# ON THE NON-ENUMERABILITY OF *L*

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In Fond Remembrance of the Salad Days
of $\alpha$-Recursion Theory 1970–1976

**Abstract.** Assume $V \neq L$. Then $L$ is not $E$-Recursively Enumerable in any member of $L$ via a forcing argument inspired by Friedberg [1]. This approach leads to an extension of Levy-Shoenfield absoluteness.

**§1. Introduction.** The notation $\{e\}(x)$ was introduced by Kleene; $x$ was a non-negative integer intended as input for a finitary effective procedure coded by $e$. Sometime later Kleene [3] redefined the notion of effective procedure to allow $x$ to be an object of finite type and computations to be infinite wellfounded trees. Finally Normann [6] and Moschovakis [5] elevated $x$ to the status of an arbitrary set. (A curious reader might consult [8].)

A partial function $F : V \to V$ is *partial* ***E*****-*recursive*** iff $\exists e \forall x$

$$F(x) \text{ is defined and equal to } y \text{ iff } \{e\}(x) \text{ converges to } y.$$

The meaning of convergence: applying $\{e\}$ to $x$ results in a wellfounded computation tree in $Ad_1(\boldsymbol{x})$, the least $\Sigma_1$ admissible set $\supseteq tc\{x\}$, the transitive closure of $\{x\}$. Notation: read $\{e\}(x)\downarrow$ as $\{e\}(x)$ converges, and $\{e\}(x)\uparrow$ as $\{e\}(x)$ diverges.

A class **A** *is* ***E*****-*Recursively Enumerable in*** ***b*** iff

$$\exists e \ \ A = \{x \mid \{e\}(x, b)\downarrow\}.$$

$A$ is $E$-Recursively Enumerable iff $\exists e \ A = \{x \mid \{e\}(x)\downarrow\}$;
$d$ is $E$-recursive in $c$ iff $\exists e \ d = \{e\}(c)$.

THEOREM 1.1. *Assume $V \neq L$. Then $L$ is not $E$-recusively enumerable in any member.of $L$.*

The standard Levy-Shoenfield absoluteness arguments are enough to prove Theorem 1 if the "member of $L$" belongs to $L(\omega_1^L)$. Otherwise something more is needed. In this paper "more" means chains of elementary substructures and forcing.

There is a conceptual gap between "constructible" and "$E$-recursively enumerable." A set $x$ is constructible iff there exists an ordinal $\gamma$ such that $x$ is a first order definable subset of $L(\gamma)$; $c$ is a member of $\{x \mid \{e\}(x)\downarrow\}$ iff there exists a

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wellfounded computation tree $E$-recursive in $c$ that witnesses the convergence of $\{e\}(c)$. In short "constructible" is less constructive than "$E$-recursively enumerable," much much less constructive. It is plausible that "$E$-recursively enumerable" is at the bottom of a hierarchy of definability with "constructibility" at the top.

§**2. Preferred nonconstructible sets.** Suppose $\lambda$ is a limit, $x \subseteq L(\lambda)$ and $x \notin L(\lambda)$. Call $x$ *amenable* iff

$$(\forall y \in L(\lambda))\ (x \cap y) \in L(\lambda).$$

It follows that $(\forall y \in L(\lambda))\ (x \nsubseteq y)$; i.e., $x$ is not a *bounded subset* of $L(\lambda)$.

The notion of amenability plays a leading role in the play of ideas below.

Suppose $V \neq L$.[1] Let $\tau$ be the least $\lambda$ such that some subset of $\lambda$ is not in $L$. Then every unbounded subset of $\tau$ is amenable, because $\tau$ is an $L$-cardinal. Some proof: For $d \in L$, let $card^L(d)$ denote the cardinality of $d$ in $L$. Suppose $\tau > card^L(\tau)$ and some $z \subseteq \tau$ is not in $L$. Let $f \in L$ inject $z$ into $card^L(\tau)$; then $f[z] \notin L$, a contradiction.

Lemma 2.1. $L \models [\tau$ *is a* regular *cardinal*$]$.[2]

Proof. Suppose $L$ thinks $\tau$ is a singular cardinal of cofinality $\rho$. In $L$ there is an ascending sequence of cardinals $\{\tau_\beta \mid \beta < \rho\}$ such that

$$\tau = sup\{\tau_\beta \mid \beta < \rho\}.$$

Every cardinal of $L$ below $\tau$ is a cardinal of $V$. Choose a nonconstructible $Z \subseteq \tau$. For each $\beta < \rho$, define $c(\beta)$ to be the least ordinal that encodes $Z \cap \tau_\beta$ as a constructible set. Note that $c(\beta) \in \tau_{\beta+1}$; $range(c)$ is a singular sequence thorough $\tau$ of length $\rho$; $range(c) \notin L$ because $Z$ is constructible from $c$.

Assume $\tau \neq \omega$. Then $O^\sharp$ does not exist. Jensen's covering theorem supplies a $K \in L$ such that $range(c) \subseteq K$ and $card^V(K) \leq \rho + \omega_1 < \tau$. Hence $card^V(K) = card^L(K)$. Let $w \in L$ map $K$ $1-1$ onto $card^L(K)$. Then $w[range(c)] \subseteq \beta$ for some $\beta < \tau$; but $w[range(c)] \notin L$. A contradiction. ⊣

§**3. The forcing setup.** Choose a cardinal $\kappa$ of $L$ such that $b, \tau \in \kappa$; $b$ will be the "member of $L$" alluded to in Theorem 1. Let $\kappa^+$ denote the $L-$cardinal succeeding $\kappa$.

The $L$-cardinal $\tau$ was defined at the beginning of Section 2. A $\Sigma_1$ admissible $L(\alpha)$ with the following properties is needed to study generic solutions of $\{e\}(x, b)\downarrow$.

(**F1**) There is an ordinal, call it $gc(\alpha)$, such that
$$L(\alpha) \models [gc(\alpha) \text{ is the greatest cardinal}].$$
(**F2**) $\kappa < gc(\alpha) < \alpha < \kappa^+$.
(**F3**) $L(\alpha) \models [gc(\alpha)$ is regular$]$.
(**F4**) $L \models [cofinality(gc(\alpha)) = cofinality(\tau)]$.
(**F5**) There is a $\Sigma_1^\alpha$ injection $i{:}\alpha \to gc(\alpha)$.

[1]This paper includes brief proofs of several elementary $\alpha$-recursion-theoretic facts for the sake of new readers.

[2]The referee supplied the much needed clincher in the proof of Lemma 1, namely the use of $O^\sharp$. The lemma is not needed in this paper, but may be handy in a sequel.

**Construction of $L(\alpha)$.** *The construction takes place inside L.* For each set $d$, $S_1(d)$ is the standard least $\Sigma_1$ substructure of $L$ with $d$ as a subset; each $\Sigma_1^{ZF}$ predicate $P(x)$ with parameters in $d$ and a solution in $L$ has one in $S_1(d)$. A chain of $\Sigma_1$ substructures of $L$ is defined by recursion on $H_{cf(\tau)}$ (cf is cofinality in $L$).

$$H_0 = S_1(\kappa \cup \{\kappa, \kappa^+\}).$$
$$H_{\delta+1} = S_1(H_\delta \cup \{H_\delta\}).$$
$$H_\gamma = \cup\{H_\delta \mid \delta < \gamma\}.(\gamma \textit{ is a limit.})$$

Let $\beta \in H_{cf(\tau)}$ be the least $\Sigma_1$ admissible ordinal $> \kappa^+$. Recall

$$t : H_{cf(\tau)} \to t[H_{cf(\tau)}],$$

the map that collapses an extensional set to a transitive set. Define $\alpha = t(\beta)$. $L(\alpha) \models [t(\kappa^+)$ is regular].

*Define* $gc(\alpha) = t(\kappa^+)$; $L(\alpha) \models [gc(\alpha)$ is the greatest cardinal]. *Define* $\boldsymbol{\rho} = cf(gc(\alpha))$ (in $L$); $\rho \leq \kappa$, so $\rho \in H_0$.

There is a *preferred sequence* $\boldsymbol{k}$ *of length* $\boldsymbol{\rho}$ cofinal in $gc(\alpha)$. For each $\delta < \rho$, $\boldsymbol{k(\delta)} = sup(H_\delta \cap \kappa^+)$.

Remark 3.1. $H_\delta \cap \kappa^+$ is an ordinal because $\kappa \cup \{\kappa\} \subseteq H_\delta$, and if $\sigma \in H_\delta \cap \kappa^+$, then an injection of $\sigma$ into $\kappa$ also belongs to $H_\delta$; $k(\delta) \subseteq H_\delta \cap \kappa^+$ so $t(k(\delta)) = k(\delta)$.

Proposition 3.2. *The range of k is an amenable subset of $gc(\alpha)$. Every member of the range of k is a limit ordinal; k is strictly increasing.*

Proof. Fix $\delta < \rho$. Then $\{\kappa(\delta') \mid \delta' < \delta\}\} \in H_\delta \cap \kappa^+$. ⊣

Proof of (F5). Keep in mind that $\alpha$ is the least $\Sigma_1$ admissible $> gc(\alpha)$. From $J$, the least $\Sigma_1$ substructure of $L(\alpha)$ that contains $gc(\alpha)+1$; $J$ is a $\Sigma_1$ admissible, initial segment of $L(\alpha)$ beyond $gc(\alpha)$, hence equal to $L(\alpha)$; $J$ is the range of a partial $\Sigma_1^\alpha$ map $j$ applied to $gc(\alpha)$; $i(\sigma)$ is the $L-$least $\upsilon$ such that $j(\upsilon) = \sigma$, if there is such a $\upsilon$.

Traditional set-forcing preserves $\Sigma_1$ admissibility; i.e., the null forcing condition forces $\Sigma_1$ admissibility. But if $\alpha$ is uncountable, a generic subset of $gc(\alpha)$ may not exist. Clause(F3) helps solve that problem in Section 4. Clause (F4) leads to a generic outside $L$. ⊣

Proposition 3.3. *Suppose $\mathcal{Q}(x)$ is a $\Sigma_1^\alpha$ predicate. Then* $\forall\eta < gc(\alpha)$, $\{x \mid \mathcal{Q}(x) \wedge x < \eta\} \in L(gc(\alpha))$.

Proof. Define $D = \eta \cup \{\eta\} \cup \{\text{parameters of } \mathcal{Q}(x)\}$. Let $H$ be the $\Sigma_1$ hull in $L(\alpha)$ containing $D$. There is a $D_1 \supseteq D$ such that $L(\alpha) - card(D_1) = L(\alpha) - card(D) < L(gc(\alpha))$, and a $\Sigma_1^\alpha$ injection of $H$ into $D_1$. Let $J$ be the transitive collapse of $H$; thus $J = t[H]$. There is a $\Sigma_1^\alpha$ injection of $J$ into $D_1$; $J \models (V = L)$. Hence $J = L(\nu)$ for some $\nu < gc(\alpha)$. Fix $x < \eta$. $Q_t(x)$ is the result of replacing each parameter $\theta$ of $\mathcal{Q}(x)$ by its collapse $t(\theta)$. Then $L(\alpha) \models \mathcal{Q}(x)$ iff $H \models \mathcal{Q}(x)$ iff $J \models Q_t(t(x))$. Fortunately $t(x) = x$. ⊣

Proof of (F4). $H_\delta \cap \kappa^+$ is an initial segment of $\kappa^+$ for all $\delta \leq cf(\tau)$; $H_{cf(\tau)} \cap \kappa^+$ is built up in $cf(\tau)$ many distinct steps. It follows that $cf(gc(\alpha)) = cf(\tau)$ in $L$. ⊣

A standard $\Sigma_1$ hull is the range of a partial $\Sigma_1$ function, hence the range of a $\Sigma_2$ function. It follows that there exists a $\Sigma_2^\alpha$ sequence of length $\rho$ through $gc(\alpha)$;

$\rho$ is said to be the $\Sigma_2^\alpha$ cofinality of $gc(\alpha)$; any $\Sigma_2^\alpha$ function with domain $\delta < gc(\alpha)$ and range $\subseteq gc(\alpha)$ is bounded below $gc(\alpha)$. A key observation of Shore [9] shows that the $\Sigma_2^\alpha$ cofinality of $gc(\alpha)$ equals the $\Sigma_2^\alpha$ cofinality of $\alpha$, and thereby enables his "blocking" method, which empowers $\Sigma_1$ admissibility to do some $\Sigma_2$ recursions.

*Forcing conditions* are denoted by $p, q, r, \ldots$; $p$ is a function from some $\beta < gc(\alpha)$ into $\{0, 1\}$; $p$ is *extended* by $q$ $(p \geq q)$ iff $q \supseteq p$. If $\gamma < gc(\alpha)$ and $\{p_\delta \mid \delta < \gamma\}$ is an increasing sequence in $L(\alpha)$ of conditions, then $\cup\{p_\delta \mid \delta < \gamma\}$ is a condition, thanks to the regularity of $gc(\alpha)$ in $L(\alpha)$;

$$G \text{ denotes a } \textit{function } \lambda\delta \mid p_\delta;$$

the domain of $G$ is $gc(\alpha)$ and its values are forcing conditions; in addition, if $\delta_1 < \delta_2$, then $p_{\delta_1} \geq p_{\delta_2}$.

Definition 3.4. *$G$ satisfies $p$* iff $p \geq G(\delta)$ for some $\delta < gc(\alpha)$.

The language $\mathcal{L}(\alpha, \mathcal{G})$ mirrors $L(\alpha, G)$ for an arbitrary $G$; terms $d, e, \ldots$ name the elements of $L(\alpha, G)$. A *ranked formula* of $\mathcal{L}(\alpha, \mathcal{G})$ has no unbounded quantifiers. A $\Sigma_1$ formula is either ranked or of the form $\exists x\mathcal{F}$ for some ranked $\mathcal{F}$.

Remark 3.5. The forcing relation, $p \Vdash \mathcal{F}$, restricted to $\mathcal{F} \in \Sigma_1^\alpha$, is $\Sigma_1^\alpha$.

Proposition 3.6. *Suppose $\mathcal{F}(u, v)$ is ranked and $p \Vdash \forall_{u\in e}\exists v\mathcal{F}(u, v)$. Then $p \Vdash \exists w\forall_{u\in e}\exists_{v\in w}\mathcal{F}(u, v)$.*

Proof. For each $\delta < \alpha$, let $v^\delta$ be a variable that ranges over $L(\delta, G)$.

$$\forall u \in e \;\; \forall q \leq p \;\; \exists r \leq q \;\; \exists\delta \;\; [r \models \exists v^\delta \mathcal{F}(u, v^\delta)].$$

For $u \in e$ and $q \leq p$, define $k(u, q)$ to be the $L$-least $< r, \delta >$ such that

$$r \leq q \;\wedge\; r \models \exists v^\delta \mathcal{F}(u, v^\delta).$$

The domain of $k$ is a member of $L(\alpha)$ and $k$ is $\Sigma_1^\alpha$. Hence $\{\delta \mid \exists r \, \langle r, \delta\rangle \in \mathit{range}(k)\}$ is bounded above by some $\delta_0 < \alpha$, and $p \Vdash \forall_{u\in e}\exists v^{\delta_0}\mathcal{F}(u, v^{\delta_0})$. ⊣

Remark 3.7. If $L(\alpha, G)$ is $\Sigma_1$ admissible, $b \in L(\alpha)$ and $\{e\}(G, b)$ converges, then its convergence is witnessed by a wellfounded computation tree in $L(\alpha, G)$; if $\{e\}(G, b)$ diverges, then some infinite descending path of the computation tree is first order definable over $L(\alpha, G)$. (For more detail see [8], [7].)

$G$ is *sufficiently generic* iff for each sentence $\mathcal{K}$ of form (i) or (ii) or (iii), there is a $p$ such that $[(G \text{ satisfies } p) \wedge (p \Vdash \mathcal{K} \vee p \Vdash \neg\mathcal{K})]$.

(i) $\mathcal{K}$ is ranked; i.e., rank less than $\alpha$;
(ii) $\mathcal{K}$ is $\exists x\mathcal{F}$ and $\mathcal{F}$ is ranked;
(iii) $\mathcal{K}$ is $\forall_{u\in e}\exists v\mathcal{F}(u, v)$.

**Note**: By Proposition 3.6, a search for a $p$ to force $\forall_{u\in e}\exists v\mathcal{F}(u, v)$ is the same as a search for a $p$ to force $\exists w\forall_{u\in e}\exists_{v\in w}\mathcal{F}(u, v)$.

The construction of a sufficiently generic $G$ will be a recursion of length $gc(\alpha)$.

Proposition 3.8. *If $G$ is sufficiently generic, then $L(\alpha, G)$ is $\Sigma_1$ admissible.*

PROOF. Suppose $\mathcal{K}$ is a ranked sentence of rank less than $\alpha$. If $\mathcal{K}$ is forced by some $p \in G$, then $L(\alpha, G) \models \mathcal{K}$. Otherwise $\neg\mathcal{K}$ is forced by some $p \in G$ and $L(\alpha, G) \models \neg\mathcal{K}$.

Suppose $\mathcal{K}$ is $\exists x \mathcal{F}(x)$, $\mathcal{F}(x)$ has rank less than $\alpha$, and $\mathcal{K}$ is forced by some $p \in G$. Then $p \Vdash \mathcal{F}(d)$ for some term $d$, and so $L(\alpha, G) \models \exists x \mathcal{F}(x)$. Suppose $\neg\mathcal{K}$ is forced by some $p \in G$. Thus $p \Vdash \forall x \neg\mathcal{F}(x)$. Then for all $d$, $(\exists q \leq p)\ \ q \in G$ and $q \Vdash \neg\mathcal{F}(d)$, and so $L(\alpha, G) \models \neg\mathcal{K}$.

Suppose $L(\alpha, G) \models \forall_{u \in e} \exists v \mathcal{F}(u, v)$, and $\mathcal{F}(u, v)$ has rank less than $\alpha$. Then $\exists p \in G\ \ p \Vdash \forall_{u \in e} \exists v \mathcal{F}(u, v)$. Hence by Proposition 3.6, $p \Vdash \exists w \forall_{u \in e} \exists_{v \in w} \mathcal{F}(u, v)$. So $L(\alpha, G) \models \exists w \forall_{u \in e} \exists_{v \in w} \mathcal{F}(u, v)$. ⊣

**§4. Forcing non-enumerability.** First the plan, and then the details, for forcing non-enumerability. Suppose in hope of a contradiction that $\exists\, b \in L$ and $e$ such that

$$L = \{x \mid \{e\}(x, b)\!\downarrow\}.$$

Section 3 sets up a suitable $\alpha$ and $gc(\alpha)$ and a notion of genericity over $L(\alpha)$ for subsets of $gc(\alpha)$. Suppose $G$ is sufficiently generic and a member of $L$. Then $\{e\}(G, b)\!\downarrow$ by Remark 3.7. There must be a forcing condition $p$ such that $p \Vdash \{e\}(\mathcal{G}, b)\!\downarrow$, i.e., $p$ forces the existence of a wellfounded computation tree in $L(\alpha, \mathcal{G})$ that witnesses the convergence of $\{e\}(\mathcal{G}, b)$. Let $D$ be an amenable, nonconstructible unbounded subset of $gc(\alpha)$ of length $cf(gc(\alpha))$. Repeat the forcing argument starting with $p$, but this time weave $D$ cofinally into $G$ in order to trick $G$ into being nonconstructible. But $p$ still forces $\{e\}(\mathcal{G}, b)\!\downarrow$, a contradiction.

*A Generic in* ***L***. Let $m \in \Sigma_1^\alpha$ be a 1–1 map of $\alpha$ onto the collection of sentences listed in (i), (ii), (iii) of the definition of sufficiently generic in Section 3; let $\mathcal{F}_\gamma$ denote the value of $m(\gamma)$. Recall $i$, the $\Sigma_1^\alpha$ injection of $\alpha$ into $gc(\alpha)$.

A sequence $p_\delta$ $(\delta < gc(\alpha))$ is defined by a $\Sigma_2^\alpha$ recursion.

CASE 0: $p_0 = \emptyset$.

CASE 1: $\lambda$ is a limit.

Then $p_\lambda = \cup\{p_\delta \mid \delta < \lambda\}$.

CASE 2: definition of $p_{\delta+1}$.

SUBCASE 2a: $\delta \notin range(i)$.

Then $p_{\delta+1} = p_\delta$.

SUBCASE 2b: $\delta \in range(i)$.

If $\exists p\, [p_\delta \geq p \wedge p \Vdash \mathcal{F}_{i^{-1}(\delta)}]$, then $p_{\delta+1}$ is the $L$-least such $p$.
If no such $p$ exists, then $p_{\delta+1} = p_\delta$.

PROPOSITION 4.1. *Suppose $\lambda$ is a limit $< gc(\alpha)$ and $\{p_\delta \mid \delta < \lambda\} \subseteq L(gc(\alpha))$. Then $\cup\{p_\delta \mid \delta < \lambda\} \in L(gc(\alpha))$.*

PROOF. Define $Q_1 = \{\delta \mid \delta < \lambda \wedge \delta \in range(i)\}$.
Define $Q_2 = \{\delta \mid \delta \in Q_1 \wedge \exists p\, [p_\delta \geq p \wedge p \Vdash \mathcal{F}_{i^{-1}(\delta)}]$.
$Q_1$ is $\Sigma_1^\alpha$ because $i$ is $\Sigma_1^\alpha$. ($i$ was introduced in (F5) above.)
$Q_2$ is $\Sigma_1^\alpha$ by Remark 2.
Hence $Q_2 \in L(gc(\alpha))$ by Proposition 3.3.

The sequence $p_\delta$ $(\delta < \lambda)$ defined above
is definable by a $\Sigma_1^\alpha$ (with $Q_2$ as a parameter) recursion.

CASE 0: $p_0 = 0$.

CASE 1: $\delta$ is a limit.

Then $p_\delta = \cup\{p_\gamma \mid \gamma < \delta\}$.

CASE 2: definition of $p_{\delta+1}$.

SUBCASE 2a. If $\delta \in Q_2$, then $p_{\delta+1}$is the $L$-least $p$
such that $p_\delta \geq p \wedge p \Vdash \mathcal{F}_{i^{-1}(\delta)}]$.

SUBCASE 2b. If $\delta \notin Q_2$, then $p_{\delta+1} = p_\delta$.

Hence $\{p_\delta \mid \delta < \lambda\} \in L(gc(\alpha))$,
since $L(\alpha) \models [gc(\alpha)$ is a regular cardinal]. ⊣

REMARK 4.2. Proposition 4.1 says, speaking informally, that the $\Sigma_2^\alpha$ recursion that defined $\{p_\delta \mid \delta < gc(\alpha)\}$ is locally $\Sigma_1^\alpha$.

*A Generic Outside* ***L***. The function $k$ of Remark 3.1 of Section 2 has domain $\rho$ and range an unbounded amenable subset of $gc(\alpha)$; $k \in L$ and *all values of k are limit ordinals* The existence of $\tau$ (Section 2) implies there is a $D$, an unbounded amenable subset of $\rho$ not in $L$. ($c_{D\cap\lambda}$ is the characteristic function of $D \cap \lambda$).)

A sequence $p_\delta$ $(\delta < gc(\alpha))$ is defined by a $\Sigma_2^\alpha$ recursion.

CASE 0: $p_0 = \emptyset$.

CASE 1: $\lambda$ is a limit.

Then $p_\lambda = \cup\{p_\delta \mid \delta < \lambda\}$.

SUBCASE 1.1: $\lambda \in range(k)$.

define $dom_\lambda =$ domain of $\cup\{p_\delta \mid \delta < \lambda\}$;

$p_\lambda = \cup\{p_\delta \mid \delta < \lambda\} \cup \{< dom_\lambda + \delta, c_{D\cap\lambda}(\delta) > \mid \delta < \lambda\}$.

SUBCASE 1.2: $\lambda \notin range(k)$.

Then $p_\lambda = \cup\{p_\delta \mid \delta < \lambda\}$.

CASE 2: definition of $p_{\delta+1}$.

SUBCASE 2.1: $\delta \notin range(i)$. Then $p_{\delta+1} = p_\delta$.

SUBCASE 2.2: $\delta \in range(i)$.

If $\exists p\, [p_\delta \geq p \wedge p \Vdash \mathcal{F}_{i^{-1}(\delta)}]$, then $p_{\delta+1}$ is the $L$-least such $p$.

Otherwise $p_{\delta+1} = p_\delta$.

PROPOSITION 4.3. *Assume* $\{p_\delta \mid \delta < \lambda\}$ *has been defined by the above* $\Sigma_2^\alpha$ *recursion for a generic outside L. Then* $\cup\{p_\delta \mid \delta < \lambda\} \in L(gc(\alpha))$ *if* $\{p_\delta \mid \delta < \lambda\} \subseteq L(gc(\alpha)$,

PROOF. Define $Q_1$ and $Q_2$ as in the proof of Proposition 4.1.

Define $Q_3 = \{\lambda' \mid \lambda'$ is a limit $<\lambda$ & $\lambda' \in range(k)\} \cup \{< 3, D \cap \lambda >\}$.

Then $Q_3 \in L(gc(\alpha))$ by Proposition 3.3 and the amenability of $D$.

Hence $\{p_\delta \mid \delta < \lambda\}$ is definable by a $\Sigma_1^\alpha$ recursion (with parameter $Q_3$) of length $\lambda$. ⊣

**Proof that $G \notin L$.** Recall $k \in L$. For each $\lambda \in range(k)$,

$$\forall \delta < \lambda \;\; c_{D\cap\lambda}(\delta) = p_\lambda(dom_\lambda + \delta).$$

Keep in mind $G$ is a function; $G(\delta) = p_\delta$ for all $\delta < gc(\alpha)$.). Then $G \notin L$; otherwise $D \in L$.

**§5. An extension of Levy-Shoenfield absoluteness.** A formula is $\Delta_0^{ZF}$ if all of its quantifiers are bounded. Suppose $\mathcal{P}(x)$ is $\Delta_0^{ZF}$ and all of its parameters belong to $L$; $s$ is said to be a *solution* of $\mathcal{P}(x)$ iff $\mathcal{P}(s)$ holds. The Levy-Shoenfield technology (LST) yields a solution of $\mathcal{P}(x)$ in $L$ if there is one in $V$ and all the parameters of $\mathcal{P}(x)$ belong to $L(\omega_1^L)$. The forcing argument of Section 4 leads to an extension of (LST) that allows parameters from $(L - L(\omega_1^L))$.

Of course some additional hypotheses are necessary. The most important one says, roughly speaking, if $\mathcal{P}(x)$ has a solution in $V$, then it has many solutions in $V$. One precedent for such a strong hypothesis is the old basis result that $A$, a $\Pi_1^1$ set of reals $\subseteq [0, 1]$, has a member in $L(\omega_1^{CK})$ if $A$ has positive Lebesgue measure.

The *cone above* ***u*** is $\{y \mid u \in L(y)\}$. Recall $Ad_1(y)$, the least $\Sigma_1$ admissible set with $y$ as a member; $Ad_1(y)$ is $L(\beta, tc(\{y\}))$; $\beta$ is the least $\delta$ such that $L(\delta, tc(\{y\}))$ is $\Sigma_1$ admissible; "*tc*" is "transitive closure."

Theorem 5.1. *Suppose $\mathcal{P}(x)$ is $\Delta_0^{ZF}$ with all parameters in $L$ and $s$ is a solution of $\mathcal{P}(x)$. Assume*:

- (A1) *$s$ is an amenable subset of $L(\lambda_s)$ ($\lambda_s$ is a limit.)*
- (A2) *for each member $y$ of the cone above $s$ there is a $z \in A_1(y)$ such that $z$ is a solution of $\mathcal{P}(x)$.*

*Then $\mathcal{P}(x)$ has a solution in $L$.*

The assumption of amenability in (A1) is an essential ingredient of the upcoming forcing argument. There are two forcing constructions below that cooperate to prove Theorem 5.1. The *first* constructs a generic in $L$, the *second* a generic in the cone above the given solution $s$.

**In brief:** By (A2) a generic $G$ in the cone above $s$ yields a solution constructible from $G$ forced by some condition $p$; then the first construction yields a generic $G \in L$ that satisfies $p$ and thereby a solution in $L$.

**Not so brief:** Let $\boldsymbol{\rho}$ *be the cofinality of* $\lambda_s$ in $L$. ($\rho = cf(\lambda_s)$.) Choose $\kappa$, a cardinal of $L$, so that $\rho$ and the parameters of $\mathcal{P}(x)$ belong to $L(\kappa)$.

*The construction of* $L(\alpha)$ *takes place inside* $L$. For each set $d$, $S_1(d)$ is the standard least $\Sigma_1$ substructure of $L$ with $d$ as a subset; each $\Sigma_1^{ZF}$ predicate $P(x)$ with parameters in $d$ and satisfiable in $L$ is satisfiable in $S_1(d)$. A chain of $\Sigma_1$ substructures of $L$ is defined by a $\Sigma_2$ recursion of length $\rho$.

$$H_0 = S_1(\kappa \cup \{\kappa, \kappa^+\}).$$
$$H_{\delta+1} = S_1(H_\delta \cup \{H_\delta\}).$$
$$H_\gamma = \cup\{H_\delta \mid \delta < \gamma\}.(\gamma \textit{ is a limit } \leq \rho).$$

Let $\eta$ be the least $\Sigma_1$ admissible ordinal $> \kappa^+$ in $H_\rho$.
Recall $t : H_\rho \to t[H_\rho]$, the collapsing map in the proof of (F5) in Section 3.
Define $\alpha = t(\eta)$. Note $L(\alpha) \models [t(\kappa^+)$ is regular].
Define $gc(\alpha) = t(\kappa^+)$; $L(\alpha) \models [gc(\alpha)$ is the greatest cardinal];
$L \models cf(gc(\alpha)) = \rho$.

Proposition 5.2. *There exists a strictly increasing function $C \in L$ with domain $\rho$ and range an unbounded set of $(gc(\alpha) \cap$ limit ordinals$)$ such that*:

$$(\forall \gamma < \rho),\ (C \restriction \gamma) \in L(gc(\alpha)).$$

Proof. By induction on $\gamma < \rho$, $H_\gamma \cap \kappa^+$ is a limit ordinal below $\kappa^+$, because if $\beta \in H_\gamma \cap \kappa^+$, then an injection of $\beta$ into $\kappa$ belongs to $H_\gamma$. Hence $t(H_\gamma \cap \kappa^+) = H_\gamma \cap \kappa^+ < gc(\alpha)$. Define $C(\gamma) = H_\gamma \cap \kappa^+$. ⊣

**First Construction**
A function $G(\delta) = p_\delta$ $(\delta < gc(\alpha))$ is defined by a $\Sigma_2^\alpha$recursion;

Case 0: $p_0 = \emptyset$.

Case 1: $\lambda$ is a limit.

Then $p_\lambda = \cup\{p_\delta \mid \delta < \lambda\}$.

Case 2: definition of $p_{\delta+1}$.

Subcase 2a: $\delta \notin range(i)$.

Then $p_{\delta+1} = p_\delta$.

Subcase 2b: $\delta \in range(i)$.

Then $p_{\delta+1}$ is the $L$-least $p$ such that
$p_\delta \geq p \Vdash \mathcal{F}_{i^{-1}(\delta)}$. If no such $p$ exists, then $p_{\delta+1} = p_\delta$.

Suppose $\lambda < gc(\alpha)$ and$\{p_\delta \mid \delta < \lambda\} \subseteq L(gc(\alpha))$. To see that
$\{p_\delta \mid \delta < \lambda\} \in gc(\alpha)$, proceed as in Proposition 4.3 of Section 4.

**Second Construction.**
A function $G(\delta) = p_\delta$ $(\delta < gc(\alpha))$ is defined by a $\Sigma_2^\alpha$ recursion;

Case 0: $p_0 = \emptyset$.

Case 1: $\lambda$ is a limit.

Subcase 1a: $\lambda \notin range(C)$.

Then $p_\lambda = \cup\{p_\delta \mid \delta < \lambda\}$.

Subcase 1b: $\lambda \in range(C)$.

Define $dom_\lambda = domain$ of $\cup\{p_\delta \mid \delta < \lambda\}$.
Define $p_\lambda = \cup\{p_\delta \mid \delta < \lambda\} \cup \{< dom_\lambda + \delta, c_{s\cap\lambda}(\delta) > \mid \delta < \lambda\}$.
($s$ is the solution of $\mathcal{P}(x)$ assumed in Theorem 2.)

Case 2: definition of $p_{\delta+1}$.

Case 2a: $\delta \notin range(i)$, then $p_{\delta+1} = p_\delta$.

Case 2b: $\delta \in range(i)$.

Then $p_{\delta+1}$ is the $L$-least such $p$ that
$p_\delta \geq p \Vdash \mathcal{F}_{i^{-1}(\delta)}$. If no such $p$ exists, then $p_{\delta+1} = p_\delta$.

Suppose $\lambda < gc(\alpha)$ and $\{p_\delta \mid \delta < \lambda\} \subseteq gc(\alpha)$. To see that
$\{p_\delta \mid \delta < \lambda\} \in gc(\alpha)$, proceed as in Proposition 4.3 of Section 4
($s$ is amenable).

**Recovery of *s* from *G*.** For each $\lambda \in range(C)$ and $\delta < \lambda$,

$$c_{s\cap\delta}(\delta) = G(\delta + 1)(dom_\lambda + \delta).$$

$C \in L$, hence $G$ belongs to the cone above $s$.

Question. *Can the leading role of amenability in this paper be diminished to a supporting part*?

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