SHELAH'S EVENTUAL CATEGORICITY CONJECTURE IN TAME AECS WITH PRIMES

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Abstract. Two new cases of Shelah's eventual categoricity conjecture are established:

Theorem 0.1. Let $\mathcal{K}$ be an AEC which is tame and has primes over sets of the form $M \cup \{a\}$. If $\mathcal{K}$ is categorical in a high-enough cardinal, then $\mathcal{K}$ is categorical on a tail of cardinals.

We do *not* assume amalgamation (however the hypotheses imply that there exists a cardinal λ so that $\mathcal{K}_{\geq \lambda}$ has amalgamation). The result had previously been established when the stronger locality assumptions of full tameness and shortness are also required.

An application of Theorem 0.1 is that Shelah's categoricity conjecture holds in the context of homogeneous model theory:

Theorem 0.2. Let D be a homogeneous diagram in a first-order theory T. If D is categorical in a $\lambda > |T|$, then D is categorical in all $\lambda' \geq \min(\lambda, \beth_{(2^{|T|})^+})$.

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1. Introduction

Shelah's categoricity conjectures are a major force in the development of classification theory for abstract elementary classes (AECs)[1]. Consider the eventual version [She09, N.4.2]:

Conjecture 1.1 (Shelah's eventual categoricity conjecture)**.** An AEC categorical in a high-enough cardinal is categorical on a tail of cardinals[2].

In [Vasc], we established the conjecture for universal classes: classes of models closed under substructures and unions of $\subseteq$-increasing chains (see [She87]). The proof starts by noting that universal classes satisfy a locality property introduced in VanDieren's 2002 Ph.D. thesis (the relevant chapter appears in [GV06b]):

Definition 1.2. Let $\mathcal{K}$ be an AEC and let $\kappa > \text{LS}(\mathcal{K})$ be an infinite cardinal. $\mathcal{K}$ is $(< \kappa)$-*tame* if for any $M \in \mathcal{K}$ and any Galois types $p, q \in \text{gS}(M)$, $p = q$ if and only if[3] $p \upharpoonright M_0 = q \upharpoonright M_0$ for any $M_0 \in \mathcal{K}_{<\kappa}$ with $M_0 \leq M$. We write κ-*tame* instead of $(< \kappa^+)$-tame. We say that $\mathcal{K}$ is *tame* if it is $(< \kappa)$-tame for some cardinal κ.

Fact 1.3 ([Bon])**.** Any universal class $\mathcal{K}$ is[4] $\text{LS}(\mathcal{K})$-tame.

The proof generalizes to give a stronger locality property introduced in [Bon14]:

Definition 1.4. Let $\mathcal{K}$ be an AEC and let $\kappa > \text{LS}(\mathcal{K})$ be an infinite cardinal. $\mathcal{K}$ is *fully* $(< \kappa)$-*tame and short* if for any $M \in \mathcal{K}$, any ordinal α, and any Galois types $p, q \in \text{gS}^\alpha(M)$ of length α, $p = q$ if and only if $p^I \upharpoonright M_0 = q^I \upharpoonright M_0$ for any $M_0 \in \mathcal{K}_{<\kappa}$ with $M_0 \leq M$ and any $I \subseteq \alpha$ with $|I| < \kappa$. We define "fully κ-tame and short" and "fully tame and short" as in Definition 1.2.

Fact 1.5. Any universal class $\mathcal{K}$ is fully $\text{LS}(\mathcal{K})$-tame and short.

Another important property of universal classes used in the proof of Shelah's eventual categoricity conjecture [Vasc, Corollary 5.22] is that

[1]For a history, see the introduction of [Vasc].

[2]More precisely, there exists a map $\mu \mapsto \lambda_\mu$ such that any AEC $\mathcal{K}$ categorical in *some* $\lambda \geq \lambda_{\text{LS}(\mathcal{K})}$ is categorical in *all* $\lambda' \geq \lambda_{\text{LS}(\mathcal{K})}$.

[3]Note that since there is no assumption of amalgamation, Galois types are defined using the transitive closure of atomic equivalence.

[4]While the main idea of the proof is due to Will Boney, the fact that it applies to universal classes is due to the author. A full proof of the Fact appears as [Vasc, Theorem 3.6].

they have primes. The definition is due to Shelah and appears in [She09, Section III.3]. For the convenience of the reader, we include it here:

Definition 1.6. Let $\mathcal{K}$ be an AEC.

(1) A *prime triple* is a triple (a, M, N) such that $M \leq N$, $a \in |N| \backslash |M|$ and for every $N' \in \mathcal{K}$, $a' \in |N'|$ such that $\mathrm{gtp}(a/M; N) = \mathrm{gtp}(a'/M; N')$, there exists $f : N \xrightarrow[M]{} N'$ so that $f(a) = a'$.
(2) We say that $\mathcal{K}$ *has primes* if for every $M \in \mathcal{K}$ and every non-algebraic $p \in \mathrm{gS}(M)$, there exists a prime triple (a, M, N) so that $p = \mathrm{gtp}(a/M; N)$.
(3) We define localizations such as "$\mathcal{K}_\lambda$ has primes" in the natural way.

By taking the closure of $|M| \cup \{a\}$ under the functions of N, we get:

Fact 1.7 (Remark 5.3 in [Vasc])**.** Any universal class has primes.

The proof of the eventual categoricity conjecture for universal classes in [Vasc] generalizes to give (note that amalgamation is not assumed, but derived from the hypotheses, see below):

Fact 1.8 (Corollary 5.20 in [Vasc])**.** Fully tame and short AECs with primes satisfy Shelah's eventual categoricity conjecture.

Many results only use the assumption of tameness (for example [GV06b, GV06c, GV06a, BKV06, Lie11, Vasa, BV]), while others use full tameness and shortness [BG, Vasb] (but it is also unclear whether it is really needed there, see [Vasb, Question 15.4]).

We asked in [Vasc, Question 5.19] whether shortness could be removed from Fact 1.8. We answer in the affirmative:

Main Theorem 3.9. Tame AECs with primes satisfy Shelah's eventual categoricity conjecture.

The proof of Fact 1.8 proceeds by first obtaining the amalgamation property and no maximal models[5]. This does not need shortness. To state the new part in the proof of Theorem 3.9, we adopt notation from [Bal09, Chapter 14].

Notation 1.9. For λ an infinite cardinal, let $h(\lambda) := \beth_{(2^\lambda)^+}$. For $\mathcal{K}$ a fixed AEC, write $H_1 := h(\mathrm{LS}(\mathcal{K}))$ and $H_2 := h(H_1) = h(h(\mathrm{LS}(\mathcal{K})))$.

[5]Joint embedding then follows from categoricity.

Theorem 3.8. Let $\mathcal{K}$ be an AEC with amalgamation and no maximal models. Assume that $\mathcal{K}$ is H_2-tame and $\mathcal{K}_{\geq H_2}$ has primes. If $\mathcal{K}$ is categorical in some $\lambda > H_2$, then $\mathcal{K}$ is categorical in all $\lambda' \geq H_2$.

This improves [Vasc, Theorem 5.18] which assumed full $\text{LS}(\mathcal{K})$-tameness and shortness (so the improvement is on two counts: "full tameness and shortness" is replaced by "tameness" and "$\text{LS}(\mathcal{K})$" is replaced by "H_2"). Compared to Grossberg and VanDieren's upward transfer [GV06a], we do *not* require categoricity in a successor cardinal, but we *do* require the categoricity cardinal to be at least H_2 and more importantly ask for the AEC to have primes.

Let us justify the assumptions of Theorem 3.8. First of all, why do we ask for $\lambda > H_2$ and not e.g. $\lambda > H_1$ or even $\lambda > \text{LS}(\mathcal{K})$? The reason is that the argument uses categoricity in *two* cardinals, so we appeal to a downward categoricity transfer implicit in [She99, II.1.6] which proves (without using primes) that classes as in the hypothesis of Theorem 3.8 must be categorical in H_2. If we know that the class is categorical in two cardinals already, then we can work above $\text{LS}(\mathcal{K})$ (provided of course we adjust the levels at which tameness and primes occur). This is Theorem 3.4. Moreover if we know that for some $\chi < \lambda$, the class of χ-saturated models of $\mathcal{K}$ has primes, then we can also lower the Hanf number from H_2 to H_1 (see Theorem 3.11). Let us now discuss the structural assumptions on $\mathcal{K}$.

Many classes occurring in practice have amalgamation and no maximal models. Grossberg conjectured [Gro02, Conjecture 2.3] that the two properties[6] should follow from categoricity, and indeed the proof of Fact 1.8 show that they do assuming the other assumptions: tameness and having primes. We now focus on these two assumptions.

A wide variety of AECs are tame (see e.g. the introduction to [GV06b] or [Bon14]), and many classes studied by algebraists have primes (one example are AECs which admit intersections, i.e. whenever $N \in \mathcal{K}$ and $A \subseteq |N|$, we have that $\bigcap\{M \leq N \mid A \subseteq |M|\} \leq N$. See [BS08] or [Vasc, Section 2]). Tameness is conjectured (see [GV06a, Conjecture 1.5]) to follow from categoricity and of course, the existence of prime models plays a key role in many categoricity transfer results including especially Shelah's proof from excellence [She83a, She83b]. Currently, no general way[7] of building prime models in AECs is known except by

[6]Trivially, a categorical AEC with amalgamation eventually has no maximal models.

[7]We discuss homogeneous model theory and more generally finitary AECs later.

going through the machinery of excellence [She09, Chapter III]. It is unknown whether excellence follows from categoricity.

In the special case of homogeneous model theory, it is easier to build prime models[8]. Let $\mathcal{K}$ be a class of models of a homogeneous diagram categorical in a $\lambda > H_2$. Clearly, $\mathcal{K}$ has amalgamation, no maximal models, and is fully $(\text{LS}(\mathcal{K}))$-tame and short. By stability and [She70, Section 5], the class of H_2-saturated models of $\mathcal{K}$ has primes. The proof of Theorem 3.8 first argues without using primes that $\mathcal{K}$ is categorical in H_2. Hence the class of H_2-saturated models of $\mathcal{K}$ is just the class $\mathcal{K}_{\geq H_2}$, so it has primes. We apply Theorem 3.8 to obtain the eventual categoricity conjecture for homogeneous model theory. Actually Theorem 3.8 is not needed for that result: [Vasc, Theorem 5.18] suffices. However we can also improve on the Hanf number H_2 and obtain Theorem 0.2 from the abstract:

Theorem 4.18. Let D be a homogeneous diagram in a first-order theory T. If D is categorical in some $\lambda > |T|$, then D is categorical in all $\lambda' \geq \min(\lambda, h(|T|))$.

This was first shown for countable languages by Lessmann [Les00][9] but the uncountable case is new. We do not know whether a similar result also holds in the framework of finitary AECs (there the categoricity conjecture has been solved for tame and *simple* finitary AECs with countable Löwenheim-Skolem number [HK06][10]).

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2. Orthogonality with primes

In the proof of Fact 1.8, shortness was used only once: to build an independence notion sufficiently nice to allow the development of orthogonality calculus. More precisely, all we used was a technical statement implicit in Chapter III of [She09] and proven in details in [Vasc, Appendix B]. We improve it in this section to Theorem 2.15. Familiarity with

[8]We thank Rami Grossberg for asking us if the methods of [Vasc] could be adapted to this context.

[9]Actually, Lessmann proved a stronger statement: for a homogeneous diagram in a countable first-order theory, categoricity in some uncountable cardinal implies categoricity in *all* uncountable cardinals, i.e. there is no need to go above H_1.

[10]The argument is similar to the proof of Morley's categoricity theorem.

Section 5 and Appendix B of [Vasc] would be helpful to understand the motivations of this section but all the relevant definitions will be given.

Recall [She09, Definition II.2.1][11] that a good λ-frame is a triple $\mathfrak{s} = (\mathcal{K}_\lambda, \mathop{\smile\!\!\!\!\!\!|}\,, \text{gS}^{\text{bs}})$ where:

(1) $\mathcal{K}$ is a nonempty AEC which has λ-amalgamation, λ-joint embedding, no maximal models, and is stable in λ.
(2) For each $M \in \mathcal{K}_\lambda$, $\text{gS}^{\text{bs}}(M)$ (called the set of *basic types* over M) is a set of nonalgebraic Galois types over M satisfying (among others) the *density property*: if $M < N$ are in $\mathcal{K}_\lambda$, there exists $a \in |N| \backslash |M|$ such that $\text{gtp}(a/M; N) \in \text{gS}^{\text{bs}}(M)$.
(3) $\mathop{\smile\!\!\!\!\!\!|}$ is an (abstract) independence relation on types of elements over models in $\mathcal{K}_\lambda$ satisfying the basic properties of first-order forking in a superstable theory: invariance, monotonicity, extension, uniqueness, transitivity, local character, and symmetry (see [She09, Definition II.2.1]).

As in [She09, Definition II.6.35], we say that a good λ-frame $\mathfrak{s}$ is *type-full* if for each $M \in \mathcal{K}_\lambda$, $\text{gS}^{\text{bs}}(M)$ consists of *all* the nonalgebraic types over M. For simplicity, we focus on type-full good frames. Given a type-full good λ-frame $\mathfrak{s} = (\mathcal{K}_\lambda, \mathop{\smile\!\!\!\!\!\!|}\,, \text{gS}^{\text{bs}})$ and $M_0 \leq M$ both in $\mathcal{K}_\lambda$, we say that a nonalgebraic type $p \in \text{gS}(M)$ *does not $\mathfrak{s}$-fork over M_0* if it does not fork over M_0 according to the abstract independence relation $\mathop{\smile\!\!\!\!\!\!|}$ of $\mathfrak{s}$. When $\mathfrak{s}$ is clear from context, we just say that *p does not fork over M_0*. We say that a good λ-frame $\mathfrak{s}$ is *on $\mathcal{K}_\lambda$* if its underlying class is $\mathcal{K}_\lambda$. We say that $\mathfrak{s}$ is *categorical* if $\mathcal{K}$ is categorical in λ and we say that it *has primes* if $\mathcal{K}_\lambda$ has primes (where we localize Definition 1.6 in the natural way).

In [She09, Section II.6], Shelah develops a theory of orthogonality for good frames. His assumptions include that the good frame is *weakly successful* [She09, Definition III.1.1], so in particular it expands to an independence relation NF for models in $\mathcal{K}_\lambda$. While weak successfulness follows from full tameness and shortness [Vasb, Theorem 11.13], it is not clear if it follows from tameness only, so we do not adopt this assumption, instead we will assume that the good frame has primes.

Hypothesis 2.1. $\mathfrak{s} = (\mathcal{K}_\lambda, \mathop{\smile\!\!\!\!\!\!|}\,, \text{gS}^{\text{bs}})$ is a categorical type-full good λ-frame which has primes. We work inside $\mathfrak{s}$.

Hypothesis 2.1 is reasonable by Fact 3.2.

[11] The definition here is simpler and more general than the original: we will *not* use the axiom (B) requiring the existence of a superlimit model of size μ. Several papers (e.g. [JS13]) define good frames without this assumption.

The definition of orthogonality we adopt is formally different from [She09, Definition III.6.2] but equivalent in Shelah's context (see [She09, Claim III.6.3]).

Definition 2.2. Let $M \in \mathcal{K}_\lambda$ and let $p, q \in \text{gS}(M)$ be nonalgebraic. We say that p is *weakly orthogonal to* q and write $p \underset{\text{wk}}{\perp} q$ if for all prime triples (b, M, N) with $q = \text{gtp}(b/M; N)$ (we say that the triple *represents* q), we have that p has a unique extension to $\text{gS}(N)$.

We say that p *is orthogonal to* q (written $p \perp q$) if for every $N \in \mathcal{K}_\lambda$ with $N \geq M$, $p' \underset{\text{wk}}{\perp} q'$, where p', q' are the nonforking extensions to N of p and q respectively.

For $p_\ell \in \text{gS}(M_\ell)$ nonalgebraic, $\ell = 1, 2$, $p_1 \perp p_2$ if and only if there exists $N \geq M_\ell$, $\ell = 1, 2$ such that the nonforking extensions to N p_1' and p_2' of p_1 and p_2 respectively are orthogonal.

Remark 2.3. Formally, the definition of orthogonality depends on the frame but $\mathfrak{s}$ will always be fixed.

As noted before, in Shelah's framework for the development of orthogonality calculus, the good frame expands to a independence relation NF for *models* in $\mathcal{K}_\lambda$. Shelah uses many times the symmetry property of NF[12] : $\text{NF}(M_0, M_1, M_2, M_3)$ if and only if $\text{NF}(M_0, M_2, M_1, M_3)$. For example it is used to prove that orthogonality is symmetric: $p \perp q$ if and only if $q \perp p$. Here we do not know that NF exists (the good frame only gives us an independence relation for types of elements, not types of models) so cannot prove that orthogonality is symmetric. This does not cause any problems for the result we are interested in. The next basic lemma says that we can replace the "for all" in Definition 2.2 by "there exists". This corresponds to [She09, Claim III.6.3].

Lemma 2.4. Let $M \in \mathcal{K}_\lambda$ and $p, q \in \text{gS}(M)$ be nonalgebraic. Then $p \underset{\text{wk}}{\perp} q$ if and only if there exists a prime triple (b, M, N) representing q such that p has a unique extension to $\text{gS}(N)$.

Proof. One direction is straightforward. Now assume (b, M, N) is a prime triple representing q such that p has a unique extension to $\text{gS}(N)$. Let (b_2, M, N_2) be a prime triple representing q. Let $p_2 \in \text{gS}(N_2)$ be an extension of p. Let $N^+ \in \mathcal{K}_\lambda$ be such that $N^+ \geq N_2$ and $a \in |N^+|$ be such that $p_2 = \text{gtp}(a/N_2; N^+)$. Extending N^+ if necessary, we may assume that $N \leq N^+$. By primeness of (b, M, N), there exists

[12]For $M_0 \leq M_\ell \leq M_3$, $\ell = 1, 2$, $\text{NF}(M_0, M_1, M_2, M_3)$ stands for "M_1 is independent of M_2 over M_0 in M_3".

$f : N_2 \underset{M}{\longrightarrow} N$ such that $f(b_2) = b$. It is enough to show that $f(p_2)$ does not fork over M, so without loss of generality f is the identity, i.e. $N_2 \leq N$ and $b = b_2$. Now $\mathrm{gtp}(a/M; N^+) = p$, so $\mathrm{gtp}(a/N; N^+)$ is an extension of p, hence does not fork over M. By monotonicity, $p_2 = \mathrm{gtp}(a/N_2; N^+)$ does not fork over M. $\square$

We now show that weak orthogonality is the same as orthogonality. Recall that we are assuming categoricity in λ, so *all the models of size λ are limit (even superlimit).* Thus we can use the following property, which Shelah proves for superlimit models $M, N \in \mathcal{K}_\lambda$:

Fact 2.5 (The conjugation property, Claim III.1.21 in [She09])**.** Let $M \leq N$ be in $\mathcal{K}_\lambda$, $\alpha < \lambda$, and let $(p_i)_{i<\alpha}$ be types in $\mathrm{gS}(N)$ that do not fork over M. Then there exists $f : N \cong M$ such that $f(p_i) = p_i \upharpoonright M$ for all $i < \alpha$.

Lemma 2.6 (Claim III.6.8.(4) in [She09])**.** For $M \in \mathcal{K}_\lambda$, $p, q \in \mathrm{gS}(M)$ nonalgebraic, $p \underset{\mathrm{wk}}{\perp} q$ if and only if $p \perp q$.

Proof. Clearly if $p \perp q$ then $p \underset{\mathrm{wk}}{\perp} q$. Conversely assume $p \underset{\mathrm{wk}}{\perp} q$ and let $N \geq M$. Let p', q' be the nonforking extensions to N of p, q respectively. We want to show that $p' \underset{\mathrm{wk}}{\perp} q'$. By the conjugation property, there exists $f : N \cong M$ such that $f(p') = p$ and $f(q) = q'$. Since weak orthogonality is invariant under isomorphism, $p' \underset{\mathrm{wk}}{\perp} q'$. $\square$

We use orthogonality to study the following class of models, see [Vasc, Section 5] for motivation:

Definition 2.7. For $\mathcal{K}$ an AEC and $M \in K$, let $\mathcal{K}_M$ be the AEC defined by adding constant symbols for the elements of M and requiring that M embeds inside every model of $\mathcal{K}_M$. That is, $L(\mathcal{K}_M) = L(\mathcal{K}) \cup \{c_a \mid a \in |M|\}$, where the c_a's are new constant symbols, and

$$\mathcal{K}_M := \{(N, c_a^N)_{a \in |M|} \mid N \in K \text{ and } a \mapsto c_a^N \text{ is a } \mathcal{K}\text{-embedding from } M \text{ into } N\}$$

We order $\mathcal{K}_M$ by $(N_1, c_a^N)_{a \in |M|} \leq (N_2, c_a^{N_2})$ if and only if $N_1 \leq N_2$ and $c_a^{N_1} = c_a^{N_2}$ for all $a \in |M|$.

Definition 2.8 (III.12.39.f in [She09])**.** Let $M \in \mathcal{K}$ and let $p \in \mathrm{gS}(M)$. We define $\mathcal{K}_{\neg^* p}$ to be the class of $N \in K_M$ such that $f(p)$ has a unique extension to $\mathrm{gS}(N \upharpoonright L(\mathcal{K}))$. Here $f : M \to N$ is given by $f(a) := c_a^N$. We order $\mathcal{K}_{\neg^* p}$ with the ordering induced from $\mathcal{K}_M$.

Remark 2.9. Let $p \in \mathrm{gS}(M)$ be nonalgebraic and suppose $M \leq N$ both are in $\mathcal{K}_\lambda$. If p has a unique extension to $\mathrm{gS}(N)$, then it must be the nonforking extension. Thus p is omitted in N. However even if p is omitted in N, p might have two nonalgebraic extensions to $\mathrm{gS}(N)$, so $\mathcal{K}_{\neg^* p}$ need not be the same as the class of models omitting p.

Using local character and uniqueness, we get (see the proof of [Vasc, Proposition 5.14]):

Proposition 2.10. Let $M \in \mathcal{K}_\lambda$, $p \in \mathrm{gS}(M)$. Then $\mathcal{K}_{\neg^* p}$ is an AEC in λ (that is, its models of size λ behave like an AEC, see [She09, Definition II.1.18])

Types in $\mathcal{K}_{\neg^* p}$ are orthgonal to p:

Lemma 2.11 (Lemma B.5 in [Vasc])**.** Fix $M \in K_\lambda$ and let $p \in \mathrm{gS}(M)$ be nonalgebraic. Let $N \in K_{\neg^* p}$ be of size λ such that the map $a \mapsto c_a^N$ is the identity (so $M \leq N \upharpoonright L(K)$). For any $N_0 \leq N \upharpoonright L(K)$ with $M \leq N_0$ and any $q \in \mathrm{gS}(N_0; N)$, $p \perp q$.

Proof. Let p' be the nonforking extension of p to N_0. By Lemma 2.6, it is enough to show that p' is weakly orthogonal to q. Let (b, N_0, N') be a prime triple such that $\mathrm{gtp}(b/N_0; N') = q$ and $N' \leq N$ (exists since we are assuming that $\mathcal{K}_\lambda$ has primes). As $N \in K_{\neg^* p}$, p has a unique extension to N, hence a unique extension to N', which must be the nonforking extension so p' also has a unique extension to N'. By Lemma 2.4, this suffices to conclude that p' and q are weakly orthogonal. $\square$

Lemma 2.12. Suppose $M \leq N$ both are in $\mathcal{K}_\lambda$. Let $p \in \mathrm{gS}(M)$ be nonalgebraic and assume that $N \in K_{\neg^* p}$ (we identify N with its canonical expansion to $\mathcal{K}_{\neg^* p}$). Let $r \in \mathrm{gS}(N)$ be such that $p \perp r$. If (a, N, N') is a prime triple representing r, then $N' \in K_{\neg^* p}$.

Proof. Write $p_N, p_{N'}$ for the nonforking extension of p to $\mathrm{gS}(N)$, $\mathrm{gS}(N')$ respectively and similarly for r. We have that $p_N \perp r$ so $p_{N'}$ is the unique extension of p_N to N'. Now if p' is an extension of p to $\mathrm{gS}(N')$, then $p' \upharpoonright N = p_N$ as $N \in K_{\neg^* p}$, so $p' = p_{N'}$ by the previous sentence. This shows that $N' \in K_{\neg^* p}$, as desired. $\square$

We want to study $\mathcal{K}_{\neg^* p}$ when $\mathcal{K}$ is not weakly uni-dimensional, where:

Definition 2.13 (Definition III.2.2.6 in [She09])**.** $\mathcal{K}_\lambda$ is *weakly uni-dimensional* if for every $M < M_\ell$, $\ell = 1, 2$ all in $\mathcal{K}_\lambda$, there is $c \in |M_2| \backslash |M|$ such that $\mathrm{gtp}(c/M; M_2)$ has more than one extension to $\mathrm{gS}(M_1)$.

The next lemma justifies the "uni-dimensional" terminology: if the class is *not* uni-dimensional, then there are two orthogonal types.

Lemma 2.14 (Lemma B.6 in [Vasc])**.** If $\mathcal{K}_\lambda$ is not weakly uni-dimensional, there are $M \in K_\lambda$ and types $p, q \in \text{gS}(M)$ such that $p \perp q$.

Proof. Assume $\mathcal{K}_\lambda$ is not weakly uni-dimensional. This means that there exists $M < M_\ell$, $\ell = 1, 2$, all in $\mathcal{K}_\lambda$ such that for any $c \in |M_2| \backslash |M|$, $\text{gtp}(c/M; M_2)$ has a unique extension to $\text{gS}(M_1)$. Pick any $c \in |M_2| \backslash |M|$ and let $p := \text{gtp}(c/M; M_2)$. Then there is a natural expansion of M_1 to $\mathcal{K}_{\neg^* p}$. So pick any $d \in |M_1| \backslash |M|$ and let $q := \text{gtp}(d/M; M_1)$. By Lemma 2.11, $p \perp q$, as desired. $\square$

We have arrived to the main theorem of this section. This generalizes [Vasc, Fact B.7] which assumed in addition that $\mathfrak{s}$ was successful and good$^+$.

Theorem 2.15. If $\mathcal{K}_\lambda$ is not weakly uni-dimensional, then there exists $M \in \mathcal{K}_\lambda$ and $p \in \text{gS}(M)$ such that $\mathfrak{s} \upharpoonright \mathcal{K}_{\neg^* p}$ (the restriction of $\mathfrak{s}$ to the models in $\mathcal{K}_{\neg^* p}$) is a type-full good λ-frame with primes.

Proof. Assume $\mathcal{K}_\lambda$ is not weakly uni-dimensional. By Lemma 2.14, there exists $M \in K_\lambda$ and types $p, q \in \text{gS}(M)$ such that $p \perp q$.

Let $\mathfrak{s}_{\neg^* p} := \mathfrak{s} \upharpoonright K_{\neg^* p}$. We check that it is a type-full good λ-frame with primes. For ease of notation, we identify a model $N \in K_{\neg^* p}$ and its reduct to $\mathcal{K}$. For $N \geq M$, we write p_N for the nonforking extension of p to $\text{gS}(N)$, and similarly for q_N.

- $\mathcal{K}_{\neg^* p}$ is not empty, since (the natural expansion of) M is in it.
- $(K_{\neg^* p})_\lambda$ is an AEC in λ by Proposition 2.10.
- Nonforking has many of the usual properties: monotonicity, invariance, disjointness, local character, continuity, and transitivity all trivially follow from the definition of $\mathcal{K}_{\neg^* p}$.
- Nonforking has the uniqueness property: Let $N \in \mathcal{K}_{\neg^* p}$ have size λ. Without loss of generality $M \leq N$. Let $N' \geq N$ be in $\mathcal{K}_{\neg p}$ of size λ and let $r_1, r_2 \in \text{gS}(N')$ be nonforking over N and such that $r_1 \upharpoonright N = r_2 \upharpoonright N$. Say $r_\ell = \text{gtp}(a_\ell / N'; N_\ell)$. Now in $\mathcal{K}$, $r_1 = r_2$, and since $\mathcal{K}_\lambda$ has primes, the equality is witnessed by an embedding $f : M_1 \xrightarrow[N]{} N_2$, with $M_1 \leq N_1$. Since $N_1 \in K_{\neg^* p}$, $M_1 \in K_{\neg^* p}$, and so $r_1 = r_2$ also in $\mathcal{K}_{\neg^* p}$.
- $(K_{\neg^* p})_\lambda$ has primes by the proof of uniqueness above.
- Nonforking has the extension property. Let $N \in K_{\neg^* p}$ have size λ. Without loss of generality, $M \leq N$. Let $r \in \text{gS}(N)$

be nonalgebraic and let $N' \geq N$ be in $\mathcal{K}_{\neg^* p}$ of size λ. Let $r' \in \mathrm{gS}(N')$ be the nonforking extension of r to N' (in $\mathcal{K}$). Let (a, N', N'') be a prime triple such that $\mathrm{gtp}(a/N'; N'') = r'$. By Lemma 2.11, $p \perp r$. By Lemma 2.12, $N'' \in K_{\neg^* p}$. Thus r' is a Galois type in $\mathcal{K}_{\neg^* p}$, as desired.

- $\mathcal{K}_{\neg^* p}$ has λ-amalgamation: because $(\mathcal{K}_{\neg^* p})_\lambda$ has the type extension property (for any Galois type $q \in \mathrm{gS}(N)$ and any $N' \geq N$, q extends to $\mathrm{gS}(N')$) and has primes, one can apply [Vasc, Theorem 4.11].
- $\mathcal{K}_{\neg^* p}$ has λ-joint embedding: since any model contains a copy of M, this is a consequence of λ-amalgamation over M.
- $\mathcal{K}_{\neg^* p}$ is stable in λ: because $\mathcal{K}_{\neg^* p}$ has "fewer" Galois types than $\mathcal{K}$, and $\mathcal{K}$ is stable in λ.
- $(\mathcal{K}_{\neg^* p})_\lambda$ has no maximal models: This is where we use the negation of weakly uni-dimensional. Let $N \in \mathcal{K}_{\neg^* p}$ be of size λ and without loss of generality assume $M \leq N$. Recall from above that there is a nonalgebraic type $q \in \mathrm{gS}(M)$ such that $p \perp q$. Let q_N be the nonforking extension of q to N and let (a, N, N') be a prime triple such that $q = \mathrm{gtp}(a/N; N')$. By Lemma 2.12, $N' \in \mathcal{K}_{\neg^* p}$. Moreover as $a \in |N'| \backslash |N|$, $N < N'$, as needed.
- $\mathfrak{s}_{\neg^* p}$ is type-full: because $\mathfrak{s}$ is.
- $\mathfrak{s}_{\neg^* p}$ has symmetry: Assume $a \underset{N_0}{\overset{N}{\smile}} N_1$, for $N_0, N_1, N \in \mathcal{K}_{\neg^* p}$, $M \leq N_0 \leq N_1 \leq N$, and $a \in |N|$. Let $b \in |N_1|$. Without loss of generality, $a \notin |N_1|$ (if $a \in |N_1|$, then $a \in |N_0|$ by disjointness and as $b \underset{N_0}{\overset{N}{\smile}} N_0$, N_0 and N witness the symmetry). By symmetry in $\mathfrak{s}$, there exists $N_0', N' \in K$ such that $N \leq N'$, $N_0 \leq N_0' \leq N'$, and $b \underset{N_0}{\overset{N'}{\smile}} N_0'$ (note that the first use of $\smile$ was in $\mathfrak{s}_{\neg^* p}$ and the second in $\mathfrak{s}$, but since the first is just the restriction of the first to models in $\mathcal{K}_{\neg^* p}$, we do not make the difference). Now let N_0'' be such that $N_0 \leq N_0'' \leq N_0'$ and (a, N_0, N_0'') is a prime triple. Since $r = \mathrm{gtp}(a/N_0; N_0'') = \mathrm{gtp}(a/N_0; N)$ and p is orthogonal to r (by Lemma 2.11), we have that $N_0'' \in \mathcal{K}_{\neg^* p}$. By monotonicity, $b \underset{N_0}{\overset{N'}{\smile}} N_0''$. Now let (b, N_0'', N'') be a prime triple with $N'' \leq N'$. By Lemma 2.12, $N'' \in K_{\neg^* p}$ and by monotonicity, $b \underset{N_0}{\overset{N''}{\smile}} N_0''$. Since all the models are in $\mathcal{K}_{\neg^* p}$, this shows that the nonforking happens in $\mathfrak{s}_{\neg^* p}$, as needed.

We have checked all the properties and therefore $\mathfrak{s}_{\neg^* p}$ is a type-full good λ-frame with primes. $\square$

Assuming tameness and existence of primes above λ, we can conclude an equivalence between uni-dimensionality and categoricity. Recall (Hypothesis 2.1) that we are assuming the existence of a categorical good λ-frame.

Theorem 2.16. Assume that $\mathcal{K}_{\geq \lambda}$ has primes and is λ-tame. The following are equivalent:

(1) $\mathcal{K}$ is weakly uni-dimensional.
(2) $\mathcal{K}$ is categorical in all $\mu > \lambda$.
(3) $\mathcal{K}$ is categorical in some $\mu > \lambda$.

Proof. Exactly as in the proof of [Vasc, Theorem 5.16], except that we use Theorem 2.15. $\square$

Remark 2.17. For the proof of Theorem 2.16 (and the other categoricity transfer theorems of this paper), the symmetry property is not needed.

3. Categoricity transfers in AECs with primes

In this section, we prove Theorem 0.1 from the abstract. We first recall that the existence of good frames follow from categoricity and tameness. We use the following notation:

Notation 3.1. For $\mathcal{K}$ an AEC with amalgamation and $\lambda > \text{LS}(\mathcal{K})$, we write $\mathcal{K}^{\lambda\text{-sat}}$ for the class of λ-saturated models in $\mathcal{K}_{\geq \lambda}$.

Fact 3.2. Let $\mathcal{K}$ be a $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and no maximal models. Let λ and μ be cardinals such that both λ and μ are strictly bigger than $\text{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in μ, then:

(1) $\mathcal{K}$ is stable in every cardinal.
(2) $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$.
(3) There exists a categorical type-full good λ-frame with underlying class $\mathcal{K}^{\lambda\text{-sat}}_{\lambda}$.

Proof. By the Shelah-Villaveces theorem ([GV, Theorem 6.3]), $\mathcal{K}$ is $\text{LS}(\mathcal{K})$-superstable (see [Vasb, Definition 10.1]), in particular it is stable in $\text{LS}(\mathcal{K})$. Now we start to use $\text{LS}(\mathcal{K})$-tameness. By [VVb, Theorem 6.8], $\mathcal{K}^{\lambda\text{-sat}}$ is an AEC with $\text{LS}(\mathcal{K}^{\lambda\text{-sat}}) = \lambda$. By [Vasb, Theorem 10.8], there is a type-full good λ-frame with underlying class $\mathcal{K}^{\lambda\text{-sat}}_{\lambda}$ (and in particular stable in λ) By uniqueness of saturated models, $\mathcal{K}^{\lambda\text{-sat}}$ is categorical in λ. $\square$

We obtain a categoricity transfer for tame AECs with primes categorical in two cardinals. First we prove a more general lemma:

Lemma 3.3. Let $\mathcal{K}$ be a $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and no maximal models. Let λ and μ be cardinals such that $\text{LS}(\mathcal{K}) < \lambda < \mu$.

If $\mathcal{K}$ is categorical in μ and $\mathcal{K}^{\lambda\text{-sat}}$ has primes, then $\mathcal{K}^{\lambda\text{-sat}}$ is categorical in all $\mu' \geq \lambda$.

Proof. By Fact 3.2, there is a categorical type-full good λ-frame $\mathfrak{s}$ with underlying class $\mathcal{K}_{\lambda}^{\lambda\text{-sat}}$. Now apply Theorem 2.16 to $\mathfrak{s}$ and $\mathcal{K}^{\lambda\text{-sat}}$. $\square$

Theorem 3.4. Let $\mathcal{K}$ be a $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and no maximal models. Let λ and μ be cardinals such that $\text{LS}(\mathcal{K}) < \lambda < \mu$. Assume that $\mathcal{K}_{\geq \lambda}$ has primes.

If $\mathcal{K}$ is categorical in both λ and μ, then $\mathcal{K}$ is categorical in all $\mu' \geq \lambda$.

Proof. By categoricity, $\mathcal{K}^{\lambda\text{-sat}} = \mathcal{K}_{\geq \lambda}$. Now apply Lemma 3.3. $\square$

Remark 3.5. What if $\lambda = \text{LS}(\mathcal{K})$? Then it is open whether $\mathcal{K}$ has a good $\text{LS}(\mathcal{K})$-frame (see the discussion in [Vasa, Section 3]). If it does, then we can use Theorem 2.16.

We present two transfers from categoricity in a single cardinal. The first uses the following downward transfer which follows from the proof of [Bal09, Theorem 14.9] (an exposition of [She99, II.1.6]).

Fact 3.6. Let $\mathcal{K}$ be an AEC with amalgamation and no maximal models. If $\mathcal{K}$ is categorical in a $\lambda > H_2$ (recall Notation 1.9) and the model of size λ is H_2^+-saturated, then $\mathcal{K}$ is categorical in H_2.

To get the optimal tameness bound, we will use[13]:

Fact 3.7 (Corollary 5.4 in [VVa])**.** Let $\mathcal{K}$ be an AEC with amalgamation and no maximal models. Let $\mu \geq H_1$ and assume that $\mathcal{K}$ is categorical in a $\lambda > \mu$ so that the model of size λ is μ^+-saturated. Then there exists a categorical type-full good μ-frame with underlying class $\mathcal{K}_{\mu}^{\mu\text{-sat}}$.

[13]For a simpler proof of Theorem 3.8 from slightly stronger assumptions, replace "H_2-tame" by "χ-tame for some $\chi < H_2$. Then in the proof one can use Fact 3.2 together with Theorem 3.4, both applied to the class $\mathcal{K}_{\geq \chi}$.

Theorem 3.8. Let $\mathcal{K}$ be an AEC with amalgamation and no maximal models. Assume that $\mathcal{K}$ is H_2-tame and $\mathcal{K}_{\ge H_2}$ has primes. If $\mathcal{K}$ is categorical in some $\lambda > H_2$, then $\mathcal{K}$ is categorical in all $\lambda' \ge H_2$.

Proof. By Fact 3.2, $\mathcal{K}$ is in particular stable in λ, hence the model of size λ is saturated. By Fact 3.6, $\mathcal{K}$ is categorical in H_2. By Fact 3.7, there is a categorical type-full good H_2-frame $\mathfrak{s}$ with underlying class $\mathcal{K}_{H_2}^{H_2\text{-sat}}$. By categoricity in H_2, $\mathcal{K}^{H_2\text{-sat}} = \mathcal{K}_{\ge H_2}$. Now apply Theorem 2.16 to $\mathfrak{s}$. $\square$

We obtain Theorem 0.1 from the abstract.

Theorem 3.9. Let $\mathcal{K}$ be a tame AECs which has primes. If $\mathcal{K}$ is categorical in a high-enough cardinal, then $\mathcal{K}$ is categorical in all high-enough cardinals.

Proof. By [Vasc, Corollary 4.13], there exists λ such that $\mathcal{K}_{\ge \lambda}$ has amalgamation. By the proof of amalgamation or for example [Vasb, Proposition 10.13]), we can assume without loss of generality that $\mathcal{K}_{\ge \lambda}$ also has no maximal models. Now apply Theorem 3.8 to $\mathcal{K}_{\ge \lambda}$. $\square$

We give a variation on Theorem 3.8 which gives a lower Hanf number but assumes that classes of saturated models have primes. We will use:

Fact 3.10 (The omitting type theorem for AECs, II.1.10 in [She99])**.** Let $\mathcal{K}$ be an AEC with amalgamation. Let $\lambda \ge \chi > \text{LS}(\mathcal{K})$ be cardinals. Assume that all the models of size λ are χ-saturated. Then all the models of size at least $\min(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$ are χ-saturated.

Theorem 3.11. Let $\mathcal{K}$ be a $\text{LS}(\mathcal{K})$-tame AEC with amalgamation and no maximal models. Let $\lambda > \text{LS}(\mathcal{K})^+$ be such that $\mathcal{K}$ is categorical in λ and let $\chi \in (\text{LS}(\mathcal{K}), \lambda)$ be such that $\mathcal{K}^{\chi\text{-sat}}$ has primes. Then $\mathcal{K}$ is categorical in all $\lambda' \ge \min(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$.

Proof. By Lemma 3.3, $\mathcal{K}^{\chi\text{-sat}}$ is categorical in all $\lambda' \ge \chi$. By Fact 3.2, $\mathcal{K}$ is stable in λ, so the model of size λ is saturated, hence χ-saturated. By Fact 3.10, all the models of size at least $\lambda_0' := \min(\lambda, \sup_{\chi_0 < \chi} h(\chi_0))$ are χ-saturated. In other words, $\mathcal{K}_{\ge \lambda_0'} = \mathcal{K}_{\ge \lambda_0'}^{\chi\text{-sat}}$. Since $\mathcal{K}^{\chi\text{-sat}}$ is categorical in all $\lambda' \ge \chi$, $\mathcal{K}$ is categorical in all $\lambda' \ge \lambda_0'$. $\square$

Remark 3.12. Theorem 3.8 and Theorem 3.11 have different strengths. It could be that we know our AEC $\mathcal{K}$ has primes but it is unclear that $\mathcal{K}^{\chi\text{-sat}}$ has primes for any χ. For example, $\mathcal{K}$ could be a universal class (or more generally an AEC admitting intersections). In this case we can use Theorem 3.8. On the other hand we may not know that $\mathcal{K}$ has primes but we could know how to build primes in $\mathcal{K}^{\chi\text{-sat}}$ (for example $\mathcal{K}$ could be an elementary class or more generally a class of homogeneous models, see the next section). There Theorem 3.11 applies.

4. Categoricity in homogeneous model theory

We use the results of the previous section to obtain Shelah's categoricity conjecture for homogeneous model theory, a nonelementary framework extending classical first-order model theory. It was introduced in [She70]. We use the presentation in [GL02] but all the results on homogeneous model theory that we use initially appeared in either [She70] or [HS00].

Definition 4.1 (Definition 2.1 in [GL02])**.** Fix a first-order theory T.

(1) A set of T-types D is a *diagram in* T if it has the form $\{\text{tp}(\bar{a}/\emptyset; M) \mid \bar{a} \in {}^{<\omega}A\}$ for a model M of T.
(2) A model M of T is a *D-model* if $D(M) := \{\text{tp}(\bar{a}/\emptyset; M) \mid \bar{a} \in {}^{<\omega}|M|\} \subseteq D$.
(3) For D a diagram of T, we let $\mathcal{K}_D$ be the class of D-models of T.
(4) For M a model of T, we write $S_D^{<\omega}(A; M)$ for the set of types of finite tuples over A which are realized in some D-model N with $N \preceq M$.

Definition 4.2 (Definition 2.2 in [GL02])**.** Let T be a first-order theory and D a diagram in T. A model M of T is *(D, λ)-homogeneous* if it is a D-model and for every $N \succeq M$, every $A \subseteq |M|$ with $|A| < \lambda$, every $p \in S_D^{<\omega}(A; N)$ is realized in M.

Definition 4.3 (Hypothesis 5.5 in [GL02])**.** We say a diagram D in T is *homogeneous* if for every λ there exists a (D, λ)-homogeneous model of T.

It is straightforward to check the following (they will be used without mention):

Proposition 4.4. For D a homogeneous diagram in T:

(1) $\mathcal{K}_D$ is an AEC with $\text{LS}(\mathcal{K}_D) = |T|$.
(2) $\mathcal{K}$ has amalgamation, no maximal models, and is fully $\text{LS}(\mathcal{K})$-tame and short (in fact syntactic and Galois types coincide).
(3) For $\lambda > |T|$, a D-model M is (D, λ)-homogeneous if and only if $M \in \mathcal{K}_D^{\lambda\text{-sat}}$.

To apply the results of the previous section, we must give conditions under which $\mathcal{K}_D^{\chi\text{-sat}}$ has primes. This is implicit in [She70, Section 5]:

Fact 4.5. Let D be a homogeneous diagram in T. If $\mathcal{K}_D$ is stable in $\chi > \text{LS}(\mathcal{K})$ then $\mathcal{K}^{\chi\text{-sat}}$ has primes.

Proof. By [She70, Theorem 5.11.(1)], D satisfies a property Shelah calls $(P, \chi, 1)$ (a form of density of isolated types). By the proof of Claim 5.2.(1) and Theorem 5.3.(1) there, this implies that the class $\mathcal{K}^{\chi\text{-sat}}$ has primes. $\square$

We immediately obtain:

Theorem 4.6. If a homogeneous diagram D in a first-order theory T is categorical in a $\lambda > |T|^+$, then it is categorical in all $\lambda' \geq \min(\lambda, h(|T|))$.

Proof. Note that $\mathcal{K}_D$ is stable in all cardinals by Fact 3.2. So we can combine Fact 4.5 and Theorem 3.11. $\square$

This proves Theorem 0.2 in the abstract modulo a small wrinkle: the case $\lambda = |T|^+$. One would like to use the categoricity transfer of Grossberg and VanDieren [GV06a] but they assume that $\mathcal{K}$ is categorical in a successor $\lambda > \text{LS}(\mathcal{K})^+$ since otherwise it is in general unclear whether there is a superlimit (see [She09, Definition N.2.4.(4)]) in $\text{LS}(\mathcal{K})$ (one can get around this difficulty if $\text{LS}(\mathcal{K}) = \aleph_0$, see [Les05]). However in the case of homogeneous model theory we can show that there *is* a superlimit, completing the proof. The key is that under stability, $(D, |T|)$-homogeneous models are closed under unions of chains. While this has not appeared in print, the argument is a repeat of a first-order proof due to Harnik. All throughout, we assume:

Hypothesis 4.7. D is a homogeneous diagram in a first-order theory T. We work inside a $(D, \bar{\kappa})$-homogeneous model $\mathfrak{C}$ for $\bar{\kappa}$ a very big cardinal.

Definition 4.8 (Definition 4.8 in [GL02])**.** A type $p \in S_D^{<\omega}(A)$ *strongly splits* over $B \subseteq A$ if there exists an indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ over B and a formula $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{x}, \bar{a}_0) \in p$ and $\neg\phi(\bar{x}, \bar{a}_1) \in p$.

Definition 4.9 (Definition 4.10 in [GL02])**.** $\kappa(D)$ is the minimal cardinal κ such that for all A and all $p \in S_D^{<\omega}(A)$, there exists $B \subseteq A$ with $|B| < \kappa$ so that p does not strongly split over B.

Fact 4.10 (Theorems 4.14, 4.15 in [GL02])**.** If D is stable in $\lambda_0 \geq |T|$ and $\lambda \geq \lambda_0$, then D is stable in λ if and only if $\lambda = \lambda^{<\kappa(D)}$.

We will use the machinery of indiscernibles and averages. Note that by [GL02, Remark 3.4, Corollary 3.12], indiscernible sequences are indiscernible sets under stability. We will use this freely. The following directly follows from the definition of strong splitting:

Fact 4.11 (Theorem 5.3 in [GL02])**.** Assume that D is stable. For all infinite indiscernible sequences I over a set A and all elements b, there exists $J \subseteq I$ with $|J| < \kappa(D)$ such that $I \backslash J$ is indiscernible over $A \cup \{b\}$.

Definition 4.12 (Definition 5.4 in [GL02])**.** For I an indiscernible sequence of cardinality at least $\kappa(D)$, let $\mathrm{Av}(I/A)$ be the set of formulas $\phi(\bar{x}, \bar{a})$ with $\bar{a} \in {}^{<\omega}A$ such that for at least $\kappa(D)$-many elements $\bar{b}$ of I, $\models \phi[\bar{b}, \bar{a}]$.

Fact 4.13 (Theorem 5.5 in [GL02])**.** If D is stable and I is an indiscernible sequence of cardinality at least $\kappa(D)$, then $\mathrm{Av}(I/A) \in S_D^{<\omega}(A)$.

Fact 4.14 (Lemma 3.2.(vi) and Lemma 3.9 in [HS00])**.** Assume that D is stable and let M be $(D, \kappa(D))$-homogeneous. Assume $p \in S_D^{<\omega}(M)$ does not strongly split over $A \subseteq |M|$ with $|A| < \kappa(D)$. Then there exists an indiscernible set I over A with $|I| \geq \kappa(D)$ such that $\mathrm{Av}(I/M) = p$.

We can conclude:

Theorem 4.15. Assume that D is stable. Let δ be a limit ordinal with $\mathrm{cf}(\delta) \geq \kappa(D)$ and let $\langle M_i : i < \delta \rangle$ be an increasing sequence of (D, λ)-homogeneous models. Then $\bigcup_{i<\delta} M_i$ is (D, λ)-homogeneous.

Proof. By cofinality consideration, we can assume without loss of generality that $\delta = \mathrm{cf}(\delta)$ and $\lambda > \delta$. In particular, $\lambda > \kappa(D)$. Also without loss of generality, λ is regular. Let $M_\delta := \bigcup_{i<\delta} M_i$. Note that M_δ is $(D, \kappa(D))$-homogeneous. Let $A \subseteq |M_\delta|$ have size less than λ and let $p \in S_D^{<\omega}(A)$. Let $q \in S_D^{<\omega}(M_\delta)$ be an extension of p and assume for sake of contradiction that q is not realized in M_δ. By definition of $\kappa(D)$, there exists $B \subseteq |M|$ such that $|B| < \kappa(D)$ and q does not strongly split over B. By making A slightly bigger we can assume without loss of generality that $B \subseteq A$. By cofinality consideration, there exists $i < \delta$ such that $B \subseteq |M_i|$.

By Fact 4.14, there exists an indiscernible set I over B with $\mathrm{Av}(I/M) = q$. Enlarging I if necessary, $|I| = \lambda$. Since M_{i+1} is (D, λ)-homogeneous, we can assume without loss of generality that $I \subseteq |M_{i+1}|$. By Fact 4.11 used $|A|$-many times (recall $|A| < \lambda$), there exists $I_0 \subseteq I$ with

$|I_0| = \lambda$ and I_0 indiscernible over A. Then $\mathrm{Av}(I_0/M) = \mathrm{Av}(I/M) = q$ so $p = \mathrm{Av}(I_0/A)$. By definition of average, if $\phi(\bar{x}, \bar{a}) \in p$, there exists $\bar{b} \in I_0$ such that $\models \phi[\bar{b}, \bar{a}]$. By indiscernibility over A, this is true for any $\bar{b} \in I_0$, hence any element of I_0 realizes p. $\square$

Remark 4.16. For a general superstable tame AEC $\mathcal{K}$, results of this form are known only when $\lambda > \mathrm{LS}(\mathcal{K})$, see [BV] and the more recent [VVb, Theorem 6.8].

We can conclude with a proof of Theorem 0.2. When $\lambda = |T|^+$, we could appeal to [GV06a] but prefer to prove a more general statement using primes:

Theorem 4.17. If a homogeneous diagram D in a first-order theory T is categorical in a $\lambda > |T|$, then the class $\mathcal{K}_D^{|T|\text{-sat}}$ of its $(D, |T|)$-homogeneous models is categorical in all $\lambda' \geq |T|$. In particular, if D is also categorical in $|T|$, then it is categorical in all $\lambda' \geq |T|$

Proof. Let $\mathcal{K} := \mathcal{K}_D$ be the class of D-models of T. By Proposition 4.4, $\mathcal{K}$ is a $\mathrm{LS}(\mathcal{K})$-tame AEC (where $\mathrm{LS}(\mathcal{K}) = |T|$) with amalgamation and no maximal models. Furthermore $\mathcal{K}$ is categorical in λ. By Fact 3.2, $\mathcal{K}$ is stable in every $\mu \geq \mathrm{LS}(\mathcal{K})$. By Fact 4.10, $\kappa(D) = \aleph_0$. By Theorem 4.15, the union of any increasing chain of $(D, |T|)$-homogeneous models is $(D, |T|)$-homogeneous. Moreover, there is a unique $(D, |T|)$-homogeneous model of cardinality $|T|$ (by stability in $|T|$ and Theorem 4.15, or directly by [GL02, Theorem 5.9]). So we get that:

(1) $\mathcal{K}_D^{|T|\text{-sat}}$ is an AEC with $\mathrm{LS}(\mathcal{K}_D^{|T|\text{-sat}}) = \mathrm{LS}(\mathcal{K})$.
(2) $\mathcal{K}_D^{|T|\text{-sat}}$ has amalgamation, no maximal models, and is $\mathrm{LS}(\mathcal{K})$-tame.
(3) $\mathcal{K}_D^{|T|\text{-sat}}$ is categorical in $\mathrm{LS}(\mathcal{K})$ and λ.

Thus the last sentence in the statement of the theorem follows from uniqueness of homogeneous models. Let us prove the first. By the methods of [HS00], there is a categorical type-full good $|T|$-frame with underlying class $(\mathcal{K}_D^{|T|\text{-sat}})_{|T|}$. By the proof of Fact 4.5, $\mathcal{K}_D^{|T|\text{-sat}}$ has primes. Now apply Theorem 2.16. $\square$

Theorem 4.18. If a homogeneous diagram D in a first-order theory T is categorical in a $\lambda > |T|$, then it is categorical in all $\lambda' \geq \min(\lambda, h(|T|))$.

Proof. If $\lambda > |T|^+$, this is Theorem 4.6. If $\lambda = |T|^+$, then we use Theorem 4.17 (by categoricity in $|T|^+$, the class $\mathcal{K}_D^{|T|\text{-sat}}$ there is such that $(\mathcal{K}_D^{|T|\text{-sat}})_{\geq |T|^+} = (\mathcal{K}_D)_{\geq |T|^+}$). $\square$

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