TRANSFERRING SYMMETRY DOWNWARD AND APPLICATIONS

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ABSTRACT. For $\mathcal{K}$ an abstract elementary class satisfying the amalgamation property, we prove a downward transfer of the symmetry property for splitting (previously isolated by the first author). This allows us to deduce uniqueness of limit models from categoricity in a cardinal of high-enough cofinality, improving on a 16-year-old result of Shelah:

Theorem 0.1. *Suppose λ and μ are cardinals so that $\operatorname{cf}(\lambda) > \mu \geq \operatorname{LS}(\mathcal{K})$ and assume that $\mathcal{K}$ has no maximal models and is categorical in λ. If $M_0, M_1, M_2 \in \mathcal{K}_\mu$ are such that both M_1 and M_2 are limit models over M_0, we have that $M_1 \cong_{M_0} M_2$.*

Another application of the symmetry transfer utilizes tameness (a locality property for types) and improves on the work of Will Boney and the second author:

Theorem 0.2. *Let $\mu \geq \operatorname{LS}(\mathcal{K})$. If $\mathcal{K}$ is μ-superstable and μ-tame, then:*

1. *If $M_0, M_1, M_2 \in \mathcal{K}_\mu$ are such that both M_1 and M_2 are limit models over M_0, then $M_1 \cong_{M_0} M_2$.*
2. *For any $\lambda > \mu$, the union of an increasing chain of λ-saturated models is λ-saturated.*
3. *There exists a unique type-full good μ^+-frame with underlying class the saturated models in $\mathcal{K}_{\mu^+}$.*

CONTENTS

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1. Introduction

In developing a classification theory for abstract elementary classes (AECs), a major focus has been to find an appropriate generalization of first-order superstability. Approximations isolated in [She99] and [SV99] have provided a mechanism for proving categoricity transfer results (see also [GV06a], [Vasc]). In Chapter IV of [She09], Shelah introduced *solvability* and claims it should be the true definition of superstability in AECs (see Discussion 2.9 in the introduction to [She09]). It seems, however, that under the assumption that the class has amalgamation, a more natural definition is a version of "$\kappa(T) = \aleph_0$", first considered by Shelah and Villaveces in [SV99]. In [GV], it is shown that this definition is equivalent to many others (including solvability and the existence of a *good frame*, a local notion of independence), provided that the AEC satisfies a locality property for types called *tameness* [GV06b].

Without tameness, progress has been made in the study of structural consequences of the Shelah-Villaveces definition of superstability such as the uniqueness of limit models (e.g. [GVV]) or the property that the union of saturated models is saturated ([BVb, Vanc]). Recently in [Vana], the first author isolated a symmetry property for splitting that turns out to be closely related to the uniqueness of limit models.

In this paper we prove a downward transfer theorem for this symmetry property. This allows us to gain insight into all of the aspects of superstability mentioned above.

Theorem 1.1. *Suppose λ and μ are cardinals so that $\lambda > \mu \geq \mathrm{LS}(\mathcal{K})$ and $\mathcal{K}$ is superstable in every $\chi \in [\mu, \lambda]$. Then λ-symmetry implies μ-symmetry.*

Theorem 1.1 (proven at the end of Section 4) improves Theorem 0.1 of [Vanb] which transfers symmetry from μ^+ to μ. We also clarify the relationship between μ-symmetry (a property of μ-splitting) and the symmetry property in good frames (see Section 3). The latter is older and has been studied in the literature: the work of Shelah in [She01] led to [She09, Theorem 3.7], which gives conditions under which a good frame (satisfying a version of symmetry) exists (but uses set-theoretic axioms beyond ZFC and categoricity in two successive cardinals). One should also mention [She09, Theorem IV.4.10] which builds a good frame (in ZFC) from categoricity in a high-enough cardinal[1]. It was observed in [BGKV, Theorem 5.14] that Shelah's proof of symmetry of first-order forking generalizes naturally to give that the symmetry property of any reasonable global independence notion follows from the assumption of no order property. This is used in [Vasa] to build a good frame from tameness and categoricity (the results there are improved in [Vasb, BVb]). As for symmetry *transfers*, Boney [Bon14a] has shown how to transfer symmetry of a good frame *upward* using tameness for types of length two. This was later improved to tameness for types of length one with a more conceptual proof in [BVc].

Theorem 1.1 differs from these works in a few ways. First, we do not assume tameness nor set-theoretic assumptions, and we do not work within the full strength of a frame or with categoricity (only with superstability). Also, we obtain a *downward* and not an upward transfer. The methods of this paper include *towers* whereas the aforementionned result of Boney and the second author use independent sequences to transfer symmetry upward.

A more notable contribution of this paper is an improvement on a 1999 result of Shelah (see [She99, Theorem 6.5]):

Fact 1.2. *Assume $\mathcal{K}$ is an AEC with amalgamation and no maximal models. Let λ and μ be cardinals such that $\mathrm{cf}(\lambda) > \mu > \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in λ, then any limit model of size μ is saturated.*

Shelah claims in a remark immediately following his result that this can be generalized to show that for $M_0, M_1, M_2 \in \mathcal{K}_\mu$, if M_1 and M_2 are limit over M_0, then $M_1 \cong_{M_0} M_2$ (that is, the isomorphism also fixes M_0). This is however not what his proof gives (see the discussion after Theorem 10.17 in [Bal09]). Here we finally prove this stronger statement (this is Theorem 0.1 of the abstract, proven as Corollary 5.3 here).

Combining Theorem 1.1 with tameness, we can also improve Hanf number bounds for several consequences of superstability. With Will Boney, the second author has shown [BVb, Theorem 7.1] that μ-superstability and μ-tameness imply that for all high-enough λ, limit models of size λ are unique (in the strong sense discussed above) and unions of chains of λ-saturated models are saturated. We transfer this behavior downward

[1]Note however the cardinal is very high and the underlying class of the frame is a smaller class of Ehrenfeucht-Mostowski models, although this can be fixed by taking an even larger cardinal. Moreover Shelah's work contains a gap, see the upcoming [BVa] for a fix and a more thorough discussion.

using our symmetry transfer theorem to get that the latter result is actually true starting from $\lambda = \mu^+$, and the former starting from $\lambda = \mu$. This is Theorem 0.2 of the abstract, proven at the end of Section 6.

Another consequence of our work is a better understanding of the relationship between superstability and symmetry. It was claimed in an early version of [GVV] that μ-superstability directly implies the uniqueness of limit models of size μ but an error was later found in the proof. Here we show that this result *is* true if we assume μ-tameness along with superstability. In fact, the stronger Theorem 6.4 states that in μ-tame AECs, μ-superstability already implies μ-symmetry. Many assumptions weaker than tameness (such as the existence of a good μ^+-frame, see Theorem 3.12) suffice to obtain such a conclusion.

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2. Background

All throughout this paper, we assume the amalgamation property:

Hypothesis 2.1. *$\mathcal{K}$ is an AEC with amalgamation.*

For convenience, we fix a big-enough monster model $\mathfrak{C}$ and work inside $\mathfrak{C}$. This is possible since by Remark 2.6, we will have the joint embedding property in addition to the amalgamation property for models of the relevant cardinalities.

Many of the pre-requisite definitions and notations used in this paper can be found in [GVV]. Here we recall the more specialized concepts that we will be using explicitly. We will use the definition of universality from [Van06, Definition I.2.1]:

Definition 2.2. *Let $M, N \in \mathcal{K}$ be such that $M \leq N$. We say that N is* μ-universal over *M if for any $M' \geq M$ with $\|M'\| \leq \mu$, there exists $f : M' \underset{M}{\longrightarrow} N$. We say that N is* universal over *M if N is $\|M\|$-universal over M.*

Towers were introduced in Shelah and Villaveces [SV99] as a tool to prove the uniqueness of limit models. A tower is an increasing sequence of length α of limit models, denoted by $\bar{M} = \langle M_i \in \mathcal{K}_\mu \mid i < \alpha \rangle$, along with a sequence of designated elements $\bar{a} = \langle a_i \in M_{i+1} \backslash M_i \mid i+1 < \alpha \rangle$ and a sequence of designated submodels $\bar{N} = \langle N_i \mid i+1 < \alpha \rangle$ for which $N_i \leq M_i$, ga-tp(a_i/M_i) does not μ-split over N_i, and M_i is universal over N_i (see Definition I.5.1 of [Van06]).

Now we recall a bit of terminology regarding towers. The collection of all towers $(\bar{M}, \bar{a}, \bar{N})$ made up of models of cardinality μ and sequences indexed by α is denoted by $\mathcal{K}^*_{\mu,\alpha}$. For $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$, if $\beta < \alpha$ then we write $(\bar{M}, \bar{a}, \bar{N}) \restriction \beta$ for the tower made of the subseqences $\bar{M} \restriction \beta = \langle M_i \mid i < \beta \rangle$, $\bar{a} \restriction \beta = \langle a_i \mid i+1 < \beta \rangle$, and $\bar{N} \restriction \beta = \langle N_i \mid i+1 < \beta \rangle$. We sometimes abbreviate the tower $(\bar{M}, \bar{a}, \bar{N})$ by $\mathcal{T}$.

Definition 2.3. *For towers $(\bar{M}, \bar{a}, \bar{N})$ and $(\bar{M}', \bar{a}', \bar{N}')$ in $\mathcal{K}^*_{\mu,\alpha}$, we say*

$$(\bar{M}, \bar{a}, \bar{N}) \leq (\bar{M}', \bar{a}', \bar{N}')$$

if for all $i < \alpha$, $M_i \leq M'_i$, $\bar{a} = \bar{a}'$, $\bar{N} = \bar{N}'$ and whenever M'_i is a proper extension of M_i, then M'_i is universal over M_i. If for each $i < \alpha$, M'_i is universal over M_i we will write $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}', \bar{N}')$.

In order to transfer symmetry from λ to μ we will need to consider a generalization of these towers where the models M_i and N_i may have different cardinalities. Fix $\lambda \geq \mu \geq \mathrm{LS}(\mathcal{K})$ and α a limit ordinal $< \mu^+$. We will write $\mathcal{K}^*_{\lambda,\alpha,\mu}$ for the collection of towers of the form $(\bar{M}, \bar{a}, \bar{N})$ where $\bar{M} = \langle M_i \mid i < \alpha \rangle$ is a sequence of models each of cardinality λ and $\bar{N} = \langle N_i \mid i+1 < \alpha \rangle$ is a sequence of models of cardinality μ. We require that for $i < \alpha$, M_i is μ-universal over N_i and ga-tp(a_i/M_i) does not μ-split over N_i.

In a natural way we order these towers by the following adaptation of Definition 2.3.

Definition 2.4. *Let $\lambda \geq \chi \geq \mu \geq \mathrm{LS}(\mathcal{K})$ be cardinals and fix $\alpha < \mu^+$ an ordinal. For towers $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\lambda,\alpha,\mu}$ and $(\bar{M}', \bar{a}', \bar{N}') \in \mathcal{K}^*_{\chi,\alpha,\mu}$, we say*

$$(\bar{M}, \bar{a}, \bar{N}) <_\mu (\bar{M}', \bar{a}', \bar{N}')$$

if for all $i < \alpha$, $M_i \leq M'_i$, $\bar{a} = \bar{a}'$, $\bar{N} = \bar{N}'$, and there is $\theta < \lambda^+$ so that M'_i is a (λ, θ)-limit model witnessed by a sequence $\langle M^\gamma_i \mid \gamma < \theta \rangle$ with $M_i < M^\gamma_0$.

Note that Definition 2.3 is defined only on towers in $\mathcal{K}^*_{\mu,\alpha}$ and is slightly weaker from the ordering $<_\mu$ when restricted to $\mathcal{K}^*_{\mu,\alpha}$. In particular, the models M'_i in the tower $(\bar{M}', \bar{a}, \bar{N})$ $<$-extending $(\bar{M}, \bar{a}, \bar{N})$ are only required to be universal over M_i and limit. It is not necessary that M'_i is limit over M_i as we require if $(\bar{M}', \bar{a}, \bar{N}) <_\mu (\bar{M}, \bar{a}, \bar{N})$.

Towers are particularly suited for superstable abstract elementary classes, in which they are known to exist and in which the union of an increasing chain of towers will be a tower. The definition below is already implicit in [SV99] and has since then been studied in many papers, e.g. [Van06, GVV, Vasb, BVb, GV]. We will use the definition from [Vasb, Definition 10.1]:

Definition 2.5. *$\mathcal{K}$ is* μ-superstable *(or* superstable in μ*) if:*

(1) $\mu \geq \text{LS}(\mathcal{K})$.
(2) $\mathcal{K}_\mu$ is nonempty, has joint embedding, and no maximal models.
(3) $\mathcal{K}$ is stable in μ[2], and:
(4) μ-splitting in $\mathcal{K}$ satisfies the following locality (sometimes called continuity) and "no long splitting chains" properties:

For all infinite α, for every sequence $\langle M_i \mid i < \alpha \rangle$ of limit models of cardinality μ with M_{i+1} universal over M_i and for every $p \in$ ga-S(M_α), where $M_\alpha = \bigcup_{i<\alpha} M_i$, we have that:

(a) If for every $i < \alpha$, the type $p \upharpoonright M_i$ does not μ-split over M_0, then p does not μ-split over M_0.
(b) There exists $i < \alpha$ such that p does not μ-split over M_i.

Remark 2.6. *By our global hypothesis of amalgamation (Hypothesis 2.1), if $\mathcal{K}$ is μ-superstable, then $\mathcal{K}_{\geq\mu}$ has joint embedding.*

Remark 2.7. *By the weak transitivity property of μ-splitting* [Vasa, Proposition 3.7]*, condition (4a) could have been omitted (i.e. it follows from the rest).*

Minor variations on this definition appear in the literature. For instance the first author uses a very similar definition without the additional requirement of no maximal models of cardinality μ [Van06]. This distinction is not critical, because the situation is uninteresting when $\mathcal{K}_\mu$ has maximal models (there are no limit models of size μ):

Proposition 2.8. *Let $\mu \geq \text{LS}(\mathcal{K})$. Assume that $\mathcal{K}_\mu$ has joint embedding and $\mathcal{K}$ is stable in μ. The following are equivalent:*

(1) $\mathcal{K}$ has a model of size μ^+.
(2) $\mathcal{K}_\mu$ is nonempty and has no maximal models.
(3) $\mathcal{K}$ has a limit model of size μ.
(4) There exists $M_0, M_1, M_2 \in \mathcal{K}_\mu$ such that $M_0 < M_1 < M_2$ and M_1 is universal over M_0.

Proof. (1) implies (2) is by joint embedding. (2) implies (3) implies (4) is straightforward. We show (4) implies (1). Assume $M_0 < M_1 < M_2$ and M_1 is universal over M_0. It is enough to show that M_2 has a proper extension. By universality, there exists $f : M_2 \xrightarrow[M_0]{} M_1$. Now extend f to $g : M_2' \cong_{M_0} M_2$. Since $M_1 < M_2$, $M_2 < M_2'$, as desired. □

The main results of this paper involve the concept of symmetry over limit models and its equivalent formulation involving towers which was identified in [Vana]:

[2]That is, $|\text{ga-S}(M)| \leq \mu$ for all $M \in \mathcal{K}_\mu$. Some authors call this "Galois-stable".

Definition 2.9. *We say that an abstract elementary class exhibits* symmetry for non-μ-splitting *if whenever models $M, M_0, N \in \mathcal{K}_\mu$ and elements a and b satisfy the conditions 1-4 below, then there exists M^b a limit model over M_0, containing b, so that* ga-tp(a/M^b) *does not μ-split over N. See Figure 1.*

(1) *M is universal over M_0 and M_0 is a limit model over N.*
(2) *$a \in M \backslash M_0$.*
(3) ga-tp(a/M_0) *is non-algebraic and does not μ-split over N.*
(4) ga-tp(b/M) *is non-algebraic and does not μ-split over M_0.*

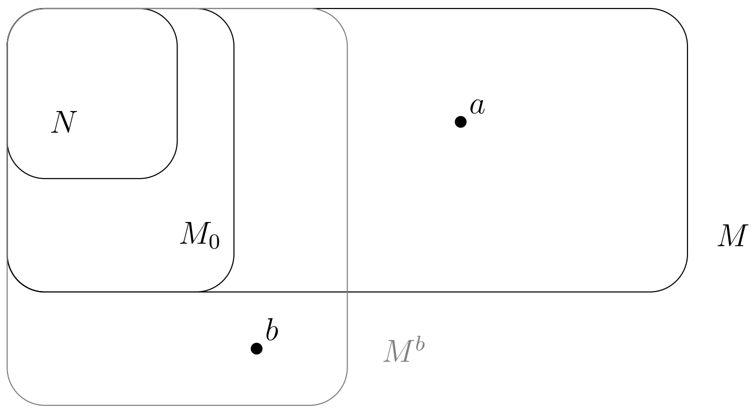

FIGURE 1. A diagram of the models and elements in the definition of symmetry. We assume the type ga-tp(b/M) does not μ-split over M_0 and ga-tp(a/M_0) does not μ-split over N. Symmetry implies the existence of M^b a limit model over M_0 containing b, so that ga-tp(a/M^b) does not μ-split over N.

We end by recalling a few results of the first author showing the importance of the symmetry property:

Fact 2.10 (Theorem 4 in [Vana]). *If $\mathcal{K}$ is μ-superstable and the union of any chain (of length less than μ^{++}) of saturated models of size μ^+ is saturated, then $\mathcal{K}$ has μ-symmetry.*

Many of the results on symmetry rely on the equivalent formulation of μ-symmetry in terms of reduced towers.

Definition 2.11. *A tower $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ is* reduced *if it satisfies the condition that for every $<$-extension $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ of $(\bar{M}, \bar{a}, \bar{N})$ and for every $i < \alpha$, $M'_i \bigcap (\bigcup_{j<\alpha} M_j) = M_i$.*

Definition 2.12. *A tower* $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ *is* continuous *if for any limit* $i < \alpha$, $M_i = \bigcup_{j<i} M_j$.

Fact 2.13 (Theorem 5 in [Vana])**.** *Assume* $\mathcal{K}$ *is* μ*-superstable. The following are equivalent:*

(1) $\mathcal{K}$ *has* μ*-symmetry.*
(2) Any reduced tower in $\mathcal{K}^*_{\mu,\alpha}$ *is continuous.*

It was previously established (in [SV99] or more explicitly in [GVV]) that the continuity of reduced towers gives uniqueness of limit models:

Fact 2.14. *Assume* $\mathcal{K}$ *is* μ*-superstable. If any reduced tower in* $\mathcal{K}^*_{\mu,\alpha}$ *is continuous (or equivalently by Fact 2.13 if* $\mathcal{K}$ *has* μ*-symmetry), then for any* $M_0, M_1, M_2 \in \mathcal{K}_\mu$*, if* M_1 *and* M_2 *are limit over* M_0*, then* $M_1 \cong_{M_0} M_2$.

Symmetry also has implications to chains of saturated models.

Fact 2.15 (Theorem 1 in [Vanc])**.** *Assume* $\mathcal{K}$ *is* μ*-superstable,* μ^+*-superstable, and has* μ^+*-symmetry. Then for any* $\theta < \mu^{++}$*, any chain of saturated models of length* θ *in* $\mathcal{K}_{\mu^+}$ *has a saturated union.*

Remark 2.16. *The proof shows that instead of* μ^+*-symmetry it is enough to assume that any limit model in* $\mathcal{K}_{\mu^+}$ *is saturated (see* [Vanc, Corollary 3]*).*

Remark 2.17. *We can use these facts to obtain the following direct proof of Theorem 1.1 when* $\lambda = \mu^+$ *(this is* [Vanb, Theorem 0.1]*). Assume that* $\mathcal{K}$ *is* μ*-superstable and* μ^+*-superstable and assume that it satisfies* μ^+*-symmetry. We want to see that it satisfies* μ*-symmetry. Indeed by Fact 2.15, the union of any chain (of length less than* μ^{++}*) of saturated models in* $\mathcal{K}_{\mu^+}$ *is saturated. By Fact 2.10,* $\mathcal{K}$ *has* μ*-symmetry.*

3. A hierarchy of symmetry properties

We discuss the relationship between the symmetry property of Definition 2.9 and other symmetry properties previously defined in the literature, especially the symmetry property in the definition of a good μ-frame. This expands on the short remark after Definition 3 of [Vana] and on Corollary 2 there. It will be convenient to use the following terminology. A minor variation (where "limit over" is replaced by "universal over") appears in [Vasa, Definition 3.8].

Definition 3.1. *Let* $M_0 \leq M \leq N$ *be models in* $\mathcal{K}_\mu$*. We say a type* $p \in \text{ga-S}(N)$ explicitly does not μ-fork over (M_0, M) *if:*

(1) M *is limit over* M_0.

(2) p does not μ-split over M_0.

We say that p does not μ-fork over M *if there exists M_0 so that p explicitly does not μ-fork over (M_0, M).*

Remark 3.2. *Assuming μ-superstability, the relation "p does not μ-fork over M" is very close to defining an independence notion with the properties of forking in a first-order superstable theory (i.e. a good μ-frame, see below). In fact using tameness it can be used to do precisely that, see* [Vasa]. *Moreover forking in any categorical good μ-frame has to be μ-forking, see Fact 3.8.*

We now give several variations on μ-symmetry. We will show that the uniform variation is equivalent to the one in Definition 2.9.

Definition 3.3. *Let $\mu \geq \mathrm{LS}(\mathcal{K})$.*

(1) $\mathcal{K}$ has uniform μ-symmetry *if for any limit models N, M_0, M in $\mathcal{K}_\mu$ where M is limit over M_0 and M_0 is limit over N, if* ga-tp(b/M) *does not μ-split over M_0, $a \in |M|$, and* ga-tp(a/M_0) *explicitly does not μ-fork over (N, M_0), there exists $M_b \in \mathcal{K}_\mu$ containing b and limit over M_0 so that* ga-tp(a/M_b) *explicitly does not μ-fork over (N, M_0).*

(2) $\mathcal{K}$ has non-uniform μ-symmetry *if for any limit models M_0, M in $\mathcal{K}_\mu$ where M is limit over M_0, if* ga-tp(b/M) *does not μ-split over M_0, $a \in |M|$, and* ga-tp(a/M_0) *does not μ-fork over M_0, there exists $M_b \in \mathcal{K}_\mu$ containing b and limit over M_0 so that* ga-tp$(a/M_b; M'')$ *does not μ-fork over M_0.*

(3) $\mathcal{K}$ has weak non-uniform μ-symmetry *if for any limit models M_0, M in $\mathcal{K}_\mu$ where M is limit over M_0, if* ga-tp(b/M) *does not μ-fork over M_0, $a \in |M|$, and* ga-tp(a/M_0) *does not μ-fork over M_0, there exists $M_b \in \mathcal{K}_\mu$ containing b and limit over M_0 so that* ga-tp$(a/M_b; M'')$ *does not μ-fork over M_0.*

The difference between the uniform and non-uniform variations is in the conclusion: in the uniform case, we start with ga-tp(a/M_0) which explicitly does not μ-fork over (N, M_0) and get get ga-tp(a/M_b) explicitly does not μ-fork over (N, M_0). Thus both types do not μ-split over N. In the non-uniform case, we start with ga-tp(a/M_0) which does not μ-fork over M_0, hence explicitly does not μ-fork over (N, M_0) for some N, but we only get that ga-tp(a/M_b) does not μ-fork over M_0, so it explicitly does not μ-fork over (N', M_0), for some N' potentially different from N.

The difference between "non-uniform" and "weak non-uniform" is in the starting assumption: in the weak case, we assume that ga-tp(b/M) does not μ-fork over M_0, hence there exists M_0' so that M_0 is limit over

M_0' and ga-tp(b/M) does not μ-split over M_0'. In the non-weak case, we assume only that ga-tp(b/M) does not μ-*split* over M_0. Even under μ-superstability, it is open whether this implies that there must exist a smaller M_0' so that ga-tp(b/M) does not μ-split over M_0'. The problem is that μ-splitting need not satisfy the transitivity property, see the discussion after Definition 3.8 in [Vasa].

Using the monotonicity property of μ-splitting, we get the easy implications:

Proposition 3.4. *Let $\mu \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ has uniform μ-symmetry, then it has non-uniform μ-symmetry. If $\mathcal{K}$ has non-uniform μ-symmetry, then it has weak non-uniform μ-symmetry.*

Playing with the definitions and monotonicity of μ-splitting (noting that cases ruled out by Definition 2.9 such as $a \in |M_0|$ are easy to handle), we also have:

Proposition 3.5. *$\mathcal{K}$ has uniform μ-symmetry if and only if it has μ-symmetry (in the sense of Definition 2.9).*

How do these definitions compare to the symmetry property in good μ-frames? Recall [She09, Definition II.2.1][3] that a good μ-frame is a triple $\mathfrak{s} = (\mathcal{K}_\mu, \mathop{\smile}\limits^{\vert}, \text{ga-S}^{\text{bs}})$ where:

1. $\mathcal{K}$ is an AEC.
2. For each $M \in \mathcal{K}_\mu$, ga-S$^{\text{bs}}(M)$ (called the set of *basic types* over M) is a set of nonalgebraic Galois types over M satisfying (among others) the *density property*: if $M < N$ are in $\mathcal{K}_\mu$, there exists $a \in |N| \backslash |M|$ such that ga-tp$(a/M; N) \in$ ga-S$^{\text{bs}}(M)$.
3. $\mathop{\smile}\limits^{\vert}$ is an (abstract) independence relation on types of length one over models in $\mathcal{K}_\mu$ satisfying several basic properties (that we will not list here) of first-order forking in a superstable theory.

As in [She09, Definition II.6.35], we say that a good μ-frame $\mathfrak{s}$ is *type-full* if for each $M \in \mathcal{K}_\mu$, ga-S$^{\text{bs}}(M)$ consists of *all* the nonalgebraic types over M. For simplicity, we focus on type-full good frames in this paper. Given a type-full good μ-frame $\mathfrak{s} = (\mathcal{K}_\mu, \mathop{\smile}\limits^{\vert}, \text{ga-S}^{\text{bs}})$ and $M_0 \leq M$ both in $\mathcal{K}_\mu$, we say that a nonalgebraic type $p \in$ ga-S(M) *does not $\mathfrak{s}$-fork over* M_0 if it does not fork over M_0 according to the abstract independence relation $\mathop{\smile}\limits^{\vert}$ of $\mathfrak{s}$. We say that a good μ-frame $\mathfrak{s}$ is *on* $\mathcal{K}_\mu$ if its underlying class is $\mathcal{K}_\mu$.

[3]We will *not* use the axiom (B) requiring the existence of a superlimit model of size μ. In fact many papers (e.g. [JS13]) define good frames without this assumption.

The existence of a good μ-frame gives quite a lot of information about the class. For $\lambda > \text{LS}(\mathcal{K})$, write $\mathcal{K}^{\lambda\text{-sat}}$ for the class of λ-saturated models in $\mathcal{K}_{\geq\lambda}$. We also define $\mathcal{K}^{0\text{-sat}} := \mathcal{K}$.

Fact 3.6. *Assume there is a good μ-frame on $\mathcal{K}_\mu^{\lambda\text{-sat}}$, for $\lambda \leq \mu$ (so in particular, unions of chains of λ-saturated models are λ-saturated). Then:*

(1) For any $M_0, M_1, M_2 \in \mathcal{K}_\mu$ such that M_1 and M_2 are limit over M_0, $M_1 \cong_{M_0} M_2$.
(2) $\mathcal{K}$ is μ-superstable.

Proof. The first part is [She09, Lemma II.4.8] (or see [Bon14a, Theorem 9.2]). Note that by the usual back and forth argument (as made explicit in the proof of [BVb, Theorem 7.1]), any limit model is isomorphic to a limit model where the models witnessing it are λ-saturated. The second part is because:

- By definition of a good μ-frame, $\mu \geq \text{LS}(\mathcal{K})$, $\mathcal{K}_\mu$ is nonempty, has amalgamation, joint embedding, and no maximal models.
- By [She09, Claim II.4.2.(1)], $\mathcal{K}$ is stable in μ.
- By the uniqueness property of $\mathfrak{s}$-forking, if a type does not $\mathfrak{s}$-fork over M_0 (where $\mathfrak{s}$ is a good μ-frame on $\mathcal{K}_\mu$), then it does not μ-split over M_0 (see [BGKV, Lemma 4.2]). Thus we obtain condition (4) in Definition 2.5 when the M_is are λ-saturated, and as noted above (or in [Vasb, Proposition 10.6]) we can do a back and forth argument to get that (4) holds in general.

□

Among the axioms a good μ-frame must satisfy is the symmetry axiom:

Definition 3.7. *The* symmetry axiom *for a good μ-frame $\mathfrak{s} = (\mathcal{K}_\mu, \mathop{\smile}\limits^{}, \text{ga-S}^{bs})$ is the following statement*[4]*: For any $M_0 \leq M$ in $\mathcal{K}_\mu$, if* ga-tp(b/M) *does not $\mathfrak{s}$-fork over M_0 and $a \in |M| \backslash |M_0|$ is so that* ga-tp$(a/M_0) \in$ ga-S$^{bs}(M_0)$*, there exists $M_b \in \mathcal{K}_\mu$ containing b and extending M_0 so that* ga-tp(a/M_b) *does not $\mathfrak{s}$-fork over M_0.*

Since the symmetry properties of Definition 3.3 are all over limit models only, we will discuss only frames whose models are the limit models. By Fact 3.6, such frames are categorical (that is, their underlying class has a single model up to isomorphism). This is not a big loss since most known general constructions of a good μ-frame (e.g.

[4]The good frame axioms imply that $\mathcal{K}$ has amalgamation in μ, so for this definition (and for simplicity only) we work inside a saturated model $\mathfrak{C}$ of size μ^+.

[She09, Theorem II.3.7], [Vasa, Theorem 1.3]) assume categoricity in μ. In the known constructions when categoricity in μ is not assumed (such as in [Vasb, Corollary 10.19]), it holds that the union of a chain of μ-saturated model is μ-saturated, so we can simply restrict the frame to the saturated models of size μ.

Recall also from [Vasb, Theorem 9.7] that categorical good μ-frames are canonical:

Fact 3.8 (The canonicity theorem for categorical good frames)**.** *Let $\mathfrak{s} = (\mathcal{K}_\mu, \perp, \text{ga-S}^{bs})$ be a categorical good μ-frame. Let $p \in \text{ga-S}^{bs}(M)$ and let $M_0 \leq M$ be in $\mathcal{K}_\mu$. Then p does not $\mathfrak{s}$-fork over M_0 if and only if p does not μ-fork*[5] *over M_0 (recall Definition 3.1).*

Remark 3.9. *The proof of the second part of Fact 3.6 and Fact 3.8 do not use the symmetry axiom (but the first part of Fact 3.6 does).*

Using the canonicity theorem, we obtain:

Theorem 3.10. *Let $\mathfrak{s}$ be a type-full categorical good μ-frame on $\mathcal{K}_\mu$, except that we do not assume that it satisfies the symmetry axiom. The following are equivalent:*

(1) $\mathfrak{s}$ satisfies the symmetry axiom (Definition 3.7).
(2) $\mathcal{K}$ has weak non-uniform μ-symmetry (Definition 3.3.(3)).

Proof. By Fact 3.8 (and Remark 3.9), μ-forking and $\mathfrak{s}$-forking coincide. Now replace $\mathfrak{s}$-forking by μ-forking in the symmetry axiom and expand the definition. □

One can ask whether weak non-uniform symmetry can be replaced by the uniform version:

Question 3.11. *Assume there is a type-full categorical good μ-frame on $\mathcal{K}_\mu$. Does $\mathcal{K}$ have μ-symmetry? More generally, if $\mathcal{K}$ is μ-superstable, does $\mathcal{K}$ have μ-symmetry?*

We will show (Theorem 6.4) that the answer is positive if $\mathcal{K}$ is μ-tame. Still much less suffices:

Theorem 3.12. *If $\mathcal{K}$ is μ-superstable and has a good μ^+-frame on $\mathcal{K}_{\mu^+}^{\lambda\text{-sat}}$ for some $\lambda \leq \mu^+$, then $\mathcal{K}$ has μ-symmetry.*

Proof. By Fact 3.6, all limit models in $\mathcal{K}_{\mu^+}$ are saturated and $\mathcal{K}$ is μ^+-superstable. By Fact 2.15, the remark following it, and Fact 2.10, $\mathcal{K}$ has μ-symmetry. □

[5]The statement of the theorem in [Vasb] uses the definition of μ-forking with "universal over" instead of "limit over", but the proof goes through also for the "limit over" definition.

We end this section with a partial answer to Question 3.11 assuming that the good frame satisfies several additional technical properties of frames introduced by Shelah (see [She09, Definitions III.1.1, III.1.3]). For this result amalgamation (Hypothesis 2.1) is not necessary.

Corollary 3.13. *Assume there is a successful good$^+$ μ-frame with underlying class $\mathcal{K}_\mu$. Then $\mathcal{K}$ has μ-symmetry.*

Proof. Let $\mathfrak{s}$ be a successful good$^+$ μ-frame with underlying class $\mathcal{K}_\mu$. By [She09, Claim II.6.36], we can assume without loss of generality that $\mathfrak{s}$ is type-full (note that by [BGKV, Theorem 6.13] there can be only one such type-full frame). By Fact 3.6, $\mathcal{K}$ is μ-superstable. By [She09, III.1.6, III.1.7, III.1.8], there is a good μ^+-frame $\mathfrak{s}^+$ on $\mathcal{K}_{\mu^+}^{\mu^+\text{-sat}}$. By Theorem 3.12, $\mathcal{K}$ has μ-symmetry. □

4. Transferring symmetry

In this section we prove Theorem 1.1 which is key to the results in the following sections. We start with a few observations which will allow us to extend the tower machinery from [GVV] and [Vanb] to include towers composed of models of different cardinalities. In particular, we derive an extension property for towers of different cardinalities, Lemma 4.7. This will allow us to adapt the arguments from [Vanb] to prove Theorem 1.1.

We start with a study of chains where each model indexed by a successor is universal over its predecessor:

Proposition 4.1. *Suppose that $\lambda \geq \mathrm{LS}(\mathcal{K})$ is a cardinal. Assume that $\mathcal{K}$ is stable in λ with no maximal models of cardinality λ. Let θ be a limit ordinal. Assume $\langle M_i \in \mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})} \mid i < \theta \rangle$ is a strictly increasing and continuous sequence of models so that for all $i < \theta$, M_{i+1} is universal over M_i. If $M := \bigcup_{i<\theta} M_i$ has size λ, then M is a (λ, θ)-limit model over some model containing M_0.*

Proof. By cardinality considerations, $\theta < \lambda^+$. Replacing θ by $\mathrm{cf}(\theta)$ if necessary, we can assume without loss of generality that θ is regular. By λ-stability and the assumption that $\mathcal{K}$ has no maximal models of cardinality λ, we can fix a (λ, θ)-limit model M^* witnessed by $\langle M_i^* \mid i < \theta \rangle$ with $M_0 \leq M_0^*$. If there exists $i < \theta$ such that $M_i \in \mathcal{K}_\lambda$, then the sequence $\langle M_j \mid j \in [i, \theta) \rangle$ witnesses that M is (λ, θ)-limit and $M_0 \leq M_i$; so assume that $\lambda > \mathrm{LS}(\mathcal{K})$ and $M_i \in K_{<\lambda}$ for all $i < \theta$. Then we must have that $\theta = \mathrm{cf}(\lambda)$. If λ is a successor, we must have that $\theta = \lambda$ and we obtain the result from [Vanb, Proposition 3.1]; so assume λ is limit. For $i < \theta$, let $\lambda_i := \|M_i\|$.

Fix $\langle a_\alpha \mid \alpha < \lambda \rangle$ an enumeration of M^*. Using the facts that M_{i+1} is universal over M_i and that M^*_{i+1} is universal over M^*_i, we can build an isomorphism $f : M \cong M^*$ inductively by defining an increasing and continuous sequence of $\mathcal{K}$-embeddings f_i so that $f_i : M_i \to M^*_i$, $f_0 = \mathrm{id}_{M_0}$, and $\{a_\alpha \mid \alpha < \lambda_i\} \subseteq \mathrm{rg}(f_{i+1})$. □

We will use the following generalization of the weak transitivity property of μ-splitting proven in [Vasa, Proposition 3.7]. The difference here is that the models are allowed to be of size bigger than μ.

Proposition 4.2. *Let $\mu \geq \mathrm{LS}(\mathcal{K})$ be such that $\mathcal{K}$ is stable in μ. Let $M_0 \leq M_1 < M_1' \leq M_2$ all be in $\mathcal{K}_{\geq\mu}$. Assume that M_1' is universal over M_1. Let $p \in \text{ga-S}(M_2)$. If $p \restriction M_1'$ does not μ-split over M_0 and p does not μ-split over some $N \in \mathcal{K}_\mu$ with $N \leq M_1$, then p does not μ-split over M_0.*

Proof. Note that by definition of μ-splitting, $M_0 \in \mathcal{K}_\mu$. Thus by making N larger if necessary we can assume that $M_0 \leq N$. By basic properties of universality we have that M_1' is universal over N, hence without loss of generality $M_1 = N$. In particular, $M_1 \in \mathcal{K}_\mu$. By stability, build $M_1'' \in \mathcal{K}_\mu$ universal over M_1 such that $M_1 < M_1'' \leq M_1'$. By monotonicity, $p \restriction M_1''$ does not μ-split over M_0. Thus without loss of generality also $M_1'' \in \mathcal{K}_\mu$. By definition of μ-splitting, it is enough to check that $p \restriction M_2'$ does not μ-split over M_0 for all $M_2' \in \mathcal{K}_\mu$ with $M_2' \leq M_2$. Thus without loss of generality again $M_2 \in \mathcal{K}_\mu$. Now use the weak transitivity property of μ-splitting [Vasa, Proposition 3.7]. □

We use the previous proposition to extend the continuity property of μ-splitting to models of size bigger than μ. This is very similar to the argument in [She09, Claim II.2.11].

Proposition 4.3. *Let $\mu \geq \mathrm{LS}(\mathcal{K})$ and assume that $\mathcal{K}$ is μ-superstable.*

Suppose $\langle M_i \in \mathcal{K}_{\geq\mu} \mid i < \delta \rangle$ is an increasing sequence of models so that, for all $i < \delta$, M_{i+1} is universal over M_i. Let $p \in \text{ga-S}(\bigcup_{i<\delta} M_i)$. If $p \restriction M_i$ does not μ-split over M_0 for each $i < \delta$, then p does not μ-split over M_0.

Proof. Without loss of generality, $\delta = \mathrm{cf}(\delta)$. Let $M_\delta := \bigcup_{i<\delta} M_i$. There are two cases to check. If $\delta > \mu$, then by [She99, Claim 3.3], there exists $N \in \mathcal{K}_\mu$ with $N \leq M_\delta$ such that p does not μ-split over N. Pick $i < \delta$ such that $N \leq M_i$. Then p does not μ-split over M_i. By Proposition 4.2 (where $(M_0, M_1 M_1', M_2, N)$ there stand for $(M_0, M_i, M_{i+1}, M_\delta, N)$ here), p does not μ-split over M_0.

Suppose then that $\delta \leq \mu$ and for sake of contradiction that M^* of cardinality μ witnesses the splitting of p over M_0, i.e. $p \restriction M^*$ μ-splits over M_0. We can find $\langle M_i^* \in \mathcal{K}_\mu \mid i < \delta \rangle$ an increasing resolution of M^* so that $M_i^* \leq M_i$ for all $i < \delta$. By monotonicity of splitting, stability in μ, and the fact that each M_{i+1} is universal over M_i, we can increase M^*, if necessary, to arrange that M_{i+1}^* is universal over M_i^*. Since $p \restriction M_i$ does not μ-split over M_0, monotonicity of non-splitting implies that $p \restriction M_i^*$ does not μ-split over M_0. Then, by μ-superstability $p \restriction M^*$-does not μ-split over M_0. This contradicts our choice of M^*. $\square$

We adapt the proof of the extension property for non-μ-splitting ([Van06, Theorem I.4.10]) to handle models of different sizes under the additional assumption of superstability in the size of the bigger model. The conclusion can also be achieved using the assumption of tameness instead of superstability (since μ-splitting and λ-splitting coincide if $\mathcal{K}$ is μ-tame and $\mu \leq \lambda$, see [BGKV, Proposition 3.12]).

Proposition 4.4. *Fix cardinals $\lambda > \mu \geq \mathrm{LS}(\mathcal{K})$. Suppose that $\mathcal{K}$ is μ-stable and λ-superstable.*

Let $M \in \mathcal{K}_\mu$ and $M^\lambda, M' \in \mathcal{K}_\lambda$ be such that $M \leq M^\lambda \leq M'$ and M^λ is limit over some model containing M. Let $p \in \text{ga-S}(M^\lambda)$ be such that p does not μ-split over M. Then there exists $q \in \text{ga-S}(M')$ extending p so that q does not μ-split over M. Moreover q is algebraic if and only if p is.

Proof. Let $\theta < \lambda^+$ and $\langle M_i^\lambda \mid i < \theta \rangle$ witness that M^λ is (λ, θ)-limit with $M \leq M_0^\lambda$. Write $p := \text{ga-tp}(a/M^\lambda)$. By λ-superstability there exists $i < \theta$ so that p does not λ-split over M_i^λ. Since M_{i+2}^λ is universal over M_{i+1}^λ there exists $f : M' \underset{M_{i+1}^\lambda}{\rightarrow} M_{i+2}^\lambda$. Extend f to $g \in \mathrm{Aut}_{M_{i+1}^\lambda}(\mathfrak{C})$. Let $q := g^{-1}(p) \restriction M' = \text{ga-tp}(g^{-1}(a)/M')$. Note that q is nonalgebraic if p is nonalgebraic (the converse will follow once we have shown that q extends p). By monotonicity, invariance, and our assumption that p does not μ-split over M, we can conclude that q does not μ-split over N. By similar reasoning also q does not λ-split over M_i^λ. In particular $\text{ga-tp}(g^{-1}(a)/M^\lambda) = q \restriction M^\lambda$ does not λ-split over M_i^λ. Since g fixes M_{i+1}^λ, we know that $g^{-1}(a)$ realizes $p \restriction M_{i+1}^\lambda$. Therefore, we get by the uniqueness of non-λ-splitting extensions that $q \restriction M^\lambda = \text{ga-tp}(f^{-1}(a)/M^\lambda) = \text{ga-tp}(a/M^\lambda) = p$. This shows that q extends p, as desired. $\square$

We can now prove an extension property for towers in $\mathcal{K}^*_{\lambda,\alpha,\mu}$.

Lemma 4.5. *Let λ and μ be cardinals satisfying $\lambda \geq \mu \geq \mathrm{LS}(\mathcal{K})$. Assume that $\mathcal{K}$ is superstable in μ and in λ. For any $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\lambda,\alpha,\mu}$, there exists $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}^*_{\lambda,\alpha,\mu}$ so that:*

$$(\bar{M}, \bar{a}, \bar{N}) <_\mu (\bar{M}', \bar{a}, \bar{N})$$

Proof. If $\lambda = \mu$, the result follows from infinitely many (for example $\mathrm{cf}(\lambda)$ many) applications of [GVV, Lemma 5.3] which is the extension property for towers. If $\lambda > \mu$, the result follows similarly from the proof of the extension property for towers using Proposition 4.4. □

We also have a continuity property:

Lemma 4.6. *Let $\mu \geq \mathrm{LS}(\mathcal{K})$ be such that $\mathcal{K}$ is μ-superstable. Let $\langle \lambda_i : i < \delta \rangle$ be an increasing sequence of cardinals with $\lambda_0 \geq \mu$. Let $\langle (\bar{M}^i, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\lambda_i, \alpha, \mu} \mid i < \delta \rangle$ be a sequence of towers such that $(\bar{M}^i, \bar{a}, \bar{N}) <_\mu (\bar{M}^{i+1}, \bar{a}, \bar{N})$ for all $i < \delta$.*

Let $\bar{M}^\delta$ be the sequence composed of models of the form $M^\delta_\beta := \bigcup_{i<\delta} M^i_\beta$ for $\beta < \alpha$. Let $\lambda := \sum_{i<\delta} \lambda_i$.

Then $(\bar{M}^\delta, \bar{a}, \bar{N}) \in \mathcal{K}_{\lambda,\alpha,\mu^*}$ and $(\bar{M}^i, \bar{a}, \bar{N}) <_\mu (\bar{M}^\delta, \bar{a}, \bar{N})$ for all $i < \delta$.

Proof. Working by induction on δ, we can assume without loss of generality that the sequence of tower is continuous. That is, for each $\beta < \alpha$ and limit $i < \delta$, $M^i_\beta = \bigcup_{j<i} M^j_\beta$. Of course, it is enough to show that $(\bar{M}^0, \bar{a}, \bar{N}) <_\mu (\bar{M}^\delta, \bar{a}, \bar{N})$. Let $\beta < \alpha$. There are two things to check: M^λ_β is a limit model over a model that contains M^0_β, and $\text{ga-tp}(a_\beta / M^\lambda_\beta)$ does not μ-split over N_β. Proposition 4.1 confirms that M^λ_β is a (λ, δ)-limit model over some model containing M^0_β. Because each $(\bar{M}^i, \bar{a}, \bar{N})$ is a tower, we know that $\text{ga-tp}(a_\beta / M^i_\beta)$ does not μ-split over N_β. This allows us to apply Proposition 4.3 to conclude that $\text{ga-tp}(a_\beta / M^\lambda_\beta)$ does not μ-split over N_β. □

We conclude an extension property for towers of different sizes

Lemma 4.7. *Let κ, λ and μ be cardinals satisfying $\lambda \geq \kappa \geq \mu \geq \mathrm{LS}(\mathcal{K})$. Assume that $\mathcal{K}$ is superstable in μ and in every $\chi \in [\kappa, \max(\kappa^+, \lambda))$*

Let $(\bar{M}^\kappa, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\kappa,\alpha,\mu}$.

1. *There exists $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\lambda,\alpha,\mu}$ so that*
$$(\bar{M}^\kappa, \bar{a}, \bar{N}) <_\mu (\bar{M}, \bar{a}, \bar{N}).$$
2. *If in addition $\mathcal{K}$ is λ-superstable, then there exists a sequence $\langle N^\lambda_\beta \mid \beta < \alpha \rangle$ so that $N_i \leq N^\lambda_\beta$ for all $\beta < \alpha$ and $(\bar{M}, \bar{a}, \bar{N}^\lambda) \in \mathcal{K}^*_{\lambda,\alpha}$.*

Proof. We prove the first statement in the lemma by induction on λ. If $\lambda = \kappa$, this is given by Lemma 4.5. Now assume that $\lambda > \kappa$. Fix an increasing continuous sequence $\langle \lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$ which is cofinal in λ and so that $\lambda_0 = \kappa$ (if $\lambda = \chi^+$ is a successor we can take $\lambda_i = \chi$ for all $i < \lambda$). We build a sequence $\langle (\bar{M}^{\lambda_i}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\lambda_i,\alpha,\mu} \mid i < \mathrm{cf}(\lambda)\rangle$ which is increasing (that is, $(\bar{M}^{\lambda_i}, \bar{a}, \bar{N}) <_\mu (\bar{M}^{\lambda_{i+1}}, \bar{a}, \bar{N})$ for all $i < \mathrm{cf}(\lambda)$) and continuous (in the obvious sense, see Lemma 4.6). This is possible by the induction hypothesis. Now by Lemma 4.6, the union of the chain of towers (defined there) is as desired.

For part (2), recall from Definition 2.4 that for each $\beta < \alpha$, M_β is a limit model over some model containing M^κ_β. Let $\langle M^*_{\beta,i} \in \mathcal{K}_\lambda \mid i < \theta_\beta\rangle$ witness this By λ-superstability, for each $\beta < \alpha$, there exists $i_\beta < \theta_\beta$ so that ga-tp$(a_\beta/M^\lambda_\beta)$ does not λ-split over M^*_{β,i_β}. By our choice of $M^*_{\beta,0}$ containing M^κ_β, and consequently N^κ_β, we can take $N^\lambda_\beta := M^*_{\beta,i_\beta}$. □

We now begin the proof of Theorem 1.1. The structure of the proof is similar to the proof of Theorem 0.1 of [Vanb]; only here we work with towers in $\mathcal{K}^*_{\lambda,\alpha,\mu}$ as opposed to only towers in $\mathcal{K}^*_{\mu,\alpha}$.

Proof of Theorem 1.1. Suppose for the sake of contradiction that $\mathcal{K}$ does not have symmetry for μ-non-splitting. By Fact 2.13 and our μ-superstability assumption, $\mathcal{K}$ has a reduced discontinuous tower in $\mathcal{K}^*_{\mu,\alpha}$ for some $\alpha < \mu^+$. Let α be the minimal ordinal for which there is a reduced, discontinuous tower in $\mathcal{K}^*_{\mu,\alpha}$. By Lemma 5.7 of [GVV], we may assume that $\alpha = \delta + 1$ for some limit ordinal δ. Fix $\mathcal{T} = (\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}^*_{\mu,\alpha}$ a reduced discontinuous tower with $b \in M_\delta \setminus \bigcup_{\beta<\alpha} M_\beta$.

Let $I := \mathrm{cf}(\lambda)$. By Lemma 4.7, we can build an increasing and continuous chain of towers $\langle \mathcal{T}^i \mid i \in I\rangle$ extending $\mathcal{T} \restriction \delta$. If $\lambda = \kappa^+$ for some κ, then select each $\mathcal{T}^i \in \mathcal{K}^*_{\kappa,\delta,\mu}$. If λ is a limit cardinal, fix $\langle \lambda_i \mid i < \mathrm{cf}(\lambda)\rangle$ to be an increasing and continuous sequence of cardinals cofinal in λ, with $\lambda_0 > \mu$ and choose $\mathcal{T}^i \in \mathcal{K}^*_{\lambda_i,\delta,\mu}$. Let $\mathcal{T}^\lambda := \bigcup_{i\in I} \mathcal{T}^i$.

Notice that by Lemma 4.6, and our assumptions on the towers $\mathcal{T}^i$, we can conclude that $\mathcal{T}^\lambda \in \mathcal{K}^*_{\lambda,\delta,\mu}$ and $\mathcal{T}^\lambda$ extends $\mathcal{T} \restriction \delta$. In particular, for each $\beta < \alpha$,

$$\text{(1)} \qquad \text{ga-tp}(a_\beta/M^\lambda_\beta) \text{ does not } \mu\text{-split over } N_\beta.$$

Furthermore by the second part of Lemma 4.7 we can find N^λ_β so that the tower defined by $(\bar{M}^\lambda, \bar{a}, \bar{N}^\lambda)$ is in $\mathcal{K}^*_{\lambda,\delta}$ and each M^λ_β is a limit over N^λ_β. We can extend this to a tower of length $\delta + 1$ by appending to $(\bar{M}^\lambda, \bar{a}, \bar{N}^\lambda)$ a model M^λ_δ of cardinality λ containing $\bigcup_{\beta<\delta} M^\lambda_\beta$ and M^δ. Call this tower $\mathcal{T}^b$, since it contains b.

By λ-symmetry and Fact 2.13, we know that all reduced towers in $\mathcal{K}^*_{\lambda,\alpha}$ are continuous. Therefore $\mathcal{T}^b$ is not reduced. However, by the density of reduced towers [GVV, Theorem 5.6], we can find a reduced, continuous extension of $\mathcal{T}^b$ in $\mathcal{K}^*_{\lambda,\delta+1}$. By λ-applications of this theorem, we may assume that for each $\beta < \alpha$, the model indexed by β in this reduced tower is a $(\lambda, \mathrm{cf}(\lambda))$-limit over M^λ_β. Refer to this tower as $\mathcal{T}^*$. See Fig. 2.

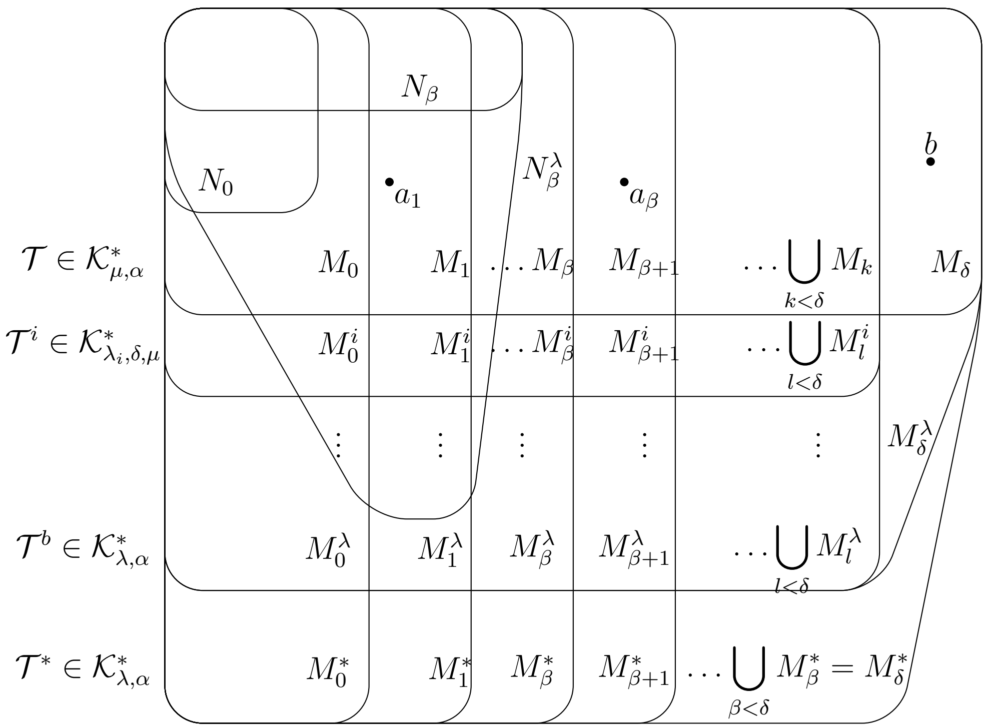

FIGURE 2. The towers in the proof of Theorem 1.1

Claim 4.8. *For every $\beta < \alpha$, $\text{ga-tp}(a_\beta/M^*_\beta)$ does not μ-split over N_β.*

Proof. Since M^λ_β and M^*_β are both limit models over N^λ_β, by λ-symmetry and Fact 2.14, there exists $f : M^\lambda_\beta \cong_{N^\lambda_\beta} M^*_\beta$. Since $\mathcal{T}^*$ is a tower extending $\mathcal{T}^b$, we know $\text{ga-tp}(a_\beta/M^*_\beta)$ does not λ-split over N^λ_β. Therefore by the definition of non-splitting, it must be the case that $\text{ga-tp}(f(a_\beta)/M^*_\beta) = \text{ga-tp}(a_\beta/M^*_\beta)$. From this equality of types we can fix $g \in \mathrm{Aut}_{M^*_\beta}(\mathfrak{C})$ with $g(f(a_\beta)) = a_\beta$. An application of $g \circ f$ to (1) yields the statement of the claim. □

We can now complete the proof of Theorem 1.1. By the continuity of $\mathcal{T}^*$ there exists $\beta < \delta$ so that $b \in M^*_\beta$. We can then use $\mathcal{T}^*$ to construct a tower $\grave{\mathcal{T}}$ in $\mathcal{K}^*_{\mu,\delta+1}$ extending $\mathcal{T}$ so that $b \in \grave{M}_\beta$ contradicting our assumption that $\mathcal{T}$ was reduced. This is possible by the downward

Löwenheim property of abstract elementary classes, μ-stability, universality of the models in $\mathcal{T}^*$, monotonicity of non-μ-splitting, and Claim 4.8.

□

Similar to the proof of [Vana, Theorem 4] we can use Lemma 4.7 to derive symmetry from categoricity

Theorem 4.9. *Suppose λ and μ are cardinals so that $\lambda > \mu \geq \mathrm{LS}(\mathcal{K})$.*

If $\mathcal{K}$ is superstable in every $\chi \in [\mu, \lambda)$, is categorical in λ, and the model of size λ is μ^+-saturated, then $\mathcal{K}$ has μ-symmetry.

Proof. Suppose that $\mathcal{K}$ does not satisfy μ-symmetry. Then by Fact 2.13 there is a reduced discontinuous tower in $\mathcal{K}^*_{\mu,\alpha}$. As in the proof of Theorem 1.1, we can find a discontinuous reduced tower $\mathcal{T} \in \mathcal{K}^*_{\mu,\alpha}$ with $\alpha = \delta + 1$ with the witness of discontinuity $b \in M_\delta \setminus \bigcup_{\beta<\delta} M_\beta$. As in the proof of Theorem 1.1, we can use Lemma 4.7 (note that we only use the first part so not assuming λ-superstability is okay) to find a tower $\mathcal{T}^\lambda \in \mathcal{K}^*_{\lambda,\mu,\delta}$ extending $\mathcal{T} \restriction \delta$.

By our assumption that the model of cardinality λ is μ^+-saturated, ga-tp$(b/\bigcup_{\beta<\delta} M_\beta)$ is realized in $\bigcup_{\beta<\delta} M^\lambda_\beta$. Let b' and $\beta' < \delta$ be such that $b' \models$ ga-tp$(b/\bigcup_{\beta<\delta} M_\beta)$ and $b' \in M^\lambda_{\beta'}$. Fix $f \in \mathrm{Aut}_{\bigcup_{\beta<\delta} M_\beta}(\mathfrak{C})$ so that $f(b') = b$. Notice that $\mathcal{T}^b := f(\mathcal{T}^\lambda)$ is a tower in $\mathcal{K}^*_{\lambda,\delta,\mu}$ extending $\mathcal{T} \restriction \delta$ with $b \in M^b_{\beta'}$.

We can now use the downward Löwenheim-Skolem property of abstract elementary classes, stability in μ, μ^+-saturation of models of cardinality λ, and monotonicity of non-μ-splitting to construct from $\mathcal{T}^b$ a discontinuous tower in $\mathcal{K}^*_{\mu,\alpha}$ extending $\mathcal{T}$ so that b appears in the model indexed by β' in the tower. This will contradict our choice of $\mathcal{T}$ being reduced. □

5. Symmetry and categoricity

Theorem 1.1 has several applications to categorical AECs. We will use the following result, an adaptation of an argument of Shelah and Villaveces [SV99, Theorem 2.2.1], to settings with amalgamation:

Fact 5.1 (The Shelah-Villaveces theorem, Theorem 6.3 in [GV])**.** *Assume that $\mathcal{K}$ has no maximal models. Let $\mu \geq \mathrm{LS}(K)$. If $\mathcal{K}$ is categorical in a $\lambda > \mu$, then $\mathcal{K}$ is μ-superstable.*

Corollary 5.2. *Suppose λ and μ are cardinals so that $\lambda > \mu \geq \mathrm{LS}(\mathcal{K})$ and assume that $\mathcal{K}$ has no maximal models. If $\mathcal{K}$ is categorical in λ and the model of size λ is μ^+-saturated, then $\mathcal{K}$ is μ-superstable and has μ-symmetry.*

Proof. By Fact 5.1, $\mathcal{K}$ is χ-superstable in every $\chi \in [\mu, \lambda)$. By Theorem 4.9, $\mathcal{K}$ has μ-symmetry. $\square$

As announced in the introduction, we can combine Corollary 5.2 with Fact 2.14 to improve on [She99, Theorem 6.5]. The following result also improves on Corollary 4.2 of [Vanb], by removing the successor assumption in the categoricity cardinal and obtaining uniqueness of limit models in much smaller cardinalities as well.

Corollary 5.3. *Suppose λ and μ are cardinals so that $\text{cf}(\lambda) > \mu \geq \text{LS}(\mathcal{K})$ and assume that $\mathcal{K}$ has no maximal models. If $\mathcal{K}$ is categorical in λ, then $\mathcal{K}$ has uniqueness of limit models of cardinality μ. That is, if $M_0, M_1, M_2 \in \mathcal{K}_\mu$ are such that both M_1 and M_2 are limit models over M_0, then $M_1 \cong_{M_0} M_2$.*

Proof. Categoricity in λ, the assumption that $\text{cf}(\lambda) > \mu$, and Fact 5.1 imply that the model of cardinality λ is μ^+-saturated. We also know by Fact 5.1 that $\mathcal{K}$ is μ-superstable. By Corollary 5.2, $\mathcal{K}$ has μ-symmetry. Then Fact 2.14 finishes the proof. $\square$

6. Symmetry and tameness

Recall that tameness is a locality property for types introduced by Grossberg and VanDieren in [GV06b] and used to prove Shelah's categoricity from a successor in [GV06c]. It has also played a key roles in the proof of many other categoricity transfers, for example [Bon14b, Vasc].

Definition 6.1 (Tameness)**.** *Let $\mu \geq \text{LS}(\mathcal{K})$. $\mathcal{K}$ is* μ-tame *if for every $M \in \mathcal{K}$ and every $p, q \in \text{ga-S}(M)$, if $p \neq q$, then there exists $M_0 \in \mathcal{K}_{\leq \mu}$ with $M_0 \leq M$ such that $p \restriction M_0 \neq q \restriction M_0$.*

Superstability has been studied together with amalgamation and tameness in works from the second author [Vasa, Vasb, BVb, GV]. We will use the following upward transfer of superstability:

Fact 6.2 (Proposition 10.10 in [Vasb])**.** *Assume $\mathcal{K}$ is μ-superstable and μ-tame. Then for all $\mu' \geq \mu$, $\mathcal{K}$ is μ'-superstable. In particular, $\mathcal{K}_{\geq \mu}$ has no maximal models and is stable in all cardinals.*

6.1. Chains of saturated models.

Our first application of tameness is to chains of saturated models. Recall from Section 3 that $\mathcal{K}^{\lambda\text{-sat}}$ denotes the class of λ-saturated models in $\mathcal{K}_{\geq \lambda}$. We would like to give conditions under which $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC. In particular unions of chains of λ-saturated models are λ-saturated. From superstability and tameness, it is known that one eventually obtains this behavior:

Fact 6.3 (Theorem 7.1 in [BVb]). *Assume $\mathcal{K}$ is μ-superstable and μ-tame. Then there exists $\lambda_0 < \beth_{(2^{\mu^+})^+}$ such that for any $\lambda \geq \lambda_0$, $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$.*

Thus we obtain another partial answer to Question 3.11:

Theorem 6.4. *If $\mathcal{K}$ is μ-superstable and μ-tame, then $\mathcal{K}$ has μ-symmetry.*

Proof. First observe that by Fact 6.2, $\mathcal{K}$ is μ'-superstable for all $\mu' \geq \mu$. By Fact 6.3, there exists $\lambda_0 \geq \mu$ such that $\mathcal{K}^{\lambda_0^+\text{-sat}}$ is an AEC. Therefore the hypothesis of Fact 2.10 are satisfied, so $\mathcal{K}$ has λ_0-symmetry. By Theorem 1.1, $\mathcal{K}$ has μ-symmetry. □

Recall that Fact 2.15 gives a condition under which certain unions of chains of saturated models are saturated. We can obtain an improvement by combining it with the following easy lemma:

Lemma 6.5. *Assume $\mathcal{K}$ is stable in μ. Let M be μ^+-saturated and let $A \subseteq |M|$ be such that $|A|$ is regular. Then there exists $M' \leq M$ such that $A \subseteq |M'|$, M' is μ^+-saturated, and $\|M'\| \leq |A| + \mu^+$.*

Proof. Without loss of generality, $|A| \geq \mu^+$. Let $\chi := |A|$. Build $\langle M_i : i \leq \mu^+ \rangle$ increasing continuous such that for all $i < \mu^+$:

(1) $A \subseteq |M_0|$.
(2) $\|M_i\| = \chi$.
(3) $M_i \leq M$.
(4) If $N \leq M_i$ has size at most μ and $p \in \text{ga-S}(N)$, then p is realized in M_{i+1}.

This is possible:

Take $M_0 \leq M$ any model containing A with $\|M_0\| = \chi$. At limits, take unions. Now given $i < \mu^+$, build $\langle N_j : j < \chi \rangle$ increasing continuous in $\mathcal{K}_\mu$ such that $M_i = \bigcup_{j<\chi} N_j$. Now take $M_{i+1} \leq M$ extending M_i such that for all $j < \chi$, any $p \in \text{ga-S}(N_j)$ is realized in M_{i+1}. This is possible by stability and saturation of M and this is enough: since χ is regular, any $N \leq M_i$ is contained in some N_j.

This is enough:

M_{μ^+} is as desired. □

Theorem 6.6. *Assume $\mathcal{K}$ is μ-superstable, μ^+-superstable, and has μ^+-symmetry. Then $\mathcal{K}^{\mu^+\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathcal{K}^{\mu^+\text{-sat}}) = \mu^+$.*

Proof. We have to show two things: that $\mathcal{K}^{\mu^+\text{-sat}}$ is closed under unions of chains, and that $\mathrm{LS}(\mathcal{K}^{\mu^+\text{-sat}}) = \mu^+$. We first show that it is closed under chains. Let $\langle M_i : i < \delta \rangle$ be an increasing chain of μ^+-saturated models in $\mathcal{K}_{\geq \mu^+}$. Let $M_\delta := \bigcup_{i<\delta} M_i$. Without loss of generality,

$\delta = \text{cf}(\delta)$. If $\delta \geq \mu^+$, the result is clear so assume $\delta < \mu^+$. Let $A \subseteq |M_\delta|$ have size μ. Using Lemma 6.5, build $\langle M_i' : i < \delta \rangle$ an increasing chain of saturated models in $\mathcal{K}_{\mu^+}$ such that $M_i' \leq M_i$ for all $i < \delta$ and $A \cap |M_i| \subseteq |M_i'|$. Then by Fact 2.15, $M_\delta' := \bigcup_{i<\delta} M_i'$ is μ^+-saturated and by definition it contains A, hence all types over A are realized in M_δ', hence in M_δ.

It remains to see that $\text{LS}(\mathcal{K}^{\mu^+\text{-sat}}) = \mu^+$. Clearly, $\text{LS}(\mathcal{K}^{\mu^+\text{-sat}}) \geq \mu^+$. We show that $\text{LS}(\mathcal{K}^{\mu^+\text{-sat}}) \leq \mu^+$. Let $M \in \mathcal{K}^{\mu^+\text{-sat}}$ and let $A \subseteq |M|$. Without loss of generality, $|A| \geq \mu^+$. We must find a μ^+-saturated $M_0 \leq M$ containing A. Work by induction on $|A|$. If $|A|$ is regular, we can use Lemma 6.5. If $|A|$ is singular, let $\delta := \text{cf}(|A|)$ and write $A = \bigcup_{i<\delta} A_i$ with $|A_i| < |A|$ and $|A_i$ regular for all $i < \delta$. Build $\langle M_i : i \leq \delta \rangle$ increasing continuous such that for all $i < \delta$, $|A_i| \subseteq |M_{i+1}|$ and $\|M_i\| \leq \text{LS}(\mathcal{K}) + |A_i|$. This is possible by the induction hypothesis and is enough: $A \subseteq |M_\delta|$ and $\|M_\delta\| \leq \text{LS}(\mathcal{K}) + |A|$. □

We also have the following:

Lemma 6.7. *Let λ be a limit cardinal and let $\lambda_0 < \lambda$. Assume that for all $\mu \in [\lambda_0, \lambda)$, $\mathcal{K}^{\mu\text{-sat}}$ is an AEC with $\text{LS}(\mathcal{K}^{\mu\text{-sat}}) = \mu$. Then $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$.*

Proof. That $\mathcal{K}^{\lambda\text{-sat}}$ is closed under chains is easy to check. To see $\text{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$, let $M \in \mathcal{K}^{\lambda\text{-sat}}$ and let $A \subseteq |M|$. Without loss of generality, $\chi := |A| \geq \lambda$. Let $\delta := \text{cf}(\lambda)$ and let $\langle \lambda_i : i < \delta \rangle$ be an increasing sequence of cardinals with limit λ. Build $\langle M_i : i \leq \delta \rangle$ increasing continuous in $\mathcal{K}_\chi$ such that for all $i < \delta$, M_{i+1} is λ_i^+-saturated and $A \subseteq |M_0|$. This is possible by assumption. Then M_δ is λ_i^+-saturated for all $i < \delta$, hence is λ-saturated. Thus it is as needed. □

Thus we obtain an improvement on the Hanf number of Fact 6.3:

Theorem 6.8. *Assume $\mathcal{K}$ is μ-superstable and μ-tame. For every $\lambda > \mu$, $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$.*

Proof. By Fact 6.2 and Theorem 6.4, $\mathcal{K}$ is λ-superstable and has λ-symmetry for any $\lambda > \mu$. By Theorem 6.6, $\mathcal{K}^{\mu^+\text{-sat}}$ is an AEC with $\text{LS}(\mathcal{K}^{\mu^+\text{-sat}}) = \mu^+$. We can replace μ^+ with any successor $\lambda > \mu$. To take care of limit cardinals λ, use Lemma 6.7. □

Note that Theorem 6.8 does not make Fact 6.3 obsolete. For one thing, the methods developped in [BVb] (in particular averages) have other applications (see for example the proof of solvability in [GV, Theorem 5.43]). Also, the proof of Theorem 6.8 *relies on* Fact 6.3 to get symmetry.

We can also say more on another result of Boney and Vasey: [BVb, Lemma 6.9.(2)] implies that, assuming μ-superstability, there is a $\lambda_0 \geq \mu$ such that if $\langle M_i : i < \delta \rangle$ is a chain of λ_0-saturated models where $\delta \geq \lambda_0$ and M_{i+1} is universal over M_i, then $\bigcup_{i<\delta} M_i$ is saturated. We can improve this too:

Theorem 6.9. *Assume $\mathcal{K}$ is μ-superstable and μ-tame. Let δ be a limit ordinal and $\langle M_i : i < \delta \rangle$ is increasing in $\mathcal{K}_{\geq \mu}$ and M_{i+1} is universal over M_i for all $i < \delta$. Let $M_\delta := \bigcup_{i<\delta} M_i$. If $\|M_\delta\| > \mu$, then M_δ is saturated.*

Proof. By Proposition 4.1, M_δ is a $(\lambda, \mathrm{cf}(\delta))$-limit model, where $\lambda = \|M_\delta\|$. By Fact 6.2, $\mathcal{K}$ is λ-superstable. By Theorem 6.4, $\mathcal{K}$ has λ-symmetry. By Fact 2.14 and our assumption on μ, M_δ is saturated. □

One can ask whether Theorem 6.8 can be improved further by also getting the conclusion for $\lambda = \mu$. If $\mu = \mathrm{LS}(\mathcal{K})$, it is not clear that $\mathrm{LS}(K)$-saturated models are the right notion so perhaps the right question is to be framed in terms of a superlimit. Recall from [She09, Definition N.2.2.4] that a superlimit model is a universal model M with a proper extension so that if $\langle M_i : i < \delta \rangle$ is an increasing chain with $M \cong M_i$ for all $i < \delta$, then (if $\delta < \|M\|^+$), $M \cong \bigcup_{i<\delta} M_i$. Note that, assuming μ-superstability and uniqueness of limit models of size μ, it is easy to see that the existence of a superlimit of size μ is equivalent to the statement that the union of an increasing chain of limit models in μ (of length less than μ^+) is limit.

Question 6.10. *Assume $\mathcal{K}$ is μ-tame and there is a type-full good μ-frame on $\mathcal{K}_\mu$ (or just that $\mathcal{K}$ is μ-superstable). Is there a superlimit model of size μ?*

6.2. **Good frames.** Recall from Section 3 that good frames are a local notion of independence for types of length one. It is known from previous work of the second author that good frames can be constructed from tameness:

Fact 6.11 (Theorem 10.8 in [Vasb])**.** *Assume $\mathcal{K}$ is μ-superstable and μ-tame. If for any $\delta < \mu^+$, any chain of length δ of saturated models in $\mathcal{K}_{\mu^+}$ has a saturated union, then there is a unique type-full good μ^+-frame with underlying class $\mathcal{K}_{\mu^+}^{\mu^+\text{-sat}}$.*

Combining this with Fact 6.3 it was proven in [BVb] that μ-superstability and μ-tameness implies the existence of a good λ-frame on the saturated models of size λ, for some $\lambda > \mu$. Now we show that we can take $\lambda = \mu^+$.

Theorem 6.12. *Assume $\mathcal{K}$ is μ-superstable and μ-tame. If $\mathcal{K}$ has a model of size μ^+, then there is a unique type-full good μ^+-frame with underlying class $\mathcal{K}_{\mu^+}^{\mu^+\text{-sat}}$.*

Proof. Combine Fact 6.11 and Theorem 6.8. The uniqueness is by the canonicity of categorical good frames (Fact 3.8). □

We end by proving Theorem 0.2:

Proof of Theorem 0.2.

(1) By Theorem 6.4 and Fact 2.14.
(2) By Theorem 6.8.
(3) By Theorem 6.12.

□

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