

UNIVERSAL CLASSES NEAR \aleph_1

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ABSTRACT. Shelah has provided sufficient conditions for an $\mathbb{L}_{\omega_1, \omega}$ -sentence ψ to have arbitrarily large models and for a Morley-like theorem to hold of ψ . These conditions involve structural and set-theoretic assumptions on all the \aleph_n 's. Using tools of Boney, Shelah, and the second author, we give assumptions on \aleph_0 and \aleph_1 which suffice when ψ is restricted to be universal:

Theorem. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be a universal $\mathbb{L}_{\omega_1, \omega}$ -sentence.*

- (1) *If ψ is categorical in \aleph_0 and $1 \leq \mathbb{I}(\psi, \aleph_1) < 2^{\aleph_1}$, then ψ has arbitrarily large models and categoricity of ψ in some uncountable cardinal implies categoricity of ψ in all uncountable cardinals.*
- (2) *If ψ is categorical in \aleph_1 , then ψ is categorical in all uncountable cardinals.*

The theorem generalizes to the framework of $\mathbb{L}_{\omega_1, \omega}$ -definable tame abstract elementary classes with primes.

1. INTRODUCTION

In a milestone paper, Shelah [Sh87a, Sh87b] gives the following classification-theoretic analysis of $\mathbb{L}_{\omega_1, \omega}$ -sentences:

Fact 1.1. *Assume that $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$. Let $\psi \in \mathbb{L}_{\omega_1, \omega}$ be a complete sentence. Assume that ψ has an uncountable model and for all $n > 0$, $\mathbb{I}(\psi, \aleph_n) < \mu_{\text{wd}}(\aleph_n)$.¹ Then ψ has arbitrarily large models and categoricity of ψ in some uncountable cardinal implies categoricity of ψ in all uncountable cardinals.*

It is provably necessary to make hypotheses on all the \aleph_n 's: a family of examples of Hart and Shelah [HaSh90] (analyzed in detail by Baldwin and Kolesnikov [BK09]) gives for each $n < \omega$ an $\mathbb{L}_{\omega_1, \omega}$ -sentence ψ_n which is categorical in $\aleph_0, \aleph_1, \dots, \aleph_n$ but not in any $\lambda > \aleph_n$.

In the present paper, we show that if we restrict the complexity of the sentence, then it suffices to make model-theoretic and set-theoretic assumptions on \aleph_0 and \aleph_1 . More precisely:

Theorem 3.3. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be a universal $\mathbb{L}_{\omega_1, \omega}$ sentence (i.e. ψ is of the form $\forall \mathbf{x} \phi(\mathbf{x})$, where ϕ is quantifier-free). If ψ is categorical in \aleph_0 and $1 \leq \mathbb{I}(\psi, \aleph_1) < 2^{\aleph_1}$, then:*

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¹See [Sh:h, VII.0.4] for a definition of μ_{wd} and [Sh:h, VII.0.5] for some of its properties. We always have that $2^{\aleph_n} \leq \mu_{\text{wd}}(\aleph_{n+1})$.

- (1) ψ has arbitrarily large models.
- (2) If ψ is categorical in some uncountable cardinal then ψ is categorical in all uncountable cardinals.

We more generally prove Theorem 3.3 for universal classes (classes of models closed under isomorphisms, substructures, and unions of \subseteq -increasing chains, see Definition 2.1 and Fact 2.2) in a countable vocabulary. The assumption of categoricity in \aleph_0 can be removed if we instead assume categoricity in \aleph_1 . In this case, we obtain the following upward categoricity transfer:

Theorem 3.5. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be a universal $\mathbb{L}_{\omega_1, \omega}$ sentence. If ψ is categorical in \aleph_1 , then it is categorical in all uncountable cardinals.*

The statements of Theorems 3.3 and 3.5 should be compared to the second author’s eventual categoricity theorem for universal classes[Vas17d]:

Fact 1.2. *Let ψ be a universal $\mathbb{L}_{\omega_1, \omega}$ -sentence. If ψ is categorical in some $\mu \geq \beth_{\omega_1}$, then ψ is categorical in all $\mu' \geq \beth_{\omega_1}$.*

Fact 1.2 is a ZFC theorem while the results of this paper use $2^{\aleph_0} < 2^{\aleph_1}$. However, Fact 1.2 is an eventual statement, valid for “big” cardinals (in fact there is a generalization to any universal class, not necessarily in a countable vocabulary), while the focus of this paper is on structural properties holding in \aleph_0 and \aleph_1 .

The proofs of Theorem 3.3 and 3.5 use powerful tools from the classification theory of abstract elementary classes (AECs): good frames [Sh:h, Chapter II], primes [Sh:h, §III.3], and tameness [GV06]. In fact the results of this paper are direct consequences of putting general facts about AECs (many only recently discovered) together: Shelah’s construction of good frames [Sh:h, §II.3], Boney’s proof of tameness in universal classes, and the second author’s proof of the eventual categoricity conjecture in tame AECs with primes [Vas17c, Vas17b].

Let us outline the proof of Theorem 3.3 with more detail. We start with K , the class of models of our universal $\mathbb{L}_{\omega_1, \omega}$ sentence ψ . This is a universal class (see Definition 2.1). We are further assuming that ψ is categorical in \aleph_0 and has one but not too many models in \aleph_1 . The first step is to show that K is well-behaved in \aleph_0 : we use machinery of Shelah (Fact 2.13) to build a *good \aleph_0 -frame* (see Definition 2.5). The second step is to observe that universal classes have a locality property for orbital types called *tameness* (see Definition 2.6): in fact orbital types are determined by their finite restrictions (this is due to Will Boney, see Fact 2.7). The third step is the easy observation that in universal classes there is a prime model over every set (see Definition 2.8): take the closure of the set under the functions of an ambient model. The fourth and final step is to use the second author’s results on AECs that have a good frame, are tame, and have primes [Vas17c, Vas17b]: any such class has arbitrarily large models and further Morley’s categoricity theorem holds of such classes.

Note that the above argument only used the structural assumption on the class in the first step (to get the good frame). Once we have a good frame, the result follows because any universal class is tame and has primes. Moreover, the argument to get the good frame works in a much more general setup than universal classes. Thus our results generalize to:

Theorem 4.4. Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let \mathbf{K} be an AEC with $\text{LS}(\mathbf{K}) = \aleph_0$. Assume that \mathbf{K} has primes, is \aleph_0 -tame, and is PC_{\aleph_0} (see [Sh:h, I.1.4], this is essentially the class of reducts of models of an $\mathbb{L}_{\omega_1, \omega}$ -sentence).

- (1) If \mathbf{K} is categorical in \aleph_0 and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$, then \mathbf{K} has arbitrarily large models and categoricity in *some* uncountable cardinal implies categoricity in *all* uncountable cardinals.
- (2) If \mathbf{K} is categorical in \aleph_1 , then \mathbf{K} is categorical in all uncountable cardinals.

To sum up, any tame AEC with primes which has good behavior in the “low” cardinals (\aleph_0 and \aleph_1) will have good behavior everywhere. If on the other hand it is not clear that the AEC is tame or has primes, Shelah’s results [Sh87a, Sh87b] and the Hart-Shelah example [HaSh90, BK09] tell us that one will need to use higher cardinals (the \aleph_n ’s) to sort out whether the AEC is well-behaved past \aleph_ω .

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2. PRELIMINARIES

We assume that the reader has some familiarity with the basics of abstract elementary classes, as presented in for example [Ba09, §4-8]. In this section, we recall the main notions that we will use.

The notion of a universal class was studied already in Tarski’s [Tar54]. Shelah [Sh300] was the first to develop classification theory for non-elementary universal classes.

Definition 2.1. *A class of structures K is a universal class if:*

- (1) K is a class of τ -structures, for some fixed vocabulary $\tau = \tau(K)$.
- (2) K is closed under isomorphisms.
- (3) K is closed under \subseteq -increasing chains.
- (4) If $M \in K$ and $N \subseteq M$, then $N \in K$.

The following basic characterization of universal classes is essentially due to Tarski [Tar54] (he proved it for finite vocabulary, but the proof generalizes). This will not be used in the present paper.

Fact 2.2 (Tarski’s presentation theorem). *Let K be a class of structures. The following are equivalent:*

- (1) K is a universal class.
- (2) K is the class of models of a universal $\mathbb{L}_{\infty, \omega}$ theory.

Universal classes are abstract elementary classes:

Definition 2.3 (Definition 1.2 in [Sh88]). *An abstract elementary class (AEC for short) is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:*

- (1) K is a class of τ -structures, for some fixed vocabulary $\tau = \tau(\mathbf{K})$.

- (2) $\leq_{\mathbf{K}}$ is a partial order (that is, a reflexive and transitive relation) on K .
- (3) $(K, \leq_{\mathbf{K}})$ respects isomorphisms: If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$. In particular (taking $M = N$), K is closed under isomorphisms.
- (4) If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.
- (5) Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq_{\mathbf{K}} M_1$;
- (6) Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\langle M_i \in K : i < \delta \rangle$ is an increasing chain. Then:
 - (a) $M_\delta := \bigcup_{i < \delta} M_i \in K$ and $M_0 \leq_{\mathbf{K}} M_\delta$.
 - (b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.
- (7) Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\text{LS}(\mathbf{K})$ for the minimal such cardinal.

Remark 2.4.

- (1) When we write $M \leq_{\mathbf{K}} N$, it is assumed that $M, N \in K$.
- (2) We write \mathbf{K} for the pair $(K, \leq_{\mathbf{K}})$, and K (no boldface) for the actual class. However we may abuse notation and write for example $M \in \mathbf{K}$ instead of $M \in K$ when there is no danger of confusion.
- (3) If K is a universal class, then $\mathbf{K} := (K, \subseteq)$ is an AEC with $\text{LS}(\mathbf{K}) = |\tau(K)| + \aleph_0$. Throughout this paper, we think of K as the AEC \mathbf{K} and may write “ \mathbf{K} is a universal class” instead of “ K is a universal class”.

In any AEC \mathbf{K} , we can define a semantic notion of type, called Galois or orbital types in the literature (such types were introduced by Shelah in [Sh300]). For $M \in \mathbf{K}$, $A \subseteq |M|$, and $\mathbf{b} \in {}^{<\omega}M$, we write $\mathbf{tp}_{\mathbf{K}}(\mathbf{b}/A; M)$ for the orbital type of \mathbf{b} over A as computed in M , see [Vas16b, 2.16] for a formal definition (usually \mathbf{K} will be clear from context and we will omit it from the notation). It is the finest notion of type respecting \mathbf{K} -embeddings. When \mathbf{K} is an elementary class, $\mathbf{tp}(\mathbf{b}/A; M)$ contains the same information as the usual notion of $\mathbb{L}_{\omega, \omega}$ -syntactic type, but in general the two notions need not coincide [HaSh90, BK09]. We will see shortly (Fact 2.7) that in universal classes the orbital types coincide with the quantifier-free types.

The *length* of $\mathbf{tp}(\mathbf{b}/A; M)$ is the length of \mathbf{b} . For $M \in \mathbf{K}$ and α a cardinal, p is a type over M of length α if there is $N \geq_{\mathbf{K}} M$ and $\mathbf{b} \in N^\alpha$ such that $p = \mathbf{tp}(\mathbf{b}/M, N)$. We write $\mathbf{S}_{\mathbf{K}}^\alpha(M) = \mathbf{S}^\alpha(M) = \{\mathbf{tp}(\mathbf{b}/M; N) \mid \mathbf{b} \in {}^\alpha N, M \leq_{\mathbf{K}} N\}$ for the set of types over M of length α . When $\alpha = 1$, we just write $\mathbf{S}(M)$. We define naturally what it means for a type to be realized inside a model, to extend another type, and to take the image of a type by a \mathbf{K} -embedding. We call an AEC \mathbf{K} λ -stable if $|\mathbf{S}(M)| \leq \lambda$ for every $M \in \mathbf{K}$ of cardinality λ .

Given an AEC \mathbf{K} and $[\lambda, \mu)$ an interval of cardinals (we allow $\mu = \infty$), let $\mathbf{K}_{[\lambda, \mu)} = \{M \in \mathbf{K} : \|M\| \in [\lambda, \mu)\}$. We write \mathbf{K}_λ for $\mathbf{K}_{\{\lambda\}}$ and $\mathbf{K}_{\geq \lambda}$ for $\mathbf{K}_{[\lambda, \infty)}$.

The notion of a good λ -frame is introduced in [Sh:h, §II.2]. As a first approximation, the reader can think of the statement “ \mathbf{K} has a good λ -frame” as saying “ \mathbf{K}

has a model of cardinality λ , amalgamation in λ , no maximal models in λ , joint embedding in λ , is stable in λ , and has a superstable-like nonforking notion for types over models of cardinality λ ".

While we will not use the definition here (only its consequence), we give it in full to give the reader a sense of the bigger picture. We will follow the simplifications and generalizations given in [Vas16a] and [BoVa17]. Shelah assumes that nonforking is only defined for a certain class of types he calls the basic types. This complicates the notation and we have no use for basic types in this paper. In Shelah's terminology, our frames will always be *type-full*.

Definition 2.5. *Let $\lambda < \mu$ where λ is an infinite cardinal and μ is an infinite cardinal or infinity. A (type-full) good $[\lambda, \mu]$ -frame is a pair (\mathbf{K}, \perp) where the following properties hold:*

- (1) \mathbf{K} is an abstract elementary class with $\lambda \geq \text{LS}(\mathbf{K})$ and $\mathbf{K}_\lambda \neq \emptyset$.
- (2) \perp is a relation on quadruples (M_0, M_1, a, N) , where $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N$, $a \in N \setminus M_1$ and $M_0, M_1, N \in \mathbf{K}_{[\lambda, \mu]}$. We write $a \underset{M_0}{\perp}^N M_1$ or $\text{tp}(a/M_1, N)$ does not fork over M_0 (this will be well-defined by the next two properties).
- (3) Invariance: If $f : N \cong N'$ and $a \underset{M_0}{\perp}^N M_1$, then $f(a) \underset{f[M_0]}{\perp}^{N'} f[M_1]$.
- (4) Monotonicity: If $a \underset{M_0}{\perp}^N M_1$ and $M_0 \leq_{\mathbf{K}} M'_0 \leq_{\mathbf{K}} M'_1 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} N' \leq N \leq_{\mathbf{K}}$ N'' with $N'' \in \mathbf{K}_{[\lambda, \mu]}$ and $a \in N'$, then $a \underset{M'_0}{\perp}^{N'} M'_1$ and $a \underset{M'_0}{\perp}^{N''} M'_1$.
- (5) $\mathbf{K}_{[\lambda, \mu]}$ has amalgamation, joint embedding and no maximal models.
- (6) Stability: $|\mathbf{S}(M)| \leq \|M\|$ for all $M \in \mathbf{K}_{[\lambda, \mu]}$.
- (7) Existence of nonforking extension: If $p \in \mathbf{S}(M)$, $M \leq_{\mathbf{K}} N$ with $M, N \in \mathbf{K}_{[\lambda, \mu]}$, then there is $q \in \mathbf{S}(N)$ that does not fork over M and extends p .
- (8) Uniqueness: If $p, q \in \mathbf{S}(N)$ do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
- (9) Symmetry: If $a_1 \underset{M_0}{\perp}^N M_2$, $a_2 \in M_2 \setminus M_0$, then there is M_1 and $N' \geq_{\mathbf{K}} N$ with $a_1 \in M_1$ and $M_1, N \in \mathbf{K}_{[\lambda, \mu]}$ such that $a_2 \underset{M_0}{\perp}^{N'} M_1$.
- (10) Local character: If δ is a regular cardinal, $\{M_i : i < \delta\} \subseteq \mathbf{K}_{[\lambda, \mu]}$ is an increasing continuous chain and $p \in \mathbf{S}(M_\delta)$, then there is an $i < \delta$ such that p does not fork over M_i .
- (11) Continuity: If δ is a limit ordinal, $\{M_i : i < \delta\} \subseteq \mathbf{K}_{[\lambda, \mu]}$ is an increasing continuous chain, $\{p_i : i < \delta\}$ with $p_i \in \mathbf{S}(M_i)$ and $i < j < \delta$ implies that $p_i = p_j \upharpoonright_{M_i}$, then there is $p \in \mathbf{S}(M_\delta)$ that extends p_i for all $i < \delta$. Moreover, if each p_i does not fork over M_0 , then neither does p .
- (12) Transitivity: If $M_0 \leq_{\mathbf{K}} M_1 \leq_{\mathbf{K}} M_2$ with $M_0, M_1, M_2 \in \mathbf{K}_{[\lambda, \mu]}$, $p \in \mathbf{S}(M_2)$ does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0 , then p does not fork over M_0 .

If $\mu = \lambda^+$, then we write good λ -frame instead of good $[\lambda, \lambda^+]$ -frame. We say that an AEC \mathbf{K} has a good $[\lambda, \mu]$ -frame if there is a good $[\lambda, \mu]$ -frame with underlying AEC \mathbf{K} .

Tameness is a locality property of orbital types (which may or may not hold), first named by Grossberg and VanDieren in [GV06]:

Definition 2.6. *We say an AEC \mathbf{K} is $(< \kappa)$ -tame if for any $M \in \mathbf{K}$ and $p \neq q \in \mathbf{S}(M)$, there is $A \subseteq |M|$ such that $|A| < \kappa$ and $p \upharpoonright A \neq q \upharpoonright A$. By κ -tame we mean $(< \kappa^+)$ -tame. If we write $(< \kappa, \lambda)$ -tame we restrict to $M \in \mathbf{K}_\lambda$. We may also talk of tameness for types of finite length, which means that we allow p, q above to be in $\mathbf{S}^{<\omega}(M)$ rather than just $\mathbf{S}(M)$ (i.e. they could be types of finite sequences rather than types of singletons).*

The following important fact is due to Will Boney. It appears in print as [Vas17c, 3.7].

Fact 2.7. *If \mathbf{K} is a universal class then \mathbf{K} is $(< \aleph_0)$ -tame for types of finite length (in fact for types of all possibly infinite lengths). Moreover, orbital types are the same as quantifier-free types.*

The final main concept used in this paper is that of prime models (here over sets of the form $M \cup \{a\}$). The appropriate definition was introduced to AECs by Shelah in [Sh:h, III.3.2]. The definition is what the reader would expect when working inside a fixed monster model, but here we may not have amalgamation, so we have to use orbital types to describe the embedding of the base set.

Definition 2.8. *Let \mathbf{K} be an AEC.*

- *A prime triple is (a, M, N) such that $M \leq_{\mathbf{K}} N$, $a \in |N|$ and for every $N' \in \mathbf{K}$ and $a' \in |N'|$ such that $\mathbf{tp}(a/M, N) = \mathbf{tp}(a'/M, N')$, there exists $f : N \xrightarrow[M]{} N'$ so that $f(a) = a'$.*
- *We say that \mathbf{K} has primes if for any $M \in \mathbf{K}$ and every $p \in \mathbf{S}(M)$, there is a prime triple (a, M, N) such that $p = \mathbf{tp}(a/M, N)$.*

By taking the closure of $M \cup \{a\}$ under the functions of an ambient model, we obtain [Vas17c, 5.3]:

Fact 2.9. *If \mathbf{K} is a universal class then \mathbf{K} has primes.*

The past two facts show that universal classes are tame and have primes. The next facts show that if we have a good frame in addition to that, then the structure of the frame transfers upward and in fact categoricity can be transferred.

We first give an approximation, due to Boney and the second author [BoVa17, 6.9], which assumes amalgamation instead of primes (an earlier result is [Bon14, 1.1], which assumes tameness for types of length two instead of just length one).

Fact 2.10. *Let \mathbf{K} be an AEC and let $\lambda \geq \text{LS}(\mathbf{K})$. If \mathbf{K} is λ -tame, \mathbf{K} has amalgamation and \mathbf{K} has a type-full good λ -frame, then \mathbf{K} has a type-full good $[\lambda, \infty)$ -frame.*

The second author showed that one could replace amalgamation by primes (in fact a weak version of amalgamation suffices) [Vas17c, 4.16]:

Fact 2.11. *Let \mathbf{K} be an AEC and let $\lambda \geq \text{LS}(\mathbf{K})$. If \mathbf{K} is λ -tame, has primes, and \mathbf{K} has a type-full good λ -frame, then $\mathbf{K}_{\geq \lambda}$ has amalgamation. Hence a type-full good $[\lambda, \infty)$ -frame by Fact 2.10.*

Finally, the second author used Fact 2.11 together with the orthogonality calculus of good frames to prove the following categoricity transfer [Vas17b, 2.8]:

Fact 2.12. *Let \mathbf{K} be an AEC and let $\lambda \geq \text{LS}(\mathbf{K})$. Assume that \mathbf{K} is λ -tame, has primes, is categorical in λ , and \mathbf{K} has a type-full good λ -frame. If \mathbf{K} is categorical in some $\mu > \lambda$, then \mathbf{K} is categorical in all $\mu' > \lambda$.*

To get the good frame, we will use the following result from the study of AECs axiomatized by $\mathbb{L}_{\omega_1, \omega}$. It is due to Shelah and already present in some form in [Sh48, Sh87a] (see also [Sh:h, II.3.4]), but we cite from other sources and sketch some details here for the convenience of the reader.

Fact 2.13. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be an $\mathbb{L}_{\omega_1, \omega}$ -sentence. If $1 \leq \mathbb{I}(\psi, \aleph_1) < 2^{\aleph_1}$, then there exists an AEC \mathbf{K} such that:*

- (1) $\tau(\mathbf{K}) = \tau(\psi)$.
- (2) Any model in \mathbf{K} satisfies ψ .
- (3) For $M, N \in \mathbf{K}$, $M \leq_{\mathbf{K}} N$ if and only if $M \preceq_{\mathbb{L}_{\infty, \omega}} N$.
- (4) \mathbf{K} is categorical in \aleph_0 .
- (5) \mathbf{K} has a type-full good \aleph_0 -frame.

One key of the proof is the following classical consequence of Keisler's omitting type theorem [Kei70, 5.10].

Fact 2.14. *Let ψ be an $\mathbb{L}_{\omega_1, \omega}$ -sentence and L^* be a countable fragment of $\mathbb{L}_{\omega_1, \omega}$. If there is a model M of ψ realizing uncountably-many L^* -types over the empty set, then $\mathbb{I}(\psi, \aleph_1) = 2^{\aleph_1}$.*

Another crucial result of Shelah will be used to obtain amalgamation from few models. See [Sh:h, I.3.8] or [Gro02, 4.3] for a proof.

Fact 2.15. *Assume $2^\lambda < 2^{\lambda^+}$. Let \mathbf{K} be an AEC. If \mathbf{K} is categorical λ and $\mathbb{I}(\mathbf{K}, \lambda^+) < 2^{\lambda^+}$ then \mathbf{K} has amalgamation in λ .*

Yet another result of Shelah will be used:

Fact 2.16 ([Sh:h, IV.1.12]). *Let \mathbf{K} be an AEC, let $\lambda \geq \text{LS}(\mathbf{K})$, and let μ be an infinite cardinal. If \mathbf{K} is categorical in λ and $\lambda = \lambda^{< \mu}$, then for any $M, N \in \mathbf{K}_{\geq \lambda}$, $M \leq_{\mathbf{K}} N$ implies $M \preceq_{\mathbb{L}_{\infty, \mu}} N$.*

Finally, we will also use [ShVas, 5.8] (we could have avoided using it at the cost of quoting more):

Fact 2.17. *If \mathbf{K} is categorical in \aleph_0 , has amalgamation and no maximal models in \aleph_0 , is $(< \aleph_0, \aleph_0)$ -tame and is stable in \aleph_0 , then \mathbf{K} has a type-full good \aleph_0 -frame.*

Proof sketch for Fact 2.13. By [Ba09, 6.3.2], there is a complete $\mathbb{L}_{\omega_1, \omega}$ sentence ψ_0 that implies ψ and has a model of cardinality \aleph_1 . Let L^* be a countable fragment containing ψ_0 and let $\mathbf{K} := (\text{Mod}(\psi), \preceq_{L^*})$.

Note that \mathbf{K} is an AEC with $\text{LS}(\mathbf{K}) = \aleph_0$, which by completeness of ψ_0 is categorical in \aleph_0 . Hence it has joint embedding in \aleph_0 . Since it has a model of cardinality \aleph_1 by assumption, \mathbf{K} also has no maximal models in \aleph_0 . Moreover, \mathbf{K} has amalgamation in \aleph_0 by Fact 2.15. Finally, by Fact 2.16 with $\lambda = \mu = \aleph_0$, $M \leq_{\mathbf{K}} N$ if and only if $M \preceq_{\mathbb{L}_{\infty, \omega}} N$.

It remains to show that \mathbf{K} has a type-full good \aleph_0 -frame. We first show:

Claim: Let $M \in \mathbf{K}_{\aleph_0}$. If $\langle p_i : i < \omega_1 \rangle$ are orbital types over M , then there exists $i < j < \omega_1$ such that $p_i \upharpoonright A = p_j \upharpoonright A$ for all finite $A \subseteq |M|$.

Proof of Claim: By amalgamation in \aleph_0 , we can find an uncountable model N extending M such that all the p_i 's are realized inside N . Say $p_i = \text{tp}(\mathbf{a}_i/M; N)$. For $A \subseteq |M|$, let τ_A denote $\tau(\mathbf{K}) \cup \{c_a \mid a \in A\}$, where the c_a 's are new constant symbols. Whenever $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$, let N'_A denote the expansion of N' to τ_A with $c_a^{N'} = a$. Observe that whenever $M \leq_{\mathbf{K}} N' \leq_{\mathbf{K}} N$ and $A \subseteq |M|$ is finite, then, since $\leq_{\mathbf{K}} = \preceq_{\mathbb{L}_{\infty, \omega}}$, we have that $M_A \preceq_{\mathbb{L}_{\infty, \omega}(\tau_A)} N'_A \preceq_{\mathbb{L}_{\infty, \omega}(\tau_A)} N_A$.

Let L^{**} be a countable fragment of $\mathbb{L}_{\omega_1, \omega}$ extending L^* and containing Scott sentences for M_A for all $A \subseteq |M|$ finite. We now apply Fact 2.14 to the following sentence:

$$\bigwedge_{n \in \omega} \{ \phi(c_{a_0}, \dots, c_{a_{n-1}}) \mid \phi \in L^{**}, a_0, \dots, a_{n-1} \in |M|, M \models \phi[a_0, \dots, a_{n-1}] \}$$

Note that the models of this sentence are essentially the extensions of M . Moreover $2^{\aleph_0} < 2^{\aleph_1}$ implies that the sentence still has few models in \aleph_1 . Thus Fact 2.14 indeed applies and we get in particular that there must exist $i < j$ such that $\text{tp}_{L^{**}}(\mathbf{a}_i/\emptyset; N_{|M|}) = \text{tp}_{L^{**}}(\mathbf{a}_j/\emptyset; N_{|M|})$. Now fix $N' \leq_{\mathbf{K}} N$ countable containing M and $\mathbf{a}_i, \mathbf{a}_j$. Also fix $A \subseteq |M|$ finite. Since $M_A \preceq_{\mathbb{L}_{\infty, \omega}(\tau_A)} N'_A$, there exists an isomorphism $f : N' \cong_A M$. Let $\mathbf{b}_i := f(\mathbf{a}_i)$, $\mathbf{b}_j := f(\mathbf{a}_j)$. By equality of the types $(N'_A, \{c_{b_k}^{N'} = a_k^i\}_{k < n}) \equiv_{L^{**} \upharpoonright \tau_{A\mathbf{b}_i}} (N'_A, \{c_{b_k}^{N'} = a_k^j\}_{k < n})$, hence $(M_A, \{c_{b_k}^{M_A} = b_k^i\}_{k < n}) \equiv_{L^{**} \upharpoonright \tau_{A\mathbf{b}_i}} (M_A, \{c_{b_k}^{M_A} = b_k^j\}_{k < n})$. Since L^{**} includes all the relevant Scott sentences, this means that there exists an automorphism g of M sending \mathbf{b}_i to \mathbf{b}_j and fixing A . Composing maps, we obtain an automorphism of N' fixing A and sending \mathbf{a}_i to \mathbf{a}_j . Thus $p_i \upharpoonright A = p_j \upharpoonright A$, as desired. \uparrow_{Claim}

Combining the Claim with [HK06, 3.12], we get that \mathbf{K} is stable in \aleph_0 and is $(< \aleph_0, \aleph_0)$ -tame for types of finite length. Therefore by Fact 2.17, \mathbf{K} has a type-full good \aleph_0 -frame. \square

3. MAIN RESULTS

In this section we prove the main theorems of this paper. We start by applying Fact 2.16 to a universal class categorical in \aleph_0 :

Lemma 3.1. *Let \mathbf{K} be a universal class in a countable vocabulary. If \mathbf{K} is categorical in \aleph_0 , then for $M, N \in \mathbf{K}$, $M \subseteq N$ if and only if $M \preceq_{\mathbb{L}_{\infty, \omega}} N$. Moreover, \mathbf{K} is the class of models of an $\mathbb{L}_{\omega_1, \omega}$ -sentence.*

Proof. Use Fact 2.16 with $\lambda = \mu = \aleph_0$ and recall that $\leq_{\mathbf{K}}$ is just the substructure relation. For the moreover part, take the Scott sentence of a countable model. \square

Applying Fact 2.13, we directly get:

Corollary 3.2. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let \mathbf{K} be a universal class in a countable vocabulary. If \mathbf{K} is categorical in \aleph_0 and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$, then \mathbf{K} has a type-full good \aleph_0 -frame.*

Proof. By Lemma 3.1, \mathbf{K} is axiomatized by an $\mathbb{L}_{\omega_1, \omega}$ sentence and the ordering on \mathbf{K} coincides with $\preceq_{\mathbb{L}_{\infty, \omega}}$. Since \mathbf{K} is already categorical, \mathbf{K} is equal to the class given by Fact 2.13, so \mathbf{K} has a type-full good \aleph_0 -frame. \square

We obtain one of our main theorem:

Theorem 3.3. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let \mathbf{K} be a universal class in a countable vocabulary. If \mathbf{K} is categorical in \aleph_0 and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$, then:*

- (1) \mathbf{K} has arbitrarily large models.
- (2) If \mathbf{K} is categorical in some uncountable cardinal then \mathbf{K} is categorical in all uncountable cardinals.

Proof. By Corollary 3.2, \mathbf{K} has a type-full good \aleph_0 -frame. By facts 2.7 and 2.9, \mathbf{K} is \aleph_0 -tame and has primes. Therefore Fact 2.11 yields (1) and Fact 2.12 yields (2). \square

Observe that the only place where we use the hypotheses “ $2^{\aleph_0} < 2^{\aleph_1}$ and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$ ” is to derive amalgamation and stability. Thus the conclusion of Theorem 3.3 also holds in ZFC if we assume that \mathbf{K} is universal, \aleph_0 -categorical, has amalgamation and no maximal models in \aleph_0 , and is stable in \aleph_0 (using Fact 2.17 to get the good frame).

We can also replace the assumption of categoricity in \aleph_0 by categoricity in \aleph_1 . To see this, we will use the following local version of Facts 2.10, 2.11, 2.12.

Fact 3.4. *Let \mathbf{K} be an AEC and $\lambda \geq \text{LS}(\mathbf{K})$. If \mathbf{K} has a type-full good λ -frame, is categorical in λ and λ^+ , is (λ, λ^+) -tame, and \mathbf{K}_{λ^+} has primes, then \mathbf{K} has a type-full good λ^+ -frame and is categorical in λ^{++} .*

Proof. The proof of Fact 2.11 is local, so \mathbf{K} has a good λ^+ -frame. That \mathbf{K} is λ^{++} -categorical follows from [Vas17a, 6.14]. \square

Theorem 3.5. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let ψ be a universal $\mathbb{L}_{\omega_1, \omega}$ sentence. If ψ is categorical in \aleph_1 , then it is categorical in all uncountable cardinals.*

Proof. Let \mathbf{K} be the class of models of ψ . Let \mathbf{K}^* be the class obtained in Fact 2.13. Note that since $\mathbf{K}_{\aleph_1}^* \neq \emptyset$ (by condition (5) of the definition of good frame), $\mathbf{K}^* \subseteq \mathbf{K}$ and \mathbf{K} is categorical in \aleph_1 , \mathbf{K}^* is also categorical in \aleph_1 . Moreover by Fact 2.16 with $\lambda = \aleph_1$ and $\mu = \aleph_0$ for $M, N \in \mathbf{K}_{\aleph_1}^*$, $M \subseteq N$ if and only if $M \preceq_{\mathbb{L}_{\infty, \omega}} N$. Hence $\mathbf{K}_{\aleph_1}^* = \mathbf{K}_{\aleph_1}$. Since the behavior of an AEC is determined by its behavior in the Löwenheim-Skolem-Tarski number, $\mathbf{K}_{\geq \aleph_1}^* = \mathbf{K}_{\geq \aleph_1}$.

Now \mathbf{K}^* has a type-full good \aleph_0 -frame and since \mathbf{K} is a universal class, \mathbf{K} is (\aleph_0, \aleph_1) -tame. Since $\mathbf{K}_{\geq \aleph_1}^* = \mathbf{K}_{\geq \aleph_1}$, one can check that \mathbf{K}^* is also (\aleph_0, \aleph_1) -tame. Furthermore, since \mathbf{K}_{\aleph_1} has primes and $\mathbf{K}_{\aleph_1}^* = \mathbf{K}_{\aleph_1}$, $\mathbf{K}_{\aleph_1}^*$ also has primes. By Fact 3.4, \mathbf{K}^* has a type-full good \aleph_1 -frame and is categorical in \aleph_2 . But this means that $\mathbf{K}_{\geq \aleph_1}$ has a type-full good \aleph_1 -frame and is categorical in \aleph_2 , so we can now apply Fact 2.12, to $\mathbf{K}_{\geq \aleph_1}$ to get the result. \square

4. OPEN QUESTIONS AND GENERALIZATIONS

The following variation on an example of Morley shows that for every countable ordinal α there are universal classes with models only up to size \beth_α .

Example 4.1. Fix $\alpha < \omega_1$. Let τ be a vocabulary consisting of unary predicates $\langle P_i : i \leq \alpha \rangle$, a binary relation E and a binary function f . Let K be the class of τ -structures M such that:

- (1) $P_i^M \subseteq P_j^M$ for all $i < j < \alpha$.
- (2) $P_0^M = \emptyset$.
- (3) $|M| = P_\alpha^M$.
- (4) $P_i^M = \bigcup_{j < i} P_j^M$ for i limit.
- (5) $x E^M y$ implies $x \in P_i^M$ and $y \in P_j^M$ for some $i < j < \alpha$.
- (6) For any $i < \alpha$ and any two distinct $y_1, y_2 \in P_i^M$, $x := f(y_1, y_2)$ satisfies:

$$(x E y_1 \wedge \neg(x E y_2)) \vee (\neg(x E y_1) \wedge x E y_2)$$

Then K is a universal class in a countable vocabulary with amalgamation, joint embedding, and a model of cardinality $\beth_\alpha(0)$ but no models of cardinality $\beth_\alpha(0)^+$.² Taking the disjoint union of K with the class of \mathbb{Q} -vector spaces, we obtain (when $\alpha \geq \omega$) a universal class in a countable vocabulary which is categorical in an infinite cardinal λ exactly when $\lambda > \beth_\alpha(0)$.

This shows that some conditions on the class are necessary to derive arbitrarily large models. However it is not clear to us that Theorem 3.3 is optimal. Indeed it is not clear to us that the hypotheses on \aleph_1 are necessary (see Baldwin-Lachlan [BL73] for a positive result when K is axiomatized by a Horn theory):

Question 4.2. If K is a universal class categorical in \aleph_0 with a model in \aleph_1 , must it be categorical in \aleph_1 ?

It would also be really nice to have a proof of Theorem 3.3 in *ZFC*, so it is natural to ask the following question.

Question 4.3. Can we drop the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ from Theorem 3.3? Can it be dropped if we add more categoricity assumptions?

Shelah [Sh:h, §I.6] has given an example of an analytic AEC which under Martin's axiom is categorical in \aleph_0 and \aleph_1 yet does not have amalgamation in \aleph_0 . It seems however plausible that there are no such examples which are universal classes.

²For a (possibly finite) cardinal μ and an ordinal α , $\beth_\alpha(\mu)$ is defined inductively by $\beth_0(\mu) = \mu$, $\beth_{\beta+1}(\mu) = 2^{\beth_\beta(\mu)}$, and $\beth_\delta(\mu) = \sup_{\beta < \delta} \beth_\beta(\mu)$ for δ limit.

We end this paper with a generalization of Theorem 3.3. The key is that we have not used the full strength of the universal assumption: all we used was tameness, having primes, and some definability. Using harder results of Shelah, Theorems 3.3 and 3.5 generalize to:

Theorem 4.4. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let \mathbf{K} be an AEC with $\text{LS}(\mathbf{K}) = \aleph_0$. Assume that \mathbf{K} has primes, is \aleph_0 -tame, and is PC_{\aleph_0} (see [Sh:h, I.1.4]).*

- (1) *If \mathbf{K} is categorical in \aleph_0 and $1 \leq \mathbb{I}(\mathbf{K}, \aleph_1) < 2^{\aleph_1}$, then \mathbf{K} has arbitrarily large models and categoricity in some uncountable cardinal implies categoricity in all uncountable cardinals.*
- (2) *If \mathbf{K} is categorical in \aleph_1 , then \mathbf{K} is categorical in all uncountable cardinals.*

Proof. As in the proof of Theorems 3.3, 3.5 but using [Sh:h, I.3.10] to derive an \aleph_0 -categorical subclass and [Sh:h, II.3.4] to derive the good \aleph_0 -frame (actually in this case we only obtain a semi-good \aleph_0 -frame with conjugation, see [JrSh875, 2.3.10], but this suffices for the proof). \square

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