1a) Let $X$ be a Riemann surface. Using the original definition of holomorphic line bundle given in class (not the description in terms of 1-cocycles), give a formal construction of the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ of two line bundles $\mathcal{L}_1$, $\mathcal{L}_2$, and of the dual $\mathcal{L}^*$ of a line bundle $\mathcal{L}$.

b) Show that the operations of tensor product and duality of line bundles turn the set of line bundles, modulo isomorphism, into an abelian group.

c) Construct a natural (i.e., independent of arbitrary choices) homomorphism

$$\mathcal{O}_X(\mathcal{L}_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(\mathcal{L}_2) \rightarrow \mathcal{O}_X(\mathcal{L}_1 \otimes \mathcal{L}_2).$$

For simplicity of notation, let $\sigma_1 \otimes \sigma_2 \in \mathcal{O}_X(\mathcal{L}_1 \otimes \mathcal{L}_2)$ denote the image under this homomorphism of the product of $\sigma_1 \in \mathcal{O}(\mathcal{L}_1)$ and $\sigma_2 \in \mathcal{O}(\mathcal{L}_2)$ – though formally ambiguous, this convention is commonly used. Define of the divisor $D(\sigma)$ of a holomorphic section $\sigma$ of a line bundle, consistently with the definition of the divisor $D(f)$ of a holomorphic function $f$ (assuming, of course, that neither $\sigma$ nor $f$ vanishes identically on any open subset of $X$). Show that $D(f \sigma) = D(f) + D(\sigma)$ and $D(\sigma_1 \otimes \sigma_2) = D(\sigma_1) + D(\sigma_2)$.

2) Let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of the Riemann surface $X$. Following the outline given in class, construct a homomorphism of abelian groups

$$H^1(\{U_\alpha\}, \mathcal{O}^*) \rightarrow \{\text{holomorphic line bundles}\}/\text{isomorphism}$$

which inverts the passage from holomorphic line bundles that are trivial on each $U_\alpha$ to cohomology classes in $H^1(\{U_\alpha\}, \mathcal{O}^*)$. 