence by the dominated convergence theorem

\[ J(fg^n) \downarrow 0. \]
\[ J(f) = K(fg) = L(fg) + J(fg) = L(fg) + L(fg^2) + J(fg^2) = \ldots = L \left( f \cdot \sum_{i=1}^{n} g^i \right) + J(fg^n). \]

We have proved that 
\[ J(fg^n) \downarrow 0. \]

Plugging this back into the above string of equalities shows (by the monotone convergence theorem for \( I \)) that \( f \sum_{i=1}^{\infty} g^n \) converges in the \( L^1(I) \) norm to \( J(f) \).
converges in the $L^1(I)$ norm to $J(f)$. In particular, since $J(1) < \infty$, we may take $f = 1$ and conclude that $\sum_{i=1}^{\infty} g^i$ converges in $L^1(I)$. So set

$$f_0 := \sum_{i=1}^{\infty} g^i \in L^1(I).$$
We have
\[ f_0 = \frac{1}{1 - g} \text{ almost everywhere} \]
so
\[ g = \frac{f_0 - 1}{f_0} \text{ almost everywhere} \]
and
\[ J(f) = I(ff_0) \]
for \( f \geq 0, f \in L^1(J) \). By breaking any \( f \in L^1(J) \) into the difference of its positive and negative parts, we conclude that (1) holds for all \( f \in L^1(J) \). The uniqueness of \( f_0 \) (almost everywhere \((I)\)) follows from the uniqueness of \( g \) in \( L^2(K) \). \( \square \)
Extensions, I.

The Radon Nikodym theorem can be extended in two directions. First of all, let us continue with our assumption that $I$ and $J$ are bounded, but drop the absolute continuity requirement. Let us say that an integral $H$ is absolutely singular with respect to $I$ if there is a set $N$ of $I$-measure zero such that $H(h) = 0$ for any $h$ vanishing on $N$.

Let us now go back to Lemma 10:

**Lemma**

*The set where $g \geq 1$ has $I$ measure zero.*
Define $J_{\text{sing}}$ by

$$J_{\text{sing}}(f) = J(1_N f).$$

Then $J_{\text{sing}}$ is singular with respect to $I$, and we can write

$$J = J_{\text{cont}} + J_{\text{sing}}$$

where

$$J_{\text{cont}} = J - J_{\text{sing}} = J(1_{N^c} \cdot).$$

Then we can apply the rest of the proof of the Radon Nikodym theorem to $J_{\text{cont}}$ to conclude that

$$J_{\text{cont}}(f) = I(ff_0)$$

where $f_0 = \sum_{i=1}^{\infty} (1_{N^c} g)^i$ is an element of $L^1(I)$ as before. In particular, $J_{\text{cont}}$ is absolutely continuous with respect to $I$. 
Extensions, II.

A second extension is to certain situations where $S$ is not of finite measure. We say that a function $f$ is **locally** $L^1$ if $f1_A \in L^1$ for every set $A$ with $\mu(A) < \infty$. We say that $S$ is **σ-finite** with respect to $\mu$ if $S$ is a countable union of sets of finite $\mu$ measure. This is the same as saying that $1 = 1_S \in \mathcal{B}$. If $S$ is σ-finite then it can be written as a disjoint union of sets of finite measure. If $S$ is σ-finite with respect to both $I$ and $J$ it can be written as the disjoint union of countably many sets which are both $I$ and $J$ finite. So if $J$ is absolutely continuous with respect to $I$, we can apply the Radon-Nikodym theorem to each of these sets of finite measure, and conclude that there is an $f_0$ which is locally $L^1$ with respect to $I$, such that $J(f) = I(ff_0)$ for all $f \in L^1(J)$, and $f_0$ is unique up to almost everywhere equality. Read the details in Loomis.
The map from $L^q \to (L^p)^*$

Recall that Hölder's inequality says that

$$\left| \int f g d\mu \right| \leq \|f\|_p \|g\|_q$$

if $f \in L^p$ and $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. For the rest of this lecture we will assume without further mention that this relation between $p$ and $q$ holds. Hölder’s inequality implies that we have a map from

$$L^q \to (L^p)^*$$

sending $g \in L^p$ to the continuous linear function on $L^p$ which sends

$$f \mapsto I(fg) = \int fg d\mu.$$
The map from $L^q \to (L^p)^*$ is injective. Furthermore, Hölder’s inequality says that the norm of this map from $L^q \to (L^p)^*$ is $\leq 1$. In particular, this map is injective.

The theorem we want to prove is that under suitable conditions on $S$ and $I$ (which are more general even that $\sigma$-finiteness) this map is surjective for $1 \leq p < \infty$.

We will first prove the theorem in the case where $\mu(S) < \infty$, that is when $I$ is a bounded integral. For this we will will use our results about the “variations of a of a bounded linear functional”:
Duality of $L^p$ and $L^q$ when $\mu(S) < \infty$.

**Theorem**

Suppose that $\mu(S) < \infty$ and that $F$ is a bounded linear function on $L^p$ with $1 \leq p < \infty$. Then there exists a unique $g \in L^q$ such that

$$F(f) = (f, g) = I(fg).$$

Here $q = p/(p - 1)$ if $p > 1$ and $q = \infty$ if $p = 1$. 

Duality of $L^p$ and $L^q$ when $\mu(S) < \infty$. 
Duality of $L^p$ and $L^q$ when $\mu(S) < \infty$.

Proof, 1.

Consider the restriction of $F$ to $L$. We know that $F = F^+ - F^-$ where both $F^+$ and $F^-$ are linear and non-negative and are bounded with respect to the $\| \cdot \|_p$ norm on $L$. The monotone convergence theorem implies that if $f_n \downarrow 0$ then $\|f_n\|_p \to 0$ and the boundedness of $F^+$ with respect to the $\| \cdot \|_p$ says that

$$\|f_n\|_p \to 0 \implies F^+(f_n) \to 0.$$

So $F^+$ satisfies all the axioms for an integral, and so does $F^-$. If $f$ vanishes outside a set of $\mu$ measure zero, then $\|f\|_p = 0$. Applied to a function of the form $f = 1_A$ we conclude that if $A$ has $\mu = \mu_1$ measure zero, then $A$ has measure zero with respect to the measures determined by $F^+$ or $F^-$. 
Duality of $L^p$ and $L^q$ when $\mu(S) < \infty$.

Proof, 2- Using Radon-Nikodym.

We can apply the Radon-Nikodym theorem to conclude that there are functions $g^+$ and $g^-$ which belong to $L^1(I)$ and such that

$$F^\pm(f) = I(fg^\pm)$$

for every $f$ which belongs to $L^1(F^\pm)$. In particular, if we set $g := g^+ - g^-$ then

$$F(f) = I(fg)$$

for every function $f$ which is integrable with respect to both $F^+$ and $F^-$, in particular for any $f \in L^p(I)$. We must show that $g \in L^q$. 
Duality of $L^p$ and $L^q$ when $\mu(S) < \infty$.

Proof, 3.

We first treat the case where $p > 1$. Suppose that $0 \leq f \leq |g|$ and that $f$ is bounded. Then

$$I(f^q) \leq I(f^{q-1} \cdot \text{sgn}(g)g) = F(f^{q-1} \cdot \text{sgn}(g)) \leq \|F\|_p \|f^{q-1}\|_p.$$ 

So

$$I(f^q) \leq \|F\|_p I(f^{(q-1)p})^{\frac{1}{p}}.$$ 

Now $(q-1)p = q$ so we have

$$I(f^q) \leq \|F\|_p I(f^q)^{\frac{1}{p}} = \|F\|_p I(f^q)^{1 - \frac{1}{q}}.$$ 

This gives

$$\|f\|_q \leq \|F\|_p$$

for all $0 \leq f \leq |g|$ with $f$ bounded. Choose such functions $f_n$ with $f_n \uparrow |g|$. It follows from the monotone convergence theorem that $|g|$ and hence $g \in L^q(I)$ proving the theorem for $p > 1$. 

Math 212a
Duality of $L^p$ and $L^q$ when $\mu(S) < \infty$.

The case $p = 1$

Let us now give the argument for $p = 1$. We want to show that $\|g\|_\infty \leq \|F\|_1$. Suppose that $\|g\|_\infty \geq \|F\|_1 + \epsilon$ where $\epsilon > 0$. Consider the function $1_A$ where

$$A := \{x : |g(x)| \geq \|F\|_1 + \frac{\epsilon}{2}\}.$$ 

Then

$$((\|F\|_1 + \frac{\epsilon}{2})\mu(A) \leq I(1_A|g|) = I(1_A\text{sgn}(g)g) = F(1_A\text{sgn}(g))$$

$$\leq \|F\|_1\|1_A\text{sgn}(g)\|_1 = \|F\|_1\mu(A)$$

which is impossible unless $\mu(A) = 0$, contrary to our assumption.

\[\square\]
The case where $\mu(S) = \infty$.

The case where $\mu(S) = \infty$, $p > 1$.

Let us first consider the case where $p > 1$. If we restrict the functional $F$ to any subspace of $L^p$ its norm can only decrease. Consider a subspace consisting of all functions which vanish outside a subset $S_1$ where $\mu(S_1) < \infty$. We get a corresponding function $g_1$ defined on $S_1$ (and set equal to zero off $S_1$) with $\|g_1\|_q \leq \|F\|_p$ and $F(f) = I(fg_1)$ for all $f$ belonging to this subspace. If $(S_2, g_2)$ is a second such pair, then the uniqueness part of the theorem shows that $g_1 = g_2$ almost everywhere on $S_1 \cap S_2$. Thus we can consistently define $g_{12}$ on $S_1 \cup S_2$. Let

$$b := \operatorname{lub}\{\|g_\alpha\|_q\}$$

taken over all such $g_\alpha$. Since this set of numbers is bounded by $\|F\|_p$ this least upper bound is finite.
We can therefore find a nested sequence of sets $S_n$ and corresponding functions $g_n$ such that

$$\|g_n\|_q \uparrow b.$$ 

By the triangle inequality, if $n > m$ then

$$\|g_n - g_m\|_q \leq \|g_n\|_q - \|g_m\|_q$$

and so, as in your proof of the $L^2$ Martingale convergence theorem in Problem set 1, this sequence is Cauchy in the $\| \cdot \|_q$ norm. Hence there is a limit $g \in L^q$ and $g$ is supported on

$$S_0 := \bigcup S_n.$$
The case where $\mu(S) = \infty$.

There can be no pair $(S', g')$ with $S$ disjoint from $S_0$ and $g' \neq 0$ on a subset of positive measure of $S'$. Indeed, if this were the case, then we could consider $g + g'$ on $S \cup S'$ and this would have a strictly larger $\| \cdot \|_q$ norm than $\|g\|_q = b$, contradicting the definition of $b$. (It is at this point in the argument that we use $q < \infty$ which is the same as $p > 1$.) Thus $F$ vanishes on any function which is supported outside $S_0$. We have thus reduced the theorem to the case where $S$ is $\sigma$-finite.
The case where $\mu(S) = \infty$.

If $S$ is $\sigma$-finite, decompose $S$ into a disjoint union of sets $A_i$ of finite measure. Let $f_m$ denote the restriction of $f \in L^p$ to $A_m$ and let $h_m$ denote the restriction of $g$ to $A_m$. Then

$$\sum_{m=1}^{\infty} f_m = f$$

as a convergent series in $L^p$ and so

$$F(f) = \sum_{m} F(f_m) = \sum_{m} \int_{A_m} f_m h_m$$

and this last series converges to $I(fg)$ in $L^1$.

So we have proved that $(L^p)^* = L^q$ in complete generality when $p > 1$, and for $\sigma$-finite $S$ when $p = 1$. 

Math 212a
212a1415 Radon Nikodym. Dualities and Riesz representation theorems. The spectral theorem, improved. Riesz redux
The case where $\mu(S) = \infty$.

It may happen (and will happen for the Haar integral on the most general locally compact group) that we don’t even have $\sigma$-finiteness. But we will have the following more complicated condition: Recall that a set $A$ is called integrable (by Loomis) if $1_A \in \mathcal{B}$. Now suppose that

$$S = \bigcup_{\alpha} S_\alpha$$

where this union is disjoint, but possibly uncountable, of integrable sets, and with the property that every integrable set is contained in at most a countable union of the $S_\alpha$. A set $A$ is called measurable if the intersections $A \cap S_\alpha$ are all integrable, and a function is called measurable if its restriction to each $S_\alpha$ has the property that the restriction of $f$ to each $S_\alpha$ belongs to $\mathcal{B}$, and further, that either the restriction of $f^+$ to every $S_\alpha$ or the restriction of $f^-$ to every $S_\alpha$ belongs to $L^1$. 
If we find ourselves in this situation, then we can find a \( g_\alpha \) on each \( S_\alpha \) since \( S_\alpha \) is \( \sigma \)-finite, and piece these all together to get a \( g \) defined on all of \( S \). If \( f \in L^1 \) then the set where \( f \neq 0 \) can have intersections with positive measure with only countably many of the \( S_\alpha \) and so we can apply the result for the \( \sigma \)-finite case for \( p = 1 \) to this more general case as well.

Our proof of Radon-Nikodym and the duality between \( L^p \) and \( L^q \) works in generality. Our proof of the Fubini theorem is much simpler in the Hausdorff setting than in general, so we will restrict our attention for the rest of this lecture to the Hausdorff case.
Fubini’s theorem in the Hausdorff setting.

**Theorem**

Let $S_1$ and $S_2$ be locally compact Hausdorff spaces and let $I$ and $J$ be non-negative linear functionals on $L(S_1)$ and $L(S_2)$ respectively. Then

$$I_x (J_y h(x, y)) = J_y (I_x (h(x, y))$$

for every $h \in L(S_1 \times S_2)$ in the obvious notation, and this common value is an integral on $L(S_1 \times S_2)$. 
Proof via Stone-Weierstrass. The equation in the theorem is clearly true if \( h(x, y) = f(x)g(y) \) where \( f \in L(S_1) \) and \( g \in L(S_2) \) and so it is true for any \( h \) which can be written as a finite sum of such functions. Let \( h \) be a general element of \( L(S_1 \times S_2) \). Then we can find compact subsets \( C_1 \subset S_1 \) and \( C_2 \subset S_2 \) such that \( h \) is supported in the compact set \( C_1 \times C_2 \). The functions of the form

\[
\sum f_i(x)g_i(y)
\]

where the \( f_i \) are all supported in \( C_1 \) and the \( g_i \) in \( C_2 \), and the sum is finite, form an algebra which separates points.
So for any $\epsilon > 0$ we can find a $k$ of the above form with

$$\|h - k\|_\infty < \epsilon.$$

Let $B_1$ and $B_2$ be bounds for $I$ on $L(C_1)$ and $J$ on $L(C_2)$ as provided by our lemma about non-negative functionals on compact sets. Then

$$|J_y h(x, y) - \sum J(g_i)f_i(x)| = |J_y (f - k)| < \epsilon B_2.$$
\[
|J_y h(x, y) - \sum J(g_i) f_i(x)| = |J_y (f - k)| < \epsilon B_2.
\]

This shows that \(J_y h(x, y)\) is the uniform limit of continuous functions supported in \(C_1\) and so \(J_y h(x, y)\) is itself continuous and supported in \(C_1\). It then follows that \(I_x (J_y h)\) is defined, and that

\[
|I_x (J_y h(x, y)) - \sum I(f)_i J(g_i)| \leq \epsilon B_1 B_2.
\]

Doing things in the reverse order shows that

\[
|I_x (J_y h(x, y)) - J_y (I_x (h(x, y)))| \leq 2\epsilon B_1 B_2.
\]

Since \(\epsilon\) is arbitrary, this gives the equality in the theorem. Since this (same) functional is non-negative, it is an integral by the first of the Riesz representation theorems above. \(\square\)
So far, in our treatment of the Daniell-Stone integral and the Riesz representation theorems, I have been following Loomis rather slavishly. But for certain applications, for example to Wiener measure, a more precise version of the Riesz representation theorem is needed. So I want to give an alternative proof of the Riesz representation theorem which will give some information about the possible \( \sigma \)-fields on which \( \mu \) is defined. In particular, I want to show that we can find a \( \mu \) (which is possibly an extension of the \( \mu \) given by our previous proof of the Riesz representation theorem) which is defined on a \( \sigma \)-field which contains the Borel field \( \mathcal{B}(X) \). Recall that \( \mathcal{B}(X) \) is the smallest \( \sigma \)-field which contains the open sets.
Regular measures.

Let \( \mathcal{F} \) be a \( \sigma \)-field which contains \( \mathcal{B}(X) \). A (non-negative valued) measure \( \mu \) on \( \mathcal{F} \) is called **regular** if

1. \( \mu(K) < \infty \) for any compact subset \( K \subset X \).
2. For any \( A \in \mathcal{F} \)
   \[
   \mu(A) = \inf \{ \mu(U) : A \subset U, \ U \text{ open} \}
   \]
3. If \( U \subset X \) is open then
   \[
   \mu(U) = \sup \{ \mu(K) : K \subset U, \ K \text{ compact} \}.
   \]

The second condition is called **outer regularity** and the third condition is called **inner regularity**.
The improved version of the Riesz representation theorem.

**Theorem**

Let $X$ be a locally compact Hausdorff space, $L$ the space of continuous functions of compact support on $X$, and $I$ a non-negative linear functional on $L$. Then there exists a $\sigma$-field $\mathcal{F}$ containing $\mathcal{B}(X)$ and a non-negative regular measure $\mu$ on $\mathcal{F}$ such that

$$I(f) = \int f \, d\mu$$  \hspace{1cm} (2)

for all $f \in L$. Furthermore, the restriction of $\mu$ to $\mathcal{B}(X)$ is unique.
The proof of this theorem hinges on some topological facts whose true place is in the chapter on metric spaces, but I will prove them here. The importance of the theorem is that it will allow us to derive some conclusions about spaces which are very huge (such as the space of “all” paths in $\mathbb{R}^n$) but are nevertheless locally compact (in fact compact) Hausdorff spaces. It is because we want to consider such spaces, that the earlier proof, which hinged on taking limits of sequences in the very definition of the Daniell integral, is insufficient to get at the results we want.
Proposition

Let $X$ be a Hausdorff space, and let $H$ and $K$ be disjoint compact subsets of $X$. Then there exist disjoint open subsets $U$ and $V$ of $X$ such that $H \subset U$ and $K \subset V$.

This we actually isprove in the chapter on metric spaces in my real variable notes.

Proposition

Let $X$ be a locally compact Hausdorff space, $x \in X$, and $U$ an open set containing $x$. Then there exists an open set $O$ such that

- $x \in O$
- $\overline{O}$ is compact, and
- $\overline{O} \subset U$. 
Propositions in topology.

**Proof.**

Choose an open neighborhood $W$ of $x$ whose closure is compact, which is possible since we are assuming that $X$ is locally compact. Let $Z = U \cap W$ so that $\overline{Z}$ is compact and hence so is $H := \overline{Z} \setminus Z$. Take $K := \{x\}$ in the preceding proposition. We then get an open set $V$ containing $x$ which is disjoint from an open set $G$ containing $\overline{Z} \setminus Z$. Take $O := V \cap Z$. Then $x \in O$ and $\overline{O} \subset \overline{Z}$ is compact and $O$ has empty intersection with $\overline{Z} \setminus Z$, and hence is is contained in $Z \subset U$.  \[\square\]
Proposition

Let $X$ be a locally compact Hausdorff space, $K \subset U$ with $K$ compact and $U$ open subsets of $X$. Then there exists a $V$ with

$$K \subset V \subset \overline{V} \subset U$$

with $V$ open and $\overline{V}$ compact.

Proof.

Each $x \in K$ has a neighborhood $O$ with compact closure contained in $U$, by the preceding proposition. The set of these $O$ cover $K$, so a finite subcollection of them cover $K$ and the union of this finite subcollection gives the desired $V$.  

Math 212a

212a1415 Radon Nikodym. Dualities and Riesz representation theorems. The spectral theorem, improved. Riesz redux
Proposition

Let $X$ be a locally compact Hausdorff space, $K \subset U$ with $K$ compact and $U$ open. Then there exists a continuous function $h$ with compact support such that

$$1_K \leq h \leq 1_U$$

and

$$\text{supp}(h) \subset U.$$

Proof.

Choose $V$ as in preceding Proposition. By Urysohn’s lemma applied to the compact space $\overline{V}$ we can find a function $h : \overline{V} \to [0, 1]$ such that $h = 1$ on $K$ and $f = 0$ on $\overline{V} \setminus V$. Extend $h$ to be zero on the complement of $\overline{V}$. Then $h$ does the trick.
Proposition

Let $X$ be a locally compact Hausdorff space, $f \in L$, i.e. $f$ is a continuous function of compact support on $X$. Suppose that there are open subsets $U_1, \ldots, U_n$ such that

$$\text{supp}(f) \subset \bigcup_{i=1}^{n} U_i.$$ 

Then there are $f_1, \ldots, f_n \in L$ such that

$$\text{supp}(f_i) \subset U_i$$

and

$$f = f_1 + \cdots + f_n.$$ 

If $f$ is non-negative, the $f_i$ can be chosen so as to be non-negative.
Proof.

By induction, it is enough to consider the case $n = 2$. Let $K := \text{supp}(f)$, so $K \subset U_1 \cup U_2$. 
Let

\[ L_1 := K \setminus U_1, \quad L_2 := K \setminus U_2. \]

So \( L_1 \) and \( L_2 \) are disjoint compact sets.
By Proposition 6.1 we can find disjoint open sets $V_1, V_2$ with

$$L_1 \subset V_1, \quad L_2 \subset V_2.$$
Propositions in topology.

Let \( L_1 \subset V_1, \) \( L_2 \subset V_2. \)

Set \( K_1 := K \setminus V_1, \) \( K_2 := K \setminus V_2. \) Then \( K_1 \) and \( K_2 \) are compact, and

\[
K = K_1 \cup K_2, \quad K_1 \subset U_1, \quad K_2 \subset U_2.
\]

Choose \( h_1 \) and \( h_2 \) as in Proposition 6.4. Then set

\[
\phi_1 := h_1, \quad \phi_2 := h_2 - h_1 \wedge h_2.
\]

Then \( \text{supp}(\phi_1) = \text{supp}(h_1) \subset U_1 \) by construction, and
\( \text{supp}(\phi_2) \subset \text{supp}(h_2) \subset U_2, \) the \( \phi_i \) take values in \([0, 1]\), and, if \( x \in K = \text{supp}(f) \)

\[
\phi_1(x) + \phi_2(x) = (h_1 \vee h_2)(x) = 1.
\]

Then set

\[
f_1 := \phi_1 f, \quad f_2 := \phi_2 f.
\]
Proof of the uniqueness and regularity of the measure.

**Proof of the uniqueness of the $\mu$ restricted to $\mathcal{B}(X)$.**

It is enough to prove that

$$\mu(U) = \sup\{l(f) : f \in L, \, 0 \leq f \leq 1_U\} \quad (3)$$

$$= \sup\{l(f) : f \in L, \, 0 \leq f \leq 1_U, \, \text{supp}(f) \subset U\} \quad (4)$$

for any open set $U$, since either of these equations determines $\mu$ on any open set $U$ and hence for the Borel field. Since $f \leq 1_U$ and both are measurable functions, it is clear that $\mu(U) = \int 1_U$ is at least as large as the expression on the right hand side of (3). This in turn is as least as large as the right hand side of (4) since the supremum in (2) is taken over as smaller set of functions that that of (3). So it is enough to prove that $\mu(U)$ is $\leq$ the right hand side of (4).
Proof of the uniqueness and regularity of the measure.

To prove: \( \mu(U) \leq \sup\{I(f) : f \in L, \ 0 \leq f \leq 1_U, \ \text{supp}(f) \subseteq U\} \). (*)

Proof.

Let \( a < \mu(U) \). Interior regularity implies that we can find a compact set \( K \subseteq U \) with

\[
a < \mu(K).
\]

Recall that we proved that there exists a continuous function \( f \) with compact support such that

\[
1_K \leq f \leq 1_U
\]

Then \( a < I(f) \), and so the right hand side of (*) is \( \geq a \). Since \( a \) was any number \( < \mu(U) \), we conclude that \( \mu(U) \) is \( \leq \) the right hand side of (*).
Outline of the proof of existence

We have proved the uniqueness part of our improved Riesz representation theorem. We now turn to the more difficult existence part. We will proceed as follows: We will

- define a function $m^*$ defined on all subsets,
- show that it is an outer measure,
- show that the set of measurable sets in the sense of Caratheodory include all the Borel sets, and that
- integration with respect to the associated measure $\mu$ assigns $I(f)$ to every $f \in L$. 
Definition of $m^*$.

Define $m^*$ on open sets by

$$m^*(U) = \sup\{l(f) : f \in L, 0 \leq f \leq 1_U, \text{ supp}(f) \subset U\}. \quad (5)$$

Clearly, if $U \subset V$ are open subsets, $m^*(U) \leq m^*(V)$. Next define $m^*$ on an arbitrary subset by

$$m^*(A) = \inf\{m^*(U) : A \subset U, \text{ U open}\}. \quad (6)$$

Since $U$ is contained in itself, this does not change the definition on open sets. It is clear that $m^*(\emptyset) = 0$ and that $A \subset B$ implies that $m^*(A) \leq m^*(B)$. So to prove that $m^*$ is an outer measure we must prove countable subadditivity.
Countable additivity of $m^*$ on open sets

We will first prove countable subadditivity on open sets, and then use the $\epsilon/2^n$ argument to conclude countable subadditivity on all sets:

Suppose $\{U_n\}$ is a sequence of open sets. We wish to prove that

$$m^* \left( \bigcup_n U_n \right) \leq \sum_n m^*(U_n).$$

(7)
Counable additivity of $m^*$ on open sets, 2

Set

$$U := \bigcup_{n} U_n,$$

and suppose that

$$f \in L, \ 0 \leq f \leq 1_U, \ \text{supp}(f) \subset U.$$ 

Since $\text{supp}(f)$ is compact and contained in $U$, it is covered by finitely many of the $U_i$. In other words, there is some finite integer $N$ such that

$$\text{supp}(f) \subset \bigcup_{n=1}^{N} U_n.$$
Counable additivity of $m^*$ on open sets, 3

There is some finite integer $N$ such that

$$\text{supp}(f) \subset \bigcup_{n=1}^{N} U_n.$$  

By our “partition of unity” Proposition we can write

$$f = f_1 + \cdots + f_N, \quad \text{supp}(f_i) \subset U_i, \quad i = 1, \ldots, N.$$  

Then

$$I(f) = \sum I(f_i) \leq \sum m^*(U_i),$$

using the original definition of $m^*$ on an open set for each $U_i$. Replacing the finite sum on the right hand side of this inequality by the infinite sum, and then taking the supremum over $f$ proves (7), where we use the original definition (5) of $m^*$ once again.
Proof of the uniqueness and regularity of the measure.

**Countable subadditivity in general.**

Let \( \{A_n\} \) be any sequence of subsets of \( X \). We wish to prove that

\[
m^* \left( \bigcup_n A_n \right) \leq \sum_n m^*(A_n).
\]

This is automatic if the right hand side is infinite. So assume that

\[
\sum_n m^*(A_n) < \infty
\]
Countable subadditivity in general, 2.

and choose open sets $U_n \supset A_n$ so that

$$m^*(U_n) \leq m^*(A_n) + \frac{\epsilon}{2^n}.$$ 

Then $U := \bigcup U_n$ is an open set containing $A := \bigcup A_n$ and

$$m^*(A) \leq m^*(U) \leq \sum m^*(U_i) \leq \sum m^*(A_n) + \epsilon.$$ 

Since $\epsilon$ is arbitrary, we have proved countable subadditivity.
Measurability of the Borel sets.

Let $\mathcal{F}$ denote the collection of subsets which are measurable in the sense of Caratheodory for the outer measure $m^*$. We wish to prove that $\mathcal{F} \supset \mathcal{B}(X)$. Since $\mathcal{B}(X)$ is the $\sigma$-field generated by the open sets, it is enough to show that every open set is measurable in the sense of Caratheodory, i.e. that

$$m^*(A) \geq m^*(A \cap U) + m^*(A \cap U^c)$$

(8)

for any open set $U$ and any set $A$ with $m^*(A) < \infty$: If $\epsilon > 0$, choose an open set $V \supset A$ with

$$m^*(V) \leq m^*(A) + \epsilon$$

which is possible by the definition (6).
Proof of the uniqueness and regularity of the measure.

**Measurability of the Borel sets, 2.**

We will show that

\[ m^*(V) \geq m^*(V \cap U) + m^*(V \cap U^c) - 2\epsilon. \quad (9) \]

This will then imply that

\[ m^*(A) \geq m^*(A \cap U) + m^*(A \cap U^c) - 3\epsilon \]

and since \( \epsilon > 0 \) is arbitrary, this will imply (8).
Proof of the uniqueness and regularity of the measure.

**Measurability of the Borel sets, 3.**

Choose an open set $V \supset A$ with

$$m^*(V) \leq m^*(A) + \epsilon$$

To prove:

$$m^*(V) \geq m^*(V \cap U) + m^*(V \cap U^c) - 2\epsilon. \quad (9)$$
Using the definition (5), we can find an $f_1 \in L$ such that

$$f_1 \leq 1_{V \cap U} \quad \text{and} \quad \text{supp}(f_1) \subset V \cap U$$

with

$$I(f_1) \geq m^*(V \cap U) - \epsilon.$$

So:
Proof of the uniqueness and regularity of the measure.

Let \( f_1 \in L \), \( f_1 \leq 1_{V \cap U} \) and \( \text{supp}(f_1) \subset V \cap U \) with

\[
I(f_1) \geq m^*(V \cap U) - \epsilon.
\]

Let \( K := \text{supp}(f_1) \). Then \( K \subset U \) and so \( K^c \supset U^c \) and \( K^c \) is open. Hence \( V \cap K^c \) is an open set and

\[
V \cap K^c \supset V \cap U^c.
\]

Using the definition (5), we can find an \( f_2 \in L \) such that

\[
f_2 \leq 1_{V \cap K^c} \quad \text{and} \quad \text{supp}(f_2) \subset V \cap K^c
\]

with

\[
I(f_2) \geq m^*(V \cap K^c) - \epsilon.
\]

But \( m^*(V \cap K^c) \geq m^*(V \cap U^c) \) since \( V \cap K^c \supset V \cap U^c \). So

\[
I(f_2) \geq m^*(V \cap U^c) - \epsilon.
\]
Proof of the uniqueness and regularity of the measure.

**Measurability of the Borel sets, concluded.**

\[ f_1 \in L, \ f_1 \leq 1_{V \cap U} \quad \text{and} \quad \text{supp}(f_1) \subset V \cap U \quad \text{with} \]
\[ I(f_1) \geq m^*(V \cap U) - \epsilon. \]

\[ f_2 \in L \quad \text{with} \]
\[ I(f_2) \geq m^*(V \cap U^c) - \epsilon. \]

So

\[ f_1 + f_2 \leq 1_K + 1_{V \cap K^c} \leq 1_V \]

since \( K = \text{supp}(f_1) \subset V \) and \( \text{supp}(f_2) \subset V \cap K^c \). Also

\[ \text{supp}(f_1 + f_2) \subset (K \cup V \cap K^c) = V. \]

Thus \( f = f_1 + f_2 \in L \) and so by (5),

\[ I(f_1 + f_2) \leq m^*(V). \]

This proves (9) and hence that all Borel sets are measurable.
Proof of the uniqueness and regularity of the measure.

Regularity of the measure.

Let $\mu$ be the measure associated to $m$ on the $\sigma$-field $\mathcal{F}$ of measurable sets. We will now prove that $\mu$ is regular. The condition of outer regularity is automatic, since this was how we defined $\mu(A) = m^*(A)$ for a general set $A \in \mathcal{F}$.

So we must prove inner regularity. We first show that compact sets have finite measure, which was our first condition for regularity.
Compact sets have finite measure.

If \( K \) is a compact subset of \( X \), we can find an \( f \in L \) such that \( 1_K \leq f \) by an earlier Proposition. Let \( 0 < \epsilon < 1 \) and set

\[
U_\epsilon := \{ x : f(x) > 1 - \epsilon \}.
\]

Then \( U_\epsilon \) is an open set containing \( K \). If \( 0 \leq g \in L \) satisfies \( g \leq 1_{U_\epsilon} \), then \( g = 0 \) on \( U_\epsilon^c \), and for \( x \in U_\epsilon \), \( g(x) \leq 1 \) while \( f(x) > 1 - \epsilon \). So

\[
g \leq \frac{1}{1 - \epsilon} f
\]

and hence \( m^*(U_\epsilon) \leq \frac{1}{1 - \epsilon} I(f) \). So,

\[
\mu(K) \leq m^*(U_\epsilon) \leq \frac{1}{1 - \epsilon} I(f) < \infty.
\]
Reviewing the preceding argument, we see that we have in fact proved the more general statement

**Proposition**

*If* $A$ *is any subset of* $X$ *and* $f \in L$ *is such that*

$$1_A \leq f$$

*then*

$$m^*(A) \leq l(f).$$
Interior regularity.

We now prove interior regularity, which is very important in certain applications. We wish to prove that

\[ \mu(U) = \sup\{ \mu(K) : K \subset U, \ K \text{ compact} \}, \]

for any open set \( U \), where, according to (5),

\[ m^*(U) = \sup\{ I(f) : f \in L, 0 \leq f \leq 1_U, \ \text{supp}(f) \subset U \}. \]

Since \( \text{supp}(f) \) is compact, and contained in \( U \), we will be done if we show that

\[ f \in L, \ 0 \leq f \leq 1 \Rightarrow I(f) \leq \mu(\text{supp}(f)). \quad (10) \]
Proof of the uniqueness and regularity of the measure.

To show:

\[ f \in L, \ 0 \leq f \leq 1 \Rightarrow I(f) \leq \mu(\text{supp}(f)). \quad (10) \]

Let \( V \) be an open set containing \( \text{supp}(f) \). By definition (5),

\[ \mu(V) \geq I(f) \]

and, since \( V \) is an arbitrary open set containing \( \text{supp}(f) \), we have

\[ \mu(\text{supp}(f)) \geq I(f) \]

using the definition (6) of \( m^*(\text{supp}(f)) \). \( \Box \)

In the course of this argument we have proved

**Proposition**

If \( g \in L, 0 \leq g \leq 1_K \) where \( K \) is compact, then

\[ I(g) \leq \mu(K). \]
Proof of the uniqueness and regularity of the measure.

**Conclusion of the proof of the improved Riesz representation theorem.**

Finally, we must show that all the elements of $L$ are integrable with respect to $\mu$ and

$$I(f) = \int f \, d\mu. \quad (11)$$

Since the elements of $L$ are continuous, they are Borel measurable. As every $f \in L$ can be written as the difference of two non-negative elements of $L$, and as both sides of (11) are linear in $f$, it is enough to prove (11) for non-negative functions.
Proof of the uniqueness and regularity of the measure.

Following Lebesgue, divide the “y-axis” up into intervals of size $\epsilon$. That is, let $\epsilon$ be a positive number, and, for every positive integer $n$ set

$$f_n(x) := \begin{cases} 0 & \text{if } f(x) \leq (n-1)\epsilon \\ f(x) - (n-1)\epsilon & \text{if } (n-1)\epsilon < f(x) \leq n\epsilon \\ \epsilon & \text{if } n\epsilon < f(x) \end{cases}$$

If $(n-1)\epsilon \geq \|f\|_{\infty}$ only the first alternative can occur, so all but finitely many of the $f_n$ vanish, and they all are continuous and have compact support so belong to $L$. Also

$$f = \sum f_n$$

this sum being finite, as we have observed, and so

$$I(f) = \sum I(f_n).$$
Proof of the uniqueness and regularity of the measure.

\[ l(f) = \sum l(f_n). \]

Set \( K_0 := \text{supp}(f) \) and

\[ K_n := \{ x : f(x) \geq n\epsilon \} \quad n = 1, 2, \ldots. \]

Then the \( K_i \) are a nested decreasing collection of compact sets, and

\[ \epsilon 1_{K_n} \leq f_n \leq \epsilon 1_{K_{n-1}}. \]

By the two preceding propositions we have

\[ \epsilon \mu(K_n) \leq l(f_n) \leq \epsilon \mu(K_{n-1}). \]

On the other hand, the monotonicity of the integral (and its definition) imply that

\[ \epsilon \mu(K_n) \leq \int f_n d\mu \leq \epsilon \mu(K_{n-1}). \]
Proof of the uniqueness and regularity of the measure.

\[ \mu(K_n) \leq I(f_n) \leq \mu(K_{n-1}) \]

and

\[ \mu(K_n) \leq \int f_n \, d\mu \leq \mu(K_{n-1}). \]

Summing these inequalities gives

\[ \epsilon \sum_{i=1}^{N} \mu(K_n) \leq I(f) \leq \epsilon \sum_{i=0}^{N-1} \mu(K_n) \]

\[ \epsilon \sum_{i=1}^{N} \mu(K_n) \leq \int f \, d\mu \leq \epsilon \sum_{i=0}^{N-1} \mu(K_n) \]

where \( N \) is sufficiently large.
Thus $I(f)$ and $\int f d\mu$ lie within a distance

$$\epsilon \sum_{i=0}^{N-1} \mu(K_n) - \epsilon \sum_{i=1}^{N} \mu(K_n) = \epsilon \mu(K_0) - \epsilon \mu(K_N) \leq \epsilon \mu(\text{supp}(f))$$

of one another. Since $\epsilon$ is arbitrary, we have proved (11) and completed the proof of the improved Riesz representation theorem.