TAMENESS AND FRAMES REVISITED

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Abstract. We combine tameness for 1-types with the existence of a good frame to obtain some amount of tameness for n-types, where n is a natural number. We use this to show how to use tameness to extend a good frame in one cardinality to a good frame in all cardinalities, improving a theorem of Boney [Bon].

Contents

1. Introduction

We show:

Theorem 1.1. Assume $\mathfrak{s}$ is a good λ-frame with underlying AEC K. If K has amalgamation and is λ-tame (for 1-types), then $\mathfrak{s}$ extends to a good $(\geq \lambda)$-frame.

This improves on [Bon], which used tameness for 2-types to prove the symmetry property. Assuming K has no maximal models, this was

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shown to be true in [Vas, Section 6]. The proof used a nonstructure result from [BGKV] which made it nonlocal: λ-tameness for types over arbitrarily large models was needed. Our proof does not need no maximal models and is completely local: it works equally well to prove for example:

Theorem 1.2. Assume $\mathfrak{s}$ is a good λ-frame with underlying AEC K. If K has amalgamation and is (λ, λ^+)-tame (for 1-types), then $\mathfrak{s}$ extends to a good $[\lambda, \lambda^+]$-frame.

While we were writing up this paper, Adi Jarden independently proved Theorem 1.2 with an additional hypothesis he called the "λ^+-continuity of serial independence property"[Jar]. A biproduct of our proofs is that his property holds in any good frame.

In the process of proving Theorem 1.1, we introduce frames with types longer than one elements and prove many general facts. In particular, we show how to start from a regular frame $\mathfrak{s}$ for 1-types and extend it to a frame $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ for types of length $< \lambda_{\mathfrak{s}}^+$

Assuming $\mathfrak{s}$ is a good λ-frame, we show that $\mathfrak{s}^{<\omega}$ is a good frame and λ-tameness for 1-types implies λ-tameness for the basic types of $\mathfrak{s}_{\geq\lambda}^{<\omega}$. We also show that $\mathfrak{s}^{<\lambda^+}$ will be a good frame without symmetry, and prove that extending a frame to bigger models and extending a frame to longer type can be done in any order. That is, we have the equation $(\mathfrak{s}_{\geq\lambda})^{<\lambda^+} = \left(\mathfrak{s}^{<\lambda^+}\right)_{\geq\lambda}$.

This paper is still in draft form and may change a lot in the future.

2. Notation and preliminaries

Type always means Galois type, and tp always stands for gtp. We assume familiarity with Galois types. For an ordinal α, we write $S^\alpha(M)$ for

$$\{\text{tp}(\bar{a}/M; N) \mid M \prec N, \bar{a} \in {}^{\alpha}N\}$$

and similarly for $S^{<\alpha}(M)$. If $p \in S^\alpha(M)$ (say $p = \text{tp}(\bar{a}/M; N)$) and $\beta < \alpha$, we write $p^\beta := \text{tp}(\bar{a}_\beta/M; N)$, where $\bar{a}_\beta = (a_i)_{i<\beta}$.

Definition 2.1. Let K be an AEC, and let $\mathcal{F}_1$, $\mathcal{F}_2$ be families of types in K. Let λ be a cardinal.

We say K is λ-tame for $(\mathcal{F}_1, \mathcal{F}_2)$-types if for any $p \in \mathcal{F}_1$ and any $q \in \mathcal{F}_2$, if $p, q \in S(M)$, and $p \neq q$, then there is $M_0 \prec M$ of size $\leq \lambda$ so that $p \upharpoonright M_0 \neq q \upharpoonright M_0$.

λ-tame for $\mathcal{F}_1$-types means λ-tame for $(\mathcal{F}_1, \mathcal{F}_1)$-types.

λ-tame for α-types when α is an ordinal means λ-tame for $\bigcup\{S^I(M) \mid M \in K\}$-types. Similarly for $< \alpha$-types. λ-tame means λ-tame for 1-types.

3. Good frames

Good frames were first defined in [She09a]. The idea is to provide a localized (i.e. only for base models of a given size λ) axiomatization of a forking-like notion for (a "nice enough" set of) 1-types. Jarden and Shelah (in [JS13]) later gave a slightly more general definition, not assuming the existence of a superlimit model and dropping some of the redundant clauses. We will use a slight variation here: we assume the models come from $K_{\mathcal{F}}$, for $\mathcal{F}$ an interval of cardinals, instead of just K_λ. We also assume that the types could be longer than just types of singletons[1]. Note that our use of frame differs from Shelah's in [She09b, Definition 0.2.1].

Definition 3.1 (Frame)**.** Let α be an ordinal and let $\mathcal{F}$ be an interval of the form $[\lambda, \mu)$, where λ is a cardinal, and $\mu > \lambda$ is either a cardinal or ∞.

A $(< \alpha, \mathcal{F})$-*frame* is a triples $\mathfrak{s} = (K, \mathop{\smile\!\!\!\!\!\!\mid}\,, S^{\text{bs}})$, where:

(1) K is an abstract elementary class with $\lambda \geq \text{LS}(K)$ and $K_{\mathcal{F}}$ has amalgamation.
(2) S^{bs} is a function with domain $K_{\mathcal{F}}$ and if $M \in K_{\mathcal{F}}$, then:

$$S^{\text{bs}}(M) \subseteq S^{<\alpha}(M)$$

(3) $\mathop{\smile\!\!\!\!\!\!\mid}$ is a relation on quadruples of the form $(M_0, M_1, \bar{a}, N)$, where $M_0 \prec M_1 \prec N$, $\bar{a} \in {}^{<\alpha}N$, and M_0, M_1, N are all in $K_{\mathcal{F}}$. We write $\mathop{\smile\!\!\!\!\!\!\mid}(M_0, M_1, \bar{a}, N)$ or $\bar{a} \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\mid}}} M_1$ instead of $(M_0, M_1, \bar{a}, N) \in \mathop{\smile\!\!\!\!\!\!\mid}$. We require $\mathop{\smile\!\!\!\!\!\!\mid}$ has the following properties:
   (a) Invariance: If $f : N \cong N'$ and $\bar{a} \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!\mid}}} M_1$, then $f(\bar{a}) \underset{f[M_0]}{\overset{N'}{\mathop{\smile\!\!\!\!\!\!\mid}}} f[M_1]$.

[1]SV: Probably this needs to be adapted to better match the notation in [She09b] (e.g. our frame should really be a pre-frame in Shelah's language), but can wait

(b) Monotonicity: If $\bar{a} \underset{M_0}{\overset{N}{\downarrow}} M_1$, $\bar{a}'$ is a subtuple of $\bar{a}$, and $M_0 \prec M_0' \prec M_1' \prec M_1 \prec N' \prec N \prec N''$ with $\bar{a}' \in N'$ and $N'' \in K_{\mathcal{F}}$, then $\bar{a}' \underset{M_0'}{\overset{N'}{\downarrow}} M_1'$ and $\bar{a}' \underset{M_0'}{\overset{N''}{\downarrow}} M_1'$.

(c) Existence: $\text{tp}(\bar{a}/M;N) \in S^{\text{bs}}(M)$ if and only if $\bar{a} \underset{M}{\overset{N}{\downarrow}} M$.

A $(\leq \alpha, \mathcal{F})$*-frame* is a $(< (\alpha+1), \mathcal{F})$-frame. An $\mathcal{F}$*-frame* is a $(\leq 1, \mathcal{F})$-frame. We write λ-frame instead of $\{\lambda\}$-frame, $(\geq \lambda)$-frame instead of $[\lambda, \infty)$-frame.

When the parameters are clear from context, or irrelevant, we just say "frame" instead of "$(< \alpha, \mathcal{F})$-frame".

Notation 3.2. Is $\mathfrak{s}$ is a $(< \alpha, \mathcal{F})$-frame, then $\alpha_{\mathfrak{s}} = \alpha$ and $\mathcal{F}_s = \mathcal{F}$.

Notation 3.3. If $\mathfrak{s} = (K, \downarrow, S^{\text{bs}})$ is a frame, we sometimes say "$\text{tp}(\bar{a}/M_1;N)$ does not fork over M_0" for $\bar{a} \underset{M_0}{\overset{N}{\downarrow}} M_1$. This makes sense by the monotonicity and invariance axioms.

Notation 3.4. If $\mathfrak{s} = (K, \downarrow, S^{\text{bs}})$ is a frame and β is an ordinal, we write $S^{\beta,\text{bs}}(M)$ for $S^{\text{bs}}(M) \cap S^{\beta}(M)$, and similarly for $S^{<\beta,\text{bs}}(M)$.

The following example gives the simplest possible frame:

Example 3.5. Let α be an ordinal. Let K be an AEC with amalgamation and let $\mathcal{F}$ be an interval of cardinals as above. Then $(K, \downarrow, S^{\text{bs}})$ is a $(< \alpha, \mathcal{F})$-frame, where:

- For $M \in K_{\mathcal{F}}$, $S^{\text{bs}}(M) := S^{<\alpha}(M)$ [$S^{\text{bs}}(M) := \emptyset$ would also work].
- For $M_0 \prec M_1 \prec N$ in $K_{\mathcal{F}}$ and $\bar{a} \in {}^{<\alpha}N$, say $\bar{a} \underset{M_0}{\overset{N}{\downarrow}} M_1$ if and only if $\text{tp}(\bar{a}/M_1;N) \in S^{\text{bs}}(M_1)$.

Definition 3.6 (Good frame)**.** Let α, $\mathcal{F}$ be as above.

A *good* $(< \alpha, \mathcal{F})$*-frame* is an $(< \alpha, \mathcal{F})$-frame $(K, \downarrow, S^{\text{bs}})$ satisfiying in addition:

(1) $K_{\mathcal{F}}$ has joint embedding and no maximal model.
(2) Disjointness: If $\bar{a} \in {}^{\beta}N$ for $\beta < \alpha$ and $\text{tp}(\bar{a}/M;N) \in S^{\text{bs}}(M)$, then $a_j \notin M$ for all $j < \beta$.
(3) *bs*-Stability: $|S^{1,\text{bs}}(M)| \leq \|M\|$ for all $M \in K_{\mathcal{F}}$.

(4) Density of basic types: If $M \precneqq N$ and $M, N \in \mathcal{F}$, then there is $a \in N$ such that $\mathrm{tp}(a/M; N) \in S^{\mathrm{bs}}(M)$.
(5) Extension: If $p \in S^{\mathrm{bs}}(M)$, $N \in K_{\mathcal{F}}$, then there is some $q \in S^{\mathrm{bs}}(N)$ that does not fork over M and extends p.
(6) Uniqueness: If $p, q \in S^{<\alpha}(M_1)$ do not fork over M_0 and $p \upharpoonright M_0 = q \upharpoonright M_0$, then $p = q$.
(7) Symmetry: If $\bar{a}_1 \overset{N}{\underset{M_0}{\downarrow}} M_2$, $\bar{a}_2 \in {}^{<\alpha}M_2$, and $\mathrm{tp}(\bar{a}_2/M_0; N) \in S^{\mathrm{bs}}(M_0)$, then there is M_1 containing $\bar{a}_1$ and $N' \succ N$ such that $\bar{a}_2 \overset{N'}{\underset{M_0}{\downarrow}} M_1$.
(8) Local character: If δ is a limit ordinal, $(M_i)_{i \leq \delta+1}$ is an increasing continuous chain in $K_{\mathcal{F}}$, and $p := \mathrm{tp}(\bar{a}/M_\delta; M_{\delta+1}) \in S^{\mathrm{bs}}(M_\delta)$ such that $\mathrm{cf}(\delta) > \ell(p)$, then there exists $i < \delta$ such that p does not fork over M_i.
(9) Continuity: If δ is a limit ordinal, $\langle M_i \in K_{\mathcal{F}} : i \leq \delta \rangle$ and $\langle \alpha_i < \alpha : i \leq \delta \rangle$ are increasing and continuous, and $p_i \in S^{\alpha,\mathrm{bs}}(M_i)$ for $i < \delta$ is such that, if $j < i < \delta$, then $p_j = p_i^{\alpha_j} \upharpoonright M_j$, then there is some $p \in S^{\alpha_\delta,\mathrm{bs}}(M_\delta)$ such that, for all $i < \delta$, $p_i = p^{\alpha_i} \upharpoonright M_i$ and this is the unique type in $S^{\alpha_\delta}(M_\delta)$ extending each p_i. Moreover, if each p_i does not fork over M_0, then neither does p.
(10) Transitivity[2]: If $M_0 \prec M_1 \prec M_2$, $p \in S^{<\alpha}(M_2)$ does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0, then p does not fork over M_0.

When we talk of a good $(< \alpha, \mathcal{F})$-frame without a given property P, we mean that the frame is not required to satisfy P: it may or may not have it. For example, a *good $(< \alpha, \mathcal{F})$-frame without stability* is a good $(< \alpha, \mathcal{F})$-frame which does not necessarily satisfy condition (3); see [JS13].

Remark 3.7. The obvious monotonicity properties hold: If $\mathfrak{s}$ is a [good] $(< \alpha, \mathcal{F})$-frame and $\beta < \alpha$, $\mathcal{F}'$ is a subinterval of $\mathcal{F}$ then the restricted frame $\mathfrak{s}'$ defined in the natural way will be a [good] $(< \beta, \mathcal{F}')$ frame.

We now show how to go the reverse way and extend frames. For simplicity[3], we focuss on going from a λ-frame to a $(< \alpha, \lambda)$-frame and from a $(< \alpha, \lambda)$-frame to a $(< \alpha, \geq \lambda)$-frame.

[2]This actually follows from uniqueness and extension, see [She09a, Claim 2.18].
[3]SV: I think we won't write much more if we actually do it for intervals, and this is useful to demonstrate locality of our methods as opposed to e.g. nonstructure arguments based on the order property.

Let's first see how to go up. This is first done in [She09a, Section 2].

Definition 3.8 (Going up)**.** Let $\mathfrak{s} := (K, \mathop{\smile\!\!\!\!\!\!|}, S^{\text{bs}})$ be a $(< \alpha, \lambda)$-frame. Define $\mathfrak{s}_{\geq \lambda} := (K, \underset{\geq \lambda}{\mathop{\smile\!\!\!\!\!\!|}}, S^{\text{bs}}_{\geq \lambda})$ as follows:

- For $M_0 \prec M_1 \prec N$ in $K_{\geq \lambda}$ and $\bar{a} \in {}^{<\alpha}N$, $\underset{\geq \lambda}{\mathop{\smile\!\!\!\!\!\!|}}(M_0, M_1, \bar{a}, N)$ if and only if there exists $M_0' \prec M_0$ in K_λ such that for all $M_0' \prec M_1' \prec N' \prec N$ with $\bar{a} \in N'$, and M_1', N' in K_λ, we have $\mathop{\smile\!\!\!\!\!\!|}(M_0', M_1', \bar{a}, N')$.
- For $M \in K_{\geq \lambda}$ and $p \in S^{<\alpha}(M)$, $p \in S^{\text{bs}}_{\geq \lambda}(M)$ if and only if there exists $N \succ M$ and $\bar{a} \in N$ such that $p = \text{tp}(\bar{a}/M; N)$ and $\underset{\geq \lambda}{\mathop{\smile\!\!\!\!\!\!|}}(M, M, \bar{a}, N)$.

Proposition 3.9. Let $\mathfrak{s}$ be a $(< \alpha, \lambda)$-frame. If $K_{\geq \lambda}$ has amalgamation, then $\mathfrak{s}_{\geq \lambda}$ is a $(< \alpha, \geq \lambda)$-frame.

Proof. Straightforward. $\square$

Shelah also observed that part of the properties of a good frame transfered:

Fact 3.10. Let $\mathfrak{s}$ be a good $(< \omega, \lambda)$-frame without symmetry. Then $\mathfrak{s}_{\geq \lambda}$ is a good $(< \omega, \geq \lambda)$-frame without bs-stability, extension, uniqueness, and symmetry.

Proof. See [She09a, Section 2]. $\square$

Later it was shown in [Bon] that all the properties transferred given enough tameness:

Fact 3.11. Assume K is λ-tame.

Let $n < \omega$, and let $\mathfrak{s}$ be a good (n, λ)-frame without symmetry, and assume $K_{\geq \lambda}$ has amalgamation and no maximal models. Then $\mathfrak{s}_{\geq \lambda}$ is a good $(n, \geq \lambda)$-frame without symmetry.

If in addition K is λ-tame for $2n$-types and $\mathfrak{s}$ is a good frame, then $\mathfrak{s}_{\geq \lambda}$ is a good frame. In this case, the no maximal models hypothesis is not needed.

Proof. See [Bon, Theorem 8.1]. $\square$

Note that the above facts were proven for good λ-frames (i.e. basic types were of length 1), but the proofs generalize to good $(< \omega, \lambda)$-frames.

Now let's see how to make a frame longer (allowing larger tuples). This similar to what is done in [She09b, Section 5] and [JS12].

Definition 3.12 (Independent sequence)**.** Let α be an ordinal.

Let $\mathfrak{s}$ be an $\mathcal{F}$-frame.

(1) $\langle a_i : i < \alpha \rangle$, $\langle M_i : i \leq \alpha \rangle$ is said to be *independent over* M when:
   (a) $(M_i)_{i \leq \alpha}$ is increasing continuous in $K_{\mathcal{F}}$.
   (b) $M \prec M_i$ for all $i \leq \alpha$, and $M \in K_{\mathcal{F}}$.
   (c) For every $i < \alpha$, $a_i \overset{M_{i+1}}{\underset{M}{\perp}} M_i$.

(2) $\bar{a} := (a_i)_{i<\alpha}$ is said to be *independent in* (M, M', N) when $M \prec M' \prec N$, $\bar{a} \in N$, and for some $(M_i)_{i \leq \alpha}$ and a model N^+ such that $M_{i^*} \prec N^+$, $N \prec N^+$, and $M' \prec M_i$ for all $i \in \hat{I}$, and $\langle a_i : i < \alpha \rangle$, $\langle M_i : i \leq \alpha \rangle$ is independent over M.

Remark 3.13. If $\alpha = 1$, then $\bar{a} := (a_0)$ is independent in (M, M', N) if and only if $\text{tp}(a_0/M'; N)$ does not fork over M.

Definition 3.14. Let α be an ordinal. Let $\mathfrak{s} := (K, \perp, S^{\text{bs}})$ be an $\mathcal{F}$-frame. Define $\mathfrak{s}^{<\alpha} := (K, \overset{<\alpha}{\perp}, S^{<\alpha,\text{bs}})$ as follows:

- For $M_0 \prec M_1 \prec N$ in $K_{\mathcal{F}}$ and $\bar{a} := (a_i)_{i<\beta}$ in N with $\beta < \alpha$, $\overset{<\alpha}{\perp}(M_0, \bar{a}, M_1, N)$ if and only if $\bar{a}$ is independent in (M_0, M_1, N).
- For $M \in K_{\mathcal{F}}$ and $p \in S^{<\alpha}(M)$, $p \in S^{<\alpha,\text{bs}}(M)$ if and only if there exists $N \succ M$ and $\bar{a} \in N$ such that $p = \text{tp}(\bar{a}/M; N)$ and $\overset{<\alpha}{\perp}(M, \bar{a}, M, N)$.

Proposition 3.15. Let α be an ordinal. Let $\mathfrak{s}$ be an $\mathcal{F}$-frame. Then $\mathfrak{s}^{<\alpha}$ is a $(<\alpha, \mathcal{F})$-frame.

Proof. Use amalgamation to see that if $\bar{a}$ is independent in (M_0, M_1, N) and $N' \succ N$, then $\bar{a}$ is independent in (M_0, M_1, N'). The rest is straightforward. $\square$

Note that when dealing with types rather than sequences, the N^+ in the definition can be avoided. That is, given $p \in S^{\beta,\text{bs}}(N)$ that does not fork over M, there is some $\langle a_i : i < \beta \rangle$, $\langle N^i : i \leq \beta \rangle$ such that $p = tp(\langle a_i : i < \beta \rangle / N; N^\beta)$ that witnesses that $\langle a_i : i < \beta \rangle$ is independent in (M, N, N^β).

We will shortly investigate what properties are preserved by the operation $\mathfrak{s} \mapsto \mathfrak{s}^{<\alpha}$, but first we show how it interacts with the going up construction:

Proposition 3.16. Let $\alpha \leq \lambda^+$.

Let $\mathfrak{s} := (K, \mathop{\smile\!\!\!\!|}, S^{\text{bs}})$ be a λ-frame. Assume $K_{\geq \lambda}$ has amalgamation. Then:

$$(\mathfrak{s}_{\geq \lambda})^{<\alpha} \subseteq (\mathfrak{s}^{<\alpha})_{\geq \lambda}$$

Where $\subseteq$ is taken componentwise.

Proof. Let $(\mathfrak{s}_{\geq \lambda})^{<\alpha} := (K, \overset{(1)}{\mathop{\smile\!\!\!\!|}}, S^{\text{bs}}_1)$, $(\mathfrak{s}^{<\alpha})_{\geq \lambda} := (K, \overset{(2)}{\mathop{\smile\!\!\!\!|}}, S^{\text{bs}}_2)$. It is enough to show $\overset{(1)}{\mathop{\smile\!\!\!\!|}} \subseteq \overset{(2)}{\mathop{\smile\!\!\!\!|}}$; Existence then implies that $S^{\text{bs}}_1 \subseteq S^{\text{bs}}_2$. Assume $\overset{(1)}{\mathop{\smile\!\!\!\!|}}(M, M^+, \bar{a}, N)$. Say $\bar{a} = (a_i)_{i<\beta}$, where $\beta < \alpha$. By definition, this means that $\bar{a}$ is independent (with respect to $\underset{\geq \lambda}{\mathop{\smile\!\!\!\!|}}$) in (M, M^+, N). Fix $(M_i)_{i \leq \alpha}$ and N^+ witnessing the independence.

In particular, we have that for every $i < \alpha$, $\underset{\geq \lambda}{\mathop{\smile\!\!\!\!|}}(M, M_i, a_i, M_j)$. By definition again, this implies in particular that we have $M'_{i,j} \prec M$ in K_λ so that $\underset{\geq \lambda}{\mathop{\smile\!\!\!\!|}}(M'_{i,j}, M_i, a_i, M_j)$. By the Löwenheim-Skolem axiom and monotonicity, since $|\beta| \leq \lambda$, we can choose $M' \prec M$ in K_λ such that for all $i < \beta$, $M_{i,j} \prec M'$ and $\underset{\geq \lambda}{\mathop{\smile\!\!\!\!|}}(M', M_i, a_i, M_j)$. In particular, $\bar{a}$ is independent (with respect to $\underset{\geq \lambda}{\mathop{\smile\!\!\!\!|}}$) in (M', M^+, N).

Now fix any $(M^+)', N' \in K_\lambda$ such that $\bar{a} \in N$, $M' \prec (M^+)' \prec M^+$, and $(M^+)' \prec N' \prec N$. We claim that $\bar{a}$ is independent (with respect to $\mathop{\smile\!\!\!\!|}$) in $(M', (M^+)', N')$, i.e. $\overset{<\alpha}{\mathop{\smile\!\!\!\!|}}(M', \bar{a}, (M^+)', N')$: construct $(M'_i \in K_\lambda)_{i \leq \alpha}$ by induction such that

- $M' \prec M'_i \prec M_i$; and
- $\bigcup_{j<i} M'_j \subset M'_i$.

Then pick $(N^+)' \in K_\lambda$ such that $M'_\alpha, N' \prec (N^+)' \prec N^+$. Then $(M'_i)_{i \leq \alpha}$ and N^+ witness the claim.

Since $(M^+)'$ was arbitrary between M' and M^+, this shows that $\overset{(2)}{\mathop{\smile\!\!\!\!|}}(M, \bar{a}, M^+, N)$, as needed. $\square$

We investigate the converse in Section 6.

The following lemmas follow from the definition of good frames and are helpful later, especially in the proof of Theorem 4.1. The first is a strengthening of to uniqueness over sets, instead of just models.

Lemma 3.17. Suppose $tp(a/M;N^*) = tp(b/M;N^*)$, $\mathop{\smile\!\!\!\!\!\!\mid}\,(M, N_1, a, N^*)$, and $\mathop{\smile\!\!\!\!\!\!\mid}\,(M, N_2, b, N^*)$. Then $tp(a/N_1 \cap N_2; N^*) = tp(b/N_1 \cap N_2; N^*)$.

Proof. Find N^+ such that $N_1, N_2 \prec N^+ \prec N^*$; this might just be N^*. Let $p' = tp(a'/N^+; N_1^+)$ and $q' = tp(b'/N^+; N_2^+)$ be nonforking extensions of $tp(a/N_1; N^*)$ and $tp(b/N_2; N^*)$, respectively, to N^+. Then $p' = q'$. Thus, we can use the various type equalities to fill out the following commutative diagram such that $g_1 \circ f_1(a) = g_2 \circ f_2(b)$.

$$\begin{tikzcd}
 & & N^{**} & & \\
 & N_1^+ \arrow[ur, "g_1"] & & N_2^+ \arrow[ul, "g_2"'] & \\
N^* \arrow[ur, "f_1"] & & N^+ \arrow[ul, "a'"'] \arrow[ur, "b'"] & & N^* \arrow[ul, "f_2"'] \\
 & N_1 \arrow[ul, "a"'] \arrow[ur] & & N_2 \arrow[ul] \arrow[ur, "b"'] & \\
 & & M \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

$\square$

The second is an equivalent way of formulating extension that is easier to work with.

Lemma 3.18. The following is equivalent to extension given in a frame:

> if we have $tp(a/M; M') \in S^{bs}(M)$ and $f : M \to N$, then there is $N' \in K_{\mathfrak{s}}$, $g : M' \to N'$, and $b \in N'$ so that $f \subset g$, $g(a) = b$, and $\mathop{\smile\!\!\!\!\!\!\mid}\,(f(M), N, b, N')$.

Proof. It is easy to see that this statement implies extension: use it with $f = \mathrm{id}$ and set $q = tp(b/N; N')$. For the other direction, let $tp(a/M; M') \in S^{bs}(M)$ and $f : M \to N$. First, extend f to $\widehat{f}$ so that all of N is in the range of $\widehat{f}$. By extension, there is some $q \in S^{bs}(\widehat{f}^{-1}(N))$ that does not fork over M so $q \restriction M = tp(a/M; M')$. Let

$q = tp(c/\widehat{f}^{-1}(N); N^+)$. Since $tp(c/M; N^+) = tp(a/M; M')$, there is some N^{++} extending N^+ and $g' : M' \to_M N^{++}$. Then, since $\widehat{f} : \widehat{f}^{-1}(N) \cong N$ and N^{++} extends $\widehat{f}^{-1}(N)$, we can copy this over; that is, find N' extending N and $\widehat{g} : N^{++} \cong N'$ so that $\widehat{f} \subset \widehat{g}$. Then we are done by taking $b = \widehat{g}(c)$ and $g = \widehat{g} \circ g'$. $\square$

Lemma 3.19 (amalgamation of independent sequences)**.** Let $\mathfrak{s}$ be a good λ-frame and $\beta < \lambda_{\mathfrak{s}}^+$. Suppose that $p, q \in S^{\beta, bs}(N)$ does not fork over M, that $p \upharpoonright M = q \upharpoonright M$, and that there are witnessing sequences $\bar{a}_\ell = \langle a_\ell^i : i < \beta \rangle$, $\langle N_\ell^i : i \leq \beta \rangle$ independent in (M, N, N_ℓ^β) for $\ell = 0, 1$ with $\bar{a}_0 \vDash p$ and $\bar{a}_1 \vDash q$. Then, there are coherent, continuous, increasing $(N_i, f_{j,i})_{j < i \leq \beta}$ and $g_\ell^i : N_\ell^i \to N_i$ such that, for all $i < j < \beta$,

$$\begin{tikzcd}
 & & N_1^j \arrow[rr] \arrow[dr, "g_1^j"'] & & N_1^i \arrow[rr] \arrow[dr, "g_1^i"'] & & N_1^\beta \arrow[dr, "g_1^\beta"'] & \\
M \arrow[r] & N \arrow[ur] \arrow[dr] & & N_j \arrow[rr, "f_{j,i}"'] & & N_i \arrow[rr, "f_{i,\beta}"'] & & N_\beta \\
 & & N_0^j \arrow[rr] \arrow[ur, "g_0^j"'] & & N_0^i \arrow[rr] \arrow[ur, "g_0^i"'] & & N_0^\beta \arrow[ur, "g_0^\beta"'] &
\end{tikzcd}$$

commutes, $g_0^{i+1}(a_0^i) = g_1^{i+1}(a_1^i)$, and $\perp(g_0^i(M), f_{i,i+1}(N_i), g_0^{i+1}(a_0^i), N_{i+1})$.

Proof: We will build

(1) models $\{N_i, M_\ell^i : i \leq \beta, \ell = 0, 1\}$;
(2) embeddings $\{f_\ell^i : N_\ell^i \to M_\ell^i, g_\ell^i : M_\ell^i \to N_{i+1} : i \leq \beta, \ell = 0, 1\}$; and
(3) coherent $\{\widehat{h}_{j,i} : N_j \to N_i, \widehat{g}_\ell^{j,i} : M_\ell^j \to M_\ell^i : i \leq \beta, \ell = 0, 1\}$

such that

(1)

$$\begin{tikzcd}
M_0^i \arrow[r, "g_0^i"'] & N_{i+1} \\
N_i \arrow[u] \arrow[r] & M_1^i \arrow[u, "g_1^i"']
\end{tikzcd}$$

commutes;

(2)

$$\begin{tikzcd}
N_\ell^{i+1} \arrow[rr, "f_\ell^{i+1}"'] & & M_\ell^{i+1} \\
N_\ell^i \arrow[u] \arrow[r, "f_\ell^i"'] & M_\ell^i \arrow[r, "g_\ell^i"'] & N_{i+1} \arrow[u]
\end{tikzcd}$$

commutes;

(3) $M_0^0 = M_1^0 = N_1$, $g_\ell^i = \mathrm{id}$, and

$$\begin{tikzcd}
N_0^0 \arrow[r, "f_0^0"'] & N_1 \\
N \arrow[u] \arrow[r] & N_1^0 \arrow[u, "f_1^0"']
\end{tikzcd}$$

commutes;

(4) $\bigcup(f_\ell^{i+1}(N_\ell^i), N_{i+1}, f_\ell^{i+1}(a_\ell^i), M_\ell^{i+1})^4$;

(5) $g_0^{i+1} \circ f_0^{i+1}(a_0^i) = g_1^{i+1} \circ f_1^{i+1}(a_1^i)$; and

(6) $(N_i, \widehat{h}_{j,i})_{j<i\leq\beta}$ and $(M_\ell^i, \widehat{g}_\ell^{j,i})_{j<i\leq\beta}$ are continuous, coherent systems generated by $\widehat{g}_\ell^{i,i+1} = g_\ell^i$ and $\widehat{h}_{i,i+1} = g_0^{i+1} = g_1^{i+1}$.

At stage i, we will construct $f_\ell^i, g_\ell^i, M_\ell^i$, and N_{i+1} for $\ell = 0, 1$. Also, at each stage, we implicitly extend the coherent system by the rule given above (at successor steps) or by taking direct limits (at limit steps).
<u>$i = 0$:</u> Amalgamate N_0^0, N_1^0 over N to get N_1.
<u>i limit:</u> We take the direct limits to get the following for $j < i$:

$$\begin{tikzcd}
 & & N_i & & \\
 & M_0^i \arrow[ur, "g_0^i"'] & & M_1^i \arrow[ul, "g_1^i"'] & \\
N_0^i \arrow[ur, "f_0^i"'] & & N_j \arrow[uu, "\widehat{h}_{j,i}"'] & & N_1^i \arrow[ul, "f_1^i"'] \\
 & M_0^j \arrow[uu, "\widehat{g}_0^{j,i}"'] \arrow[ur, "g_0^j"'] & & M_1^j \arrow[uu, "\widehat{g}_1^{j,i}"'] \arrow[ul, "g_1^j"'] & \\
N_0^j \arrow[uu] \arrow[ur, "f_0^j"'] & & & & N_1^j \arrow[uu] \arrow[ul, "f_1^j"']
\end{tikzcd}$$

[4] Note that $f_\ell^{i+1}(N_\ell^i) = g_\ell^i \circ f_\ell^i(N_\ell^i)$

Set $N_{i+1} = N_i$.

$\underline{i = j+1}$: Use the above Lemma–replace (M_0, M_1, a, f, N_0) there with $(N_\ell^i, N_ell^{i+1}, a_\ell^i, g_\ell^i \circ f_\ell^i, N_i)$ here–to get $(f_\ell^{i+1}, M_\ell^{i+1})$ here, written as (g, N_1) there. From the hypothesis, we have $\smile(M, N_\ell^i, a_\ell^i, N_\ell^{i+1})$. Applying f_ℓ^{i+1} and stretching the ambient model from $f_\ell^{i+1}(N_\ell^{i+1})$ to M_ℓ^{i+1}, we get $\smile(f_\ell^{i+1}(M), f_\ell^{i+1}(N_\ell^i), f_\ell^{i+1}(a_\ell^i), M_\ell^{i+1})$. Using (T) for $\mathfrak{s}$, this gives

$$\smile(f_\ell^{i+1}(M), N_{i+1}, f_\ell^{i+1}(a_\ell^i), M_\ell^{i+1})$$

By the commutative diagrams, $f_0^{i+1} \upharpoonright M = f_1^{i+1} \upharpoonright M$, so, since a_0^i and a_1^i have the same type over M, we have that

$$tp(f_0^{i+1}(a_o^i)/f_0^{i+1}(M); M_0^{i+1}) = tp(f_1^{i+1}(a_1^i)/f_1^{i+1}(M); M_1^{i+1}).$$

By uniqueness for $\mathfrak{s}$, these imply that

$$tp(f_0^{i+1}(a_o^i)/N_{i+1}; M_0^{i+1}) = tp(f_1^{i+1}(a_1^i)/N_{i+1}; M_1^{i+1}).$$

We can witness this with $g_\ell^{i+1} : M_\ell^{i+1} \to N_{i+2}$ for $\ell = 0, 1$; that is, $g_0^{i+1} \upharpoonright N_{i+1} = g_1^{i+1} \upharpoonright N_{i+1}$ and $g_0^{i+1} \circ f_0^{i+1}(a_0^i) = g_1^{i+1} \circ f_1^{i+1}(a_1^i)$.

This construction completes the proof; set $f_{j,i} = \widehat{h}_{j,i}$. †

Corollary 3.20. Suppose $M_0 \prec M \prec N$ and $\alpha \leq \beta$ such that there are $p \in S^{\alpha,bs}(M)$ and $q \in S^{\beta,bs}(N)$ such that $q^\alpha \upharpoonright M = p$ and p, q do not fork over M_0. If $\bar{a}_p = \langle a_p^i : i < \alpha \rangle$, $\langle N_p^i : i \leq \alpha \rangle$ is independent in (M_0, M, N_p^α) such that $\bar{a}_p \vDash p$ and $\bar{a}_q = \langle a_q^i : i < \beta \rangle$, $\langle N_q^i : i \leq \beta \rangle$ is independent in (M_0, N, N_q^β) such that $\bar{a}_q \vDash q$, then there is $\langle M_q^i : i \leq \beta \rangle$ and $h_i : N_p^i \to M_q^i$ for $i \leq \alpha$ such that

(1) $\bar{a}_q$, $\langle M_q^i : i \leq \beta \rangle$ is independent in (M_0, N, M_q^β);
(2) $N_q^i \prec M_q^i$ for all $i \leq \beta$; and
(3) $h_{i+1}(a_p^i) = a_q^i$ and $\mathrm{id}_M \subset h_i \subset h_{i+1}$.

Proof: First, extend the p-sequence to $\langle a_p^i : i < \beta \rangle$, $\langle N_p^i : i \leq \beta \rangle$ independent in (M_0, M, N_p^β). Then, we can amalgamate there sequences over M using Lemma 3.19: there is $(N_i, f_{j,i})_{j<i\leq\beta}$ and $g_x^i : N_x^i \to N_i$ for $x = p, q$ and $i \leq \beta$ as above. Since we have $g_q^\beta : N_q^\beta \cong g_q^\beta(N_q^\beta) \prec N_\beta$, we can extend g_q^β to an $L(K)$-isomorphism h with N_β in its range. Set $M_q^i := h^{-1\prime\prime} N_i$ for $i \leq \beta$. Note that $h_i := h^{-1} \circ g_q^i : N_q^i \to M_q^i$ is the identity. †

Lemma 3.21. Suppose we have $M \prec M_0 \prec M_\ell$ for $\ell = 1, 2$ such that $a \in M_1$ and $\smile\!\!\!\!|\,(M, M_0, a, M_1)$. Then there is $N \succ M_1$ and $f : M_2 \to_{M_0} N$ such that $\smile\!\!\!\!|\,(M, f(M_2), a, N)$.

$$\begin{tikzcd}
 & M_1 \arrow[r, dashed] & N \\
 & M_0 \arrow[u, "a"] \arrow[r] & M_2 \arrow[u, dashed, "f"'] \\
M \arrow[ur] & &
\end{tikzcd}$$

Proof: By extension, there is some $q \in S^{bs}(M_2)$ that extends $tp(a/M_0; M_1)$ such that q does not fork over M_0. Realize q as $tp(b/M_2; N^-)$. Since $tp(a/M_0; M_1) = tp(b/M_0; N^+)$, there is some $N \succ M_1$ and a mapping $f : N^- \xrightarrow[M_0]{} N$ such that $f(b) = a$. Since q does not fork over M_0, we can just rewrite this as $\smile\!\!\!\!|\,(M_0, M_2, b, N^+)$. Applying f to this, and applying invariance and monotonicity, we get that $\smile\!\!\!\!|\,(M_0, f(M_2), a, N)$. From the hypothesis, we have $\smile\!\!\!\!|\,(M, M_0, a, M_1)$ and we can combine these with transitivity to get $\smile\!\!\!\!|\,(M, f(M_2), a, N)$, as desired. †

Lemma 3.22. Suppose we have $M \prec M_0 \prec M_1$ and $f : M_0 \to M_2$ such that $a \in M_1$ and $\smile\!\!\!\!|\,(M, M_0, a, M_1)$. Then, there is $N \succ M_2$ and $g : M_1 \to N$ extending f such that $\smile\!\!\!\!|\,(f(M), M_2, g(a), N)$.

$$\begin{tikzcd}
 & M_1 \arrow[r, dashed, "g"'] & N \\
 & M_0 \arrow[u, "a"] \arrow[r, "f"'] & M_2 \arrow[u, dashed] \\
M \arrow[ur] & &
\end{tikzcd}$$

Proof: Extend f to an $L(K)$-isomorphism $\widehat{f}$ with range M_2. By extension, there is some $q \in S^{bs}(\widehat{f}^{-1}(M_2))$ that extends $tp(a/M_0; M_1)$ and does not fork over M_0. Write q as $tp(b/\widehat{f}^{-1}(M_2); N^+)$. Since $tp(a/M_0; M_1) = tp(b/M_0; N^+)$, there is $N^{++} \succ N^+$ and $h : M_1 \to_{M_0} N^{++}$ such that $h(a) = b$. Then, since N^++ extends $\widehat{f}^{-1}(M_2)$, we can find an $L(K)$-isomorphism $\widehat{f}^+$ that extends $\widehat{f}$ such that N^++ is the domain of $\widehat{f}^+$. Set $N := \widehat{f}^+(N^{++})$ and $g = \widehat{f}^+ \circ g$. Some nonforking calculus shows that this works. †

$$
\begin{tikzcd}
& M_1 \arrow[r, "h"] & N^{++} \arrow[r, Rightarrow, "\widehat{f}^+"] & N \\
& & N^+ \arrow[u] & \\
& M_0 \arrow[uu, "a"'] \arrow[r] & \widehat{f}^{-1}(M_2) \arrow[u, "b"'] \arrow[r, Rightarrow, "\widehat{f}"] & M_2 \arrow[uu] \\
M \arrow[ur] & & &
\end{tikzcd}
$$

Lemma 3.23. Given $\bar{a} = \langle a_i : i < \alpha \rangle$ independent in (M, M_0, M_1) and $M_2 \succ M_0$ containing $\bar{b}$ such that $tp(\bar{a}/M_0; M_1) = tp(\bar{b}/M_0; M_2)$, we have that $\bar{b}$ is independent in (M, M_0, M_2).

Proof: Let $\langle N_i : i \leq \alpha \rangle$ and N^+ witness the independence of $\bar{a}$. First , use the type equality to find $M^* \succ M_2$ and $f : M_1 \to_{M_0} M^*$ such that $f(a_i) = b_i$. Then, we use amalgamation to find N^* and g such that $N^* \succ M^*$ and $g : N^+ \to N^*$ extends f.

Set $N_i' := g(N_i)$ and $N^{++} := N^*$. We claim that this witnesses $\bar{b}$ is independent in (M, M_0, M_2).

- $\langle N_i' : i \leq \alpha \rangle$ is increasing and continuous because $\langle N_i : i \leq \alpha \rangle$ is.
- $M_0 \prec N_i' \prec N^{++}$ because $M_0 \prec N_i \prec N^+$; g fixes M_0; and $g(N^+) \prec N^{++}$.
- $M_2 \prec N^{++}$ by the amalgamation construction.
- $\smile(M, N_i', b_i, N_{i+1}')$ because we know that $\smile(M, N_i, a_i, N_{i+1})$ and we can apply g to this. †

The following easy consequence of invariance is often useful.

Proposition 3.24. Let $\mathfrak{s}$ be a $(< \alpha, \mathcal{F})$-frame with amalgamation and extension. Assume $p := \text{tp}(\bar{a}/M; N) \in S^{<\alpha}(M)$ does not fork over $M_0 \prec M$. Let $M \prec M' \prec N$. Then there exists $M'' \cong_M M'$ and $N' \succ N$ such that $\text{tp}(\bar{a}/M''; N')$ does not fork over M_0.

Proof. Let $q \in S^{<\alpha}(M')$ be an extension of p that does not fork over M_0. Extending N if necessary, we can assume without loss of generality $q := \mathrm{tp}(\bar{a}'/M'; N)$. In particular, $p = \mathrm{tp}(\bar{a}'/M; N)$. Therefore there is $N' \succ N$ and $f : N \to N'$ fixing M such that $f(\bar{a}') = \bar{a}$. Let $M'' := f[M']$. By invariance, $f(q) = \mathrm{tp}(\bar{a}/M''; N')$ does not fork over M_0. $\square$

Proposition 3.25. Let $\mathfrak{s}$ be a $(<\alpha, \mathcal{F})$-frame with amalgamation and extension. Let $M_0 \prec M_\ell \prec N$, $\ell = 1, 2$, be models in $K_{\mathcal{F}}$. Let $\bar{a} \in {}^{<\alpha}N$ be given so that $\mathrm{tp}(\bar{a}/M_\ell; N)$ does not fork over M_0 for $\ell = 1, 2$.

Then there exists $N_3 \succ N$, $N_3 \succ M_3 \succ M_1$ in $K_{\mathcal{F}}$ and $f : M_2 \to M_3$ fixing M_0 such that $\mathrm{tp}(\bar{a}/M_3; N_3)$ does not fork over M_0.

$$\begin{tikzcd}
M_1 \arrow[r, dotted] & M_3 \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[u, dotted, "f"']
\end{tikzcd}$$

Proof. By amalgamation, there is $N_3' \succ N$, $M_1 \prec M_3' \prec N_3'$, and $g : M_2 \to M_3$ an embedding fixing M_0. By Proposition 3.24, there is $N_3 \succ N_3'$ and $h : M_3 \cong_{M_1} M_3'$ such that $\mathrm{tp}(\bar{a}/M_3; N_3)$ does not fork over M_0. Take $f := h^{-1} \circ g$. $\square$

An interesting fact is that when α is finite, the ordering does not matter:

Fact 3.26. Let $\mathfrak{s}$ be a good λ-frame without stability (but with symmetry). If $\bar{a}$ is a finite tuple independent in (M, M', N), then any permutation of $\bar{a}$ is independent in (M, M', N).

Proof. See [JS12, Theorem 4.2.(a)]. $\square$

We can also concatenate two sequences together. This is reproven later (Theorem 5.1) without the symmetry assumption.

Fact 3.27. Let $\mathfrak{s}$ be a good λ-frame without stability. Let $M \prec M_0 \prec M_1 \prec M_2$ such that $\bar{a} = \langle a_i : i < \alpha \rangle$ is independent in (M, M_0, M_1) and $\bar{b} = \langle b_i : i < \beta \rangle$ is independent in (M, M_1, M_2). Then $\bar{a}\bar{b}$ is independent in (M, M_0, M_2).

Proof. This follows from the definition of frames if the sequences are of length 1 and is [JS12, Proposition 4.1] for longer sequences. Note that this relies on [JS12, Proposition 2.6], which is proved as [JS13, Proposition 3.1.8] and uses Symmetry in an essential way. $\square$

4. Transfering various properties

We get that many properties transfer from the frame to its "elongation" immediately. This is claimed (for finite tuples) by Shelah in [She09b, Exercise 9.4.1] but a proof has never appeared anywhere. Here we show that all the properties transfer to the elongation to types of size $< \lambda_{\mathfrak{s}}^+$, except symmetry which we can only manage to transfer to $\mathfrak{s}^{<\omega}$. Jarden and Sitton [JS12] have essentially shown that symmetry also transfers to $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ assuming some strong continuity properties.

Theorem 4.1. Let $\mathfrak{s} := (K, \perp, S^{\text{bs}})$ be a good λ-frame without stability or symmetry.

(1) $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ has local character.
(2) $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ has uniqueness.
(3) $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ has extension.
(4) $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ has continuity.
(5) If $\mathfrak{s}$ has symmetry, then $\mathfrak{s}^{<\omega}$ has symmetry

In particular, $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ is a good frame without stability or symmetry and if $\mathfrak{s}$ has symmetry then $\mathfrak{s}^{<\omega}$ is a good frame without stability.

Note that this result can be improved: if $\mathfrak{s}$ is a good $\mathcal{F}$-frame without stability or symmetry, then (1) – (4) above hold in $\mathfrak{s}^{<\mu}$, where $\mu = \sup_{\lambda \in \mathcal{F}}(\lambda^+)$. The same is true of Lemma 3.19 and other results.[5]

Proof.

(1) Let $p \in S^{<\lambda^+, bs}(N)$ and $N = \bigcup_{i<\delta} N_i$ with $\ell(p) < \delta = \text{cf}\,\delta < \lambda_{\mathfrak{s}}^+$. Thus, there is some $\bar{a} = \langle a_i : i < \beta \rangle$ and increasing, continuous $\langle N^i : i \leq \beta \rangle$ such that $\beta < \delta$, $p = tp(\bar{a}/N; N^\beta)$, and, for all $i < \beta$, $\perp(N, N^i, a_i, N^{i+1})$. By (M) for $\mathfrak{s}$, $tp(a_i/N; N^{i+1}) \in S^{bs}_{\mathfrak{s}}(N)$. By local character for $\mathfrak{s}$, for all $i < \beta$, there is some $j_i < \delta$ such that $\perp(N_{j_i}, N, a_i, N^{i+1})$. By (T) for $\mathfrak{s}$, $\perp(N_{j_i}, N^i, a_i, N^{i+1})$. Set $j_* := \sup_{i<\beta} j_i$; since cf $\delta > \beta$, we have that $j_* < \beta$. By (M) for $\mathfrak{s}$, $\perp(N_{j_*}, N^i, a_i, N^{i+1})$ for all $i < \beta$. This is exactly what we need to conclude that $\bar{a}$ is independent in (N_{j_*}, N, N^β). Thus, $p = tp(\bar{a}/N; N^\beta)$ does not fork over N_{j_*}.
(2) This follows directly from Lemma 3.19.
(3) We prove two extension results separately: extending the domain and extending the length. Combining these gives that $\mathfrak{s}^{<\lambda_{\mathfrak{s}}^+}$ has extension.

[5] WB: I added this remark.

For extending the domain, let $p \in S^{<\lambda_{\mathfrak{s}}^+, bs}(M)$ and $N \succ M$. By definition of this frame, there is some $\bar{a} = \langle a_i : i < \beta \rangle$ and increasing, continuous $\langle N^i : i \leq \beta \rangle$ such that $\perp(M, N^i, a_i, N^{i+1})$ for all $i < \beta$. We wish to construct increasing and continuous $\langle M^i : i \leq \beta \rangle$ and $\langle f_i : N^i \to M^i : i \leq \beta \rangle$ such that

(a) $f_0 \restriction N = \mathrm{id}$; and

(b) $\perp(M, M^i, f_i(a_i), M^{i+1})$.

This is done by induction by taking unions at limits and by using Lemma 3.18 at all successor steps. Since $\beta < \lambda_s^+$, M^i is of size $\lambda_{\mathfrak{s}}$ at all steps and the induction can continue. Then $tp(\bar{a}/M; N^\beta) = tp(f(\bar{a})/M; M^\beta)$ as witnessed by f and $f(\bar{a})$ is independent in (M, N, M^β). Thus, $q = tp(f(\bar{a})/N, M^\beta)$ is as desired.

To extend the length, suppose that $\beta < \alpha < \lambda_{\mathfrak{s}}^+$ and $p \in S^{\beta, bs}(N)$ does not fork over M. This means that there is $\langle a_i : i < \beta \rangle$, $\langle N^i : i \leq \beta \rangle$ independent in (M, N, N^β) such that $p = tp(\langle a_i : i < \beta \rangle / N; N^\beta)$. We will extend this sequence to be of length α by induction. At limit steps, simply taking the union of the extensions works. If we have $\beta \leq \gamma < \alpha$ and have already extended to γ (i.e., $\langle a_i : i < \gamma \rangle$, $\langle N^i : i \leq \gamma \rangle$ is defined), then let $r \in S^{bs}(M)$ be arbitrary. Let $r^+ \in S^{bs}(N^\gamma)$ be its nonforking extension. Thus, there is $a_\gamma \in N^{\gamma+1}$ such that $\perp(M, N^\gamma, a_\gamma, N^{\gamma+1})$. Then $\langle a_i : i < \gamma + 1 \rangle$, $\langle N^i : i \leq \gamma + 1 \rangle$ is independent in $(M, N, N^{\gamma+1})$, as desired.

(4) For all $i < \delta$, there is some $\bar{a}_i = \langle a_i^k : k < \alpha_i \rangle$, $\langle N_i^k : k \leq \alpha_i \rangle$ independent in $(M_i, M_i, N_i^{\alpha_i})$ such that $p_i = tp(\bar{a}_i/M_i; N_i^{\alpha_i})$. To show uniqueness, if we already have $q \in S^{\alpha_\delta}(M_\delta)$ that extends each p_i, then we can do this so $a_i^k = a_j^k = a^k$ for all $i, j < \delta$ with $k < \alpha_i, \alpha_j$ such that $\langle a^k : k < \alpha_\delta \rangle$ realizes q. For the moreover clause, if each p_i does not fork over M_0, we can pick the independent sequences to witness this. Then change the rest of the proof so M_0 is aways the model that types and tuples do not fork over.

We will construct $\langle M_i^k : i < \delta, k \leq \alpha_i \rangle$ and $\{ f_{j,i}^k : M_j^k \to M_i^k : k \leq \alpha_j, j < i < \alpha_\delta \}$ such that

(a) $N_i^k \prec M_i^k$ and $\bar{a}_i, \langle M_i^k : k < \alpha_i \rangle$ is independent in $(M_i, M_i, M_i^{\alpha_i})$;

(b) for each $k \leq \alpha_j$, $(M_i^k, f_{l,i}^k)_{j \leq l \leq i < \alpha_\delta}$ is a coherent, direct system such that

$$\begin{tikzcd}
M_{i_2} \arrow[r] & M_{i_2}^{k_0} \arrow[r] & M_{i_2}^{k_1} \\
M_{i_1} \arrow[r] \arrow[u] & M_{i_1}^{k_0} \arrow[r] \arrow[u, "f_{i_1,i_2}^{k_0}"] & M_{i_1}^{k_1} \arrow[u, "f_{i_1,i_2}^{k_1}"] \\
M_{i_0} \arrow[r] \arrow[u] & M_{i_0}^{k_0} \arrow[r] \arrow[u, "f_{i_0,i_1}^{k_0}"] & M_{i_0}^{k_1} \arrow[u, "f_{i_0,i_1}^{k_1}"] \arrow[uu, bend right, "f_{i_0,i_2}^{k_1}"']
\end{tikzcd}$$

commutes; and

(c) $f_{j,i}^k(a_j^k) = a_i^k$.

This is enough. For each $k < \alpha_\delta$, set $(M_\delta^k, f_{i,\delta}^k)_{i<\delta, k \leq \alpha_i} = \varinjlim(M_i^k, f_{j,i}^k)$. Then $\langle M_\delta^k : k < \alpha_\delta \rangle$ is increasing and continuous because each $\langle M_i^k : k < \alpha_i \rangle$ is. Set $M_\delta^{\alpha_\delta} := \cup_{k<\alpha_\delta} M_\delta^k$. For $k < \alpha_i, \alpha_j$, we have that $f_{i,\delta}^{k+1}(a_i^k) = f_{j,\delta}^{k+1}(a_j^k)$. Thus, there is no confusion in setting $a_\delta^k = f_{i,\delta}^{k+1}(a_i^k)$ for some/any $k < \alpha_i$. Set $p = tp(\bar{a}_\delta / M_\delta, M_\delta^{\alpha_\delta})$.

Note that $M_\delta \prec M_\delta^0$; indeed $f_{i,\delta}^k \upharpoonright M_i$ is the identity for all $k \leq \alpha_i$. Thus, we have that

$$p_i = tp(\bar{a}_i / M_i; M_i^{\alpha_i}) = tp(\langle a_\delta^k : k < \alpha_i \rangle / M_i; M_\delta^{\alpha_\delta}) = p^{\alpha_i} \upharpoonright M_i$$

If we are showing uniqueness as well, then we have that $f_{i,j}^{k+1}$ sends a^k to itself, so $\bar{a}_\delta = \langle a^k : k < \alpha_\delta \rangle$ realizes q. Thus, $p = q$.

Claim: For all $k < \alpha_\delta$, $\perp(M_\delta, M_\delta^k, a_\delta^k, M_\delta^{k+1})$[6].

Proof of Claim: Given $i < \delta$ and $k < \alpha_i$, we have by construction that $\perp(M_i, M_i^k, a_i^k, M_i^{k+1})$. Applying $f_{i,\delta}^k$ to this, we get $\perp(M_i, f_{i,\delta}^k(M_i^k), a_\delta^k, f_{i,\delta}^{k+1}(M_i^{k+1}))$. By construction,

$$M_\delta^k = \bigcup_{i<\delta} f_{i,\delta}^k(M_i^k) \text{ and } M_\delta^{k+1} = \bigcup_{i<\delta} f_{i,\delta}^{k+1}(M_i^{k+1})$$

Thus, by Continuity for $\mathfrak{s}$, we have, for all $i < \delta$, $\perp(M_i, M_\delta^k, a_\delta^k, M_\delta^{k+1})$. By Monotonicity for $\mathfrak{s}$, we get $\perp(M_\delta, M_\delta^k, a_\delta^k, M_\delta^{k+1})$.

Thus, $\bar{a}_\delta, \langle M_\delta^k : k \leq \alpha_\delta \rangle$ is independent in $(M_\delta, M_\delta, M_\delta^{\alpha_\delta})$. So $p \in S^{\alpha_\delta, bs}(M_\delta)$ and extends each p_i as desired.

Construction: Repeated applications of Corollary 3.20.

(5) Assume $\bar{a}_1 \underset{M_0}{\overset{N}{\perp}} M_2$, $\bar{a}_2 \in {}^{<\omega}M_2$, and $\text{tp}(\bar{a}_2 / M_0; N) \in S^{\text{bs}}(M_0)$.

[6] Recall that, in the moreover clause, this is $\perp(M_0, M_\delta^k, a_\delta^k, M_\delta^{k+1})$

By existence, $\bar{a}_2 \underset{M_0}{\overset{M_2}{\downarrow}} M_0$. By concatenation (Fact 3.27), $\bar{a}_1\bar{a}_2 \underset{M_0}{\overset{N}{\downarrow}} M_0$. By Fact 3.26, $\bar{a}_2\bar{a}_1 \underset{M_0}{\overset{N}{\downarrow}} M_0$. By definition of being independent, this means that there exists M_1 containing $\bar{a}_1$ and $N' \succ N$ such that $\bar{a}_2 \underset{M_0}{\overset{N'}{\downarrow}} M_1$, as needed.

□

Note that the continuity above proves Jarden's λ^+-continuity of serial independence (see [Jar, Definition 4.0.20]). This allows Jarden's proof of symmetry ([Jar, Theorem 4.0.22]) to go through without any extra hypotheses.

Also, continuity and the independence of order for finite tuples (see Fact 3.26) is enough to prove the *finite continuity property* (see [JS12, Definition 8.2]) from just the assumption of a good λ-frame without stability. This improves [JS12, Proposition 8.4], which proves this with the additional assumptions that $\mathfrak{s}$ satisfies the conjugation property and that $K^{3,uq}$ is dense with respect to $\prec_{bs}$.

5. Concatenation without Symmetry

A key property in the proof that symmetry transfers (Theorem 4.1.(5)) was the ability to concatenate two independent sequences. This has already been stated in Fact 3.27, but the existing proof of Jarden and Sitton uses symmetry. Here, we improve this to just requiring that $\mathfrak{s}$ is a frame that also satisfies extension and uniqueness (and, thus, transitivity). We avoid any use of symmetry or nonforking amalgamation.

This is not crucial for our main result, but shows that the situation is somewhat similar to the first-order context, where concatenation holds in any theory (See e.g. [GIL02, Lemma 1.6]).

Theorem 5.1 (Concatenation)**.** Let $\mathfrak{s}$ be a frame with extension, transitivity[7], and continuity. Let $M \prec M_0 \prec M_1 \prec M_2$ such that $\bar{a} = \langle a_i : i < \alpha \rangle$ is independent in (M, M_0, M_1) and $\bar{b} = \langle b_i : i < \beta \rangle$ is independent in (M, M_1, M_2). Then $\bar{a}\bar{b}$ is independent in (M, M_0, M_2).

A[8] diagram of the proof is included at the end.

[7]WB: I agree that this is all that the proof seems to need, but worth triple checking

[8]WB: very much in need of better drawing

Proof: From the independence of $\bar{a}$, there is continuous, increasing $\langle M_0^i : i \leq \alpha \rangle$ and N_0^+ such that

- $M_0 \prec M_0^i \prec N_0^+$;
- $M_1 \prec N_0^+$; and
- $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}(M, M_0^i, a_i, M_0^{i+1})$.

From the independence of $\bar{b}$, there is continuous, increasing $\langle M_1^i : i \leq \beta \rangle$ and N_1^+ such that

- $M_1 \prec M_1^i \prec N_1^+$;
- $M_2 \prec N_1^+$; and
- $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}(M, M_1^i, b_i, M_1^{i+1})$.

Step I: There is a coherent, continuous $\langle N_i, f_{j,i} : N_j \to N_i : j < i \leq \alpha \rangle$ and $f : M_1^0 \to_{M_0} N_0$ such that

- $M_0^i \prec N_i$ and f_{i+1} fixes M_0^i; and
- $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}(M, f_{i,i+1}(N_i), a_0^i, N_{i+1})$.

We proceed by induction using Lemma 3.21. At limit stages, we will take direct limits and this will be enough. For the base case, we simply amalgamate M_1^0 and M_0^0 over M_0; this gives $N_0 \succ M_0^0$ and $f_0 : M_1^0 \to_{M_0} N_0$. For successor steps (moving from i to $i+1$), we apply the above claim to

$$\begin{tikzcd}
 & M_0^{i+1} \arrow[r, dashed] & N_{i+1} \\
 & M_0^i \arrow[u, "a_0^i"] \arrow[r] & N_i \arrow[u, dashed, "a_0^i"', "f_{i,i+1}"] \\
M \arrow[ur] & &
\end{tikzcd}$$

and generate the rest of the direct system from that.

Step II: There is increasing continuous $\langle N_1^j : j \leq \beta \rangle$ and increasing continuous $\langle g_j : M_1^j \to N_1^j : j \leq \beta \rangle$ such that

- $N_1^0 = N_\alpha$ and $g_0 = f_{0,\alpha} \circ f$; and
- $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}(M, N_1^j, g_{j+1}(b_j), N_1^{j+1})$.

We proceed by induction using Lemma 3.22. At limit stages, we take unions and this will be enough. The base case is set. At successor steps (moving from j to $j+1$), we apply the above claim to

$$\begin{tikzcd}
& N_1^j \arrow[r, dashed, "g_{j+1}(a_1^j)"] & N_1^{j+1} \\
& M_1^j \arrow[u, "g_i"] \arrow[r, "a_1^j"] & M_1^{j+1} \arrow[u, dashed, "g_{j+1}"'] \\
M \arrow[ur] & &
\end{tikzcd}$$

Set $g := \bigcup_{i \leq \beta} g_i$. Now we amalgamate N_1^+ and N_1^β (via g) over M_1^β to get $N^{++} \succ N_1^\beta$ and $h : N_1^+ \to N^{++}$ extending g.

Define the sequence $\langle N^i : i \leq \alpha + \beta \rangle$ by

$$N^i := \begin{cases} M_0^i & \text{if } i \leq \alpha \\ N_1^j & \text{if } i = \alpha + j \in (\alpha, \beta] \end{cases}$$

Claim: This sequences witnesses that $\bar{c} := \bar{a}^\frown g(\bar{b})$ is independent in (M, M_0, N^{++}).

Proof of Claim: It is easy to see that this sequence is of the proper type, ie, it is increasing and continuous and $M_0 \prec N^i \prec N^{++}$.

If $i < \alpha$, then we need to show that $\mathop{\smile\hspace{-0.9em}\mid}(M, N^i, c_i, N^{i+1})$, which is the same as $\mathop{\smile\hspace{-0.9em}\mid}(M, M_0^i, a_i, M_0^{i+1})$. This just follows from the definition of the independent sequence for $\bar{a}$.

If $i = \alpha$, then we need to show that $\mathop{\smile\hspace{-0.9em}\mid}(M, N^\alpha, c_\alpha, N^{\alpha+1})$, which is the same as $\mathop{\smile\hspace{-0.9em}\mid}(M, M_0^\alpha, g_1(b_0), N_1^1)$. We know that $\mathop{\smile\hspace{-0.9em}\mid}(M, N_1^0, g_1(b_0), N_1^1)$ holds from the construction in **Step II** and we know that $M \prec M_0^\alpha \prec N_1^1$. Thus, by Monotonicity, we have the desired nonforking.

If $i = \alpha + j > \alpha$, then we need to show that $\mathop{\smile\hspace{-0.9em}\mid}(M, N^i, c_i, N^{i+1})$, which is the same as $\mathop{\smile\hspace{-0.9em}\mid}(M, N_1^j, g_{j+1}(b_j), N_1^{j+1})$. This holds directly by the construction in **Step II**. $\dagger_{\textbf{Claim}}$

Notice that the map h, which extends the map g, shows that $tp(\bar{a}(g(\bar{b})/M_0; N_1^\beta)) = tp(\bar{a}\bar{b}/M_0; M_2)$. Thus, by Lemma 3.23, we have that $\bar{a}\bar{b}$ is independent in (M, M_0, M_2). $\dagger$

$$
\begin{tikzcd}
 & & & & & & & & & & & N^{++} \\
 & & & & & & & & & & N_1^\beta \arrow[ur] & \\
 & & & & & & & & & N_1^2 \arrow[ur] & & \\
 & & & & & & & & N_1^1 \arrow[ur] & & & N_1^+ \arrow[uuu, "h"'] \\
 & & & & & M_{0,\alpha} \arrow[r] & N_\alpha = N_1^0 \arrow[urr] & & & & M_1^\beta \arrow[uuu, "g"'] \arrow[ur] & \\
 & & & & M_0^2 \arrow[ur] \arrow[rr] & & N_2 \arrow[u, "f_{2,\alpha}"] & & & & & \\
 & & & M_0^1 \arrow[ur] \arrow[rrr] & & & N_1 \arrow[u, "f_{1,2}"] & & & M_1^2 \arrow[uuuu, "g_2"'] \arrow[uur] & & \\
 & & M_0^0 \arrow[ur] \arrow[rrrr] & & & & N_0 \arrow[u, "f_{0,1}"] & & M_1^1 \arrow[uuuu, "g_1"'] \arrow[ur] & & & \\
 & & & & & & & M_1^0 \arrow[ul, "f_0"] \arrow[uuuul, "g_0"'] \arrow[ur] & & & & \\
M \arrow[r] & M_0 \arrow[uur] \arrow[rrrrr] & & & & & M_1 \arrow[ur] \arrow[rrrrr] & & & & & M_2 \arrow[uuuuuu]
\end{tikzcd}
$$

6. "Up" and "long" commute

Theorem 6.1. Let $\mathfrak{s}$ be a good λ-frame without stability or symmetry and assume that $\mathfrak{s}_{\geq \lambda}$ is also a good λ-frame without stability or symmetry. Then:

$$(\mathfrak{s}_{\geq \lambda})^{<\lambda^+} = \left(\mathfrak{s}^{<\lambda^+}\right)_{\geq \lambda}$$

Proof. Let $\mathfrak{s}_{\geq \lambda} := (K, \mathop{\smile\!\!\!\!\!\;|}, S^{\text{bs}})$. Write $(\mathfrak{s}_{\geq \lambda})^{<\lambda^+} := (K, \overset{(1)}{\mathop{\smile\!\!\!\!\!\;|}}, S_1^{\text{bs}})$, $\left(\mathfrak{s}^{<\lambda^+}\right)_{\geq \lambda} := (K, \overset{(2)}{\mathop{\smile\!\!\!\!\!\;|}}, S_2^{\text{bs}})$. By Proposition 3.16 and existence, it is enough to show $\overset{(2)}{\mathop{\smile\!\!\!\!\!\;|}} \subseteq \overset{(1)}{\mathop{\smile\!\!\!\!\!\;|}}$. Assume $\overset{(2)}{\mathop{\smile\!\!\!\!\!\;|}}(M, \bar{a}, N, \widehat{N})$. By definition of $\overset{(2)}{\mathop{\smile\!\!\!\!\!\;|}}$ and monotonicity, we can assume without loss of generality that $M \in K_\lambda$. We know that for all $N' \leq N$ and $\widehat{N}' \leq \widehat{N}$ of size λ, $\bar{a}$ is independent (with respect to $\mathop{\smile\!\!\!\!\!\;|}$) in $(M, N', \widehat{N}')$. We want to see that $\bar{a}$ is independent (with respect to $\mathop{\smile\!\!\!\!\!\;|}$) in $(M, N, \widehat{N})$.

Say $N, \widehat{N} \in K_\mu$, $\mu \geq \lambda$. Work by induction on μ. We already have what we want if $\mu = \lambda$, so assume $\mu > \lambda$. Let $(N_i)_{i \leq \mu}$ be an increasing continuous resolution of N such that $N_\mu = N$, $N_0 = M$, $\|N_i\| = \lambda + |i|$.

By the induction hypothesis, $\bar{a}$ is independent (with respect to $\mathop{\smile\!\!\!\!\!\;|}$) in $(M, N_i, \widehat{N})$ for all $i < \mu$. In other words, for any $i < \mu$, $\text{tp}(\bar{a}/N_i; \widehat{N})$ does not fork (in the sense of $(\mathfrak{s}_{\geq \lambda})^{<\lambda^+}$) over M. By Theorem 4.1 (and the remark in the statement), we know that $(\mathfrak{s}_{\geq \lambda})^{<\lambda^+}$ has the continuity property. Thus $\text{tp}(\bar{a}/N; \widehat{N})$ also does not fork (in the sense of $(\mathfrak{s}_{\geq \lambda})^{<\lambda^+}$) over M. This is exactly what we needed to prove. $\square$

7. Extending frames revisited

Lemma 7.1. Let $\mathfrak{s} := (K, \mathop{\smile\!\!\!\!\!\;|}, S^{\text{bs}})$ be a $(< \alpha, \geq \lambda)$-frame such that if $\bar{a} \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\;|}}} M_1$, then there is $M_0' \prec M_0$ in K_λ with $\bar{a} \underset{M_0'}{\overset{N}{\mathop{\smile\!\!\!\!\!\;|}}} M_1$. Assume $\mathfrak{s}$ has uniqueness.

Then K is λ-tame for basic $(< \alpha)$-types, i.e. for any $M \in K$ and $p, q \in S^{\text{bs}}(M)$, if $p \neq q$, then there is $M_0 \prec M$ of size $\leq \lambda$ such that $p \upharpoonright M_0 \neq q \upharpoonright M_0$.

Proof. Fix $p, q \in S^{\mathrm{bs}}(M)$ as above. By monotonicity, one can find $M_0 \prec M$ such that both p and q do not fork over M_0. By uniqueness, M_0 is as desired. $\square$

Note that the "such that" in the hypothesis follows from local character if $\alpha \leq \omega$.

Lemma 7.2. Let $\mathfrak{s}$ be a good λ-frame without stability. Write $(\mathfrak{s}^{<\omega})_{\geq\lambda} := (K, \mathop{\smile\!\!\!\!\!\!|}, S^{\mathrm{bs}})$. If $\bar{a} \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!|}}} M_1$, and $\bar{a}'$ is a permutation of $\bar{a}$, then $\bar{a}' \underset{M_0}{\overset{N}{\mathop{\smile\!\!\!\!\!\!|}}} M_1$.

Proof. Fix $M_0' \prec M_0$ in K_λ such that for any $M_0' \prec M_1' \prec M_1$, $M_1' \prec N' \prec N$ with $\bar{a} \in N'$, M_1' and N' in K_λ, we have $\bar{a} \underset{M_0'}{\overset{N}{\mathop{\smile\!\!\!\!\!\!|}}} M_1'$. By Fact 3.26, this also means $\bar{a}' \underset{M_0'}{\overset{N}{\mathop{\smile\!\!\!\!\!\!|}}} M_1'$. The result follows. $\square$

Theorem 7.3. Assume $\mathfrak{s}$ is a good $\mathcal{F}$-frame without stability and symmetry, where λ is an interval of the form $[\lambda, \mu)$, and $\mu > \lambda$ is either a cardinal or ∞. Then $\mathfrak{s}$ has symmetry if and only if $\mathfrak{s} \upharpoonright \lambda$ has symmetry.

Proof. Let $\mathfrak{s} \upharpoonright \lambda := (K, \mathop{\smile\!\!\!\!\!\!|}, S^{\mathrm{bs}})$ and let $\mathfrak{s} := (K, \underset{\geq\lambda}{\mathop{\smile\!\!\!\!\!\!|}}, S^{\mathrm{bs}}_{\geq\lambda})$. If $\mathfrak{s}$ has symmetry, then in particular $\mathfrak{s} \upharpoonright \lambda$ has symmetry. Now assume $\mathfrak{s} \upharpoonright \lambda$ has symmetry. By [She09a, Section 2], there is at most one good $\mathcal{F}$-frame without symmetry extending $\mathfrak{s} \upharpoonright \lambda$. Now $\mathfrak{s}$ is such a frame, and since $\mathfrak{s}$ gives us some tameness, [Bon] proves that $\mathfrak{s}_{\mathcal{F}}$ is also a good frame without symmetry, so $\mathfrak{s} = \mathfrak{s}_{\mathcal{F}}$.

Recall that [Bon, Theorem 6.1] proves symmetry for $\mathfrak{s} = \mathfrak{s}_{\mathcal{F}}$ assuming λ-tameness for 2-types. We revisit this proof and use the same notation. Suppose $\underset{\geq\lambda}{\mathop{\smile\!\!\!\!\!\!|}}(M_0, M_2, a_1, M_3)$, $a_2 \in M_2$ with $\mathrm{tp}(a_2/M_0; M_3) \in S^{\mathrm{bs}}_{\geq\lambda}(M_0)$. Let $M_0 \prec M_1 \prec M_3$ be a model containing a_1. By extension, there is $M_3' \succ M_3$ and $a' \in M_3'$ such that $\underset{\geq\lambda}{\mathop{\smile\!\!\!\!\!\!|}}(M_0, M_1, a', M_3')$ and $\mathrm{tp}(a'/M_0; M_3') = \mathrm{tp}(a_2/M_0; M_3)$. Boney argues it is enough to see that $p := \mathrm{tp}(a_1 a_2/M_0; M_3) = \mathrm{tp}(a_1 a'/M_0; M_3') =: p'$, shows that this equality holds for all restrictions to models of size λ, and then uses tameness for 2-types. Our hypotheses only include tameness for 1-types, but by Theorem 4.1, $(\mathfrak{s}_{\geq\lambda})^{\leq 2}$ has uniqueness, so by Lemma 7.1, it is enough to see that p, p' are basic types of $(\mathfrak{s}_{\geq\lambda})^{\leq 2}$.

First, let's see that a_1a_2 is independent (with respect to $\underset{\geq\lambda}{\mathop{\smile}\limits^{\vert}}$) in (M_0, M_0, M_3). The increasing chain (M_0, M_2, M_3) witnesses that a_2a_1 is independent in (M_0, M_0, M_3). Now from Theorem 6.1, $(\mathfrak{s}_{\geq\lambda})^{\leq 2} = \left(\mathfrak{s}^{\leq 2}\right)_{\geq\lambda}$ so we can use Lemma 7.2 to see that a_1a_2 is independent in (M_0, M_0, M_3). Similarly, (M_0, M_1, M_3') witnesses that a_1a' is independent in (M_0, M_0, M_3'). Thus p and p' are basic types of $(\mathfrak{s}_{\geq\lambda})^{\leq 2}$, as desired. □

We can now prove the announced theorem.

Corollary 7.4. Assume there is a good λ-frame $\mathfrak{s}$, K has amalgamation, and is λ-tame for 1-types. Then $\mathfrak{s}_{\geq\lambda}$ (and in fact even $\mathfrak{s}_{\geq\lambda}^{<\omega}$) is a good $(\geq\lambda)$-frame.

Proof. Let $\mathfrak{s} := (K, \mathop{\smile}\limits^{\vert}, S^{\text{bs}})$ and let $\mathfrak{s}_{\geq\lambda} := (K, \underset{\geq\lambda}{\mathop{\smile}\limits^{\vert}}, S^{\text{bs}}_{\geq\lambda})$. By Fact 3.11, $\mathfrak{s}_{\geq\lambda}$ is a good frame without symmetry or no maximal models. Symmetry follows from the previous theorem, and [Bon, Theorem 7.1] now gives us no maximal models. □

Corollary 7.5. Let $\mathfrak{s} := (K, \mathop{\smile}\limits^{\vert}, S^{\text{bs}})$ be a good λ-frame. Assume K is λ-tame for 1-types.

Then K is λ-tame for the basic types of $(\mathfrak{s}_{\geq\lambda})^{<\lambda^+} = \left(\mathfrak{s}^{<\lambda^+}\right)_{\geq\lambda}$.

Proof. Follows easily from Corollary 7.4, Theorem 6.1, and Lemma 7.1. □

Note that for types of infinite length, we don't have Theorem 6.1 to tell us that order doesn't matter in extending the length and domain size.

We can improve Corollary 7.5 to types of even longer length. Suppose that $\mathfrak{s}$ is good λ-frame for 1-types. Define

$$\mathfrak{s}^\infty := (\mathfrak{s}_{\geq\lambda})^{<\infty} = \bigcup_{\alpha\in\mathbf{ON}} (\mathfrak{s}_{\geq\lambda})^{<\alpha}$$

Note that this order is the only one that makes sense: there are no independent sequences of length μ^+ or longer in a μ-frame. Since $\mathfrak{s}$ is a good λ-frame, extensions of Theorem 4.1 will transfer many of the frame properties to $\mathfrak{s}^\infty$, although *not* symmetry. However, we have enough for a tameness result.

Corollary 7.6. Let $\mathfrak{s}$ be a good λ-frame. Assume K is λ-tame for 1-types. Then a basic type of p of $\mathfrak{s}^\infty$ is determined by its restrictions to models of size $\lambda + |\ell(p)|$.

Proof. Let $p, q \in S^{bs}_{\mathfrak{s}^\infty}(M)$ such that $p \upharpoonright M^- = q \upharpoonright M^-$ for all $M^- \in K_\mu$ such that $M^- \prec M$ where $\mu := \lambda + |\ell(p)|$. If $\mu \geq \|M\|$, then this just gives $p = q$. Otherwise, set $\mathfrak{t} = \mathfrak{s}_{\geq\lambda} \upharpoonright \mu$; this is good μ-frame without symmetry. Since, p, q are basic for s^∞, they are basic for $(\mathfrak{t}_{\geq\mu})^{<\mu^+}$. By Theorem 6.1, this means they are basic for $(\mathfrak{t}^{<\mu^+})_{\geq\mu}$. Then, we can find some $M_0 \prec M$ of size μ such that p and q do not fork over M_0. We know that $p \upharpoonright M_0 = q \upharpoonright M_0$ and that $(\mathfrak{t}_{\geq\mu})^{<\mu^+}$ has uniqueness (see Theorem 4.1); thus $p = q$. $\square$

References

[BGKV] Will Boney, Rami Grossberg, Alexei Kolesnikov, and Sebastien Vasey, *Canonical forking in AECs*, Preprint. URL: `http://arxiv.org/abs/1404.1494v1`.

[Bon] Will Boney, *Tameness and extending frames*, Preprint. URL: `http://arxiv.org/abs/1308.6006v2`.

[GIL02] Rami Grossberg, José Iovino, and Olivier Lessmann, *A primer of simple theories*, Archive for Mathematical Logic **41** (2002), no. 6, 541–580.

[Jar] Adi Jarden, *A semi-good frame with amalgamation and tameness in* λ^+, Preprint. URL: `http://arxiv.org/abs/1406.4251v1`.

[JS12] Adi Jarden and Alon Sitton, *Independence, dimension and continuity in non-forking frames*, The Journal of Symbolic Logic **78** (2012), no. 2, 602–632.

[JS13] Adi Jarden and Saharon Shelah, *Non-forking frames in abstract elementary classes*, Annals of Pure and Applied Logic **164** (2013), 135–191.

[She09a] Saharon Shelah, *Categoricity in abstract elementary classes: going up inductively*, Classification theory for abstract elementary classes, College Publications, 2009, pp. 224–377.

[She09b] ———, *Toward classification theory of good λ-frame and abstract elementary classes*, Classification theory for abstract elementary classes, College Publications, 2009, pp. 378–644.

[Vas] Sebastien Vasey, *Forking and superstability in tame AECs*, Preprint. URL: `http://arxiv.org/abs/1405.7443v1`.

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