TAMENESS AND FRAMES REVISITED

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Abstract. We combine tameness for types of singleton with the existence of a good frame to obtain some amount of tameness for types of longer sequences. We use this to show how to use tameness to extend a good frame in one cardinality to a good frame in all cardinalities, improving a theorem of Boney. Along the way, we prove many general results on the independent sequences induced by the good frame. In particular, we show that tameness and a good frame imply Shelah's notion of dimension is well-behaved, complementing previous work of Jarden and Sitton.

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1. Introduction

We show:

Theorem 1.1. Assume $\mathfrak{s}$ is a good λ-frame with underlying AEC K. If K has amalgamation and is λ-tame (for 1-types), then $\mathfrak{s}$ extends to a good $(\geq \lambda)$-frame.

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See Corollary 6.9 for the proof. This improves on Boney [Bonb], who additionally used tameness for 2-types to prove the symmetry property. As observed by Vasey in [Vas, Section 6], the result can also be proven using a nonstructure theorem if we assume in addition that K has no maximal models. However, this makes the proof nonlocal: λ-tameness for types over arbitrarily large models is needed. Our proof does not need the no maximal models hypothesis and is completely local; it works equally well to prove, for example,

Theorem 1.2. Assume $\mathfrak{s}$ is a good λ-frame with underlying AEC K. If K has amalgamation in λ^+ and is (λ, λ^+)-tame (for 1-types), then $\mathfrak{s}$ extends to a good $[\lambda, \lambda^+]$-frame.

While we were writing up this paper, Adi Jarden [Jar] independently proved Theorem 1.2 with an additional hypothesis he called the "λ^+-continuity of serial independence property", which he then deduced from the existence property for uniqueness triples (a version of domination). A byproduct of our proofs is that his continuity property holds in any good frame (see Corollary 4.10).

In the process of proving Theorem 1.1, we analyze frames with types longer than one elements and prove many general facts. In particular, we show how to start from a regular frame $\mathfrak{s}$ for 1-types and extend it to a frame $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ for types of length $< \lambda_{\mathfrak{s}}^+$. Our starting point is the fact that in a stable first-order theory, $ab \underset{A}{\perp} B$ if and only if both $a \underset{Ab}{\perp} Bb$ and $b \underset{A}{\perp} B$. Thus the nonforking relation is uniquely determined by its restriction to singleton types. This was exploited by Grossberg and Lessmann in [GL00] to obtain an independence relation from a pregeometry. In a good frame, the nonforking relation is unfortunately not defined for any base, but only for models. Thus we do not define nonforking for all sequences, but only for *independent* sequences. Extending nonforking to all sequences seems harder: Shelah showed how to do it in [She09, Section II.6] using the existence property for uniqueness triples. While independent sequences in good frames were first introduced and studied by Shelah in [She09, Section III.5], here we investigate for the first time their connection with tameness.

Assuming tameness, we show that $\mathfrak{s}^{<\lambda^+}$ is a good frame and conclude that any two infinite maximal independent sets have the same size (this was proven by Shelah under different hypotheses, and later improved on by Jarden and Sitton in [JS12, Theorem 1.1]). All our main results from tameness are summarized and proven in Corollary 6.10.

The paper is structured as follows. In Section 2, we review background in the theory of AECs. In Section 3, we give the definition of good frames and prove some easy general facts. In Section 4, we define independent sequences and show how to use them to extend a frame for types of singletons to a frame for longer types. We show all properties are preserved in the process, except perhaps symmetry. In Section 5, we give conditions under which symmetry also transfers and show how to use it to define a well-behaved notion of dimension. In Section 6, we prove Theorems 1.1 and 1.2 by studying how to extend a frame to bigger models, and how this construction interacts with independent sequences.

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2. Preliminaries

2.1. **Abstract elementary classes.** We assume the reader is familiar with the definition of an abstract elementary class (AEC) and the basic related concepts. See Grossberg's [Gro02] or Baldwin's [Bal09] for an introduction to AECs. A more advanced introduction to frames can be found in [She09, Chapter II]

For the rest of this section, fix an AEC K. We denote the partial ordering on K by $\prec$, and write $M \precneqq N$ if $M \prec N$ and $M \neq N$.

For K an abstract elementary class and $\mathcal{F}$ an interval[1] of cardinals of the form $[\lambda, \theta)$, where $\theta > \lambda \geq \text{LS}(K)$ is either a cardinal or ∞, let $K_{\mathcal{F}} := \{M \in K \mid \|M\| \in \mathcal{F}\}$. We write K_λ instead of $K_{\{\lambda\}}$, $K_{\geq\lambda}$ instead of $K_{[\lambda,\infty)}$ and $K_{\leq\lambda}$ instead of $K_{[\text{LS}(K),\lambda]}$.

The following properties of AECs are classical:

Definition 2.1. Let $\mathcal{F}$ be an interval of cardinals as above.

(1) $K_{\mathcal{F}}$ has *amalgamation* if for any $M_0 \prec M_\ell \in K_{\mathcal{F}}$, $\ell = 1, 2$ there exists $N \in K_{\mathcal{F}}$ and $f_\ell : M_\ell \xrightarrow[M_0]{} N$, $\ell = 1, 2$.
(2) $K_{\mathcal{F}}$ has *joint embedding* if for any $M_\ell \in K_{\mathcal{F}}$, $\ell = 1, 2$ there exists $N \in K_{\mathcal{F}}$ and $f_\ell : M_\ell \to N$, $\ell = 1, 2$.

[1] The definitions that follow make sense for an arbitrary set of cardinals $\mathcal{F}$, but the proofs of most of the facts below require that $\mathcal{F}$ is an interval.

(3) $K_{\mathcal{F}}$ has *no maximal models* if for any $M \in K_{\mathcal{F}}$ there exists $N \gneqq M$ in $K_{\mathcal{F}}$.

2.2. **Galois types, stability, and tameness.** We assume familiarity with Galois types (see [Gro02, Section 6]). For $M \in K$ and α an ordinal, we write $S^\alpha(M)$ for the set of Galois types of sequences of length α over M. We write $S^{<\alpha}(M)$ for $\bigcup_{\beta<\alpha} S^\beta(M)$ and $S^{<\infty}(M)$ for $\bigcup_{\beta\in \mathrm{OR}} S^\beta(M)$. We write $S(M)$ for $S^1(M)$ and $S^{\mathrm{na}}(M)$ for the set of *nonalgebraic* 1-types over M, that is:

$$S^{\mathrm{na}}(M) := \{\mathrm{gtp}(a/M;N) \mid a \in N \backslash M, M \prec N \in K\}$$

From now on, we will write $\mathrm{tp}(a/M;N)$ for $\mathrm{gtp}(a/M;N)$. If $p \in S^\alpha(M)$, we define $\ell(p) := \alpha$ and $\mathrm{dom}(p) := M$.

Say $p = \mathrm{tp}(\bar{a}/M;N) \in S^\alpha(M)$, where $\bar{a} = \langle a_i : i < \alpha\rangle$. For $X \subseteq \alpha$ and $M_0 \prec M$, write $p^X \upharpoonright M_0$ for $\mathrm{tp}(\bar{a}_X/M_0;N)$, where $\bar{a}_X := \langle a_i : i \in X\rangle$. We say $p \in S^{\alpha,\mathrm{na}}(M)$ if $a_i \notin M$ for all $i < \alpha$, and similarly define $S^{<\alpha,\mathrm{na}}(M)$ (it is easy to check these definitions do not depend on the choice of $\bar{a}$ and N).

We briefly review the notion of tameness. Although it appears implicitly (for saturated models) in Shelah [She99], tameness as a property of AECs was first introduced in Grossberg and VanDieren [GV06b] and used to prove a stability spectrum theorem there. It was later used in Grossberg and VanDieren [GV06c, GV06a] to prove an upward categoricity transfer, which Boney [Bonc] used to prove Shelah's categoricity conjecture for successors from class-many strongly compact cardinals.

Definition 2.2 (Tameness)**.** Let $\theta > \lambda \geq \mathrm{LS}(K)$ and let $\mathcal{G} \subseteq \bigcup_{M\in K} S^{<\infty}(M)$ be a family of types. We say that K is (λ,θ)*-tame for* $\mathcal{G}$ if for any $M \in K_{\leq\theta}$ and any $p, q \in \mathcal{G} \cap S^{<\infty}(M)$, if $p \neq q$, then there exists $M_0 \prec M$ of size $\leq \lambda$ such that $p \upharpoonright M_0 \neq q \upharpoonright M_0$. We define similarly $(\lambda, <\theta)$-tame, $(<\lambda,\theta)$-tame, etc. When $\theta = \infty$, we omit it. (λ,θ)-tame for α-types means (λ,θ)-tame for $\bigcup_{M\in K} S^\alpha(M)$, and similarly for $<\alpha$-types. When $\alpha = 1$, we omit it and simply say (λ,θ)-tame.

We also recall that we can define a notion of stability:

Definition 2.3 (Stability)**.** Let $\lambda \geq \mathrm{LS}(K)$ and α be cardinals. We say that K is α*-stable in* λ if for any $M \in K_\lambda$, $|S^\alpha(M)| \leq \lambda$.

We say that K is *stable* in λ if it is 1-stable in λ.

We say that K is *α-stable* if it is α-stable in λ for some $\lambda \geq \mathrm{LS}(K)$. We say that K is *stable* if it is 1-stable in λ for some $\lambda \geq \mathrm{LS}(K)$. We write "unstable" instead of "not stable".

We define similarly stability for $K_{\mathcal{F}}$, e.g. $K_{\mathcal{F}}$ is stable if and only if K is stable in λ for some $\lambda \in \mathcal{F}$.

Remark 2.4. If $\alpha < \beta$, and K is β-stable in λ, then K is α-stable in λ.

The following follows from [Bona, Theorem 3.1].

Fact 2.5. Let $\lambda \geq \mathrm{LS}(K)$. Let α be a cardinal. Assume K is stable in λ and $\lambda^\alpha = \lambda$. Then K is α-stable in λ.

2.3. **Commutative Diagrams.** Since a picture is worth a thousand words, we make extensive use of commutative diagrams to illustrate the proofs. Most of the notation is standard. When we write

$$\begin{tikzcd} M_0 \arrow[r, "f", "{[a]}"'] & M_1 \arrow[r, dotted, "g", "{[\bar{b}]}"'] & M_2 \end{tikzcd}$$

The functions f and g, typically written above arrows, are always K-embeddings; that is, $f : M_0 \cong f[M_0] \prec M_1$. Writing no functions means that the K-embedding is the identity. The elements in square brackets a and $\bar{b}$, typically written below arrows, are elements that exist in the target model, but not the source model; that is, $a \in M_1 - f[M_0]$. Writing no element simply means that there are no elements *that we wish to draw the reader's attention to* in the difference. In particular, it does *not* mean that the two models are isomorphic. We sometimes make a distinction between embeddings appearing in the hypothesis of a statement (denoted by solid lines), and those appearing in the conclusion (denoted by dotted lines).

3. Good frames

Good frames were first defined in [She09, Chapter II]. The idea is to provide a localized (i.e. only for base models of a given size λ) axiomatization of a forking-like notion for (a "nice enough" set of) 1-types. These axioms are similar to the properties of first-order forking in a superstable theory. Jarden and Shelah (in [JS13]) later gave a slightly more general definition, not assuming the existence of a superlimit model and dropping some of the redundant clauses. We give a slightly more general variation here: following [Vas], we assume the models come from $K_{\mathcal{F}}$, for $\mathcal{F}$ an interval, instead of just K_λ. We also assume that the types could be longer than just types of singletons. We first adapt the definition of a pre-λ-frame from [She09, Definition III.0.2.1]:

Definition 3.1 (Pre-frame)**.** Let α be an ordinal and let $\mathcal{F}$ be an interval of the form $[\lambda, \theta)$, where λ is a cardinal, and $\theta > \lambda$ is either a cardinal or ∞.

A *pre-*$(< \alpha, \mathcal{F})$*-frame* is a triple $\mathfrak{s} = (K, \perp, S^{\text{bs}})$, where:

(1) K is an abstract elementary class with $\lambda \geq \text{LS}(K)$, $K_\lambda \neq \emptyset$.
(2) $S^{\text{bs}} \subseteq \bigcup_{M \in K_{\mathcal{F}}} S^{<\alpha,\text{na}}(M)$. For $M \in K_{\mathcal{F}}$ and β an ordinal, we write $S^{\beta,\text{bs}}(M)$ for $S^{\text{bs}} \cap S^{\beta,\text{na}}(M)$ and similarly for $S^{<\beta,\text{bs}}(M)$.
(3) $\perp$ is a relation on quadruples of the form $(M_0, M_1, \bar{a}, N)$, where $M_0 \prec M_1 \prec N$, $\bar{a} \in {}^{<\alpha}N$, and M_0, M_1, N are all in $K_{\mathcal{F}}$. We write $\perp(M_0, M_1, \bar{a}, N)$ or $\bar{a} \underset{M_0}{\overset{N}{\perp}} M_1$ instead of $(M_0, M_1, a, N) \in \perp$.
(4) The following properties hold:
  (a) Invariance: If $f : N \cong N'$ and $\bar{a} \underset{M_0}{\overset{N}{\perp}} M_1$, then $f(\bar{a}) \underset{f[M_0]}{\overset{N'}{\perp}} f[M_1]$. If $\text{tp}(\bar{a}/M_1; N) \in S^{\text{bs}}(M_1)$, then $\text{tp}(f(\bar{a})/f[M_1]; N') \in S^{\text{bs}}(f[M_1])$.
  (b) Monotonicity: If $\bar{a} \underset{M_0}{\overset{N}{\perp}} M_1$, $\bar{a}'$ is a subsequence of $\bar{a}$, $M_0 \prec M_0' \prec M_1' \prec M_1 \prec N' \prec N \prec N''$ with $\bar{a}' \in N'$, and $N'' \in K_{\mathcal{F}}$, then $\bar{a}' \underset{M_0'}{\overset{N'}{\perp}} M_1'$ and $\bar{a}' \underset{M_0'}{\overset{N''}{\perp}} M_1'$. If $\text{tp}(\bar{a}/M_1; N) \in S^{\text{bs}}(M_1)$ and $\bar{a}'$ is a subsequence of $\bar{a}$, then $\text{tp}(\bar{a}'/M_1; N) \in S^{\text{bs}}(M_1)$.
  (c) Nonforking types are basic: If $\bar{a} \underset{M}{\overset{N}{\perp}} M$, then $\text{tp}(\bar{a}/M; N) \in S^{\text{bs}}(M)$.

A *pre-*$(\leq \alpha, \mathcal{F})$*-frame* is a pre-$(< (\alpha + 1), \mathcal{F})$-frame. When $\alpha = 1$, we drop it. We write pre-$(< \alpha, \lambda)$-frame instead of pre-$(< \alpha, \{\lambda\})$-frame or pre-$(< \alpha, [\lambda, \lambda^+))$-frame; and pre-$(< \alpha, (\geq \lambda))$-frame instead of pre-$(< \alpha, [\lambda, \infty))$-frame. We sometimes drop the $(< \alpha, \mathcal{F})$ when it is clear from context.

For $\mathfrak{s}$ a pre-$(< \alpha, \mathcal{F})$-frame, $\beta \leq \alpha$, and $\mathcal{F}' \subseteq \mathcal{F}$ an interval, we let $\mathfrak{s}_{\mathcal{F}'}^{<\beta}$ denote the pre-$(< \beta, \mathcal{F}')$-frame defined in the obvious way by restricting the basic types and $\perp$ to models in $K_{\mathcal{F}'}$ and elements of length $< \beta$. If $\mathcal{F}' = \mathcal{F}$ or $\beta = \alpha$, we omit it. For $\lambda' \in \mathcal{F}$, we write $\mathfrak{s}_{\lambda'}^{<\beta}$ instead of $\mathfrak{s}_{\{\lambda'\}}^{<\beta}$.

Notation 3.2. If $\mathfrak{s} = (K, \perp, S^{\text{bs}})$ is a pre-$(< \alpha, \mathcal{F})$-frame, then $\alpha_{\mathfrak{s}} := \alpha$, $\mathcal{F}_s := \mathcal{F}$, $K_{\mathfrak{s}} := K$, $\underset{\mathfrak{s}}{\perp} := \perp$, and $S^{\text{bs}}_{\mathfrak{s}} := S^{\text{bs}}$. If $\mathcal{F} = [\lambda, \theta)$, then let $\lambda_{\mathfrak{s}} := \lambda$, $\theta_{\mathfrak{s}} := \theta$.

By the invariance and monotonicity properties, $\perp$ is really a relation on types. This justifies the next definition.

Definition 3.3. If $\mathfrak{s} = (K, \perp, S^{\text{bs}})$ is a pre-$(< \alpha, \mathcal{F})$-frame, $p \in S^{<\alpha}(M_1)$ is a type, we say p *does not fork over* M_0 if $\bar{a} \overset{N}{\underset{M_0}{\perp}} M_1$ for some (equivalently any) $\bar{a}$ and N such that $p = \text{tp}(\bar{a}/M_1; N)$. If $\mathfrak{s}$ is not clear from context, we add "with respect to $\mathfrak{s}$".

Remark 3.4. We could have started from $(K, \perp)$ and defined the basic types as those that do not fork over their own domain. Since we are sometimes interested in studying frames that only satisfy existence over a certain class of models (like the saturated models), we will not adopt this approach.

Remark 3.5. We could also have specified only $K_{\mathcal{F}}$ or even only K_λ instead of the full AEC K. This is completely equivalent since, by [She09, Section II.2], K_λ fully determines K.

Definition 3.6 (Good frame)**.** Let α, $\mathcal{F}$ be as above.

A *good* $(< \alpha, \mathcal{F})$*-frame* is a pre-$(< \alpha, \mathcal{F})$-frame $(K, \perp, S^{\text{bs}})$ satisfying in addition:

(1) $K_{\mathcal{F}}$ has amalgamation, joint embedding, and no maximal models.
(2) <u>bs-Stability</u>: $|S^{1,\text{bs}}(M)| \leq \|M\|$ for all $M \in K_{\mathcal{F}}$.
(3) <u>Density of basic types</u>: If $M \lneq N$ are in $K_{\mathcal{F}}$, then there is $a \in N$ such that $\text{tp}(a/M; N) \in S^{\text{bs}}(M)$.
(4) <u>Existence of nonforking extension</u>: If $p \in S^{\text{bs}}(M)$, $N \succ M$ is in $K_{\mathcal{F}}$, and $\beta < \alpha$ is such that $\ell(p) \leq \beta$, then there is some $q \in S^{\beta,\text{bs}}(N)$ that does not fork over M and extends p, i.e. $q^\beta \upharpoonright M = p$.
(5) <u>Uniqueness</u>: If $p, q \in S^{<\alpha}(N)$ do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
(6) <u>Symmetry</u>: If $\bar{a}_1 \overset{N}{\underset{M_0}{\perp}} M_2$, $\bar{a}_2 \in {}^{<\alpha}M_2$, and $\text{tp}(\bar{a}_2/M_0; N) \in S^{\text{bs}}(M_0)$, then there is M_1 containing $\bar{a}_1$ and $N' \succ N$ such that $\bar{a}_2 \overset{N'}{\underset{M_0}{\perp}} M_1$.

(7) Local character: If δ is a limit ordinal, $\langle M_i \in K_{\mathcal{F}} : i \leq \delta \rangle$ is increasing continuous, and $p \in S^{\text{bs}}(M_\delta)$ is such that $\ell(p) < \text{cf}(\delta)$, then there exists $i < \delta$ such that p does not fork over M_i.
(8) Continuity: If δ is a limit ordinal, $\langle M_i \in K_{\mathcal{F}} : i \leq \delta \rangle$ and $\langle \alpha_i < \alpha : i \leq \delta \rangle$ are increasing and continuous, and $p_i \in S^{\alpha_i, \text{bs}}(M_i)$ for $i < \delta$ are such that $j < i < \delta$ implies $p_j = p_i^{\alpha_j} \restriction M_j$, then there is some $p \in S^{\alpha_\delta, \text{bs}}(M_\delta)$ such that for all $i < \delta$, $p_i = p^{\alpha_i} \restriction M_i$, and this is the unique type in $S^{\alpha_\delta}(M_\delta)$ extending each p_i. Moreover, if each p_i does not fork over M_0, then neither does p.
(9) Transitivity: If $M_0 \prec M_1 \prec M_2$, $p \in S(M_2)$ does not fork over M_1 and $p \restriction M_1$ does not fork over M_0, then p does not fork over M_0.

We will sometimes refer to "existence of nonforking extension" as simply "existence".

For $\mathbb{L}$ a list of properties, a good$^{-\mathbb{L}}$ $(< \alpha, \mathcal{F})$-frame is a pre-$(< \alpha, \mathcal{F})$-frame that satisfies all the properties of good frames except possibly the ones in $\mathbb{L}$. In this paper, $\mathbb{L}$ will only contain symmetry and/or bs-stability. We abbreviate symmetry by S, bs-stability by St, and write good$^-$ for good$^{-(S,St)}$.

We say that K has a good $(< \alpha, \mathcal{F})$-frame if there is a good $(< \alpha, \mathcal{F})$-frame where K is the underlying AEC (and similarly for good$^-$).

Remark 3.7. Transitivity follows directly from existence and uniqueness by [She09, Claim II.2.18].

Remark 3.8. The obvious monotonicity properties hold: If $\mathfrak{s}$ is a good $(< \alpha, \mathcal{F})$-frame, $\beta \leq \alpha$, and $\mathcal{F}'$ is a subinterval of $\mathcal{F}$, then $\mathfrak{s}^{<\beta}_{\mathcal{F}'}$ is a good $(< \beta, \mathcal{F}')$ frame (and similarly for good$^-$).

Remark 3.9. If T is a superstable first-order theory, then forking induces a good $(\geq |T|)$-frame on the class of models of T ordered by elementary submodel. In the non-elementary context, Shelah showed in [She09, Theorem II.3.7] how to build a good frame from local categoricity hypotheses and GCH-like assumptions, while Vasey [Vas] showed how to build a good frame in ZFC from categoricity, tameness, and a monster model. Note that a family of examples due to Hart and Shelah [HS90] demonstrates that, in the absence of tameness, an AEC could have a good λ-frame but no good $(\geq \lambda)$-frame (see [Bonb, Section 10] for a detailed writeup).

Note that for types of finite length, local character implies that nonforking is witnessed by a model of small size:

Proposition 3.10. Let $\alpha \leq \omega$. Assume $\mathfrak{s} = (K, \mathop{\smile\mkern-10mu\mid}, S^{\text{bs}})$ is a pre-$(< \alpha, \mathcal{F})$-frame satisfying local character and transitivity. If $M \in K_{\mathcal{F}}$ and $p \in S^{\text{bs}}(M)$, then there exists $M' \in K_\lambda$ such that p does not fork over M'.

Proof. Same proof as [Vas, Proposition 2.21]. $\square$

We conclude this section with an easy variation on the existence property that will be used later.

Lemma 3.11. Assume $\mathfrak{s} = (K, \mathop{\smile\mkern-10mu\mid}, S^{\text{bs}})$ is a pre-$(< \alpha, \mathcal{F})$-frame with amalgamation, existence, and transitivity. Suppose $M \prec M_0 \prec M_1$ are in $K_{\mathcal{F}}$ and $f : M_0 \to M_2$ is given with $M_2 \in K_{\mathcal{F}}$. Assume also that we have $\bar{a} \in M_1$ such that $\bar{a} \underset{M}{\overset{M_1}{\mathop{\smile\mkern-10mu\mid}}} M_0$.

There is $N \succ M_2$ and $g : M_1 \to N$ extending f such that $g(\bar{a}) \underset{g[M]}{\overset{N}{\mathop{\smile\mkern-10mu\mid}}} M_2$. A diagram is below.

$$\begin{tikzcd}
 & M_1 \arrow[r, dotted, "g"'] & N \\
 & M_0 \arrow[u, "{[\bar{a}]}"] \arrow[r, "f"'] & M_2 \arrow[u, dotted] \\
M \arrow[ur] & &
\end{tikzcd}$$

Proof. Extend f to an $L(K)$-isomorphism $\widehat{f}$ with range M_2. By existence, there is some $q \in S^{\text{bs}}(\widehat{f}^{-1}[M_2])$ that extends $\text{tp}(\bar{a}/M_0; M_1)$ and does not fork over M_0. Realize q as $\text{tp}(\bar{b}/\widehat{f}^{-1}[M_2]; N^+)$. Since $\text{tp}(\bar{a}/M_0; M_1) = \text{tp}(\bar{b}/M_0; N^+)$, there is $N^{++} \succ N^+$ and $h : M_1 \xrightarrow[M_0]{} N^{++}$ such that $h(\bar{a}) = \bar{b}$. Then, since N^++ extends $\widehat{f}^{-1}[M_2]$, we can find an $L(K)$-isomorphism $\widehat{f}^+$ that extends $\widehat{f}$ such that N^++ is the domain of $\widehat{f}^+$. Set $N := \widehat{f}^+[N^{++}]$ and $g := \widehat{f}^+ \circ g$. Some nonforking calculus shows that this works. $\square$

4. Independent sequences form a good$^-$ frame

In this section, we show how to make a good^{-S} $\mathcal{F}$-frame longer (i.e. extend the nonforking relation to longer sequences). This is done by using independent sequences, introduced by Shelah [She09, Definition

III.5.2] and also studied by Jarden and Sitton [JS12], to define basic types and nonforking. Preservation of the symmetry property will be studied in Section 5, and in Section 6 we will review how to make the frame larger (i.e. extend the nonforking relation to larger *models*).

Note that Shelah already claims many of the results of this section (for finite tuples) in [She09, Exercise III.9.4.1] but the proofs have never appeared anywhere.

Definition 4.1 (Independent sequence)**.** Let α be an ordinal and let $\mathfrak{s}$ be a pre-$\mathcal{F}$-frame.

(1) $\langle a_i : i < \alpha \rangle$, $\langle M_i : i \leq \alpha \rangle$ is said to be *independent in* (M, M', N) when:
   (a) $(M_i)_{i \leq \alpha}$ is increasing continuous in $K_{\mathcal{F}}$.
   (b) $M \prec M' \prec M_0$ and $M, M' \in K_{\mathcal{F}}$.
   (c) $M_\alpha \prec N$ is in $K_{\mathcal{F}}$.
   (d) For every $i < \alpha$, $a_i \overset{M_{i+1}}{\underset{M}{\mathop{\smile}\limits^{\mid}}} M_i$.

   $\langle a_i : i < \alpha \rangle$, $\langle M_i : i \leq \alpha \rangle$ is said to be *independent over* M when it is independent in (M, M_0, M_α).
(2) $\bar{a} := \langle a_i : i < \alpha \rangle$ is said to be *independent in* (M, M_0, N) when $M \prec M_0 \prec N$, $\bar{a} \in N$, and for some $\langle M_i : i \leq \alpha \rangle$ and a model N^+ such that $M_\alpha \prec N^+$, $N \prec N^+$, and $\langle a_i : i < \alpha \rangle$, $\langle M_i : i \leq \alpha \rangle$ is independent over M.

Remark 4.2. If $\alpha = 1$, then $\bar{a} := \langle a_0 \rangle$ is independent in (M, M', N) if and only if $\text{tp}(a_0/M'; N)$ does not fork over M.

This motivates the next definition:

Definition 4.3. Let $\mathfrak{s} := (K, \mathop{\smile}\limits^{\mid}, S^{\text{bs}})$ be a pre-$\mathcal{F}$-frame, where $\mathcal{F} = [\lambda, \theta)$. Let $\alpha \leq \theta$. Define $\mathfrak{s}^{<\alpha} := (K, \overset{<\alpha}{\mathop{\smile}\limits^{\mid}}, S^{<\alpha,\text{bs}})$ as follows:

- For $M_0 \prec M_1 \prec N$ in $K_{\mathcal{F}}$ and $\bar{a} := (a_i)_{i<\beta}$ in N with $\beta < \alpha$, $\overset{<\alpha}{\mathop{\smile}\limits^{\mid}}(M_0, M_1, \bar{a}, N)$ if and only if $\bar{a}$ is independent in (M_0, M_1, N).
- For $M \in K_{\mathcal{F}}$ and $p \in S^{<\alpha}(M)$, $p \in S^{<\alpha,\text{bs}}(M)$ if and only if there exists $N \succ M$ and $\bar{a} \in N$ such that $p = \text{tp}(\bar{a}/M; N)$ and $\overset{<\alpha}{\mathop{\smile}\limits^{\mid}}(M, M, \bar{a}, N)$.

Lemma 4.4 (Invariance)**.** Let $\mathfrak{s} := (K, \mathop{\smile}\limits^{\mid}, S^{\text{bs}})$ be a pre-$\mathcal{F}$-frame, where $\mathcal{F} = [\lambda, \theta)$. Let $\alpha \leq \theta$. Assume $K_{\mathcal{F}}$ has amalgamation. Given $\bar{a} = \langle a_i : i < \alpha \rangle$ independent in (M, M_0, M_1) and $M_2 \succ M_0$ containing $\bar{b}$ such that $\text{tp}(\bar{a}/M_0; M_1) = \text{tp}(\bar{b}/M_0; M_2)$, we have that $\bar{b}$ is independent in (M, M_0, M_2).

Proof. Let $\langle N_i : i \leq \alpha \rangle$ and N^+ witness the independence of $\bar{a}$. First , use the type equality to find $M^* \succ M_2$ and $f : M_1 \xrightarrow[M_0]{} M^*$ such that $f(a_i) = b_i$. Then, we use amalgamation to find N^* and g such that $N^* \succ M^*$ and $g : N^+ \to N^*$ extends f.

Set $N_i' := g(N_i)$ and $N^{++} := N^*$. We claim that this witnesses $\bar{b}$ is independent in (M, M_0, M_2).

- $\langle N_i' : i \leq \alpha \rangle$ is increasing and continuous because $\langle N_i : i \leq \alpha \rangle$ is.
- $M_0 \prec N_i' \prec N^{++}$ because $M_0 \prec N_i \prec N^+$; g fixes M_0; and $g(N^+) \prec N^{++}$.
- $M_2 \prec N^{++}$ by the amalgamation construction.
- $b_i \overset{N_{i+1}'}{\underset{M}{\mathop{\smile}\limits^{\vert}}} N_i'$ because we know that $a_i \overset{N_{i+1}}{\underset{M}{\mathop{\smile}\limits^{\vert}}} N_i$ and we can apply g to this.

$\square$

Remark 4.5. When dealing with types rather than sequences, the N^+ in the definition can be avoided. That is, given $p \in S^{\beta,\mathrm{bs}}(N)$ that does not fork over M, there is some $\langle a_i : i < \beta \rangle$, $\langle N^i : i \leq \beta \rangle$ such that $p = \mathrm{tp}(\langle a_i : i < \beta \rangle / N; N^\beta)$ that witnesses that $\langle a_i : i < \beta \rangle$ is independent in (M, N, N^β).

Lemma 4.6. Let $\mathfrak{s} := (K, \mathop{\smile}\limits^{\vert}, S^{\mathrm{bs}})$ be a pre-$\mathcal{F}$-frame, where $\mathcal{F} = [\lambda, \theta)$. Let $\alpha \leq \theta$. If $K_\mathcal{F}$ has amalgamation, then $\mathfrak{s}^{<\alpha}$ is a pre-$(< \alpha, \mathcal{F})$-frame.

Proof. Invariance is Lemma 4.4. For monotonicity, one can also use invariance to see that if $\bar{a}$ is independent in (M_0, M_1, N) and $N' \succ N$, then $\bar{a}$ is independent in (M_0, M_1, N'). The rest is straightforward. $\square$

The next result shows that local character and existence are preserved when elongating a frame:

Theorem 4.7. Assume $\mathfrak{s} := (K, \mathop{\smile}\limits^{\vert}, S^{\mathrm{bs}})$ is a good$^-$ $\mathcal{F}$-frame, where $\mathcal{F} = [\lambda, \theta)$. Then:

(1) $\mathfrak{s}^{<\theta}$ has local character. Moreover, if $p \in S^{\alpha,\mathrm{bs}}(N)$, then there exists $M \prec N$ in $K_{\leq \lambda + |\alpha|}$ such that p does not fork over M.
(2) $\mathfrak{s}^{<\theta}$ has existence.

Proof.

(1) Assume $p \in S^{<\theta,\mathrm{bs}}(N)$ and $N = \bigcup_{i<\delta} N_i$ with $\ell(p) < \delta < \theta$. Without loss of generality, $\delta = \mathrm{cf}(\delta)$. Then, there is some $\bar{a} =$

$\langle a_i : i < \alpha \rangle$ and increasing, continuous $\langle N^i : i \leq \alpha \rangle$ such that $\alpha < \delta$, $p = \text{tp}(\bar{a}/N; N^\alpha)$, and, for all $i < \alpha$, $a_i \underset{N}{\overset{N^{i+1}}{\perp}} N^i$. By Monotonicity for $\mathfrak{s}$, $\text{tp}(a_i/N; N^{i+1}) \in S^{\text{bs}}(N)$. By Local Character for $\mathfrak{s}$, for all $i < \alpha$ there is some $j_i < \delta$ such that $a_i \underset{N_{j_i}}{\overset{N^{i+1}}{\perp}} N$. By Transitivity for $\mathfrak{s}$, $a_i \underset{N_{j_i}}{\overset{N^{i+1}}{\perp}} N^i$. Set $j_* := \sup_{i<\alpha} j_i$; since $\text{cf}(\delta) > \alpha$, we have that $j_* < \alpha$. By Monotonicity for $\mathfrak{s}$, $a_i \underset{N_{j_*}}{\overset{N^{i+1}}{\perp}} N^i$ for all $i < \alpha$. This is exactly what we need to conclude that $\bar{a}$ is independent in (N_{j_*}, N, N^α). Thus $p = \text{tp}(\bar{a}/N; N^\alpha)$ does not fork over N_{j_*}.

The moreover part is proved similarly: By Proposition 3.10, for each $i < \alpha$ there exists $M^i \prec N$ in K_λ such that $N \underset{M^i}{\overset{N^\alpha}{\perp}} a_i$. By Transitivity, $N^i \underset{M^i}{\overset{N^\alpha}{\perp}} a_i$. Now by the Löwenheim-Skolem axiom, there exists $M \prec N$ in $K_{\leq \lambda + |\alpha|}$ such that $\bigcup_{i<\alpha} M^i \prec M$. By Monotonicity, $N^i \underset{M}{\overset{N^\alpha}{\perp}} a_i$, so $\bar{a}$ is independent in (M, N, N^α), so p does not fork over M, as needed.

(2) We prove two extension results separately: extending the domain and extending the length. Combining these two results shows that $\mathfrak{s}^{<\theta}$ has existence.

For extending the domain, let $p \in S^{<\theta,\text{bs}}(M)$ and $N \succ M$. By definition of this frame, there is some $\bar{a} = \langle a_i : i < \beta \rangle$ and increasing, continuous $\langle M^i : i \leq \beta \rangle$ such that $a_i \underset{M}{\overset{M^{i+1}}{\perp}} M^i$ for all $i < \beta$. We wish to construct increasing and continuous $\langle N^i : i \leq \beta \rangle$ and $\langle f_i : M^i \to N^i : i \leq \beta \rangle$ such that

(a) $f_0 \restriction M = \text{id}$; and

(b) $f_i(a_i) \underset{M}{\overset{N^{i+1}}{\perp}} N^i$.

This is done by induction by taking unions at limits and by using Lemma 3.11 at all successor steps. Since $\beta < \theta$, N^i is in $K_{\mathcal{F}}$ at all steps and the induction can continue. Then $\text{tp}(\bar{a}/M; M^\beta) = \text{tp}(f(\bar{a})/M; N^\beta)$ as witnessed by f and $f(\bar{a})$ is independent in (M, N, N^β). Thus, $q = \text{tp}(f(\bar{a})/N, N^\beta)$ is as desired.

$$\begin{tikzcd}
N \arrow[r, dotted] & N^i \arrow[r, dotted] & N^{i+1} \arrow[r, dotted] & N^\beta \\
M \arrow[u] \arrow[r] & M^i \arrow[r] \arrow[u, dotted, "f_i"] & M^{i+1} \arrow[r] \arrow[u, dotted, "f_{i+1}"] & M^\beta \arrow[u, dotted, "f_\beta"]
\end{tikzcd}$$

To extend the length, suppose that $\beta < \alpha < \theta$ and $p \in S^{\beta,\mathrm{bs}}(N)$ does not fork over M. This means that there is $\langle a_i : i < \beta\rangle$, $\langle N^i : i \leq \beta\rangle$ independent in (M, N, N^β) such that $p = \mathrm{tp}(\langle a_i : i < \beta\rangle/N; N^\beta)$. We will extend this sequence to be of length α by induction. At limit steps, simply taking the union of the extensions works. If we have $\beta \leq \gamma < \alpha$ and have already extended to γ (i.e., $\langle a_i : i < \gamma\rangle$, $\langle N^i : i \leq \gamma\rangle$ is defined), then let $r \in S^{\mathrm{bs}}(M)$ be arbitrary (use no maximal models and density of basic types). Let $r^+ \in S^{\mathrm{bs}}(N^\gamma)$ be its nonforking extension. Thus, there is $a_\gamma \in N^{\gamma+1}$ that realizes r^+ such that $a_\gamma \underset{M}{\overset{N^{\gamma+1}}{\perp}} N^\gamma$. Then $\langle a_i : i < \gamma{+}1\rangle$, $\langle N^i : i \leq \gamma{+}1\rangle$ is independent in $(M, N, N^{\gamma+1})$, as desired.

$\square$

The next technical lemma is key in showing that uniqueness and continuity are preserved when making a frame longer. This allows us to put together two independent sequences into one.

Lemma 4.8 (Amalgamation of independent sequences)**.** Let $\mathfrak{s}$ be a good$^-$ $\mathcal{F}$-frame, and $\beta < \theta_{\mathfrak{s}}$. Suppose that $p, q \in S^{\beta,\mathrm{bs}}(N)$ do not fork over M, that $p \restriction M = q \restriction M$, and that there are witnessing sequences $\bar{a}_\ell = \langle a^i_\ell : i < \beta\rangle$, $\langle N^i_\ell : i \leq \beta\rangle$ independent in (M, N, N^β_ℓ) for $\ell = 0, 1$ with $\bar{a}_0 \vDash p$ and $\bar{a}_1 \vDash q$. Then, there are coherent, continuous, increasing $(N_i, f_{j,i})_{j<i\leq\beta}$ and $g^i_\ell : N^i_\ell \to N_i$ such that, for all $j < i < \beta$,

$$\begin{tikzcd}
 & & N^j_1 \arrow[rr] \arrow[dr, dotted, "g^j_1"] & & N^i_1 \arrow[rr] \arrow[dr, dotted, "g^i_1"] & & N^\beta_1 \arrow[dr, dotted, "g^\beta_1"] & \\
M \arrow[r] & N \arrow[ur] \arrow[dr] & & N_j \arrow[rr, dotted, "f_{j,i}"'] & & N_i \arrow[rr, dotted, "f_{i,\beta}"'] & & N_\beta \\
 & & N^j_0 \arrow[rr] \arrow[ur, dotted, "g^j_0"'] & & N^i_0 \arrow[rr] \arrow[ur, dotted, "g^i_0"'] & & N^\beta_0 \arrow[ur, dotted, "g^\beta_0"'] &
\end{tikzcd}$$

commutes, $g_0^{i+1}(a_0^i) = g_1^{i+1}(a_1^i)$, and[2] $g_0^{i+1}(a_0^i) \underset{g_0^i[M]}{\overset{N_{i+1}}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}}} f_{i,i+1}[N_i]$.

Proof. We will build:

(1) models $\{N_i, M_\ell^i : i \leq \beta, \ell = 0, 1\}$;
(2) embeddings $\{h_\ell^i : N_\ell^i \to M_\ell^i, r_\ell^i : M_\ell^i \to N_i : i \leq \beta, \ell = 0, 1\}$; and
(3) coherent embeddings $\{f_{j,i} : N_j \to N_i, \widehat{r}_\ell^{j,i} : M_\ell^j \to M_\ell^i : i \leq \beta, \ell = 0, 1\}$

such that, for $i < \beta$:

(1)

$$\begin{tikzcd}
M_0^{i+1} \arrow[r, "r_0^{i+1}"'] & N_{i+1} \\
N_i \arrow[u] \arrow[r] & M_1^{i+1} \arrow[u, "r_1^{i+1}"']
\end{tikzcd}$$

commutes;

(2)

$$\begin{tikzcd}
N_\ell^{i+1} \arrow[rr, "h_\ell^{i+1}"'] & & M_\ell^{i+1} \\
N_\ell^i \arrow[u] \arrow[r, "h_\ell^i"'] & M_\ell^i \arrow[r, "r_\ell^i"'] & N_i \arrow[u]
\end{tikzcd}$$

commutes;

(3) $M_\ell^0 = N_0$, $r_\ell^0 = \mathrm{id}_{N_0}$ for $\ell = 0, 1$, and

$$\begin{tikzcd}
N_0^0 \arrow[r, "h_0^0"'] & N_0 \\
N \arrow[u] \arrow[r] & N_1^0 \arrow[u, "h_1^0"']
\end{tikzcd}$$

commutes;

(4) $h_\ell^{i+1}(a_\ell^i) \underset{h_\ell^{i+1}[N_\ell^i]}{\overset{M_\ell^{i+1}}{\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}}} N_i$;
(5) $r_0^{i+1} \circ h_0^{i+1}(a_0^i) = r_1^{i+1} \circ h_1^{i+1}(a_1^i)$; and
(6) $(N_i, f_{j,i})_{j<i\leq\beta}$ and $(M_\ell^i, \widehat{r}_\ell^{j,i})_{j<i\leq\beta}$ are continuous, coherent systems generated by $\widehat{r}_\ell^{i,i+1} = r_\ell^i$ and $f_{i,i+1} = r_0^i \upharpoonright N_i = r_1^i \upharpoonright N_i$.

[2]Note that $g_0^i[M] = g_1^i[M]$ by commutativity of the diagram.

Once these objects have been constructed we will have the following commutative diagram for $j < i \leq \beta$:

$$\begin{tikzcd}
 & & N_i & & \\
 & M_0^i \arrow[ur, "r_0^i"] & & M_1^i \arrow[ul, "r_1^i"'] & \\
N_0^i \arrow[ur, "h_0^i"'] & & N_j \arrow[uu, "f_{j,i}"] \arrow[ul] \arrow[ur] & & N_1^i \arrow[ul, "h_1^i"'] \\
 & M_0^j \arrow[uu, "\widehat{r}_0^{j,i}"] \arrow[ur, "r_0^j"'] & & M_1^j \arrow[uu, "\widehat{r}_1^{j,i}"] \arrow[ul, "r_1^j"] & \\
N_0^j \arrow[uu] \arrow[ur, "h_0^j"'] & & & & N_1^j \arrow[uu] \arrow[ul, "h_1^j"]
\end{tikzcd}$$

We can then take $g_\ell^i := r_\ell^i \circ h_\ell^i$. This gives the desired diagram by removing the M_i^ℓs. The function equality is given by (5) and the nonforking is given by applying $f_{i,i+1}$ to (4).

The construction proceeds by induction. At stage i, we will construct $h_\ell^i, r_\ell^i, M_\ell^i$, and N_i for $\ell = 0, 1$. Also, at each stage, we implicitly extend the coherent system by the rule given in (6) above (at successor steps) or by taking direct limits (at limit steps).

<u>$i = 0$:</u> Amalgamate N_0^0, N_1^0 over N to get N_0. Also set $M_\ell^0 := N_0$ and $r_\ell^0 := \mathrm{id}_{N_0}$ for $\ell = 0, 1$.

<u>i limit:</u> Take direct limits and use continuity to see everything is preserved.

<u>$i = j + 1$:</u> Use Lemma 3.11 –replace $(M, M_0, M_1, \bar{a}, f, M_2)$ there with $(M, N_\ell^j, N_\ell^{j+1}, a_\ell^j, r_\ell^j \circ h_\ell^j, N_j)$ here–to get $(h_\ell^{j+1}, M_\ell^{j+1})$ here, written as (g, N) there; this gives (4):

$$h_\ell^{j+1}(a_\ell^j) \underset{h_\ell^{j+1}[M]}{\overset{M_\ell^{j+1}}{\perp}} N_j$$

By the commutative diagrams, $h_0^{j+1} \restriction M = h_1^{j+1} \restriction M$, so, since a_0^j and a_1^j have the same type over M, we have that:

$$\mathrm{tp}(h_0^{j+1}(a_0^j)/h_0^{j+1}[M]; M_0^{j+1}) = \mathrm{tp}(h_1^{j+1}(a_1^j)/h_1^{j+1}[M]; M_1^{j+1})$$

By Uniqueness for $\mathfrak{s}$, these imply that:

$$\mathrm{tp}(h_0^{j+1}(a_0^j)/N_j; M_0^{j+1}) = \mathrm{tp}(h_1^{j+1}(a_1^j)/N_j; M_1^{j+1})$$

We can witness this with $r_\ell^{j+1} : M_\ell^{j+1} \to N_{j+1}$ for $\ell = 0, 1$; that is, $r_0^{j+1} \restriction N_j = r_1^{j+1} \restriction N_j$ and $r_0^{j+1} \circ h_0^{j+1}(a_0^j) = r_1^{j+1} \circ h_1^{j+1}(a_1^j)$. $\square$

Corollary 4.9. Let $\mathfrak{s} = (K, \mathop{\smile\!\!\!\!\!\!|}, S^{\mathrm{bs}})$ be a good$^-$ $\mathcal{F}$-frame. Suppose $M_0 \prec M \prec N$ are in $K_\mathcal{F}$ and $\alpha \leq \beta < \theta_\mathfrak{s}$ are such that there are $p \in S^{\alpha,\mathrm{bs}}(M)$ and $q \in S^{\beta,\mathrm{bs}}(N)$ such that $q^\alpha \restriction M = p$ and p, q do not fork over M_0. If $\bar{a}_p = \langle a_p^i : i < \alpha \rangle$, $\langle N_p^i : i \leq \alpha \rangle$ is independent in (M_0, M, N_p^α) such that $\bar{a}_p \vDash p$ and $\bar{a}_q = \langle a_q^i : i < \beta \rangle$, $\langle N_q^i : i \leq \beta \rangle$ is independent in (M_0, N, N_q^β) such that $\bar{a}_q \vDash q$, then there is $\langle M_q^i : i \leq \beta \rangle$ and $h_i : N_p^i \to M_q^i$ for $i \leq \alpha$ such that:

(1) $\bar{a}_q$, $\langle M_q^i : i \leq \beta \rangle$ is independent in (M_0, N, M_q^β);
(2) $N_q^i \prec M_q^i$ for all $i \leq \beta$; and
(3) $h_{i+1}(a_p^i) = a_q^i$ and $\mathrm{id}_M \subseteq h_i \subseteq h_{i+1}$.

Proof. First, extend the p-sequence to $\langle a_p^i : i < \beta \rangle$, $\langle N_p^i : i \leq \beta \rangle$ independent in (M_0, M, N_p^β) (use that $\mathfrak{s}^{<\theta_\mathfrak{s}}$ has existence). We can then amalgamate these sequences over M using Lemma 4.8: there is $(N_i, f_{j,i})_{j<i\leq\beta}$ and $g_x^i : N_x^i \to N_i$ for $x = p, q$ and $i \leq \beta$ as above. Since we have $g_q^\beta : N_q^\beta \cong g_q^\beta[N_q^\beta] \prec N_\beta$, we can extend g_q^β to an $L(K)$-isomorphism h with N_β in its range. Set $M_q^i := h^{-1}[N_i]$ for $i \leq \beta$. Note that $h_i := h^{-1} \circ g_q^i : N_q^i \to M_q^i$ is the identity. $\square$

Corollary 4.10. Assume $\mathfrak{s} := (K, \mathop{\smile\!\!\!\!\!\!|}, S^{\mathrm{bs}})$ is a good$^-$ $\mathcal{F}$-frame, where $\mathcal{F} = [\lambda, \theta)$. Then:

(1) $\mathfrak{s}^{<\theta}$ has uniqueness.
(2) $\mathfrak{s}^{<\theta}$ has continuity.

Proof.

(1) This follows directly from Lemma 4.8.
(2) For all $i < \delta$, there is some $\bar{a}_i = \langle a_i^k : k < \alpha_i \rangle$, $\langle N_i^k : k \leq \alpha_i \rangle$ independent in $(M_i, M_i, N_i^{\alpha_i})$ such that $p_i = \mathrm{tp}(\bar{a}_i / M_i; N_i^{\alpha_i})$.
   We will construct $\langle M_i^k : i < \delta, k \leq \alpha_i \rangle$ and $\{f_{j,i}^k : M_j^k \to M_i^k : k \leq \alpha_j, j < i < \alpha_\delta\}$ such that
   (a) $N_i^k \prec M_i^k$ and $\bar{a}_i, \langle M_i^k : k < \alpha_i \rangle$ is independent in $(M_i, M_i, M_i^{\alpha_i})$;

(b) for each $k \leq \alpha_j$, $(M_i^k, f_{l,i}^k)_{j \leq l \leq i < \alpha_\delta}$ is a coherent, direct system such that

$$\begin{tikzcd}
M_{i_2} \arrow[r] & M_{i_2}^{k_0} \arrow[r] & M_{i_2}^{k_1} \\
M_{i_1} \arrow[r] \arrow[u] & M_{i_1}^{k_0} \arrow[r] \arrow[u, "f^{k_0}_{i_1,i_2}"] & M_{i_1}^{k_1} \arrow[u, "f^{k_1}_{i_1,i_2}"] \\
M_{i_0} \arrow[r] \arrow[u] & M_{i_0}^{k_0} \arrow[r] \arrow[u, "f^{k_0}_{i_0,i_1}"] & M_{i_0}^{k_1} \arrow[u, "f^{k_1}_{i_0,i_1}"] \arrow[uu, bend right, "f^{k_1}_{i_0,i_2}"']
\end{tikzcd}$$

commutes; and

(c) $f_{j,i}^k(a_j^k) = a_i^k$.

This is possible: just apply Corollary 4.9 at successors and take direct limits at limits.

This is enough. For each $k < \alpha_\delta$, set $(M_\delta^k, f_{i,\delta}^k)_{i<\delta, k \leq \alpha_i} = \varinjlim (M_i^k, f_{j,i}^k)$. Then $\langle M_\delta^k : k < \alpha_\delta \rangle$ is increasing and continuous because each $\langle M_i^k : k < \alpha_i \rangle$ is. Set $M_\delta^{\alpha_\delta} := \cup_{k<\alpha_\delta} M_\delta^k$. For $k < \alpha_i, \alpha_j$, we have that $f_{i,\delta}^{k+1}(a_i^k) = f_{j,\delta}^{k+1}(a_j^k)$. Thus, there is no confusion in setting $a_\delta^k = f_{i,\delta}^{k+1}(a_i^k)$ for some/any $k < \alpha_i$. Set $p = \mathrm{tp}(\bar{a}_\delta / M_\delta, M_\delta^{\alpha_\delta})$.

Note that $M_\delta \prec M_\delta^0$; indeed $f_{i,\delta}^k \restriction M_i$ is the identity for all $k \leq \alpha_i$. Thus, we have that

$$p_i = \mathrm{tp}(\bar{a}_i / M_i; M_i^{\alpha_i}) = \mathrm{tp}(\langle a_\delta^k : k < \alpha_i \rangle / M_i; M_\delta^{\alpha_\delta}) = p^{\alpha_i} \restriction M_i$$

Claim: For all $k < \alpha_\delta$, $a_\delta^k \underset{M_\delta}{\overset{M_\delta^{k+1}}{\smile\!\!\!\!\!\!\mid}} M_\delta^k$.[3]

Proof of Claim: Given $i < \delta$ and $k < \alpha_i$, we have by construction that $a_i^k \underset{M_i}{\overset{M_i^{k+1}}{\smile\!\!\!\!\!\!\mid}} M_i^k$. Applying $f_{i,\delta}^k$ to this, we get $a_\delta^k \underset{M_i}{\overset{f_{i,\delta}^{k+1}(M_i^{k+1})}{\smile\!\!\!\!\!\!\mid}} f_{i,\delta}^k(M_i^k)$. By construction,

$$M_\delta^k = \bigcup_{i<\delta} f_{i,\delta}^k(M_i^k) \text{ and } M_\delta^{k+1} = \bigcup_{i<\delta} f_{i,\delta}^{k+1}(M_i^{k+1})$$

[3]Recall that, in the moreover clause, this is $a_\delta^k \underset{M_0}{\overset{M_\delta^{k+1}}{\smile\!\!\!\!\!\!\mid}} M_\delta^k$

Thus, by Continuity for $\mathfrak{s}$, we have, for all $i < \delta$, $a_\delta^k \overset{M_\delta^{k+1}}{\underset{M_i}{\mathop{\smile}\limits^{\mid}}} M_\delta^k$.

By Monotonicity for $\mathfrak{s}$, we get $a_\delta^k \overset{M_\delta^{k+1}}{\underset{M_\delta}{\mathop{\smile}\limits^{\mid}}} M_\delta^k$.

Thus, $\bar{a}_\delta, \langle M_\delta^k : k \leq \alpha_\delta \rangle$ is independent in $(M_\delta, M_\delta, M_\delta^{\alpha_\delta})$. So $p \in S^{\alpha_\delta, \text{bs}}(M_\delta)$ and extends each p_i as desired.

To show uniqueness, if we already have $q \in S^{\alpha_\delta}(M_\delta)$ that extends each p_i, then we can assume $a_i^k = a_j^k = a^k$ for all $i, j < \delta$ with $k < \alpha_i, \alpha_j$ and $\langle a^k : k < \alpha_\delta \rangle$ realizes q. We have that $f_{i,j}^{k+1}$ sends a^k to itself, so $\bar{a}_\delta = \langle a^k : k < \alpha_\delta \rangle$ realizes q. Thus, $p = q$. For the moreover clause, if each p_i does not fork over M_0, we can pick the independent sequences to witness this, and change the rest of the proof so M_0 is aways the model that types and tuples do not fork over.

$\square$

Remark 4.11. Note that a special case (when $\mathcal{F} = [\lambda, \lambda^+]$) of the continuity property above is Jarden's λ^+-continuity of serial independence (see [Jar, Definition 3.0.13]). This allows Jarden's proof that symmetry transfers up ([Jar, Proposition 4.0.31]) to go through without any extra hypotheses.

Putting everything together, we obtain that all the property of a good$^-$ frame transfer to the elongation; recall that good$^-$ frames are good frames except they might fail stability and/or symmetry. We will later see that symmetry transfers to finite sequences and give conditions under which it transfers to all sequences.

Corollary 4.12. Assume $\mathfrak{s}$ is a good$^-$ $\mathcal{F}$-frame. Then $\mathfrak{s}^{<\theta_\mathfrak{s}}$ is a good$^-$ $(< \theta_\mathfrak{s}, \mathcal{F})$-frame.

Proof. Set $\theta := \theta_\mathfrak{s}$. $\mathfrak{s}^{<\theta_\mathfrak{s}}$ is a pre-$(< \theta, \mathcal{F})$-frame by Lemma 4.6. Amalgamation, joint-embedding, no maximal models, and density of basic types hold since they hold in $\mathfrak{s}$. Existence and local character hold by Theorem 4.7, uniqueness and continuity hold by Corollary 4.10. Finally, transitivity follows from Remark 3.7. $\square$

Note that bs-stability only mentions basic 1-types, so it transfers immediately. Thus, the only property left is symmetry, which is discussed in the next two sections.

We conclude by proving a concatenation lemma for independent sequences. This is already proved for good frames in [JS12, Proposition

4.1], but the proof relies on [JS12, Proposition 2.6], which is proved as [JS13, Proposition 3.1.8] and uses symmetry in an essential way. Here, we improve this to just requiring that $\mathfrak{s}$ is a pre-frame that also satisfies amalgamation, existence, continuity, and transitivity. In particular, we avoid any use of symmetry or nonforking amalgamation. This shows that the situation is somewhat similar to the first-order context, where concatenation holds in any theory (see e.g. [GIL02, Lemma 1.6]).

Theorem 4.13 (Concatenation)**.** Assume $\mathfrak{s}$ is a pre-$\mathcal{F}$-frame with amalgamation, existence, transitivity, and continuity. Let $M \prec M_0 \prec M_1 \prec M_2$ be such that $\bar{a} = \langle a_i : i < \alpha \rangle$ is independent in (M, M_0, M_1) and $\bar{b} = \langle b_i : i < \beta \rangle$ is independent in (M, M_1, M_2). Then $\bar{a}\bar{b}$ is independent in (M, M_0, M_2).

Proof. From the independence of $\bar{a}$ in (M, M_0, M_1), there is a continuous, increasing $\langle M_0^i : i \leq \alpha \rangle$ and N_0^+ such that

- $M_0 \prec M_0^i \prec N_0^+$;
- $M_1 \prec N_0^+$; and
- $a_i \overset{M_0^{i+1}}{\underset{M}{\downarrow}} M_0^i$.

From the independence of $\bar{b}$ in (M, M_1, M_2), there is a continuous, increasing $\langle M_1^i : i \leq \beta \rangle$ and N_1^+ such that

- $M_1 \prec M_1^i \prec N_1^+$;
- $M_2 \prec N_1^+$; and
- $b_i \overset{M_1^{i+1}}{\underset{M}{\downarrow}} M_1^i$.

Define increasing and continuous $\langle N_1^i : i \leq \beta \rangle$ and $\langle g_i : M_1^i \to N_i : i \leq \beta \rangle$ such that:

- $N_0^+ \prec N_1^0$ and $g_0 \restriction M_1 = \mathrm{id}_{M_1}$; and
- For all $i < \beta$, $g_{i+1}(b_i) \overset{N_1^{i+1}}{\underset{M}{\downarrow}} N_1^i$.

This can easily be constructed by inductions: amalgamate M_1^0 and N_0^+ over M_1 to get N_1^0 and g_0. At successor steps, apply Lemma 3.11 and take unions at limit stages.

After this construction, amalgamate N_1^+ and N_1^β over M_1^β to get N^{++} and g so the following diagram commutes for $j < \beta$:

$$\begin{tikzcd}
 & & & N_0^+ \arrow[r, dotted] & N_1^0 \arrow[r, dotted] & N_1^j \arrow[r, dotted] & N_1^{j+1} \arrow[r, dotted] & N^{++} \\
 & M_0^0 \arrow[r] & M_0^\alpha \arrow[ur] & & M_1^0 \arrow[r] \arrow[u, dotted, "g_0"'] & M_1^j \arrow[r] \arrow[u, dotted, "g_j"'] & M_1^{j+1} \arrow[r] \arrow[u, dotted, "g_{j+1}"'] & N_1^+ \arrow[u, dotted, "g"'] \\
M \arrow[r] & M_0 \arrow[rr] \arrow[u] & & M_1 \arrow[rrrr] \arrow[uu] \arrow[ur] & & & & M_2 \arrow[u]
\end{tikzcd}$$

Define the sequence $\langle N^i : i \leq \alpha + \beta \rangle$ by

$$N^i := \begin{cases} M_0^i & \text{if } i \leq \alpha \\ N_1^j & \text{if } i = \alpha + j \in (\alpha, \beta] \end{cases}$$

Claim: This sequences witnesses that $\bar{c} := \bar{a}^\frown g(\bar{b})$ is independent in (M, M_0, N^{++}).

Proof of Claim: It is easy to see that this sequence is of the proper type, i.e. it is increasing and continuous and $M_0 \prec N^i \prec N^{++}$ for all $i \leq \alpha + \beta$.

If $i < \alpha$, then we need to show that $c_i \overset{N^{i+1}}{\underset{M}{\downarrow}} N^i$, which is the same as $a_i \overset{M_0^{i+1}}{\underset{M}{\downarrow}} M_0^i$. This just follows from independence of $\bar{a}$.

If $i = \alpha$, then we need to show that $c_\alpha \overset{N^{\alpha+1}}{\underset{M}{\downarrow}} N^\alpha$, which is the same as $g_1(b_0) \overset{M_0^\alpha}{\underset{M}{\downarrow}} N_1^0$. We know that $g_1(b_0) \overset{N_1^1}{\underset{M}{\downarrow}} N_1^0$ holds from the construction and we know that $M \prec M_0^\alpha \prec N_1^1$. Thus, by Monotonicity, we have the desired nonforking.

If $i = \alpha + j > \alpha$, then we need to show that $c_i \overset{N^{i+1}}{\underset{M}{\downarrow}} N^i$, which is the same as $g_{j+1}(b_j) \overset{N_1^{j+1}}{\underset{M}{\downarrow}} N_1^j$. This holds directly by the construction. $\dagger_{\textbf{Claim}}$

Notice that the map g shows that $\text{tp}(\bar{a} g(\bar{b})/M_0; N_1^\beta) = \text{tp}(\bar{a}\bar{b}/M_0; M_2)$. Thus, by Invariance (Lemma 4.4), we have that $\bar{a}\bar{b}$ is independent in (M, M_0, M_2). $\square$

5. Symmetry in long frames

In this section, we discuss when symmetry transfers from a good frame to its elongation. First we show that it transfers to finite tuples. The key is:

Fact 5.1 (Theorem 4.2.(a) in [JS12]). Let $\mathfrak{s}$ be a good^{-St} $\mathcal{F}$-frame. If $\bar{a}$ is a finite tuple independent in (M, M', N), then any permutation of $\bar{a}$ is independent in (M, M', N).

This property is crucial in our analysis, so we generalize it to infinite tuples. To make the next definition easier to state, we introduce new terminology:

Definition 5.2. A set I is said to be *independent in* (M, M_0, N) if *some* enumeration of I is independent in (M, M_0, N).

Definition 5.3. Let $\mathfrak{s}$ be a pre-$\mathcal{F}$-frame and $\mu \leq \theta_{\mathfrak{s}}$ be a cardinal. We say that $\mathfrak{s}$ has *μ-symmetry of independence* if for any $M_0 \prec M \prec N$ in $K_{\mathcal{F}}$ and any $I \subseteq N$ with $|I| < \mu$, I is independent in (M_0, M, N) if and only if *every* enumeration of I is independent in (M_0, M, N).

If $\mu = \theta_{\mathfrak{s}}$, we omit it.

Thus a restatement of Fact 5.1 is that any good^{-St} frame has $\aleph_0$-symmetry of independence.

Remark 5.4. Continuity and $\aleph_0$-symmetry of independence are enough to prove that what Jarden and Sitton call the *finite continuity property* (see [JS12, Definition 8.2]) holds in any good^{-St} λ-frame. This improves on [JS12, Proposition 8.4], which proves this with the additional assumptions that $\mathfrak{s}$ satisfies the conjugation property and density of uniqueness triples (there is no reason to define these terms here).

Theorem 5.5. Let $\mathfrak{s}$ be a good^{-} $\mathcal{F}$-frame and let $\mu \leq \theta_{\mathfrak{s}}$ be a cardinal. The following are equivalent:

(1) $\mathfrak{s}^{<\mu}$ has symmetry.
(2) For any $M_0 \prec M \prec N$ in $K_{\mathcal{F}}$ and $\bar{a}\bar{b} \in N$ such that $\ell(\bar{a}\bar{b}) < \mu$, $\bar{a}\bar{b}$ is independent in (M_0, M, N) if and only if $\bar{b}\bar{a}$ is independent in (M_0, M, N).
(3) $\mathfrak{s}$ has μ-symmetry of independence.

Proof. We first show (1) is equivalent to (2). Assume $\mathfrak{s}^{<\mu}$ has symmetry, and let $M_0 \prec M \prec N$ in $K_{\mathcal{F}}$ and $\bar{a}\bar{b} \in N$ be such that $\ell(\bar{a}\bar{b}) < \mu$ and $\bar{a}\bar{b}$ is independent in (M_0, M, N). Then there exists $\langle M^i : i \leq$

$\ell(\bar{a}\bar{b})\rangle$ and $N^+ \succ N$ witnessing it. Say $\alpha := \ell(\bar{a})$. Then $\bar{a} \in M^\alpha$, $tp(\bar{a}/M; M^\alpha) \in S^{\alpha,\text{bs}}(M_0)$, and $\bar{b}$ is independent in (M, M^α, N^+), i.e. $\bar{b} \overset{N^+}{\underset{M}{\mathop{\smile}\limits^{\mid}}} M^\alpha$. By Symmetry, there must exist a model M' containing $\bar{b}$ and $N^{++} \succ N^+$ such that $\bar{a} \overset{N^{++}}{\underset{M}{\mathop{\smile}\limits^{\mid}}} M'$. By Monotonicity, $\bar{a} \overset{N^{++}}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$, so by Transitivity, $\bar{a} \overset{N^{++}}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M'$. By Monotonicity, $\bar{b} \overset{M'}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$. By concatenation (Theorem 4.13), $\bar{b}\bar{a} \overset{N^{++}}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$ and so by Monotonicity, $\bar{b}\bar{a} \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M$, as needed.

Conversely, assume (2). Assume $\bar{a}_1 \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M_2$ with $\bar{a}_1 \in {}^{<\mu}N$, and $\bar{a}_2 \in {}^{<\mu}M_2$ is such that $\text{tp}(\bar{a}_2/M_0; N) \in S^{<\mu,\text{bs}}(M_0)$. By existence, $\bar{a}_2 \overset{M_2}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M_0$. By concatenation, $\bar{a}_1\bar{a}_2 \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M_0$. By (2), $\bar{a}_2\bar{a}_1 \overset{N}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M_0$. By definition of independent, there exists M_1 containing $\bar{a}_1$ and $N' \succ N$ such that $\bar{a}_2 \overset{N'}{\underset{M_0}{\mathop{\smile}\limits^{\mid}}} M_1$, as needed.

Next, we show that (2) is equivalent to (3). It is clear that (3) implies (2), so we assume (2) and we prove (3) as follows: we prove the following by induction on $\alpha < \mu$:

$(*)_\alpha$ Let $M_0 \prec M \prec N$ from $K_{\mathcal{F}}$ and $I \subset N$ of size $< \mu$. If I is independent in (M_0, M, N), then *every* enumeration of I of order type α is independent in (M_0, M, N).

So let $\alpha < \mu$ and assume $(*)_\beta$ holds for all $\beta < \alpha$. Suppose I as above is independent in (M_0, M, N) and let $\langle a_i : i < \alpha\rangle$ be an enumeration of I of type α.

First, suppose α is finite. Then I is finite so Fact 5.1 gives the result.

Second, suppose $\alpha = \beta + 1$ is an infinite successor. Then $\langle a_\beta\rangle^\frown\langle a_i : i < \beta\rangle$ has order type β and so (by $(*)_\beta$) is independent in (M_0, M, N). Since (2) implies (1), the original sequence must also be independent.

Finally, suppose that α is limit. By monotonicity, every subset of I is independent in (M_0, M, N). In particular, for each $\beta < \alpha$ $\{a_i : i < \beta\}$ is independent in (M_0, M, N), and so by $(*)_\beta$ $\langle a_i : i < \beta\rangle$ is also independent in (M_0, M, N). Thus by continuity (Corollary 4.10) $\langle a_i : i < \alpha\rangle$ is independent in (M_0, M, N).

□

As a corollary, we manage to solve Exercise III.9.4.1 in [She09]:

Corollary 5.6. Let $\mathfrak{s}$ be a good [good^{-St}] $\mathcal{F}$-frame. Then $\mathfrak{s}^{<\omega}$ is a good [good^{-St}] $\mathcal{F}$-frame.

Proof. By Corollary 4.12, $\mathfrak{s}^{<\omega}$ is a good^{-} $\mathcal{F}$-frame. By Fact 5.1, $\mathfrak{s}$ has $\aleph_0$-symmetry of independence. By Theorem 5.5, $\mathfrak{s}^{<\omega}$ has symmetry, as needed. Since bs-stability only refers to basic 1-types, $\mathfrak{s}$ satisfies it if and only if $\mathfrak{s}^{<\omega}$ does. □

Unfortunately, we do not know whether in general ω above can be replaced by an uncountable cardinal. To do this, Jarden and Sitton [JS12, Definition 3.4] introduced the following strong version of continuity:

Definition 5.7. Let $\mathfrak{s}$ be a pre-$\mathcal{F}$-frame and $\mu \leq \theta_{\mathfrak{s}}$ be a cardinal. We say that *μ-independence in $\mathfrak{s}$ is finitely witnessed* if for any $M_0 \prec M \prec N$ in $K_{\mathcal{F}}$ and any $I \subseteq N$ with $|I| < \mu$, I is independent in (M_0, M, N) if and only if all its finite subsets are independent in (M_0, M, N).

If $\mu = \theta_{\mathfrak{s}}$, we omit it.

Remark 5.8. In [JS12, Theorem 9.3] shows that independence is finitely witnessed in a good λ-frame assuming the conjugation property, categoricity in λ, and density of uniqueness triples. Earlier, Shelah had proven the same result under stronger hypotheses [She09, Theorem III.5.4].

We will eventually show that independence is finitely witnessed assuming λ-tameness (see Corollary 6.9). First observe that independence being finitely witnessed is also equivalent to symmetry in the elongation:

Theorem 5.9. Let $\mathfrak{s}$ be a good^{-St} $\mathcal{F}$-frame and let $\mu \leq \theta_{\mathfrak{s}}$ be a cardinal. Then $\mathfrak{s}^{<\mu}$ has symmetry if and only if μ-independence in $\mathfrak{s}$ is finitely witnessed.

Proof. By Fact 5.1, $\mathfrak{s}$ has $\aleph_0$-symmetry of independence. If μ-independence in $\mathfrak{s}$ is finitely witnessed, then $\aleph_0$-symmetry of independence is easily seen to be equivalent to μ-symmetry of independence and the result follows from Theorem 5.5. Conversely, if $\mathfrak{s}^{<\mu}$ has symmetry, then by Theorem 5.5 $\mathfrak{s}$ has μ-symmetry of independence, and by Corollary 4.12 $s^{<\mu}$ has continuity. Independence can then easily be proven to be finitely witnessed using induction on the size of the set I. □

This shows the importance of transferring symmetry across the elongation: then independence is finitely witnessed and as a result [JS12, Theorem 1.1], showing we can define a well-behaved notion of dimension, follows:

Corollary 5.10. Let $\mathfrak{s}$ be a good^{-St} λ-frame and assume $\mathfrak{s}^{<\lambda^+}$ has symmetry. Let $M \prec M_0 \prec N$ be in K_λ. If:

(1) $P \subseteq S^{\mathrm{bs}}(M_0)$
(2) I_1, I_2 are each $\subseteq$-maximal sets in

$$\{I : I \text{ is independent in } (M, M_0, N) \text{ and } a \in I \Rightarrow \mathrm{tp}(a/M_0; N) \in P\}$$

(3) One of I_1, I_2 is infinite.

Then I_1 and I_2 are both infinite and $|I_1| = |I_2|$.

Proof. By Theorem 5.9, independence in $\mathfrak{s}$ is finitely witnessed. Thus what Jarden and Sitton call the continuity property (see [JS12, Definition 5.5]) hold, and so [JS12, Theorem 1.1] gives us the result. □

Remark 5.11. In [She09, Definition III.5.12, Theorem III.5.14], Shelah had already introduced this particular notion of dimension and proven Corollary 5.10 under different hypotheses.

Next, we show symmetry indeed transfers to the elongation if the original frame is "sufficiently global":

Lemma 5.12. Assume $\mathfrak{s}$ is a good$^-$ $\mathcal{F}$-frame and $\mathcal{F} = [\lambda, \theta)$. If $\theta \geq \beth_{(2^\lambda)^+}$, then $\mathfrak{s}_\lambda^{<\lambda^+}$ has symmetry.

Proof. By [Vas, Proposition 6.3] and Fact 2.5, $K_{\mathcal{F}}$ is λ-stable in 2^λ. By the same nonstructure proof as [Vas, Corollary 6.10], it follows that $\mathfrak{s}_\lambda^{<\lambda^+}$ must have symmetry. □

Note that the methods of [Vas, Proposition 6.3] give that if $\chi := \mathfrak{tb}_\lambda^1 := \sup_{M \in K_\lambda} |S(M)|$, and $\mathfrak{s}$ is a good$^-$ $[\lambda, \chi]$-frame, then $\mathfrak{s}_\chi$ will satisfy bs-stability (and hence be a good^{-S}-frame).

We now apply the lemma to the maximal elongation of a $(\geq \lambda)$-frame $\mathfrak{s}$, namely $\mathfrak{s}^{<\infty} := \cup_{\alpha \in \mathbf{ON}} \mathfrak{s}^{<\alpha}$.

Corollary 5.13. Assume $\mathfrak{s}$ is a good$^-$ $(\geq \lambda)$-frame. Then $\mathfrak{s}^{<\infty}$ has symmetry.

Proof. Use Lemma 5.12 with each $\lambda' \in [\lambda, \infty)$. □

Corollary 5.14. Assume $\mathfrak{s}$ is a good^{-S} [good^{-}] $(\geq \lambda)$-frame. Then $\mathfrak{s}^{<\infty}$ is a good [good^{-St}] $(<\infty, \geq \lambda)$-frame.

Proof. Combine Corollary 4.12 and Corollary 5.13. □

6. Going up and transferring symmetry

We finally go back to our original motivation, which was to make a frame *larger* by defining it for types of the same length but over bigger models. This was first done in [She09, Section II.2]:

Definition 6.1 (Going up)**.** Let $\mathfrak{s} := (K, \mathop{\smile\!\!\!\!\!\!\mid}, S^{\text{bs}})$ be a pre-$(<\alpha, \lambda)$-frame, and let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals as usual. Define $\mathfrak{s}_{\mathcal{F}} := (K, \underset{\mathcal{F}}{\mathop{\smile\!\!\!\!\!\!\mid}}, S^{\text{bs}}_{\mathcal{F}})$ as follows:

- For $M_0 \prec M_1 \prec N$ in $K_{\mathcal{F}}$ and $\bar{a} \in {}^{<\alpha}N$, $\underset{\mathcal{F}}{\mathop{\smile\!\!\!\!\!\!\mid}}(M_0, M_1, \bar{a}, N)$ if and only if there exists $M_0' \prec M_0$ in K_λ such that for all $M_0' \prec M_1' \prec N' \prec N$ with $\bar{a} \in N'$, and M_1', N' in K_λ, we have $\bar{a} \overset{N'}{\underset{M_0'}{\mathop{\smile\!\!\!\!\!\!\mid}}} M_1'$.
- For $M \in K_{\mathcal{F}}$ and $p \in S^{<\alpha}(M)$, $p \in S^{\text{bs}}_{\mathcal{F}}(M)$ if and only if there exists $N \succ M$ and $\bar{a} \in N$ such that $p = \text{tp}(\bar{a}/M; N)$ and $\underset{\mathcal{F}}{\mathop{\smile\!\!\!\!\!\!\mid}}(M, M, \bar{a}, N)$.

Lemma 6.2. If $\mathfrak{s}$ be a pre-$(<\alpha, \lambda)$-frame, $\mathcal{F} = [\lambda, \theta)$ is an interval of cardinals as usual, then $\mathfrak{s}_{\mathcal{F}}$ is a pre-$(<\alpha, \mathcal{F})$-frame.

Proof. Straightforward. □

Shelah also observed that part of the properties of a good frame transferred:

Fact 6.3. Let $\mathfrak{s}$ be a good^{-} λ-frame, and let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals as usual. Then $\mathfrak{s}_{\mathcal{F}}$ satisfies all the properties of a good $\mathcal{F}$-frame except perhaps bs-stability, existence, uniqueness, and symmetry.

Proof. See [She09, Section II.2]. □

Transferring the rest of the properties from a good λ-frame to a good $[\lambda, \lambda^+]$-frame was the project of the rest of [She09, Section II] and involved combinatorial set-theoretic hypotheses and shrinking the AEC under consideration. Later it was shown in [Bonb] that all the properties transferred given enough tameness and amalgamation:

Fact 6.4 (Theorem 8.1 in [Bonb])**.** Let $\mathfrak{s}$ be a good$^-$ [good^{-S}] λ-frame, and let $\mathcal{F} = [\lambda, \theta)$ be an interval of cardinals where $\theta > \lambda$ can be ∞. If $K_{\mathcal{F}}$ has amalgamation and no maximal models, the following are equivalent:

(1) K is λ-tame for the basic types of $\mathfrak{s}_{\mathcal{F}}$.
(2) $\mathfrak{s}_{\mathcal{F}}$ is a good$^-$ [good^{-S}] $\mathcal{F}$-frame.

Moreover, if $\mathfrak{s}$ has symmetry and K is (λ, θ)-tame for 2-length types, then $\mathfrak{s}_{\mathcal{F}}$ has symmetry. In this case, the no maximal models hypothesis is not needed.

We aim to improve the moreover part of the above result. This is done in three steps. First, as observed in [Bonb], we will show that tameness for the basic types of $\left(s^{\leq 2}\right)_{\mathcal{F}}$ is enough. Second, we will also see that tameness for the basic types of $(s_{\mathcal{F}})^{\leq 2}$ follows for free from tameness of the basic types of $\mathfrak{s}_{\mathcal{F}}$. Third, to close the loop, we will prove that the operations $\mathfrak{s} \mapsto \mathfrak{s}^{\leq 2}$ and $\mathfrak{s} \mapsto \mathfrak{s}_{\mathcal{F}}$ commute. One direction is easy:

Proposition 6.5. Let $\mathfrak{s} := (K, \perp, S^{\text{bs}})$ be a pre-λ-frame, and let $\mathcal{F} := [\lambda, \theta)$ be an interval of cardinals as usual. Assume $K_{\mathcal{F}}$ has amalgamation. Then:

$$(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+} \subseteq \left(\mathfrak{s}^{<\lambda^+}\right)_{\mathcal{F}}$$

Where $\subseteq$ is taken componentwise.

Proof. Assume we know that $\underset{(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}}{\perp}(M_0, M, \bar{a}, N)$. We show that $\underset{(\mathfrak{s}^{<\lambda^+})_{\mathcal{F}}}{\perp}(M_0, M, \bar{a}, N)$. The proof of inclusion of the basic types is completely similar.

Let $\bar{a} := \langle a_i : i < \beta \rangle$, for $\beta < \lambda^+$. By assumption, $\bar{a}$ is independent (with respect to $\underset{\mathcal{F}}{\perp}$) in (M_0, M, N). Fix $\langle M^i : i \leq \beta \rangle$ and N^+ witnessing the independence. In particular, for every $i < \beta$, $\underset{\mathcal{F}}{\perp}(M_0, M^i, a_i, N^+)$. By definition of $\underset{\mathcal{F}}{\perp}$, this implies in particular that for each $i < \beta$, there exists $M_i^0 \prec M_0$ in K_λ such that $\underset{\mathcal{F}}{\perp}(M_i^0, M^i, a_i, N^+)$. Using the Löwenheim-Skolem axiom and the fact that $|\beta| \leq \lambda$, we can choose $M^* \prec M_0$ in K_λ such that for all $i < \beta$,we have $M_i^0 \prec M^*$. Thus,

$\underset{\mathcal{F}}{\perp}(M^0, M_i, a_i, N^+)$ for all $i < \beta$. In particular, $\bar{a}$ is independent (with respect to $\underset{\mathcal{F}}{\perp}$) in (M^*, M, N).

Now fix any $M_0', N' \in K_\lambda$ such that $\bar{a} \in N'$, $M^* \prec M_0' \prec M$, and $M_0' \prec N' \prec N$. We claim that $\bar{a}$ is independent (with respect to $\perp$) in (M^*, M_0', N'), i.e. $\overset{<\lambda^+}{\perp}(M^*, M_0', \bar{a}, N')$. To see this, construct $\langle M_i' \in K_\lambda : i \leq \beta\rangle$ increasing continuous such that for all $i \leq \beta$, $M^* \prec M_i' \prec M^i$ and $a_i \in M_{i+1}'$. Finally, pick $(N^+)' \in K_\lambda$ such that $M_\beta', N' \prec (N^+)' \prec N^+$. Then $\langle M_i' : i \leq \beta\rangle$ and $(N^+)'$ witness our claim. By definition, this means exactly that $\underset{(\mathfrak{s}^{<\alpha})_{\mathcal{F}}}{\perp}(M_0, M, \bar{a}, N)$, as needed. $\square$

The converse needs more hypotheses and relies on Corollary 4.12:

Theorem 6.6. Let $\mathfrak{s} := (K, \perp, S^{\text{bs}})$ be a good$^-$ λ-frame, and let $\mathcal{F} := [\lambda, \theta)$ be an interval of cardinals as usual. Assume that $\mathfrak{s}_{\mathcal{F}}$ is a good$^-$ $\mathcal{F}$-frame. Then:

$$(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+} = \left(\mathfrak{s}^{<\lambda^+}\right)_{\mathcal{F}}$$

Proof. By Proposition 6.5 and existence, it is enough to show $\underset{(\mathfrak{s}^{<\lambda^+})_{\mathcal{F}}}{\perp} \subseteq \underset{(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}}{\perp}$. Assume $\underset{(\mathfrak{s}^{<\lambda^+})_{\mathcal{F}}}{\perp}(M, N, \bar{a}, \widehat{N})$. By definition of $\underset{(\mathfrak{s}^{<\alpha})_{\mathcal{F}}}{\perp}$ and monotonicity, we can assume without loss of generality that $M \in K_\lambda$. We know that for all $N' \prec N$ and $\widehat{N}' \prec \widehat{N}$ in K_λ with $M \prec N \prec \widehat{N}'$ and $\bar{a} \in \widehat{N}'$, $\bar{a}$ is independent (with respect to $\perp$) in $(M, N', \widehat{N}')$. We want to see that $\bar{a}$ is independent (with respect to $\underset{\mathcal{F}}{\perp}$) in $(M, N, \widehat{N})$.

Let $\mu \geq \lambda$ be such that $N, \widehat{N} \in K_{\leq \mu}$. Work by induction on μ. We already have what we want if $\mu = \lambda$, so assume $\mu > \lambda$. Let $(N_i)_{i \leq \mu}$ be an increasing continuous resolution of N such that $N_\mu = N$, $N_0 = M$, $\|N_i\| = \lambda + |i|$.

By the induction hypothesis and monotonicity, $\bar{a}$ is independent (with respect to $\underset{\mathcal{F}}{\perp}$) in $(M, N_i, \widehat{N})$ for all $i < \mu$. In other words, for any $i < \mu$, $\text{tp}(\bar{a}/N_i; \widehat{N}))$ does not fork (in the sense of $(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}$) over M. By Corollary 4.12, we know that $(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}$ has continuity. Thus $\text{tp}(\bar{a}/N; \widehat{N})$ also does not fork (in the sense of $(\mathfrak{s}_{\mathcal{F}})^{<\lambda^+}$) over M. This is exactly what we needed to prove. $\square$

Tameness for the basic types of large good frames follows from uniqueness as in [Bonb, Theorem 3.2]. We now show that having a large good frame implies some tameness for long types:

Proposition 6.7. Assume $\mathfrak{s} := (K, \mathop{\smile\!\!\!\!\!\!|}, S^{\text{bs}})$ is a good$^-$ $\mathcal{F}$-frame. Let $\mathcal{F} := [\lambda, \theta)$.

For each $\alpha < \theta$, K is $(\lambda + |\alpha|, < \theta)$-tame for the basic types of $\mathfrak{s}^{<\theta}$ of length $\leq \alpha$.

Proof. Let $\alpha < \theta$, and let $p, q \in S^{\leq \alpha, \text{bs}}(M)$ be distinct. By the moreover part of Theorem 4.7.(1), one can find $M_0 \prec M$ in $K_{\leq \lambda + |\alpha|}$ such that both p and q do not fork over M_0. By uniqueness, we must have $p \upharpoonright M_0 \neq q \upharpoonright M_0$, as needed. □

We are now ready to prove the required symmetry transfer. We first state it abstractly without mentioning tameness:

Theorem 6.8. Assume $\mathfrak{s}$ is a good$^-$ $\mathcal{F}$-frame. Let $\mathcal{F} := [\lambda, \theta)$.

Then $\mathfrak{s}$ has symmetry if and only if $\mathfrak{s}_\lambda$ has symmetry.

Proof. Of course, symmetry for $\mathfrak{s}$ implies in particular symmetry for $\mathfrak{s}_\lambda$. Now assume symmetry for $\mathfrak{s}_\lambda$.

First note that $\mathfrak{s} = (\mathfrak{s}_\lambda)_{\mathcal{F}}$. This is because by the methods of [She09, Section II.2] (see especially Claim 2.14 and the remark preceding it)

Let $\mathfrak{t} := \mathfrak{s}_\lambda := (K, \mathop{\smile\!\!\!\!\!\!|}, S^{\text{bs}})$. Thus $\mathfrak{s} = \mathfrak{t}_{\mathcal{F}}$. Recall that [Bonb, Theorem 6.1] proves symmetry for $\mathfrak{s}$ assuming $(\lambda, < \theta)$-tameness for 2-types. We revisit this proof and use the same notation.

Suppose $\underset{\mathcal{F}}{\mathop{\smile\!\!\!\!\!\!|}}(M_0, M_2, a_1, M_3)$, $a_2 \in M_2$ with $\text{tp}(a_2/M_0; M_3) \in S^{\text{bs}}_{\mathcal{F}}(M_0)$. Let $M_0 \prec M_1 \prec M_3$ be a model containing a_1. By existence, there is $M_3' \succ M_3$ and $a' \in M_3'$ such that $\underset{\mathcal{F}}{\mathop{\smile\!\!\!\!\!\!|}}(M_0, M_1, a', M_3')$ and $\text{tp}(a'/M_0; M_3') = \text{tp}(a_2/M_0; M_3)$. Boney argues it is enough to see that $p := \text{tp}(a_1 a_2/M_0; M_3) = \text{tp}(a_1 a'/M_0; M_3') =: p'$, shows that this equality holds for all restrictions to models of size λ, and then uses tameness for 2-types. This is not part of our hypotheses, but by Proposition 6.7, it is enough to see that p, p' are basic types of $\mathfrak{s}^{\leq 2}$.

First, let's see that $a_1 a_2$ is independent (with respect to $\underset{\mathcal{F}}{\mathop{\smile\!\!\!\!\!\!|}}$) in (M_0, M_0, M_3).

The increasing chain (M_0, M_2, M_3) witnesses that $a_2 a_1$ is independent (with respect to $\underset{\mathcal{F}}{\mathop{\smile\!\!\!\!\!\!|}}$ again) in (M_0, M_0, M_3). Thus $\text{tp}(a_2 a_1/M_0; M_3) \in$

$S^{\mathrm{bs}}_{\mathfrak{s}^{\leq 2}}(M_0)$, and $\mathfrak{s}^{\leq 2} = (\mathfrak{t}_{\mathcal{F}})^{\leq 2} = \left(\mathfrak{t}^{\leq 2}\right)_{\mathcal{F}}$ by Theorem 6.6. Thus there exists $M_0' \prec M_0$ in K_λ such that for all $M_0'' \succ M_0'$ in K_λ with $M_0'' \prec M_3$, $\mathrm{tp}(a_2 a_1 / M_0''; M_3)$ does not fork (in the sense of $\mathfrak{t}^{\leq 2}$) over M_0'. Since we have symmetry in $\mathfrak{t}$, we have (by Fact 5.1) that also $\mathrm{tp}(a_1 a_2 / M_0''; M_3)$ does not fork over M_0' for all $M_0'' \succ M_0'$, $M_0'' \prec M_3$ in K_λ. Thus by definition and Theorem 6.6 again, $a_1 a_2$ is independent (with respect to $\underset{\mathcal{F}}{\perp}$) in (M_0, M_0, M_3), as needed. Similarly, (M_0, M_1, M_3') witnesses that $a_1 a'$ is independent in (M_0, M_0, M_3'). Thus p and p' are basic types of $\mathfrak{s}^{\leq 2}$, as needed. $\square$

We can now prove the announced theorem.

Corollary 6.9. Let $\mathfrak{s} := (K, \perp, S^{\mathrm{bs}})$ be a good λ-frame, and let $\mathcal{F} := [\lambda, \theta)$ be an interval of cardinals, where $\theta > \lambda$ is either a cardinal or ∞. Assume $K_{\mathcal{F}}$ has amalgamation and K is $(\lambda, < \theta)$-tame. Then $\mathfrak{s}_{\mathcal{F}}$ is a good $\mathcal{F}$-frame.

Proof. By the proof of Fact 6.4, $\mathfrak{s}_{\mathcal{F}}$ has all the properties of a good frame, except perhaps no maximal models and symmetry. Symmetry follows from the previous theorem and [Bonb, Theorem 7.1] now gives us no maximal models. $\square$

We conclude by summarizing what our results give from a good frame, amalgamation, and tameness:

Corollary 6.10. Let $\mathfrak{s} := (K, \perp, S^{\mathrm{bs}})$ be a good λ-frame. If $K_{\geq \lambda}$ has amalgamation and is λ-tame, then:

(1) $\mathfrak{s}_{\geq \lambda}$ is a good $(\geq \lambda)$-frame, and in fact even $\mathfrak{t} := (s_{\geq \lambda})^{<\infty}$ is a good $(< \infty, \geq \lambda)$-frame.
(2) For all α, K is $(\lambda + |\alpha|)$-tame for the basic types of $\mathfrak{t}$ of length $\leq \alpha$.
(3) $\left(\mathfrak{s}^{<\lambda^+}\right)_{\geq \lambda} = (\mathfrak{s}_{\geq \lambda})^{<\lambda^+}$.
(4) $\mathfrak{t}$ has symmetry of independence and independence in $\mathfrak{s}_{\geq \lambda}$ is finitely witnessed (see Definitions 5.3 and 5.7).
(5) We have a well-behaved notion of dimension: For $M \prec M_0 \prec N$ in K_λ, if:
 (a) $P \subseteq S^{\mathrm{bs}}(M_0)$
 (b) I_1, I_2 are $\subseteq$-maximal sets in

$$\{I : I \text{ is independent in } (M, M_0, N) \text{ and } a \in I \Rightarrow \mathrm{tp}(a/M_0; N) \in P\}$$

 (c) One of I_1, I_2 is infinite.

Then I_1 and I_2 are both infinite and $|I_1| = |I_2|$.

Proof.

(1) $\mathfrak{s}_{\geq \lambda}$ is a good $(\geq \lambda)$-frame by Corollary 6.9. $\mathfrak{t}$ is a good $(< \infty, \geq \lambda)$-frame by Corollary 5.14.
(2) By Proposition 6.7.
(3) By Theorem 6.6.
(4) By Theorem 5.5, Proposition 5.9, and Corollary 5.13.
(5) By Corollary 5.10.

□

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