1) Let $\Gamma \subset G_Z$ be a subgroup of finite index. Following the outline given in class, carefully state and prove the existence of a compactification of $\Gamma \backslash H$, obtained by adding one point at each cusp. Show that this compactification has a natural structure of Riemann surface.

2) Recall the notion of an equivariant line bundle $\mathcal{L} \to X$ – equivariant with respect to the action of a group $G$ that acts on the Riemann surface $X$: an action of $G$, by holomorphic maps, on the total space of $\mathcal{L}$, lying over the action of $G$ on $X$ (in the sense that the projection $p : \mathcal{L} \to X$ relates the two actions), such that for each $g \in G$ and $x \in X$, the map $\mathcal{L}_x \to \mathcal{L}_{gx}$ induced by $g$ is linear. Since this linear map has an inverse of the same type, it is necessarily a linear isomorphism. Recall also that the line bundles $\mathcal{L}_k \to \mathbb{C}P^1$ are $G_{\mathbb{C}}$-equivariant. In particular, the restricted line bundle $\mathcal{L}_k \to H$ is $G_R$-equivariant, and therefore equivariant with respect to the action of any subgroup $\Gamma \subset G_R$.

a) Let $\Gamma \subset G_R$ be a discrete subgroup which acts on $H$ without fixed points in the geometric sense – i.e., $\Gamma$ may contain $-1$, but if so, $\Gamma / \{\pm 1\}$ acts without fixed points. If $k$ is odd, suppose $-1 \notin \Gamma$. Show that $\mathcal{L}_k$ “drops” to $\Gamma \backslash H$. In other words, there exists a holomorphic line bundle over $\Gamma \backslash H$, whose space of holomorphic sections over any open subset $U \subset \Gamma \backslash H$ is naturally isomorphic to the space of holomorphic sections of $\mathcal{L}_k$ over the inverse image $\tilde{U} \subset H$ of $U$. Naturality means in particular that the isomorphism is compatible with respect to restriction to open subsets. With a slight abuse of notation, we denote that line bundle on $\Gamma \backslash H$ by the same symbol $\mathcal{L}_k$.

b) In addition to the assumptions made in a), suppose that $\Gamma \subset G_Z$ is a subgroup of finite index. Show that the line bundle $\mathcal{L}_k \to \Gamma \backslash H$ constructed in a) extends naturally to the compactification of $\Gamma \backslash H$. Caution: a holomorphic line bundle $\mathcal{L} \to \Delta^*$ over the punctured disk $\Delta^*$ can be extended across the puncture in more than one way, since the line bundle $\mathcal{L}_{\{0\}} \to \Delta$ corresponding to the divisor $\{0\}$ is trivial over $\Delta^*$ (here $\Delta$ and $\Delta^*$ should be thought of as open subsets of an ambient compact Riemann surface; all holomorphic line bundles over the disk itself are trivial). One way to get a “distinguished” extension across the puncture is to have a “distinguished” section over $\Delta^*$ without zeroes; if that section can be continued to a non-zero section of the extension of $\mathcal{L}$ across the puncture, the extension becomes uniquely determined.