BUILDING PRIME MODELS IN FULLY GOOD ABSTRACT ELEMENTARY CLASSES

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Abstract. We show how to build primes models in classes of saturated models of abstract elementary classes (AECs) having a well-behaved independence relation:

Theorem 0.1. Let $\mathcal{K}$ be an almost fully good AEC that is categorical in $\text{LS}(\mathcal{K})$ and has the $\text{LS}(\mathcal{K})$-existence property for domination triples. For any $\lambda > \text{LS}(\mathcal{K})$, the class of Galois saturated models of $\mathcal{K}$ of size λ has prime models over every set of the form $M \cup \{a\}$.

This generalizes an argument of Shelah, who proved the result when λ is a successor cardinal. As a corollary, we deduce an equivalence between eventual categoricity and having primes in AECs satisfying a locality property (note that (1) below implies (2) appears elsewhere; here we prove (2) implies (1)).

Corollary 0.2. Let $\mathcal{K}$ be a fully $\text{LS}(\mathcal{K})$-tame and short AEC with amalgamation. Write $H_1 := \beth_{\left(2^{\text{LS}(\mathcal{K})}\right)^+}$ and assume that $\mathcal{K}$ is categorical in *some* $\lambda \geq H_1$. The following are equivalent:

(1) $\mathcal{K}_{\geq H_1}$ has primes over sets of the form $M \cup \{a\}$.
(2) $\mathcal{K}$ is categorical in *all* $\lambda' \geq H_1$.

Contents

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1. Introduction

Prime models are a crucial ingredient in the proof of Morley's categoricity theorem [Mor65]. Morley's construction gives a *primary model*: a model whose universe can be enumerated so that the type of each element is isolated over the previous ones. This construction can be generalized to certain non-elementary context such as homogeneous model theory [She70] and even finitary abstract elementary classes [HK06].

In general abstract elementary classes (AECs), it seems that the construction breaks down due to the lack of compactness. Shelah [She09, Section III.4] works around this difficulty by assuming that the class satisfies an axiomatization of superstable forking for its models of size λ (in Shelah's terminology, $\mathcal{K}$ has a successful good λ-frame) and uses *domination* to build for every saturated M of size λ^+ and every element a a saturated model N containing $M \cup \{a\}$ and prime in the class of saturated models of $\mathcal{K}$ size λ^+. Here, saturation is defined in terms of Galois (orbital) types.

Shelah shows [She09, Chapter II] that the assumption of existence of a successful good λ-frame follows from strong *local* hypotheses: categoricity in λ, λ^+, a medium number of models in λ^{++}, and set-theoretic hypotheses such as $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. In [Vas16, Vasb], we showed that successful good frames can also be built assuming that the class satisfies *global hypotheses*: amalgamation, categoricity in some high-enough cardinal, and a locality property called full tameness and shortness. It is known that amalgamation and the locality property both follow from categoricity and a large cardinal axiom [MS90, Bon14]. The global hypotheses actually enable us to build the global generalization of a successful good λ-frame: what we call an almost fully good independence relation (see Definition 2.13). In this paper, we show that Shelah's argument generalizes to this global setup and λ^+ can be replaced by a limit cardinal. Thus we obtain a general construction of primes (in an appropriate class of saturated models) that works assuming only the existence of a well-behaved independence notion (this is Theorem 0.1 from the abstract).

Recently, [Vase, Theorem 0.2] showed that assuming the global hypotheses above, existence of primes over *every* set of the form $M \cup \{a\}$ implies categoricity on a tail of cardinals. Unfortunately, we cannot use the construction of prime models of this paper to deduce a new categoricity transfer in the global framework (we only get existence of primes in a subclass of saturated models). However we can use it to obtain that in the global framework, categoricity on a tail of cardinals implies the existence of primes (this is Corollary 0.2 from the abstract). This gives a converse to [Vase, Theorem 0.2] (we asked if such a converse was true in [Vase, Conjecture 5.22]).

The background required to read this paper is a basic knowledge of AECs (for example Chapters 4-12 of Baldwin's book [Bal09]). Some familiarity with good frames and their generalizations, in particular the beginning of [Vasb] and Shelah's construction of primes [She09, Section III.4], would be helpful but we state all the necessary definitions here. We rely on several results from [SV99, GV06a, Jar16, Vasb, VV, Vase] but they are used as black boxes: little understanding of the material there is needed. We rely on a recent preprint [Vasa] to get a threshold of H_1 in Corollary 0.2. These recent results are *not* needed if one replaces H_1 by H_2^+ (see Notation 4.3) there.

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2. Background

We give some background on superstability and independence that will be used in the next section. We assume familiarity with the basics of AECs as laid out in e.g. [Bal09] or the forthcoming [Gro]. We will use the notation from the preliminaries of [Vasc]. All throughout this section, we assume:

Hypothesis 2.1. $\mathcal{K}$ is an AEC with amalgamation.

This will mostly be assumed throughout the paper (Hypothesis 3.1 implicitly implies it by Definition 2.13.(1b)). Note however that assuming high-enough categoricity and a large cardinal axiom, it will hold on a tail (see the beginning of the proof of Corollary 4.10).

We recall the definition of the locality properties we will use. Tameness was isolated in [GV06b] from an argument of [She99]. It says that (Galois) types are determined by small restrictions of their domain. Type-shortness (which we just call shortness) says that types are determined by small restrictions of their parameter set. It was first isolated by Boney in [Bon14].

Definition 2.2 (Tameness and shortness)**.** Let κ be an infinite cardinal (usually $\kappa > \text{LS}(\mathcal{K})$).

(1) [GV06b, Definition 3.2] $\mathcal{K}$ is $(<\kappa)$-*tame* if for any $M \in \mathcal{K}$ and any *distinct* types $p, q \in \text{gS}(M)$, there exists $A \subseteq |M|$ with $|A| < \kappa$ such that $p \upharpoonright A \neq q \upharpoonright A$.
(2) [Bon14, Definition 3.1] $\mathcal{K}$ is *fully* $(<\kappa)$-*tame* if for any $M \in \mathcal{K}$, any ordinal α, and any *distinct* types $p, q \in \text{gS}^\alpha(M)$ (so p and q can have *any, possibly infinite, length*), there exists $A \subseteq |M|$ with $|A| < \kappa$ such that $p \upharpoonright A \neq q \upharpoonright A$.
(3) [Bon14, Definition 3.2] $\mathcal{K}$ is *fully* $(<\kappa)$-*short* if for any $M \in \mathcal{K}$, any ordinal α, and any *distinct* types $p, q \in \text{gS}^\alpha(M)$, there exists $I \subseteq \alpha$ with $|I| < \kappa$ such that $p^I \neq q^I$.

We say that $\mathcal{K}$ is *fully* $(<\kappa)$-*tame and short* if it is fully $(<\kappa)$-tame and fully $(<\kappa)$-short. κ-tame means $(<\kappa^+)$-tame, and similarly for κ-short. When we omit the parameter κ, we mean that there exists κ such that the property holds.

Several classes of interests are tame (and often fully tame and short), see the upcoming survey [BVb] for examples. Here we note that full tameness and shortness directly follow from a large cardinal axiom [Bon14].

Fact 2.3. Let $\mathcal{K}$ be an AEC. If $\kappa > \text{LS}(\mathcal{K})$ is a strongly compact cardinal, then $\mathcal{K}$ is fully $(<\kappa)$-tame and short.

Remark 2.4. In this paper, we sometimes assume that $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-tame (or fully $\text{LS}(\mathcal{K})$-tame and short). Restricting the tameness cardinal to be $\text{LS}(\mathcal{K})$ simplifies notation and usually does not lose any generality: if instead we know that $\mathcal{K}$ is κ-tame for $\kappa > \text{LS}(\mathcal{K})$, then we can replace $\mathcal{K}$ by $\mathcal{K}_{\geq \kappa}$.

The definition of superstability below is already implicit in [SV99] and has since then been studied in several papers, e.g. [Van06, GVV, Vasb, BVa, GV, VV]. We will use the definition from [Vasb, Definition 10.1]:

Definition 2.5. $\mathcal{K}$ is μ-*superstable* (or *superstable in* μ) if:

(1) $\mu \geq \text{LS}(\mathcal{K})$.
(2) $\mathcal{K}_\mu$ is nonempty, has joint embedding, and no maximal models.
(3) $\mathcal{K}$ is stable in μ (that is, $|\text{gS}(M)| \leq \mu$ for all $M \in \mathcal{K}_\mu$. Some authors call this "Galois-stable").
(4) μ-splitting in $\mathcal{K}$ satisfies the following locality property: for all limit ordinal $\delta < \mu^+$ and every increasing continuous sequence

$\langle M_i : i \leq \delta \rangle$ in $\mathcal{K}_\mu$ with M_{i+1} universal over M_i for all $i < \delta$, if $p \in \mathrm{gS}(M_\delta)$, then there exists $i < \delta$ so that p does not μ-split over M_i.

Remark 2.6. By the global hypothesis of amalgamation (Hypothesis 2.1), if $\mathcal{K}$ is μ-superstable, then $\mathcal{K}_{\geq \mu}$ has joint embedding.

We will use the following notation to describe classes of saturated models:

Definition 2.7. For $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}^{\lambda\text{-sat}}$ is the class of λ-saturated (according to Galois types) models in $\mathcal{K}_{\geq \lambda}$. We order $\mathcal{K}^{\lambda\text{-sat}}$ with the strong substructure relation induced from $\mathcal{K}$.

We will also make use of uniqueness of limit models (see [GVV] for history and motivation on limit models). We will use a global definition of limit models, where we permit the limit model and the base to have different sizes. This extra generality is used: in (4) in Lemma 3.6, M_i^ℓ and M_{i+1}^ℓ may have different sizes.

Definition 2.8. Let $M_0 \leq M$ be models in $\mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$.

(1) M is *universal over* M_0 if for any $N \in \mathcal{K}_{\|M_0\|}$ with $M_0 \leq N$, there exists $f : N \xrightarrow[M_0]{} M$.
(2) M is *limit over* M_0 if there exists a limit ordinal δ and a strictly increasing continuous sequence $\langle N_i : i \leq \delta \rangle$ such that:
  (a) $N_0 = M_0$.
  (b) $N_\delta = M$.
  (c) For all $i < \delta$, N_{i+1} is universal over N_i.
  We say that M is *limit* if it is limit over some $M' \leq M$.

We will use the following consequences of superstability and tameness without comments:

Fact 2.9.

(1) Assume that $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-tame and $\mathrm{LS}(\mathcal{K})$-superstable. Then:
  (a) [Vasb, Proposition 10.10] $\mathcal{K}$ is superstable in every $\mu \geq \mathrm{LS}(\mathcal{K})$. In particular, $\mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$ has no maximal models and is stable in every $\mu \geq \mathrm{LS}(\mathcal{K})$.
  (b) [VV, Corollary 6.10] For every $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$.
  (c) [VV, Corollary 6.9] If $\mu \geq \mathrm{LS}(\mathcal{K})$, $M_0, M_1, M_2 \in \mathcal{K}_\mu$ are such that both M_1 and M_2 are limit over M_0, then $M_1 \cong_{M_0} M_2$.

(2) The Shelah-Villaveces theorem [SV99][1]: If $\mathcal{K}$ has no maximal models and is categorical in a $\lambda > \text{LS}(\mathcal{K})$, then $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable.

Remark 2.10. Facts 2.9.(1b,(1c) are an improvement on the threshold cardinal in [BVa] (e.g. there it is shown that $\text{LS}(\mathcal{K})$-superstability and $\text{LS}(\mathcal{K})$-tameness implies that $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC for all $\lambda \geq \beth_{\left(2^{\text{LS}(\mathcal{K})}\right)^+}$).

Also observe that limit models are saturated:

Proposition 2.11. Assume that $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable and $\text{LS}(\mathcal{K})$-tame. Let $M \in \mathcal{K}_{>\text{LS}(\mathcal{K})}$. The following are equivalent:

(1) M is saturated.
(2) M is limit over every $M_0 \in K_{<\|M\|}$ with $M_0 \leq M$.
(3) M is limit.
(4) M is limit over some M_0 with $M_0 \in \mathcal{K}_{\|M\|}$.

Proof. (1) implies (2) implies (3) is straightforward. (3) implies (4) is [VV, Proposition 3.1]. (4) implies (1) follows from uniqueness of limit models (Fact 2.9). $\square$

We obtain uniqueness of limit models in a generalized sense:

Proposition 2.12. Assume that $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable and $\text{LS}(\mathcal{K})$-tame. If M_1, M_2 are limit over M_0 and $\|M_1\| = \|M_2\|$, then $M_1 \cong_{M_0} M_2$.

Proof. Let $\mu := \|M_1\| = \|M_2\|$. If $\|M_0\| = \mu$, then $M_1 \cong_{M_0} M_2$ by Fact 2.9. If $\|M_0\| < \mu$, note that by Proposition 2.11, M_1 and M_2 are both saturated, hence by uniqueness of saturated models $M_1 \cong_{M_0} M_2$. $\square$

We will work use a global forking-like independence notion that has the basic properties of forking in a superstable first-order theory. This is a stronger notion than Shelah's good frame [She09, Chapter II] because in good frames forking is only defined for types of length one. We invite the reader to consult [Vasb] for more explanations and motivations on global and local independence notions.

Definition 2.13 (Definition 8.1 in [Vasb])**.** $\mathfrak{i} = (\mathcal{K}, \perp)$ is an *almost fully good independence relation* if:

(1) $\mathcal{K}$ is an AEC satisfying the following structural assumptions:

[1]In the context of AECs with amalgamation, this is discussed in [GV, Theorem 6.3].

(a) $\mathcal{K}_{<\mathrm{LS}(K)} = \emptyset$ and $\mathcal{K} \neq \emptyset$.
(b) $\mathcal{K}$ has amalgamation, joint embedding, and no maximal models.
(c) $\mathcal{K}$ is stable in all cardinals.

(2) (a) $\mathfrak{i}$ is a $(<\infty, \geq \mathrm{LS}(K))$-independence relation (see [Vasb, Definition 3.6]). That is, $\mathop{\smile\!\!\!\!\!\!|}$ is a relation on quadruples (M, A, B, N) with $M \leq N$ and $A, B \subseteq |N|$ satisfying invariance, monotonicity, and normality. We write $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} B$ instead of $\mathop{\smile\!\!\!\!\!\!|}(M, A, B, N)$, and we also say $\mathrm{gtp}(\bar{a}/B; N)$ does not fork over M for $\mathrm{ran}(\bar{a}) \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} B$.

(b) $\mathfrak{i}$ has base monotonicity, disjointness ($A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} B$ implies $A \cap B \subseteq |M|$), symmetry, uniqueness, and the local character properties:
   (i) If $p \in \mathrm{gS}^\alpha(M)$, there exists $M_0 \leq M$ with $\|M_0\| \leq |\alpha| + \mathrm{LS}(K)$ such that p does not fork over M_0.
   (ii) If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in \mathrm{gS}^\alpha(M_\delta)$ and $\mathrm{cf}(\delta) > \alpha$, then there exists $i < \delta$ such that p does not fork over M_i.

(c) $\mathfrak{i}$ has the following weakening of the extension property: for any $M \leq N$ and any $p \in \mathrm{gS}^\alpha(M)$, there exists $q \in \mathrm{gS}^\alpha(N)$ that extends p and does not fork over M provided at least one of the following conditions hold:
   (i) M is saturated.
   (ii) $\|M\| = \mathrm{LS}(\mathcal{K})$.
   (iii) $\alpha \leq \mathrm{LS}(\mathcal{K})$.

(d) $\mathfrak{i}$ has the left and right $(\leq \mathrm{LS}(K))$-witness properties: $A \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} B$ if and only if for all $A_0 \subseteq A$ and $B_0 \subseteq B$ with $|A_0| + |B_0| \leq \mathrm{LS}(K)$, we have that $A_0 \overset{N}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} B_0$.

(e) $\mathfrak{i}$ has full model continuity: if for $\ell < 4$, all limit ordinals δ, $\langle M_i^\ell : i \leq \delta \rangle$ are increasing continuous such that for all $i < \delta$, $M_i^0 \leq M_i^\ell \leq M_i^3$ for $\ell = 1, 2$ and $M_i^1 \overset{M_i^3}{\underset{M_i^0}{\mathop{\smile\!\!\!\!\!\!|}}} M_i^2$, then $M_\delta^1 \overset{M_\delta^3}{\underset{M_\delta^0}{\mathop{\smile\!\!\!\!\!\!|}}} M_\delta^2$.

We say that K is *almost fully good* if there exists $\mathop{\smile}\limits^{\vert}$ such that $(K, \mathop{\smile}\limits^{\vert})$ is almost fully good.

Remark 2.14. We call such relations "almost" fully good because we do not assume the full extension property, only the weakening above. The problem is that it is not known how to get the full extension property in the context of this paper (see the discussion in Section 15 of [Vasb]).

Remark 2.15. Let $\mathcal{K}$ be almost fully good. Then:

(1) $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-tame (this follows from local character and uniqueness, see [GK, p. 15] or [Vasb, Proposition 4.6.(1)]).
(2) $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable (because non-forking implies non-splitting, see the proof of [VV, Fact 4.8.(2)]).

We will use this freely.

In [Vasb, Theorem 15.6], it was shown that fully tame and short superstable AECs with amalgamation are (on a tail) almost fully good. The threshold cardinals were improved in Appendix A of [Vase].

Fact 2.16 (Corollary A.16 in [Vase])**.** Assume that $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable and fully $\text{LS}(\mathcal{K})$-tame and short. Then $\mathcal{K}^{\lambda\text{-sat}}$ is almost fully good, where $\lambda := \left(2^{\text{LS}(\mathcal{K})}\right)^{+5}$.

We will make use of Shelah's uniqueness triples [She09, Definition II.5.3]. In our framework, they have an easier definition:

Definition 2.17. Let $\mathfrak{i} = (\mathcal{K}, \mathop{\smile}\limits^{\vert})$ be an almost fully good independence relation. (a, M, N) is a *domination triple* if $M \leq N$, $a \in |N| \backslash |M|$, and for any $N' \geq N$ and any $B \subseteq |N'|$, if $a \mathop{\smile}\limits^{\vert}{}_{M}^{N'} B$, then $N \mathop{\smile}\limits^{\vert}{}_{M}^{N'} B$.

Remark 2.18. By [Vasb, Lemma 11.7], uniqueness triples and domination triples coincide in our framework.

In [She09, Definition III.1.1], Shelah defines a good λ-frame to be *weakly successful* if it has the existence property for uniqueness triples. We give an analogous definition for domination triples:

Definition 2.19. Let $\mathfrak{i} = (\mathcal{K}, \mathop{\smile}\limits^{\vert})$ be an almost fully good independence relation and let $\lambda \geq \text{LS}(\mathcal{K})$. We say that $\mathfrak{i}$ *has the λ-existence property for domination triples* if for every $M \in \mathcal{K}_\lambda$ and every nonalgebraic $p \in \text{gS}(M)$, there exists a domination triple (a, M, N) so that $p = \text{gtp}(a/M; N)$.

The existence property for domination triples is a reasonable hypothesis: if the independence relation does not have it, we can restrict ourselves to a subclass of saturated models.

Fact 2.20 (Lemma 11.12 in [Vasb])**.** Let $\mathfrak{i} = (\mathcal{K}, \perp)$ be an almost fully good independence relation. For every $\lambda > \text{LS}(\mathcal{K})$, $\mathfrak{i} \upharpoonright \mathcal{K}^{\lambda\text{-sat}}$ (the restriction of $\mathfrak{i}$ to λ-saturated models) has the λ-existence property for domination triples.

Finally, we recall the definition of prime models in the framework of abstract elementary classes. This does not need amalgamation and is due to Shelah [She09, Section III.3]. While it is possible to define what it means for a model to be prime over an arbitrary set (see [Vase, Definition 5.1]), here we focus on primes over sets of the form $M \cup \{a\}$. The technical point in the definition is that since we are not working inside a monster model, how $M \cup \{a\}$ is embedded matters. Thus we use a formulation in terms of Galois types: instead of saying that N is prime over $M \cup \{a\}$, we say that (a, M, N) is a *prime triple*:

Definition 2.21. Let $\mathcal{K}$ be an AEC (not necessarily with amalgamation).

(1) A *prime triple* is (a, M, N) such that $M \leq N$, $a \in |N| \backslash |M|$ and for every $N' \in \mathcal{K}$, $a' \in |N'|$ such that $\text{gtp}(a/M; N) = \text{gtp}(a'/M; N')$, there exists $f : N \underset{M}{\longrightarrow} N'$ so that $f(a) = a'$.
(2) We say that $\mathcal{K}$ *has primes* if for $M \in \mathcal{K}$ and every nonalgebraic $p \in \text{gS}(M)$, there exists a prime triple representing p, i.e. there exists a prime triple (a, M, N) so that $p = \text{gtp}(a/M; N)$.
(3) We define localizations such as "$\mathcal{K}_\lambda$ has primes" or "$\mathcal{K}_\lambda^{\lambda\text{-sat}}$ has primes" in the natural way (in the second case, we ask that all models in the definition be saturated).

3. Building primes over saturated models

We show that in almost fully good AECs, there exists primes among the saturated models (see Definition 2.21). For models of successor size, this is shown in [She09, Claim III.4.9] (or in [Jar] with slightly weaker hypotheses). We generalizes Shelah's proof to limit sizes here. This is the core of the paper. Throughout this section, we assume:

Hypothesis 3.1.

(1) $\mathcal{K}$ is an almost fully good AEC, as witnessed by $\mathfrak{i} = (\mathcal{K}, \perp)$.
(2) $\mathcal{K}$ is categorical in $\text{LS}(\mathcal{K})$.

(3) $\mathfrak{i}$ has the $\text{LS}(\mathcal{K})$-existence property for domination triples (see Definition 2.19).

This is reasonable by Facts 2.16 and 2.20 (in fact, it is plausible that the $\text{LS}(\mathcal{K})$-existence property for domination triples directly follows from the other two hypotheses, but we do not know how to prove it in general).

Remark 3.2. When we refer to forking, we mean forking in the sense of $\perp$. Strictly speaking, this depends on the choice of the witness $\mathfrak{i}$ to $\mathcal{K}$ being almost fully good. However by [BGKV, Corollary 5.19] and the $(\leq \text{LS}(\mathcal{K}))$-witness property (Definition 2.13.(2d)), there is a unique such witness.

Remark 3.3. In Section III of [She09] Shelah works with a more local hypothesis: the existence of a successful good μ-frame ([She09, Definition III.1.1]). It is implicit in Shelah's work (and is made precise in [Vasb, Theorem 12.16]) that this implies that $\mathcal{K}$ is almost fully good (except perhaps for full model continuity) for models of size μ. Shelah shows that this implies that $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ has primes when $\lambda := \mu^+$. Since here we want to show the same result for a limit $\lambda > \mu$, we will use that the independence relation is well-behaved for models of sizes in $[\lambda, \mu)$, hence not only in one size. We could still have made our hypothesis more local (i.e. only requiring that the independence relation behaves well for types of length less than μ and models in $\mathcal{K}_{<\mu}$) but for notational simplicity we do not adopt this approach.

We start by showing that domination triples are closed under unions. This is a key consequence of full model continuity.

Lemma 3.4. Let $\langle M_i : i < \delta \rangle, \langle N_i : i < \delta \rangle$ be increasing and assume that (a, M_i, N_i) are domination triples for all $i < \delta$. Then $(a, \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i)$ is a domination triple.

Proof. For ease of notation, we work inside a monster model $\mathfrak{C}$ and write $A \underset{M}{\perp} B$ for $A \overset{\mathfrak{C}}{\underset{M}{\perp}} B$. Let $M_\delta := \bigcup_{i<\delta} M_i$, $N_\delta := \bigcup_{i<\delta} N_i$. Assume that $a \underset{M_\delta}{\perp} N$ with $M_\delta \leq N$ (by extension for types of length one, we can assume this without loss of generality). By local character, for all sufficiently large $i < \delta$, $a \underset{M_i}{\perp} N$. By definition of domination triples, $N_i \underset{M_i}{\perp} N$. By full model continuity, $N_\delta \underset{M_\delta}{\perp} N$. $\square$

The conclusion of the next fact is a key step in Shelah's construction of a successor frame in [She09, Chapter II]. The fact says that if $M^0 \leq M^1$ are of the same successor size, then their resolutions satisfy a natural independence property on a club. In the framework of this paper, this is due to Jarden [Jar16]. To give the reader a feeling for the difficulties encountered, we first explain in the proof how the (straightforward) first-order argument fails to generalize.

Fact 3.5. For every $\mu \geq \text{LS}(\mathcal{K})$, for every $M^0 \leq M^1$ in $\mathcal{K}_{\mu^+}$, if $\langle M_i^\ell : i < \mu^+ \rangle$ are increasing continuous resolutions of M^ℓ and all are limit models[2] in $\mathcal{K}_\mu$, $\ell = 0, 1$, then the set of $i < \mu^+$ so that $M^0 \underset{M_i^0}{\overset{M^1}{\downarrow}} M_i^1$ is a club.

Proof. Let us first see how the first-order argument would go. By local character, for every $i < \mu^+$, there exists $j_i < \mu^+$ such that $M_i^1 \underset{M_{j_i}^0}{\overset{M^1}{\downarrow}} M^0$. Pick $i^* < \mu^+$ such that $j_i < i^*$ for every $i < i^*$. Using symmetry and the finite character of (first-order) forking, it is then straightforward to see that $M^0 \underset{M_{i^*}^0}{\overset{M^1}{\downarrow}} M_{i^*}^1$. Thus i^* has the desired property, and the argument shows we can find a closed unbounded subset of such i^*. Here however we do not have the finite character (only the $\text{LS}(\mathcal{K})$-witness property, see Definition 2.13.(2d)).

Full model continuity (Definition 2.13.(2e)) seems to be the replacement we are looking for, but in the argument above we do *not* have that $M_{j_i}^0 \leq M_i^1$ so cannot use it! It is open whether the appropriate generalization of full model continuity holds here.

On to the actual proof. Assume without loss of generality that $\mu = \text{LS}(\mathcal{K})$ (if $\mu > \text{LS}(\mathcal{K})$, replace $\mathcal{K}$ with $\mathcal{K}^{\mu\text{-sat}}$). We now want to apply [Jar16, Theorem 7.8]. The conclusion there is that for any model $M^0, M^1 \in \mathcal{K}_{\mu^+}$, $M^0 \leq_{\mu^+}^{\text{NF}} M^1$ if and only if $M^0 \leq M^1$, where $M^0 \leq_{\mu^+}^{\text{NF}} M^1$ is defined to hold if and only if there exists increasing continuous resolutions of M^0 and M^1 as here.

Let us check that the hypotheses of Jarden's theorem are satisfied. First, amalgamation in μ^+ and μ-tameness hold (by definition of an almost fully good AEC and Remark 2.15). Second, [Jar16, Hypothesis 6.5] holds: $\mathcal{K}$ is categorical in $\text{LS}(\mathcal{K})$, has a semi-good $\text{LS}(\mathcal{K})$-frame (this is weaker than the existence of an almost fully good independence

[2] And hence if $\mu > \text{LS}(\mathcal{K})$ are saturated (Proposition 2.11).

relation, in fact the frame will be good), satisfies the conjugation property (by [She09, III.1.21] which tells us that conjugation holds in any good $\mathrm{LS}(\mathcal{K})$-frame categorical in $\mathrm{LS}(\mathcal{K})$), and has the existence property for domination triples by Hypothesis 3.1. Therefore the hypotheses of Jarden's theorem are satisfied so its conclusion holds. $\square$

We can now generalize the proof of [She09, Claim III.4.3] to limit cardinals. Roughly, it tells us that every nonalgebraic type over a saturated model has a resolution into domination triples.

Lemma 3.6. Let $\lambda > \mathrm{LS}(\mathcal{K})$ and let $\delta := \mathrm{cf}(\lambda)$. Let $M^0 \in \mathcal{K}_\lambda$ be saturated and let $p \in \mathrm{gS}(M_0)$ be nonalgebraic. Then there exists a saturated $M^1 \in \mathcal{K}_\lambda$, an element $a \in |M^1|$, and increasing continuous resolutions $\langle M_i^\ell : i \leq \delta \rangle$ of M^ℓ, $\ell = 0, 1$ such that for all $i < \delta$:

(1) $p = \mathrm{gtp}(a/M^0; M^1)$.
(2) $a \in |M_0^1|$.
(3) p does not fork over M_0^0.
(4) For $\ell = 0, 1$, $M_i^\ell \in \mathcal{K}_{[\mathrm{LS}(\mathcal{K}), \lambda)}$ and M_{i+1}^ℓ is limit over M_i^ℓ.
(5) (a, M_i^0, M_i^1) is a domination triple.

Proof. For $\ell = 0, 1$, we first choose by induction $\langle N_i^\ell : i \leq \lambda \rangle$ increasing continuous and an element a that will satisfy some weaker requirements. In the end, we will rename the N_i^ℓ's to get the desired M_i^ℓ's. We require that for all $i < \lambda$:

(1) $N_0^0 \leq M^0$ and p does not fork over M_0^0.
(2) $a \in |N_0^1|$.
(3) For $\ell = 0, 1$, $N_i^\ell \in \mathcal{K}_{|i| + \mathrm{LS}(\mathcal{K})}$ and $N_i^0 \leq N_i^1$.
(4) $\mathrm{gtp}(a/N_i^0; N_i^1)$ does not fork over N_0^0.
(5) If i is odd, and $\ell = 0, 1$, then N_{i+1}^ℓ is limit over N_i^ℓ.
(6) If i is even and (a, N_i^0, N_i^1) is *not* a domination triple, then $N_i^1 \overset{N_{i+1}^1}{\underset{N_i^0}{\not\downarrow}} N_{i+1}^0$.

<u>This is possible</u>. First pick $N_0^0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ such that $N_0^0 \leq M^0$ and p does not fork over M_0^0. This is possible by local character. Now pick $N_0^1 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ such that $N_0^0 \leq N_0^1$ and there is $a \in |N_0^1|$ with $\mathrm{gtp}(a/N_0^0; N_0^1) = p \restriction N_0^0$. This takes care of the case $i = 0$. For i limit, take unions. Now assume that $i = j + 1$ is a successor. We consider several cases:

- If j is even and (a, N_j^0, N_j^1) is *not* a domination triple, then there must exist witnesses $N_{j+1}^0, N_{j+1}^1 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K}) + |j|}$ such that $N_j^0 \leq$

N^0_{j+1}, $N^0_{j+1} \leq N^1_{j+1}$, $N^1_j \leq N^1_{j+1}$, $a \underset{N^0_j}{\overset{N^1_{j+1}}{\perp}} N^0_{j+1}$ but $N^1_j \underset{N^0_j}{\overset{N^1_{j+1}}{\not\perp}} N^0_{j+1}$.
This satisfies all the conditions (we know that $\text{gtp}(a/N^0_j; N^1_j)$ does not fork over N^0_0, so by transitivity also $\text{gtp}(a/N^0_{j+1}; N^1_{j+1})$ does not fork over N^0_0).
- If j is even and (a, N^0_j, N^1_j) *is* a domination triple, take $N^\ell_{j+1} := N^\ell_j$, for $\ell = 0, 1$.
- If j is odd, pick $N^0_i \in \mathcal{K}_{\text{LS}(\mathcal{K})+|j|}$ limit over N^0_j and N^1_i limit over N^0_i and N^1_j so that $\text{gtp}(a/N^0_i; N^1_i)$ does not fork over N^0_0. This is possible by the extension property for types of length one.

This is enough. By the odd stages of the construction, and basic properties of universality, for all $i < \lambda$, $\ell = 0, 1$, N^ℓ_{i+2} is universal over N^ℓ_i. Thus for $\ell = 0, 1$ and $i \leq \lambda$ a limit ordinal, N^ℓ_i is limit. In particular, by Proposition 2.11, N^ℓ_λ is saturated. By uniqueness of saturated models, $N^0_\lambda \cong_{N^0_0} M^0$. By uniqueness of the nonforking extension, without loss of generality $N^0_\lambda = M^0$. Now let C be the set of *limit* $i < \lambda$ such that (a, N^0_i, N^1_i) is a domination triple. We claim that C is a club:

- C is closed by Lemma 3.4.
- C is unbounded: given $\alpha < \lambda$, let $\mu := |\alpha| + \text{LS}(\mathcal{K})$. Let E_μ be the set of $i < \mu^+$ such that i is limit and $N^0_{\mu^+} \underset{N^0_i}{\overset{N^1_{\mu^+}}{\perp}} N^1_i$. By Fact 3.5, E_μ is a club. The even stages of the construction imply that for $i \in E_\mu$, (a, N^0_i, N^1_i) is a domination triple. In other words, $E_\mu \subseteq C$. Now pick $\beta \in E_\mu \backslash (\alpha + 1)$. We have that $\alpha < \beta$ and $\beta \in E_\mu \subseteq C$. This completes the proof that C is unbounded.

Let $\langle \alpha_i : i < \delta \rangle$ (recall that $\delta = \text{cf}(\lambda)$) be a cofinal strictly increasing continuous sequence of elements of C. For $i < \delta$, $\ell = 0, 1$, let $M^\ell_i := N^\ell_{\alpha_i}$. This works: Clauses (1), (2), (3) are straightforward to check using monotonicity of forking. Clause (5) holds by definition of C. As for (4), we have observed above that for $\ell = 0, 1$, for all $i < \lambda$, N^ℓ_{i+2} is universal over N^ℓ_i. Hence for all limit ordinals $i < j < \lambda$, N^ℓ_j is limit over N^ℓ_i. In particular because C contains only limit ordinals, for all $i < \delta$, $N^\ell_{\alpha_{i+1}}$ is limit over $N^\ell_{\alpha_i}$, as desired. $\square$

In [She09, Claim III.4.9], Shelah observes that triples as in the conclusion of Lemma 3.6 are prime triples. For the convenience of the reader, we include the proof here. We will use the following fact which follows from the uniqueness property of forking and some renaming.

Fact 3.7 (Lemma 12.6 in [Vasb])**.** For $\ell < 2$, $i < 4$, let $M_i^\ell \in \mathcal{K}$ be such that for $i = 1, 2$, $M_0^\ell \leq M_i^\ell \leq M_3^\ell$.

If $M_1^\ell \overset{M_3^\ell}{\underset{M_0^\ell}{\perp}} M_2^\ell$ for $\ell < 2$, $f_i : M_i^1 \cong M_i^2$ for $i = 0, 1, 2$, and $f_0 \subseteq f_1$, $f_0 \subseteq f_2$, then $f_1 \cup f_2$ can be extended to $f_3 : M_3^1 \to M_4^2$, for some M_4^2 with $M_3^2 \leq M_4^2$.

We can now give a proof of Theorem 0.1 from the abstract.

Theorem 3.8. For any $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ has primes (see Definition 2.21).

Proof. Let $M \in \mathcal{K}_\lambda$ be saturated and let $p \in \mathrm{gS}(M)$ be nonalgebraic. We must find a triple (a, M, N) such that $M \leq N$, $N \in \mathcal{K}_\lambda$ is saturated, $p = \mathrm{gtp}(a/M; N)$, and (a, M, N) is a prime triple among the saturated models of size λ.

Set $M^0 := M$ and let $\delta := \mathrm{cf}(\lambda)$. Let M^1, a, $\langle M_i^\ell : i \leq \delta \rangle$ be as described by the statement of Lemma 3.6. Recall (this is key) that $\|M_i^\ell\| < \lambda$ for any $i < \delta$. We show that (a, M^0, M^1) is as desired. By assumption, $M^0 \leq M^1$, $p = \mathrm{gtp}(a/M^0; M^1)$, and $M^1 \in \mathcal{K}_\lambda$ is saturated. It remains to show that (a, M^0, M^1) is a prime triple in $\mathcal{K}_\lambda^{\lambda\text{-sat}}$. Let $M' \in \mathcal{K}_\lambda^{\lambda\text{-sat}}$, $a' \in |M'|$ be given such that $\mathrm{gtp}(a'/M^0; M') = \mathrm{gtp}(a/M^0; M^1)$. We want to build $f : M^1 \underset{M^0}{\longrightarrow} M'$ so that $f(a) = a'$.

We build by induction an increasing continuous chain of embeddings $\langle f_i : i \leq \delta \rangle$ so that for all $i \leq \delta$:

(1) $f_i : M_i^1 \underset{M_i^0}{\longrightarrow} M'$.
(2) $f_i(a) = a'$.

This is enough since then $f := f_\delta$ is as required. This is possible: for $i = 0$, we use that M' is saturated, hence realizes $p \upharpoonright M_0^0$, so there exists $f_0 : M_0^1 \underset{M_0^0}{\longrightarrow} M'$ witnessing it, i.e. $f_0(a) = a'$. At limits, we take unions. For $i = j + 1$ successor, let $\mu := \|M_j^1\| + \|M_i^0\|$. Pick $N_j \leq M'$ with $N_j \in \mathcal{K}_\mu$ and N_j containing both $f_j[M_j^1]$ and M_i^0.

By assumption, p does not fork over M_0^0 and by assumption $p = \mathrm{gtp}(a'/M^0; M')$, so by monotonicity of forking, $a' \overset{N_j}{\underset{M_j^0}{\perp}} M_{j+1}^0$. We know that (a, M_j^0, M_j^1) is a domination triple, hence applying f_j and using invariance, $(a', M_j^0, f_j[M_j^1])$ is a domination triple. Therefore $f_j[M_j^1] \underset{M_j^0}{\overset{N_j}{\perp}} M_{j+1}^0$. By a similar argument, we also have $M_j^1 \underset{M_j^0}{\overset{M_{j+1}^1}{\perp}} M_{j+1}^0$. By Fact 3.7, the map $f_j \cup \mathrm{id}_{M_{j+1}^0}$ can be extended to a K-embedding $g : M_{j+1}^\alpha \to N_j'$ for some $N_j' \geq N_j$ of size μ. Since $\mu < \lambda$ and M' is saturated, there exists $h : N_j' \xrightarrow[N_j]{} M'$. Let $f_{j+1} := h \circ g$. □

4. Primes in fully tame and short AECs

Using Theorem 3.8, we obtain that certain prime models can be built in any superstable fully tame and short AEC with amalgamation:

Corollary 4.1. Let $\mathcal{K}$ be a $\mathrm{LS}(\mathcal{K})$-superstable fully $\mathrm{LS}(\mathcal{K})$-tame and short AEC with amalgamation. For any $\lambda \geq \left(2^{\mathrm{LS}(\mathcal{K})}\right)^{+7}$, $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ has primes.

Proof. By Facts 2.16 and 2.20, Hypothesis 3.1 holds for the AEC $\mathcal{K}^{\mu\text{-sat}}$ (it is an AEC by Fact 2.9), where $\mu := \left(2^{\mathrm{LS}(\mathcal{K})}\right)^{+6}$. Now apply Theorem 3.8. □

We do not know if prime models can be built assuming just tameness, namely:

Question 4.2. Can "fully tame and short" be replaced by only "tame"?

Several variations on Corollary 4.1 can be given using a large cardinal axiom instead of the locality hypotheses (Fact 2.3) or categoricity instead of superstability (see Fact 2.9.(2)). In particular, we can prove (2) implies (1) of Corollary 0.2 in the abstract. We will use the notation from [Bal09, Chapter 14]:

Notation 4.3. For an infinite cardinal λ, $h(\lambda) := \beth_{\left(2^\lambda\right)^+}$. For a fixed AEC $\mathcal{K}$, we write $H_1 := h(\mathrm{LS}(\mathcal{K}))$, $H_2 := h(H_1) = h(h(\mathrm{LS}(\mathcal{K})))$.

Corollary 4.4. Let $\mathcal{K}$ be a fully $\mathrm{LS}(\mathcal{K})$-tame and short AEC with amalgamation. If $\mathcal{K}$ is categorical in *all* $\lambda \geq H_1$, then $\mathcal{K}_{\geq H_1}$ has primes.

Proof. By partitioning the AEC into disjoint classes that each have joint embedding and working within the class that has arbitrarily large models, we can assume without loss of generality that $\mathcal{K}$ has no maximal models. By the Shelah-Villaveces theorem (Fact 2.9.(2)), $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable. By Fact 2.9, any model in $\mathcal{K}_{\geq H_1}$ is saturated, so the result follows from Corollary 4.1. $\square$

We briefly discuss (1) implies (2) of Corollary 0.2. By [Vase, Theorem 0.2], this holds when H_1 is replaced by H_2^+:

Fact 4.5. Let $\mathcal{K}$ be fully $\text{LS}(\mathcal{K})$-tame and short AEC with amalgamation and primes. If $\mathcal{K}$ is categorical in *some* $\lambda \geq H_2^+$, then $\mathcal{K}$ is categorical in *all* $\lambda' \geq H_2^+$.

Remark 4.6. "fully $\text{LS}(\mathcal{K})$-tame and short" can be replaced with just "$\text{LS}(\mathcal{K})$-tame" [Vasd].

Using recent results, we can improve the threshold cardinal:

Fact 4.7. Let $\mathcal{K}$ be an $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and arbitrarily large models.

(1) [Vasa, Theorem 3.3][3] If δ is a limit ordinal that is divisible by $\left(2^{\text{LS}(\mathcal{K})}\right)^+$, then $\mathcal{K}$ is categorical in $\beth_\delta$. If $\mathcal{K}$ is categorical in a $\lambda > \text{LS}(\mathcal{K})$, then $\mathcal{K}$ is categorical in all cardinals of the form $\beth_\delta$, where $\left(2^{\text{LS}(\mathcal{K})}\right)^+$ divides δ.
(2) [GV06a] If $\mathcal{K}$ is categorical in a successor $\lambda > \text{LS}(\mathcal{K})^+$, then $\mathcal{K}$ is categorical in all $\lambda' \geq \lambda$.
(3) [Vasa, Corollary 7.4] If $\text{LS}(\mathcal{K}) < \lambda_0 < \lambda_1$ are such that $\mathcal{K}$ is categorical in both λ_0 and λ_1 and λ_1 is a successor, then $\mathcal{K}$ is categorical in all $\lambda \in [\lambda_0, \lambda_1]$.

We obtain the following more general version of Corollary 0.2:

Corollary 4.8. Let $\mathcal{K}$ be a fully $\text{LS}(\mathcal{K})$-tame and short AEC with amalgamation and arbitrarily large models. Assume that $\mathcal{K}$ is categorical in *some* $\lambda_0 > \text{LS}(\mathcal{K})$. The following are equivalent:

(1) $\mathcal{K}$ is categorical in some successor $\lambda_1 > \text{LS}(\mathcal{K})^+$.
(2) $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda_0, H_1)$.
(3) $\mathcal{K}_{\geq H_1}$ has primes.
(4) There exists μ such that $\mathcal{K}_{\geq \mu}$ has primes.

[3] The version for classes of models axiomatized by an $\mathbb{L}_{\kappa,\omega}$ theory, κ strongly compact, appears in [MS90]. In tame AECs, a little bit more work has to be done to show that the model in the categoricity cardinal is saturated (one uses Fact 2.9).

Proof. By Fact 4.7.(1), $\mathcal{K}$ is categorical in a proper class of cardinals, and in particular in H_1.

Now assume (1). By Fact 4.7.(2), $\mathcal{K}$ is categorical in all $\lambda' \geq \lambda_1$. By Fact 4.7.(3) (with λ_0 there standing for $\min(\lambda_0, H_1)$ here), $\mathcal{K}$ is categorical in any $\lambda' \geq \min(\lambda_0, H_1)$. So (2) holds. Assume (2). Then (3) holds by Corollary 4.4. If (3) holds, then (4) trivially holds. Assume (4). Since $\mathcal{K}$ is categorical in a proper class of cardinals, we can apply Fact 4.5 to $\mathcal{K}_{\geq \mu}$ and obtain that $\mathcal{K}$ is categorical on a tail of cardinals. In particular, (1) holds. $\square$

Remark 4.9. In condition (3), we can replace H_1 by:

$$\min\left(H_1, \lambda_0 + \left(2^{\mathrm{LS}(\mathcal{K})}\right)^{+7}\right)$$

Moreover we can also prove minor improvements such as, instead of (2), "there exists $\chi < H_1$ such that $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda_0, \chi)$". Finally if $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$, we can accept $\lambda_1 = \mathrm{LS}(\mathcal{K})^+$ and add to the list of conditions that $\mathcal{K}$ is categorical in every $\mu \geq \mathrm{LS}(\mathcal{K})$ (see [Vasa, Remark 7.5]).

We end by stating a version of Corollary 4.8 using large cardinals. This shows that, assuming the existence of a proper class of strongly compact cardinals, Shelah's eventual categoricity conjecture for AECs [She09, Conjecture N.4.2] is *equivalent* to the statement that every AEC categorical in a high-enough cardinals eventually has primes.

Corollary 4.10. Let $\mathcal{K}$ be an AEC and let $\kappa > \mathrm{LS}(\mathcal{K})$ be a strongly compact cardinal. Assume that $\mathcal{K}$ is categorical in *some* $\lambda \geq h(\kappa)$. The following are equivalent:

(1) $\mathcal{K}_{\geq h(\kappa)}$ has primes.
(2) K is categorical in *all* $\lambda' \geq h(\kappa)$.

Proof. By [MS90, Proposition 1.13] (this is a result about classes of models of $\mathbb{L}_{\kappa,\omega}$, but one can adapt the proofs to AECs as pointed out in [Bon14, Section 7]), $\mathcal{K}_{\geq \kappa}$ has amalgamation. By Fact 2.3, $\mathcal{K}$ is fully $(< \kappa)$-tame and short. Now apply Corollary 4.8 to $\mathcal{K}_{\geq \kappa}$. $\square$

Even for (2) implies (1), we do not know whether the large cardinal assumption is necessary. Grossberg [Gro02, Conjecture 2.3] has conjectured that an AEC categorical in a high-enough cardinal should have amalgamation on a tail, but even the following weakening is open:

Question 4.11. Let $\mathcal{K}$ be an AEC categorical in every $\mu \geq \mathrm{LS}(\mathcal{K})$. Does there exists a cardinal λ such that $\mathcal{K}_{\geq \lambda}$ has amalgamation?

The answer is positive when $2^\mu < 2^{\mu^+}$ for every cardinal μ (by [She01, Claim 1.10.(0]).

One can similarly ask whether full tameness and shortness follows from categoricity: Shelah has shown [She99, Main Claim II.2.3] that any AEC with amalgamation that is categorical in every cardinal is tame, but we do not know if this can be strengthened to "fully tame and short":

Question 4.12. Let $\mathcal{K}$ be an AEC with amalgamation. If $\mathcal{K}$ is categorical in every $\mu \geq \mathrm{LS}(\mathcal{K})$, is $\mathcal{K}$ fully tame and short?

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