MODELS OF LONG SENTENCES I

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For Professor Gert Müller (in memoriam)
and
For Professor Chi Tat Chong on his 60th birthday

Abstract. Let $\mathcal{L}$ be a countable first order language. Let A be a Σ_1 admissible set such that $\mathcal{L} \in A$ and the cardinality of A is ω_1. Let $T \subseteq \mathcal{L}_{A,\omega}$ be a set of sentences such that $< A, T >$ is Σ_1 admissible. T is **consistent** iff no contradiction can be derived from T via a deduction in A. T is **complete** iff for each sentence $\mathcal{F} \in \mathcal{L}_{A,\omega}$, either $\mathcal{F} \in T$ or $(\neg\mathcal{F}) \in T$. An ***n*-type** p of T is a consistent, complete set of formulas of arity $\leq n$ such that $< A, p >$ is Σ_1 admissible; ST is the set of all types of T.

Note that sentences of uncountable length may belong to T.

Say T is **degenerate** iff T has a countable, ω-homogeneous model that realizes every type in ST.

A typical instance of **type-completeness** is: if $\exists y\mathcal{F}(\overline{x}, y) \in p(\overline{x}) \in ST$, then there is a $q(\overline{x}, y) \in ST$ such that $p(\overline{x}) \subseteq q(\overline{x}, y)$ and $\mathcal{F}(\overline{x}, y) \in q(\overline{x}, y)$.

T is **type-admissible** iff $< A, \overline{p} >$ is Σ_1 admissible for each finite, coherent pair $\overline{p}$ of types.

Main Results: If T is consistent, complete, type-complete, type-admissible, and not degenerate, then T has a model of cardinality ω_1; **Mild stability** implies type-completeness and type-admissibilty.

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1. Introduction

This paper stands on its own four feet[1] but also serves as an introduction to the arguments of its sequel [5], where longer sentences, larger models and more complex countable approximations are studied. The proof of the **Main Result** below was inspired by the work of Barwise [1] and his students on admissible sets and their application to model theory [4]. Jensen's proof [2] of the gap-2 conjecture in L plays a part, behind the scenes in this paper, but on stage in its sequel [5].

Let $\mathcal{L}$ be a countable first order language. Recall that $\mathcal{L}_{\infty,\omega}$ is an extension of first order logic that allows arbitrary conjunctions and disjunctions of formulas subject to the restriction that a formula can contain only finitely many free variables. On the other hand a formula can mention arbitrarily many individual constants.

Recall that a set A is $\mathbf{\Sigma}_1$ **admissible** iff A is transitive, closed under pairing and unary unions, and satisfies Δ_0 separation and Δ_0 collection (or bounding); Σ_1 admissibility implies Δ_1 separation and Σ_1 collection.

From now on assume A is a Σ_1 admissible set such that $\mathcal{L} \in A$ and the cardinality of A is ω_1.

(Note: the uncountability of A does not imply $\omega_1 \subseteq A$.) For any $Z \subseteq A$, define

$$A[Z] = \cup\{L(\alpha, tc(\{a\}); Z) \mid a \in A\}, \tag{1.1}$$

where α is the least ordinal not in A, and $L(\alpha, tc(\{a\}); Z)$ is the result of iterating first order definability, with $x \in Z$ as an additional Δ_0 formula, through the ordinals less than α, and with $tc(\{a\})$ as the starting set (tc is transitive closure). The structure $< A[Z], Z >$ is said to be $\mathbf{\Sigma}_1$ **admissible** iff $A[Z]$ is Σ_1 admissible with $x \in Z$ as an additional Δ_0 formula. Define $\mathcal{L}_{A,\omega}$ to be be the restriction of $\mathcal{L}_{\infty,\omega}$ to formulas with standard codes in A.

Assume $T \subseteq \mathcal{L}_{A,\omega}$ is a set of sentences such that $< A[T], T >$ is Σ_1 admissible.

Definition 1. *Suppose $Z \subseteq A$; Z is* ***amenable*** *iff $(Z \cap b) \in A$ for every $b \in A$.*

[1] With thanks to Theodore Slaman.

Remark 2. *Recall $A[Z] = A$ iff Z is amenable.*

Assume T is amenable. Thus $< A, T >$ is Σ_1 admissible.

Definition 3. *T is* ***consistent*** *iff no contradiction can be derived from T using the axioms and rules of $\mathcal{L}_{\infty,\omega}$ via a deduction that belongs to A. (The axioms and rules of $\mathcal{L}_{\infty,\omega}$ extend first order logic primarily by adding an infinitary conjunction rule: if $\mathcal{F}_i$ is deducible for each $i \in I$, then $\wedge\{\mathcal{F}_i \mid i \in I\}$ is deducible.) As a rule, below a set $Z \subseteq \mathcal{L}_{A,\omega}$ will be said to be* ***consistent*** *iff $< A, Z >$ is Σ_1 admissible (this implies Z is amenable), and no deduction in A from Z yields a contradiction. (Also the set of free variables occurring in Z is finite.)*

Remark 4. *Let $Z \subseteq \mathcal{L}_{A,\omega}$ be as in* ***Definition 3.*** *By Barwise Z is consistent in the strongest syntactical sense: no deduction in V, the class of all sets, from Z using the axioms and rules of $\mathcal{L}_{\infty,\omega}$ yields a contradiction.*

Definition 5. *T is* ***complete*** *iff for each sentence $\mathcal{F} \in \mathcal{L}_{A,\omega}$, $\mathcal{F} \in T$ or $(\neg\mathcal{F}) \in T$.*

Definition 6. *Let $\overline{x}$ denote a sequence $x_1, ...x_n$ of n distinct free variables. A formula has arity n iff the number of distinct free variables occurring in it is n. An* **n-*type*** *$p(\overline{x})$ of T is a set of formulas whose free variables occur in $\overline{x}$ and such that:*
(i) $p(\overline{x}) \subseteq \mathcal{L}_{A,\omega}$ and $p(\overline{x})$ is amenable;
(ii) For each $\mathcal{G}(\overline{x}) \in \mathcal{L}_{A,\omega}$ of arity $\leq n$, either $\mathcal{G}(\overline{x}) \in p(\overline{x})$ or $(\neg\mathcal{G}(\overline{x})) \in p(\overline{x})$;
(iii) The structure $< A, p(\overline{x}) >$ is Σ_1 admissible;
(iv) $T \subseteq p(\overline{x})$ and $p(\overline{x})$ is consistent.

Definition 7. *ST is the set of all n-types of T for all $n > 0$.*

Proposition 8. *Suppose $\vee\{\mathcal{G}_i(\overline{x}) \mid i \in I\} \in p(\overline{x})$. Then for some $i_0 \in I$, $\mathcal{G}_{i_0}(\overline{x}) \in p(\overline{x})$.*

Proof. Suppose not. Then $(\neg\mathcal{G}_i(\overline{x})) \in p(\overline{x})$ for all $i \in I$. By clause (iii) of Definition 6,

$$\wedge\{\neg\mathcal{G}_i(\overline{x}) \mid i \in I\} \tag{1.2}$$

is deducible from $p(\overline{x})$ via a deduction in A. But then $p(\overline{x})$ is inconsistent. □

A type is presented as a set $p(\overline{x})$ of formulas whose free variables belong to $\overline{x}$. The choice of $\overline{x}$ matters.

For $\overline{v}$ a subsequence of $\overline{x}$ ($\overline{v} \subseteq \overline{x}$), define

$$p(\overline{v}) = \{\mathcal{F}(\overline{v}) \mid \mathcal{F}(\overline{v}) \in p(\overline{x})\}. \tag{1.3}$$

Definition 9. *Suppose* $p_1(\overline{x^1}), p_2(\overline{x^2}) \in ST$. *Let* $\overline{v}$ $(= \overline{x^1} \cap \overline{x^2})$ *be the sequence of variables common to* $\overline{x^1}$ *and* $\overline{x^2}$. *The pair,* $p_1(\overline{x^1}), p_2(\overline{x^2})$, *is said to be* ***coherent*** *iff* $p_1(\overline{v}) = p_2(\overline{v})$.

Definition 10. T *is* ***type-admissible*** *iff* $< A, \overline{p} >$ *is* Σ_1 *admissible for each coherent pair* $\overline{p}$ *of types in* ST.

Type-admissibility is needed for the amalgamation of types during the construction of the model $\mathcal{B}_{\omega_1}$ in the proof of the Main Result. In some situations it can be dropped, cf. Subsections 5.2 and 5.3.

Proposition 11. *Suppose* T *is amenable, consistent, complete and type-admissible. If* $p_1(\overline{x^1}), p_2(\overline{x^2})$ *is a coherent pair of types, then* $p_1(\overline{x^1}) \cup p_2(\overline{x^2})$ *is consistent.*

Proof. Suppose not. Then there is a deduction in A of a contradiction from $\mathcal{F}_1(\overline{x^1}) \wedge \mathcal{F}_2(\overline{x^2})$ for some $\mathcal{F}_i(\overline{x^i}) \in p_i(\overline{x^i})$ $(i = 1, 2)$. Let $\overline{v}$ be $\overline{x^1} \cap \overline{x^2}$, and $\overline{u^i}$ be $\overline{x^i} - \overline{v}$. Then

$$\exists \overline{u^1} \mathcal{F}_1(\overline{u^1}, \overline{v}) \wedge \exists \overline{u^2} \mathcal{F}_2(\overline{u^2}, \overline{v}) \tag{1.4}$$

yields a contradiction. But the coherence of p_1 and p_2 implies formula (1.4) belongs to $p_1(\overline{x^1})$. □

Notation 12. *Let* $\overline{x^1 \cup x^2}$ *denote a sequence of distinct free variables, every one of which occurs in* $\overline{x^1}$ *or* $\overline{x^2}$.

Definition 13. T *is* ***type-complete*** *iff:*
Suppose $p_1(\overline{x^1}), p_2(\overline{x^2}) \in ST$ *and*

$$\{\exists y \mathcal{G}(\overline{x^1 \cup x^2}, y)\} \cup p_1(\overline{x^1}) \cup p_2(\overline{x^2}) \tag{1.5}$$

is consistent. Then there exists an $r(\overline{x^1 \cup x^2}, y) \in ST$ *such that* $\mathcal{G}(\overline{x^1 \cup x^2}, y) \in r(\overline{x^1 \cup x^2}, y)$ *and*

$$p_1(\overline{x^1}), p_2(\overline{x^2}) \subseteq r(\overline{x^1 \cup x^2}, y). \tag{1.6}$$

Every consistent, complete theory contained in a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ is type-complete. In the uncountable case a type-complete theory has advantages similar to those of an atomic theory, cf. Subsection 5.2.

Proposition 14. *Suppose* T *is amenable, consistent, complete, type-admissible and type-complete. If* $p_1(\overline{x^1}) \cup p_2(\overline{x^2})$ *is consistent, then there exists an* $r(\overline{x^1 \cup x^2}) \in ST$ *such that* $p_1(\overline{x^1}), p_2(\overline{x^2}) \subseteq r(\overline{x^1 \cup x^2})$.

Proof. An instance of type-completeness with

$$[(\overline{x^1 \cup x^2} = \overline{x^1 \cup x^2})] \wedge (y = y) \tag{1.7}$$

as $\mathcal{G}(\overline{x^1 \cup x^2}, y)$. □

Definition 15. *T is **degenerate** iff T has a countable, ω-homogeneous model that realizes every type in ST.*

Main Result (MR). If T is amenable, consistent, complete, type-complete, type-admissible, and not degenerate, then T has a model $\mathcal{B}$ of cardinality ω_1.

MR+. In addition: if $X \subseteq ST$ and $card(X) = \omega_1$, then $\mathcal{B}$ can be made to realize all the types in X.

The proof of MR given below, after a minor adjustment in Subsection 5.2, yields an atomic model when T is atomic; in that case the assumptions of amenability, type-completeness and type-admissibility can be dropped.

Let $\mathcal{B}_{\omega_1}$ be the model whose existence is claimed in the main result. The structure $\mathcal{B}_{\omega_1}$ is a Henkin-style model that is the limit of a chain of countable partial Henkin models, $\mathcal{B}_\gamma$ $(\gamma < \omega_1)$. $\mathcal{B}_\gamma$ is said to be "partial," because it is the result of a Henkin-type construction $\mathbf{C}_\gamma$ of length ω carried out on a countable set of formulas that may lack the subformula property. The construction $\mathbf{C}_\gamma$ builds a partial model of T_γ, the intersection of T with a countable Σ_1 hull H_γ. The theory T_γ may have a sentence $\mathcal{G}$ of uncountable length; consequently $\mathcal{G}$ will be declared true by $\mathbf{C}_\gamma$, but if $\mathcal{G}$ has an existential subsentence not in H_γ, then that subsentence will not be assigned an existential witness by $\mathbf{C}_\gamma$. On the other hand each n-tuple of Henkin constants of $\mathcal{B}_\gamma$ is assigned to some n-type of T. The hull H_γ ensures that T_γ is a good approximation of T, an approximation that gets better as γ increases. The embeddability of $\mathcal{B}_\gamma$ in $\mathcal{B}_{\gamma+1}$ results from a property of $\mathcal{B}_{\gamma+1}$ akin to ω-saturation.

2. Σ_1 Substructures

Let H be a Σ_1 substructure of the universe V $(H \preceq_1 V)$ such that the cardinality of H is ω_1, $A \subseteq H$,and $A, T, ST \in H$. Define

$$ST_H = ST \cap H. \tag{2.1}$$

Proposition 16. *If ST_H is countable, then $ST_H = ST$.*

Proof. Suppose ST is uncountable. In H there is an one-one map of A into ST. But then ST_H is uncountable since $A \subseteq H$. Thus ST is countable. In H there is a map of ω onto ST, so $ST \subseteq H$. □

Proposition 17. *There exists a chain H_γ ($\gamma < \omega_1$) of countable structures such that*

$$A, T \in H_0, \tag{2.2}$$
$$H_\gamma \preceq_1 H_{\gamma+1} \quad (\gamma < \omega_1), \tag{2.3}$$
$$H_\lambda = \cup\{H_\gamma \mid \gamma < \lambda\} \quad (\textit{limit } \lambda < \omega_1), \tag{2.4}$$
$$H = \cup\{H_\gamma \mid \gamma < \omega_1\}. \tag{2.5}$$

3. Akin to ω-Saturation

For $\gamma < \omega_1$, define

$$\mathcal{L}_{\gamma,\omega} = \mathcal{L}_{A,\omega} \cap H_\gamma, \tag{3.1}$$
$$T_\gamma = T \cap H_\gamma, \tag{3.2}$$
$$ST_\gamma = ST_H \cap H_\gamma. \tag{3.3}$$

Then $\mathcal{L}_{A,\omega}$ is $\cup\{\mathcal{L}_{\gamma,\omega} \mid \gamma < \omega_1\}$, T is $\cup\{T_\gamma \mid \gamma < \omega_1\}$, and ST_H is $\cup\{ST_\gamma \mid \gamma < \omega_1\}$.

From an intuitive point of view, the structure $\mathcal{B}_\gamma$ is the result of building a countable ω-homogeneous structure that realizes all the types in ST_γ and no others.

Construction of $\mathcal{B}_\gamma$. Fix $s < \omega$. Prior to stage s of the construction, a sequence $\overline{c^{\gamma,s}}$ of distinct individual constants $c_1^\gamma, ..., c_s^\gamma$ was developed ($\overline{c^{\gamma,0}}$ is null). A type $r_{\gamma,s}(\overline{x^{\gamma,s}}) \in ST_\gamma$ was assigned to $\overline{c^{\gamma,s}}$; thus $\overline{x^{\gamma,s}}$ denotes $x_1^\gamma, ..., x_s^\gamma$, and the result of the construction prior to stage s is the set of sentences $r_{\gamma,s}(\overline{c^{\gamma,s}})$ ($r_{\gamma,0}$ is T). Suppose $\overline{v}$ is a subsequence of $\overline{x^{\gamma,s}}$ and $\overline{d}$ is a subsequence of $\overline{c^{\gamma,s}}$ that realizes $r_{\gamma,s}(\overline{v})$; i.e. $r_{\gamma,s}(\overline{d}) \in r_{\gamma,s}(\overline{c^{\gamma,s}})$. Let y be a variable not occurring in $\overline{x^{\gamma,s}}$.

Case I (existential witnesses). Suppose $\mathcal{G}(\overline{v}, y) \in \mathcal{L}_{\gamma,\omega}$ and

$$\exists y \mathcal{G}(\overline{d}, y) \in r_{\gamma,s}(\overline{d}). \tag{3.4}$$

Then $\{\exists y \mathcal{G}(\overline{v}, y)\} \cup r_{\gamma,s}(\overline{v})$ is consistent. By the type-completeness of T, there is an $r(\overline{v}, y) \in ST$ such that

$$\mathcal{G}(\overline{v}, y) \in r(\overline{v}, y) \text{ and } r_{\gamma,s}(\overline{v}) \subseteq r(\overline{v}, y). \tag{3.5}$$

By Propositions 11 and 14 there is an $r'(\overline{x^{\gamma,s}}, y) \in ST$ such that $r_{\gamma,s}(\overline{x^{\gamma,s}}), r(\overline{v}, y) \subseteq r'(\overline{x^{\gamma,s}}, y)$. And $r'(\overline{x^{\gamma,s}}, y)$ can be taken from ST_γ, since $H_\gamma \preccurlyeq_1 H \preccurlyeq_1 V$.

Let e be an individual constant not occurring in $\overline{c^{\gamma,s}}$ or in $\mathcal{L}$ or in $\mathcal{B}_\delta$ for any $\delta < \gamma$ Define:

$$c^\gamma_{s+1} = e; \tag{3.6}$$

$$\overline{c^{\gamma,s+1}} \text{ is the sequence } \overline{c^{\gamma,s}}, c^\gamma_{s+1}; \tag{3.7}$$

$$r_{\gamma,s+1}(\overline{x^{\gamma,s+1}}) = r'(\overline{x^{\gamma,s}}, y). \tag{3.8}$$

The result of the construction at the end of stage s is the set of sentences $r_{\gamma,s+1}(\overline{c^{\gamma,s+1}})$. And $\mathcal{G}(\overline{d}, c^\gamma_{s+1}) \in r_{\gamma,s+1}(\overline{c^{\gamma,s+1}})$.

Case 2 (homogeneity and universality). Suppose $q(\overline{v}, y) \in ST_\gamma$ and $r_{\gamma,s}(\overline{v}) \subseteq q(\overline{v}, y)$. As in Case 1, there is an $r'(\overline{x^{\gamma,s}}, y) \in ST_\gamma$ such that

$$r_{\gamma,s}(\overline{x^{\gamma,s}}), q(\overline{v}, y) \subseteq r'(\overline{x^{\gamma,s}}, y). \tag{3.9}$$

Let e and $r_{\gamma,s+1}(\overline{x^{\gamma,s+1}})$ be as in Case 1. Then $\overline{d}, e$ realizes $q(\overline{v}, y)$ and the result of the construction at the end of stage s is $r_{\gamma,s+1}(\overline{c^{\gamma,s}}, e)$.

Define

$$\mathcal{B}_\gamma = \cup\{r_{\gamma,s}(\overline{c^{\gamma,s}}) \mid s < \omega\} \tag{3.10}$$

$$c^\gamma \text{ is } c^\gamma_1, c^\gamma_2, ...c^\gamma_s, ...(s < \omega). \tag{3.11}$$

The universe of $\mathcal{B}_\gamma$ is c^γ. Let $\overline{d}$ denote an n-tuple of $\mathcal{B}_\gamma$ $(n \geq 0)$.

Proposition 18. *(i) $\mathcal{B}_\gamma$ is a set of sentences of the form $\mathcal{F}(\overline{d})$, where $\mathcal{F}(\overline{x})$ is a formula of $\mathcal{L}_{A,,\omega}$ and $\overline{d}$ is a subsequence of c^γ. Every sentence of this form, or its negation, belongs to $\mathcal{B}_\gamma$.*
(ii) Each $\overline{d}$ has been assigned a type $p(\overline{x}) \in ST_\gamma$, and $\mathcal{B}_\gamma$ is the union of all such $p(\overline{d})$'s.
(iii) Suppose $\exists y \mathcal{G}(\overline{x}, y) \in \mathcal{L}_{\gamma,\omega}$ and $\exists y \mathcal{G}(\overline{d}, y) \in \mathcal{B}_\gamma$; then $\mathcal{G}(\overline{d}, c^\gamma_s) \in \mathcal{B}_\gamma$ for some s.
(iv) Suppose $p(\overline{x})$ has been assigned to $\overline{d}$ and $p(\overline{x}) \subseteq q(\overline{x}, y) \in ST_\gamma$; then $q(\overline{x}, y)$ has been assigned to $\overline{d}, e$ for some $e \in c^\gamma$ ($p(\overline{x})$ can be null).

Remark 19. *In the light of Proposition 18 it is reasonable to say: $\mathcal{B}_\gamma$ is ω-**homogeneous** and the set of types **realized** in $\mathcal{B}_\gamma$ is ST_γ.*

4. Proof of The Main Result

An injective, type preserving map

$$m_{\gamma,\gamma+1} : c^\gamma \to c^{\gamma+1} \tag{4.1}$$

is defined by recursion on $s < \omega$. Prior to stage s of the recursion, a sequence

$$\overline{b^{\gamma+1,s}} = b^{\gamma+1}_1, ..., b^{\gamma+1}_s \tag{4.2}$$

was developed so that $r_{\gamma,s}(\overline{b^{\gamma+1,s}}) \subseteq \mathcal{B}_{\gamma+1}$. ($\overline{b^{\gamma+1,0}}$ is null.) By Proposition 18(iv),

$$r_{\gamma,s+1}(\overline{b^{\gamma+1,s}}, e) \subseteq \mathcal{B}_{\gamma+1} \tag{4.3}$$

for some $e \in c^{\gamma+1}$. Define $m_{\gamma,\gamma+1}(c^{\gamma}_{s+1}) = e = b^{\gamma+1}_{s+1}$.

The map $m_{\gamma,\gamma+1} : \mathcal{B}_{\gamma} \to \mathcal{B}_{\gamma+1}$ is defined by

$$m_{\gamma,\gamma+1}(\mathcal{F}(\overline{d})) = \mathcal{F}(m_{\gamma,\gamma+1}(\overline{d})). \tag{4.4}$$

Thus the type assigned to $\overline{d}$ in $\mathcal{B}_{\gamma}$ equals the type assigned to $m_{\gamma,\gamma+1}(\overline{d})$ in $\mathcal{B}_{\gamma+1}$.

A direct system, $\{\mathcal{B}_{\gamma}, m_{\gamma,\delta} \mid \gamma < \delta < \omega\}$ is defined by recursion on ω_1.

Define $m_{\gamma,\delta+1} = m_{\delta,\delta+1} m_{\gamma,\delta}$. If λ is a countable limit, then let $\mathcal{B}'_{\lambda}, m'_{\gamma,\lambda}$ $(\gamma < \lambda)$ be the direct limit of $\{\mathcal{B}_{\gamma}, m_{\gamma,\delta} \mid \gamma < \delta < \lambda\}$. Recall the terminology suggested in Remark 19. By Proposition 18, both $\mathcal{B}'_{\lambda}$ and $\mathcal{B}_{\lambda}$ are ω-homogeneous and realize the same set of types, ST_{λ}. A back-and-forth argument produces an isomorphism

$$i_{\lambda} : \mathcal{B}'_{\lambda} \to \mathcal{B}_{\lambda}. \tag{4.5}$$

Define $m_{\gamma,\lambda} = i_{\lambda} m'_{\gamma,\lambda}$.

Definition 20. *$\mathcal{B}_{\omega_1}$ is the direct limit of $\{\mathcal{B}_{\gamma}, m_{\gamma\delta} \mid \gamma < \delta < \lambda < \omega_1\}$.*

Lemma 21. *(i) $\mathcal{B}_{\omega_1}$ is finitarily consistent; i.e. no deduction from $\mathcal{B}_{\omega_1}$ of finite length yields a contradiction.*
(ii) If a disjunction belongs to $\mathcal{B}_{\omega_1}$, then some term of the disjunction belongs to $\mathcal{B}_{\omega_1}$.
(iii) $\mathcal{B}_{\omega_1}$ has existential witnesses.
(iv) (completeness) Suppose $\mathcal{F}(\overline{x})$ belongs to $\mathcal{L}_{A,\omega}$ and $\overline{d}$ is a sequence of individual constants of $\mathcal{B}_{\omega_1}$, constants of the form $m_{\gamma,\omega_1}(c^{\gamma}_{s})$; then $\mathcal{F}(\overline{d})$, or its negation, belongs to $\mathcal{B}_{\omega_1}$.
(v) Let $\mathcal{B}_{\omega_1}$ ambiguously denote the Henkin-style model defined by the sentences of $\mathcal{B}_{\omega_1}$. Then $\mathcal{B}_{\omega_1}$ is ω-homogeneous and the set of types realized in $\mathcal{B}_{\omega_1}$ is ST_H.
(vi) $\mathcal{B}_{\omega_1}$ has cardinality ω_1.

Proof. Assertions (i)-(iv) follow from Proposition 18. A Henkin-style model is determined by $\mathcal{B}_{\omega_1}$ as in first order logic thanks to (i)-(iv). Suppose $\mathcal{B}_{\omega_1}$ is countable. Hence $\mathcal{B}_{\omega_1} = \mathcal{B}_{\gamma_0}$ for some countable γ_0. Then ST_H is countable by Lemma 21(iii), hence $ST_H = ST$ by Proposition 16, and so T is degenerate. □

MR+. In addition: if $X \subseteq ST$ and $card(X) = \omega_1$, then $\mathcal{B}$ can be made to realize all the types in X.

Proof. A slight modification of Sections 2 and 3. Let H^X be a Σ_1 substructure of V such that $card(H^X) = \omega_1$; $X, A \subseteq H^X$; and $X, A, T, ST \in H^X$. Define

$$ST^X_H = (ST \cap H^X). \tag{4.6}$$

There exists a chain H^X_γ $(\gamma < \omega_1)$ of countable structures such that

$$X, A, T \in H^X_0, \tag{4.7}$$

$$H^X_\gamma \preceq_1 H^X_{\gamma+1}, \tag{4.8}$$

$$H^X_\lambda = \cup\{H^X_\gamma \mid \gamma < \lambda\} \ (limit\ \lambda), \tag{4.9}$$

$$H^X = \cup\{H^X_\gamma \mid \gamma < \omega_1\}. \tag{4.10}$$

Define $ST^X_\gamma = ST^X_H \cap H^X_\gamma$ $(\gamma < \omega_1)$. Then $ST^X_H = \cup\{ST^X_\gamma \mid \gamma < \omega_1\}$. Now proceed as in Section 3. The set of types realized in $\mathcal{B}_\gamma$ will be ST^X_γ. □

5. Extensions of MR and MR+

5.1. The Number of Models.

Corollary 22. *Assume T is amenable, consistent, complete, type-complete, type-admissible, and not degenerate. If $card(ST) > \omega_1$, then the number of models of T of cardinality ω_1 is at least $card(ST)$.*

Proof. Let $\{p_\gamma \mid \gamma < card(ST)\}$ be an enumeration of ST. Let $\mathcal{B}^\gamma$ be a model of T of cardinality ω_1 that realizes p_γ. Then the number of models of T (up to isomorphism) in $\{\mathcal{B}^\gamma \mid \gamma < card(ST)\}$ is at least $card(ST)$; otherwise $card(ST) \leq \omega_1$. □

5.2. **Atomic Theories.** (In this subsection, as in the Introduction, $\mathcal{L}$ is a countable first order language, A is a Σ_1 admissible set of cardinality ω_1, $T \subseteq \mathcal{L}_{A,\omega}$, and $< A, T >$ is Σ_1 admissible.)

Corollary 23. *Assume T is consistent, complete and atomic. Suppose T does not have a countable atomic model. Then T has an atomic model of cardinality ω_1.*

Proof. A small modification of Sections 2, 3 and 4. Define aT to be the set of atoms of T. (There are no repetitions in aT; each atom of T has just one formula representing it in aT.) Replace ST by aT in the definition of H in Section 2. Define H_γ as in Section 2. Let $aT_\gamma = aT \cap H_\gamma$. Then $aT = \cup\{aT_\gamma \mid \gamma < \omega_1\}$. Now proceed as in Section 3. In both cases of the construction of $\mathcal{B}_\gamma$, the type r' is an atom. As in Remark 19, it is reasonable to say $\mathcal{B}_\gamma$ is ω-homogeneous and the set of types realized in $\mathcal{B}_\gamma$ is aT_γ. If $\mathcal{B}_{\omega_1}$ were countable, then T would have a countable atomic model. □

5.3. **$\mathcal{L}_{\omega_1,\omega}$.** Let HC be the set of hereditarily countable sets. Let $\mathcal{L}$ be countable and $\in HC$. Then $\mathcal{L}_{\omega_1,\omega}$ is $\mathcal{L}_{HC,\omega}$. For any $Z \subseteq HC$: Z is amenable and $< HC, Z >$ is Σ_1 admissible.

Corollary 24. *Assume the Continuum Hypothesis. If $T \subseteq \mathcal{L}_{\omega_1,\omega}$ is consistent, complete, type-complete and not degenerate, then T has a model of cardinality ω_1.*

6. Stability,Type-Completeness and Type-Admissibility

This section outlines some of the points made in [5]. Suppose $\mathcal{L}$ is a countable first order language, A is a Σ_1 admissible set, and $T \subseteq \mathcal{L}_{A,\omega}$ is a set of sentences such that $< A, T >$ is Σ_1 admissible and T is consistent and complete as in Section 1. Note that no assumption is made about the cardinality of A.

Does T have a model? A seemingly simpler question is: Does T have any types? The latter can be answered with the help of a suitable notion of stability. Call T **mildly stable** if ST', the set of types of T', is countable whenever T' is a countable subtheory of T.

A sketch of a proof that $ST \neq \emptyset$. There exists a countable $H \prec_1 V$ such that T, $ST \in H$. Then

$$T_H = T \cap H$$

is countable. So $S(T_H)$ is countable by mild stability of T. Let

$$m : H \longrightarrow m[H]$$

be the Mostowski collapse. Thus $m[T_H] = m(T)$ and $S(T_H)$ is $\{m^{-1}[p] \mid p \in S(m(T))\}$. Hence $S(m(T))$ is countable. Hence

$$S(m(T)) \in m[H]$$

by arguments in effective descriptive set theory (similar to showing a countable Δ^1_1 set of hyperarithmetic reals is a member of $L(\omega_1^{CK})$). Then $S(m(T)) \neq \emptyset$ because $m(T)$ is countable. Choose $p \in S(m(T))$. Then $m^{-1}(p) \in ST$.

Lemma 25. *If T is mildly stable, then T is type-complete.*

Mild stability also helps to resolve the question of type-admissibility.

Let A^+ be be the least Σ_1 admissible set with A as a member. Call A **strongly admissible** if $< A, Z \cap A >$ is Σ_1 admissible for all $Z \in A^+$.

Lemma 26. *If T is mildly stable, A is strongly admissible and $T \in A^+$, then T is type-admissible.*

References

[1] Barwise, J., *Admissible Sets and Structures. An Approach to Definability Theory*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin 1975.
[2] Devlin, K. J., *Aspects of Constructibility*, Lecture Notes in Mathematics. Springer-Verlag, Berlin 1973.
[3] Knight, Julia F., *Prime and Atomic Models*, Jour. Symb. Logic **43** (1978), no 3, 385-393.
[4] Nadel, M., $\mathcal{L}_{\omega_1,\omega}$ *and Admissible Fragments*, In: *Model-Theoretic Logics*, Edited by J. Barwise & S. Feferman, Perspectives in Mathematical Logic. Springer-Verlag, Berlin 1985, 271-316.
[5] Sacks, G. E., *Models of Long Sentences II*, forthcoming.

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