SHELAH'S EVENTUAL CATEGORICITY CONJECTURE IN UNIVERSAL CLASSES. PART II

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Abstract. We prove that a universal class categorical in a high-enough cardinal is categorical on a tail of cardinals. As opposed to other results in the literature, we work in ZFC, do not require the categoricity cardinal to be a successor, do not assume amalgamation, and do not use large cardinals. Moreover we give an explicit bound on the "high-enough" threshold:

Theorem 0.1. Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$ sentence. If ψ is categorical in *some* $\lambda \geq \beth_{\beth_{\omega_1}}$, then ψ is categorical in *all* $\lambda' \geq \beth_{\beth_{\omega_1}}$.

As a byproduct of the proof, we show that a conjecture of Grossberg holds in universal classes:

Corollary 0.2. Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$ sentence that is categorical in some $\lambda \geq \beth_{\beth_{\omega_1}}$, then the class of models of ψ has the amalgamation property for models of size at least $\beth_{\beth_{\omega_1}}$.

We also establish generalizations of these two results to uncountable languages. As part of the argument, we develop machinery to transfer model-theoretic properties between two different classes satisfying a compatibility condition. This is used as a bridge between Shelah's milestone study of universal classes (which we use extensively) and a categoricity transfer of the author for abstract elementary classes that have amalgamation, are tame, and have primes over sets of the form $M \cup \{a\}$.

Contents

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1. Introduction

In 1969, Morley and Keisler told Shelah that first-order model theory was essentially done and the future lied in the model theory of infinitary logics (see the beginning of §6 in [She00]). In 1976, Shelah proposed [She90, Open Problem D.(3a)] the following far-reaching conjecture:

Conjecture 1.1 (Shelah's categoricity conjecture for $\mathbb{L}_{\omega_1,\omega}$)**.** Let ψ be an $\mathbb{L}_{\omega_1,\omega}$ sentence. If ψ is categorical in *some* cardinal $\lambda \geq \beth_{\omega_1}$, then ψ is categorical in *all* cardinals $\lambda' \geq \beth_{\omega_1}$.

This is now recognized as the central test question in nonelementary model theory. In 1977, Shelah introduced abstract elementary classes (AECs) [She87a], an abstract framework encompassing classes of models of an $\mathbb{L}_{\lambda^+,\omega}(Q)$ theory and several other examples of interests. Shelah has stated [She09b, N.4.2] the following version of the conjecture there:

Conjecture 1.2 (Shelah's eventual categoricity conjecture for AECs)**.** If an AEC is categorical in a high-enough cardinal, then it is categorical on a tail of cardinals.

While many pages of approximations exist (see e.g. the introduction of [Vase] for a history), both conjectures are still open.

In this paper, we prove an approximation of Conjecture 1.1 when ψ is a *universal*[1] sentence ($\beth_{\omega_1}$ is replaced by $\beth_{\beth_{\omega_1}}$, see more below). More generally, we confirm Conjecture 1.2 for universal classes: classes of models of a universal $\mathbb{L}_{\infty,\omega}$ sentence, or equivalently classes of models

[1]That is, ψ is of the form $\forall x_0 \forall x_1 \dots \forall x_n \phi(x_0, x_1, \dots, x_n)$, where ϕ is a quantifier-free $\mathbb{L}_{\omega_1,\omega}$ formula.

K in a fixed vocabulary $\tau(K)$ closed under isomorphisms, substructure, and unions of $\subseteq$-increasing chains.

Main Theorem 7.3. Let K be a universal class. If K is categorical in *some* $\lambda \geq \beth_{\beth_{\left(2^{|\tau(K)|+\aleph_0}\right)^+}}$, then K is categorical in *all* $\lambda' \geq \beth_{\beth_{\left(2^{|\tau(K)|+\aleph_0}\right)^+}}$.

Let us compare the main theorem to earlier approximations to Shelah's eventual categoricity conjecture[2]: In a series of papers [GV06b, GV06c, GV06a], Grossberg and VanDieren isolated *tameness*, a locality properties of AECs, and (using earlier work of Shelah [She99]) proved Shelah's eventual categoricity conjecture in tame AECs with amalgamation assuming that the starting categoricity cardinal is a successor. Boney [Bon14] later showed (building on work of Makkai-Shelah [MS90]) that tameness (as well as amalgamation, if in addition categoricity in a high-enough cardinal is assumed) follows from a large cardinal axiom (a proper class of strongly compact cardinals exists). Therefore the eventual categoricity conjecture follows from the following two extra assumptions: the categoricity cardinal is a successor, and a large cardinal axiom holds. In [She09a, IV.7.12], Shelah claims to be able to remove the successor hypothesis assuming amalgamation[3] and the generalized continuum hypothesis (GCH)[4]. Shelah's proof is clarified in [Vasa, Section 8], but it relies on a claim which Shelah has yet to publish a proof of. In any case, all known categoricity transfers (which do not make model-theoretic assumptions on the AEC) rely on the existence of large cardinals together with either GCH or the assumption that the categoricity cardinal is a successor.

In the prequel to this paper [Vase] we showed that some of these limitations could be overcome in the case of universal classes[5]:

Fact 1.3 (Corollary 5.27 in [Vase])**.** Let K be a universal class.

(1) If K is categorical in cardinals of arbitrarily high cofinality, then K is categorical on a tail of cardinals.
(2) If $\kappa > |\tau(K)| + \aleph_0$ is a measurable cardinal and K is categorical in *some* $\lambda \geq \beth_{\beth_\kappa}$ then K is categorical in *all* $\lambda' \geq \beth_{\beth_\kappa}$.

[2]We do not present a complete history or an exhaustive list of recent results here. See the introduction of [Vase] for the former and [BV] for the latter.

[3]By [Bon14, Theorem 7.6], this can also be replaced by a large cardinal axiom.

[4]It is enough to assume the existence of a suitable family of cardinals θ such that $2^\theta < 2^{\theta^+}$.

[5]In earlier versions of [Vase] we claimed to prove the main theorem here but a mistake was later discovered.

Still, requirements on the categoricity cardinal in the first case and the existence of large cardinals in the second case could not be completely eliminated. These hypotheses were made to prove the amalgamation property, which is known to be the only obstacle:

Fact 1.4 (Corollary 7.15 in [Vasa]). Let K be a universal class with amalgamation. If K is categorical in *some* $\lambda \geq \beth_{\left(2^{|\tau(K)|+\aleph_0}\right)^+}$, then K is categorical in *all* $\lambda' \geq \beth_{\left(2^{|\tau(K)|+\aleph_0}\right)^+}$.

Note that (see [Vase, Vasd]) all the facts stated above hold in a much wider context than universal classes: tame AECs with primes. However for the specific case of universal classes there is a well-developed structure theory [She87c]. This paper uses it to remove the assumption of amalgamation from Fact 1.4 and prove the main theorem. Further, a conjecture of Grossberg [Gro02, Conjecture 2.3] says that any AEC categorical in a high-enough cardinal should have amalgamation on a tail. A byproduct of this paper is that Grossberg's conjecture holds in universal classes (see the proof of Theorem 7.3). Note that the behavior of amalgamation in universal classes is nontrivial: Kolesnikov and Lambie-Hanson have shown [KLH] that for each $\alpha < \omega_1$, there is a universal class in a countable vocabulary that has amalgamation up to $\beth_\alpha$ but fails amalgamation everywhere above $\beth_{\omega_1}$ (the example is not categorical in any uncountable cardinal).

The argument to establish Grossberg's conjecture is somewhat indirect: first we use Shelah's structure theory of universal classes to show that there exists an ordering $\leq$ (potentially different from substructure) such that $(K, \leq)$ has amalgamation and other structural properties. We then work inside $(K, \leq)$ to transfer categoricity and only then deduce that $\leq$ is actually substructure (on a tail of cardinals).

The main obstacle to the argument above is that it is unclear that $(K, \leq)$ is an AEC (it may fail the smoothness axiom). The hard part of this paper is proving that it actually *is* an AEC. This is done by working inside a framework for forking-like independence in $(K, \leq)$ that Shelah calls AxFri_1 and proving new results for that framework, including Theorem 5.39 telling us how to copy a chain witnessing the failure of smoothness into an independent tree of models.

It should be noted that these new results (in Section 5) are really the only new pieces needed to prove the main theorem. The rest of the paper is about combining the structure theory of universal classes developed by Shelah [She09b, Chapter V] with the categoricity transfers appearing in [Vasa]. Another contribution of this paper is Section 3 which considers two weak AECs $\mathbf{K}^1, \mathbf{K}^2$ satisfying a compatibility condition (the isomorphism types of models in a categoricity cardinal is the same). The motivation here is the idea outlined above: we may want to study an AEC $\mathbf{K}^1$ by changing its ordering, giving a new class $\mathbf{K}^2$ which has certain properties P of $\mathbf{K}^1$ together with some new properties P' that $\mathbf{K}^1$ may not have. We may know a theorem telling us that a single class that has both P and P' is well-behaved. Section 3 gives tools to generalize the original theorem to the case when we do *not* have a single class (i.e. $\mathbf{K}^1 = \mathbf{K}^2$) but instead have potentially different classes $\mathbf{K}^1$ and $\mathbf{K}^2$.

Note in passing that this paper does not make [Vase] obsolete: the results there hold for a wider context than universal classes, whereas we do not know how to generalize the proof of the main theorem here. Furthermore, we rely heavily here on [Vase].

A natural question is why, the threshold in Theorem 0.1 is $\beth_{\beth_{\omega_1}}$ and not $\beth_{\omega_1}$ as in Conjecture 1.1. The $\beth_{\beth_{\omega_1}}$ comes from the fact that in the argument outlined above the class $(K, \leq)$ has Löwenheim-Skolem-Tarski number χ, for some $\chi < \beth_{\omega_1}$. After proving that it is an AEC, we apply known categoricity transfers to this class, hence the final threshold for categoricity is of order $\beth_{(2^\chi)^+} \leq \beth_{\beth_{\omega_1}}$ (a similar phenomenon occurs in [She99], where Shelah proves that the class $\mathbf{K}$ is χ-weakly tame for some $\chi < \beth_{(2^{\text{LS}(\mathbf{K})})^+}$ and then obtains a threshold of $\beth_{(2^\chi)^+}$). It is unknown whether this argument can be refined to lower the threshold to $\beth_{\omega_1}$.

Let us discuss the background required to read this paper. It is assumed that the reader has a solid knowledge of AECs (including at minimum the material in [Bal09]). Still, except for the basic concepts, we have tried to explicitly state all the definitions and facts. Only little understanding of [Vase, Vasd, Vasa] is required: they are used only as black boxes. While some results in [Vase] rely on deep results of Shelah from the first sections of Chapter IV of [She09a], we do not use them[6]. At one point (Lemma 3.4) we rely on Shelah's construction of a certain linear order [She09a, IV.5]. This can also be taken as a black box. Last but not least, we rely on part of Shelah's original study of universal classes [She87c] (we quote from the updated version in Chapter V of [She09b]). All the results that we use from there have full proofs. We do not rely on any of Shelah's nonstructure results.

[6]The one exception is [She09a, IV.1.12.(2)] (see Fact 2.11), but the proof is short and elementary.

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2. Preliminaries

We state definitions and facts that will be used later. All throughout this paper, we use the letters M, N for models and write $|M|$ for the universe of a model M and $\|M\|$ for the cardinality of its universe. We may abuse notation and write e.g. $a \in M$ when we really mean $a \in |M|$.

Recall the definition of a universal class (for examples, see e.g. [Vase, Example 2.2]).

Definition 2.1 ([She87c])**.** A class of structure K is *universal* if:

(1) It is a class of τ-structures for a fixed vocabulary $\tau = \tau(K)$, closed under isomorphisms.
(2) If $\langle M_i : i < \delta \rangle$ is $\subseteq$-increasing in K, then $\bigcup_{i<\delta} M_i \in K$.
(3) If $M \in K$ and $M_0 \subseteq M$, then $M_0 \in K$.

Remark 2.2. Notice the following fundamental property of a universal class K. Given a subset A of $N \in K$, $\mathrm{cl}^N(A)$, the closure of A under the functions of N (or equivalently $\bigcap\{N_0 \in K \mid A \subseteq |N|, N_0 \subseteq N\}$) is in K.

The following is a well-known syntactic characterization, due to Tarski (only the "if" part will be used):

Fact 2.3. K is a universal class if and only if $K = \mathrm{Mod}(\psi)$, for some universal $\mathbb{L}_{\infty,\omega}$ sentence ψ.

Universal classes are abstract elementary classes:

Definition 2.4 (Definition 1.2 in [She87a]). An *abstract elementary class* (AEC for short) is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:

(1) K is a class of τ-structures, for some fixed vocabulary $\tau = \tau(\mathbf{K})$.
(2) $\leq_{\mathbf{K}}$ is a partial order (that is, a reflexive and transitive relation) on K.
(3) $(K, \leq_{\mathbf{K}})$ respects isomorphisms: If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$. In particular (taking $M = N$), K is closed under isomorphisms.
(4) If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.
(5) Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq_{\mathbf{K}} M_1$;
(6) Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\langle M_i \in K : i < \delta \rangle$ is an increasing chain. Then:
    (a) $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_0 \leq_{\mathbf{K}} M_\delta$.
    (b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.
(7) Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\text{LS}(\mathbf{K})$ for the minimal such cardinal.

Remark 2.5.

(1) When we write $M \leq_{\mathbf{K}} N$, we implicitly also mean that $M, N \in K$.
(2) We write $\mathbf{K}$ for the pair $(K, \leq_{\mathbf{K}})$, and K (no boldface) for the actual class. However we may abuse notation and write for example $M \in \mathbf{K}$ instead of $M \in K$ when there is no danger of confusion. Note that in this paper we will sometimes work with two AECs $\mathbf{K}^1$, $\mathbf{K}^2$ that happen to have the same underlying class K but not the same ordering.

Notice that if K is a universal class, then $\mathbf{K} := (K, \subseteq)$ is an AEC with $\text{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$. Throughout this paper we will use the following notation:

Notation 2.6. Let K be a universal class. We think of K as the AEC $\mathbf{K} := (K, \subseteq)$, and may write "$\mathbf{K}$ is a universal class" instead of "K is a universal class".

We will also have to deal with AECs that may not satisfy the smoothness axiom:

Definition 2.7 (I.1.2.(2) in [She09a])**.** A *weak AEC* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$ satisfying all the axioms of AECs except perhaps smoothness ((6b) in Definition 2.4).

Shelah introduced the following parametrized version of smoothness:

Definition 2.8 (V.1.18.(3) in [She09b])**.** Let $\mathbf{K}$ be a weak AEC. Let $\lambda \geq \text{LS}(\mathbf{K})$ and let δ be a limit ordinal. We say that $\mathbf{K}$ is $(\leq \lambda, \delta)$-*smooth* if for any increasing chain $\langle M_i : i \leq \delta \rangle$ with $\|M_i\| \leq \lambda$ for all $i < \delta$ and $\|M_\delta\| \leq \lambda + \delta$, we have that $\bigcup_{i<\delta} M_i \leq_{\mathbf{K}} M_\delta$. $(\leq \lambda, \leq \kappa)$-smooth means $(\leq \lambda, \delta)$-smooth for all $\delta \leq \kappa$, and similarly for the other variations.

Remark 2.9. Above, we could have allowed $\|M_\delta\| > \lambda + \delta$ and gotten an equivalent definition. Indeed, if $M_i \leq_{\mathbf{K}} M_\delta$ for all $i < \delta$ and we want to see that $\bigcup_{i<\delta} M_i \leq_{\mathbf{K}} M_\delta$, we can use the Löwenheim-Skolem-Tarski axiom to take $N \leq_{\mathbf{K}} M_\delta$ containing $\bigcup_{i<\delta} |M_i|$ and having size at most $\lambda + \delta$. Then we can use coherence to see that $M_i \leq_{\mathbf{K}} N$ for all $i < \delta$, hence by smoothness, $\bigcup_{i<\delta} M_i \leq_{\mathbf{K}} N$ and so by transitivity of $\leq_{\mathbf{K}}$, $\bigcup_{i<\delta} M_i \leq_{\mathbf{K}} M_\delta$.

We now list a several known facts about AECs that we will use. First, recall that an AEC $\mathbf{K}$ is determined by its restriction $\mathbf{K}_{\text{LS}(\mathbf{K})}$ to models of size $\text{LS}(\mathbf{K})$. More precisely:

Fact 2.10 (II.1.23 in [She09a])**.** Assume $\mathbf{K}^1$ and $\mathbf{K}^2$ are AECs with $\lambda := \text{LS}(\mathbf{K}^1) = \text{LS}(\mathbf{K}^2)$. If $\mathbf{K}^1_\lambda = \mathbf{K}^2_\lambda$, then $\mathbf{K}^1_{\geq\lambda} = \mathbf{K}^2_{\geq\lambda}$.

We will use the relationship between the ordering of any AEC and elementary equivalence in a sufficiently powerful infinitary logic:

Fact 2.11. Let $\mathbf{K}$ be an AEC and let $M, N \in \mathbf{K}$.

(1) [Kue08, Theorem 7.2.(b)] If $M \preceq_{\mathbb{L}_{\infty,\text{LS}(\mathbf{K})^+}} N$, then $M \leq_{\mathbf{K}} N$.
(2) [She09a, IV.1.12.(2)] Let λ be an infinite cardinal such that $\mathbf{K}$ is categorical in λ and $\lambda = \lambda^{\text{LS}(\mathbf{K})}$. If $M, N \in \mathbf{K}_\lambda$ and $M \leq_{\mathbf{K}} N$, then $M \preceq_{\mathbb{L}_{\infty,\text{LS}(\mathbf{K})^+}} N$.

We will also use that AECs have a Hanf number. Below, we write $\delta(\lambda)$ for the pinning down ordinal at λ: the first ordinal that is not definable in $\mathbb{L}_{\lambda^+,\omega}$. We will also deal with the more general $\delta(\lambda, \kappa)$ (see [She90, VII.5.5.1] for a precise definition). Recall the following well-known facts about this ordinal (see e.g. [She90, VII.5]):

Fact 2.12.

(1) (Lopez-Escobar) $\delta(\aleph_0) = \omega_1$.
(2) (Morley and C.C. Chang) For any infinite cardinals λ and κ, $\delta(\lambda, \kappa) \le (2^\lambda)^+$.

Definition 2.13. Let $\mathbf{K}$ be an AEC.

(1) Let $\lambda(\mathbf{K})$ be the least cardinal $\lambda \ge \mathrm{LS}(\mathbf{K})$ such that there exists a vocabulary $\tau_1 \supseteq \tau(\mathbf{K})$, a first-order τ_1-theory T_1, and a set of T_1-types Γ such that:
    (a) $\mathbf{K} = \mathrm{PC}(T_1, \Gamma, \tau(\mathbf{K}))$.
    (b) For $M, N \in \mathrm{EC}(T_1, \Gamma)$, if $M \subseteq N$, then $M \upharpoonright \tau(\mathbf{K}) \le_{\mathbf{K}} N \upharpoonright \tau(\mathbf{K})$.
    (c) $|T_1| + |\tau_1| \le \mathrm{LS}(\mathbf{K})$ and $|\Gamma| \le \lambda$.
(2) Let $\delta(\mathbf{K}) := \delta(\mathrm{LS}(\mathbf{K}), \lambda(\mathbf{K}))$.
(3) Let $h(\mathbf{K}) := \beth_{\delta(\mathbf{K})}$.

Remark 2.14. By Chang's presentation theorem [Cha68], if $\mathbf{K}$ is axiomatized by an $\mathbb{L}_{\lambda^+,\omega}$ sentence, and the ordering is just substructure (as for universal classes), then $\lambda(\mathbf{K}) \le \lambda$.

It makes sense to talk of $\lambda(\mathbf{K})$ because of Shelah's presentation theorem:

Fact 2.15 (I.1.9 in [She09a])**.** For any AEC $\mathbf{K}$, there exists a vocabulary $\tau_1 \supseteq \tau(\mathbf{K})$, a first-order τ_1-theory T_1, and a set of T_1-types Γ such that (1a) and (1b) in Definition 2.13 hold and $|T_1| + |\tau_1| \le \mathrm{LS}(\mathbf{K})$. Thus $\lambda(\mathbf{K}) \le 2^{\mathrm{LS}(\mathbf{K})}$.

Definition 2.16. For an infinite cardinal λ, let $h(\lambda) := \beth_{(2^\lambda)^+}$.

Remark 2.17. By Facts 2.12 and 2.15, For any AEC $\mathbf{K}$, $h(\mathbf{K}) \le h(\mathrm{LS}(\mathbf{K}))$.

The reason $h(\mathbf{K})$ is interesting is because it is a Hanf number for $\mathbf{K}$ (this follows from Chang's result on the Hanf number of PC classes [Cha68]).

Fact 2.18. Let $\mathbf{K}$ be an AEC. If $\mathbf{K}$ has a model of size $h(\mathbf{K})$, then $\mathbf{K}$ has arbitrarily large models.

In the rest of this section, we quote categoricity transfer results that we will use. We assume that the reader is familiar with notions such as amalgamation, joint embedding, Galois types, Ehrenfeucht-Mostowski models, and tameness (see for example [Bal09]). The notation we use is standard and is described in details at the beginning of [Vasc] (for Ehrenfeucht-Mostowski models, we use the notation in [She09a, IV.0.8][7]). For example, we write $\mathrm{gtp}(\bar{b}/M; N)$ for the Galois type of $\bar{b}$ over M, as computed in N. This assumes that we are working inside an AEC $\mathbf{K}$ that is clear from context. When we want to emphasize $\mathbf{K}$, we will write $\mathrm{gtp}_{\mathbf{K}}(\bar{b}/M; N)$.

The following result is implicit in the proof of [GV06c, Corollary 4.3]. For completeness, we sketch a proof.

Fact 2.19. If $\mathbf{K}$ is an AEC with amalgamation and arbitrarily large models, then the categoricity spectrum (i.e. the class of cardinals $\lambda \geq \mathrm{LS}(\mathbf{K})$ such that $\mathbf{K}$ is categorical in λ) is closed.

Proof. Let $\lambda > \mathrm{LS}(\mathbf{K})$ be a limit cardinal such that $\mathbf{K}$ is categorical in unboundedly many $\lambda_0 < \lambda$. We show that $\mathbf{K}$ is categorical in λ. Since $\mathbf{K}$ has arbitrarily large models, we can use Ehrenfeucht-Mostowski models to see that $\mathbf{K}$ is (Galois) stable in every $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$ (see e.g. [She, Claim I.1.7]). Therefore for every categoricity cardinal $\lambda_0 \in [\mathrm{LS}(\mathbf{K}), \lambda)$, the model of size λ_0 is (Galois) saturated. Hence it is straightforward to see that any model of size λ is saturated. The result now follows from the uniqueness of saturated models. $\square$

To state the next categoricity transfer, we first recall Shelah's notion of an AEC having primes. The intuition is that the AEC has prime models over every set of the form $M \cup \{a\}$, for $M \in \mathbf{K}$. This is described formally using Galois types.

Definition 2.20 (III.3.2 in [She09a])**.** Let $\mathbf{K}$ be an AEC.

(1) (a, M, N) is a *prime triple* if $M \leq_{\mathbf{K}} N$, $a \in |N| \backslash |M|$, and for every $N' \in \mathbf{K}$, $a' \in |N'|$, such that $\mathrm{gtp}(a/M; N) = \mathrm{gtp}(a'/M; N')$, there exists $f : N \underset{M}{\longrightarrow} N'$ with $f(a) = a'$.
(2) $\mathbf{K}$ *has primes* if for any nonalgebraic Galois type $p \in \mathrm{gS}(M)$ there exists a prime triple (a, M, N) such that $p = \mathrm{gtp}(a/M; N)$.

By taking the closure of the relevant set under the functions of an ambient model, we obtain:

Fact 2.21 (Remark 5.3 in [Vase])**.** Any universal class has primes.

The following is a ZFC approximation of Shelah's eventual categoricity conjecture in tame AECs with amalgamation. It combines works of Makkai-Shelah [MS90], Shelah [She99], Grossberg and VanDieren [GV06c, GV06a], and the author [Vase, Vasd, Vasa].

[7]For $\mathbf{K}$ an AEC, we call Φ an *EM blueprint for* $\mathbf{K}$ if $\Phi \in \Upsilon^{\mathrm{or}}_{\mathbf{K}}$.

Fact 2.22. Let $\mathbf{K}$ be a $\mathrm{LS}(\mathbf{K})$-tame AEC with amalgamation and arbitrarily large models. Let $\lambda > \mathrm{LS}(\mathbf{K})$ be such that $\mathbf{K}$ is categorical in λ.

(1) [Vasa, Theorem 3.3][8] If δ is a limit ordinal that is divisible by $\left(2^{\mathrm{LS}(\mathbf{K})}\right)^+$, then $\mathbf{K}$ is categorical in $\beth_\delta$.
(2) $\mathbf{K}$ is categorical in all $\lambda' \geq \min(\lambda, h(\mathrm{LS}(\mathbf{K})))$ when at least one of the following holds:
   (a) [Vasa, 7.4, 7.7] There exists a *successor* cardinal $\mu > \mathrm{LS}(\mathbf{K})^+$ such that $\mathbf{K}$ is categorical in μ.
   (b) [Vasa, Theorem 7.14] $\mathbf{K}$ has primes.

Remark 2.23. In Fact 2.22, we do not use that $\mathbf{K}$ has joint embedding: we can find a sub-AEC $\mathbf{K}^0$ of $\mathbf{K}$ that has joint embedding and work within $\mathbf{K}^0$. See Definition 6.11.

Remark 2.24. If in Fact 2.22 we start instead with a χ-tame AEC (with $\chi > \mathrm{LS}(\mathbf{K})$), the same conclusions hold for $\mathbf{K}_{\geq \chi}$.

3. Compatible pairs of AECs

Let K be a universal class. A central result of Shelah [She09b, V.B] is that if K does not have the order property, there is an ordering $\leq$ such that $(K, \leq)$ has several structural properties, including amalgamation. The downside is that $(K, \leq)$ might loose the smoothness axiom, i.e. it may only be a weak AEC. We will give the precise statement of Shelah's result and discuss its implications in the next sections.

Here, we look at the situation abstractly: we consider pairs of weak AECs $\mathbf{K}^1 = (K^1, \leq_{\mathbf{K}^1})$ and $\mathbf{K}^2 = (K^2, \leq_{\mathbf{K}^2})$ satisfying a compatibility condition. The case of interest is $\mathbf{K}^1 = (K, \subseteq)$ and $\mathbf{K}^2 = (K, \leq)$.

Definition 3.1. For $\ell = 1, 2$, let $\mathbf{K}^\ell = (K^\ell, \leq_{\mathbf{K}^\ell})$ be weak AECs. $\mathbf{K}^1$ and $\mathbf{K}^2$ are *compatible* if:

(1) $\tau(\mathbf{K}^1) = \tau(\mathbf{K}^2)$.
(2) For any $\lambda > \mathrm{LS}(\mathbf{K}^1) + \mathrm{LS}(\mathbf{K}^2)$, if either $\mathbf{K}^1$ or $\mathbf{K}^2$ is categorical in λ, then $K^1_\lambda = K^2_\lambda$.

We write $\mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2)$ instead of $\mathrm{LS}(\mathbf{K}^1) + \mathrm{LS}(\mathbf{K}^2)$.

[8] The version for classes of models axiomatized by an $\mathbb{L}_{\kappa,\omega}$ theory, κ strongly compact, appears in [MS90]. In tame AECs, a little bit more work has to be done to show that the model in the categoricity cardinal is saturated (one has to use the Shelah-Villaveces theorem [SV99, Theorem 2.2.1] together with the upward superstability transfer in [Vasb, Proposition 10.10]).

Remark 3.2. This definition is really only useful when one of the classes is categorical. Note that in (2), we only ask for $K^1_\lambda = K^2_\lambda$, i.e. the isomorphism type of the model of size λ must be the same in both classes, but the orderings need not agree.

For the rest of this section, we assume:

Hypothesis 3.3. $\mathbf{K}^1 = (K^1, \leq_{\mathbf{K}^1})$ and $\mathbf{K}^2 = (K^2, \leq_{\mathbf{K}^2})$ are compatible weak AECs. We set $\tau := \tau(\mathbf{K}^1) = \tau(\mathbf{K}^2)$.

Assume that $\mathbf{K}^1$ is categorical in a $\lambda > \text{LS}(\mathbf{K}^1, \mathbf{K}^2)$. What can we say about $\mathbf{K}^2$? If $\mathbf{K}^1$ is a universal class and $\mathbf{K}^2$ is as above, $\mathbf{K}^1$ is an AEC, and one of our ultimate goal is to show that $\mathbf{K}^2$ is also an AEC. The following result will turn out to be key. Under some assumptions, $\mathbf{K}^2$ is stable below the categoricity cardinal.

Lemma 3.4. Assume:

(1) $\mathbf{K}^1$ is an AEC with arbitrarily large models.
(2) $\mathbf{K}^2$ has amalgamation and joint embedding.

Let $\lambda > \text{LS}(\mathbf{K}^1, \mathbf{K}^2)$. If $\mathbf{K}^2$ is categorical in λ, then $\mathbf{K}^2$ is $(< \omega)$-stable in all $\mu \in [\text{LS}(\mathbf{K}^1, \mathbf{K}^2), \lambda)$ such that $\mu^+ < \lambda$. That is, for any such μ and any $M \in \mathbf{K}^2_\mu$, $|\text{gS}^{<\omega}_{\mathbf{K}^2}(M)| \leq \mu$

Before starting the proof, a few comments are in order. First note that the case $\mathbf{K}^1 = \mathbf{K}^2$ is a classical result that can be traced back to Morley [Mor65, Theorem 3.7]. It appears explicitly as [She99, Claim I.1.7]. The proof uses Ehrenfeucht-Mostowski (EM) models. Here, we have additional difficulties since the EM models are well-behaved really only for $\mathbf{K}^1$ and not for $\mathbf{K}^2$ (in fact, $\mathbf{K}^2$ may be only a weak AEC, so may not have any suitable EM blueprint). More precisely, if Φ is an EM blueprint for $\mathbf{K}^1$ and $I \subseteq J$ are linear orders, then $\text{EM}_\tau(I, \Phi) \leq_{\mathbf{K}^1} \text{EM}_\tau(J, \Phi)$ but possibly $\text{EM}_\tau(I, \Phi) \not\leq_{\mathbf{K}^2} \text{EM}_\tau(J, \Phi)$. Thus a Galois type of $\mathbf{K}^2$ computed inside $\text{EM}_\tau(I, \Phi)$ may not be the same as one computed in $\text{EM}_\tau(J, \Phi)$. For this reason, we want to use only that Galois types are invariant under isomorphisms in the proof, and hence want to use the existence of certain linear orderings with many automorphisms.

Fortunately, Shelah gives a proof of the case $\mathbf{K}^1 = \mathbf{K}^2$ in [She, Claim I.1.7] (the online version of [She99]) that we can imitate. It uses the following fact, which Shelah claims follows from the appendix of [She87b]. Unfortunately, we have not been able to derive a proof from the material there, but a more explicit construction also appears in [She09a, IV.5.1.(2)].

Fact 3.5. Let $\theta < \lambda$ be infinite cardinals with θ regular. There exists a linear order I of size λ such that for every $I_0 \subseteq I$ of size less than θ, there is $J \subseteq I$ such that:

(1) $I_0 \subseteq J$.
(2) $\|J\| = \|I_0\| + \aleph_0$.
(3) For any $\bar{a} \in {}^{<\omega}I$, there is $f \in \mathrm{Aut}_{I_0}(I)$ such that $f(\bar{a}) \in {}^{<\omega}J$.

This fact at hand, the proof is straightforward:

Proof of Lemma 3.4. Since $\mathbf{K}^1$ has arbitrarily large models and is an AEC, it has an Ehrenfeucht-Mostowski blueprint Φ. Let $\mu \in [\mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2), \lambda)$ and let $M \in K^2_\mu$. We want to see that $|\mathrm{gS}_{\mathbf{K}^2}(M)| \leq \mu$. Let I be as described by Fact 3.5 (where θ there stands for μ^+ here, we are using that $\mu^+ < \lambda$). Suppose for a contradiction that $|\mathrm{gS}^{<\omega}_{\mathbf{K}^2}(M)| > \mu$. Then using amalgamation we can find $N \in K^2$ with $M \leq_{\mathbf{K}^2} N$ and a sequence $\langle \bar{a}_i \in {}^{<\omega}|N| : i < \mu^+\rangle$ such that for $i < j < \mu^+$, $\mathrm{gtp}_{\mathbf{K}^2}(\bar{a}_i/M; N) \neq \mathrm{gtp}_{\mathbf{K}^2}(\bar{a}_j/M; N)$.

By joint embedding and categoricity, without loss of generality $N = \mathrm{EM}_\tau(I, \Phi)$. Now let $I_0 \subseteq I$ be such that $|I_0| = \mu$ and $M \subseteq \mathrm{EM}_\tau(I_0, \Phi)$. Let J be as given by the definition of I and let $M_1 := \mathrm{EM}_\tau(J, \Phi)$. We have that for each $i < \mu^+$, there is a finite linear order $I_i \subseteq I$ generating $\bar{a}_i$, so pick $f_i \in \mathrm{Aut}_{I_0}(I)$ such that $f_i[I_i] \subseteq J$. Let $\widehat{f_i} \in \mathrm{Aut}_{M_1}(N)$ be the automorphism of $N = \mathrm{EM}_\tau(I, \Phi)$ naturally induced by f_i. Then $\widehat{f_i}(\bar{a}_i) \in |M_1|$. By the pigeonhole principle, without loss of generality there is $\bar{b} \in |M_1|$ such that for all $i < \mu^+$, $\widehat{f_i}(\bar{a}_i) = \bar{b}$. But this means that for $i < \mu^+$:

$$\mathrm{gtp}_{\mathbf{K}^2}(\bar{a}_i/M; N) = \mathrm{gtp}_{\mathbf{K}^2}(\widehat{f_i}(\bar{a}_i)/M; N) = \mathrm{gtp}_{\mathbf{K}^2}(\bar{b}/M; N)$$

So for $i < j < \mu^+$, $\mathrm{gtp}_{\mathbf{K}^2}(\bar{a}_i/M; N) = \mathrm{gtp}_{\mathbf{K}^2}(\bar{a}_j/M; N)$, a contradiction. $\square$

For the rest of this section, we assume that $\mathbf{K}^1$ and $\mathbf{K}^2$ are both AECs and discuss categoricity transfers (generalizing Fact 2.22) to this setup. First, we show that categoricity in a suitable cardinal implies that the two classes (and their ordering) are equal on a tail.

Lemma 3.6. Assume $\mathbf{K}^1$ and $\mathbf{K}^2$ are AECs. Let λ be an infinite cardinal such that:

(1) $\mathbf{K}^1$ is categorical in λ.
(2) $\lambda = \lambda^{\mathrm{LS}(\mathbf{K}^1, \mathbf{K}^2)}$.

Then $\mathbf{K}^1_{\ge\lambda} = \mathbf{K}^2_{\ge\lambda}$.

Proof. By compatibility, $K^1_\lambda = K^2_\lambda$. By Fact 2.10 (where $\mathbf{K}^1, \mathbf{K}^2$ there stand for $\mathbf{K}^1_{\ge\lambda}$, $\mathbf{K}^2_{\ge\lambda}$ here), it is enough to show that the orderings of $\mathbf{K}^1$ and $\mathbf{K}^2$ coincide on K^1_λ. So let $M, N \in K^1_\lambda$. We show that $M \le_{\mathbf{K}^1} N$ implies $M \le_{\mathbf{K}^2} N$ (the converse is symmetric).

So assume that $M \le_{\mathbf{K}^1} N$. By Fact 2.11.(2) (where $\mathbf{K}$, λ there stand for $\mathbf{K}^1_{\ge \mathrm{LS}(\mathbf{K}^1,\mathbf{K}^2)}$, λ here), $M \preceq_{\mathbb{L}_{\infty,\mathrm{LS}(\mathbf{K}^1,\mathbf{K}^2)^+}} N$. By Fact 2.11.(1) (where $\mathbf{K}$ there stands for $\mathbf{K}^2_{\ge \mathrm{LS}(\mathbf{K}^1,\mathbf{K}^2)}$ here), $M \le_{\mathbf{K}^2} N$, as desired. $\square$

The next result shows that if one of the classes has amalgamation, we can find a categoricity cardinal satisfying the condition of the previous lemma.

Theorem 3.7. Assume $\mathbf{K}^1$ and $\mathbf{K}^2$ are AECs categorical in a proper class of cardinals. If $\mathbf{K}^1$ has amalgamation, then there exists λ such that $\mathbf{K}^1_{\ge\lambda} = \mathbf{K}^2_{\ge\lambda}$.

Proof. Because $\mathbf{K}^1$ is categorical in a proper class of cardinals, it has arbitrarily large models, so by Fact 2.19, $\mathbf{K}^1$ is categorical on a closed unbounded class of cardinals. In particular, one can find an infinite cardinal λ such that $\mathbf{K}^1$ is categorical in λ and $\lambda = \lambda^{\mathrm{LS}(\mathbf{K}^1,\mathbf{K}^2)}$. By Lemma 3.6, $\mathbf{K}^1_{\ge\lambda} = \mathbf{K}^2_{\ge\lambda}$. $\square$

We end this section with a categoricity transfer. Intuitively, this shows that if we start with an AEC $\mathbf{K}^1$ with primes, it is enough to change its ordering (getting an AEC $\mathbf{K}^2$) so that $\mathbf{K}^2$ has amalgamation and is tame (it may lose existence of primes). This is especially relevant to universal classes, since they always have primes (Fact 2.21). Note that Fact 2.22.(2b) is the case $\mathbf{K}^1 = \mathbf{K}^2$.

Theorem 3.8. Assume $\mathbf{K}^1$ and $\mathbf{K}^2$ are AECs such that:

(1) $\mathbf{K}^1$ has primes.
(2) $\mathbf{K}^2$ has amalgamation, arbitrarily large models, and is $\mathrm{LS}(\mathbf{K}^2)$-tame.

If $\mathbf{K}^2$ is categorical in a $\lambda > \mathrm{LS}(\mathbf{K}^2)$, then $\mathbf{K}^2$ is categorical in all $\lambda' \ge \min(\lambda, h(\mathrm{LS}(\mathbf{K}^2)))$.

Proof. By Fact 2.22.(1), $\mathbf{K}^2$ is categorical in a proper class of cardinals. By Theorem 3.7 (where the role of $\mathbf{K}^1$ and $\mathbf{K}^2$ is switched), we can fix a cardinal λ_0 such that $\mathbf{K}^1_{\ge\lambda_0} = \mathbf{K}^2_{\ge\lambda_0}$. In particular, $\mathbf{K}^2_{\ge\lambda_0}$ has primes. By Fact 2.22.(2b), $\mathbf{K}^2_{\ge\lambda_0}$ is categorical on a tail, and in particular in a successor cardinal. Applying Fact 2.22.(2a) to $\mathbf{K}^2$, this implies that $\mathbf{K}^2$ is categorical in all $\lambda' \geq \min(\lambda, h(\mathrm{LS}(\mathbf{K}^2)))$, as desired. $\square$

4. Independence in weak AECs

AxFri_1 is an axiomatic framework for independence in weak AECs that Shelah introduces in [She87c]. The main motivation for the axioms is that if K is a universal class that does not have the order property, then there is an ordering $\leq$ such that $(K, \leq)$ satisfies AxFri_1 (see Section 6). Here, we repeat the definition and state some facts that we will use. We quote from Chapter V of [She09b], an updated version of [She87c].

Definition 4.1 (AxFri_1, V.B in [She09b])**.** $(\mathbf{K}, \mathop{\smile}\limits^{\mid}, \mathrm{cl})$ satisfies[9] AxFri_1 if:

(1) $\mathbf{K}$ is a weak AEC.
(2) For each $N \in \mathbf{K}$, cl^N is a function from $\mathcal{P}(|N|)$ to $\mathcal{P}(|N|)$. Often, $\mathrm{cl}^N(A)$ induces a $\tau(\mathbf{K})$-substructure M of N. In this case, we identify $\mathrm{cl}^N(A)$ with M. We require cl to satisfy the following axioms: For $N, N' \in \mathbf{K}$, $A, B \subseteq |N|$:
   (a) Invariance: If $f : N \cong N'$, then $\mathrm{cl}^{N'}(f[A]) = f[\mathrm{cl}^N(A)]$.
   (b) Monotonicity 1: If $A \subseteq B$, then $\mathrm{cl}^N(A) \subseteq \mathrm{cl}^N(B)$.
   (c) Monotonicity 2: If $N \leq_{\mathbf{K}} N'$, then $\mathrm{cl}^N(A) = \mathrm{cl}^{N'}(A)$.
   (d) Idempotence: $\mathrm{cl}^N(\mathrm{cl}^N(A)) = \mathrm{cl}^N(A)$.
(3) $\mathop{\smile}\limits^{\mid}$ is a 4-ary relation on $\mathbf{K}$. We write $M_1 \mathop{\mathop{\smile}\limits^{\mid}}\limits_{M_0}^{M_3} M_2$ instead of $\mathop{\smile}\limits^{\mid}(M_0, M_1, M_2, M_3)$. We require that $\mathop{\smile}\limits^{\mid}$ satisfies the following axioms:
   (a) $M_1 \mathop{\mathop{\smile}\limits^{\mid}}\limits_{M_0}^{M_3} M_2$ implies that for $\ell = 1, 2$, $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$.
   (b) Invariance: If $f : M_3 \cong M_3'$ and $M_1 \mathop{\mathop{\smile}\limits^{\mid}}\limits_{M_0}^{M_3} M_2$, then $f[M_1] \mathop{\mathop{\smile}\limits^{\mid}}\limits_{f[M_0]}^{M_3'} f[M_2]$.
   (c) Monotonicity 1: If $M_1 \mathop{\mathop{\smile}\limits^{\mid}}\limits_{M_0}^{M_3} M_2$ and $M_3 \leq_{\mathbf{K}} M_3'$, then $M_1 \mathop{\mathop{\smile}\limits^{\mid}}\limits_{M_0}^{M_3'} M_2$.
   (d) Monotonicity 2: If $M_1 \mathop{\mathop{\smile}\limits^{\mid}}\limits_{M_0}^{M_3} M_2$ and $M_0 \leq_{\mathbf{K}} M_2' \leq_{\mathbf{K}} M_2$, then $M_1 \mathop{\mathop{\smile}\limits^{\mid}}\limits_{M_0}^{M_3} M_2'$.

[9] In order to be consistent with [Vase], we write cl rather than Shelah's $\langle\rangle_{\mathrm{gn}}$.

(e) Base enlargement: If $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\;\;|}}} M_2$ and $M_0 \leq_{\mathbf{K}} M_2' \leq_{\mathbf{K}} M_2$, then $\mathrm{cl}^{M_3}(M_2' \cup M_1) \overset{M_3}{\underset{M_2'}{\mathop{\smile\!\!\!\!\!\!\;\;|}}} M_2$.

(f) Symmetry: If $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\;\;|}}} M_2$, then $M_2 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\;\;|}}} M_1$.

(g) Existence: If $M_0 \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$, then there exists $N \in \mathbf{K}$ and $f_\ell : M_\ell \underset{M_0}{\longrightarrow} N$, $\ell = 1, 2$, such that $f[M_1] \overset{N}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\;\;|}}} f[M_2]$.

(h) Uniqueness: If for $\ell = 1, 2$, $M_1^\ell \overset{M_3^\ell}{\underset{M_0^\ell}{\mathop{\smile\!\!\!\!\!\!\;\;|}}} M_2^\ell$ and $fori < 3$, $f_i : M_i^1 \cong M_i^2$ are such that $f_0 \subseteq f_1$, $f_0 \subseteq f_2$, then there exists $N \in \mathbf{K}$ with $M_3^2 \leq_{\mathbf{K}} N$ and $h : M_3^1 \to N$ such that $f_1 \cup f_2 \subseteq h$.

(i) Finite character: If δ is a limit ordinal, $\langle M_{2,i} : i \leq \delta \rangle$ is increasing and continuous, $M_0 \leq_{\mathbf{K}} M_{1,0}$, and $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!\;\;|}}} M_{2,\delta}$, then $\mathrm{cl}^{M_3}(M_1 \cup M_{2,\delta}) = \bigcup_{i<\delta} \mathrm{cl}^{M_3}(M_1 \cup M_{2,i})$.

We say that a weak AEC $\mathbf{K}$ *satisfies AxFri*$_1$ if there exists $\mathop{\smile\!\!\!\!\!\!\;\;|}$ and cl such that $(\mathbf{K}, \mathop{\smile\!\!\!\!\!\!\;\;|}, \mathrm{cl})$ satisfies AxFri$_1$.

Remark 4.2. The definition we give is slightly different from Shelah's: Shelah does not assume that $\mathbf{K}$ has a Löwenheim-Skolem-Tarski number. We do not need the extra generality, although there are places (e.g. Section 5) where the existence of a Löwenheim-Skolem-Tarski number is not used.

Remark 4.3. There is an example (derived from the class of metric graphs, see [She09b, V.B.1.22]) of a triple $(\mathbf{K}, \mathop{\smile\!\!\!\!\!\!\;\;|}, \mathrm{cl})$ that satisfies AxFri$_1$ but where $\mathbf{K}$ is not an AEC.

Remark 4.4. If a weak AEC $\mathbf{K}$ satisfies AxFri$_1$, then by the existence property for $\mathop{\smile\!\!\!\!\!\!\;\;|}$, $\mathbf{K}$ has amalgamation.

In the rest of this section, we assume:

Hypothesis 4.5. $(\mathbf{K}, \mathop{\smile\!\!\!\!\!\!\;\;|}, \mathrm{cl})$ satisfies AxFri$_1$.

The following is easy to see from the definition of the closure operator.

Fact 4.6. Let $N \in \mathbf{K}$ and let $\langle A_i : i \in I \rangle$ be a sequence of subsets of $|N|$, $I \neq \emptyset$. Then:

(1) $\bigcup_{i \in I} \mathrm{cl}^N(A_i) \subseteq \mathrm{cl}^N(\bigcup_{i \in I} A_i)$.

(2) $\text{cl}^N(\bigcup_{i \in I} A_i) = \text{cl}^N(\bigcup_{i \in I} \text{cl}^N(A_i))$.

The following are consequences of the axioms and will all be used in the rest of this paper.

Fact 4.7.

(1) [She09b, V.B.1.21.(1)] If $M_1 \overset{M_3}{\underset{M_0}{\perp}} M_2$, then $\text{cl}^{M_3}(M_1 \cup M_2) \leq_{\mathbf{K}} M_3$ and $M_1 \overset{\text{cl}^{M_3}(M_1 \cup M_2)}{\underset{M_0}{\perp}} M_2$.
(2) [She09b, V.C.1.3] Transitivity: If $M_1 \overset{M_3}{\underset{M_0}{\perp}} M_2$ and $M_3 \overset{M_5}{\underset{M_2}{\perp}} M_4$, then $M_1 \overset{M_5}{\underset{M_0}{\perp}} M_4$.
(3) [She09b, V.C.1.6] Let δ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ be $\subseteq$-increasing continuous chains such that for all $i < j < \delta$, $M_j \overset{N_j}{\underset{M_i}{\perp}} N_i$. Then for all $i \leq \delta$, $M_\delta \overset{N_\delta}{\underset{M_i}{\perp}} N_i$.
(4) [She09b, V.C.1.10.(1)] Let δ be a limit ordinal. Let $\langle M_i : i \leq \delta+1 \rangle$, $\langle N_i^a : i \leq \delta \rangle$, $\langle N_i^b : i \leq \delta \rangle$ be increasing continuous chains such that for all $i < \delta$, $N_i^a \overset{N_i^b}{\underset{M_i}{\perp}} M_{\delta+1}$ and $N_i^b = \text{cl}^{N_i^b}(M_{\delta+1} \cup N_i^a)$. Then $N_\delta^a \overset{N_\delta^b}{\underset{M_\delta}{\perp}} M_{\delta+1}$.
(5) Let δ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ be increasing continuous so that for $i, j < \delta$, $M_j \overset{N_j}{\underset{M_i}{\perp}} N_i$. Let $M \in \mathbf{K}$ be such that $M_i \leq_{\mathbf{K}} M$ for all $i < \delta$ (but possibly $M_\delta \not\leq_{\mathbf{K}} M$). Then there exists $N \in \mathbf{K}$ and an embedding $f : M \underset{M_\delta}{\longrightarrow} N$ such that for all $i < \delta$:
   (a) $N_i \leq_{\mathbf{K}} N$.
   (b) $f[M] \overset{N}{\underset{M_i}{\perp}} N_i$.
   (c) $N = \text{cl}^N(f[M] \cup N_\delta)$.

Proof of (5). This is given by the proof of [She09b, V.C.1.11], but Shelah omits the end of the proof. We give it here. We build $\langle N_i^a, N_i^b, f_i : i \leq \delta \rangle$ such that:

(1) $\langle N_i^x : i \leq \delta \rangle$ is increasing continuous for $x \in \{a, b\}$.

(2) For $i < \delta$, $M \overset{N_i^b}{\underset{M_i}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}} N_i^a$.
(3) For $i < \delta$, $N_i^b = \text{cl}^{N_i^b}(M \cup N_i^a)$.
(4) For $i \leq \delta$, $f_i : N_i \cong_{M_i} N_i^a$.

This is possible by the proof of [She09b, V.C.1.11]. Let us see that it is enough. Find $N \in \mathbf{K}$ and $f : N_\delta^b \cong N$ that extends f_δ^{-1}. We claim that this works. First observe that $f \upharpoonright M : M \xrightarrow[M_\delta]{N}$ as f fixes M_i for each $i < \delta$ and $M \leq_{\mathbf{K}} N_0^b \leq_{\mathbf{K}} N_\delta^b$. Now:

(1) For all $i < \delta$, $N_i \leq_{\mathbf{K}} N$, since $N_i^a \leq_{\mathbf{K}} N_i^b \leq_{\mathbf{K}} N_\delta^b$ and $f_i^{-1} : N_i^a \cong N_i$.
(2) For all $i < \delta$, we have that $M \overset{N_i^b}{\underset{M_i}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}} N_i^a$ by construction, so applying f to this we get $f[M] \overset{f[N_i^b]}{\underset{M_i}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}} f[N_i^a]$, i.e. $f[M] \overset{f[N_i^b]}{\underset{M_i}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}} N_i$, so $f[M] \overset{N}{\underset{M_i}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}} N_i$ by monotonicity.
(3) $N = \text{cl}^N(f[M] \cup N_\delta)$: Why? Note that by continuity $N_\delta^b = \bigcup_{i<\delta} N_i^b$ and the latter is $\bigcup_{i<\delta} \text{cl}^{N_\delta^b}(M \cup N_i^a)$ by construction. Now, $N_\delta^b = \text{cl}^{N_\delta^b}(N_\delta^b) = \text{cl}^{N_\delta^b}(\bigcup_{i<\delta} \text{cl}^{N_\delta^b}(M \cup N_i^a))$. By Fact 4.6, this is $\text{cl}^{N_\delta^b}(\bigcup_{i<\delta} M \cup N_i^a) = \text{cl}^{N_\delta^b}(M \cup N_\delta^a)$. We have shown that $N_\delta^b = \text{cl}^{N_\delta^b}(M \cup N_\delta^a)$. Applying f to this equation, we obtain $N = \text{cl}^N(f[M] \cup N_\delta)$, as desired.

$\square$

The next notion is studied explicitly in [She09b, [V.E.1.2] and [BGKV, Definition 3.4] (where it is called the minimal closure of $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}$). It is a way to extend $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}$ to take sets on the left and right hand side.

Definition 4.8. We write $A \overset{M_3}{\underset{M_0}{\overline{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}}} B$ if $M_0 \leq_{\mathbf{K}} M_3$, $A \cup B \subseteq |M_3|$, and there exists $M_3' \geq_{\mathbf{K}} M_3$, $M_1 \leq_{\mathbf{K}} M_3'$, and $M_2 \leq_{\mathbf{K}} M_3'$ such that $A \subseteq |M_1|$, $B \subseteq |M_2|$, and $M_1 \overset{M_3'}{\underset{M_0}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}} M_2$.

Lemma 4.9.

(1) $M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}} M_2$ if and only if $M_1 \overset{M_3}{\underset{M_0}{\overline{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.8em}}}}} M_2$ and $M_0 \leq_{\mathbf{K}} M_\ell \leq_{\mathbf{K}} M_3$ for $\ell = 1, 2$.

(2) Invariance: if $A \overset{M_3}{\underset{M_0}{\mathop{\perp}}} B$ and $f : M_3 \cong M_3'$, then $f[A] \overset{M_3'}{\underset{f[M_0]}{\mathop{\perp}}} f[B]$.
(3) Monotonicity: if $A \overset{M_3}{\underset{M_0}{\mathop{\perp}}} B$ and $A_0 \subseteq A$, $B_0 \subseteq B$, and $M_3 \leq_{\mathbf{K}} M_3'$, then $A_0 \overset{M_3'}{\underset{M_0}{\mathop{\perp}}} B_0$.

Proof. Straight from the definitions. $\square$

Notation 4.10.

(1) When $N \in \mathbf{K}$, $M \leq_{\mathbf{K}} N$, $B \subseteq |N|$, and $\bar{a} \in {}^{<\infty}|N|$, we write $\bar{a} \overset{N}{\underset{M}{\mathop{\perp}}} B$ for $\operatorname{ran}(\bar{a}) \overset{N}{\underset{M}{\mathop{\perp}}} B$.
(2) For $p \in \mathrm{gS}^{<\infty}(B; N)$ and $M \leq_{\mathbf{K}} N$, we say p *does not fork over* M if whenever $p = \mathrm{gtp}(\bar{a}/B; N)$, we have that $\bar{a} \overset{N}{\underset{M}{\mathop{\perp}}} B$. Note that this does not depend on the choices of representatives by Lemma 4.9.

The following properties all appear either in [BGKV, Section 5.1] or [Vasb, Sections 4,12]. We will use them without comments.

Fact 4.11.

(1) Normality: If $A \overset{M_3}{\underset{M_0}{\mathop{\perp}}} B$, then $A \cup |M_0| \overset{M_3}{\underset{M_0}{\mathop{\perp}}} B \cup |M_0|$.
(2) Base monotonicity: if $A \overset{M_3}{\underset{M_0}{\mathop{\perp}}} B$ and $M_0 \leq_{\mathbf{K}} M_0' \leq_{\mathbf{K}} M_3$ is such that $|M_0'| \subseteq B$, then $A \overset{M_3}{\underset{M_0'}{\mathop{\perp}}} B$.
(3) Symmetry: If $A \overset{M_3}{\underset{M_0}{\mathop{\perp}}} B$, then $B \overset{M_3}{\underset{M_0}{\mathop{\perp}}} A$.
(4) Extension: Let $M \leq_{\mathbf{K}} N$ and $B \subseteq C \subseteq |N|$ be given. If $p \in \mathrm{gS}^{<\infty}(B; N)$ does not fork over M, then there exists $N' \geq_{\mathbf{K}} N$ and $q \in \mathrm{gS}^{<\infty}(C; N')$ extending p and not forking over M.
(5) Uniqueness: Let $M \leq_{\mathbf{K}} N$ and let $|M| \subseteq B \subseteq |N|$. If $p, q \in \mathrm{gS}^{<\infty}(B; N)$ do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
(6) Transitivity: If $A \overset{N}{\underset{M_0}{\mathop{\perp}}} M$, $A \overset{N}{\underset{M}{\mathop{\perp}}} B$, and $M_0 \leq_{\mathbf{K}} M$, then $A \overset{N}{\underset{M_0}{\mathop{\perp}}} B$.

The following is a form of local character that $\mathop{\copy\noforkbox}\limits$ may have:

Definition 4.12 (V.C.3.7 in [She09b]). We say that $\mathop{\smile\!\!\!\!\!\!|}\,$ *is χ-based* if whenever $M \leq_{\mathbf{K}} M^*$ and $A \subseteq |M^*|$ then there are N_0 and N_1 so that $\|N_1\| \leq |A| + \chi$, $N_0 = M \cap N_1$, $A \subseteq |N_1|$, and $N_1 \underset{N_0}{\overset{M^*}{\mathop{\smile\!\!\!\!\!\!|}\,}} M$.

Interestingly, if $\mathop{\smile\!\!\!\!\!\!|}\,$ is based then smoothness for small lengths implies smoothness for all lengths.

Fact 4.13 (V.D.1.2 in [She09b]). If $\mathbf{K}$ is $(\leq \mathrm{LS}(\mathbf{K}), \leq \mathrm{LS}(\mathbf{K})^+)$-smooth (recall Definition 2.8) and $\mathop{\smile\!\!\!\!\!\!|}\,$ is $\mathrm{LS}(\mathbf{K})$-based, then $\mathbf{K}$ is smooth, i.e. it is an AEC.

A consequence of $\mathop{\smile\!\!\!\!\!\!|}\,$ being based is that the class is tame. The argument appears already in [GK, p. 15]. We include a proof for completeness.

Lemma 4.14. Assume that $\mathop{\smile\!\!\!\!\!\!|}\,$ is $\mathrm{LS}(\mathbf{K})$-based.

(1) Set local character: if $p \in \mathrm{gS}^{<\infty}(M)$, then there are $M_0 \leq_{\mathbf{K}} M$ such that $\|M_0\| \leq |\ell(p)| + \mathrm{LS}(\mathbf{K})$ and p does not fork over M_0.
(2) $\mathbf{K}$ is $\mathrm{LS}(\mathbf{K})$-tame.

Proof.

(1) Straight from the definitions.
(2) Let $p_1, p_2 \in \mathrm{gS}(M)$ and assume that $p_1 \restriction A = p_2 \restriction A$ for all $A \subseteq |M|$ of size at most $\mathrm{LS}(\mathbf{K})$. Since $\mathop{\smile\!\!\!\!\!\!|}\,$ is $\mathrm{LS}(\mathbf{K})$-based, (1) implies that there exists $M_0 \leq_{\mathbf{K}} M$ such that $M_0 \in \mathbf{K}_{\leq \mathrm{LS}(\mathbf{K})}$ and both p_1 and p_2 do not fork over M_0. By assumption, $p_1 \restriction M_0 = p_2 \restriction M_0$. By uniqueness, $p_1 = p_2$.

$\square$

5. Enumerated trees and generalized symmetry

Hypothesis 5.1. $(\mathbf{K}, \mathop{\smile\!\!\!\!\!\!|}\,, \mathrm{cl})$ satisfies AxFri_1. Eventually, we will also assume Hypotheses 5.7 and 5.25

Consider a minimal failure of smoothness: an increasing chain $\langle M_i : i \leq \delta \rangle$ that is continuous below δ but so that $\bigcup_{i<\delta} M_i \not\leq_{\mathbf{K}} M_\delta$. We would like to copy this chain into a tree indexed by ${}^{\leq \delta}\lambda$. The branches of the tree should be as independent as possible.

The main theorem of this section, Theorem 5.39, shows that it can be done. The main difficulty in the proof is that we cannot assume smoothness when we construct the tree, so we have difficulties at limits. We work around this by studying trees enumerated in some order, giving a definition of a closed subset of such tree (Definition 5.8) and proving a generalized symmetry theorem for these sets (Theorem 5.34). Generalized symmetry says intuitively (as in [She83a, She83b]) that whether a tree is independent does not depend on its enumeration, so closed sets will be as independent of each other as possible. Once generalized symmetry is proven, the construction of the desired tree can be carried out.

This section draws a lot of inspiration from [She09b, V.C.4], where Shelah defines a notion of stable construction which is supposed to accomplish similar goals than here. Shelah even states Theorem 5.39 as an exercise [She09b, V.C.4.14]. However, we cannot solve it when smoothness fails. It seems that clause (vi) in [She09b, Definition V.C.4.2] is too restrictive and precisely prevents us from copying a non-smooth chain into a tree.

We start by setting up the notation of this section for trees. The universe of the trees we will use is always an ordinal α, and we think of $(\alpha, \in)$ as giving the order in which the tree is enumerated.

Definition 5.2. An *enumerated tree* is a pair $(\alpha, \trianglelefteq)$, where α is an ordinal and $\trianglelefteq$ is a partial order on α such that for all $i, j < \alpha$:

(1) $0 \trianglelefteq i$.
(2) $i \trianglelefteq j$ implies $i \leq j$.
(3) $(\{k < \alpha \mid k \trianglelefteq i\}, \trianglelefteq)$ is a well-ordering.

Definition 5.3. Let $(\alpha, \trianglelefteq)$ be an enumerated tree.

(1) For $i < \alpha$, and $R \in \{\triangleleft, \trianglelefteq\}$, let $\text{pred}_R(i) := \{k \leq i \mid kRi\}$. When $R = \triangleleft$, we omit it.
(2) A *branch* of $(\alpha, \trianglelefteq)$ is a set $b \subseteq \alpha$ such that:
    (a) $\trianglelefteq$ linearly orders b.
    (b) $i \in b$ implies $\text{pred}(i) \subseteq b$.
(3) A branch $b \subseteq \alpha$ is *bounded (in $(\alpha, \trianglelefteq)$)* if either it has a maximum or $b = \text{pred}(i)$ for some $i < \alpha$. It is unbounded otherwise. We say that a set $u \subseteq \alpha$ is *bounded* if any branch $b \subseteq u$ is bounded.
(4) We say that $(\alpha, \trianglelefteq)$ is *continuous* when for any $i, j < \alpha$, if $\text{pred}(i) = \text{pred}(j)$ and $\text{pred}(i)$ does not have a maximum, then $i = j$.
(5) When $(\alpha, \trianglelefteq)$ is continuous and $b \subseteq \alpha$ is a bounded branch, we let:

$$\text{top}(b) := \begin{cases} \max(b) & \text{if } b \text{ has a maximum} \\ \text{The unique } i < \alpha \text{ such that } b = \text{pred}(i) & \text{otherwise.} \end{cases}$$

(6) When $u \subseteq \alpha$, let:

$$B(u) := \{b \subseteq u \mid b \text{ is a branch and for any branch } b', b \subseteq b' \subseteq u \text{ implies } b' = b\}$$

be the set of maximal branches in u.

(7) When $(\alpha, \trianglelefteq)$ is continuous and $u \subseteq \alpha$ is a bounded set, we let $\text{top}(u) := \sup_{b \in B(u)} \text{top}(b)$ (this will only be used when $B(u)$ is finite, so in that case the supremum is actually a maximum).

Lemma 5.4. If $u \subseteq v$ and $b \in B(u)$, then there is a branch $b' \in B(v)$ such that $b \subseteq b'$. Consequently, $|B(u)| \leq |B(v)|$.

Proof. Straightforward from the definition of $B(u)$. The last sentence is because the map $b \mapsto b'$ (for some choice of b') is an injection from $B(u)$ to $B(v)$. $\square$

We now combine the definition of a tree with the class $\mathbf{K}$. Note that continuity of chains of models is only required when the chain is smooth (see (5) below).

Definition 5.5. A *continuous enumerated tree of models* is a tuple $(\langle M_i : i < \alpha \rangle, N, \alpha, \trianglelefteq)$ satisfying:

(1) $(\alpha, \trianglelefteq)$ is a continuous enumerated tree.
(2) $N \in \mathbf{K}$.
(3) For all $i < \alpha$, $M_i \leq_{\mathbf{K}} N$.
(4) For all $i, j < \alpha$, $i \trianglelefteq j$ implies $M_i \leq_{\mathbf{K}} M_j$.
(5) For all $i < \alpha$, if $\text{pred}(i)$ has no maximum *and* $\bigcup_{j \triangleleft i} M_j \leq_{\mathbf{K}} N$, then $M_i = \bigcup_{j \triangleleft i} M_j$.

Remark 5.6. By coherence, for all $i < \alpha$, $\bigcup_{j \triangleleft i} M_j \leq_{\mathbf{K}} N$ if and only if $\bigcup_{j \triangleleft i} M_j \leq_{\mathbf{K}} M_i$

From now on, we work inside a continuous enumerated tree of models.

Hypothesis 5.7. $\mathcal{T} := (\langle M_i : i < \alpha \rangle, N, \alpha, \trianglelefteq)$ is a continuous enumerated tree of models.

The following is a key definition. Intuitively, a set is closed if it is closed under initial segments *and* all its branches smoothly embed inside N.

Definition 5.8. $u \subseteq \alpha$ is *closed* if:

(1) $i \in u$ implies $\mathrm{pred}(i) \subseteq u$.
(2) $b \in B(u) \backslash \{\emptyset\}$ implies $\bigcup_{i \in b} M_i \leq_{\mathbf{K}} N$.

Lemma 5.9.

(1) An arbitrary intersection of closed sets is closed.
(2) A finite union of closed sets is closed.

Proof.

(1) Let $\langle u_i : i < \gamma \rangle$ be closed, $\gamma > 0$. Let $u := \bigcap_{i<\gamma} u_i$. We show that u is closed. It is easy to check that u satisfies (1) from the definition of a closed set. We check (2). Let $b \in B(u) \backslash \{\emptyset\}$. We want to see that $\bigcup_{j \in b} M_j \leq_{\mathbf{K}} N$. By Lemma 5.4, for each $i < \gamma$ there exists $b_i \in B(u_i)$ such that $b \subseteq b_i$. Since u_i is closed, we have that $\bigcup_{j \in b_i} M_j \leq_{\mathbf{K}} N$. If there exists $j < \gamma$ such that $b = b_j$, we are done so assume that this is not the case. This implies that b is bounded. Let $k := \mathrm{top}(b)$. We know that $b \subsetneq b_j$ for all $j < \gamma$, so by downward closure we must have that $k \in b_j$ for all $j < \gamma$. But then this means that $k \in u$, so $k \in b$, a contradiction.
(2) Let u, v be closed. We show that $u \cup v$ is closed. As before, (1) is straightforward to see. As for (2), let $b \in B(u \cup v)$. It is straightforward to see that either $b \in B(u)$ or $b \in B(v)$. In either case we get that $\bigcup_{i \in b} M_i \leq_{\mathbf{K}} N$, as desired.

□

Remark 5.10. Lemma 5.9 almost tells us that closed sets induce a topology on α. While it is easy to check that the empty set is closed, α itself may not be closed (think of a chain $\langle M_i : i \leq \delta \rangle$ where $\bigcup_{i<\delta} M_i \not\leq_{\mathbf{K}} M_\delta$. The tree could consist of $\langle M_i : i < \delta \rangle$ and $N = M_\delta$). However α *will* be closed when all the maximal branches of the tree have a maximum (e.g. if $(\alpha, \trianglelefteq)$ looks like ${}^{\leq \delta}\lambda$ for some cardinal $\lambda \geq 2$ and limit ordinal δ).

The next definition describes the model M^u generated by a set $u \subseteq \alpha$. Typically, u will be closed and in case the tree is sufficiently independent (see Definition 5.24), M^u will be in $\mathbf{K}$.

Definition 5.11. For $u \subseteq \alpha$, $M^u := \mathrm{cl}^N(|M_0| \cup \bigcup_{i \in u} |M_i|)$.

Lemma 5.12. Let $u, v \subseteq \alpha$. $M^{u \cup v} = \mathrm{cl}^N(M^u \cup M^v)$.

Proof. By Fact 4.6. □

Lemma 5.13. If b is a closed and bounded branch, then $M^b = M_i$, where $i := \text{top}(b)$.

Proof. If i is a maximum of b or b is empty, this is clear. If not, we know since b is closed that $\bigcup_{j \lhd i} M_j = \bigcup_{j \in b} M_j \leq_{\mathbf{K}} N$. By (5) in Definition 5.5, $M_i = \bigcup_{j \lhd i} M_j$. Note that $\bigcup_{j \lhd i} M_j = \text{cl}^N(\bigcup_{j \lhd i} M_j)$ and by Fact 4.6, this is equal to M^b. So $\bigcup_{j \lhd i} M_j = M^b$, as desired. $\square$

The next definition describes when two (typically closed) sets u and v are "as independent as possible", i.e. the model generated by u is independent of the one generated by v over the model generated by $u \cap v$. There are two variations depending on whether the ambient model is N or the model generated by $u \cup v$. Generalized symmetry (Theorem 5.34) will say that under appropriate conditions, if the tree is independent then any closed sets u and v are as independent as possible.

Definition 5.14. Let $u, v \subseteq \alpha$.

(1) We write uv for $u \cup v$.
(2) We write $u \mathop{\smile\!\!\!\!\!\!|}\, v$ if $M^u \underset{M^{u \cap v}}{\overset{M^{uv}}{\mathop{\smile\!\!\!\!\!\!|}}} M^v$.
(3) We write $u \overset{N}{\mathop{\smile\!\!\!\!\!\!|}} v$ if $M^u \underset{M^{u \cap v}}{\overset{N}{\mathop{\smile\!\!\!\!\!\!|}}} M^v$.

The following will be used without comments.

Lemma 5.15. If $u \overset{N}{\mathop{\smile\!\!\!\!\!\!|}} v$ if and only if $u \mathop{\smile\!\!\!\!\!\!|}\, v$ and $M^{uv} \leq_{\mathbf{K}} N$.

Proof. If $u \overset{N}{\mathop{\smile\!\!\!\!\!\!|}} v$, then by Fact 4.7.(1), $M^{uv} \leq_{\mathbf{K}} N$ and $u \mathop{\smile\!\!\!\!\!\!|}\, v$. The converse is by the monotonicity 2 property of $\mathop{\smile\!\!\!\!\!\!|}$ in Definition 4.1. $\square$

If $u \subseteq v$, there is an easy way to determine whether $u \mathop{\smile\!\!\!\!\!\!|}\, v$.

Lemma 5.16. If $u \subseteq v$, $M^u \leq_{\mathbf{K}} N$, and $M^v \leq_{\mathbf{K}} N$, then $u \overset{N}{\mathop{\smile\!\!\!\!\!\!|}} v$.

Proof. Straight from the definition. $\square$

We now translate the properties of Section 4 into properties of the relations $u \mathop{\smile\!\!\!\!\!\!|}\, v$ and $u \overset{N}{\mathop{\smile\!\!\!\!\!\!|}} v$.

Lemma 5.17. Let $u, v, w \subseteq \alpha$ be closed.

(1) Symmetry: If $u \mathop{\smile\!\!\!\!\!\!|}\, v$, then $v \mathop{\smile\!\!\!\!\!\!|}\, u$. If $u \overset{N}{\mathop{\smile\!\!\!\!\!\!|}} v$, then $v \overset{N}{\mathop{\smile\!\!\!\!\!\!|}} u$.

(2) Base enlargement: If $u \overset{N}{\mathop{\unicode{x2AEB}}} v$, $u \cap v \subseteq w \subseteq v$, and $M^w \leq_{\mathbf{K}} M^v$, then $uw \overset{N}{\mathop{\unicode{x2AEB}}} v$.
(3) Transitivity: If $u \overset{N}{\mathop{\unicode{x2AEB}}} v$, $uv \overset{N}{\mathop{\unicode{x2AEB}}} w$, $uv \cap w = u$, and $M^{v \cap w} \leq_{\mathbf{K}} M^w$, then $v \overset{N}{\mathop{\unicode{x2AEB}}} w$.

Proof.

(1) Straightforward from the symmetry axiom.
(2) Directly from the base enlargement axiom.
(3) Let $M_0 := M^{u \cap v}$, $M_1 := M^v$, $M_2 := M^u$, $M_3 := M^{uv}$, $M_4 := M^w$, $M_5 := N$. We know that $u \overset{N}{\mathop{\unicode{x2AEB}}} v$, so $M^{uv} \leq_{\mathbf{K}} N$ and $u \mathop{\unicode{x2AEB}} v$, hence $M_1 \underset{M_0}{\overset{M_3}{\mathop{\unicode{x2AEB}}}} M_2$ holds. We know that $uv \overset{N}{\mathop{\unicode{x2AEB}}} w$ (so in particular $M^{uvw} \leq_{\mathbf{K}} N$) and $uv \cap w = u$, i.e. $M^{uv \cap w} = M^u = M_2$, so $M_3 \underset{M_2}{\overset{M_5}{\mathop{\unicode{x2AEB}}}} M_4$ holds. Applying Fact 4.7.(2), we obtain $M_1 \underset{M_0}{\overset{M_5}{\mathop{\unicode{x2AEB}}}} M_4$, i.e. $M^v \underset{M^{u \cap v}}{\overset{N}{\mathop{\unicode{x2AEB}}}} M^w$. Now since $uv \cap w = u$, we must have that $u \subseteq w$ and $v \cap w \subseteq u$. Therefore $u \cap v \subseteq w \cap v$. By coherence, $M^{u \cap v} \leq_{\mathbf{K}} M^{v \cap w} \leq_{\mathbf{K}} M^w$. By base enlargement, $M^v \underset{M^{v \cap w}}{\overset{N}{\mathop{\unicode{x2AEB}}}} M^w$, i.e. $v \overset{N}{\mathop{\unicode{x2AEB}}} w$.

$\square$

A key part of the proof of generalized symmetry is a concatenation property telling us when $uv \overset{N}{\mathop{\unicode{x2AEB}}} w$ if we know something about u and v separately. We start with the following result:

Lemma 5.18. Let $u, v, w \subseteq \alpha$ be closed. If:

(1) $u(v \cap w) \overset{N}{\mathop{\unicode{x2AEB}}} w$.
(2) $v \overset{N}{\mathop{\unicode{x2AEB}}} uw$.
(3) $M^{u(v \cap w)} \leq_{\mathbf{K}} M^{uw}$.
(4) $M^{uv \cap w} \leq_{\mathbf{K}} M^{uv}$.

Then $uv \overset{N}{\mathop{\unicode{x2AEB}}} w$.

Proof. We apply base enlargement with u, v, w there standing for $v, uw, u(v \cap w)$ here. The hypotheses hold by (2) and (3). We obtain $uv \overset{N}{\perp} uw$. We want to apply transitivity, where u, v, w there stand for $u(v \cap w)$, w, uv here. The conditions there are:

- $u \overset{N}{\perp} v$, which translates to $u(v \cap w) \overset{N}{\perp} w$ here (holds by (1)).
- $uv \overset{N}{\perp} w$, which translates to $uw \overset{N}{\perp} uv$ here (holds by the paragraph above).
- $uv \cap w = u$, which translates to $u(v \cap w)w \cap uv = u(v \cap w)$, i.e. $uw \cap uv = u(v \cap w)$, which is true.
- $M^{v \cap w} \leq_{\mathbf{K}} M^w$, which translates to $M^{w \cap uv} \leq_{\mathbf{K}} M^{uv}$, which is true by (4).

Therefore the conclusion of transitivity holds. In our case, this means that $w \overset{N}{\perp} uv$. By symmetry, $uv \overset{N}{\perp} w$, as desired. □

Lemma 5.19. Let $u, v, w \subseteq \alpha$ be closed. If:

(1) $u \overset{N}{\perp} w$.
(2) $M^{uv \cap w} \leq_{\mathbf{K}} N$.
(3) $M^{u(v \cap w)} \leq_{\mathbf{K}} M^{u(v \cap w)}$.

Then $u(v \cap w) \overset{N}{\perp} w$.

Proof. We use Lemma 5.18 with u, v, w there standing for $w \cap v$, u, w here. Let us check the hypotheses:

- (1) there translates to $(w \cap v)(u \cap w) \overset{N}{\perp} w$ here. So it is enough to see that $M^w \leq_{\mathbf{K}} N$ and $M^{uv \cap w} \leq_{\mathbf{K}} N$. This holds by (1) and (2).
- (2) there translates to $u \overset{N}{\perp} w$ here, which is (1).
- (3) there translates to $M^{uv \cap w} \leq_{\mathbf{K}} M^w$ here. This holds by (1), (2), and coherence.
- (4) there translates to $M^{uv \cap w} \leq_{\mathbf{K}} M^{u(v \cap w)}$ here. This holds by (3).

The hypotheses hold, so we obtain that $u(v \cap w) \overset{N}{\perp} w$, as needed. □

Finally, we obtain a usable concatenation property.

Lemma 5.20 (Concatenation)**.** Let $u, v, w \subseteq \alpha$ be closed. If:

(1) $u \overset{N}{\mathop{\smile}\limits^{\vert}} w$.
(2) $v \overset{N}{\mathop{\smile}\limits^{\vert}} uw$.
(3) $M^{uv \cap w} \leq_{\mathbf{K}} N$.
(4) $M^{u(v \cap w)} \leq_{\mathbf{K}} N$.
(5) $M^{uv} \leq_{\mathbf{K}} N$.

Then $uv \overset{N}{\mathop{\smile}\limits^{\vert}} w$.

Proof. We use Lemma 5.18. Let us check the hypotheses:

- (1) says $u(v \cap w) \overset{N}{\mathop{\smile}\limits^{\vert}} w$. This holds by Lemma 5.19. Note that (1) there holds by (1), (2) there holds by (3), and (3) there holds by (3), (4), and coherence.
- (2) there is (2) here.
- (3) there is given by (1), (4), and coherence.
- (4) there is given by (3), (5), and coherence.

The hypotheses hold, so we obtain that $uv \overset{N}{\mathop{\smile}\limits^{\vert}} w$, as needed. $\square$

Another key ingredient of the proof of generalized symmetry is a continuity property that tells us how to deal with increasing chains $\langle u_i : i < \delta \rangle$ of closed sets. At that point, the following hypothesis will appear in some of the statements (we do *not* assume it globally).

Definition 5.21. We say that cl *is algebraic* if for any $M, N \in \mathbf{K}$ with $M \subseteq N$ and any $A \subseteq |M|$, $\mathrm{cl}^M(A) = \mathrm{cl}^N(A)$.

Recall that we are working under Hypothesis 5.1, so cl is in particular a fixed operator satisfying Monotonicity 2 (Definition 4.1.(2c)). The difference here is that we assume that closure is the same whenever $M \subseteq N$ (not only under the stronger condition $M \leq_{\mathbf{K}} N$).

Note that if $\mathrm{cl}^N(A)$ is the closure of A under the functions of N, then cl is algebraic. This will be the closure operator when we study universal classes, so we do not lose much by assuming it here.

Lemma 5.22. Assume that cl is algebraic. Let δ be a limit ordinal and let $\langle u_i : i \leq \delta \rangle$ be an increasing continuous chain of closed sets. If for all $i < \delta$, M^{u_i} is a $\tau(\mathbf{K})$-structure, then $M^{u_\delta} = \bigcup_{i<\delta} M^{u_i}$.

Proof. Let $M_\delta := \bigcup_{i<\delta} M^{u_i}$.

First observe that $M_\delta \subseteq N$, because for all $i < \delta$, $M^{u_i} \subseteq N$ (as we are assuming it is a $\tau(\mathbf{K})$-structure and by definition it must inherit the function symbols from N). Therefore because cl is algebraic, $\mathrm{cl}^N(M_\delta) = \mathrm{cl}^{M_\delta}(M_\delta) = M_\delta$. But $\mathrm{cl}^N(M_\delta) = \mathrm{cl}^N\left(\bigcup_{i<\delta} \mathrm{cl}^N(M_0 \cup \bigcup_{j \in u_i} M_j)\right)$. By Fact 4.6, this is just $\mathrm{cl}^N\left(\bigcup_{i<\delta}(M_0 \cup \bigcup_{j\in u_i} M_i)\right) = \mathrm{cl}^N(M_0 \cup \bigcup_{i \in u_\delta} M_i) = M^{u_\delta}$. Combining the chains of equalities, we have the result. $\square$

Lemma 5.23 (Continuity)**.** Assume that cl is algebraic.

Let δ be a limit ordinal and let $\langle u_i : i \leq \delta\rangle$, $\langle v_i : i \leq \delta\rangle$ be increasing continuous chains of closed sets. If for all $i, j < \delta$:

(1) $u_i \perp v_j$.
(2) $u_\delta \cap v_i \perp v_j$.
(3) $v_\delta \cap u_i \perp u_j$.

Then $u_\delta \perp v_\delta$.

Proof. Claim 1: $M^{u_\delta \cap v_\delta} \leq_{\mathbf{K}} M^{v_\delta}$.

Proof of Claim 1: We use Fact 4.7.(3) where M_i, N_i there stand for $M^{u_\delta \cap v_i}$, M^{v_i} here. Why is $\langle M^{u_\delta \cap v_i} : i \leq \delta\rangle$ $\subseteq$-increasing and continuous? Note that $M^{u_\delta \cap v_i}$ is a member of $\mathbf{K}$ for each $i < \delta$ (by (2)), and the chain is increasing by definition of $M^{u_\delta \cap v_i}$. The continuity is because $\langle v_i : i \leq \delta\rangle$ is itself continuous (use Lemma 5.22). Similarly, $\langle M^{v_i} : i \leq \delta\rangle$ is $\subseteq$-increasing continuous. Also, (2) ensures that the independence hypothesis of Fact 4.7.(3) is satisfied. Therefore we have in particular that $M_\delta \leq_{\mathbf{K}} N_\delta$ there. That is, $M^{u_\delta \cap v_\delta} \leq_{\mathbf{K}} M^{v_\delta}$. $\dagger_{\text{Claim}}$.

Claim 2: For all $j < \delta$, $u_j \perp v_\delta$.

Proof of Claim 2: Fix $j < \delta$. We use Fact 4.7.(4) where M_i, $M_{\delta+1}$, N_i^a, N_i^b there stand for $M^{u_j \cap v_{j+i}}$, M^{u_j}, $M^{v_{j+i}}$, $M^{u_j v_{j+i}}$ here. By (1), all the hypotheses of Fact 4.7 are satisfied. In details, we have to check that there $M_\delta \leq_{\mathbf{K}} M_{\delta+1}$, which here translates to $M^{i_j \cap v_\delta} \leq_{\mathbf{K}} M^{u_j}$, but this holds by (3). Also, $N_i^b = \mathrm{cl}^{N_i^b}(M_{\delta+1} \cup N_i^a)$ there translates to $M^{u_j v_{j+i}} = \mathrm{cl}^{M_{u_j v_{j+i}}}(M_{u_j} \cup M^{v_{j+i}})$. This holds because $M^{u_j v_{j+i}} \subseteq N$ (by (1), $M^{u_j v_{j+i}} \in \mathbf{K}$, and hence by definition it must be a substructure of N), and hence because cl is algebraic, $\mathrm{cl}^N(A) = \mathrm{cl}^{M^{u_j v_{j+i}}}(A)$ for any set A. The other conditions are checked similarly. Applying Fact 4.7.(4), we obtain the desired result. $\dagger_{\text{Claim 2}}$.

To prove that $u_\delta \perp v_\delta$, we use Fact 4.7.(4) where $M_i, M_{\delta+1}$, N_i^a, N_i^b there stand for $M^{u_i \cap v_i}$, M^{v_δ}, M^{u_i}, $M^{u_i v_\delta}$ here. We need to know there that $M_\delta \leq_{\mathbf{K}} M_{\delta+1}$, i.e. $M^{u_\delta \cap v_\delta} \leq_{\mathbf{K}} M^{v_\delta}$, but this is given by Claim 1. $\square$

With the forking calculus out of the way, we are ready to start proving generalized symmetry. First, we state what it means for a tree to be independent. The intuition is that for any $i \trianglelefteq j$, M_j is independent over M_i of as much as possible that comes before j in the enumeration of the tree.

Definition 5.24. $\mathcal{T}$ is *independent* if for any $i \trianglelefteq j < \alpha$:

$$M_j \overset{N}{\underset{M_i}{\overline{\downarrow}}} \bigcup_{k \in A_{i,j}} M_k$$

where $\overline{\downarrow}$ is from Definition 4.8 and:

$$A_{i,j} := \{k < j \mid \text{pred}_{\trianglelefteq}(k) \cap \text{pred}_{\trianglelefteq}(j) \subseteq \text{pred}_{\trianglelefteq}(i)\}$$

Hypothesis 5.25. $\mathcal{T}$ (from Hypothesis 5.7) is independent.

Our aim is to prove Theorem 5.34 which gives conditions under which $u \downarrow v$ for any closed sets u and v. We prove increasingly stronger approximations to this result, each time using the previously proven approximations. First, we prove it when u and v are closed bounded branches.

Lemma 5.26. If a and b are closed bounded branches, then $a \overset{N}{\downarrow} b$.

Proof. Let $i := \text{top}(a)$, $j := \text{top}(b)$. By Lemma 5.13, $M^a = M_i$, $M^b = M_j$. By Definition 5.5.(3), $M^a \leq_{\mathbf{K}} N$ and $M^b \leq_{\mathbf{K}} N$. Note that $a \cap b$ is also a closed bounded branch so $M^{a \cap b} \leq_{\mathbf{K}} N$ also. By coherence, $M^{a \cap b} \leq_{\mathbf{K}} M^x$ for $x \in \{a, b\}$. By symmetry, we can assume without loss of generality that $j \leq i$. Furthermore, if $i = j$ then Lemma 5.16 gives the result, so assume $j < i$. Let $k := \text{top}(a \cap b)$. By Lemma 5.13 again, $M^{a \cap b} = M_k$. Now by Definition 5.24, we must have that $M_i \overset{N}{\underset{M_k}{\overline{\downarrow}}} M_j$. By what we have argued, we must actually have $M_i \overset{N}{\underset{M_k}{\downarrow}} M_j$, i.e. $a \overset{N}{\downarrow} b$, as needed. $\square$

Next, we prove it when u is a closed and bounded branch and v is a bounded finite union of closed branches that comes before u in the enumeration of the tree.

Lemma 5.27. If:

(1) a is a closed and bounded branch.
(2) v is a closed and bounded set with $B(v)$ finite.
(3) $\text{top}(a) \geq \text{top}(v)$.

Then $a \overset{N}{\perp} v$.

Proof. Let $n := |B(v)|$. We work by induction on n. If $n = 1$, the result holds by Lemma 5.26. Otherwise, say $B(v) = \{b_0, \dots, b_{n-1}\}$, where without loss of generality $\text{top}(b_0) < \text{top}(b_1) < \dots < \text{top}(b_{n-1})$. By the induction hypothesis, $b_{n-1} \overset{N}{\perp} b_0 \dots b_{n-2}$. In particular, $M^v = M^{b_0 \dots b_{n-1}} \leq_{\mathbf{K}} N$. Now using Definition 5.24 (or Lemma 5.16 if $\text{top}(a) = \text{top}(v)$, so $a \subseteq v$), it is easy to check that $M^a \underset{M^{a \cap v}}{\overset{N}{\perp}} M^v$, so the result follows. $\square$

Next, we can show that $M^u \leq_{\mathbf{K}} N$ when u is a bounded finite union of closed branches.

Lemma 5.28. If u and v are bounded closed sets with $B(u)$ and $B(v)$ both finite, then:

(1) $M^u \leq_{\mathbf{K}} N$.
(2) $u \subseteq v$ implies $M^u \leq_{\mathbf{K}} M^v$.

Proof. The second part follows from the first and coherence. For the first part, let $n := |B(u)|$ and write $B(u) = \{b_0, \dots, b_{n-1}\}$ with $\text{top}(b_0) < \dots < \text{top}(b_{n-1})$. If $n = 1$, the result follows from Lemma 5.26 (where a, b there stand for u, u here) so assume that $n \geq 2$. Apply Lemma 5.27 where a, v there stand for b_{n-1}, $b_0 \dots b_{n-2}$ here. $\square$

We now use the previous result together with concatenation to show that $u \overset{N}{\perp} v$ when u and v are bounded finite union of closed branches.

Lemma 5.29. If u and v are closed bounded sets with $B(u)$ and $B(v)$ both finite, then $u \overset{N}{\perp} v$.

Proof. Work by induction on $|B(u)| + |B(v)|$. By symmetry, without loss of generality $\text{top}(u) \geq \text{top}(v)$. Let $n := |B(u)|$. Write $B(u) = \{a_0, \dots, a_{n-1}\}$ with $\text{top}(a_0) < \dots < \text{top}(a_{n-1})$. If $n = 1$, the result is given by Lemma 5.27, so assume now that $n \geq 2$. We use concatenation (Lemma 5.20) with u, v, w there standing for $a_0 \dots a_{n-2}$, a_{n-1}, v here. Let us check the hypotheses:

- (1) there translates to $a_0 \dots a_{n-2} \overset{N}{\mathop{\smile\!\!\!\!\!\!|}\;} v$ here. This holds by the induction hypothesis.
- (2) there translates to $a_{n-1} \overset{N}{\mathop{\smile\!\!\!\!\!\!|}\;} a_0 \dots a_{n-2} v$ here. This holds by Lemma 5.27.
- (3)-(5) there hold by Lemma 5.28.

The hypotheses hold, so we obtain $a_0 \dots a_{n-1} \overset{N}{\mathop{\smile\!\!\!\!\!\!|}\;} v$, as desired. $\square$

Next, we can use the continuity property to prove generalized symmetry for all closed bounded sets.

Lemma 5.30. Assume that cl is algebraic. If u and v are closed bounded sets, then $u \mathop{\smile\!\!\!\!\!\!|}\; v$.

Proof. Let $\lambda := |B(u \cup v)|$. We work by induction on λ. If $\lambda < \aleph_0$, then this is taken care of by Lemma 5.29. Otherwise, say $B(u) = \langle a_i : i < \lambda \rangle$ and $B(v) = \langle b_i : i < \lambda \rangle$ (we allow repetition in the enumerations). For $i \leq \lambda$, let $u_i := \bigcup_{j<i} b_j$ and $v_i := \bigcup_{j<i} b_j$. It is easy to check that $\langle u_i : i \leq \lambda \rangle$, $\langle v_i : i \leq \lambda \rangle$ are increasing continuous resolutions of u and v respectively. Moreover, each member of the chain is a closed bounded set. We apply Lemma 5.23 (where δ there stands for λ here). Its hypotheses hold by the induction hypothesis. We obtain that $u_\lambda \mathop{\smile\!\!\!\!\!\!|}\; v_\lambda$, as desired. $\square$

When u or v is not bounded, we will make an additional hypothesis which says that branches do not have too many non-smooth points. In the case we are interested in (see Theorem 5.39), each branch will have at most one smooth point, so this hypothesis is reasonable. Note again that we *do not* assume this globally, only in some statements.

Definition 5.31. $\mathcal{T}$ is *resolvable* if for any branch $b \subseteq \alpha$, $\{i \in b \mid \bigcup_{j \lhd i} M_j \not\leq_{\mathbf{K}} N\}$ is finite.

Definition 5.32. For $u \subseteq \alpha$, let $B'(u) := \{b \in B(u) \mid b \text{ is unbounded}\}$.

Assuming that $\mathcal{T}$ is resolvable, we show that every closed set has a resolution with fewer unbounded branches than the original set. This will allow us to do a proof by induction on $|B'(u)|$.

Lemma 5.33. Assume that $\mathcal{T}$ is resolvable.

(1) Let b be a closed branch. Then there is a limit ordinal δ and an increasing continuous sequence of closed bounded branches $\langle b_i : i \leq \delta \rangle$ such that $b = b_\delta$.

(2) Let u be a closed unbounded set. Then there is a limit ordinal δ and an increasing continuous sequence $\langle u_i : i \le \delta \rangle$ of closed sets such that $u_\delta = u$ and for all $i < \delta$, $|B'(u_i)| < |B'(u)|$.

Proof.

(1) If b is bounded, we can take $b = b_i$ for all $i \le \delta$, so assume that b is unbounded. Since $\mathcal{T}$ is resolvable, we know that there exists $i \in b$ such that for all $i' \ge i$, $\bigcup_{j \lhd i'} M_j \le_{\mathbf{K}} N$. In other words, $\text{pred}(i')$ is closed. So let $\delta := \text{otp}(b)$ and write $b \backslash i = \langle i_j : j < \delta \rangle$. For $j < \delta$, let $b_j := \text{pred}(i_j)$.

(2) Say $B'(u) = \{b_i : i < \lambda\}$. Let $v := u \backslash \bigcup_{i<\lambda} b_i$. Note that v is closed and bounded. If λ is infinite, we can let $\delta := \lambda$ and for $i \le \delta$, $u_i := v \cup \bigcup_{j<i} b_j$. So assume that λ is finite. By the first part, for each $i < \lambda$ there exists a limit ordinal δ_i and a resolution $\langle b_i^j : j < \delta_i \rangle$ of b_i into closed bounded branches. Let $\delta := \sum_{i<\lambda} \delta_i$. Now for $j < \delta$, there are unique $i < \lambda$ and $k < \delta_i$ such that $j = \sum_{i_0<i} \delta_{i_0} + k$. Set $u_j := \bigcup_{i_0<i} b_{i_0} \cup b_i^k$. It is straightforward to check that this works.

□

Theorem 5.34 (Generalized symmetry)**.** Assume that $\mathcal{T}$ is resolvable and cl is algebraic. If u and v are closed sets, then $u \mathop{\smile}\limits^{\mid} v$.

Proof. Work by induction on $\lambda := |B'(u)| + |B'(v)|$. If $\lambda = 0$, this is given by Lemma 5.30. If λ is infinite, we can use an argument analogous to the proof of Lemma 5.30, so assume that λ is finite and non-zero.

By Lemma 5.33, we can find limit ordinals δ_1, δ_2 and $\langle u_i : i \le \delta_1 \rangle$, $\langle v_i : i \le \delta_2 \rangle$ that are increasing continuous resolutions of u and v respectively so that each member in the chain is closed, and for all $i < \delta_1$, $|B'(u_i)| < |B'(u)|$, and similarly for v.

By symmetry, without loss of generality, $\delta_1 \le \delta_2$. We first use Lemma 5.23 with δ there standing for δ_1 here. The hypotheses hold by the induction hypothesis. So we obtain $u \mathop{\smile}\limits^{\mid} v_{\delta_1}$. If $\delta_1 = \delta_2$, we are done. Otherwise by the induction hypothesis (using that λ is finite) we have that $u \mathop{\smile}\limits^{\mid} v_i$ for all $i < \delta_2$. So we use Lemma 5.23 a second time with δ, u_i, v_i there standing for δ_2, u, v_i here. We obtain that $u \mathop{\smile}\limits^{\mid} v_{\delta_2}$, as desired. □

For the remainder of this section, we focus on building independent trees. We drop Hypotheses 5.7 and 5.25. It will be convenient to have the tree enumerated in a particular order:

Definition 5.35. An enumerated tree $(\alpha, \trianglelefteq)$ is *in pre-order* if for any $i < \alpha$ and any $b \in B(i)$, either $b = \text{pred}(i)$ or $b \in B(\alpha)$.

The idea is that (Lemma 5.37) if the tree is in pre-order, then the set $A_{i,j}$ from Definition 5.24 is closed, so we can use the generalized symmetry theorem on it. Before proving this, we show that the tree we care about has an enumeration in pre-order.

Lemma 5.36. Let δ be a limit ordinal and let λ be a cardinal with $\lambda \geq 2$. Then there exists an enumeration $\langle \eta_i : i < \alpha \rangle$ of ${}^{\leq \delta}\lambda$ such that defining $i \trianglelefteq j$ if and only if η_i is an initial segment of η_j, we have that $(\alpha, \trianglelefteq)$ is a continuous enumerated tree which is in pre-order.

Proof. Let $\langle \nu_j : j < \beta \rangle$ be an enumeration (without repetitions) of ${}^{\leq \delta}\lambda$ such that if ν_j is an initial segment of $\nu_{j'}$, then $j \leq j'$. We define α and $\langle \eta_i : i < \alpha \rangle$ by induction on i such that:

1. $(i, \trianglelefteq)$ is a continuous enumerated tree.
2. If $b \in B(i)$, then either there is $j \in b$ such that $\eta_j \in {}^{\delta}\lambda$, or $b = \text{pred}(i)$.

There are three cases:

- $\{\eta_j : j < i\} = \{\nu_j : j < \beta\}$. Then we are done and let $\alpha := i$.
- If there is $b \in B(i)$ such that for some $j < \beta$, $\bigcup_{k \in b} \eta_k$ is an initial segment of ν_j but $\nu_j \notin \{\eta_k \mid k \in b\}$, then pick any such b and the least such j, and let $\eta_i := \nu_j$.
- Otherwise, let $j < \beta$ be least such that $\nu_j \neq \eta_k$ for any $k < i$. Let $\eta_i := \nu_j$.

It is straightforward to see that this works. $\square$

Lemma 5.37. Let $\mathcal{T} := (\langle M_i : i < \alpha \rangle, N, \alpha, \trianglelefteq)$ be a continuous enumerated tree of models. If:

1. $(\alpha, \trianglelefteq)$ is in pre-order.
2. For any $b \in B(\alpha)$, b is bounded.

Then for any $i \trianglelefteq j < \alpha$, $A_{i,j} = \{k < j \mid \text{pred}_{\trianglelefteq}(k) \cap \text{pred}_{\trianglelefteq}(j) \subseteq \text{pred}_{\trianglelefteq}(i)\}$ (see Definition 5.24) is closed.

Proof. Let $b \in B(A_{i,j})$. We have to see that $\bigcup_{k \in b} M_k \leq_{\mathbf{K}} N$. Now either $b = \text{pred}_{\trianglelefteq}(i)$, in which case $\bigcup_{k \in b} M_k = M_i \leq_{\mathbf{K}} N$, or $b \not\subseteq \text{pred}_{\trianglelefteq}(j)$. In this case, it is easy to check that $b \in B(j)$ (otherwise we could just extend the branch), so since $(\alpha, \trianglelefteq)$ is in pre-order, either $b = \text{pred}(j)$ or $b \in B(\alpha)$. The first case was dealt with before and in the second case, b is bounded so has a maximum j' (otherwise it would not be in $B(\alpha)$) and so $\bigcup_{k \in b} M_k = M_{j'} \leq_{\mathbf{K}} N$. $\square$

We can now prove that any reasonable tree can be "made independent". This can be seen as a generalization of the existence property of independence. Note that generalized symmetry is used in the proof.

Lemma 5.38. Assume that cl is algebraic and we are given a resolvable continuous enumerated tree of models $\mathcal{T}^0 := (\langle M_i^0 : i < \alpha \rangle, N^0, \alpha, \trianglelefteq)$. If:

(1) $(\alpha, \trianglelefteq)$ is in pre-order.
(2) For any $b \in B(\alpha)$, b is bounded.

Then we can find $\langle M_i : i < \alpha \rangle$, N, and $\langle f_i : i < \alpha \rangle$ such that:

(1) $\mathcal{T} := (\langle M_i : i < \alpha \rangle, N, \alpha, \trianglelefteq)$ is a resolvable *independent* continuous enumerated tree.
(2) For all $i, j < \alpha$, $f_i : M_i^0 \cong M_i$ and $i \trianglelefteq j$ implies $f_i \subseteq f_j$.
(3) $N = M^\alpha := \mathrm{cl}^N(\bigcup_{i<\alpha} M_i)$.

Proof. We build $\langle N_i : i < \alpha \rangle$, $\langle M_i : i < \alpha \rangle$, $\langle f_i : i < \alpha \rangle$ such that:

(1) $\langle N_i : i \leq \alpha \rangle$ is increasing.
(2) $\langle f_i : i < \alpha \rangle$ satisfies (2).
(3) For all $i \in (0, \alpha)$, $\mathcal{T}_i := (\langle M_j : j < i \rangle, \bigcup_{j<i} N_j, i, \trianglelefteq)$ is a resolvable independent continuous enumerated tree of models.
(4) For all $i < \alpha$, $N_i = \mathrm{cl}^{N_i}(M_0 \cup \bigcup_{j<i} M_j)$ $(= M^i)$.

This is enough, as we can then take $N := \bigcup_{i<\alpha} N_i$. This is possible. When $i = 0$, set $N_0 := M_0 := M_0^0$, $f_0 := \mathrm{id}_{M_0^0}$. Now assume that $i > 0$. Let $N_i' := \bigcup_{j<i} N_j$. There are two cases:

- Case 1: pred(i) has a maximum: Let $j := \max(\mathrm{pred}(i))$. Use the existence axiom to find f_i extending f_j and $N_i \geq_{\mathbf{K}} N_i'$ so that $f_i : M_i^0 \cong M_i$, $M_i \overset{N_i}{\underset{M_j}{\perp}} N_i'$, and $N_i = \mathrm{cl}^{N_i}(M_i \cup N_i')$. It is easy to check that this works.
- Case 2: pred(i) does not have a maximum: Let $M_i' := \bigcup_{j \triangleleft i} M_j$, $(M_i^0)' := \bigcup_{j \triangleleft i} M_j^0$, $f_i' := \bigcup_{j \triangleleft i} f_j$. Let M_i'', $g : M_i^0 \cong M_i''$ be such that g extends f_i'.

  Let $\delta := \mathrm{otp}(\mathrm{pred}(i))$. Note that δ is a limit ordinal. Let $\langle i_j : j < \delta \rangle$ list pred(i) in increasing order. For $j < i$, let $u_j := A_{i,j}$, where $A_{i,j}$ is as in Definition 5.24. Note that $\bigcup_{j<\delta} u_j = i$. By

Lemma 5.37, u_j is closed in $\mathcal{T}^0$, hence (taking the image of $\mathcal{T}^0$ by $\bigcup_{j<i} f_j$) in $\mathcal{T}_i$. We use Fact 4.7.(5) with M_j, N_j, M there standing for M_{i_j}, $M^{u_{i_j}}$, M_i'' here. The hypotheses are satisfied by Theorem 5.34 (applied to $\mathcal{T}_i$) and monotonicity. We obtain $N_i \in \mathbf{K}$ and a map $f : M_i'' \xrightarrow[M_i']{} N_i$ such that for all $j < \delta$:

(1) $N_{i_j} \leq_{\mathbf{K}} N_i$.
(2) $f[M_i''] \overset{N_i}{\underset{M_{i_j}}{\perp}} M^{u_{i_j}}$.
(3) $N_i = \mathrm{cl}^{N_i}(f[M_i''] \cup M^i)$.

Let $f_i := f \circ g$ and let $M_i := f[M_i'']$. This works by the above properties.

□

A specialization of Lemma 5.38 yields the main theorem of this section.

Theorem 5.39 (Tree construction)**.** Assume that cl is algebraic. Let δ be a limit ordinal and let $\lambda \geq 2$ be a cardinal. Let $\langle M_i : i \leq \delta \rangle$ be an increasing chain.

If $\langle M_i : i < \delta \rangle$ is continuous but $\bigcup_{i<\delta} M_i \not\leq_{\mathbf{K}} M_\delta$, then there is $\langle M_\eta \mid \eta \in {}^{\leq \delta}\lambda \rangle$, $\langle f_\eta \mid \eta \in {}^{\leq \delta}\lambda \rangle$ and $N \in \mathbf{K}$ such that for all $\eta, \nu \in {}^{\leq \delta}\lambda$:

(1) $M_\eta \leq_{\mathbf{K}} N$, $f_\eta : M_{\ell(\eta)} \cong M_\eta$.
(2) If η is an initial segment of ν, then $M_\eta \leq_{\mathbf{K}} M_\nu$ and $f_\eta \subseteq f_\nu$.
(3) If $\eta \neq \nu$ have length δ and $\alpha < \delta$ is least such that $\eta \restriction (\alpha+1) \neq \nu \restriction (\alpha + 1)$, then $M_\eta \overset{N}{\underset{M_{\nu \restriction \alpha}}{\perp}} M_\nu$.

Proof. By Lemma 5.36, we can find an enumeration $\langle \eta_i : i < \alpha \rangle$ of ${}^{\leq \delta}\lambda$ such that $(\alpha, \trianglelefteq)$ is a continuous enumerated tree in pre-order and $i \trianglelefteq j < \alpha$ implies that η_i is an initial segment of η_j. For $i < \alpha$, let $M_i^0 := M_{\ell(\eta_i)}$ and let $N^0 := M_\delta$. Then it is straightforward to check that $\mathcal{T}^0 := (\langle M_i^0 : i < \alpha \rangle, N^0, \alpha, \trianglelefteq)$ satisfies the hypotheses of Lemma 5.38. We obtain $\langle M_i : i < \alpha \rangle$, N, and $\langle f_i : i < \alpha \rangle$ there that correspond to $\langle M_{\eta_i} : i < \alpha \rangle$, N, and $\langle f_{\eta_i} : i < \alpha \rangle$ here. Since the resulting tree is independent, we obtain the independence condition via Lemma 5.26. □

6. Structure theory of universal classes

In this section, we precisely state a result of Shelah saying that for a universal class K which does not have the order property there is an ordering $\leq$ so that $(K, \leq)$ satisfies AxFri_1. We then use the tree construction theorem (Theorem 5.39) to show that failure of smoothness in that class implies unstability at certain cardinals.

We start by specializing the order property from [She09b, Definition V.A.1.1] to the quantifier-free version for universal classes:

Definition 6.1. A universal class $\mathbf{K}$ has the *order property of length* χ if there exists a quantifier-free first-order formula $\phi(\bar{x}, \bar{y}, \bar{z})$, a model $M \in K$, a sequence $\bar{c} \in {}^{\ell(\bar{z})}|M|$, and sequences $\langle \bar{a}_i : i < \chi \rangle$, $\langle \bar{b}_i : i < \chi \rangle$ from M (with $\ell(\bar{a}_i) = \ell(\bar{x})$, $\ell(\bar{b}_i) = \ell(\bar{y})$ for all $i < \chi$) so that for all $i, j < \chi$, $M \models \phi[\bar{a}_i; \bar{b}_j; \bar{c}]$ if and only if $i < j$. We say that $\mathbf{K}$ has the *order property* if it has the order property of length χ for all cardinals χ.

Remark 6.2. In the next section, we will show (Lemma 7.1) that categoricity in some $\lambda > \text{LS}(\mathbf{K})$ implies failure of the order property.

The following result is proven (in a more general form) in §2 of [GS86].

Fact 6.3. Let $\mathbf{K}$ be a universal class. If $\mathbf{K}$ does not have the order property, then there exists $\chi < h(\mathbf{K})$ (recall Definition 2.13) such that $\mathbf{K}$ does not have the order property of length χ.

From failure of the order property, Shelah shows that there exists a certain ordering $\leq^{\chi^+, \mu^+}$ on K such that $(K, \leq^{\chi^+, \mu^+})$ satisfies AxFri_1 (recall Definition 4.1). We now proceed to define this ordering.

Definition 6.4 (Averages, V.A.2 in [She09b])**.** Let $\mathbf{K}$ be a universal class. Let $M \in \mathbf{K}$, let I be an index set, and let $\mathbf{I} := \langle \bar{a}_i : i \in I \rangle$ be a sequence of elements of M of the same finite arity $n < \omega$. Let $\chi \leq \mu$ be infinite cardinals such that[10] $|\mathbf{I}| \geq \chi$.

(1) For $A \subseteq |M|$, we let $\text{Av}_\chi(\mathbf{I}/A; M)$ (the χ-average of $\mathbf{I}$ over A in M) be the set of quantifier-free first-order formulas $\phi(\bar{x})$ over A such that $\ell(\bar{x}) = n$ and $|\{i \in I \mid M \models \phi[\bar{a}_i]\}| < \chi$.
(2) We say that $\mathbf{I}$ is (χ, μ)-*convergent in* M if $|\mathbf{I}| \geq \mu$ and for every $A \subseteq |M|$, $p := \text{Av}_\chi(\mathbf{I}/A; M)$ is complete over A (i.e. for every quantifier-free formula $\phi(\bar{x})$ over A with $\ell(\bar{x}) = n$, either $\phi(\bar{x}) \in p$ or $\neg\phi(\bar{x}) \in p$).
(3) Let $A, B \subseteq |M|$ and let p be a set of quantifier-free formulas over B (all of the same arity $n < \omega$). We say that p is (χ, μ)-*averageable over* A *in* M if there exists a sequence $\mathbf{I} \subseteq {}^n A$ that is (χ, μ)-convergent in M so that $p = \text{Av}_\chi(\mathbf{I}/B; M)$.

[10] We sometimes think of $\mathbf{I}$ as just the set of its elements (i.e. as if it was only $\text{ran}(\mathbf{I})$), e.g. we write $|\mathbf{I}|$ instead of $|\text{ran}(\mathbf{I})|$ and $\mathbf{I} \subseteq {}^n A$ instead of $\text{ran}(\mathbf{I}) \subseteq {}^n A$.

Remark 6.5. In the above notation, the usual notion of average from the first-order framework [She90, Definition III.1.5] can be written $\mathrm{Av}_{\aleph_0}(\mathbf{I}/A; \mathfrak{C})$, modulo the fact that here all the formulas are quantifier-free. This can be remedied (as Shelah does) by adding a parameter Δ containing the formulas of interest, but we have no use for it here.

Remark 6.6 (Monotonicity)**.**

(1) Since the formulas under consideration are quantifier-free, we have the following monotonicity properties: if $M_0 \subseteq M$ and $A, \mathbf{I} \subseteq |M_0|$, then $\mathrm{Av}_\chi(\mathbf{I}/A; M_0) = \mathrm{Av}_\chi(\mathbf{I}/A; M)$. Similarly, if $\mathbf{I}$ is (χ, μ)-convergent in M, then it is (χ, μ)-convergent in M_0, and if $A \subseteq B \subseteq |M_0|$ and p is a set of quantifier-free type over B that is (χ, μ)-averageable over A in M, then it is (χ, μ)-averageable over A in M_0.
(2) If p above is (χ, μ)-averageable over A in M, then whenever $A \subseteq A' \subseteq B_0 \subseteq B$, we have that $p \upharpoonright B_0$ is (χ, μ)-averageable over A' in M.

Definition 6.7 (V.A.4.1 in [She09b])**.** Let $\mathbf{K}$ be a universal class and let $\chi \leq \mu$ be infinite cardinals. For $M, N \in \mathbf{K}$, we write $M \leq^{\chi,\mu} N$ if $M \subseteq N$ and for every $\bar{c} \in {}^{<\omega}|N|$, the quantifier-free type of $\bar{c}$ over M in N, $\mathrm{tp}_{\mathrm{qf}}(\bar{c}/M; N)$, is (χ, μ)-averageable over M.

From now on we assume:

Hypothesis 6.8.

(1) $\mathbf{K} = (K, \subseteq)$ is a universal class with arbitrarily large models.
(2) $\chi \geq \mathrm{LS}(\mathbf{K})$ is such that $\mathbf{K}$ does not have the order property of length χ^+.
(3) Set $\mu := 2^{2^\chi}$.

Definition 6.9. Let $\mathbf{K}^0 := (K, \leq^{\chi^+, \mu^+})$.

The following is the key structure theorem for universal classes without the order property. It is due to Shelah [She09b, Chapter V.B]:

Fact 6.10.

(1) $\mathbf{K}^0$ is a weak AEC with $\mathrm{LS}(\mathbf{K}^0) \leq \mu^+$.
(2) For $M \in K$ and $A \subseteq |M|$, let $\mathrm{cl}^M(A)$ be the closure of A under the functions of M. We can define a 4-ary relation $\downarrow$ on K by $M_1 \overset{M_3}{\underset{M_0}{\downarrow}} M_2$ if and only if all of the following conditions are satisfied:

(a) $M_0 \leq_{\mathbf{K}^0} M_1$ and $M_0 \leq_{\mathbf{K}^0} M_2$.
(b) $M_1 \subseteq M_3$ and $M_2 \subseteq M_3$.
(c) $\mathrm{cl}^{M_3}(M_1 \cup M_2) \leq_{\mathbf{K}^0} M_3$.
(d) For any $\bar{c} \in {}^{<\omega}|M_1|$, $\mathrm{tp}_{\mathrm{qf}}(\bar{c}/M_2; M_3)$ is (χ^+, μ^+)-averageable over M_0.

We then have that $(\mathbf{K}^0, \mathop{\smile}\limits^{\mid}, \mathrm{cl})$ satisfies AxFri_1. Moreover cl is algebraic (see Definition 5.21) and $\mathop{\smile}\limits^{\mid}$ is μ^+-based (see Definition 4.12).

Proof. That $(\mathbf{K}^0, \mathop{\smile}\limits^{\mid}, \mathrm{cl})$ satisfies AxFri_1 and has Löwenheim-Skolem-Tarski number bounded by μ^+ is the content of [She09b, V.B.2.9]. Since cl is just closure under the functions, it is clearly algebraic. That $\mathop{\smile}\limits^{\mid}$ is μ^+-based is observed (but not explicitly proven) in [She09b, V.C.5.7]. We give the proof here. First, we show:

<u>Claim</u>: For any cardinal λ, $\mathbf{K}^0$ is $(\leq \lambda, \mu^+)$-smooth. That is, if $\langle M_i : i < \mu^+\rangle$ is increasing in $\mathbf{K}^0$ and $M \in \mathbf{K}^0$ is such that $M_i \leq_{\mathbf{K}^0} M$ for all $i < \mu^+$, then $\bigcup_{i<\mu^+} M_i \leq_{\mathbf{K}^0} M$.

<u>Proof of claim</u>:

By [She09b, V.A.4.4], for any $N, N' \in \mathbf{K}^0$, $N \leq_{\mathbf{K}^0} N'$ if and only if $N \preceq_\Delta N'$, where Δ is a certain fragment of $\mathbb{L}_{\mu^+,\mu^+}$. The result now follows from the basic properties of Δ-elementary substructure. $\dagger_{\text{Claim}}$

Let $M \leq_{\mathbf{K}^0} M^*$ and let $A \subseteq |M^*|$ be given. By definition of $\leq_{\mathbf{K}^0} = \leq^{\chi^+,\mu^+}$, for each $\bar{c} \in {}^{<\omega}|M^*|$ there exists $\mathbf{I}^{\bar{c}} \subseteq {}^{\ell(\bar{c})}|M|$ that is (χ^+, μ^+)-convergent and so that $\mathrm{Av}(\mathbf{I}^{\bar{c}}/M; M^*) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}/M; M^*)$. Without loss of generality, $|\mathbf{I}^{\bar{c}}| \leq \mu^+$.

We build increasing $\langle M_i^0 : i < \mu^+\rangle$, $\langle M_i^1 : i < \mu^+\rangle$ such that for all $i < \mu^+$:

(1) $M_i^0 \leq_{\mathbf{K}^0} M$.
(2) $M_i^0 \leq_{\mathbf{K}^0} M_i^1 \leq_{\mathbf{K}^0} M^*$.
(3) $\|M_i^1\| \leq |A| + \mu^+$.
(4) $|M| \cap |M_i^1| \subseteq |M_0^{i+1}|$.
(5) For all $\bar{c} \in {}^{<\omega}M_i^1$, $\mathbf{I}^{\bar{c}} \subseteq |M_{i+1}^0|$.

This is enough: let $N_0 := \bigcup_{i<\mu^+} M_i^0$, $N_1 := \bigcup_{i<\mu^+} M_i^1$. By the claim, $N_0 \leq_{\mathbf{K}^0} N_1 \leq_{\mathbf{K}^0} M^*$ and by requirements (1) and (4), $M \cap N_1 = N_0$. Finally, $N_1 \mathop{\smile}\limits^{\mid}{}_{N_0}^{M^*} M$ by definition of $\mathbf{I}^{\bar{c}}$ and requirement (5).

This is possible: assume that $\langle M_j^\ell : j < i\rangle$ have been defined for $\ell = 0, 1$. Let $M_{i,0}^0 := \bigcup_{j<i} M_j^0$, $M_{i,0}^1 := \bigcup_{j<i} M_j^1$. Use that $\mathrm{LS}(\mathbf{K}^0) \leq \mu^+$ to

pick M_i^0 such that $M_i^0 \leq_{\mathbf{K}^0} M$, $|M| \cap M_{i,0}^1 \subseteq |M_i^0|$, $\mathbf{I}^{\bar{c}} \subseteq |M_i^0|$ for all $\bar{c} \in {}^{<\omega}|M_{i,0}^1|$, and $\|M_i^0\| \leq |A| + \mu^+$. Note that by coherence, $M_j^0 \leq_{\mathbf{K}^0} M_i^0$. Now pick M_i^1 such that $M_i^1 \leq_{\mathbf{K}^0} M^*$, $A \cup |M_{i,0}^1| \cup |M_i^0| \subseteq M_i^1$, and $\|M_i^1\| \leq |A| + \mu^+$. It is easy to check that this satisfies all the requirements. $\square$

Note that $\mathbf{K}^0$ has amalgamation (Remark 4.4). However, we do not know if it satisfies joint embedding, so we partition $\mathbf{K}^0$ into disjoint AECs, each of which has joint embedding. We will then concentrate on just one of these AECs. This trick appears in [She87c, Section II.3].

Definition 6.11. For $M, N \in \mathbf{K}^0$, write $M \sim N$ if they can be $\leq_{\mathbf{K}^0}$-embedded inside a common model. This is an equivalence relation and the equivalence classes partition $\mathbf{K}^0$ into disjoint weak AECs $\langle \mathbf{K}_i^0 : i \in I \rangle$ that have amalgamation and joint embedding. There is only a set of such classes, so there exists[11] $i \in I$ such that $\mathbf{K}_i^0$ has arbitrarily large models. Let $\mathbf{K}^* := (\mathbf{K}_i^0)_{\geq \mu^+}$.

From now on, we will work with $\mathbf{K}^*$. We note a few trivial properties of independence there:

Lemma 6.12.

(1) $\mathbf{K}^*$ is a weak AEC with amalgamation, joint embedding, and arbitrarily large models.
(2) $\mathrm{LS}(\mathbf{K}^*) = \mu^+$.
(3) $\mathbf{K}$ and $\mathbf{K}^*$ are compatible (recall Definition 3.1).
(4) $(\mathbf{K}^*, \mathop{\smile\!\!\!\!\mid} \restriction \mathbf{K}^*, \mathrm{cl})$ satisfies AxFri_1, where for $M \in \mathbf{K}^*$, cl^M is closure under the functions of M and $\mathop{\smile\!\!\!\!\mid} \restriction \mathbf{K}^*$ is the natural restriction of $\mathop{\smile\!\!\!\!\mid}$ (from Fact 6.10) to $\mathbf{K}^*$.

Proof. Straightforward. $\square$

Notation 6.13. We abuse notation and write $\mathop{\smile\!\!\!\!\mid}$ for $\mathop{\smile\!\!\!\!\mid} \restriction \mathbf{K}^*$ (where again $\mathop{\smile\!\!\!\!\mid}$ is from Fact 6.10).

Lemma 6.14.

(1) If $A \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\mid}}} B$ and $\bar{c} \in {}^{<\omega}A$, then $\mathrm{tp}_{\mathrm{qf}}(\bar{c}/B; M_3)$ is (χ^+, μ^+)-averageable over M_0.
(2) cl is algebraic.
(3) $\mathop{\smile\!\!\!\!\mid}$ is $\mathrm{LS}(\mathbf{K})$-based.

[11]There could be many and for our purpose the choice of i does not matter. Moreover i is unique if $\mathbf{K}$ is categorical in some $\lambda \geq \mu^+$.

Proof. (1) follows directly from the definition of $\mathop{\smile\!\!\!\!\!\!\mid}\,$. For the rest, cl is algebraic because cl satisfies this property in $\mathbf{K}^0$ (Fact 6.10). Similarly, in $\mathbf{K}^0$ $\mathop{\smile\!\!\!\!\!\!\mid}\,$ is μ^+-based (Fact 6.10) and it is straightforward to check that this carries over to $\mathbf{K}^*$. □

Next, we study what happens if smoothness fails in $\mathbf{K}^*$. Recall that our goal is to see that this is incompatible with categoricity (in a high-enough cardinal). Shelah has shown [She09b, V.C.2.6], that failure of smoothness implies that $\mathbf{K}^*$ has 2^λ-many nonisomorphic models at every high-enough *regular* cardinal λ. So in particular $\mathbf{K}^*$ cannot be categorical in a regular cardinal. However we are also interested in the singular case. Shelah states as an exercise [She09b, V.C.4.13] that categoricity is also contradicted. However we have been unable to prove it.

Instead, we aim to see that failure of smoothness implies that $\mathbf{K}^*$ has many *types*, i.e. it is Galois unstable in some suitable cardinals. This will contradict Lemma 3.4. The argument is similar to [She09b, V.E.3.15], which shows that failure of superstability (in the sense that there is an increasing chain $\langle M_i : i < \delta \rangle$ and a type $p \in \mathrm{gS}(\bigcup_{i<\delta} M_i)$ that forks over every M_i, $i < \delta$) implies unstability at suitable cardinals. The extra difficulty here is that smoothness fails, but the hard work in constructing the tree has already been done in Theorem 5.39.

First observe that any failure of smoothness must be witnessed by a small chain:

Lemma 6.15. If $\mathbf{K}^*$ is $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \mathrm{LS}(\mathbf{K}^*)^+)$-smooth (recall Definition 2.8), then $\mathbf{K}^*$ is smooth, i.e. it is an AEC.

Proof. By Lemma 6.14.(3), $\mathbf{K}^*$ is $\mathrm{LS}(\mathbf{K}^*)$-based, so apply Fact 4.13. □

We now show that failure of smoothness implies unstability at some not too high cardinal.

Theorem 6.16. Assume that $\mathbf{K}^*$ is not $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \mathrm{LS}(\mathbf{K}^*)^+)$-smooth. Let $\kappa \leq \mathrm{LS}(\mathbf{K}^*)^+$ be least such that $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \kappa)$-smoothness fails. If $\lambda \geq \mathrm{LS}(\mathbf{K}^*)^+$ is such that $\lambda = \lambda^{<\kappa}$ and $\lambda < \lambda^\kappa$, then $\mathbf{K}^*$ is $(< \omega)$-unstable in λ.

Proof. Fix an increasing sequence $\langle M_i : i \leq \kappa \rangle$ such that $\|M_i\| \leq \mathrm{LS}(\mathbf{K}^*)^+$ for all $i \leq \kappa$ and $\bigcup_{i<\kappa} M_i \not\leq_{\mathbf{K}^*} M_\kappa$. Without loss of generality (using minimality of κ) the sequence is continuous below κ, i.e. $M_i = \bigcup_{j<i} M_j$ for every $i < \kappa$. Let $N \in \mathbf{K}^*$ and $\langle M_\eta, f_\eta \mid \eta \in {}^{\leq \kappa}\lambda \rangle$ be as given by Theorem 5.39 (where $\delta, \mathbf{K}$ there stands for $\kappa, \mathbf{K}^*$ here; note that cl is algebraic by Lemma 6.14.(2) so the hypotheses of the theorem hold).

By definition of $\leq_{\mathbf{K}^*}$ (so really of $\leq_{\mathbf{K}^0}$, see Definitions 6.9 and 6.7), we have that $\bigcup_{i<\kappa} M_i \not\leq^{\chi^+,\mu^+} M_\kappa$. By definition of $\leq^{\chi^+,\mu^+}$, there exists $\bar{c} \in {}^{<\omega}|M_\kappa|$ such that $q := \mathrm{tp}_{\mathrm{qf}}(\bar{c}/\bigcup_{i<\kappa} M_i; M_\kappa)$ is not (χ^+, μ^+)-averageable over $\bigcup_{i<\kappa} M_i$ in M_κ. For $\eta \in {}^\kappa\lambda$, let $\bar{c}_\eta := f_\eta(\bar{c})$.

Note that by (1) in Theorem 5.39, for all $\eta \in {}^{\leq\kappa}\lambda$, $\|M_\eta\| = \|M_{\ell(\eta)}\| \leq \mathrm{LS}(\mathbf{K}^*)^+ \leq \lambda$, so fix $M \leq_{\mathbf{K}^*} N$ such that $\|M\| = \lambda$ and $\bigcup_{\eta \in {}^{<\kappa}\lambda} |M_\eta| \subseteq |M|$. For $\eta \in {}^\kappa\lambda$, let $p_\eta := \mathrm{gtp}_{\mathbf{K}^*}(\bar{c}_\eta/M; N)$.

Because $\lambda < \lambda^\kappa$, it is enough to prove the following:

<u>Claim</u>: For $\eta, \nu \in {}^\kappa\lambda$, if $\eta \neq \nu$, then $p_\eta \neq p_\nu$.

<u>Proof of claim</u>: Let $\alpha < \kappa$ be least such that $\eta \restriction (\alpha + 1) \neq \nu \restriction (\alpha + 1)$. By (3) in Theorem 5.39 and the monotonicity property of $\overline{\downarrow}$ (see Lemma 4.9) we have that $\bar{c}_\eta \underset{M_{\nu\restriction\alpha}}{\overset{N}{\downarrow}} M_\nu$. By monotonicity again, $\bar{c}_\eta \underset{M_{\nu\restriction\alpha}}{\overset{N}{\downarrow}} \bigcup_{\beta<\kappa} M_{\nu\restriction\beta}$. Now assume for a contradiction that $p_\eta = p_\nu$. Then by monotonicity and invariance, $\bar{c}_\nu \underset{M_{\nu\restriction\alpha}}{\overset{N}{\downarrow}} \bigcup_{\beta<\kappa} M_{\nu\restriction\beta}$ so $\bar{c}_\nu \underset{M_{\nu\restriction\alpha}}{\overset{M_\nu}{\downarrow}} \bigcup_{\beta<\kappa} M_{\nu\restriction\beta}$. Applying f_ν^{-1} to this, we get that $\bar{c} \underset{M_\alpha}{\overset{M_\kappa}{\downarrow}} \bigcup_{i<\kappa} M_i$. In particular, by Lemma (6.14).(1), q is (χ^+, μ^+)-averageable over M_α in M_κ. By Remark 6.6, q is (χ^+, μ^+)-averageable over $\bigcup_{i<\kappa} M_i$ in M_κ. This contradicts the choice of $\bar{c}$. $\dagger_{\mathrm{Claim}}$. $\square$

7. Categoricity in universal classes

In this section, we derive the main theorem of this paper. First, we explain why, in a universal class, categoricity (in some $\lambda > \mathrm{LS}(\mathbf{K})$) implies failure of the order property. Note that Shelah argues [She09b, Claim V.B.2.6] that if K has the order property, then it has 2^μ-many models of size μ (for any $\mu > \mathrm{LS}(\mathbf{K})$), contradicting categoricity. Since Shelah's construction of many models is very technical, we sketch an easier proof.

Lemma 7.1. Assume that a universal class $\mathbf{K}$ is categorical in a $\lambda > \mathrm{LS}(K)$. Then $\mathbf{K}$ does not have the order property (recall Definition 6.1).

Proof. If $\mathbf{K}$ does not have arbitrarily large models, then $\mathbf{K}$ does not have the order property. Now assume that $\mathbf{K}$ has arbitrarily large models. We can use Ehrenfeucht-Mostowski models and the standard argument (due to Morley, see [Mor65, Theorem 3.7]) shows that if $M \in K_\lambda$, $\mu \in [\mathrm{LS}(\mathbf{K}), \lambda)$, and $A \subseteq |M|$ is such that $|A| \leq \mu$, then M realizes at most μ-many first-order syntactic quantifier-free types over A. However if $\mathbf{K}$ had the order property, we would be able to build a set $A \subseteq |M|$ with $|A| \leq \mathrm{LS}(\mathbf{K})$ but with at least $\mathrm{LS}(\mathbf{K})^+$ types over A realized in M (using Dedekind cuts, see e.g. the proof of [BGKV, Fact 5.13]). This is a contradiction. □

Next, we deduce more structure from categoricity:

Theorem 7.2. Let $\mathbf{K}$ be a universal class. If $\mathbf{K}$ is categorical in some $\lambda \geq \beth_{h(\mathbf{K})}$, then there exists $\mathbf{K}^*$ such that:

(1) $\mathbf{K}^*$ is an AEC.
(2) $\mathrm{LS}(\mathbf{K}) \leq \mathrm{LS}(\mathbf{K}^*) < h(\mathbf{K})$.
(3) $\mathbf{K}$ and $\mathbf{K}^*$ are compatible (recall Definition 3.1).
(4) $\mathbf{K}^*$ has amalgamation, joint embedding, and arbitrarily large models.
(5) $\mathbf{K}^*$ is $\mathrm{LS}(\mathbf{K}^*)$-tame.

Proof. Let $\mathbf{K}$ be a universal class and let $\lambda \geq \beth_{h(\mathbf{K})}$ be such that $\mathbf{K}$ is categorical in λ. By Fact 2.18, $\mathbf{K}$ has arbitrarily large models. By Lemma 7.1, $\mathbf{K}$ does not have the order property. By Fact 6.3, we can fix $\chi \in [\mathrm{LS}(\mathbf{K}), h(\mathbf{K}))$ such that $\mathbf{K}$ does not have the order property of length χ^+. Thus Hypothesis 6.8 is satisfied, and we can let $\mathbf{K}^*$ be as in Definition 6.11. We have to check that it has all the required properties. First, $\mathbf{K}^*$ is a weak AEC with amalgamation, joint embedding, and arbitrarily large models (Lemma 6.12.(1)). Moreover (Lemma 6.12.(2)), $\mathrm{LS}(\mathbf{K}) \leq \mathrm{LS}(\mathbf{K}^*) = \mu^+ = \left(2^{2^\chi}\right)^+ < h(\mathbf{K})$. Also, $\mathbf{K}$ and $\mathbf{K}^*$ are compatible (Lemma 6.12.(3)). This takes care of (2), (3), and (4) in the statement of Theorem 7.2. Combining Lemma 4.14.(2) and Lemma 6.14, we obtain that $\mathbf{K}^*$ is $\mathrm{LS}(\mathbf{K}^*)$-tame, so (5) also holds.

It remains to see (1): $\mathbf{K}^*$ is an AEC, i.e. it satisfies the smoothness axiom. Suppose not. Then by Lemma 6.15, there is a small counterexample: $\mathbf{K}^*$ is not $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \mathrm{LS}(\mathbf{K}^*)^+)$-smooth. Let $\kappa \leq \mathrm{LS}(\mathbf{K}^*)^+$ be least such that $\mathbf{K}^*$ is not $(\leq \mathrm{LS}(\mathbf{K}^*), \leq \kappa)$-smooth. Note that κ is regular. Let $\lambda_0 := \beth_\kappa(\mathrm{LS}(\mathbf{K}^*))$. Note:

- $\lambda_0 \geq \mathrm{LS}(\mathbf{K}^*)^+$.
- $\lambda_0 = \lambda_0^{<\kappa}$ and $\lambda_0 < \lambda_0^\kappa$ (because $\mathrm{cf}(\lambda_0) = \kappa$).

- Since $\kappa \leq \text{LS}(\mathbf{K}^*) < h(\mathbf{K})$, we have that $\lambda_0 \leq \beth_{\text{LS}(\mathbf{K}^*)+\kappa} < \beth_{h(\mathbf{K})} \leq \lambda$. Similarly, $\lambda_0^+ < \lambda$.

By Lemma 3.4 (where $\mathbf{K}^1, \mathbf{K}^2, \mu, \lambda$ there stand for $\mathbf{K}, \mathbf{K}^*, \lambda_0, \lambda$ here, note that we are using that $\lambda_0^+ < \lambda$), $\mathbf{K}^*$ is $(<\omega)$-stable in λ_0. However Theorem 6.16 (where λ there stands for λ_0 here) says that $\mathbf{K}^*$ is $(<\omega)$-unstable in λ_0, a contradiction. □

Finally, we have all the results we need to prove the main theorem:

Theorem 7.3. Let $\mathbf{K}$ be a universal class. If $\mathbf{K}$ is categorical in *some* $\lambda \geq \beth_{h(\mathbf{K})}$, then there exists $\chi < \beth_{h(\mathbf{K})}$ such that $\mathbf{K}$ is categorical in *all* $\lambda' \geq \chi$. Moreover, $\mathbf{K}_{\geq \chi}$ has amalgamation.

Proof. Let $\mathbf{K}^*$ be as given by Theorem 7.2. By Fact 2.21, $\mathbf{K}$ has primes, so by Theorem 3.8 (where $\mathbf{K}^1$, $\mathbf{K}^2$ there stand for $\mathbf{K}$, $\mathbf{K}^*$ here), $\mathbf{K}^*$ is categorical in all $\lambda' \geq \chi := h(\text{LS}(\mathbf{K}^*))$. By compatibility (recalling that $\text{LS}(\mathbf{K}) \leq \text{LS}(\mathbf{K}^*)$), $\mathbf{K}$ is also categorical in all $\lambda' \geq \chi$. Finally, since $\text{LS}(\mathbf{K}^*) < h(\mathbf{K})$, we have that $\chi = h(\text{LS}(\mathbf{K}^*)) = \beth_{\left(2^{\text{LS}(\mathbf{K}^*)}\right)^+} < \beth_{h(\mathbf{K})}$.

For the moreover part, note that $\chi^{\text{LS}(\mathbf{K},\mathbf{K}^*)} = \chi^{\text{LS}(\mathbf{K}^*)} = \chi$ so by Lemma 3.6, $\mathbf{K}_{\geq \chi} = \mathbf{K}^*_{\geq \chi}$. Since the latter has amalgamation, so does the former. □

Remark 7.4. In fact, $\mathbf{K}_{\geq \chi}$ satisfies much more than amalgamation. This is because $\mathbf{K}_{\geq \chi}$ is a locally universal class (see [Vase, Definition 2.20]). Thus it is fully χ-tame and short (see [Vase, Corollary 3.8]) and admits a global notion of independence (for types over arbitrary sets) that is similar to forking in a first-order superstable theory (see [Vase, Appendix C]).

Proof of Theorem 0.1 and Corollary 0.2. Let ψ be a universal $\mathbb{L}_{\omega_1,\omega}$ sentence. The class $\mathbf{K}$ of models of ψ is a universal class (Fact 2.3) with $h(\mathbf{K}) = \beth_{\omega_1}$ (see Remark 2.14 and Fact 2.12). Now apply Theorem 7.3. □

Remark 7.5. Theorem 0.1 applies more generally to universal classes that are axiomatized by a (not necessarily universal) $\mathbb{L}_{\omega_1,\omega}$ sentence.

References

[Bal09] John T. Baldwin, *Categoricity*, University Lecture Series, vol. 50, American Mathematical Society, 2009.

[BGKV] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Preprint. URL: `http://arxiv.org/abs/1404.1494v2`.

[Bon14] Will Boney, *Tameness from large cardinal axioms*, The Journal of Symbolic Logic **79** (2014), no. 4, 1092–1119.

[BV] Will Boney and Sebastien Vasey, *A survey on tame abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1512.00060v3`.

[Cha68] C.C. Chang, *Some remarks on the model theory of infinitary languages*, The syntax and semantics of infinitary languages (Jon Barwise, ed.), Lecture Notes in Mathematics, vol. 72, Springer, 1968, pp. 36–63.

[GK] Rami Grossberg and Alexei Kolesnikov, *Superior abstract elementary classes are tame*, Preprint. URL: `http://www.math.cmu.edu/~rami/AtameP.pdf`.

[Gro02] Rami Grossberg, *Classification theory for abstract elementary classes*, Contemporary Mathematics **302** (2002), 165–204.

[GS86] Rami Grossberg and Saharon Shelah, *On the number of nonisomorphic models of an infinitary theory which has the infinitary order property. Part A*, The Journal of Symbolic Logic **51** (1986), no. 2, 302–322.

[GV06a] Rami Grossberg and Monica VanDieren, *Categoricity from one successor cardinal in tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 2, 181–201.

[GV06b] ———, *Galois-stability for tame abstract elementary classes*, Journal of Mathematical Logic **6** (2006), no. 1, 25–49.

[GV06c] ———, *Shelah's categoricity conjecture from a successor for tame abstract elementary classes*, The Journal of Symbolic Logic **71** (2006), no. 2, 553–568.

[KLH] Alexei Kolesnikov and Christopher Lambie-Hanson, *The Hanf number for amalgamation of coloring classes*, The Journal of Symbolic Logic, To appear. URL: `http://arxiv.org/abs/1409.2508v2`.

[Kue08] David W. Kueker, *Abstract elementary classes and infinitary logics*, Annals of Pure and Applied Logic **156** (2008), 274–286.

[Mor65] Michael Morley, *Categoricity in power*, Transactions of the American Mathematical Society **114** (1965), 514–538.

[MS90] Michael Makkai and Saharon Shelah, *Categoricity of theories in $L_{\kappa,\omega}$, with κ a compact cardinal*, Annals of Pure and Applied Logic **47** (1990), 41–97.

[She] Saharon Shelah, *Categoricity for abstract classes with amalgamation (updated)*, Oct. 29, 2004 version. URL: `http://shelah.logic.at/files/394.pdf`.

[She83a] ———, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in \mathbb{L}_{\omega_1,\omega}$. Part A*, Israel Journal of Mathematics **46** (1983), no. 3, 214–240.

[She83b] ———, *Classification theory for non-elementary classes I: The number of uncountable models of $\psi \in \mathbb{L}_{\omega_1,\omega}$. Part B*, Israel Journal of Mathematics **46** (1983), no. 4, 241–273.

[She87a] ———, *Classification of non elementary classes II. Abstract elementary classes*, Classification Theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 419–497.

[She87b] ———, *Existence of many $\mathbb{L}_{\infty,\lambda}$-equivalent, non-isomorphic models of T of power λ*, Annals of Pure and Applied Logic **34** (1987), 291–310.

[She87c] ———, *Universal classes*, Classification theory (Chicago, IL, 1985) (John T. Baldwin, ed.), Lecture Notes in Mathematics, vol. 1292, Springer-Verlag, 1987, pp. 264–418.

[She90] ———, *Classification theory and the number of non-isomorphic models*, 2nd ed., Studies in logic and the foundations of mathematics, vol. 92, North-Holland, 1990.

[She99] ———, *Categoricity for abstract classes with amalgamation*, Annals of Pure and Applied Logic **98** (1999), no. 1, 261–294.

[She00] ———, *On what I do not understand (and have something to say), model theory*, Math. Japonica **51** (2000), 329–377.

[She09a] ———, *Classification theory for abstract elementary classes*, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.

[She09b] ———, *Classification theory for abstract elementary classes 2*, Studies in Logic: Mathematical logic and foundations, vol. 20, College Publications, 2009.

[SV99] Saharon Shelah and Andrés Villaveces, *Toward categoricity for classes with no maximal models*, Annals of Pure and Applied Logic **97** (1999), 1–25.

[Vasa] Sebastien Vasey, *A downward categoricity transfer for tame abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1510.03780v3`.

[Vasb] ———, *Independence in abstract elementary classes*, Preprint. URL: `http://arxiv.org/abs/1503.01366v5`.

[Vasc] ———, *Infinitary stability theory*, Preprint. URL: `http://arxiv.org/abs/1412.3313v4`.

[Vasd] ———, *Shelah's eventual categoricity conjecture in tame AECs with primes*, Preprint. URL: `http://arxiv.org/abs/1509.04102v3`.

[Vase] ———, *Shelah's eventual categoricity conjecture in universal classes. Part I*, Preprint. URL: `http://arxiv.org/abs/1506.07024v9`.

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