E-RECURSIVE INTUITIONS

GERALD E. SACKS

In Memoriam
Professor Joseph R. Shoenfield

ABSTRACT. An informal sketch (with intermittent details) of parts of E-Recursion theory, mostly old, some new, that stresses intuition. The lack of effective unbounded search is balanced by the availability of divergence witnesses. A set is E-closed iff it is transitive and closed under the application of partial E-recursive functions. Some finite injury, forcing, and model theoretic constructions can be adapted to E-closed sets that are not Σ_1 admissible. Reflection plays a central role.

CONTENTS

1. INITIAL INTUITIONS

One of the central intuitions of classical recursion theory is the effectiveness of unbounded search. Let A be a nonempty recursively enumerable set of nonnegative integers. A member of A can be selected by simply enumerating A until some member appears. This procedure, known as unbounded search, consists of following instructions until a termination point is reached. What eventually appears is not merely

Date: December 10, 2010.
My thanks to the organizers of "Effective Mathematics of the Uncountable," CUNY Graduate Center, August 2009, for their kind invitation.

some number $n \in A$ but a computation that reveals $n \in A$. Unbounded search in its full glory consists of enumerating all computations until a suitable computation, if it exists, is found. It follows there exists a partial recursive function g such that for all e:

$$g(e)\downarrow \ \longleftrightarrow\ W_e \neq \varnothing$$
$$g(e)\downarrow \ \longrightarrow\ g(e) \in W_e.$$

(W_e is the e-th recursively enumerable set; $g(e) \downarrow$ means $g(e)$ **converges**, i.e. has a value. The symbol $\uparrow$ indicates **divergence**.)

In E-recursion theory unbounded search is not effective. An E-recursive enumeration of all computations is available, but sifting through them for some desired outcome is demonstrably not effective in most circumstances. The exceptions are called selection principles. Finding them is an important part of the subject. Unbounded search is legal in the setting of α-recursion theory. Both E-recursion and α-recursion, restricted to ω, are equivalent to classical recursion theory. For α-recursion the equivalence is almost immediate, for E-recursion some proof is needed.

In the classical theory unbounded search is enabled by Kleene's least number operator scheme:

$$f(x) \simeq (\mathit{least}\ y)(g(x,y)=0); \qquad (1.1)$$

The symbol $\simeq$ denotes strong equality. Say $f(x)$ is strongly equal to $g(x)$ iff neither is defined or both have the same value. Then in (1.1) f is partial recursive iff g is. Nothing resembling Kleene's least number operator appears among the Normann [4] schemes[1] for E-recursion. Below x, y, w, z are arbitrary sets, and e, i, j, n are nonnegative integers. The first 3 are finitary in nature.

(1) projection

$$\{e\}(x_1, ..., x_n) = x_i \qquad if\ e = <1, n, i> .$$

(2) difference

$$\{e\}(x_1, ..., x_n) = x_i - x_j \qquad if\ e = <2, n, i, j> .$$

(3.1) pairing

$$\{e\}(x_1, ..., x_n) = \{x_i, x_j\} \qquad if\ e = <3, 1, n, i, j> .$$

(3.2) union

$$\{e\}(x_1, ..., x_n) = \cup\{y \mid y \in x_1\} \qquad if\ e = <3, 2, n> .$$

[1] Developed independently by Y. N. Moschovakis.

Scheme (4) is the only scheme that is potentially infinitary. If x_1 is infinite, then the computation of (4) entails infinitely many subcomputations.

(4) E-recursive bounding

$$\{e\}(x_1, ..., x_n) \simeq \{\{c\}(y, x_2, ..., x_n) \mid y \in x_1\} \quad if\ e =< 4, n, c > .$$

The left side of scheme (4) converges iff $\{c\}(y, x_2, ..., x_n) \downarrow$ for all $y \in x_1$.

(5) composition

$$\begin{aligned} \{e\}(x_1, ..., x_n) &\simeq \{c\}(\{d_1\}(x_1, ..., x_n), ..., \{d_m\}(x_1..., x_n)) \\ if\ e &= < 5, n, m, c, d_1, , , d_m > . \end{aligned}$$

(6) enumeration

$$\{e\}(c, x_1, ..., x_n, y_1, ..., y_m) \simeq \{c\}(x_1, ..., x_n)\ if\ e =< 6, m, n > .$$

The enumeration scheme leads to Kleene's **fixed point theorem**: let f be a total recursive function; then there exists some e such that

$$\{e\} \simeq \{f(e)\}.$$

(The partial functions $\{e\}$ and $\{f(e)\}$ have the same graph.) The fixed point theorem implies definition by effective transfinite recursion is effective (Section 3).

Each Normann scheme is a closure condition in the inductive definition of E, **the class of E-recursive evaluations**. Each member of E is of the form

$$< e, < x_1, ..., x_n >, y > .$$

The above tuple is put in E iff the schemes determine a value y for $\{e\}(x_1, ..., x_n)$. The definition of E is a Σ_1 transfinite recursion on the ordinals σ.

Stage $\sigma = 0$: $< e, < x_1, ..., x_n >, x_i >$ is put in E iff $e =< 1, n, i >$. Schemes (2) and (3) are treated similarly.

Stage $\sigma > 0$: $< e, < x_1, ..., x_n >, z >$ is put in E iff
it was not put in before stage σ,
$e =< 4, n, c >$,
$\forall y_{y \in x_1} \exists w[< c, < y, x_2, ..., x_n >, w >$ put in E before stage $\sigma]$,
and $z = \{w \mid \exists y_{y \in x_1}[< c, < y, x_2, ..., x_n >, w >$ is already in $E]\}$.
Schemes (5) and (6) are treated similarly.

Define **$\{e\}(x_1, ..., x_n)$ converges to y** iff

$$< e, < x_1, ..., x_n >, y > \in E.$$

A function f is **partial E-recursive** iff there is an e such that

$$f(x_1, ..., x_n) \simeq \{e\}(x_1, ..., x_n)$$

for all $x_1, ..., x_n$. A class of sets is **E-recursively enumerable** iff it is the domain of a partial E-recursive function. The graph of a partial E-recursive function is E-recursively enumerable. One consequence of the lack of unbounded search in E-recursion is: a function whose graph is E-recursive need not be E-recursive; an example is $O(x)$, where $x \in L$ and $O(x)$ is the least δ such that x is a first order definable subset of$L(\delta)$. Another consequence is: the range of a E-recursive function on the ordinals need not be E-recursively enumerable, cf. Proposition 3.1.

E, the class of E-recursive evaluations, is E-recursively enumerable thanks to the enumeration scheme.

A set b is **E-closed** iff b is transitive and for all $e < \omega$,

$$\{e\}(x_1, ..., x_n) \in b$$

whenever $x_1, ..., x_n \in b$ and $\{e\}(x_1, ..., x_n) \downarrow$.

The above six schemes restricted to the nonnegative integers define the partial recursive functions of classical recursion theory. Recall HF, the set of hereditarily finite sets, defined by

$$x \in HF \leftrightarrow [x \ is \ finite \wedge \forall y \in x (y \in HF)].$$

A Δ_0 predicate is said to be **lightface** if all its parameters are finite ordinals; Δ_0 means all quantifiers are bounded. The set of nonnegative integers, ω, is a lightface Δ_0 -definable subset of HF, hence E-recursive by Proposition 2.5. Let $f(x_1, ..., x_n)$ be a partial function from ω^n into ω. Then f is a classical partial recursive function iff it is E-recursive. This follows from Gandy's selection principle, Theorem 4.1, which legitimizes the search for a nonnegative integer in E-recursion theory.

2. E-Recursively Enumerable Versus Σ_1

Every E-recursively enumerable class is Δ_1 definable; the converse is false. These results can be derived from the notion of computation. A **computation instruction** is an $(n+1)$-tuple $< e, x_1, ..., x_n >$ or more simply $< e, \overline{x} >$. Associated with $< e, \overline{x} >$ is a tree, $T_{<e,\overline{x}>}$. Every node of the tree is a computation instruction; its top node is $< e, \overline{x} >$ and it branches downward as determined by the schemes of Section 1.

If e is an index for one of the first three schemes, then $< e, \overline{x} >$ is a terminal node.

If e is $< 4, n, c >$, then $< c, y, x_2, ..., x_n, >$ is an immediate subcomputation instruction of $< e, \overline{x} >$ for each $y \in x_1$.

If e is $< 5, n, m, c, d_1, ..., d_m >$, then $< d_j, \overline{x} >$ is an immediate subcomputation instruction of $< e, \overline{x} >$ for $1 \leq j \leq m$; in addition

$$\text{if } \{d_j\}(\overline{x}) \text{ converges to } y_j \text{ for } 1 \leq j \leq m,$$

then $< c, y_1, ..., y_m >$ is an immediate subcomputation instruction of $< e, \overline{x} >$.

If e is not interpretable as a scheme, then $< e, \overline{x} >$ has just one immediate subcomputation instruction, a repeat of $< e, \overline{x} >$.

Define $b >_U a$ to be a is an immediate subcomputation instruction of b. The predicate $b >_U a$ is E-recursively enumerable. The predicate

$$\exists c[b >_U c >_U a]$$

is not E-recursively enumerable.

Proposition 2.1. $\{e\}(\overline{x})\downarrow \longleftrightarrow T_{<e,\overline{x}>}$ *is wellfounded.*

Both directions of 2.1 are proved by transfinite induction.

Suppose $\{e\}(x)\downarrow$; its **length of computation,** denoted by $| \{e\}(x) |$, is the ordinal height of $T_{<e,x>}$. Otherwise its length is ∞.

Proposition 2.2. *If A is E-recursively enumerable, then A is Δ_1 definable.*

Proof. A is Σ_1 because E, the class of E-recursive evaluations, is Σ_1. Suppose for some e,

$$A = \{x \mid \{e\}(x)\downarrow\}.$$

Then for all x,

$$x \notin A \longleftrightarrow T_{<e.x>} \text{ is illfounded.}$$

□

Proposition 2.3. *There exists a Δ_1 definable class that is not E-recursively enumerable.*

2.3 follows from 2.2 and a diagonal argument.

The proof of 2.2 makes a point whose importance is not readily apparent. Suppose $\{e\}(x)\uparrow$ (diverges). Then the tree $T_{<e,x>}$ has an infinite descending path. Any such path witnesses the divergence of $\{e\}(x)$. Moschovakis [2] was the first to realize the importance of **divergence witnesses** and to apply them fruitfully. They are essential ingredients of priority constructions and forcing arguments in E-recursion theory. Their power is ample compensation for the failures of unbounded search.

Proposition 2.4. *If A and $V - A$ are E-recursively enumerable, then A is E-recursive.*

The usual proof of the counterpart of 2.4 in classical recursion uses unbounded search and so is not applicable in E-recursion; 2.4 follows from Gandy Selection (Theorem 4.1).

A Δ_0 predicate is said to be **lightface** if all its parameters are finite ordinals; Δ_0 means all quantifiers are bounded.

Proposition 2.5. *Every lightface Δ_0 predicate is E-recursive.*

3. E-Recursion Versus α-Recursion

Recall how transfinite recursion (TR) works in set theory. Let $I : V \longrightarrow V$. Consider the equation

$$f(\gamma) = I(f \restriction \gamma). \tag{3.1}$$

$f \restriction \gamma$ is the graph of f restricted to γ. There exists a unique f that satisfies (3.1) for all γ. If I is Σ_1 definable, then f is Σ_1 definable.

Effective Transfinite Recursion (ETR): If I is E-recursive, then the unique f that satisfies (3.1) is E-recursive.

ETR is a consequence of Kleene's fixed point theorem (Section 1).

Thus $L(\gamma)$, the γ-th initial segment of Gödel's L, as a function of γ, is E-recursive. Gödel enumerated L one set at a time by means of an E-recursive function.

Proposition 3.1. *$V = L$ iff L is E-recursively enumerable.*

Proof. Suppose $\forall x(x \in L \longleftrightarrow \{e\}(x) \downarrow)$ and $V \neq L$. Then for some $b \notin L$, $\{e\}(b) \uparrow$. By Proposition 2.2, divergence is Σ_1 definable. Then by Levy-Shoenfield absoluteness, $\{e\}(x) \uparrow$ for some $x \in L$. □

Proposition 3.2. *The predicates, $\mid \{e\}(x) \mid < \gamma$ and $\mid \{e\}(x) \mid = \gamma$, are E-recursive.*

Proof. By effective transfinite recursion. □

For any set x, define $E(x)$ to be the least transitive, E-closed set with x as a member. The schemes for E-recursion map $L(tc(\{x\}))$ into $L(tc(\{x\}))$. And $L(\gamma, tc(\{x\}))$ is an E-recursive function of γ and x. It follows there is an ordinal, κ^x, such that

$$E(x) = L(\kappa^x, tc(\{x\})).$$

Also $\gamma < \kappa^x$ iff $\gamma = \{e\}(x, a_1, ...a_n)$ for some $e < \omega$ and $a_1, ...a_n \in tc(x)$.

For any set x, define $Ad_1(x)$ to be the least Σ_1 admissible set with x as a member. Then

$$Ad_1(x) = L(\alpha^x, tc(\{x\})),$$

where α^x is the least β such that $L(\beta, tc(\{x\}))$ satisfies Σ_1 bounding. Proposition 3.5 implies $\kappa^{\omega_1} < \alpha^{\omega_1}$.

The ordinal κ^x can be regarded as the least β such that $L(\beta, tc(\{x\}))$ satisfies E-recursive bounding. An induction on the length of computations shows

Proposition 3.3. $E(x) \subseteq Ad_1(x)$.

Proposition 3.4. *There exists a partial E-recursive function g such that for all $d < \omega$ and all x;*

(i) $\{d\}(x) \downarrow \longleftrightarrow g(d, x) \downarrow$
(ii) $\{d\}(x) \downarrow \longrightarrow g(d, x) = T_{<d,x>}$.

Proof. By effective transfinite recursion. □

Suppose B is E-closed and $x \in B$. If $\{e\}(x) \downarrow$, then $T_{<e,x>}$ is wellfounded, hence E-recursive in x and so a member of B. If $\{e\}(x) \uparrow$, then $T_{<e,x>}$ is illfounded and might not be in B; nonetheless some infinite descending path through $T_{<e,x>}$ might be in B.

Say B **admits divergence witnesses** iff for all e, $x \in B$: if $\{e\}(x)$ diverges, then some witness to the divergence belongs to B.

Proposition 3.5. $E(\omega_1)$ *is not* Σ_1 *admissible.*

Proof. For some κ, $E(\omega_1) = L(\kappa)$. It suffices to show that $L(\kappa)$ admits divergence witnesses, because then there is a map m from $\omega \times \omega_1$ into $L(\kappa)$ whose graph is a Σ_1 definable subset of $L(\kappa)$ and whose range is unbounded in $L(\kappa)$. The value of $m(e, \beta)$ is either the value of$\{e\}(\omega_1, \beta)$ or the L-least witness to the divergence of $\{e\}(\omega_1, \beta\}$.

The definition of $E(\omega_1)$ implies there is an injective map f of $L(\kappa)$ into ω_1 in L. Suppose $\{e\}(x) \uparrow$ for some $x \in L(\kappa)$; it follows from Proposition 4.1 and Lemma 4.2 that a witness to its divergence is first order definable over $L(\kappa)$. The witness, t, has domain ω. For all n, $t(n) \in L(\kappa)$; f maps the graph of t to a countable subset of ω_1, hence to a member of $L(\omega_1)$. But then $t \in L(\kappa)$. □

Proposition 3.6. *If* $b \subseteq \omega$, *then* $E(b)$ *is* Σ_1 *admissible.*

Proof. If b is finite, then $E(b) = L(\omega)$. Assume b is infinite. Then $E(b) = L(\omega_1^b, b)$, where ω_1^b is the least ordinal not recursive in b. $L(\omega_1^b, b)$ is Σ_1 admissible by a Σ_1^1 bounding argument from hyperarithmetic theory. ($L(\omega_1^b, b) \cap 2^\omega$ is the set of all subsets of ω hyperarithmetic in b.) □

4. Selection and Reflection

Let $W(e,x)$ be $\{n \mid n \in \omega \wedge \{e\}(n,x)\downarrow\}$. Thus $W(e,x)$ is the e-th set (of nonnegative integers) E-recursively enumerable in x. Gandy selection is a uniform method for sifting through computations to find one, if there are any, that puts a non-negative integer into $W(e,x)$; "uniform" means the method is the same for all e and x. The notion of **uniformity** has applications, as in Corollary 4.1.

Theorem 4.1. *(Gandy Selection) There exists a partial E-recursive function $\phi(e,x)$ such that for all $e > \omega$ and all x:*
(i) $(\exists n < \omega)[\{e\}(n,x)\downarrow] \longrightarrow \phi(e,x)\downarrow$.
(ii) $\phi(e,x)\downarrow \longrightarrow [\phi(e,x) \in \omega \wedge \{e\}(\phi(e,x),x)\downarrow]$.

The proof of Gandy Selection requires a preparatory lemma.

Lemma 4.1. *(Moschovakis) Suppose $\{d\}(x)\downarrow$ or $\{e\}(y)\downarrow$. Then*

$$min(\mid \{d\}(x) \mid, \mid \{e\}(y) \mid)$$

is E-recursive uniformly in d, e, x, y.

Proof. By effective transfinite recursion on min. A rough approximation of the recursion equation is: $min\,(\mid u \mid, \mid v \mid) =$

$$max\{min(\mid a \mid, \mid b \mid) \mid a <_U u \wedge b <_U v\}$$

where u, v, a, b are computation instructions, and $a <_U b$ means a is an immediate subcomputation instruction of b. The above recursion is not as effective as it might be because if $u\uparrow$, then $\{a \mid a <_U u\}$ may not be E-recursive in u. But enough a's are explicit to make the recursion (slightly modified) effective, hence successful. □

Proof of Gandy Selection. For simplicity drop the "x" in the "$\{e\}(n,x)$" of Theorem 4.1. Kleene's fixed point theorem yields a partial recursive function $t(e,n)$ whose definition has two cases.

Case 1: $t(e,n+1)\downarrow$ and $\mid t(e,n+1) \mid \leq \mid \{e\}(n) \mid$. Then

$$t(e,n) \simeq t(e,n+1)+1.$$

Case 2: $\mid \{e\}(n) \mid < \mid t(e,n+1) \mid$. Then $t(e,n) = 0$.

The above split into cases is effective by Lemma 4.1 and Proposition 3.2. Assume $\{e\}(n)\downarrow$ for some n. Then $t(e,0)\downarrow$ and

$$\mid \{e\}(n) \mid < \mid t(e,n+1) \mid$$

for some n; let n_0 be the least such. Then $t(e,0) = n_0$.

Read $w \leq_E z$ as: w is ***E*-recursive in** z. And define it by:

$$\exists e[w = \{e\}(z)].$$

There is a $c < \omega$ and a recursive function h such that for all e, w, and z: $\{h(e)\}(w,z)\downarrow$ iff $[\{e\}(z)$ converges to $w]$ iff $\{c\}\{e,w,z)\downarrow$.

Gandy can select an e, if there is one, uniformly in w and z such that $\{c\}(e,z,w)\downarrow$. It follows that $w \leq_E z$ is an E-recursively enumerable predicate of w and z.

Corollary 4.1. *Suppose P(x,y) is E-recursively enumerable and*

$$\forall_{x\in z}\exists y[y \leq_E x \ \wedge \ P(x,y)]$$

Then there exists a partial E-recursive f such that

$$\forall_{x\in z}[f(x)\downarrow \ \wedge \ P(x,f(x))].$$

The concepts of reflection and divergence are closely linked in E-recursion theory. Define

$$\kappa_0^x = \sup\{\gamma \mid \gamma \leq_E x\}.$$

An ordinal δ is said to be x-reflecting if

$$[L(\delta, tc(\{x\})) \models \mathcal{F}] \rightarrow [L(\kappa_0^x, tc(\{x\})) \models \mathcal{F}]$$

for every Σ_1 sentence $\mathcal{F}$ whose only parameter is x. The predicate, δ is x-reflecting, is E-recursively enumerable in x.

If $\mathcal{F}$ reflects down to κ_0^x, then it reflects down to an ordinal E-recursive in x, because the least member of a nonempty set y of ordinals is E-recursive in x if $y \leq_E x$.

Define

$$\kappa_r^x = \text{ the greatest } x\text{-reflecting ordinal.}$$

Clearly $\kappa_0^x \leq \kappa_r^x$. Define $\kappa_r^{x,a}$ to be $\kappa_r^{<x,a>}$.

Proposition 4.1. $\kappa_r^{x,a} \leq \kappa^x$ *for all* $a \in tc(x)$.

Proof. Suppose not; $E(<x,a>) = E(x) = L(\kappa^x, tc(\{x\}))$. Then

$$E(<x,a>) \in L(\kappa_r^{x,a}, tc(\{x\})).$$

The latter reflects below $\kappa_0^{x,a}$, because there is a Π_3 sentence $\mathcal{F}$ such that for every transitive class A

$$A \text{ is } E\text{-closed} \longleftrightarrow <A,\in> \models \mathcal{F}.$$

Thus $E(<x,a>) \in L(\kappa_0^{x,a}, tc(\{x\})$. But then $\kappa^{<x,a>} < \kappa_o^{<x,a>}$. □

The situation of greatest interest is when $\kappa_0^x < \kappa_r^x < \kappa^\times$. Harrington in his study of $E(2^\omega)$ made a breakthrough when he proved: if $a \in 2^\omega$ and $\{e\}(a, 2^\omega)$ diverges, then some witness to the divergence appears at level $\kappa_r^{2^\omega,a}$ of $E(2^\omega)$. His result inspired

Lemma 4.2. *Assume some wellordering of $tc(x)$ is E-recursive in x. If $\{e\}(x)$ diverges, then some witness to the divergence is first order definable over $L(\kappa_r^x, tc(\{x\}))$.*

A witness to divergence is an infinite descending path in $T_{<e,x>}$. It has the form $\lambda n \mid t(n)$, where $t(0) =< e, x >$ and for each n: $t(n+1)$ is a subcomputation instruction of $t(n)$. In the proof of Lemma 4.2, $t(n)$ is defined by recursion on n and Lemma 4.3 is used to insure that each $t(n) \in L(\kappa_r^x, tc(\{x\}))$.

Lemma 4.3. *(Kechris's Basis Theorem) Suppose $y \leq_E x$ and A is E-recursively enumerable in x. If $y - A$ is nonempty, then*

$$\exists b[b \in y - A \;\wedge\; \kappa_r^{x,b} \leq \kappa_r^x.$$

Proof. It suffices to find a $b \in y - A$ such that $\kappa_0^{x,b} \leq \kappa_r^x$. Suppose no such b exists. Then

$$y \subseteq A \cup \{b \mid \kappa_r^x < \kappa_0^{x,b}\}.$$

The predicate, $\kappa_r^x < \kappa_0^{x,b}$, is E-recursively enumerable in x, because it is equivalent to

$$\exists\delta[\delta \leq_E x, b \;\wedge\; \delta \text{ is not } x\text{-reflecting}].$$

(The predicate, δ is not x-reflecting, is E-recursive in x.) Thus y is a subset of a set E-recursively enumerable in x, and so for each $b \in y$, there is a δ_b such that

$$\delta_b \leq_E x, b$$

and (i) or (ii) holds:

(i) δ_b is the length of a computation that shows $y \in A$;
(ii) δ_b is not x-reflecting.

It follows from Gandy selection, as in Corollary 4.1, that δ_b can be construed as a partial E-recursive function of x, b defined for all $b \in y$. Let

$$\delta^\infty = \sup\{\delta_b \mid b \in y\}.$$

Then $\delta^\infty \leq_E x$ by E-recursive bounding, scheme (4) of Section 1, and so $\delta^\infty < \kappa_r^x$. But then (i) holds for every $b \in y$; hence $y - A$ is empty. □

The Kechris Basis Theorem is analogous to one of Gandy's basis theorems: if D is a nonempty Σ_1^1 (with real parameter x) set of reals, then $\exists b \in D$ such that $\omega_1^{x,b} = \omega_1^x$. In both an element b of a "co-recursively enumerable" set is found such that some ordinal generated by b is minimized. The key step in Kechris's proof is showing (ii) cannot happen (i.e. $\delta_b > \kappa_r^x$) if the desired b does not exist. The corresponding step in Gandy's proof is showing $\omega_1^{x,b} > \omega_1^x$ cannot happen if the desired b does not exist. Analogies can be misleading, but in this case it is reasonable to say that aspects of ω_1^x are shared by κ_0^x together with κ_r^x.

Theorem 4.2. *Let x be a set of ordinals. Then (i)$\longleftrightarrow$(ii).*
(i) $E(x)$ is not Σ_1 admissible.
(ii) $E(x)$ admits divergence witnesses.

The proof of (ii)$\longrightarrow$(i) is similar to that of Proposition 3.5.

The proof of $\neg$(ii)$\longrightarrow$ $\neg$(i) is more difficult. For some κ, $E(x) = L(x, \kappa)$. Let (I) be the sentence

$$\exists y_{y \in L(\kappa,x)}[\kappa_r^{x,y} \geq \kappa].$$

Proof of (I)$\longrightarrow$ $\neg$(i). Suppose $y \in L(\kappa, x)$ and $\kappa_r^{x,y} \geq \kappa$. Assume

$$L(\kappa, x) \models \forall u_{u \in d} \exists v \mathcal{F}(u, v, p),$$

$d, p \in L(\kappa, x)$ and $\mathcal{F}$ is Δ_0. It can be shown that

$$\kappa_r^{x,y,p,b} \geq \kappa_r^{x,y}$$

by a painstaking examination of the details of the proof of Lemma 4.2; in particular there is a preferred divergence witness, namely the "leftmost" infinite branch of $T_{<e.x>}$. By reflection

$$\forall b_{bed} \exists c[c \leq_E x, y, p, b \wedge \ \mathcal{F}(b, c, p)].$$

By Corollary 4.1 c can be construed as a partial E-recursive function of x, y, p, b defined for all $b \in d$. Then E-recursive bounding yields

$$L(\kappa, x) \models \exists v \forall u_{u \in d} \mathcal{F}(u, v, p).$$

Proof of $\neg$(ii)$\longrightarrow$(I). Assume $\neg$(I) and obtain (ii) via Lemma 4.2.

Let $A \subseteq L(\kappa)$. Define A is ***E*-recursively enumerable on** $L(\kappa)$ by

$$\exists e \exists b_{b \in L(\kappa)} \forall y_{y e L(\kappa)}[y \in A \longleftrightarrow \{e\}(b, y) \downarrow].$$

Define A is ***E*-recursive on** $L(\kappa)$ by $\exists e \exists b_{b \in L(\kappa)}$ such that

$$\forall y_{y \in L(\kappa)}[\{e\}(b, y) \downarrow \ \wedge \ (y \in A \longleftrightarrow \{e\}(b, y) = 1)].$$

The interaction above between Σ_1 admissibility and divergence leads to

The Divergence-Admissibility Split. Every E-recursively closed $L(\kappa)$ belongs to **just one** of two classes.

Class I: $L(\kappa)$ admits divergence witnesses.

Class II: $L(\kappa)$ is Σ_1 admissible; and for all $A \subseteq L(\kappa)$, A is Σ_1 definable over $L(\kappa)$ iff A is E-recursively enumerable on $L(\kappa)$.

5. Finite Injury Arguments and Post's Problem

The title of this section is misleading. In the setting of E-recursion a classical finite injury argument (or an α-finite injury argument) becomes a wait-and-see argument. The standard approach to Post's Problem seeks to preserve inequalities. Negative requirements arise and are violated for the sake of positive requirements of higher priority. In E-recursion inequalities are still welcome but divergence witnesses are also sought. With their assistance injuries can be avoided when $L(\kappa)$ is E-closed but not Σ_1 admissible.

Let $\mathcal{E}$ be E-closed. Suppose $B \subseteq \mathcal{E}$. The relativisation of E-recursiveness to B is simply a matter of adding a seventh scheme

$$\{c\}^B(x_1, ...x_n) = B \cap x_i \qquad (c =< 7, n, i >)$$

to the original six for E-recursion. The additional scheme has the same effect as adding the characteristic function of B to the list of finitary functions (projection, difference etc.). Say f is partial E-recursive relative to B if $f \simeq \{e\}^B$ for some e. And $\mathcal{E}$ is E-closed relative to B iff

$$\{e\}^B(x) \downarrow \longrightarrow \{e\}^B(x) \in \mathcal{E}$$

for all $e < \omega$ and $x \in \mathcal{E}$.

Suppose $L(\kappa)$ is E-closed, but not Σ_1 admissible, in order to guarantee that $L(\kappa)$ admits divergence witnesses. A subset of κ is E-recursively enumerable on $L(\kappa)$ iff it equals

$$\{x \mid x < \kappa \wedge \{e)(x, u) \downarrow\}$$

for some $e < \omega$ and $u < \kappa$; to solve Post's problem, two such subsets, A and B, are constructed. As usual there is a list of requirements. Each requirement is settled before proceeding to the next. In the following A and B ambiguously denote sets and characteristic functions. Let requirement 0 be

$$A(w_0^0) \neq \{e_0\}^B(w_0^0, u_0) \quad (w_0^0, u_0 \in \kappa)$$

At stage 0, $B = \emptyset$. Enumerate all E-recursive computations in $L(\kappa)$.At some stage $\sigma_0 < \kappa$ of the enumeration either (i) or (ii) will happen.

(i) A computation appears that defines a value v for $\{e_0\}^{\emptyset}(w_0^0, u_0)$.The computation is based on positive and negative membership facts that B must now satisfy forever. $A(w_0)$ is set equal to 1 if $v = 0$, and to 0 otherwise. All the witnesses w associated with remaining requirements of the form $B(w) \neq \{e\}^A(w, u)$ are given values large enough to insure they will not injure the negative commitments made for B in requirement 0.

(ii) Computations appear, as in the proof of Lemma 4.2, that establish a divergence witness for $\{e\}^{\emptyset}(w_0, u_0)$.The computations are based on positive and negative membership facts that B must now satisfy forever. And some witness values must be increased as in case (i).

Note that $\sigma_0 \leq \kappa_r^{w_0^0, u_0}$ by Lemma 4.2. Requirements can be indexed by triples of the form $< e_\delta, w_\delta^{i^\delta}, u_\delta >$, where $\delta < \kappa$, $e_\delta < \omega$, and $i^\delta \in \{0, 1\}$. With this long indexing, time may run out before all the work is done. There is enough time if

$$\sup\{\kappa_r^{w_\delta^{i^\delta}, u_\delta} \mid \delta < \gamma\} < \kappa$$

for all $\gamma < \kappa$. If not, a shorter indexing of requirements is needed. Define ρ^κ to be the least $\beta \leq \kappa$ such that some f is a *partial* E-recursive on $L(\kappa)$ map of β onto $L(\kappa)$.

Slaman [8] proved splitting and density theorems for sets E-recursively enumerable on $L(\kappa)$ by means of κ-finite injury arguments.

Theorem 5.1. *If $\gamma < \rho^\kappa$ and $p < \kappa$, then* $\sup\{\kappa_r^{\delta,p} \mid \delta < \gamma\} < \kappa$.

The proof of Theorem 5.1 is a collapsing (or condensation) argument as in the fine structure of L. A device rightly known as Shore Blocking together with the above theorem buys enough time to satisfy all the requirements.

6. Forcing: C.C.C. Versus Countably Closed

Suppose $\mathcal{P} \in L(\kappa)$ is a notion of set forcing. (A $\mathcal{P}$-generic G is a subset of some member of $L(\kappa)$.) If $L(\kappa)$ is Σ_1 admissible and G is $\mathcal{P}-$ *generic*, then $L(\kappa, G)$ is Σ_1 admissible. In short, set forcing preserves Σ_1 admissibility.

Now suppose $L(\kappa)$ is not Σ_1 admissible. If $L(\kappa)$ is E-closed and G is $\mathcal{P} - generic$,then $L(\kappa, G)$ need not be E-closed. Set forcing does not in general preserve E-closure.

For example consider $E(\omega_1)$ ($= L(\kappa)$ for some κ). Let G be a generic map of ω onto ω_1. Then

$$L(\kappa, G) = L(\kappa, b)$$

for some $b \subseteq \omega$. Suppose $L(\kappa, G)$ is E-closed. Then $L(\kappa, b)$ is $E(b)$.By Proposition 3.6, $E(b)$ is Σ_1 admissible and consequently $L(\kappa)$ is Σ_1 admissible. But Proposition 3.5 says $L(\kappa)$ is not Σ_1 admissible.

An instance of set forcing $\mathcal{P}$ consists of a set P of forcing conditions p, q, r, ..., and an extension relation $\geq$. If $p \geq q$, then q says as much as, or more than, p says about the generic object. Assume for the rest of this Section that $L(\kappa)$ is E-closed but not Σ_1 admissible.

For an arbitrary (not necessarily generic) G, each element of $L(\kappa, G)$ is the value of $\{e\}(a, G)$ for some $a \in L(\kappa)$ and $e < \omega$.To show that $L(\kappa,\ G)$ is E-closed for a $\mathcal{P}$-generic G, there are two approaches.

(1) For all $e < \omega$ and $a \in L(\kappa)$, try to force $\mid \{e\}(a, G) \mid$ to be as small as possible in the hope of forcing a value less than κ. This approach succeeds when $\mathcal{P}$ is c.c.c. (**countable chain condition**); c.c.c means that every antichain is countable (any two distinct elements of an antichain have no common extension).

(2) For all $e < \omega$ and $a \in L(\kappa)$, again try to force $\mid \{e\}(a, G) \mid$ to be as small as possible, but allow for the possibility of failure and exploit that possibility to try and force a divergence witness for $\{e\}(a, G)$ into $L(\kappa, G)$. This approach succeeds when $\mathcal{P}$ is countably closed:

$$\forall n(p_n \geq p_{n+1}) \rightarrow \exists q \forall n(p_n \geq q).$$

Success is plausible in this case because a divergence witness is an infinite path through an illfounded computation tree with countably many levels. Reflection plays a major role in forcing the existence of divergence witnesses with the help of Lemma 4.2 and Kechris's Basis Theorem (Lemma 4.3).

The first approach is conceptually simpler than the second, but more combinatoric. Suppose there exist r and δ such that

$$p \geq r \text{ and } r \Vdash (\mid \{e\}(a, G) \mid = \delta).$$

Then it can be shown that there exist such r and δ E-recursive in p, e, a (and some background parameters such as ω and $\mathcal{P}$). Define

$$\min(p, e, a) \simeq \min_{\delta} \exists r_{p \geq r}(r \Vdash (\mid \{e\}(a, G) \mid = \delta)).$$

An effective transfinite recursion on $\min(p, e, a)$ shows $\min(p, e, a)$ is E-recursive in p, e, a uniformly. The recursion manipulates conditions directly and draws heavily on Gandy selection and the countable chain condition.

7. Model-Theoretic Completeness and Compactness

Suppose $L(\kappa)$ is E-closed but not Σ_1 admissible. Let $\mathcal{L} \subseteq L(\kappa)$ denote an E-recursive on $L(\kappa)$ set of atomic symbols for a first order language, and let $\mathcal{L}_{\kappa,\omega}$ be the restriction of $\mathcal{L}_{\infty,\omega}$ to $L(\kappa)$. The rules and axiom schemes of $\mathcal{L}_{\infty,\omega}$ are in essence the same as those of infinitary logic with one notable addition: a set containing a deduction of $\mathcal{F}_i$ for each $i \in I$ qualifies as a deduction of the conjunction

$$\wedge\{\mathcal{F}_i \mid i \in I\}.$$

If $L(\kappa)$ is not a union of Σ_1 admissible sets, then there a choice for $\mathcal{L}$ such that some sentence of $\mathcal{L}_{\kappa,\omega}$ has a proof in $\mathcal{L}_{\infty,\omega}$ but not in $\mathcal{L}_{\kappa,\omega}$. Let $\Delta \subseteq \mathcal{L}_{\kappa,\omega}$ denote an E -recursive on $L(\kappa)$ set of sentences throughout the present Section. There exist $\mathcal{L}$ and Δ such that every κ-finite subset of Δ is consistent in the sense of $\mathcal{L}_{\infty,\omega}$ but Δ is not. (A set is κ-finite iff it belongs to $L(\kappa)$.) The lack of Σ_1 admissibility makes life on $L(\kappa)$ difficult for a model theorist But there is hope because some forcing arguments (Section 6) succeed on $L(\kappa)$ despite the lack of Σ_1 admissibility.

Let $\mathcal{F}$ denote a sentence of $\mathcal{L}$; $\mathcal{F}$ is said to be a **logical consequence** of Δ (in symbols $\Delta \vdash \mathcal{F}$) iff $\mathcal{F}$ is deducible from Δ via the axioms and rules of $\mathcal{L}_{\infty,\omega}$.

Define $\Delta \vdash_\kappa \mathcal{F}$ to mean $\Delta \vdash \mathcal{F}$ via a deduction in $L(\kappa)$.

Define $\Delta \vdash^E_\kappa \mathcal{F}$ to mean $\Delta \vdash \mathcal{F}$ via a deduction E-recursive in $\mathcal{F}$.

Say Δ is **κ-consistent** iff no contradiction is deducible from Δ via a deduction in $L(\kappa)$.

Say Δ **admits effectivization of deductions** iff for every sentence $\mathcal{F} \in \mathcal{L}$,

$$\Delta \vdash_\kappa \mathcal{F} \quad \to \quad \Delta \vdash^E_\kappa \mathcal{F}\ .$$

Let $\vee\{\mathcal{F}_i \mid i \in I\}$ be a typical disjunction of $\mathcal{L}_{\kappa,\omega}$; "typical" means $I \in L(\kappa)$ and $\mathcal{F}_i \leq_E i, I$ uniformly in i.

Proposition 7.1. *Suppose Δ admits effectivization of deductions and*

$$\Delta, \vee\{\mathcal{F}_i \mid i \in I\} \textit{ is } \kappa\textit{-consistent.}$$

then $\Delta, \mathcal{F}_i$ is κ-consistent for some $i \in I$.

Proof. Suppose not. Then $\Delta \vdash^E_\kappa \neg\mathcal{F}_i$ for all $i \in I$. Thus for each $i \in I$, there is an e such that

$$\{e)(i, I) \text{ converges to a deduction of } \neg\mathcal{F}_i \text{ from } \Delta.$$

For each i, Gandy selection makes it possible to find such an e effectively; Gandy's method is uniform in i, hence there is one e, call it e_0, that works for all i. (The existence of e_0 via Gandy also needs the fact that the class of all deductions from Δ in $L(\kappa)$ is E-recursively enumerable on $L(\kappa)$; that fact follows from the assumption that Δ is E-recursive on $L(\kappa)$) The E-recursive bounding scheme implies

$$\{\{e_0\}(i, I) \mid i \in I\} \in L(\kappa).$$

But then $\Delta \vdash_\kappa \neg \vee \{\mathcal{F}_i \mid i \in I\}$. □

Lemma 7.1. *Suppose κ is countable, Δ is κ-consistent, and $L(\kappa)$ admits effectivization of deductions. Then Δ has a model.*

The proof of Lemma 7.1 is a standard model theoretic construction with ω steps. Each step adds one sentence and preserves the κ-consistency of the previous step. Proposition 7.1 plays a essential part in the construction.

Lemma 7.2. *If Δ is κ-consistent and admits effectivization of deductions, then Δ is consistent in the sense of $\mathcal{L}_{\infty,\omega}$.*

If Δ is countable, then Lemma 7.2 follows from Lemma 7.1. If Lemma 7.1 failed for some uncountable Δ, then by absoluteness it would fail for some countable Δ.

The above formulation of model theory on a E-closed, but not Σ_1 admissible, $L(\kappa)$ is provisional until some questions are answered.

(Q1) Are there some non-trivial examples of κ-consistent Δ's that admit effectivization of deductions?

(Q2) Can some established results such as those in [7], be obtained from Lemma 7.1?

(Q3) Does some form of type omitting make sense in the above formulation?

A partial affirmative answer to (Q1) can be extracted from Section 6. Let $\mathcal{P} \in L(\kappa)$ be a *c.c.c* set forcing relation There exists a κ-consistent Δ that captures the essential properties of set forcing with $\mathcal{P}$ and that admits effectivization of deductions.

References

[1] Harrington, L. (1973), *Contributions to recursion theory in higher types,* Ph.D. Thesis, MIT, Cambridge, Massachusetts,[2]

[2] Moschovakis, Y. N. (1967), *Hyperanalytic predicates,* Trans. Amer. Math. Soc., 129, 249-282.

[3] ———————— (1980), *Descriptive Set Theory,* North-Holland, Amsterdam.

[4] Normann, D. (1978), *Set recursion*, in Generalized Recursion Theory II, North-Holland, Amsterdam, 303-320.

[5] Sacks, G. E. (1971), *The k-section of a type n-object*, Amer. J. Math. 99, 901-917.

[6] ————— (1990), *Higher Recursion Theory,* Springer-Verlag, Berlin.

[7] Slaman, T. A. (1985), *Reflection and forcing in E-recursion theory,* Ann. Pure and Appl. Log. 29, 79-106.

[8] ————— (1985), *The E-recursively enumerable degrees are dense,* in Proceedings of Symposia in Pure Math., Amer. Math. Soc. 42, 195-213.

Harvard University
E-mail address: `Sacks@math.harvard.edu`

[2]There is more to E-recursion theory than meets the eye above. To see more consult the References.