THE LAZY MODEL THEORETICIAN'S GUIDE TO SHELAH'S EVENTUAL CATEGORICITY CONJECTURE IN UNIVERSAL CLASSES

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Abstract. We give a short overview of the proof of Shelah's eventual categoricity conjecture in universal classes in [Vasd].

1. Introduction

We sketch a proof of:

Theorem 1.1. A universal class that is categorical in a proper class of cardinals is categorical on a tail of cardinals.

The reader should see the introduction of [Vasd] for motivation and history. Theorem 1.1 is a weaker statement than what is proven in [Vasd]: a universal class K which is categorical in[1] a $\lambda \geq \beth_{h(\text{LS}(K))}$ is categorical in all $\lambda' \geq \beth_{h(\text{LS}(K))}$. At the appropriate point, we will comment on where we have to do more work to get the stronger statement. Note that this is not a self-contained argument, we simply attempt to outline the proof and quote extensively from elsewhere. For another exposition, see the upcoming [BVa].

We attempt to use as few prerequisites as possible and make what we use explicit. We do not discuss generalizations to tame AECs with primes [Vasc], although we end up using part of the proof there.

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[1]Here and below, we write $h(\theta) := \beth_{(2^\theta)^+}$. We see universal classes as AECs so that for K a universal class, $\text{LS}(K) = |L(K)| + \aleph_0$. For K a fixed AEC, we write $H_1 := h(\text{LS}(K))$ and $H_2 := h(H_1)$.

We assume familiarity with a basic text on AECs such as [Bal09] or the upcoming [Gro]. We also assume the reader is familiar with the definition of a good $\mathcal{F}$-frame (see [She09a, Chapter II] for the original definition of a good λ-frame and [Vasa, Definition 2.21]) for good $\mathcal{F}$-frames), and the definition of superstability (implicit in [SV99], but we use the definition in [Vasb, Definition 10.1]). All the good frames we will use are *type-full*, i.e. their basic types are the nonalgebraic types, and we will omit the "type-full".

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2. The proof

The first step in the proof is to get amalgamation from categoricity:

Theorem 2.1. If K is a universal class categorical in some $\lambda \geq H_1$, then $K_{\geq \lambda}$ has amalgamation and no maximal models.

The proof relies on some results on EM model and the order property implicit in [She99] and proven explicitly in [BGKV], Chapters V.A and V.B of [She09b], as well as an argument of Shelah in [She09a, Chapter IV] which is fully proven there (the proof is only half a page).

Proof of Theorem 2.1. By categoricity, K_λ has joint embedding. Moreover K has arbitrarily large models, so once it is shown that $K_{\geq \lambda}$ has amalgamation, we will have that $K_{\geq \lambda}$ has joint embedding, and hence no maximal models. The proof of amalgamation has three steps:

(1) K does not have the order property[2].
    [Why? If it did, then using a Dedekind cut argument, we can prove that the model of size λ realizes $\text{LS}(K)^+$-many syntactic types over a set of size $\text{LS}(K)$ (this is [She99, Claim 4.7] proven as [BGKV, Fact 5.13]). However by the standard argument (using categoricity), M realizes only $\text{LS}(K)$-many syntactic type over any set of size $\text{LS}(K)$.

[2]In this context, K has the order property means that there exists a quantifier-free first-order formula $\phi(\bar{x}, \bar{y})$ such that for every μ, there is $M \in K$ and a sequence $\langle \bar{a}_i : i < \mu \rangle$ in M such that $M \models \phi[\bar{a}_i, \bar{a}_j]$ if and only if $i < j$.

(2) K can be ordered with[3] a partial order $\leq^*$ so that $(K, \leq^*)$ has amalgamation. Moreover, $\leq^*$ is $\preceq_\Phi$ for a certain set Φ of formulas in $L_{\infty,\omega}$.
   [Why? This is true for any universal class without the order property. The first sentence is by [She09b, V.B.2.8, V.B.2.9] and the second is by [She09b, V.A.4.4].
(3) For $M, N \in K_{\geq\lambda}$, $M \subseteq N$ implies $M \preceq_{L_{\infty,\omega}} N$. Thus $\leq^*$ restricted to models of size at least λ is just $\subseteq$. In other words, $K_{\geq\lambda}$ has amalgamation.
   [Why? By [She09a, IV.1.12.(1)]]

□

Once we have obtained amalgamation and no maximal models, it is enough[4] to show:

Theorem 2.2. Let K be a universal class with amalgamation and no maximal models. If K is categorical in some $\lambda > H_2$, then K is categorical in all $\lambda' \geq H_2$.

The argument depends on [She99], on the construction of a good frame and related results in [Vasa], on Boney's theorem on extending good frames using tameness [Bon14] (the subsequent paper [BVb] is not needed here), and on the Grossberg-VanDieren categoricity transfer [GV06b]. The argument also depends on some results about unidimensionality in III.2 of [She09a] (these results have short full proofs, and have appeared in other forms elsewhere, most notably in [GV06b, GV06a]).

There is a dependency on the Shelah-Villaveces theorem ([SV99, Theorem 2.2.1]), which can be removed in case one is willing to assume that $\text{cf}(\lambda) > \text{LS}(K)$. This is reasonable: if K as above is categorical in a proper class of cardinals, then by amalgamation and no maximal models, the categoricity spectrum will contain a club, hence cardinals of arbitrarily high cofinality.

Proof of Theorem 2.2. We proceed in several steps.

[3]The proof of the eventual categoricity conjecture from categoricity in a single cardinal in [Vasd, Section 6] proceeds by working inside $(K, \leq^*)$ and using more of the results of [She09b, Chapter V]. Note that $(K, \leq^*)$ is an AEC, but not a universal class.

[4]Really, we want to show the result below when K is *locally* universal (see [Vasd, Definition 2.20]), since tails of universal classes need not be universal. For simplicity, we ignore this detail.

(1) K is $\text{LS}(K)$-superstable.

[Why? By [SV99, Theorem 2.2.1], or really the variation using amalgamation stated explicitly in [GV, Theorem 6.3]. Alternatively, if one is willing to assume that $\text{cf}(\lambda) > \text{LS}(K)$, one can directly apply [She99, Lemma 6.3].]

(2) K is $(< \aleph_0)$-tame.

[Why? See [Vasd, Section 3] (this does not use the categoricity hypothesis).]

(3) K is stable in λ.

[Why? By [Vasa, Theorem 5.6], $\text{LS}(K)$-superstability and $\text{LS}(K)$-tameness imply stability everywhere above $\text{LS}(K)$.]

(4) The model of size λ is saturated.

[Why? Use stability to build a μ^+-saturated model of size λ for each $\mu < \lambda$. Now apply categoricity.]

(5) K is categorical in H_2.

[Why? By the proof of [She99, II.1.6], or see [Bal09, 14.8].]

(6) K has a good H_2-frame.

[Why? By [Vasa, Theorem 7.3] which tells us how to construct a good frame at a categoricity cardinal assuming tameness and superstability below it.]

(7) For $M \in K_{H_2}$, $p \in \text{gS}(M)$, let $K_{\neg^* p}$ be defined as in [Vasd, Definition 5.7]: roughly, it is the class of N so that p has a unique extension to $\text{gS}(N)$ (so in particular p is omitted in N), but we add constant symbols for M to the language to make it closed under isomorphisms. Then $K_{\neg^* p}$ is a universal class.

[Why? That it is closed under substructure is clear. That it is closed under unions of chains is because universal classes are $(< \aleph_0)$-tame, so if a type has two distinct extensions over the union of a chain, it must have two distinct extension over an element of the chain. Here is an alternate, more general, argument: K_{H_2} is $\aleph_0$-local (by the existence of the good frame), so using tameness it is not hard to see that $K_{\geq H_2}$ is $\aleph_0$-local. Now proceed as before.]

(8) If K is not categorical in H_2^+, then there exists $M \in K_{H_2}$ and $p \in \text{gS}(M)$ so that $K_{\neg^* p}$ has a good H_2-frame.

[Why? See [Vasc, Theorem 2.15][5]: it shows that if K_{H_2} is weakly unidimensional (a property that Shelah introduces in III.2 of [She09a] and shows is equivalent to categoricity in H_2^+), then the good H_2-frame that K has, restricted to $K_{\neg^* p}$ (for a

[5]The original argument in [Vasd] is harder, as it requires building a global independence relation.

suitable p) is a good H_2-frame. The definition of weak unidimensionality is essentially the negation of the fact that there exists two types $p \perp q$ (for a notion of orthogonality defined using prime models).]

(9) If K is not categorical in H_2^+, $K_{\neg^* p}$ above has arbitrarily large models.

[Why? By Theorem 2.3 below (recalling that $K_{\neg^* p}$ is a universal class), $K_{\neg^* p}$ has a good ($\geq H_2$)-frame. Part of the definition of such a frame requires existence of a model in every cardinal $\mu \geq H_2$.

(10) If K is not categorical in H_2^+, the model of size λ is not saturated. This contradicts (4) above, therefore K is categorical in H_2^+.

[Why? Take $M \in K_{\neg^* p}$ of size λ (exists by the previous step). Then M omits p and the domain of p has size $H_2 < \lambda$.]

(11) K is categorical in all $\lambda' \geq H_2$.

[Why? We know that K is categorical in H_2 and H_2^+, so apply the upward transfer of Grossberg and VanDieren [GV06b, Theorem 0.1].

□

To complete the proof, we need the following:

Theorem 2.3. Let K be a universal class. Let $\lambda \geq \text{LS}(K)$. If K has a good λ-frame, then K has a good ($\geq \lambda$)-frame.

Proof.

(1) K is λ-tame for types of length two.

[Why? See [Vasd, Section 3].]

(2) K has weak amalgamation: if[6] $\text{gtp}(a_1/M; N_1) = \text{gtp}(a_2/M; N_2)$, there exists $N_1' \leq N_1$ containing a_1 and M and $N \geq N_1'$, $f : N_2 \underset{M}{\longrightarrow} N$ so that $f(a_2) = a_1$.

[Why? By the isomorphism characterization of Galois types in AECs which admit intersections, see [BS08, Lemma 2.6] or [Vasd, Proposition 2.17]. More explicitly, set $N_1' := \text{cl}^{N_1}(a_1 M)$, where cl^{N_1} denotes closure under the functions of N_1. Then chase the definition of equality of Galois types.]

(3) K has amalgamation.

[Why? By [Vasd, Theorem 4.14].]

[6]Since we do not assume amalgamation, Galois types are defined using the transitive closure of atomic equivalence, see e.g. [She09a, Definition II.1.9].

(4) K has a good $(\geq \lambda)$-frame.

[Why? By Boney's upward frame transfer [Bon14] which tells us that amalgamation, λ-tameness for types of length two, and a good λ-frame imply that the frame can be extended to a good $(\geq \lambda)$-frame.]

□

Proof of Theorem 1.1. Let K be a universal class categorical in a proper class of cardinals. Pick a categoricity cardinal $\lambda \geq H_1$. By Theorem 2.1, $K_{\geq \lambda}$ has amalgamation and no maximal models. By Theorem 2.2 applied to $K_{\geq \lambda}$ (ignoring for simplicity that $K_{\geq \lambda}$ is not quite a universal class, see footnote 4), $K_{\geq \lambda}$, and hence K, is categorical in all $\lambda' \geq h(h(\lambda))$. □

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