ON PRIME MODELS IN TOTALLY CATEGORICAL ABSTRACT ELEMENTARY CLASSES

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Abstract. We show:

Theorem 0.1. Let $\mathcal{K}$ be a fully $\text{LS}(\mathcal{K})$-tame and short abstract elementary class (AEC) with amalgamation. Write $H_1 := \beth_{\left(2^{\text{LS}(\mathcal{K})}\right)^+}$ and assume that $\mathcal{K}$ is categorical in *some* $\lambda \geq H_1$. The following are equivalent:

(1) $\mathcal{K}_{\geq H_1}$ has primes over sets of the form $M \cup \{a\}$.
(2) $\mathcal{K}$ is categorical in *all* $\lambda' \geq H_1$.

Note that (1) implies (2) appears in an earlier paper. Here we prove (2) implies (1), generalizing an argument of Shelah who proved the existence of primes at successor cardinals. Assuming a large cardinal axiom, we deduce an equivalence between Shelah's eventual categoricity conjecture and the statement that every AEC categorical in a proper class of cardinals eventually has prime models over sets of the form $M \cup \{a\}$.

Corollary 0.2. Assume there exists a proper class of almost strongly compact cardinals. Let $\mathcal{K}$ be an AEC categorical in a proper class of cardinals. The following are equivalent:

(1) There exists λ_0 such that $\mathcal{K}_{\geq \lambda_0}$ has primes over sets of the form $M \cup \{a\}$.
(2) There exists λ_1 such that $\mathcal{K}$ is categorical in all $\lambda \geq \lambda_1$.

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1. Introduction

Shelah's eventual categoricity conjecture for abstract elementary classes (AECs) [She09, Conjecture N.4.2] is the statement that there exists a function $\mu \mapsto \lambda_\mu$ so that an AEC $\mathcal{K}$ categorical in *some* $\lambda \geq \lambda_{\text{LS}(\mathcal{K})}$ is categorical in *every* $\lambda \geq \lambda_{\text{LS}(\mathcal{K})}$. By a similar argument as for the existence of Hanf numbers [Han60] (see [Bal09, Conclusion 15.13]), this is equivalent to:

Conjecture 1.1 (Shelah's eventual categoricity conjecture)**.** If an AEC is categorical in a proper class of cardinals, then it is categorical on a tail of cardinals.

The conjecture has been the major driving force in developing a classification theory for AECs[1]. In this paper, we discuss the following closely related question:

Question 1.2. What is a natural property P of AECs such that an AEC $\mathcal{K}$ categorical in a proper class of cardinals has P if and only if $\mathcal{K}$ is categorical on a tail of cardinals?

In other words, what is a property P so that Shelah's eventual categoricity conjecture holds in AECs satisfying P and conversely any AEC categorical on a tail satisfies P? As a non-example, by Shelah's generalization of Morley's categoricity theorem [She74] elementary classes satisfy Shelah's eventual categoricity conjecture (in fact much more), but we cannot take P above to be "$\mathcal{K}$ is an elementary class", since not all totally categorical AECs are elementary.

On the other hand, consider another special case of Shelah's eventual categoricity conjecture: assuming a large cardinal axiom (the existence of a proper class of strongly compact cardinals), it is known from the work of Makkai-Shelah [MS90], Grossberg-VanDieren [GV06c, GV06a], and Boney [Bon14b] that if an AEC is categorical in a proper class of *successor* cardinals, then it is categorical on a tail of cardinals [Bon14b, Theorem 7.5]. So assuming large cardinals, we could take P to be "$\mathcal{K}$ is categorical in a proper class of successor cardinals". However

[1]For a more complete history of the conjecture, see the introduction of [Vase].

we are looking for a more natural hypothesis that does not mention categoricity.

Recently, we proved (in ZFC) that Shelah's eventual categoricity conjecture holds in universal classes with amalgamation [Vase] and showed more generally[2] [Vase, Theorem 5.18] that it holds in AECs which have amalgamation, are fully tame and short (a locality property for Galois types generalizing Grossberg and VanDieren's tameness [GV06b], see [Bon14b, Definition 3.3]), and have primes over sets of the form $M \cup \{a\}$ (we just say that the AEC *has primes*, see Definition 2.17).

It is natural to ask whether a converse holds (see [Vase, Conjecture 5.22]): does any fully tame and short *totally categorical* AEC with amalgamation have prime on a tail? Theorem 0.1 (proven as Theorem 4.3) gives a positive answer. We emphasize again that (1) implies (2) appears elsewhere [Vasa, Theorem 7.14]. This paper focuses on (2) implies (1).

Using the methods of [MS90] to derive the amalgamation property from categoricity and [Bon14b] to obtain full tameness and shortness[3], we can replace the model-theoretic assumptions on $\mathcal{K}$ by a large cardinal axiom and obtain Corollary 0.2 from the abstract (proven here as Corollary 4.14).

In other words, assuming a large cardinal axiom, we can take P to be "There exists λ such that $K_{\geq \lambda}$ has primes" in Question 1.2. We do not know whether the large cardinal assumption is necessary, see the end of Section 4 for a discussion.

We prove (2) implies (1) in Theorem 0.1 by showing more generally that in every AEC $\mathcal{K}$ with a superstable-like global independence notion, the class of *saturated* models in $\mathcal{K}$ has primes (see Theorem 3.10). This is done by generalizing an argument of Shelah who proved this in [She09, Section III.4] for saturated models of successor sizes (see also [Jar]). As another application of the existence of primes in classes of saturated models, we show (see Theorem 4.3) that a certain non-uniform version of having primes suffices for (1) implies (2) of Theorem 0.1.

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[2]It has since been shown [Vasd] that full tameness and shortness can be replaced by tameness. Hanf numbers have also further been improved [Vasa, Theorem 7.14], giving the precise statement of (1) implies (2) in Theorem 0.1.

[3]We use [SK96, BTR] to replace "strongly compact" by "almost strongly compact".

like to thank Professor Grossberg for his guidance and assistance in my research in general and in this work specifically.

2. Background

We give some background on superstability and independence that will be used in the next section. We assume familiarity with the basics of AECs as laid out in e.g. [Bal09] or the forthcoming [Gro]. We will use the notation from the preliminaries of [Vasc]. All throughout this section, we assume:

Hypothesis 2.1. $\mathcal{K}$ is an AEC with amalgamation.

The definition of superstability below is already implicit in [SV99] and has since then been studied in several papers, e.g. [Van06, GVV, Vasb, BVb, GV, VV]. We will use the definition from [Vasb, Definition 10.1]:

Definition 2.2. $\mathcal{K}$ is *μ-superstable* (or *superstable in μ*) if:

(1) $\mu \geq \mathrm{LS}(\mathcal{K})$.
(2) $\mathcal{K}_\mu$ is nonempty, has joint embedding, and no maximal models.
(3) $\mathcal{K}$ is stable in μ[4], and:
(4) μ-splitting in $\mathcal{K}$ satisfies the following locality property: for all limit ordinal $\delta < \mu^+$ and every increasing continuous sequence $\langle M_i : i \leq \delta \rangle$ in $\mathcal{K}_\mu$ with M_{i+1} universal over M_i for all $i < \delta$, if $p \in \mathrm{gS}(M_\delta)$, then there exists $i < \delta$ so that p does not μ-split over M_i.

Remark 2.3. By our global hypothesis of amalgamation (Hypothesis 2.1), if $\mathcal{K}$ is μ-superstable, then $\mathcal{K}_{\geq \mu}$ has joint embedding.

We will use the following notation to describe classes of saturated models:

Definition 2.4. For $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}^{\lambda\text{-sat}}$ is the class of λ-saturated models in $\mathcal{K}_{\geq \lambda}$. We order $\mathcal{K}^{\lambda\text{-sat}}$ with the strong substructure relation induced from $\mathcal{K}$.

We will also make use of uniqueness of limit models (see [GVV] for history and motivation on limit models). First, we give a global definition of limit models, where we permit the limit model and the base to have different sizes:

Definition 2.5. Let $M_0 \leq M$ be models in $\mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$.

[4] That is, $|\mathrm{gS}(M)| \leq \mu$ for all $M \in \mathcal{K}_\mu$. Some authors call this "Galois-stable".

(1) M is *universal over* M_0 if for any $N \in \mathcal{K}_{\|M_0\|}$ with $M_0 \leq N$, there exists $f : N \xrightarrow[M_0]{} M$.
(2) M is *limit over* M_0 if there exists a limit ordinal δ and a strictly increasing continuous sequence $\langle N_i : i \leq \delta \rangle$ such that:
   (a) $N_0 = M_0$.
   (b) $N_\delta = M$.
   (c) For all $i < \delta$, N_{i+1} is universal over N_i.

   We say that M is *limit* if it is limit over some $M' \leq M$.

We will use the following consequences of superstability and tameness without comments:

Fact 2.6.

(1) Assume that $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-tame and $\mathrm{LS}(\mathcal{K})$-superstable. Then:
   (a) [Vasb, Proposition 10.10] $\mathcal{K}$ is superstable in every $\mu \geq \mathrm{LS}(\mathcal{K})$. In particular, $\mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})}$ has no maximal models and is stable in every $\mu \geq \mathrm{LS}(\mathcal{K})$.
   (b) [VV, Theorem 6.8] For every $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$.
   (c) [VV, Theorem 6.4] If $\mu \geq \mathrm{LS}(\mathcal{K})$, $M_0, M_1, M_2 \in \mathcal{K}_\mu$ are such that both M_1 and M_2 are limit over M_0, then $M_1 \cong_{M_0} M_2$.
(2) The Shelah-Villaveces theorem [SV99][5]: If $\mathcal{K}$ has no maximal models and is categorical in a $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-superstable.

Also observe that limit models are saturated:

Proposition 2.7. Assume that $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-superstable and $\mathrm{LS}(\mathcal{K})$-tame. Let $M \in \mathcal{K}_{>\mathrm{LS}(\mathcal{K})}$. The following are equivalent:

(1) M is saturated.
(2) M is limit over every $M_0 \in K_{<\|M\|}$ with $M_0 \leq M$.
(3) M is limit.
(4) M is limit over some M_0 with $M_0 \in \mathcal{K}_{\|M\|}$.

Proof. (1) implies (2) implies (3) is straightforward. (3) implies (4) is [VV, Proposition 4.1]. (4) implies (1) follows from uniqueness of limit models (Fact 2.6). □

We obtain uniqueness of limit models in a generalized sense:

[5]In the context of AECs with amalgamation, this is discussed in [GV, Theorem 6.3].

Proposition 2.8. Assume that $\mathcal{K}$ is $\mathrm{LS}(\mathcal{K})$-superstable and $\mathrm{LS}(\mathcal{K})$-tame. If M_1, M_2 are limit over M_0 and $\|M_1\| = \|M_2\|$, then $M_1 \cong_{M_0} M_2$.

Proof. Let $\mu := \|M_1\| = \|M_2\|$. If $\|M_0\| = \mu$, then $M_1 \cong_{M_0} M_2$ by Fact 2.6. If $\|M_0\| < \mu$, note that by Proposition 2.7, M_1 and M_2 are both saturated, hence by uniqueness of saturated models $M_1 \cong_{M_0} M_2$. $\square$

We will work use a global forking-like independence notion that has the basic properties of forking in a superstable first-order theory. This is a stronger notion than Shelah's good frame [She09, Chapter II] because in good frames the forking is only defined for types of length one. We invite the reader to consult [Vasb] for more explanations and motivations on global and local independence notions.

Definition 2.9 (Definition 8.1 in [Vasb])**.** $\mathfrak{i} = (K, \mathop{\smile\hskip-0.9em^{\mid}\,})$ is a *fully good independence relation* if:

1. K is an AEC with $K_{<\mathrm{LS}(K)} = \emptyset$ and $K \neq \emptyset$.
2. K has amalgamation, joint embedding, and no maximal models.
3. K is stable in all cardinals.
4. $\mathfrak{i}$ is a $(<\infty, \geq \mathrm{LS}(K))$-independence relation (see [Vasb, Definition 3.6]). That is, $\mathop{\smile\hskip-0.9em^{\mid}\,}$ is a relation on quadruples (M, A, B, N) with $M \leq N$ and $A, B \subseteq |N|$ satisfying invariance, monotonicity, and normality. We write $A \overset{N}{\underset{M}{\mathop{\smile\hskip-0.9em^{\mid}\,}}} B$ instead of $\mathop{\smile\hskip-0.9em^{\mid}\,}(M, A, B, N)$, and we also say $\mathrm{gtp}(\bar{a}/B; N)$ does not fork over M for $\mathrm{ran}(\bar{a}) \overset{N}{\underset{M}{\mathop{\smile\hskip-0.9em^{\mid}\,}}} B$.
5. $\mathfrak{i}$ has base monotonicity, disjointness ($A \overset{N}{\underset{M}{\mathop{\smile\hskip-0.9em^{\mid}\,}}} B$ implies $A \cap B \subseteq |M|$), symmetry, uniqueness, extension, and the local character properties:
   (a) If $p \in \mathrm{gS}^{\alpha}(M)$, there exists $M_0 \leq M$ with $\|M_0\| \leq |\alpha| + \mathrm{LS}(K)$ such that p does not fork over M_0.
   (b) If $\langle M_i : i \leq \delta \rangle$ is increasing continuous, $p \in \mathrm{gS}^{\alpha}(M_\delta)$ and $\mathrm{cf}(\delta) > \alpha$, then there exists $i < \delta$ such that p does not fork over M_i.
6. $\mathfrak{i}$ has the left and right $(\leq \mathrm{LS}(K))$-witness properties: $A \overset{N}{\underset{M}{\mathop{\smile\hskip-0.9em^{\mid}\,}}} B$ if and only if for all $A_0 \subseteq A$ and $B_0 \subseteq B$ with $|A_0| + |B_0| \leq \mathrm{LS}(K)$, we have that $A_0 \overset{N}{\underset{M}{\mathop{\smile\hskip-0.9em^{\mid}\,}}} B_0$.

(7) $\mathfrak{i}$ has full model continuity: if for $\ell < 4$, $\langle M_i^\ell : i \leq \delta \rangle$ are increasing continuous such that for all $i < \delta$, $M_i^0 \leq M_i^\ell \leq M_i^3$ for $\ell = 1, 2$ and $M_i^1 \underset{M_i^0}{\overset{M_i^3}{\downarrow}} M_i^2$, then $M_\delta^1 \underset{M_\delta^0}{\overset{M_\delta^3}{\downarrow}} M_\delta^2$.

We say that $\mathfrak{i}$ is *good* if it has all the properties above except full model continuity. We say that K is *[fully] good* if there exists $\downarrow$ such that $(K, \downarrow)$ is [fully] good.

It is open in general whether fully tame and short AECs with amalgamation categorical in high-enough cardinals will be good. The problem is with the extension property (see the discussion in Section 15 of [Vasb]). Thus we will use the following weakening:

Definition 2.10. $\mathfrak{i} = (K, \downarrow)$ is *almost fully good* if it satisfies Definition 2.9 except that only the following types are required to have a nonforking extension:

(1) Types that do not fork over *saturated* models.
(2) Type that do not fork over models of size $\text{LS}(K)$.
(3) Types of length at most $\text{LS}(K)$.

As before, we say that K is *almost fully good* if there exists $\downarrow$ such that $(K, \downarrow)$ is almost fully good. If we drop "fully" we mean that full model continuity need not hold.

Remark 2.11. Let $\mathcal{K}$ be almost good. Then:

(1) $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-tame (this follows from local character and uniqueness, see the proof of [Bon14a, Theorem 3.2]).
(2) $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable (because non-forking implies non-splitting, see the proof of [VV, Fact 3.6]).

We will use this freely.

In [Vasb, Theorem 15.6.(3)], it was shown that fully tame and short categorical AECs with amalgamation are (on a tail) almost fully good. Hanf numbers were improved in Appendix A of [Vase].

Fact 2.12 (Corollary A.16 in [Vase])**.** Assume that $\mathcal{K}$ is fully $\text{LS}(\mathcal{K})$-tame and short. If $\mathcal{K}$ is categorical in a $\theta > \text{LS}(\mathcal{K})$, then $\mathcal{K}^{\lambda\text{-sat}}$ is almost fully good, where $\lambda := \left(2^{\text{LS}(\mathcal{K})}\right)^{+5}$.

We will make use of Shelah's uniqueness triples [She09, Definition II.5.3]. In our framework, they have an easier definition:

Definition 2.13. Let $\mathfrak{i} = (\mathcal{K}, \perp)$ be an almost good independence relation. (a, M, N) is a *domination triple* if $M \leq N$, $a \in |N| \backslash |M|$, and for any $N' \geq N$ and any $B \subseteq |N'|$, if $a \underset{M}{\overset{N'}{\perp}} B$, then $N \underset{M}{\overset{N'}{\perp}} B$.

Remark 2.14. By [Vasb, Lemma 11.7], uniqueness triples and domination triples coincide in our framework.

In [She09, Definition III.1.1], Shelah defines a good λ-frame to be *weakly successful* if it has the existence property for uniqueness triples. We give an analogous definition for domination triples:

Definition 2.15. Let $\mathfrak{i} = (\mathcal{K}, \perp)$ be an almost good independence relation and let $\lambda \geq \text{LS}(\mathcal{K})$. We say that $\mathfrak{i}$ *has the λ-existence property for domination triples* if for every $M \in \mathcal{K}_\lambda$ and every nonalgebraic $p \in \text{gS}(M)$, there exists a domination triple (a, M, N) so that $p = \text{gtp}(a/M; N)$.

Fact 2.16 (Lemma 11.12 in [Vasb])**.** Let $\mathfrak{i} = (\mathcal{K}, \perp)$ be an almost good independence relation. For every $\lambda > \text{LS}(\mathcal{K})$, $\mathfrak{i} \upharpoonright \mathcal{K}^{\lambda\text{-sat}}$ (the restriction of $\mathfrak{i}$ to λ-saturated models) has the λ-existence property for domination triples.

Finally, we recall the definition of prime models in the framework of abstract elementary classes. This does not need amalgamation and is due to Shelah [She09, Section III.3]. While it is possible to define what it means for a model to be prime over an arbitrary set (see [Vase, Definition 5.1]), here we focus on primes over sets of the form $M \cup \{a\}$. The technical point in the definition is that since we are not working inside a monster model, how $M \cup \{a\}$ is embedded matters. Thus we use a formulation in terms of Galois types: instead of saying that N is prime over $M \cup \{a\}$, we say that (a, M, N) is a *prime triple*:

Definition 2.17. Let $\mathcal{K}$ be an AEC (not necessarily with amalgamation).

(1) A *prime triple* is (a, M, N) such that $M \leq N$, $a \in |N| \backslash |M|$ and for every $N' \in \mathcal{K}$, $a' \in |N'|$ such that $\text{gtp}(a/M; N) = \text{gtp}(a'/M; N')$, there exists $f : N \underset{M}{\longrightarrow} N'$ so that $f(a) = a'$.
(2) We say that $\mathcal{K}$ *has primes* if for $M \in \mathcal{K}$ and every nonalgebraic $p \in \text{gS}(M)$, there exists a prime triple representing p, i.e. there exists a prime triple (a, M, N) so that $p = \text{gtp}(a/M; N)$.
(3) We define localizations such as "$\mathcal{K}_\lambda$ has primes" or "$\mathcal{K}_\lambda^{\lambda\text{-sat}}$ has primes" in the natural way (in the second case, we ask that all models in the definition be saturated).

3. Building primes over saturated models

We show that in almost fully good AECs, there exists primes among the saturated models (see Definition 2.17). For models of successor size, this is shown in [She09, Claim III.4.9] (or in [Jar] with slightly weaker hypotheses). We generalizes Shelah's proof to limit sizes here. By Fact 2.12 this will show that fully tame and short totally categorical AECs with amalgamation have primes (on a tail). Throughout this section, we assume:

Hypothesis 3.1.

(1) $\mathcal{K}$ is an almost fully good AEC, as witnessed by $\mathfrak{i} = (\mathcal{K}, \mathop{\smile\!\!\!\!\!\!|}\,)$.
(2) $\mathfrak{i}$ has the $\text{LS}(\mathcal{K})$-existence property for domination triples (see Definition 2.15).

Remark 3.2. When we refer to forking, we mean forking in the sense of $\mathop{\smile\!\!\!\!\!\!|}\,$. Strictly speaking, this depends on the choice of the witness $\mathfrak{i}$ to $\mathcal{K}$ being almost fully good. However by [BGKV, Corollary 5.19] and the $(\leq \text{LS}(\mathcal{K}))$-witness property, there is a unique such witness.

Remark 3.3. In Section III of [She09] Shelah works with a more local hypothesis: the existence of a successful good μ-frame ([She09, Definition III.1.1]). It is implicit in Shelah's work (and is made precise in [Vasb, Theorem 12.16]) that this implies that $\mathcal{K}$ is good for models of size μ. Shelah shows that this implies that $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ has primes when $\lambda := \mu^+$. Since here we want to show the same result for a limit $\lambda > \mu$, we will use that the independence relation is well-behaved for models of sizes in $[\lambda, \mu)$, hence not only in one size. We could still have made our hypothesis more local (i.e. only requiring that the independence relation behaves well for types of length less than μ and models in $\mathcal{K}_{<\mu}$) but for notational simplicity we do not adopt this approach.

We start by showing that domination triples are closed under unions:

Lemma 3.4. Let $\langle M_i : i < \delta \rangle, \langle N_i : i < \delta \rangle$ be increasing and assume that (a, M_i, N_i) are domination triples for all $i < \delta$. Then $(a, \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i)$ is a domination triple.

Proof. For ease of notation, we work inside a monster model $\mathfrak{C}$ and write $A \mathop{\smile\!\!\!\!\!\!|}_{M} B$ for $A \overset{\mathfrak{C}}{\underset{M}{\mathop{\smile\!\!\!\!\!\!|}}} B$. Let $M_\delta := \bigcup_{i<\delta} M_i$, $N_\delta := \bigcup_{i<\delta} N_i$. Assume that $a \mathop{\smile\!\!\!\!\!\!|}_{M_\delta} N$ with $M_\delta \leq N$ (by extension for types of length one, we can assume this without loss of generality). By local character, for all sufficiently large $i < \delta$, $a \underset{M_i}{\perp} N$. By definition of domination triples, $N_i \underset{M_i}{\perp} N$. By full model continuity, $N_\delta \underset{M_\delta}{\perp} N$. $\square$

The conclusion of the next fact is a key step in Shelah's construction of a successor frame in [She09, Chapter II]. It is not as straightforward as it seems, because full model continuity applies only to chains of models $\langle M_i^\ell : i \leq \delta \rangle$ with $M_i^0 \leq M_i^\ell$, $\ell = 0, 1, 2$. Here we can use amalgamation, tameness, and Fact 2.16 saying that for every $\mu > \text{LS}(\mathcal{K})$, $\mathfrak{i} \upharpoonright \mathcal{K}^{\mu\text{-sat}}$ has the μ-existence property for domination triples to get:

Fact 3.5 (Theorem 7.8 in [Jar16])**.** For every $\mu \geq \text{LS}(\mathcal{K})$, for every $M^0 \leq M^1$ in $\mathcal{K}_{\mu^+}$, if $\langle M_i^\ell : i < \mu^+ \rangle$ are increasing continuous resolutions of M^ℓ and all are limit models[6] in $\mathcal{K}_\mu$, $\ell = 0, 1$, then the set of $i < \mu^+$ so that $M^0 \overset{M^1}{\underset{M_i^0}{\perp}} M_i^1$ is a club.

We can now generalize the proof of [She09, Claim III.4.3] to limit cardinals. Roughly, it tells us that every nonalgebraic type over a saturated model has a resolution into domination triples.

Lemma 3.6. Let $\lambda > \text{LS}(\mathcal{K})$ and let $\delta := \text{cf}(\lambda)$. Let $M^0 \in \mathcal{K}_\lambda$ be saturated and let $p \in \text{gS}(M_0)$ be nonalgebraic. Then there exists a saturated $M^1 \in \mathcal{K}_\lambda$, an element $a \in |N|$, and increasing continuous resolutions $\langle M_i^\ell : i \leq \delta \rangle$ of M^ℓ, $\ell = 0, 1$ such that for all $i < \delta$:

(1) $p = \text{gtp}(a/M^0; M^1)$.
(2) $a \in |M_0^1|$.
(3) p does not fork over M_0^0.
(4) For $\ell = 0, 1$, $M_i^\ell \in \mathcal{K}_{[\text{LS}(\mathcal{K}), \lambda)}$ and M_{i+1}^ℓ is limit over M_i^ℓ.
(5) (a, M_i^0, M_i^1) is a domination triple.

Proof. For $\ell = 0, 1$, we choose by induction $\langle N_i^\ell : i \leq \lambda \rangle$ increasing continuous and an element a such that for all $i < \lambda$:

(1) $N_0^0 \leq M^0$ and p does not fork over M_0^0.
(2) $a \in |N_0^1|$.
(3) For $\ell = 0, 1$, $N_i^\ell \in \mathcal{K}_{|i| + \text{LS}(\mathcal{K})}$ and $N_i^0 \leq N_i^1$.
(4) $\text{gtp}(a/N_i^0; N_i^1)$ does not fork over N_0^0.
(5) If i is odd, and $\ell = 0, 1$, then N_{i+1}^ℓ is limit over N_i^ℓ.

[6] And hence if $\mu > \text{LS}(\mathcal{K})$ are saturated (Proposition 2.7).

(6) If i is even and (a, N_i^0, N_i^1) is *not* a domination triple, then $N_i^1 \underset{N_i^0}{\overset{N_{i+1}^1}{\not\perp}} N_{i+1}^0$.

This is possible. First pick $N_0^0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ such that $N_0^0 \leq M^0$ and p does not fork over M_0^0. This is possible by local character. Now pick $N_0^1 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ such that $N_0^0 \leq N_0^1$ and there is $a \in |N_0^1|$ with $\mathrm{gtp}(a/N_0^0; N_0^1) = p \upharpoonright N_0^0$. This takes care of the case $i = 0$. For i limit, take unions. Now assume that $i = j + 1$ is a successor. We consider several cases:

- If j is even and (a, N_j^0, N_j^1) is *not* a domination triple, then there must exist witnesses $N_{j+1}^0, N_{j+1}^1 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})+|j|}$ such that $N_j^0 \leq N_{j+1}^0$, $N_{j+1}^0 \leq N_{j+1}^1$, $N_j^1 \leq N_{j+1}^1$, $a \underset{N_j^0}{\overset{N_{j+1}^1}{\perp}} N_{j+1}^0$ but $N_j^1 \underset{N_j^0}{\overset{N_{j+1}^1}{\not\perp}} N_{j+1}^0$. This satisfies all the conditions (we know that $\mathrm{gtp}(a/N_j^0; N_j^1)$ does not fork over N_0^0, so by transitivity also $\mathrm{gtp}(a/N_{j+1}^0; N_{j+1}^1)$ does not fork over N_0^0).
- If j is even and (a, N_j^0, N_j^1) *is* a domination triple, take $N_{j+1}^\ell := N_j^\ell$, for $\ell = 0, 1$.
- If j is odd, pick $N_i^0 \in \mathcal{K}_{\mathrm{LS}(\mathcal{K})+|j|}$ limit over N_j^0 and N_i^1 limit over N_i^0 and N_j^1 so that $\mathrm{gtp}(a/N_i^0; N_i^1)$ does not fork over N_0^0. This is possible by the extension property for types of length one.

This is enough. By the odd stages of the construction, and basic properties of universality, for all $i < \lambda$, $\ell = 0, 1$, N_{i+2}^ℓ is universal over N_i^ℓ. Thus for $\ell = 0, 1$ and $i \leq \lambda$ a limit ordinal, N_i^ℓ is limit. In particular, by Proposition 2.7, N_λ^ℓ is saturated. By uniqueness of saturated models, $N_\lambda^0 \cong_{N_0^0} M^0$. By uniqueness of the nonforking extension, without loss of generality $N_\lambda^0 = M^0$. Now let C be the set of *limit* $i < \lambda$ such that (a, N_i^0, N_i^1) is a domination triple. We claim that C is a club:

- C is closed by Lemma 3.4.
- C is unbounded: given $\alpha < \lambda$, let $\mu := |\alpha| + \mathrm{LS}(\mathcal{K})$. Let E_μ be the set of $i < \mu^+$ such that i is limit and $N_{\mu^+}^0 \underset{N_i^0}{\overset{N_{\mu^+}^1}{\perp}} N_i^1$. By Fact 3.5, E_μ is a club. The even stages of the construction imply that for $i \in E_\mu$, (a, N_i^0, N_i^1) is a domination triple. In other words, $E_\mu \subseteq C$. Now pick $\beta \in E_\mu \backslash (\alpha + 1)$. We have that $\alpha < \beta$ and $\beta \in E_\mu \subseteq C$. This completes the proof that C is unbounded.

Let $\langle \alpha_i : i < \delta \rangle$ (recall that $\delta = \text{cf}(\lambda)$) be a cofinal strictly increasing continuous sequence of elements of C. For $i < \delta$, $\ell = 0, 1$, let $M_i^\ell := N_{\alpha_i}^\ell$. This works: Clauses (1), (2), (3) are straightforward to check using monotonicity of forking. Clause (5) holds by definition of C. As for (4), we have observed above that for $\ell = 0, 1$, for all $i < \lambda$, N_{i+2}^ℓ is universal over N_i^ℓ. Hence for all limit ordinals $i < j < \lambda$, N_j^ℓ is limit over N_i^ℓ. In particular because C contains only limit ordinals, for all $i < \delta$, $N_{\alpha_{i+1}}^\ell$ is limit over $N_{\alpha_i}^\ell$, as desired. $\square$

In [She09, Claim III.4.9], Shelah observes that triples as in the conclusion of Lemma 3.6 are prime triples. For the convenience of the reader, we include the proof here. We will use the following fact which follows from the uniqueness property of forking and some renaming.

Fact 3.7 (Lemma 12.6 in [Vasb])**.** For $\ell < 2$, $i < 4$, let $M_i^\ell \in \mathcal{K}$ be such that for $i = 1, 2$, $M_0^\ell \leq M_i^\ell \leq M_3^\ell$.

If $M_1^\ell \overset{M_3^\ell}{\underset{M_0^\ell}{\perp}} M_2^\ell$ for $\ell < 2$, $f_i : M_i^1 \cong M_i^2$ for $i = 0, 1, 2$, and $f_0 \subseteq f_1$, $f_0 \subseteq f_2$, then $f_1 \cup f_2$ can be extended to $f_3 : M_3^1 \to M_4^2$, for some M_4^2 with $M_3^2 \leq M_4^2$.

Theorem 3.8. For any $\lambda > \text{LS}(\mathcal{K})$, $\mathcal{K}_\lambda^{\lambda\text{-sat}}$ has primes (see Definition 2.17).

Proof. Let $M \in \mathcal{K}_\lambda$ be saturated and let $p \in \text{gS}(M)$ be nonalgebraic. We must find a triple (a, M, N) such that $M \leq N$, $N \in \mathcal{K}_\lambda$ is saturated, $p = \text{gtp}(a/M; N)$, and (a, M, N) is a prime triple among the saturated models of size λ.

Set $M^0 := M$ and let $\delta := \text{cf}(\lambda)$. Let M^1, a, $\langle M_i^\ell : i \leq \delta \rangle$ be as described by the statement of Lemma 3.6. We show that (a, M^0, M^1) is as desired. By assumption, $M^0 \leq M^1$, $p = \text{gtp}(a/M^0; M^1)$, and $M^1 \in \mathcal{K}_\lambda$ is saturated. It remains to show that (a, M^0, M^1) is a prime triple in $\mathcal{K}_\lambda^{\lambda\text{-sat}}$. Let $M' \in \mathcal{K}_\lambda^{\lambda\text{-sat}}$, $a' \in |M'|$ be given such that $\text{gtp}(a'/M^0; M') = \text{gtp}(a/M^0; M^1)$. We want to build $f : M^1 \underset{M^0}{\longrightarrow} M'$ so that $f(a) = a'$.

We build by induction an increasing continuous chain of embeddings $\langle f_i : i \leq \delta \rangle$ so that for all $i \leq \delta$:

(1) $f_i : M_i^1 \underset{M_i^0}{\longrightarrow} M'$.
(2) $f_i(a) = a'$.

This is enough since then $f := f_\delta$ is as required. This is possible: for $i = 0$, we use that M' is saturated, hence realizes $p \restriction M_0^0$, so there exists $f_0 : M_0^1 \xrightarrow[M_0^0]{} M'$ witnessing it, i.e. $f_0(a) = a'$. At limits, we take unions. For $i = j+1$ successor, let $\mu := \|M_j^1\| + \|M_i^0\|$. Pick $N_j \leq M'$ with $N_j \in \mathcal{K}_\mu$ and N_j containing both $f_j[M_j^1]$ and M_i^0.

By assumption, p does not fork over M_0^0 and by assumption $p = \text{gtp}(a'/M^0; M')$, so by monotonicity of forking, $a' \overset{N_j}{\underset{M_j^0}{\downarrow}} M_{j+1}^0$. We know that (a, M_j^0, M_j^1) is a domination triple, hence applying f_j and using invariance, $(a', M_j^0, f_j[M_j^1])$ is a domination triple. Therefore $f_j[M_j^1] \overset{N_j}{\underset{M_j^0}{\downarrow}} M_{j+1}^0$.

By a similar argument, we also have $M_j^1 \overset{M_{j+1}^1}{\underset{M_j^0}{\downarrow}} M_{j+1}^0$. By Fact 3.7, the map $f_j \cup \text{id}_{M_{j+1}^0}$ can be extended to a K-embedding $g : M_{j+1}^\alpha \to N_j'$ for some $N_j' \geq N_j$ of size μ. Since $\mu < \lambda$ and M' is saturated, there exists $h : N_j' \xrightarrow[N_j]{} M'$. Let $f_{j+1} := h \circ g$. $\square$

To prove the main theorem of this section, we will use two categoricity transfer results which hold in a much weaker framework than good AECs (the independence relation need only be defined for types of elements).

Fact 3.9. Assume that $\mathcal{K}$ is categorical in $\text{LS}(\mathcal{K})$.

(1) [Vasa, Corollary 6.16] For any $\mu \geq \text{LS}(\mathcal{K})$, If $\mathcal{K}^{\mu\text{-sat}}$ (where we define $\mathcal{K}^{\text{LS}(\mathcal{K})\text{-sat}} := \mathcal{K}$) is categorical in a *successor* $\lambda > \mu$, then $\mathcal{K}$ is categorical in all $\lambda' > \text{LS}(\mathcal{K})$.
(2) [Vasd, Theorem 2.16] if $\mathcal{K}$ is categorical in *some* $\lambda > \text{LS}(\mathcal{K})$ and $\mathcal{K}_{[\text{LS}(\mathcal{K}),\lambda)}$ has primes, then $\mathcal{K}$ is categorical in $\text{LS}(\mathcal{K})^+$ (and hence by the above in all $\lambda' > \text{LS}(\mathcal{K})$).

We obtain the promised converse to [Vase, Theorem 5.16]. This answers Question 5.21 there.

Theorem 3.10. Assume that $\mathcal{K}$ is categorical in $\text{LS}(\mathcal{K})$. The following are equivalent:

(1) $\mathcal{K}^{\text{LS}(\mathcal{K})^+\text{-sat}}$ has primes and $\mathcal{K}$ is categorical in *some* $\lambda > \text{LS}(\mathcal{K})$.
(2) There exists $\lambda > \lambda_0 > \text{LS}(\mathcal{K})$ such that $\mathcal{K}_{[\lambda_0,\lambda)}^{\lambda_0\text{-sat}}$ has primes and $\mathcal{K}^{\lambda_0\text{-sat}}$ is categorical in λ.
(3) For some $\lambda \geq \text{LS}(\mathcal{K})$, $\mathcal{K}^{\lambda\text{-sat}}$ is categorical in λ^+.
(4) $\mathcal{K}$ is categorical in *all* $\lambda > \text{LS}(\mathcal{K})$.

Proof. Assume first that (4) holds. Clearly, (3) holds. Also, by categoricity, $\mathcal{K}^{\lambda'\text{-sat}} = \mathcal{K}_{\geq \lambda'}$ for all $\lambda' > \mathrm{LS}(\mathcal{K})$ so by Theorem 3.8, $\mathcal{K}_{\geq \mathrm{LS}(\mathcal{K})^+}$ has primes. This shows that both (1) and (2) hold. Hence (4) implies all the other statements. Conversely, (3) implies (4) by Fact 3.9.(1).

Now assume (1). If $\lambda = \mathrm{LS}(\mathcal{K})^+$, then (3) (and hence (4) and (2) by the above) holds. If $\lambda > \mathrm{LS}(\mathcal{K})^+$, then we can take $\lambda_0 := \mathrm{LS}(\mathcal{K})^+$ in (2), so (2) holds. This shows that (1) implies (2).

It remains to show that (2) implies (3). Assume (2) and let λ_0 and λ witness it. By Fact 3.9.(2) (with $\mathcal{K}, \mathrm{LS}(\mathcal{K})$ there standing for $\mathcal{K}^{\lambda_0\text{-sat}}, \lambda_0$ here), $\mathcal{K}^{\lambda_0\text{-sat}}$ is categorical in λ_0^+, so (3) holds. □

3.1. **A categoricity transfer for good and nice AECs.** In this subsection, we drop Hypothesis 3.1. We want to see that instead of having primes, we can ask only for a non-uniform version (called weakly having primes below). The reader should see [Vase] for motivation on the definitions below.

Definition 3.11. Let $\mathcal{K}$ be an arbitrary AEC.

(1) [Vase, Definition 4.11] $\mathcal{K}$ has *weak amalgamation* if whenever $\mathrm{gtp}(a_1/M; N_1) = \mathrm{gtp}(a_2/M; N_2)$, there exists $M_1 \leq N_1$ with $M \leq M_1$, $a_1 \in |M_1|$ and $N \geq N_2$, $f : M_1 \underset{M}{\longrightarrow} N$ so that $f(a_1) = a_2$.
(2) [Vase, Definition 5.1.(4)] $\mathcal{K}$ *weakly has primes* if whenever $\mathrm{gtp}(a_1/M; N_1) = \mathrm{gtp}(a_2/M; N_2)$, there exists $M_1 \leq N_1$ with $M \leq M_1$, $a_1 \in |M_1|$ and $f : M_1 \underset{M}{\longrightarrow} N_2$ so that $f(a_1) = a_2$.
(3) For $M \in \mathcal{K}$, let $\mathcal{K}_M$ be the AEC defined by adding constant symbols for the elements of M and requiring that M embeds inside every model of $\mathcal{K}_M$. That is, $L(\mathcal{K}_M) = L(\mathcal{K}) \cup \{c_a \mid a \in |M|\}$, where the c_a's are new constant symbols, and:

$$\mathcal{K}_M := \{(N, c_a^N)_{a \in |M|} \mid N \in \mathcal{K} \text{ and } a \mapsto c_a^N \text{ is a } \mathcal{K}\text{-embedding from } M \text{ into } N\}$$

We order $\mathcal{K}_M$ by $(N_1, c_a^N)_{a \in |M|} \leq (N_2, c_a^{N_2})$ if and only if $N_1 \leq N_2$ and $c_a^{N_1} = c_a^{N_2}$ for all $a \in |M|$.
(4) [She09, III.12.39.(d)] For $M \in \mathcal{K}$ and $p \in \mathrm{gS}(M)$, we define $\mathcal{K}_{\neg^* p}$ to be the class of $N \in \mathcal{K}_M$ such that $f(p)$ has a unique extension to $\mathrm{gS}(N \restriction L(K))$. Here $f : M \to N$ is given by $f(a) := c_a^N$. We order $\mathcal{K}_{\neg^* p}$ with the strong substructure relation induced from $\mathcal{K}_M$.
(5) [Vase, Definition 5.13] $\mathcal{K}$ is *nice* if:

(a) $\mathcal{K}$ has weak amalgamation.
(b) For any $M \in \mathcal{K}$ and any $p \in \mathrm{gS}(M)$, $\mathcal{K}_{\neg^* p}$ has weak amalgamation and if $\mathcal{K}$ is $\|M\|$-tame, then so is $\mathcal{K}_{\neg^* p}$.
(c) We say $\mathcal{K}$ is λ-*nice* if in the previous definition, we ask that $M \in \mathcal{K}_\lambda$ (so $\mathcal{K}$ is nice if and only if it is λ-nice for all λ). We define other localizations such as "$\mathcal{K}_{[\lambda,\theta)}$ weakly has primes" or "$\mathcal{K}_{[\lambda,\theta)}$ is λ-nice" in the natural way.

Weakly having primes is a non-uniform version of having primes, since we require only that the "weak prime model" M_1 in Definition 3.11.(2) embeds inside N_2. The choice of M_1 could be different for another model N_2. The next result is easy to check.

Fact 3.12 (Proposition 5.14 in [Vase])**.** Let $\mathcal{K}$ be an arbitrary AEC. If $\mathcal{K}$ weakly has primes, then $\mathcal{K}$ is nice.

Using Theorem 3.8, we obtain a categoricity transfer for nice classes that have a global independence relation. This relies on:

Fact 3.13 (Theorem 5.16 in [Vase])**.** Assume Hypothesis 3.1 and that $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in some $\lambda > \mathrm{LS}(\mathcal{K})$, $\mathcal{K}_{\mathrm{LS}(\mathcal{K})}$ has primes, and $\mathcal{K}_{[\mathrm{LS}(\mathcal{K}),\lambda)}$ is $\mathrm{LS}(\mathcal{K})$-nice[7], then $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})^+$ (and hence by Fact 3.9 in all $\lambda' > \mathrm{LS}(\mathcal{K})$).

Theorem 3.14. To the list of equivalent conditions in Theorem 3.10, we can add:

(5) There exists $\lambda > \lambda_0 > \mathrm{LS}(\mathcal{K})$ such that $\mathcal{K}^{\lambda_0\text{-sat}}_{[\lambda_0,\lambda)}$ is λ_0-nice and $\mathcal{K}^{\lambda_0\text{-sat}}$ is categorical in λ.
(6) There exists $\lambda > \lambda_0 > \mathrm{LS}(\mathcal{K})$ such that $\mathcal{K}^{\lambda_0\text{-sat}}_{[\lambda_0,\lambda)}$ weakly has primes and $\mathcal{K}^{\lambda_0\text{-sat}}$ is categorical in λ.

Proof. By Fact 3.12, (6) implies (5). Moreover, (2) (in Theorem 3.10) implies (6). We close the loop by showing that (5) implies (3). Assume (5). By Theorem 3.8, $\mathcal{K}^{\lambda_0\text{-sat}}_{\lambda_0}$ has primes. By Fact 3.13 (where $\mathcal{K}, \mathrm{LS}(\mathcal{K})$ there stand for $\mathcal{K}^{\lambda_0\text{-sat}}, \lambda_0$ here), $\mathcal{K}^{\lambda_0\text{-sat}}$ is categorical in λ_0^+, as desired. $\square$

Remark 3.15. We can further weaken the definition of nice and still get that the proof of Theorem 3.14 goes through: instead of in Definition 3.11.(5) requiring the condition for all $p \in \mathrm{gS}(M)$, it is enough

[7] [Vase, Theorem 5.16] assumes $\mathcal{K}$ is nice, but the proof shows that the more local condition given here is enough.

to require it for *some* minimal $p \in \mathrm{gS}(M)$. This follows from the definition of unidimensionality in [Vasa, Definition 5.2] and the proof of Fact 3.13 which works whenever we have $p \perp q$ in $\mathcal{K}_{\lambda_0}$ and $\mathcal{K}_{\neg^* p}$ nice. Note that when p is minimal, $\mathcal{K}_{\neg^* p}$ is simply the class of models in $\mathcal{K}_M$ which omit their copy of p.

4. The main theorems

In this section, we use the notation from [Bal09, Chapter 14]:

Notation 4.1. For a fixed AEC $\mathcal{K}$, we write $H_1 := \beth_{\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+}$.

Before proving the main theorems of this paper, we state some more facts:

Fact 4.2. Let $\mathcal{K}$ be an $\mathrm{LS}(\mathcal{K})$-tame AEC with amalgamation and arbitrarily large models.

(1) [Vasa, Theorem 3.3][8] If $\mathcal{K}$ is categorical in a $\lambda > \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is categorical in all cardinals of the form $\beth_\delta$, where $\left(2^{\mathrm{LS}(\mathcal{K})}\right)^+$ divides δ.
(2) [GV06a] If $\mathcal{K}$ is categorical in a successor $\lambda > \mathrm{LS}(\mathcal{K})^+$, then $\mathcal{K}$ is categorical in all $\lambda' \geq \lambda$.
(3) [Vasa, Corollary 7.4] If $\mathrm{LS}(\mathcal{K}) < \lambda_0 < \lambda_1$ are such that $\mathcal{K}$ is categorical in both λ_0 and λ_1 and λ_1 is a successor, then $\mathcal{K}$ is categorical in all $\lambda \in [\lambda_0, \lambda_1]$.

Theorem 4.3. Let $\mathcal{K}$ be a fully $\mathrm{LS}(\mathcal{K})$-tame and short AEC with amalgamation and arbitrarily large models. Assume that $\mathcal{K}$ is categorical in *some* $\lambda_0 > \mathrm{LS}(\mathcal{K})$. The following are equivalent:

(1) $\mathcal{K}$ is categorical in some successor $\lambda_1 > \mathrm{LS}(\mathcal{K})^+$.
(2) $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda_0, H_1)$.
(3) $\mathcal{K}_{\geq H_1}$ has primes.
(4) There exists μ such that $\mathcal{K}_{\geq \mu}$ has primes.
(5) There exists μ such that $\mathcal{K}_{\geq \mu}$ weakly has primes (see Definition 3.11.(2)).
(6) There exists μ such that $\mathcal{K}_{\geq \mu}$ is nice (see Definition 3.11.(5)).

Proof. By partitioning the AEC into disjoint classes that each have joint embedding, we can assume without loss of generality that $\mathcal{K}$ has no maximal models (see for example [Vasa, Remark 7.2]). By Fact 4.2, $\mathcal{K}$ is categorical in H_1. Moreover by Fact 2.12, there exists $\mu < H_1$

[8] This can be traced back to an omitting type theorem of Shelah [She99, II.1.6].

such that $\mathcal{K}^{\mu\text{-sat}}$ is almost fully good. By Fact 2.16, we can assume without loss of generality that the independence relation witnessing almost full goodness has the μ-existence property for domination triples. Therefore Hypothesis 3.1 holds of $\mathcal{K}^{\mu\text{-sat}}$. By categoricity (and Fact 2.6), $\mathcal{K}^{H_1\text{-sat}} = \mathcal{K}_{\geq H_1}$. By Fact 4.2, $\mathcal{K}$ is categorical in a proper class of cardinals. Now combine the last two parts of Fact 4.2 with Theorems 3.10 and 3.14. $\square$

Remark 4.4. In condition (3), we can replace H_1 by:

$$\min\left(H_1, \lambda_0 + \left(2^{\mathrm{LS}(\mathcal{K})}\right)^{+6}\right)$$

Moreover we can also prove minor improvements such as, instead of (2), "there exists $\chi < H_1$ such that $\mathcal{K}$ is categorical in all $\lambda' \geq \min(\lambda_0, \chi)$". Finally if $\mathcal{K}$ is categorical in $\mathrm{LS}(\mathcal{K})$, we can accept $\lambda_1 = \mathrm{LS}(\mathcal{K})^+$ and add to the list of conditions that $\mathcal{K}$ is totally categorical (see [Vasa, Remark 7.5]).

We can get an amalgamation-free version by assuming that $\mathcal{K}$ is eventually syntactically characterizable:

Definition 4.5 (Definition 2.1 in [BVa])**.** An AEC $\mathcal{K}$ is $L_{\infty,\theta}$*-syntactically characterizable* if whenever $M, N \in \mathcal{K}$, if $M \leq N$ then $M \preceq_{L_{\infty,\theta}} N$. We say that $\mathcal{K}$ is *eventually syntactically characterizable* if for every infinite cardinal θ, there exists λ such that $\mathcal{K}_{\geq\lambda}$ is $L_{\infty,\theta}$-syntactically characterizable.

The main facts about being eventually syntactically characterizable are:

Fact 4.6.

(1) [BVa, Proposition 1.3] If K has amalgamation and is categorical in unboundedly many cardinals, then K is eventually syntactically characterizable.
(2) [She09, Claim IV.1.12.(1)] Let θ be an infinite[9] cardinal. If K is categorical in a $\lambda = \lambda^{<\theta} \geq \mathrm{LS}(K)$, then $K_{\geq\lambda}$ is $L_{\infty,\theta}$-syntactically characterizable.
(3) [She09, Conclusion IV.2.12.(1)] If K is categorical in cardinals of arbitrarily large cofinality (that is, for every θ there exists λ such that K is categorical in λ and $\mathrm{cf}(\lambda) \geq \theta$), then K is eventually syntactically characterizable.

[9]Shelah assumes in addition that $\theta > \mathrm{LS}(K)$, but the proof shows that it is not necessary.

(4) [Vase, Corollary 4.16] If $\mathcal{K}$ is a tame AEC with weak amalgamation (see Definition 3.11.(1)) that is eventually syntactically characterizable and categorical in a proper class of cardinals, then there exists λ such that $\mathcal{K}_{\geq \lambda}$ has amalgamation.

Remark 4.7. Shelah claims [She09, Conclusion IV.2.14] that an AEC categorical in a proper class of cardinals is eventually syntactically characterizable. However Will Boney and the author have found a gap in the proof, see [BVa].

Theorem 4.8. Let $\mathcal{K}$ be a fully tame and short AEC. Assume further that $\mathcal{K}$ is eventually syntactically characterizable and is categorical in a proper class of cardinals. The following are equivalent:

(1) There exists λ_0 such that $\mathcal{K}_{\geq \lambda_0}$ has primes[10].
(2) There exists λ_1 such that $\mathcal{K}$ is categorical in all $\lambda \geq \lambda_1$ and $\mathcal{K}_{\geq \lambda_1}$ has amalgamation.

Proof. Note that a categorical AEC with amalgamation eventually has no maximal models. Therefore by Fact 4.6.(4), assuming any of the two conditions, there exists λ_2 such that $\mathcal{K}_{\geq \lambda_2}$ has amalgamation and no maximal models. Now apply Theorem 4.3. $\square$

We do not know if prime models can be built assuming just tameness, namely:

Question 4.9. Can "fully tame and short" be replaced by only "tame"?

To replace the model-theoretic hypotheses of Theorem 4.3 by large cardinals, we will use almost strongly compact cardinals (first isolated by Bagaria and Magidor [BM14]):

Definition 4.10. An uncountable limit cardinal κ is *almost strongly compact* if for every $\mu < \kappa$, every κ-complete filter can be extended to a μ-complete ultrafilter.

Remark 4.11. If κ is almost strongly compact, then taking an ultrapower of the universe (see [Jec03, Chapter 17]), we can show that for every $\kappa_0 < \kappa$, there exists $\kappa' \in (\kappa_0, \kappa]$ so that κ' is measurable.

By [SK96] (this is a result about classes of models of $L_{\kappa,\omega}$, but one can adapt the proofs to AECs as pointed out in [Bon14b, Section 7]):

[10] Or just weakly has primes or is nice.

Fact 4.12. Let $\mathcal{K}$ be an AEC and let $\kappa > \mathrm{LS}(\mathcal{K})$ be a measurable cardinal. If $\mathcal{K}$ is categorical in a $\lambda > \kappa$, then $\mathcal{K}_{\geq \kappa}$ has no maximal models and amalgamation in every $\mu \in [\kappa, \lambda)$.

The next fact is due to Will Boney when κ is strongly compact [Bon14b]. The improvement to almost strongly compact is due to Brooke-Taylor and Rosický [BTR]. In [BU], it is shown that the large cardinal bound is optimal.

Fact 4.13. Let $\mathcal{K}$ be an AEC with amalgamation. If $\kappa > \mathrm{LS}(\mathcal{K})$ is almost strongly compact, then $\mathcal{K}$ is fully $(< \kappa)$-tame and short.

Corollary 4.14. Let $\mathcal{K}$ be an AEC categorical in a proper class of cardinals. Assume that there exists a proper class of almost strongly compact cardinals. The following are equivalent:

(1) There exists λ_0 such that $\mathcal{K}_{\geq \lambda_0}$ has primes[11].
(2) There exists λ_1 such that $\mathcal{K}$ is categorical in all $\lambda \geq \lambda_1$.

Proof. Let $\kappa > \mathrm{LS}(\mathcal{K})$ be almost strongly compact. By Remark 4.11, there exists $\kappa' \in (\mathrm{LS}(\mathcal{K}), \kappa]$ so that κ' is measurable. By[12] Fact 4.12 and unbounded categoricity, $\mathcal{K}_{\geq \kappa'}$ has amalgamation. By Fact 4.13, $\mathcal{K}_{\geq \kappa}$ is fully tame and short. Now apply Theorem 4.3 to $\mathcal{K}_{\geq \kappa}$. $\square$

We do not know whether the large cardinal assumption is necessary. Grossberg [Gro02, Conjecture 2.3] has conjectured that an AEC categorical in a high-enough cardinal should have amalgamation on a tail, but even the following weakening is open:

Question 4.15. Let $\mathcal{K}$ be a totally categorical AEC. Does there exists a cardinal λ such that $\mathcal{K}_{\geq \lambda}$ has amalgamation?

The answer is known to be positive when $2^{\mu} < 2^{\mu^+}$ for every cardinal μ. This can be combined with the best currently known approximation to the categoricity conjecture in this context:

Theorem 4.16. Assume $2^{\mu} < 2^{\mu^+}$ for every cardinal μ, as well as the result of an unpublished claim of Shelah[13]. Let $\mathcal{K}$ be an AEC that is categorical in a proper class of cardinals. The following are equivalent:

(1) There exists λ_0 such that $\mathcal{K}_{\geq \lambda_0}$ has amalgamation.

[11]Again, this can be replaced by weakly having primes or being nice.

[12]It may also be possible to adapt the methods of [MS90] to an almost strongly compact, but we just want to quote here.

[13]See [Vasa, Claim 8.2] and the discussion around it.

(2) There exists λ_1 such that $\mathcal{K}$ is categorical in all $\lambda \geq \lambda_1$.

Proof. (2) implies (1) is by [She01, Claim 1.10.(0)] which shows that amalgamation in μ follows from $2^\mu < 2^{\mu^+}$ and categoricity in μ, μ^+. (1) implies (2) is by the last theorem of Shelah's book [She09, Theorem IV.7.12], see [Vasa, Corollary 8.19] for an exposition. $\square$

One can similarly ask whether full tameness and shortness follows from categoricity: Shelah has shown [She99, Main Claim II.2.3] that totally categorical AECs with amalgamation are tame, but we do not know if this can be strengthened to "fully tame and short":

Question 4.17. Let $\mathcal{K}$ be a totally categorical AEC with amalgamation. Is $\mathcal{K}$ fully tame and short?

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