Math 212a Lecture 1

Introduction.

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The official title of this course is

“Theory of functions of a real variable”.

I assume that you know what the real numbers are - that they are the completion of the rational numbers, and that they are complete in the sense that any Cauchy sequence of real numbers converges to a real number limit.

In the notes to this course I review the concept of “completion” in the more general context of metric spaces. But the treatment depends on existence of the real number system.
1 Some history.
   • Irrationality of $\pi$.
   • Transcendental numbers.

2 What is the definition of a function?

3 Some dubious 18th century sums.
   • Toeplitz’s theorem.
   • Cesaro summability.
   • Abel summability

4 Fourier and Fourier series.

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History: irrational numbers.

The Greeks know that $\sqrt{2}$ is irrational. A proof appears in a scholium to Euclid’s elements. To the best of my knowledge, no such statement was made about $\pi$ in Greek mathematics.
The great Islamic mathematician, astronomer, and historian, al Biruni (937-1018) states in *al-Qanun al-Masudi* (3 vols. Hyderabad 1954-56) vol., Part III ch. 5 p.303 “the perimeter of the circle to its diameter is a certain ratio; its number (meaning the measure of the perimeter) to its number (that of the diameter) is a ratio which is surd (Arabic: samm)”. I am indebted to Prof. Tony Levy for this reference. So al Biruni “knew” that $\pi$ is irrational.

The great Jewish jurist and philosopher, Maimonides (1135-1205), also states in his commentary to the Mishnah (Eruvin I-5) that $\pi$ is irrational. Neither of these authors offer a proof of this fact.

The first proof was given by Lambert in 1761. Lambert’s proof depended on the integral calculus.
Ivan Niven’s version of the proof.

Lambert’s proof was chewed over and over for several centuries. The proof I am about to present due to Niven seems to have reduced the argument to a bare minimum.

To prove: that \( \pi = \frac{a}{b} \) where \( a \) and \( b \) are (positive) integers is impossible.

Suppose the contrary. For any positive integer \( n \) define \( f = f_n \) by

\[
f(x) = \frac{x^n(a - bx)^n}{n!} \text{ so } 0 < f(x) < \frac{\pi^n a^n}{n!}
\]
on \( 0 < x < \pi \). Define

\[
F := f - f'' + f^iv - f^vi + \cdots + (-1)^n f^{(2n)}.
\]
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\[
f(x) = \frac{x^n(a-bx)^n}{n!}\] so \( f^{(k)}(0) = 0 \) for \( k < n \) and \( = \) the \( (k-n) \)th derivative of \( (a-bx)^n \) at zero for \( k \geq n \) which is an integer. Hence \( F(0) \) is an integer. Also

\[
f(\pi - x) = \frac{1}{n!} \left( \frac{a}{b} - x \right)^n (a - a + bx)^n = f(x)
\]

so \( F(\pi) \) is an integer.
We have shown that $F(0)$ and $F(\pi)$ are integers. Now

$$F = f - f'' + f^iv - f^vi + \cdots + (-1)^n f^{(2n)}$$

so

$$F'' + F = f.$$

We have

$$(F' \sin - F \cos)' = F'' \sin + F' \cos - F' \cos + F \sin = (F'' + F) \sin = f \sin$$

so by the fundamental theorem of the calculus,

$$\int_0^\pi f(x) \sin(x) dx = (F' \sin - F \cos)|_0^\pi = F(\pi) + F(0)$$

is a positive integer as the integrand is positive on $(0, \pi)$. But by choosing $n$ large enough we can make the integral as small as we like. This contradiction shows that $\pi = \frac{a}{b}$ is impossible.
The first person to show that there exist numbers which are “transcendental”, i.e. not solutions of an algebraic equation with rational coefficients, was Liouville, who showed in 1844 that the number

$$\sum_{n=1}^{\infty} 10^{-n!}$$

is transcendental.

In 1874 Cantor showed the “most” real numbers are transcendental by proving the the algebraic numbers are countable and that the real numbers are uncountable.
Hermite proved in 1873 that $e$ is transcendental, and Lindemann proved in 1884 that $\pi$ is transcendental using the methods developed by Hermite. A consequence of Lindemann’s result is that it is impossible to “square the circle” by means of straightedge and compass, for such a construction would lead to an algebraic number $\theta$ with $\theta^2 = \pi$, and if $\theta$ is algebraic so is $\theta^2$.

I will present a proof that $e$ is transcendental in the appendix to this lecture.
It is therefore somewhat surprising that it was not until 1872 that a definition was given of the real numbers. In fact two definitions, one via ”Dedekind cuts” given by Dedekind and one by Cantor via equivalence classes of Cauchy sequences of rational numbers.
The meaning of the word “function” however will not be settled in this course. Here is an example of the kind of trouble we will have to face: The first person to give a correct proof of the convergence of Fourier series to the function it should represent (for a reasonable class of functions) was Dirichlet in 1829. In the course of his analysis of what Fourier series mean he introduced the so-called “function” $1_{\mathbb{Q}}$ which takes the value 1 on the rational numbers and the value 0 on the irrational numbers.
It can not be evaluated or graphed on any computer since computers use floating point arithmetic and see only rational numbers. From the point of view of a computer $1 \mathbb{Q}$ is identically one.

From the point of view of Lebesgue integration theory which we will study later on this semester, two functions which agree except on a set of “measure zero” are to be regarded as the same. The rational numbers have measure zero in the Lebesgue theory. So from the point of view of Lebesgue’s theory $1 \mathbb{Q}$ is identically zero.
Can we actually apply the definition? For example, Euler introduced the number $\gamma$ defined as the limit as $n \to \infty$ of $\sum_{k=1}^{n} \frac{1}{k} - \log n$. To this day we do not know if $\gamma$ is rational or irrational. So what is $1_{\mathbb{Q}}(\gamma)$?

Conclusion: We should not admit Dirichlet’s $1_{\mathbb{Q}}$ into our menagerie of functions.
What about Dirac’s “δ-function”? 

This was defined by Dirac to be the “function” $\delta$ such that $\delta(x) = 0$ for $x \neq 0$ and is “so infinite at 0” that 

$$\int_{-\infty}^{\infty} \delta(x)g(x)dx = g(0)$$

for any continuous function $g$. From the point of view of Lebesgue’s (or Riemann’s) integration theory, this statement is nonsense: the value of a function at a single point (even if infinite) can not affect the value of an integral. When I was a student learning about Lebesgue integration, we would laugh at the physicists for believing in the existence of such a function.
Yet Dirac’s $\delta$ function is not only important in physics, we will be making a lot of use of it in this course. It is easily implemented on the computer. It is the rule which assigns to each continuous function $g$ the value $g(0)$. In other words, it is the map

$$g \mapsto g(0).$$

It is not a function of a real variable but rather a function of a function of a real variable. We will definitely include the Dirac $\delta$ function into our menagerie.
Since many of the “functions” we will study involve infinite series, we start by looking at some infinite series from the viewpoint of 18th century mathematicians:

Let \( a := 1 - 1 + 1 - 1 + \cdots \). Then

\[
a = 1 - (1 - 1 + 1 - 1 + \cdots) = 1 - a.
\]

So \( a = \frac{1}{2} \).

Let \( s := 1 - 2 + 3 - 4 + 5 - 6 + \cdots \). Then

\[
s = 1 + (-2 + 3 - 4 + \cdots)
= 1 - (1 - 1 + 1 - 1 + \cdots) - (1 - 2 + 3 - 4 + 5 - 6 + \cdots)
= 1 - a - s
\]

So \( s = \frac{1}{4} \).
Let $Z := 1 + 2 + 3 + 4 + \cdots$. Then

$$Z - s = 0 + 4 + 0 + 8 + \cdots = 4Z.$$  

So $3Z = -s$ or

$$Z = -\frac{1}{12}.$$
Euler: “ich glaube, dass jede series einem bestimmten Wert haben müsse. Um aber allen Schwierigkeiten, welche dagegen gemacht worden, zu begegnen, so sollte dieser Wert nicht mit dem Namen der Summe belegt werden, weil man mit dieser Wort gemeiniglich eine solchen Begriff zu verknipfen pflegt, als wenn die Summe durch eine wirkliche Summierung herausgebracht würde: welche Idee bei den seribus divergentibus nicht stattfindet...”
Leonhard Euler

Born: 15 April 1707 in Basel, Switzerland
Died: 18 Sept 1783 in St Petersburg, Russia
Hardy *Divergent series* (p.5) “it does not occur to a modern mathematician that a collection of symbols should have a ‘meaning’ until one has been assigned to it by definition. [This] was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, “by $X$ we mean $Y$”. … mathematicians before Cauchy asked not ‘How shall we define $1 - 1 + 1 - 1 + \cdots$?’ but ‘What is $1 - 1 + 1 - 1 + \cdots$?’, and this habit led them into unnecessary perplexities and controversies which were often really verbal.”
Some justifications.

The geometric series

\[ \frac{1}{1 - z} = 1 + z + z^2 + z^3 + z^4 + \cdots \]

converges for \(|z| < 1\). The function \(z \mapsto \frac{1}{1-z}\) exists and is holomorphic for all \(z \neq 1\), and its value at \(-1\) is \(\frac{1}{2}\).

So if we define the **Abel sum** of a series \(\sum a_n\) to be the limit (if it exists)

\[ \lim_{x \to 1} \sum a_n x^n \]

then we can say that the Abel sum of \(1 - 1 + 1 - 1 + \cdots\) is \(\frac{1}{2}\).
Differentiating the above geometric series for $|z| < 1$ gives

$$\frac{1}{(1 - z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots.$$ 

Again, the function $z \mapsto \frac{1}{(1 - z)^2}$ is holomorphic for all $z \neq 1$ and its value at $-1$ is $\frac{1}{4}$. So the Abel sum of $1 - 2 + 3 - 4 + \cdots$ is $\frac{1}{4}$. 
Justification of the formula $1 + 2 + 3 + \cdots = -\frac{1}{12}$.

This lies a bit deeper: The series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

converges for $\Re s > 1$ and so defines a holomorphic function $s \mapsto \zeta(s)$ there. It turns out that $\zeta$ has a singularity at $s = 1$ but can be analytically continued to all $z \neq 1$. Its value at $-1$ is $-\frac{1}{12}$.

In fact, Euler computes this value by means of what is known today as the Euler-MacLaurin summation formula.

We shall get these results in lecture 6 after we develop the Fourier transform - especially the Mellin transform.
Toeplitz’s theorem.

Let $M = (m_{nk})$ be an infinite matrix (indexed by the positive integers $\mathbb{N}$) which satisfies the following three conditions:

1. $\lim_{n \to \infty} m_{nk} = 0$ for all $k$ (the terms in every column $\to 0$).

2. There is an $H$ such that $\sum_k |m_{nk}| \leq H$ for every $n$. (The rows are uniformly absolutely summable.)

3. $\lim_{n \to \infty} \sum_k m_{nk} = 1$. (The row sums tend to 1.)

Let $s_k \to s$ be a convergent sequence of real numbers. (For example $s_k$ might be the partial sums of a convergent series.) Since the $s_k$ are bounded, condition 2) implies that the series

$$\sigma_n := \sum_k m_{nk}s_k$$

converges for each $n$. 

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Theorem

\[ \sigma_n \rightarrow s. \]

Write \( s_k - s = r_k \) so \( \sigma_n = s \sum_k m_{nk} + \sum_k m_{nk} r_k \). By hypothesis 3), \( s \sum_k m_{nk} \rightarrow s \). Since \( r_k \rightarrow 0 \) we are reduced to proving the theorem for the case \( s = 0 \).

So we need to prove the following:
Hypotheses:

1. \( \lim_{n \to \infty} m_{nk} = 0 \) for all \( k \) (the terms in every column \( \to 0 \)).

2. There is an \( H \) such that \( \sum_k |m_{nk}| \leq H \) for every \( n \). (The rows are uniformly absolutely summable.)

3. \( \lim_{n \to \infty} \sum_k m_{nk} = 1 \). (The row sums tend to 1.)

4. \( s_k \to 0 \).

Let

\[ \sigma_n := \sum_k m_{nk}s_k. \]

To prove:

\[ \sigma_n \to 0. \]
Proof.

For any $\epsilon > 0$ choose $N = N(\epsilon)$ so large that $|s_k| < \frac{\epsilon}{2H}$ for $k > N$. By 1) we may then choose $P = P(\epsilon)$ so large that

$$|m_{nk}| < \frac{\epsilon}{2 \sum_{k=1}^{N} |s_k|}$$

for $k = 1, 2, \ldots, N$, for $n > P$.

Then for $n > P$ we have

$$|\sigma_n| \leq \sum_{k=1}^{N} |m_{nk}s_k| + \frac{\epsilon}{2H} \sum_{k>N} |m_{nk}|$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2H} \sum_{k=1}^{\infty} |m_{nk}| \leq \epsilon.$$
Given any matrix $M$ satisfying the conditions of Toeplitz’s theorem, it is possible that the sequence $M \cdot s$ makes sense for some sequence $s$ without $s$ being convergent. That is, it is conceivable that the series $\sigma_n = \sum_k m_{nk}s_k$ all converge. It is also possible that the $\sigma_n$ converge to some limit $s$. Then we will say that

$$s_n \xrightarrow{M} s,$$

or that “$\sum a_j$ is $M$ - summable” to $s$, if $s_k = \sum_{1}^{k} a_j$ is the partial sum of a series $\sum a_j$. 
Cesaro summability.

For example, the matrix

\[ C := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

clearly satisfies the hypotheses of Toeplitz’s theorem. Then

\[ \sigma_n = \frac{1}{n} \left( s_1 + s_2 + \cdots + s_n \right). \]

If \( s_1 = 1, s_2 = 0, s_3 = 1 \) etc., then \( \sigma_n \to \frac{1}{2} \). These values of \( s_k \) are the partial sums of the series \( 1 - 1 + 1 - 1 + \cdots \). We say that this series is Cesaro summable to \( \frac{1}{2} \).
Abel summability.

Let $0 \leq r_n < 1$, $r_n \to 1$. Let $a_{nk} := (1 - r_n)r_n^k$. (Here we are indexing over the non-negative integers.) Clearly $\lim_{n \to \infty} a_{nk} = 0$ for each fixed $k$, and $\sum_k |a_{nk}| = \sum_k a_{nk} = 1$ by the geometric series. So

$$A = (a_{nk}) = ((1 - r_n)r_n^k)$$

satisfies the hypotheses of Toeplitz’s theorem.

For this choice of matrix we have:
\[ \sigma_n = (1 - r_n) \sum_{k=0}^{\infty} s_k r_n^k \]

\[ = s_0 + (s_0 - s_1) r_n + (s_2 - s_1) r_n^2 + \cdots. \]

So if the \( s_n \) are partial sums:

\( s_0 = c_0, \ s_1 = c_0 + c_1, \ s_2 = c_0 + c_1 + c_2 \) etc., then

\[ \sigma_n = \sum_{k \geq 0} c_k r_n^k. \]

If the \( \sigma_n \) approach a limit we have “Abel summability relative to this subsequence”. If the limit as \( r \to 1 \) of \( \sum_k c_k r^k \) exists (not merely for a subsequence) then we have we have what is known as **Abel summability**. (It is easy to check that Cesaro summability implies Abel summability.)
Some history.

Daniel Bernoulli(1753) proposed a trigonometric series as a solution for the problem of vibrating strings and Euler found the formula for the “Fourier coefficients”

\[ f(x) = \sum a_n e^{inx}, \quad a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx \]

in 1757.
The problem of how "heat diffuses" in a continuous medium defied the mathematics of early 1800’s. This problem was solved by Fourier using “intuitive methods”. He submitted his manuscript *Theorie de la propagation de la chaleur* to the Institut de France on 21 December 1807. The prize committee consisted of Laplace, Lagrange, Lacroix and Monge, with Poisson acting as secretary. They rejected the manuscript. Lagrange was particularly firm, questioning the meaning of the equality sign in

$$f(x) = \sum a_n e^{inx}, \quad a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx.$$
For example for the “square wave”

\[
    f(x) := \begin{cases} 
        -1 & -\pi \leq x < 0, \\
        1 & 0 \leq x < \pi 
    \end{cases}
\]

the series is

\[
    \frac{4}{\pi} \sum_{n \geq 1} \frac{\sin(2n - 1)x}{2n - 1}.
\]

How could this series converge when the series of its coefficients

\[
    \frac{4}{\pi} \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right)
\]

diverges?
Poisson summarized the rejection in 5 pages saying essentially that the paper was rejected on the grounds that it contained nothing new or interesting. Fourier submitted a revised version to essentially the same group of judges in 1811 as a candidate for the Grand prix de mathematiques for 1812. Although they awarded him the prize, they refused to publish the paper in the *Mémoires de l’Académie des Sciences*. In 1824 Fourier became the Secretary of the Academy and he then had his 1811 paper published.
Fourier’s influence.

In addition to stimulating the development of mathematics and physics for the next two centuries, the work of Fourier changed the notion of a function. Previously a function had been a natural object, such as a polynomial or exponential or trigonometric function. In short, a function was something given by a closed expression. They were like biological species to be discovered and classified. Now, after Fourier, a function could be invented and be quite arbitrary. Also, there was some controversy as to the meaning of the definite integral, say as it appears in the formula above for the Fourier coefficients. In the 18th century, an integral was thought of as an anti-derivative. This is the way an operation like “int” is handled by a symbolic manipulation program like Maple today.
The fact that the integral is the difference of the anti-derivatives at the end-points was a definition, not a theorem. Since the anti-derivatives of some very important functions, such as $e^{-x^2/2}$ can not be expressed in terms of elementary known functions, what meaning does an expression such as $\int_a^b e^{-x^2/2} \, dx$ have? It was Fourier’s important idea to bypass the anti-derivative, and define the definite integral as the area under the curve. Thus the definite integral becomes primary, and the indefinite integral or anti-derivative becomes secondary.
Jean Baptiste Joseph Fourier

Born: 21 March 1768 in Auxerre, Bourgogne, France
Died: 16 May 1830 in Paris, France
Dirichlet.

Getting back to Fourier series, the problem of establishing their convergence was open. Poisson published an erroneous proof in 1820, and Cauchy published a flawed proof in 1826 - the error pointed out by Dirichlet in 1829 where he, Dirichlet, gave the first correct proof of sufficient conditions for convergence - that a piecewise smooth integrable function converges at every point to the average of its right and left hand limits. This result was extended by Jordan in 1881 to all functions of bounded variation.

Dirichlet also realized the once we extend the notion of a function, there will be certain functions for which “the area under the curve” definition of the definite integral makes no sense, such as his “function” $1_\mathbb{Q}$. 
Johann Peter Gustav Lejeune Dirichlet

Born: 13 Feb 1805 in Düren, French Empire (now Germany)
Died: 5 May 1859 in Göttingen, Hanover (now Germany)
Riemann.

But the “area under the curve” definition did make sense for continuous functions, or functions which were continuous with only finitely many jumps. This was greatly extended by Riemann in his Habilitationschrift entitled On the representability of a function by trigonometric series written in 1854, but not published until after his death in 1867. He used his new concept of integration (today known as the Riemann integral) to conclude that the Fourier coefficients of any Riemann integrable function tend to zero as $n \to \infty$. This was generalized by Lebesgue, to what is today known as the Riemann-Lebesgue lemma.
Riemann’s notion of the integral prevailed until the beginning of the twentieth century with the arrival of the Lebesgue integral. Lebesgue was also very interested in Fourier series, and the first application his new integral were to this subject. Lebesgue’s notion of integral extended to a much wider class of functions - Riemann integrability is equivalent to continuity almost everywhere. There was much more freedom in taking limits under the integral sign. The theory of Lebesgue involves the concept of the “measure” of a set and the associated notion of an integral of a function.
Lebesgue’s notion of measure was extended by Caratheodory in 1905 to account for the area of $k$-dimensional subsets of $n$-dimensional space, where $k$ is an integer, and this was generalized by Hausdorff in 1917 to allow the possibility of $k$ being any real number, thus introducing the notion of “fractional dimension”. This line of work was continued by a small group of researchers (mainly Besicovitch and his coworkers) but exploded into the public consciousness by the work of Mandelbrot in the 1980’s with the advent of computer graphics. “Fractals” are now part of the everyday language.
Daniel and Stone.

The integration aspect of Lebesgue’s theory, starting with the idea that an integral is a linear function on a certain class of functions, and that this notion was primary, with measure theory secondary was introduced by Daniell in 1911 and brought to perfection by Stone in 1942. In the meanwhile, Hilbert spaces such as the space $L^2$ of square integrable functions, and more generally the $L^p$ spaces consisting of functions for which $\int_{\mathbb{R}} |f|^p < \infty$, $p \geq 1$ were being studied. A key theorem proved by F. Riesz in the 1930’s says that $L^p$, $p > 1$ is the dual space of $L^q$ where $p$ and $q$ are related by

$$\frac{1}{p} + \frac{1}{q} = 1.$$
In other words, an element of $L^p$ can be thought of (and is the most general) bounded linear function on $L^q$ where we use the norm
\[ \|g\|_q := \left( \int_{\mathbb{R}} |g|^q \right)^{1/q} \]
on $L^q$. In other words, instead of thinking of an element of $L^p$ as a function (on its domain of definition, for example the real numbers, so assigning a complex number to every real number), we can think of it as a *functional*, a rule which assigns numbers to functions.
For example, an element of $L^2$ can be thought of as assigning a number to any element of $L^2$ in a way which is continuous with respect to $L^2$ convergence. We will exploit this “duality”, and the Riesz representation theorem in its various guises will be a key ingredient in this course.
Back to the square wave.

Recall that the “square wave” is defined as

\[ f(x) := \begin{cases} 
-1 & \quad -\pi \leq x < 0, \\
1 & \quad 0 \leq x < \pi 
\end{cases} \]

Its Fourier series is

\[ \frac{4}{\pi} \sum_{n \geq 1} \frac{\sin(2n - 1)x}{2n - 1}. \]

Indeed:
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$a_n = \frac{1}{2\pi} \left[ \int_0^\pi e^{-in\pi} \, dx - \int_{-\pi}^0 e^{-in\pi} \, dx \right]$

$= \frac{1}{2\pi} \frac{2}{-in} \left[ e^{-in\pi} - 1 \right].$

This last expression in brackets vanishes when $n$ is even, and equals $-2$ if $n$ is odd. So the even terms of the Fourier series vanish, while the sum of the terms involving $e^{inx}$ for odd $n$ is

$\frac{1}{2\pi} \frac{4}{i|n|} \left[ e^{inx} - e^{-inx} \right] = \frac{4}{\pi|n|} \sin |n|x.$
Let $s_n$ denote the $n$-th partial sum. Here are the graphs of $s_n$ for some values of $n$: Notice that there is blip overshooting the square wave whose distance from the horizontal (of about .18) does not decrease as $n$ increases. Rather the width of the blip appears to go to zero.
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Figure 1: The graphs of $s$ and $s_3$, $s_7$ and $s_{15}$ over $[-\pi, \pi]$. 
Let us verify these facts: To locate the maxima or minima we compute

\[
\frac{ds_n}{dx} = \frac{4}{\pi} \sum_{k=1}^{n} \cos(2k - 1)x
\]

which we can sum as a geometric sum (for \(x \neq n\pi\)) as

\[
\frac{4}{\pi} \frac{1}{2} \left( e^{ix} \sum_{k=0}^{n-1} e^{2kix} + e^{-ix} \sum_{k=0}^{n-1} e^{-2kix} \right)
\]

\[
= \frac{4}{\pi} \frac{1}{2} \left( e^{ix} \frac{1 - e^{2inx}}{1 - e^{2ix}} + e^{-ix} \frac{1 - e^{-2inx}}{1 - e^{-2ix}} \right) = \frac{2 \sin 2nx}{\pi \sin x}.
\]
We have shown that \( \frac{ds_n}{dx} = \frac{2}{\pi} \frac{\sin 2nx}{\sin x} \) for \( x \neq n\pi \). But this extends by continuity to all values of \( x \). This function vanishes at \( x = \pm \frac{\pi}{2n} \), so these are the extrema closest to the origin. Differentiating one more time we check that we have a maximum to the right and a minimum to the left of the origin. Evaluating \( s_n \) at its maximum near the origin gives:

\[
\begin{align*}
    s_n \left( \frac{\pi}{2n} \right) &= \frac{4}{\pi} \sum_{k=1}^{n} \frac{\sin[(2k-1)\frac{\pi}{2n}]}{2k-1} \\
    &= \frac{2}{\pi} \sum_{k=1}^{n} \frac{\sin[(2k-1)\frac{\pi}{2n}]}{(2k-1)\frac{\pi}{2n}} \frac{\pi}{n} \to \frac{2}{\pi} \int_{0}^{\pi} \frac{\sin t}{t} dt.
\end{align*}
\]
Evaluating the integral gives approximately 1.18. Indeed, we have the power series expansion

$$\frac{\sin t}{t} = \sum_{r=0}^{\infty} \frac{(-1)^r t^{2r}}{(2r + 1)!}$$

which is valid for all $t$. We may integrate term by term. We get

$$\int_0^{\pi} \frac{\sin t}{t} \, dt = \pi \left( 1 - \frac{\pi^2}{3!3} + \frac{\pi^4}{5!5} - \frac{\pi^6}{6!6} + \cdots \right).$$

As the series is an oscillating series with decreasing terms, the error involved in truncating the series is at most the first term neglected, and stopping at an odd term overshoots the mark while stopping at an even term undershoots the mark. Summing the first five terms gives 1.179384 while adding the next term gives 1.178957.
So this overshoot never disappears. This is the Gibbs phenomenon, Gibbs, 1899, first discovered by Wilbraham in 1848 and proved as a general phenomenon - that the overshoot at the jump is about 9 percent of the total jump by Bocher in 1906. So naturally it is called the Gibbs phenomenon. It appears from the figure that we have uniform convergence to the square wave as long as we stay a positive distance away from the jump. This is an illustration of Dirichlet’s theorem which we will discuss in the next lecture. We also will learn that the square wave belongs to $L^2$ and that its Fourier series converges to it in the $L^2$ norm. So the Gibbs phenomenon shows something about the subtlety of $L^2$ convergence.

In the next two slides I will reproduce the original Gibbs paper which appeared in *NATURE* on April 27, 1899. Gibbs worked with the zigzag function rather than the square wave function.
Fourier's Series.

I should like to correct a careless error which I made (Nature, December 29, 1898) in describing the limiting form of the family of curves represented by the equation

\[ y = 2 \left( \sin x - \frac{1}{2} \sin 2x \ldots \pm \frac{1}{n} \sin nx \right) \ldots (1) \]

as a zigzag line consisting of alternate inclined and vertical portions. The inclined portions were correctly given, but the vertical portions, which are bisected by the axis of X, extend beyond the points where they meet the inclined portions, their total lengths being expressed by four times the definite integral

\[ \int_0^n \frac{\sin u}{u} du. \]

If we call this combination of inclined and vertical lines \( C_n \), and the graph of equation (1) \( C_n \), and if any finite distance \( d \) be specified, and we take for \( n \) any number greater than \( 100/d^2 \), the distance of every point in \( C_n \) from \( C \) is less than \( d \), and the distance of every point in \( C \) from \( C_n \) is also less than \( d \). We may therefore call \( C \) the limit (or limiting form) of the sequence of curves of which \( C_n \) is the general designation.

But this limiting form of the graphs of the functions expressed by the sum (1) is different from the graph of the function expressed by the limit of that sum. In the latter the vertical portions are wanting, except their middle points.
I think this distinction important; for (with exception of what relates to my unfortunate blunder described above), whatever differences of opinion have been expressed on this subject seem due, for the most part, to the fact that some writers have had in mind the limit of the graphs, and others the graph of the limit of the sum. A misunderstanding on this point is a natural consequence of the usage which allows us to omit the word limit in certain connections, as when we speak of the sum of an infinite series. In terms thus abbreviated, either of the things which I have sought to distinguish may be called the graph of the sum of the infinite series.

J. Willard Gibbs.

New Haven, April 12.
Josiah Willard Gibbs

Born: 11 Feb 1839 in New Haven, Connecticut, USA
Died: 28 April 1903 in New Haven, Connecticut, USA
Proof that $e$ is transcendental.

The proof given here is taken from *Topics in Algebra* by I.N. Herstein, pages 216 -219. Hermite’s proof was simplified by Hilbert, and further streamlined by Hurwitz. Herstein follows Hurwitz’s proof.
Let $f$ be a polynomial of degree $r$ with real coefficients. Let

$$F := f + f' + f'' + \cdots + f^{(r)}.$$  

For any differentiable function $g$ we have

$$\frac{d}{dx}(e^{-x}g(x)) = e^{-x}(g' - g).$$  

So

$$\frac{d}{dx}(e^{-x}F(x)) = -e^{-x}f(x).$$  

The mean value theorem asserts that if $g$ is continuously differentiable on an interval $[x_1, x_2]$ then

$$\frac{g(x_2) - g(x_1)}{x_2 - x_1} = g'(x_1 + \theta(x_2 - x_1))$$

for some $\theta$ with $0 < \theta < 1$.

We apply this to the function $e^{-x}F(x)$ and the interval $[0, k]$, to obtain
\[ e^{-k} F(k) - F(0) = -ke^{-\theta_k} f(k\theta_k) \]

where \( 0 < \theta_k < 1 \). Multiply this by \( e^k \) to obtain
\[ F(k) - e^k F(0) = -ke^{(1-\theta_k)} f(k\theta_k) \quad (Eq_k). \]

Call the right hand side of this equation \( \epsilon_k \). It depends, of course, on \( f \). Now suppose that \( e \) satisfies an algebraic equation
\[ c_n e^n + c_{n-1} e^{n-1} + \cdots + c_0 = 0 \quad (*) \]

where the \( c_j \) are integers and \( c_0 \neq 0 \). Multiply \((Eq_k)\) by \( c_k \) and sum from \( k = 1 \) to \( n \), to obtain
\[ c_1 F(1) + c_2 F(2) + \cdots + c_n F(n) - F(0)\left[c_1 e + c_2 e^2 + \cdots + c_n e^n\right] \]
\[ = c_1 \epsilon_1 + \cdots + c_n \epsilon_n. \]

In view of the supposed equation \((*)\), the expression in brackets equals \(-c_0\). So we obtain
This is true for any polynomial \( f \) (where, recall, the \( \epsilon_k \) depend on \( f \)). We will derive a contradiction by showing that we can find a prime number \( p > n \), \( p > c_0 \), and a polynomial \( f = f_p \) such that

1. \( F(i) \) is an integer divisible by \( p \) for \( i = 1, \ldots, n \).
2. \( F(0) \) is an integer not divisible by \( p \). Since \( p \not| c_0 \), this implies that
3. \( c_0 F(0) + c_1 F(1) + \cdots c_n F(n) \) is an integer not divisible by \( p \).
4. \( |c_1 \epsilon_1 + \cdots + c_n \epsilon_n| < 1 \).

The last two items contradict one another in view of (1). Indeed, the left hand side of (1) is an integer, and the only integer less than one in absolute value is 0. But 0 is divisible by \( p \).
So we want to show that we can choose a large $p$ and a polynomial $f_p$ so that items 1, 2, and 4 hold.

The polynomial (used by Hermite) is $f = f_p$ where

$$f(x) := \frac{1}{(p-1)!} x^{p-1} (1-x)^p (2-x)^p \cdots (n-x)^p.$$

For $i < p$ the polynomial $f$ and its first $p-1$ derivatives vanish at $1, 2, \ldots, n$. If we multiply out we get the expansion

$$f(x) = \frac{1}{(p-1)!} \left[ (n!)^p x^{p-1} + a_0 x^p + a_1 x^{p+1} + \cdots \right]$$

where the $a_j$ are integers.
Now if $g$ is any polynomial with integer coefficients then, for $i \geq p$, the $i$-th derivative of $\frac{1}{(p-1)!} g$ is a polynomial with integer coefficients divisible by $p$, since the $i$-th derivative of $x^k$ vanishes for $k < i$ and is $k(k-1) \cdots (k-i+1)x^{k-i}$ for $k \geq i$. So for the $f = f_p$ above, and $i \geq p$, the $i$-th derivative of $f$ is a polynomial with integer coefficients. Hence $F$ when evaluated at $1, 2, \ldots, n$ is an integer divisible by $p$. This establishes item 1).
As for item 2), the first $p - 2$ derivatives of $f$ vanish at 0, and for $i \geq p$, the $i$-th derivative of $f$ is an integer divisible by $p$. But $f^{(p-1)}(0) = (n!)^p$ and since $p > n$ this is not divisible by $p$. This establishes item 2) and hence item 3).
As to item 4), we have

$$
\epsilon_k = -ke^{(1-\theta_k)k}f(k\theta_k) = -\frac{e^{(1-\theta_k)k}(1 - \theta_k)^p \cdots (n - k\theta_k)^p(k\theta_k)^p}{(p - 1)!}
$$

Since $0 < \theta_k < 1$ and $k \leq n$ we see that

$$
|\epsilon_k| \leq \frac{e^n n^p (n!)^p}{(p - 1)!}
$$

which we can make as small as we like by choosing $p$ sufficiently large. This establishes 4) and with it proves that $e$ is transcendental.