BOUNDS ON WEAK SCATTERING

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In Memory of Jon Barwise

Contents

1. Introduction

This paper has two themes less disparate than they seem at first reading:

Extending classical descriptive set theoretic results that impose bounds on suitably defined functions from ω^ω into ω_1.

Extending and clarifying some early results on Scott ranks of countable structures sketched in [12][1].

Let F be a function, possibly partial, from ω^ω into ω_1. A typical *classical bounding* theorem says the range of F is bounded by a countable ordinal if the graph of F has a suitable definition. For example, the graph of F is boldface Σ_1^1 with real parameter p; in this formulation the graph of F is viewed as a subset of $\omega^\omega \times \omega_1$ by requiring each value of F to be a well ordering of ω. Let $F(X)$ ambiguously denote the well ordering and also the

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[1][12] was a hasty writeup of a talk given at the 1971 meeting of the International Congress of Logic, Methodology and Philosophy of Science. Some details absent from [12] but needed here are presented below..

ordinal represented by the well ordering. For each X, $F(X)$ is the unique solution of a Σ^1_1 formula with parameters p, X. Consequently $F(X)$ (the well ordering) is hyperarithmetic in p, X, and so

$$F(X) < \omega_1^{p,X}, \tag{1.1}$$

the least ordinal not recursive in p, X. The effective version of the theorem says that the bound on the range of F is an ordinal less than ω_1^p.

A recursion-theoretic approach to the effective bound originated by Kleene is as follows. (See Sacks[13] for details.) Suppose

$$(\forall\gamma < \omega_1^p)(\exists X)[F(X) > \gamma]. \tag{1.2}$$

Let R_e^p be the linear ordering of ω recursive in p via index e. Define W^p to be the set of all e such that R_e^p is a well ordering. Then

$$e \in W^p \longleftrightarrow \exists X \exists g[g \textit{ is a } 1-1 \textit{ order preserving map of } R_e^p \textit{ into } F(X)]. \tag{1.3}$$

But then W^p is boldface Σ^1_1 with parameter p. This last is false according to a Kleene hierarchy result that says W^p is universal boldface Π^1_1 with parameter p, hence not boldface Σ^1_1 with parameter p.

A model theoretic approach to effective bounds is the path taken in this paper. A sketch may help to clarify later sections. Let $A(p)$ be the least Σ_1 admissible set with p as a member. Let Z be a $\Sigma_1^{A(p)}$ definable set of sentences of $\mathcal{L}_{\omega_1,\omega}$ coded by elements of $A(p)$ such that every model M of Z has the following properties:

(1) The ordinals recursive in p form a proper initial segment of the ordinals in the sense of M.
(2) There is an $X_0 \in M$ such that for all $\gamma < \omega_1^p$, $F(X_0) > \gamma$.
(3) $p \in M$ and M is a Σ_1 admissible structure.

Assume the range of F is not bounded by an ordinal below ω_1^p. Then each $A(p)$-finite subset of Z (i.e. each subset of Z that is a member of $A(p)$) is consistent, and so Z has a model by Barwise Compactness. With the addition of "effective" type omitting, as in Grilliot[2] or Keisler[4], Z has a model M that omits ω_1^p, but has non-standard ordinals greater than all standard ordinals less than ω_1^p. Then

$$\omega_1^{p,X_0} \leq \omega_1^p, \tag{1.4}$$

otherwise ω_1^p is recursive in $\langle p, X_0\rangle$ and so $\omega_1^p \in M$. But then $\omega_1^{p,X_0} = \omega_1^p$ and $F(X_0) \geq \omega_1^{p,X_0}$ by property (2) of Z, which contradicts (1.1).

The search for a bounding theorem that extends the classical result seems hopeless at first. An extension has to talk about an F that allows $F(X) \geq \omega_1^{X,p}$, but $\omega_1^{X,p}$, as a function of X, is unbounded. Model theory comes to the rescue. Every countable structure $\mathcal{A}$ has a Scott rank[14], sr($\mathcal{A}$), an ordinal that can be as high as $\omega_1^{\mathcal{A}} + 1$ (see Section 2 for elaboration).

Let T be a countable theory. A reasonable starting assumption on T is

$$\forall\mathcal{A}[\mathcal{A} \models T \longrightarrow sr(\mathcal{A}) \leq \omega_1^{\mathcal{A}}]. \tag{1.5}$$

An ingenious example (MA) devised by Makkai[7] shows that (1.5) is not enough. Examination of (MA) and its illuminative extensions in Knight & Young[5] leads to two further assumptions on T. The first, *effective k-splitting*, is technical and perhaps peripheral and is discussed further in sections 9 and 10. The second, *weakly scattered*, is central. The theory T_M associated with (MA) satisfies (1.5) and has properties similar to effective k-splitting. In addition for every Σ_1 admissible countable α, T_M has a model $\mathcal{A}$ such that

$$\omega_1^{\mathcal{A}} = \alpha = sr(\mathcal{A}). \tag{1.6}$$

Corollary 9.2 says: if T is weakly scattered, satisfies (1.5), and has effective k-splitting, then there is a countable bound on the Scott ranks of the countable models of T; the effective version provides a bound less than the first Σ_2 admissible ordinal relative to T in contrast to the classical case above, where the effective bound on the range of F is less than ω_1^p, the first Σ_1 admissible ordinal relative to p.

The notion of weakly scattered is inspired by Morley's concept of scattered. Let $\mathcal{L}$ be a countable first order language, $\mathcal{L}_0$ a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and $T \subseteq \mathcal{L}_0$ a theory (i.e. a set of sentences) with a model. For (a) and (b) below, let $\mathcal{L}'$ be any countable fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_0$, and T' any finitarily consistent, ω-complete theory contained in $\mathcal{L}'$ and extending T. (The notions of finitary consistency and ω-completeness for fragments are reviewed at the beginning of Section 4.) T is said to be **scattered** iff (a) and (b) hold.

(a) For all $n > 0$ and all T', S_nT', the set of all n-types over T', is countable.

(b) For all $\mathcal{L}'$, the set $\{T' \mid T' \subseteq \mathcal{L}'\}$is countable.

The above definition of scattered is equivalent to the one in Morley's ground breaking [10]. The theory T is said to be **weakly scattered** iff (a) holds. By [10] a scattered theory can have at most ω_1 many countable models. In contrast a weakly scattered theory can have 2^ω many countable models.

Robin Knight[6] has announced a counterexample to Vaught's Conjecture (VC), a scattered first order theory with ω_1 many countable models. VC has a precise formulation in Section 5.

In [12] the following bounding result was established: if T is scattered and satisfies (1.5), then T has only countably many countable models; furthermore every countable model of T has a countable copy in $L(\beta, T)$ for some $\beta < \sigma_2^T$, the least α such that $L(\alpha, T)$ is Σ_2 admissible. Hence Vaught's conjecture holds for T if T satisfies (1.5). The proofs given in [12] were somewhat sketchy, so missing details needed in later sections of this paper are given in Sections 3 through 5. If Vaught's Conjecture is false, then results for scattered theories yield information about models of counterexamples to VC. Theorem 4.9(vii) says: if VC fails for T, then T has a model of cardinality ω_1 not elementarily equivalent in the sense of $\mathcal{L}_{\omega_1,\omega}$ to any countable model (Harnik & Makkai[3]). Theorem 5.3 describes an ω_1-sequence of atomic and saturated models that every counterexample must possess. Section 5 includes a related absoluteness result implicit in Morley[10]: VC(T), Vaught's Conjecture for T, is a $\Sigma_1^{L(\omega_1^{L(T)},T)}$ predicate of T, hence Σ^1_2.

Steel[15], as reported in [7], used an assumption stronger than (1.5) to prove VC(T). In Section 2 an arbitrary countable structure $\mathcal{A}$ is associated with a theory $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ contained in a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ canonically generated from $\mathcal{A}$. By an argument of Nadel[11], $\mathcal{A}$ is a homogeneous model of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$. Steel's assumption, is equivalent to: for every $\mathcal{A}$ a model of T, $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ is ω-categorical. Assumption (1.5) is equivalent to: for every $\mathcal{A}$ a model of T, $\mathcal{A}$ is an atomic model of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$. Sacks & Young[9] produced a structure $\mathcal{A}$ such that $\mathcal{A}$ is an atomic model of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$, but $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ is not ω-categorical. (In addition $\omega_1^{\mathcal{A}} = \omega_1^{CK}$ and $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ is a Δ_1 subset of $L(\omega_1^{CK})$.)

Sections 7 through 9 are devoted to bounding for weakly scattered theories.

2. Scott Analysis and Rank

This section revisits [12] as promised in Section 1. Scott[14] showed that an arbitrary countable structure $\mathcal{A}$ with underlying first order language $\mathcal{L}$ can be characterized up to isomorphism by a single sentence of $\mathcal{L}_{\omega_1,\omega}$. In essence there is a countable fragment $\mathcal{L}^{\mathcal{A}}$ of $\mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{A}$ is the atomic model of $T^{\mathcal{A}}$, the complete theory of $\mathcal{A}$ in $\mathcal{L}^{\mathcal{A}}$. Nadel[11] pointed the way to a canonical choice for $\mathcal{L}^{\mathcal{A}}$.

The admissible set $L(\omega_1^{\mathcal{A}}, \mathcal{A})$ is Gödel's L relativised to $\mathcal{A}$ as an element[2], and chopped off at $\omega_1^{\mathcal{A}}$, the least γ such that $L(\gamma, \mathcal{A})$ is Σ_1 admissible. Let

$$\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega} = \mathcal{L}_{\omega_1,\omega} \cap L(\omega_1^{\mathcal{A}}, \mathcal{A}). \tag{2.1}$$

Nadel[11] showed that:

$$\textit{$\mathcal{A}$ is a homogeneous model of its complete theory } T^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega} \textit{ in } \mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}. \tag{2.2}$$

It follows that $\mathcal{A}$ is the atomic model of its complete theory in

$$\mathcal{L}_{\omega_1,\omega} \cap L(\omega_1^{\mathcal{A}} + 1, \mathcal{A}), \tag{2.3}$$

since the types over $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ realized in $\mathcal{A}$ are first order definable over $L(\omega_1^{\mathcal{A}}, \mathcal{A})$ and so become atoms of the complete theory of $\mathcal{A}$ contained in (2.3).

A Σ_1 recursion defines a canonical choice for $\mathcal{L}^{\mathcal{A}}$ and yields the definition of Scott rank for $\mathcal{A}$.

(1) $\mathcal{L}^{\mathcal{A}}_0 = \mathcal{L}$.
(2) $\mathcal{L}^{\mathcal{A}}_\lambda = \cup\{\mathcal{L}^{\mathcal{A}}_\delta \mid \delta < \lambda\}$ for limit λ.
(3) $T^{\mathcal{A}}_\delta =$ complete theory of $\mathcal{A}$ in $\mathcal{L}^{\mathcal{A}}_\delta$.

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[2] Strictly speaking, the relativisation is to the transitive closure of $\mathcal{A}$.

(4) $\mathcal{L}^{\mathcal{A}}_{\delta+1}$ = least fragment $\mathcal{L}^{+}$ of $\mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{L}^{+} \supseteq \mathcal{L}^{\mathcal{A}}_{\delta}$, and for each $n > 0$, if $p(\overrightarrow{x})$ is a non-principal n-type *of* $T^{\mathcal{A}}_{\delta}$ realized in $\mathcal{A}$, then the conjunction

$$\wedge\{\mathcal{F}(\overrightarrow{x}) \mid \mathcal{F}(\overrightarrow{x}) \in p(\overrightarrow{x})\}$$

is a member of $\mathcal{L}^{+}$.

Note that if $\mathcal{A}$ is isomorphic to $\mathcal{B}$, then $\mathcal{L}^{\mathcal{A}}_{\delta} = \mathcal{L}^{\mathcal{B}}_{\delta}$ and $T^{\mathcal{A}}_{\delta} = T^{\mathcal{B}}_{\delta}$ for all δ. For some $\delta < \omega_1$, all the $n-types$ of $T^{\mathcal{A}}_{\delta}$ realized in $\mathcal{A}$ are principal. To see this, fix γ and suppose some non-principal type $p_{\gamma+1}$ of $T^{\mathcal{A}}_{\gamma+1}$ is realized in $\mathcal{A}$. Let p_{γ} be the restriction of $p_{\gamma+1}$ to $T^{\mathcal{A}}_{\gamma}$. Since $p_{\gamma+1}$ is non-principal, there is a formula $\mathcal{G}(\overrightarrow{x})$ of $\mathcal{L}^{\mathcal{A}}_{\gamma+1}$ such that both

$$\exists \overrightarrow{x}[p_{\gamma}(\overrightarrow{x}) \wedge \mathcal{G}(\overrightarrow{x})] \text{ and } \exists \overrightarrow{x}[p_{\gamma}(\overrightarrow{x}) \wedge \neg\mathcal{G}(\overrightarrow{x})]$$

belong to $T^{\mathcal{A}}_{\gamma+1}$. Then there are $n-tuples$ $\overrightarrow{b}$ and $\overrightarrow{c}$ of $\mathcal{A}$ such that

$$\mathcal{A} \models [p_{\gamma}(\overrightarrow{b}) \wedge \mathcal{G}(\overrightarrow{b})], \text{ and } \mathcal{A} \models [p_{\gamma}(\overrightarrow{c}) \wedge \neg\mathcal{G}(\overrightarrow{c})].$$

Thus a distinction between $\overrightarrow{b}$ and $\overrightarrow{c}$ is made by a formula of $\mathcal{L}^{\mathcal{A}}_{\gamma+1}$ but not by any formula of $\mathcal{L}^{\mathcal{A}}_{\gamma}$. Since $\mathcal{A}$ is countable, only countably many distinctions can be made.

Let $d_{\mathcal{A}}$ be the the least $\delta < \omega_1$ such that every distinction ever made is made by a formula of $\mathcal{L}^{\mathcal{A}}_{\delta}$. Then

$$\mathcal{A} \textit{ is the atomic model of } T^{\mathcal{A}}_{d_{\mathcal{A}}+1}. \tag{2.4}$$

The **Scott Rank** of $\mathcal{A}$ is defined by

$$sr(\mathcal{A}) = \textit{least } \alpha[\mathcal{A} \textit{ is the atomic model of } T^{\mathcal{A}}_{\delta}]. \tag{2.5}$$

If $\mathcal{A}$ is isomorphic to $\mathcal{B}$, then $sr(\mathcal{A}) = sr(\mathcal{B})$. Nadel's proof of (2.2)(pg. 273 of [11]), sketched below, also shows

$$\mathcal{A} \textit{ is a homogeneous model of } T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}. \tag{2.6}$$

Hence $d_{\mathcal{A}} \leq \omega_1^{\mathcal{A}}$, and so

$$sr(\mathcal{A}) \leq \omega_1^{\mathcal{A}} + 1. \tag{2.7}$$

Note that $\mathcal{L}^{\mathcal{A}}_{\delta}$ and $T^{\mathcal{A}}_{\delta}$, as functions of $\delta < \omega_1^{\mathcal{A}}$, are $\Sigma_1^{L(\omega_1^{\mathcal{A}},\mathcal{A})}$, i.e. their graphs are Σ_1 definable subsets of $L(\omega_1^{\mathcal{A}}, \mathcal{A})$. Since the formulas of $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ and $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ are "enumerated" in increasing order of complexity,

$$\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}}} \text{ and } T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}} \text{ are } \Delta_1^{L(\omega_1^{\mathcal{A}},\mathcal{A})}. \tag{2.8}$$

To prove (2.6), let $p(\overrightarrow{x})$ be an $n-type$, and $q(\overrightarrow{x}, y)$ an $(n+1)-type$, of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$, and $\overrightarrow{a}$, $\overrightarrow{b}$ $n-tuples$ of $\mathcal{A}$. Suppose $p(\overrightarrow{x}) \subseteq q(\overrightarrow{x}, y)$ and

$$\mathcal{A} \models [p(\overrightarrow{a}) \wedge p(\overrightarrow{b}) \wedge \exists y q(\overrightarrow{a}, y)]. \tag{2.9}$$

For homogeneity, a $d \in \mathcal{A}$ is required so that $\mathcal{A} \models q(\overrightarrow{b}, d)$. Suppose no such d exists. Let $q_\delta(x, y)$ be the restriction of $q(x, y)$ to $\mathcal{L}_\delta^{\mathcal{A}}$. Then

$$\{q_\delta(x, y) \mid \delta < \omega_1^{\mathcal{A}}\} \text{ is } \Sigma_1^{L(\omega_1^{\mathcal{A}}, \mathcal{A})}. \tag{2.10}$$

For each $d \in \mathcal{A}$, there is a $\delta < \omega_1^{\mathcal{A}}$ such that $\mathcal{A} \models \neg q_\delta(\overrightarrow{b}, d)$. Since δ can be defined as a $\Sigma_1^{L(\omega_1^{\mathcal{A}}, \mathcal{A})}$ function of d, the Σ_1 admissibility of $L(\omega_1^{\mathcal{A}}, \mathcal{A})$ implies there is a $\delta_\infty < \omega_1^{\mathcal{A}}$ such that $\mathcal{A} \models \forall y \neg q_{\delta_\infty}(\overrightarrow{b}, y)$. But then

$$\mathcal{A} \models \forall y \neg q(\overrightarrow{a}, y). \tag{2.11}$$

A typical use of Scott rank in conjunction with Barwise Compactness and Grilliot type omitting is as follows.

Proposition 2.1. *Suppose $L(\alpha, T)$ is countable and Σ_1 admissible. If for each $\beta < \alpha$, T has a model of Scott rank $\geq \beta$, then T has a countable model such that.*

$$sr(\mathcal{A}) \geq \omega_1^{T,\mathcal{A}} = \alpha. \tag{2.12}$$

Note that the $\mathcal{A}$ of (2.12) must have Scott rank either α or $\alpha + 1$ by (2.7). Forcing the outcome to be $\alpha + 1$ is a problem addressed in this paper but far from resolved.

3. Small Δ_0^{ZF} Sets

The following is one of many variations (e.g. Makkai[8]) on a theme initiated by Barwise[1], an extension of a recursion theoretic fact needed for the enumeration of models of both scattered and weakly scattered theories. The variation below was mentioned and used in [12]. The recursion theoretic fact is: if a set S of reals is Σ_1^1 and has cardinality less that 2^ω, then there exists a hyperarithmetic real H such that every member of S is Turing reducible to H; in addition an index for H can be computed uniformly from an index for S. The latter uniformity is key to establishing the Σ_1 character of the enumeration of models in Sections 4 and 8. Recall that a Δ_0^{ZF} formula is a formula in the language of set theory with only bounded quantifiers "$(\forall x \epsilon y)$" and "$(\exists u \epsilon v)$". Let $D(x, y)$ be a Δ_0^{ZF} lightface formula, and A a countable Σ_1 admissible set. Suppose $p, b \in A$. Define

$$S_{p,b} = \{x \mid x \epsilon V \wedge x \subseteq b \wedge D(x, p)\} \tag{3.1}$$

Theorem 3.1. *If $S_{p,b} \notin A$, then the cardinality of $S_{p,b}$ is 2^ω.*

Proof. Let the language $\mathcal{L}$ consist of: $\in$, bounded quantifiers $\forall x \in y$ and $\exists x \in y$, an individual constant $\underline{e}$ for each $e \in A$, and a special individual constant $\underline{c}$ different from all the $\underline{e}$'s. Let Z be the following Δ_1^A set of sentences of $\mathcal{L}$.

(1) the atomic diagram of A: $\underline{d} \in \underline{e} \leftrightarrow d \in e$; $\underline{d} \notin \underline{e} \leftrightarrow d \notin e$ for $d, e \in A$.
(2) $\underline{c} \subseteq \underline{b}$, $D(\underline{c}, \underline{p})$, and $\underline{c} \neq \underline{e}$ for all $e \in A$.

Suppose Z is not consistent in the sense of $\mathcal{L}_{\omega_1,\omega}$. Then there is a $z_0 \in A$ such that $z_0 \subseteq Z$ and z_0 is not consistent. And z_0 consists of some $A_0 \in A$ such that A_0 is a subset of the atomic diagram of A, and

$$\underline{c} \subseteq \underline{b},\ D(\underline{c}, \underline{p}),\ \text{and}\ \{\underline{c} \neq \underline{e} \mid e \in f\} \tag{3.2}$$

for some $f \in A$. Since z_0 is inconsistent, there is a deduction $E \in A$ of

$$[\underline{c} \subseteq \underline{b} \wedge D(\underline{c}, \underline{p})] \longrightarrow \underline{c} \in f \tag{3.3}$$

from A_0. But then $S_{p,b} \subseteq f$ and so $S_{p,b} \in A$.

Suppose Z is consistent. Then a Henkin style construction in ω many stages yields a model of Z, hence an actual $c \in (S_{p,b} - A)$. At stage j, a sentence σ of $\mathcal{L}$ is considered, and σ_j is either σ or $\neg\sigma$ so long as $Z \cup \{\sigma_i \mid i \leq j\}$ is consistent. If σ_j is an infinite disjunction (e.g. σ_j begins with "$\exists x \in \underline{e}$"), then some component of σ_j is added immediately.

The construction can be varied so 2^ω many c's are produced. Let t be a one-one map of ω onto $\{\underline{g} \mid g \in b\}$. After σ_j is chosen, and before σ_{j+1} is chosen, create a split as follows. Choose an n so that $(t(n) \in \underline{c})$ and $(t(n) \notin \underline{c})$ are each consistent with $Z \cup \{\sigma_i \mid i \leq j\}$. Then the construction takes 2^ω different paths, and different paths produce different c's. Such splits always exist. Otherwise there is a j such that $Z \cup \{\sigma_i \mid i \leq j\}$ is consistent and for each n there is a deduction $D_n \in \mathcal{A}$ from $Z \cup \{\sigma_i \mid i \leq j\}$ of either $(t(n) \in \underline{c})$ or $(t(n) \notin c)$. The Σ_1 admissibility of A puts all the D_n's in some $D \in A$. This D decides which elements of $\underline{b}$ belong to $\underline{c}$. Hence there is an $e \in A$ such that $(\underline{c} = \underline{e})$ is deducible from $Z \cup \{\sigma_i \mid i \leq j\}$, a contradiction. □

Corollary 3.2. *$S_{p,b}$ is countable $\longleftrightarrow$ $S_{p,b} \in A$.*

Theorem 3.3. *There exists a lightface Σ_1^{ZF} formula $\mathcal{F}(u, v, w)$ such that for any countable Σ_1 admissible set A and any $p, b, s \in A$:*

$$S_{p,b} \text{ is countable} \longrightarrow A \models \exists w \mathcal{F}(\underline{p}, \underline{b}, w) \tag{3.4}$$

$$(\forall s \in A)\ \{[A \models \mathcal{F}(\underline{p}, \underline{b}, \underline{s})] \longrightarrow s = S_{p,b}\}. \tag{3.5}$$

Proof. The existence of $\mathcal{F}$ is implicit in the proof of Theorem 3.1. Thus Z is inconsistent iff $S_{p,b}$ is countable iff $S_{p,b} \in \mathcal{A}$. The statement

$$A \models \mathcal{F}(\underline{p}, \underline{b}, \underline{s}) \tag{3.6}$$

says: (i) there exist $A_0 \in A$ and E such that $A_0 \subseteq$ atomic diagram of A, and E is a deduction of (3.3) from A_0; and (ii)

$$s = \{x \mid x \in f \wedge x \subseteq b \wedge D(x, p)\}. \tag{3.7}$$

□

4. Enumeration of Models for Scattered Theories

Let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ for some countable first order language $\mathcal{L}$, and $T \subseteq \mathcal{L}_0$ a theory with a model. Throughout this section T is scattered as defined in Section 1. For convenience assume T mentions all formulas of $\mathcal{L}_0$; thus $\mathcal{L}_0$ and $\mathcal{L}$ are recoverable from T.

Review of $\boldsymbol{\omega}$-completeness and finitary consistency for fragments. Let $\mathcal{L}'$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, and $T' \subseteq \mathcal{L}'$ a set of sentences. T' is **$\boldsymbol{\omega}$-complete** in $\mathcal{L}'$ iff (1) and (2) hold.

(1) For every sentence $\mathcal{F} \in \mathcal{L}'$, either $\mathcal{F} \in T'$ or $(\neg\mathcal{F}) \in T'$.
(2) For any sentence $(\vee_i \mathcal{F}_i) \in T'$, there is an i such that $\mathcal{F}_i \in T'$.

Say T' is **finitarily consistent** iff no contradiction can be derived from T' using only the finitary rules of $\mathcal{L}_{\omega_1,\omega}$. The infinitary step being avoided is deriving an infinite conjunction by deriving each of its components. Say T' is **$\boldsymbol{\omega}$-consistent** iff for any sentence $(\vee_i \mathcal{F}_i) \in \mathcal{L}'$, if $T' \cup \{\vee_i \mathcal{F}_i\}$ is finitarily consistent, then there is an i such that $T' \cup \{\mathcal{F}_i\}$ is finitarily consistent.

Proposition 4.1. *If T' is finitarily consistent and ω-complete, then T' has a model.*

Proof. Note that T' is ω-consistent. The model is constructed by extending T' to a finitarily consistent and ω-complete set of sentences that includes Henkin axioms. At each stage of the construction, the set of sentences up to that point is ω-consistent. □

Proposition 4.2. *Suppose for all $\beta \leq \gamma < \lambda$, T_β is finitarily consistent and ω-complete in the fragment $\mathcal{L}_\beta$, $T_\beta \subseteq T_\gamma$, and $\mathcal{L}_\beta \subseteq \mathcal{L}_\gamma$. Then $\cup\{T_\beta \mid \beta < \lambda\}$ is finitarily consistent and ω-complete in the fragment $\cup\{\mathcal{L}_\beta \mid \beta < \lambda\}$.*

End of Review of $\boldsymbol{\omega}$-completeness and finitary consistency for fragments.

Morley[10] showed that the scatteredness of T implies the countable models of T can be arranged in a hierarchy of height at most ω_1 based on Scott rank with at most countably many models on each level. The current section revisits [12] and presents a Σ_1 enumeration of the countable models of T with a recursion-theoretic eye on some constructive details. The enumeration is a continuous tree $\mathcal{TR}(\mathcal{T})$ with at most ω_1 levels, and at most countably many nodes on each level. Each node is a theory T' finitarily consistent and ω-complete in a fragment $\mathcal{L}_{T'}$ with $T \subseteq T'$ and $\mathcal{L}_0 \subseteq \mathcal{L}_{T'}$. Each T' has an atomic model, and the class of all such models is the class of all countable models of T.

The *enumeration* of $\mathcal{TR}(T)$ is as follows.

Level 0. Call T' a node iff T' is a finitarily consistent and ω-complete extension of T in the fragment $\mathcal{L}_0$ $(= \mathcal{L}_{T'})$.

Level λ (limit). Call $T^{'}$ a node iff there is a sequence T_β $(\beta < \lambda)$ such that: T_β is on level β; $T_\beta \subseteq T_\gamma$ $(\beta < \gamma < \lambda)$; and $T^{'} = \cup\{T_\beta \mid \beta < \lambda\}$. $\mathcal{L}_{T'} = \cup\{\mathcal{L}_{T_\beta} \mid \beta < \lambda\}$.

Level $\delta + 1$. Suppose S is a node on level δ, i.e. a finitarily consistent theory ω-complete in its fragment $\mathcal{L}_S$. If S is ω-categorical, then S has no successors on level $\delta + 1$. Otherwise S has a non-principal n-type $p(\overrightarrow{x})$. Let $\mathcal{L}'_S$ be the least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_S$ and containing the conjunction

$$\wedge\{\mathcal{F}(\overrightarrow{x}) \mid \mathcal{F}(\overrightarrow{x}) \in p(\overrightarrow{x})\} \tag{4.1}$$

for every non-principal n-type $p(\overrightarrow{x})$ of S for all $n > 0$. Say T' is a successor of S on level $\delta + 1$ if T' is a finitarily consistent and ω-complete extension of S in the fragment $\mathcal{L}'_S$ $(= \mathcal{L}_{T'})$.

Proposition 4.3. *If $\beta < \omega_1$, then $\mathcal{TR}(T)$ has only countably many nodes on level β.*

Proof. By induction on β. Level 0 is countable by clause (b) of the definition of scattered. Suppose S is on level δ. Assume $\mathcal{L}_S$ is countable. The set of all non-principal n-types of S is countable by clause (a) of the definition of scattered, hence $\mathcal{L}'_S$ is countable. The set of all successors of S on level $\delta+1$ is countable by clause (b) of the definition of scattered.

Let T' be any node on the countable limit level λ. Let $\mathcal{L}_\lambda$ be the least fragment extending all the $\mathcal{L}_S$'s for all theories S on all levels below λ. By induction $\mathcal{L}_\lambda$ is countable. Let T'' be any finitarily consistent and ω-complete extension of T' in $\mathcal{L}_\lambda$. The set of all T'''s is countable, so the set of all T''s is countable. □

Let $\mathcal{TR}(T) \restriction \beta$ be the restriction of $\mathcal{TR}(T)$ to the levels *below* β.

Proposition 4.4. *(i) If $\beta < \alpha < \omega_1$ and $L(\alpha, T)$ is Σ_1 admissible, then*

$$(\mathcal{TR}(T) \restriction \beta) \in L(\alpha, T).$$

(ii) There exists a lightface Σ_1^{ZF} formula $\mathcal{G}(u, v, w)$ such that for all scattered T, all countable Σ_1 admissible $L(\alpha, T)$, and all $b \in L(\alpha, T)$:

$$(\mathcal{TR}(T) \restriction \beta) = b \iff L(\alpha, T) \models \mathcal{G}(T, \beta, b).$$

Proof. By a $\Sigma_1^{L(\alpha,T)}$ recursion that relies on Theorem 3.3.

Suppose

$$(\mathcal{TR}(T) \restriction (\delta + 1)) \in L(\alpha, T), \tag{4.2}$$

and theory S is on level δ. The set of non-principal types of S is the unique $s \in L(\alpha, T)$ that satisfies the Σ_1 $\mathcal{F}$ of Theorem 3.3 with p and b both equal to S. The statement "q is a non-principal type of S" is lightface Δ_0^{ZF} and corresponds to the formula $D(x, y)$ of (3.1). The fragment $\mathcal{L}^{'}_S$ was defined just before equation (4.1). The set of successors of S on level $\delta+1$ is obtained from Theorem 3.3 with parameters $\langle p, b\rangle$ equal to $\langle S, \mathcal{L}^{'}_S\rangle$. □

Let $\mathcal{A}$ be a countable model of T (a scattered theory as above). The Scott analysis of $\mathcal{A}$ differs little from its **tree analysis**:

(1) $T(0,\mathcal{A})$ = theory of $\mathcal{A}$ in $\mathcal{L}_0$, and $\mathcal{L}_{T(0,\mathcal{A})} = \mathcal{L}_0$.
(2) $T(\lambda,\mathcal{A}) = \cup\{T(\beta,\mathcal{A}) \mid \beta < \lambda\}$.
(3) $\mathcal{L}_{T(\lambda,\mathcal{A})} = \cup\{\mathcal{L}_{T(\beta,\mathcal{A})} \mid \beta < \lambda\}$.
(4) $\mathcal{L}_{T(\delta+1,\mathcal{A})} = \mathcal{L}'_{T(\delta,A)}$ (defined similarly to $\mathcal{L}'_S$ on level $\delta+1$ of $\mathcal{TR}(T)$ above).
(5) $T(\delta+1,\mathcal{A})$ = theory of $\mathcal{A}$ in $\mathcal{L}_{T(\delta+1,\mathcal{A})}$.

Recall from Section 2 the definition of $d_{\mathcal{A}}$, the distinction rank of $\mathcal{A}$, and the argument that the Scott rank of $\mathcal{A}$ is either $d_{\mathcal{A}}$ or $d_{\mathcal{A}}+1$. Clearly there is a $\delta < \omega_1$ such that for all n, any distinction made between n-tuples of $\mathcal{A}$ by a formula of $\mathcal{L}_{T(\omega_1,\mathcal{A})}$ is made by a formula of $\mathcal{L}_{T(\delta,\mathcal{A})}$. The *tree rank of* $\mathcal{A}$, is defined by

$$tr(\mathcal{A}) = \text{least } \delta[\mathcal{A} \text{ is the atomic model of } T(\delta,\mathcal{A})]. \tag{4.3}$$

Proposition 4.5. $tr(\mathcal{A}) \leq sr(\mathcal{A})$.

Proof. $\mathcal{L}^A_\delta$ was defined just after Equation 2.3. By induction on δ, $\mathcal{L}^A_\delta \subseteq \mathcal{L}_{T(\delta,\mathcal{A})}$. Thus $T^{\mathcal{A}}_{sr(\mathcal{A})} \subseteq T(sr(\mathcal{A}),\mathcal{A})$. But $\mathcal{A}$ is an atomic, hence homogeneous model of $T^{\mathcal{A}}_{sr(\mathcal{A})}$, and so $\mathcal{A}$ is an atomic model of $T(sr(\mathcal{A}),\mathcal{A})$. □

Proposition 4.6. *Suppose* $\mathcal{A} \models T$ *and* $\mathcal{L}(\alpha, < T, \mathcal{A} >)$ *is* Σ_1 *admissible. Then*

$$tr(\mathcal{A}) < \alpha \longrightarrow sr(\mathcal{A}) < \alpha.$$

Proof. Suppose not. Then D, the set of all distinctions between n-tuples (all $n > 0$) of $\mathcal{A}$ made by formulas of $\mathcal{L}_{T(tr(\mathcal{A}),\mathcal{A})}$, belongs to $\mathcal{L}(\alpha, < T, \mathcal{A} >)$ by Proposition 4.4. And there is an unbounded $\Sigma_1^{L(\alpha,\langle T,\mathcal{A}\rangle)}$ map of D into α, a violation of the Σ_1 admissibility of $\mathcal{L}(\alpha, < T, \mathcal{A} >)$. The map carries each distinction $d \in D$ to the least δ such that d is made by some formula of $\mathcal{L}^{\mathcal{A}}_\delta$. □

A theory T can be scattered up to a point. The tree $\mathcal{TR}(T)$ is said to be **scattered below $\boldsymbol{\beta}$** if the notion of scattered enumeration succeeds for T on all levels below β. To be more precise, $\mathcal{TR}(T)$ has only countably many nodes (perhaps none) on each level below β.

Proposition 4.7. *Suppose* $\alpha < \omega_1$, $L(\alpha, T)$ *is* Σ_1 *admissible,* T *is scattered below* $(\alpha+1)$, *and* T *has a model of Scott rank* $\geq \beta$ *for each* $\beta < \alpha$. *Then there exists a theory* T_α *on level* α *of* $\mathcal{TR}(T)$ *such that* T_α *is* $\Delta_1^{L(\alpha,T)}$.

Proof. By Proposition 4.6 $\mathcal{TR}(T)$ has nodes on all levels below α, if an $\mathcal{A}$ can be found that satisfies the hypotheses of Proposition 4.6 and also $sr(\mathcal{A}) \geq \alpha$. To find $\mathcal{A}$ through Barwise Compactness, consider the following set Z of sentences.

$(Z1)$ Introduce a constant $\underline{e}$ to name each $e \in L(\alpha, T)$. Add the atomic diagram (in the sense of $\mathcal{L}_{\omega_1,\omega}$) of $L(\alpha, T)$ to Z. For each $\beta < \alpha$,

$$\forall x[x \in \underline{\beta} \longleftrightarrow \vee\{x = \underline{\gamma} \mid \gamma < \beta\}] \tag{4.4}$$

is a typical member of $(Z1)$. Any model of $(Z1)$ is an end extension of $L(\alpha, T)$.

$(Z2)$ Introduce a new constant $\underline{d}$, and add sentences saying $\underline{d}$ is an ordinal greater than $\underline{\beta}$ for each $\beta < \alpha$.

$(Z3)$ Add $\underline{\mathcal{A}} \models T$ and $sr(\mathcal{A}) > \underline{\beta}$ for each $\beta < \alpha$.

$(Z4)$ Add the axioms for Σ_1 admissibility.

Let M be a model of Z that omits α but extends $L(\alpha, T)$ as in [2] or[4]. $L(\alpha, < T, \mathcal{A} >)$ is Σ_1 admissible, otherwise $\alpha \in M$. Condition $(Z3)$ insures $sr(\mathcal{A}) \geq \alpha$.

Let T' denote an arbitrary node below level α. Call T' unbounded if T' has extensions to theories on arbitrarily high levels below α. Then T can be regarded as an unbounded node.

Suppose T' is an unbounded node below level β for some $\beta < \alpha$; then T' has an unbounded extension on level β. Otherwise the Σ_1 admissibility of $L(\alpha, T)$ implies T' is bounded.

There exists a $\beta_0 < \alpha$ and an unbounded node T_{β_0} on level β_0 such for all $\beta \in (\beta_0, \alpha)$, T_{β_0} has a unique unbounded extension on level β. Otherwise a tree $\mathcal{U}$ of unbounded nodes can be constructed such that $\mathcal{U}$ is isomorphic to the binary branching tree $2^{<\omega}$, and the branches of $\mathcal{U}$ define a continuum of nodes on some level $\alpha_0 \leq \alpha$ of $\mathcal{TR}(T) \restriction (\alpha + 1)$.

The set S_{ub} of unbounded nodes above T_{β_0} forms an expanding sequence whose union is the desired T_α. To see S_{ub} is $\Delta_1^{L(\alpha,T)}$, let N_γ be the set of all nodes on level γ extending T_{β_0} for each $\gamma \in (\beta_0, \alpha)$. Then N_γ, as a function of γ, is $\Sigma_1^{L(\alpha,T)}$ by Proposition 4.4, and $(N_\gamma - S_{ub}) \in L(\alpha, T)$ since $N_\gamma \cap S_{ub}$ has just one element. There is a $\Sigma_1^{L(\alpha,T)}$ function that takes each node $e \in (N_\gamma - S_{ub})$ to a bound on the levels occupied by extensions of e. But then there is a strict upper bound $b < \alpha$ on the levels occupied by extensions of members of $(N_\gamma - S_{ub})$. Any such b singles out the unique member of $N_\gamma \cap S_{ub}$. □

Proposition 4.8. *Suppose $\alpha \leq \omega_1$, $L(\alpha, T)$ is Σ_2 admissible, T is scattered below α, and T has models of arbitrarily high Scott rank less than α. Then there exists a theory T_α on level α of $\mathcal{TR}(T)$ such that T_α is $\Delta_1^{L(\alpha,T)}$.*

Proof. The proof is similar to that of Proposition 4.7. The only difference is in the handling of $\mathcal{U}$. Then and now $\mathcal{U}$ can be defined by a $\Sigma_2^{L(\alpha,T)}$ recursion of length ω, since the set of unbounded nodes is $\Pi_1^{L(\alpha,T)}$. But now the Σ_2 admissibility of $L(\alpha, T)$ implies $\mathcal{U} \in L(\alpha, T)$, and so the branches of $\mathcal{U}$ define a continuum of nodes on some level $\alpha_0 < \alpha$ of $\mathcal{TR}(T)$. □

Two $\mathcal{L}$-structures are said to be $\mathcal{L}_{\omega_1,\omega}$**−equivalent** if they satisfy the same sentences of $\mathcal{L}_{\omega_1,\omega}$. (Recall: if $\mathcal{A}$ is countable and $\mathcal{L}_{\omega_1,\omega}$−equivalent to $\mathcal{B}$, then $\mathcal{A}$ is $\mathcal{L}_{\infty,\omega}$−equivalent to $\mathcal{B}$.)

Theorem 4.9. *Suppose Vaught's Conjecture fails for T. Then there exist T_β, $\mathcal{A}_\beta$ and $\mathcal{L}_\beta$ ($\beta \leq \omega_1$) such that:*
(i) If $\beta < \omega_1$, then T_β is an ω-complete theory in the countable fragment $\mathcal{L}_\beta$.
(ii If $\beta \leq \gamma \leq \omega_1$, then $T_\beta \subseteq T_\gamma$, $\mathcal{A}_\beta \subseteq \mathcal{A}_\gamma$ and $\mathcal{L}_\beta \subseteq \mathcal{L}_\gamma$.
(iii) If λ(limit) $\leq \omega_1$, then $T_\lambda = \cup\{T_\beta \mid \beta < \lambda\}$ and $\mathcal{A}_\lambda = \cup\{\mathcal{A}_\beta \mid \beta < \lambda\}$.
(iv) T_{ω_1} is $\Delta_1^{L(\omega_1,T)}$ definable.
(v) If $\beta \leq \omega_1$, then $\mathcal{A}_\beta$ is an atomic model of T_β.
(vi) If $\beta < \omega_1$, then $\mathcal{A}_{\beta+1}$ realizes a non-principal type of T_β.
(vii) (Harnik & Makkai[3]) The cardinality of $\mathcal{A}_{\omega_1}$ is ω_1, and $\mathcal{A}_{\omega_1}$ is not $\mathcal{L}_{\omega_1,\omega}$-equivalent to any countable model.

Proof. An uncountable model $\mathcal{A}_{\omega_1}$ of T is constructed that is not $\mathcal{L}_{\omega_1,\omega}$-equivalent to any countable model. By Proposition 4.8, there is a theory T_{ω_1} on level ω_1 of $\mathcal{TR}(\omega_1)$ such that T_{ω_1} is $\Delta_1^{L(\omega_1,T)}$. Thus $T_{\omega_1} = \cup\{T_\gamma \mid \gamma < \omega_1\}$, and $(\gamma \leq \delta) \rightarrow (T_\gamma \subseteq T_\delta)$. p, the parameter used in the $\Delta_1^{L(\alpha,T)}$ definition of T_{ω_1}, belongs to $L(\alpha_0, T)$ for some $\alpha_0 < \omega_1$. Define

$$K = \{\beta \mid \alpha_0 < \beta < \omega_1 \wedge L(\beta,T) \preccurlyeq_1 L(\omega_1,T)\}.$$

(Recall that $X \preccurlyeq_1 Y$ means X is a Σ_1^{ZF} substructure of Y.) Let $\{\gamma_\delta \mid \delta < \omega_1\}$ be an increasing enumeration of K. Then $L(\gamma_\delta, T)$ is Σ_1 admissible, and so

$$T_{\gamma_\delta} = T_{\omega_1} \cap L(\gamma_\delta, T)$$

by Proposition 4.4(i). Also T_{γ_δ} is $\Delta_1^{L(\gamma_\delta,T)}$ definable via the same Δ_1 definition that works for T_{ω_1}, since $p \in L(\gamma_\delta, T) \preccurlyeq_1 L(\omega_1, T)$.

Structures $\mathcal{A}_\delta$ ($\delta \leq \omega_1$) and inclusion maps $i_{\beta,\delta} : \mathcal{A}_\beta \longrightarrow \mathcal{A}_\delta$ ($\beta < \delta$) are defined by recursion on δ. The map $i_{\beta,\delta}$ will be elementary with respect to the language $\mathcal{L}_{\gamma_\beta}$; i.e. any sentence of $\mathcal{L}_{\gamma_\beta}$ with parameters in $\mathcal{A}_\beta$ and true in $\mathcal{A}_\beta$ will also be true in $\mathcal{A}_\delta$.

Stage 0. Structure $\mathcal{A}_0$ is the countable atomic model of T_{γ_0}.

Stage $\delta+1$. Assume $\mathcal{A}_\delta$ is the countable atomic model of T_{γ_δ}. Extend $\mathcal{A}_\delta$ to $\mathcal{A}_{\delta+1}$, the countable atomic model of $T_{\gamma_{\delta+1}}$, so that the inclusion map, $i_{\delta,\delta+1}$ is $\mathcal{L}_{\gamma_\delta}$-elementary.

Stage λ (limit $\leq \omega_1$). Let

$$\mathcal{A}_\lambda = \cup\{\mathcal{A}_\delta \mid \delta < \lambda\}$$

For all $\delta < \delta' < \lambda$, assume the inclusion map $i_{\delta,\delta'}$ is $\mathcal{L}_{\gamma_\delta}$-elementary. Then for each $\delta < \lambda$, $\mathcal{A}_\lambda$ is an $\mathcal{L}_{\gamma_\delta}$-elementary extension of $\mathcal{A}_\delta$, and so is a model of T_{γ_δ}. Thus $\mathcal{A}_\lambda$ is a model of $T\gamma_\lambda$.

To see that $\mathcal{A}_\lambda$ is an atomic model of $T\gamma_\lambda$, let $\overrightarrow{a}$ be an n-tuple of $\mathcal{A}_\lambda$. For some $\delta < \lambda$, $\overrightarrow{a}$ is an n-tuple of $\mathcal{A}_\delta$, and realizes some atom $\mathcal{F}(\overrightarrow{x})$ of

T_{γ_δ}. Then $\mathcal{F}(\overrightarrow{x})$ is an atom of T_λ, because $L(\gamma_\delta, T) \preccurlyeq_1 L(\lambda, T)$. Hence $\overrightarrow{a}$ realizes $\mathcal{F}(\overrightarrow{x})$ in $\mathcal{A}_\lambda$, since $i_{\delta,\lambda}$ is $\mathcal{L}_\delta$-elementary.

If $\mathcal{A}_{\omega_1}$ were $\mathcal{L}_{\omega_1,\omega}$-equivalent to some countable model, then it would be an atomic model of T_{γ_δ} for some $\delta < \omega_1$. But $\mathcal{A}_{\delta+1}$, hence $\mathcal{A}_{\omega_1}$, realizes a non-principal type of T_{γ_δ}. □

5. Absoluteness of Vaught's Conjecture

Let $VC(T)$ be the predicate: Vaught's conjecture holds for T. Morley's work [10] implies that $VC(T)$ is absolute. The enumeration tree, $\mathcal{TR}(T)$, of Section 4 is applied below to make the statement of $VC(T)$ more precise and to see in some detail how T can satisfy Vaught's Conjecture. Suppose an attempt is made to develop $\mathcal{TR}(T)$ and the attempt fails to produce a tree with only countably many nodes on each level and ω_1 many non-empty levels Then there must be a countable β such that one of the following holds:

(1) $\beta = 0$ and T has uncountably many finitarily consistent, ω-complete extensions in $\mathcal{L}_0$.
(2) $\beta = \delta + 1$, some theory S is on level δ, and for some n, the set of n-types of S is uncountable.
(3) $\beta = \delta + 1$ some theory S is on level δ, for all n the set of n-types of S is countable, and the set of all finitarily consistent, ω-complete extensions of S in $\mathcal{L}'_S$ is uncountable. $\mathcal{L}'_S$ is defined just before 4.1.
(4) $\beta = \lambda$ and the set of nodes on level λ is uncountable.
(5) Level β is empty.

Define the **Vaught Rank** of $T, vr(T)$, to be the least countable β that satisfies one of 1-5 above. (If there is no such β, let $vr(T)$ be ω_1.)

Define the predicate $VC(T)$ by $vr(T) < \omega_1$.

Suppose $vr(T) = \beta < \omega_1$. If $\beta = 0$, then T has 2^ω finitarily consistent, ω-complete extensions in $\mathcal{L}_0$ by Theorem 3.1, hence 2^ω many countable models. The same holds in cases 3 and 4. If 5 holds, then T has only countably many countable models, and each one is the atomic model of a theory on some level of $\mathcal{TR}(T)$ below level β. Suppose case 2 holds. Then for some n, there are 2^ω n-types of S by Theorem 3.1, hence 2^ω many countable models of T.

Recall that

$$\omega_1^{L(T)} = \mathit{least}\ \gamma[L(T) \models (\gamma\ \mathit{is\ uncountable})]. \tag{5.1}$$

Proposition 5.1. *The predicate, Vaught's Conjecture holds for T, is $\Sigma_1^{L(\omega_1^{L(T)},T)}$, hence Σ^1_2.*

Proof. By Proposition 4.4, $\mathcal{TR}(T) \subseteq L(\omega_1, T)$ and is $\Sigma_1^{L(\omega_1,T)}$. The statement $VC(T)$ says: at some level $\gamma < \omega_1$, either (a) $\mathcal{TR}(T)$ ends or (b) "blows up", i.e. a perfect kernel of theories or types is manifest. Let α_0 be the least $\alpha > \gamma$ such that $L(\alpha, T)$ is Σ_1 admissible.

Suppose (a) holds. Then Levy-Shoenfield Absoluteness implies $\alpha_0 < \omega_1^{L(T)}$, and there is a $\mathcal{L}_{\omega_1,\omega}$ sentence $\mathcal{K} \in L(\alpha_0, T)$ that expresses the fact that every model of T is an atomic model of some theory on some level at or below γ of $\mathcal{TR}(T)$.

Suppose (b) holds. Theorem 3.1 implies the existence of a perfect kernel of theories or types. A coding of some such perfect kernel by a real is constructible from any counting of α_0. The proof of Theorem 3.1 relies on the consistency of a certain set Z of axioms. Z is $\Sigma_1^{L(\alpha_0,T)}$, and the consistency of Z is $\Pi_1^{L(\alpha_0,T)}$. Hence Levy-Shoenfield Absoluteness implies $\alpha_0 < \omega_1^{L(T)}$, and so a code for the perfect kernel belongs to $L(\omega_1^{L(T)}, T)$. □

Proposition 5.2. *Suppose T is a counterexample to Vaught's Conjecture. Then there is a theory T_{ω_1} on level ω_1 of $\mathcal{TR}(T)$ such that T_{ω_1} is $\Delta_1^{L(\omega_1,T)}$. For all countable β: T_β, the restriction of T_{ω_1} to level β, has an atomic model whose Scott rank is β.*

Proof. By Proposition 4.8. □

Suppose $L(\alpha, T)$ is Σ_1 admissible, $\mathcal{A}$ is a countable model of T, and $\omega_1^{\mathcal{A}} = \alpha$. According to (2.6), $\mathcal{A}$ is a homogenous model of $T_\alpha^{\mathcal{A}}$; $\mathcal{A}$ is said to be ***α*-saturated** if every n-type ($n \geq 1$) of $T_\alpha^{\mathcal{A}}$ is realized in $\mathcal{A}$.

Theorem 5.3. *Suppose T is a counterexample to Vaught's conjecture. Then there is a $\Delta_1^{L(\omega_1,T)}$ theory T_{ω_1} on level ω_1 of $\mathcal{TR}(T)$ and a closed unbounded set $C \subseteq \omega_1$ such that $\forall \alpha \in C$: T_α, the restriction of T_{ω_1} to level α, has an atomic model $\mathcal{A}_\alpha$ of Scott rank α and an α-saturated model $\mathcal{B}_\alpha$ of Scott rank $\alpha + 1$.*
The atomic models form an expanding chain and each inclusion $\mathcal{A}_\beta \subset \mathcal{A}_\gamma$ ($\beta < \gamma$) is elementary with respect to the language of T_β.

Proof. Proposition 4.8 provides T_{ω_1}. Let $p \in L(\omega_1, T)$ be the parameter needed for the $\Delta_1^{L(\omega_1,T)}$ definition of T_{ω_1}. For any α, let α^+ be the least $\beta > \alpha$ such that $L(\beta, T)$ is Σ_1 admissible.

For $x \in L(T)$, let $H_1(x)$ be the Σ_1 hull of x in $L(T)$. Recall that

$$x \subseteq H_1(x) \preccurlyeq_1 L(T)$$

and that x and $H_1(x)$ have the same cardinality in $L(T)$.

An expanding sequence of countable Σ_1 hulls, H^δ ($\delta < \omega_1$), is defined by recursion on δ.

Stage 0. H^0 is $H_1(\{tc(p), \omega_1, tc(T)\})$. ($tc$ is transitive closure.) Note: ω_1^+, $\omega \in H^0$; if $d < e < \omega_1$ and $e \in H^0$, then $d \in H^0$. Let c_0 be the lub of the countable ordinals in H^0. Let $L(\beta_0, T)$ be the transitive collapse of H^0. Then

$$c_0 = \omega_1^{L(\beta_0,T)} \text{ and } L(c_0^+, T) \subseteq L(\beta_0, T). \tag{5.2}$$

Stage $\boldsymbol{\delta + 1}$. Assume H^δ is countable in V. Then $H^\delta \cap \omega_1$ is a proper initial segment of ω_1. Let c_δ be the least countable ordinal not in H^δ. $H^{\delta+1}$ is $H_1(H^\delta \cup \{c_\delta\})$.

Stage $\boldsymbol{\lambda}$ (limit). Let H^λ be $\cup\{H_\delta \mid \delta < \lambda\}$.
Then $C = \{c_\delta \mid \delta < \omega_1\}$ is a closed unbounded set.
Let $L(\beta_\delta, T)$ be the transitive collapse of H^δ. Then

$$c_\delta = \omega_1^{L(\beta_\delta,T)} \text{ and } L(c_\delta^+, T) \subseteq L(\beta_\delta, T). \tag{5.3}$$

Let T_{c_δ} be the restriction of T_{ω_1} to level c_δ of $\mathcal{TR}(T)$. Then T_{c_δ} is $\Delta_1^{L(c_\delta,T)}$ via parameter p, and N, the set of non-principal types of T_{c_δ}, is non-empty and countable in V. Then $T_{c_\delta} \in L(c_\delta^+, T)$, and so $N \in L(c_\delta^+, T)$ by Theorem 3.1. Hence the structure $L[c_\delta, T; T_{c_\delta}, N]$ (i.e. $L(c_\delta, T)$ with $x \in T_{c_\delta}$ and $x \in N$ as additional atomic predicates) is Σ_1 admissible because no subset of c_δ in $L(\beta_\delta, T)$ can define a counting of $\omega_1^{L(\beta_\delta,T)}$. Now the construction of M in the proof of Theorem 6.1 can be imitated to produce a model $\mathcal{B}$ of T_{c_δ} such that $\mathcal{B}$ realizes all the types in N and $\omega_1^{\mathcal{B}} = c_\delta$.

The atomic $\mathcal{A}_\beta$'s are supplied by Theorem 4.9. □

6. Bounds on Scattered Theories

Once again $\mathcal{L}$ is a countable first order language, $\mathcal{L}_0$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, and $T \subseteq \mathcal{L}_0$ has a model. Both $\mathcal{L}$ and $\mathcal{L}_0$ are effectively recoverable from T_0. "Scattered below β" was defined just before Proposition 4.7.

Theorem 6.1. *Suppose $\alpha < \omega_1$, $L(\alpha, T)$ is Σ_2 admissible, The theory T is scattered below α, and for each $\beta < \alpha$, T has a model of Scott rank $\geqslant \beta$. Then T has a model $\mathcal{A}$ such that $\omega_1^{\mathcal{A}} = \alpha$ and $sr(\mathcal{A}) = \alpha + 1$.*

Proof. By Proposition 4.8 $\mathcal{TR}(A)$ has a theory T_α on level α such that T_α is Δ_1^α and T_α is $\cup\{T_\beta \mid \beta < \alpha\}$, where T_β is a node on level β. Let Z be the following set of sentences.

$(Z1)$ The atomic diagram of $L(\alpha, T)$ in the sense of $\mathcal{L}_{\omega_1,\omega}$.

$(Z2)$ Add $(\underline{d} > \underline{\beta})$ for all $\beta < \alpha$. $\underline{d}$ is a constant not occurring in $(Z1)$.

$(Z3)$ Let $T_{\underline{d}}$ be a theory on level $\underline{d}$ of $\mathcal{TR}(T)$. Add $\mathcal{A}$ is the countable atomic model of $T_{\underline{d}}$ and $\mathcal{F} \in T_{\underline{d}}$ for each sentence $\mathcal{F} \in T_\alpha$.

$(Z4)$ Add $(b(\overrightarrow{x})$ is an atom of $T_{\underline{d}})$ for each $b(\overrightarrow{x})$ that is an atom of T_α, i.e. $b(\overrightarrow{x})$ generates a principal type of T_α.

$(Z5)$ Add the axioms of Σ_1 admissibility.

The set Z is $\Sigma_2^{L(\alpha,T)}$, since the set of atoms of T_α is $\Pi_1^{L(\alpha,T)}$.

Suppose $\beta < \alpha$, $L(\beta, T)$ is Σ_1 admissible, and Z_β is $Z \cap L(\beta, T)$. A fundamental fact of forcing in the setting of set theory is: the Levy collapse of a cardinal to V preserves replacement; furthermore the preservation of Σ_n replacement needs only Σ_n replacement in V. To check the consistency of Z_β, augment $L(\alpha, T)$ by adding a generic counting of $L(\beta, T)$ to $L(\alpha, T)$ that preserves the Σ_2 admissibility of $L(\alpha, T)$. The set Z_β can be modeled by the augmented $L(\alpha, T)$. By Proposition 4.4, $T_\beta \subseteq L(\beta, T)$. Interpret $\underline{d}$ as β. Interpret $\mathcal{A}$ as the atomic model of T_β. Such an $\mathcal{A}$ belongs to the augmented $L(\alpha, T)$ because there T_β is countable. If $b(\overrightarrow{x})$ is an atom of T_α and belongs to $L(\beta, T)$, then $b(\overrightarrow{x})$ is an atom of T_β.

The set Z has a model M that is a proper end extension of $L(\alpha, T)$ but omits α. We have $\omega_1^{\mathcal{A}} \leq \alpha$, for otherwise α is recursive in $\mathcal{A}$, and then $\alpha \in M$. By design $\mathcal{A} \models T_\beta$ for all $\beta < \alpha$, hence $sr(\mathcal{A}) \geq \alpha$ by Proposition 4.5, and so $\omega_1^{\mathcal{A}} = \alpha$ by (2.6).

Suppose $sr(\mathcal{A}) = \alpha$. Then $\alpha \in M$ as follows. By supposition $\mathcal{A}$ is the atomic model of T_α. The rank of an atom $b(\overrightarrow{x})$ of T_α is the least $\beta < \alpha$ such that $b(\overrightarrow{x})$ is an atom of T_β. Let f be the function that carries each $\overrightarrow{a} \in \mathcal{A}$ to the rank of an atom of T_α that generates the principal type realized by $\overrightarrow{a}$ in $\mathcal{A}$. Thanks to $(Z4)$ f is definable from T_d, and so $f \in M$. Then $lub(range\ f) = \alpha \in M$. □

Corollary 6.2. *([12]) Suppose for every countable model $\mathcal{A}$ of T, the Scott rank of $\mathcal{A}$ is less than or equal to $\omega_1^{\mathcal{A}}$. Then Vaught's Conjecture holds for T.*

Proof. Suppose $VC(T)$ fails. Then T is scattered below ω_1, and $\mathcal{TR}(T)$ has nodes on every countable level. Choose an $\alpha < \omega_1$ such that $L(\alpha, T)$ is Σ_2 admissible. Then T has a countable model $\mathcal{A}$ such that $\omega_1^{\mathcal{A}} = \alpha$ and $sr(\mathcal{A}) = \alpha + 1$. □

A more effective version of Corollary 6.2 is as follows. Define

$$\sigma_2^T = least\ \alpha[L(\alpha, T)\ is\ \Sigma_2\ admissible]. \tag{6.1}$$

The Vaught rank of T, denoted by $vr(T)$, was defined at the beginning of Section 6.

Corollary 6.3. *Suppose T does not have a countable model $\mathcal{A}$ such that*

$$\omega_1^{\mathcal{A}} = \sigma_2^T\ and\ sr(\mathcal{A}) = \sigma_2^T + 1. \tag{6.2}$$

Then $vr(T) < \sigma_2^T$.

Proof. If $vr(T) \geqslant \sigma_2^T$, then T is scattered below σ_2^T and $\mathcal{TR}(T)$ has nodes on every level below σ_2^T, □

As a warm-up to the main bounding results of the paper (Section 8), the above is recast as an effective bounding theorem.

Corollary 6.4. *Suppose T is scattered and*

$$sr(\mathcal{A}) \leq \omega_1^{\mathcal{A}}\ for\ every\ countable\ \mathcal{A} \models T. \tag{6.3}$$

Then $\exists \beta < \sigma_2^T$ such that

$$sr(\mathcal{A}) < \beta\ for\ every\ \mathcal{A} \models T. \tag{6.4}$$

Let $SA(T)$ say: for every countable model $\mathcal{A}$ of T, the theory $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ is ω-categorical. Steel [15], as reported in Makkai[7], showed that $VC(T)$ follows from $SA(T)$. Theorem 6.5 is an effective version of Steel's result.

The set $L(\alpha, T)$ is said to be **recursively Mahlo** if $L(\alpha, T)$ is Σ_1 admissible and every $\Delta_1^{L(\alpha,T)}$ closed unbounded subset of α has a member β such that $L(\beta, T)$ is Σ_1 admissible. Define

$$rm(T) = \textit{ least } \gamma\ [L(\gamma, T) \textit{ is recursively Mahlo}]. \tag{6.5}$$

Note that $rm(T) < \sigma_2^T$.

Theorem 6.5. *Suppose T is scattered and*

$$T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}} \textit{ is } \omega - \textit{categorical for every countable } \mathcal{A} \models T. \tag{6.6}$$

Then $\exists \beta < rm(T)$ such that

$$sr(\mathcal{A}) < \beta \textit{ for every countable } \mathcal{A} \models T. \tag{6.7}$$

Proof. Suppose there is no such β. Let α be $rm(T)$. Then Proposition 4.7 supplies a $\Delta_1^{L(\alpha,T)}$ theory T_α on level α of $\mathcal{TR}(T)$. Then $T_\alpha = \cup\{T_\beta \mid \beta < \alpha\}$, and T_β, as a function of β, is $\Sigma_1^{L(\alpha,T)}$.

There is a $\Sigma_1^{L(\alpha,T)}$ function f_0 such that $T_\beta \subseteq L(f_0(\beta), T)$ for all $\beta < \alpha$. Iteration of f_0 leads to a $\Delta_1^{L(\alpha,T)}$ closed unbounded set

$$C_0 = \{\gamma \mid T_\gamma \subseteq L(\gamma, T)\}. \tag{6.8}$$

A similar argument produces a $\Delta_1^{L(\alpha,T)}$ closed unbounded set C_1 such that

$$\forall \gamma \in C_1[(T_\alpha \cap L(\gamma, T)) \textit{ is } \Delta_1^{L(\gamma,T)}]. \tag{6.9}$$

Then there is a $\Delta_1^{L(\alpha,T)}$ closed unbounded set K such that

$$\forall \gamma \in K[T_\gamma \subseteq L(\gamma, T) \text{ and } T_\gamma \text{ is } \Delta_1^{L(\gamma,T)}]. \tag{6.10}$$

Hence for some $\gamma_0 \in K$, $L(\gamma_0, T)$ is Σ_1 admissible. Consequently T_{γ_0} has a model $\mathcal{B}$ such that $\omega_1^{\mathcal{B}} = \gamma_0$. But then $T^{\mathcal{B}}_{\omega_1^{\mathcal{B}}}$, hence T_{γ_0}, is ω-categorical, and so has no extension to a node on level α. □

7. Iterated Classical Bounding

In this section classical bounding (reviewed in Section 1) is translated into the language of Σ_1 admissible sets and revised to allow for iterated use in Σ_1 recursive definitions in Section 8.

Let $B(x)$ be a Δ_0^{ZF} formula with parameter p_0. The formula $B(x)$ is **β-bounded** iff :

$$\forall c[B(c) \iff L[\beta, p_0; c] \models B(\underline{c})]. \tag{7.1}$$

The set $L[\beta, p_0; c]$ is the result of iterating first order definability with $y \in c$ as an additional atomic predicate through the ordinals less than β starting with the transitive closure (tc) of $\{p_0\}$. Assume $B(x)$ is β-bounded. Define

$$c_\beta = c \cap L[\beta, p_0; c]. \tag{7.2}$$

Then $B(c) \iff B(c_\beta)$. For all z let A_z be the least Σ_1 admissible set with z as a member; thus

$$A_z = L(\omega_1^z, tc(\{z\})). \tag{7.3}$$

Let $\mathcal{F}(u,v)$ be a Σ_1^{ZF} formula with parameter p_1, and let p be $\{p_0, p_1\}$. Suppose for all c: if $B(c)$, then there exists a unique $\delta \in A_{\{p,\beta,c_\beta\}}$ such that

$$A_{\{p,\beta,c_\beta\}} \models \mathcal{F}(\underline{c_\beta}, \underline{\delta}); \tag{7.4}$$

designate δ by $\delta_{p,\beta,c}$.

Theorem 7.1. *(i) There exists a $\delta_{p,\beta} \in A_{\{p,\beta\}}$ such that for all c:*

$$B(c) \longrightarrow \delta_{p,\beta,c} \leq \delta_{p,\beta}. \tag{7.5}$$

(ii) $\delta_{p,\beta}$ can be construed as a partial function of p and β whose restriction to any Σ_1 admissible A has a Σ_1^A definition uniformly in A, i.e. one Σ_1 formula works for all A.

Proof. Let Z be the following $\Sigma_1^{A_{\{p,\beta\}}}$ set of sentences. Let $\alpha = \omega_1^{\{p,\beta\}}$.

($Z1$) Introduce constants $\underline{c}$ and $\underline{c_\beta}$, and put $\underline{c_\beta} = \underline{c} \cap L[\beta, p_0; \underline{c}]$ and $B(\underline{c_\beta})$ in Z.

($Z2$) Add constants that name the elements of

$$L(\alpha, tc(\{p, \beta, c_\beta\})). \tag{7.6}$$

and sentences of $\mathcal{L}_{\omega_1,\omega}$ that define each element in terms of elements of lower definability rank.

($Z3$) Let $\mathcal{F}(u,v)$ be $\exists w \mathcal{G}(u,v,w)$ for some Δ_0^{ZF} formula $\mathcal{G}(u,v,w)$. Add $\neg\mathcal{G}(\underline{c_\beta}, \underline{\delta}, \underline{r})$ for all $\delta < \alpha$ and every $\underline{r}$ that names an element of (7.6).

($Z4$) Add axioms for Σ_1 admissibility.

Suppose Z is consistent. Assume for a moment that

$$Z \text{ is countable.} \tag{7.7}$$

As in the proof of Proposition 4.7, Z has a model M that is a proper end extension of (7.6) but omits α. Then (7.6) is Σ_1 admissible, and so

$$A_{\{p,\beta,c_\beta\}} = L(\alpha, tc(\{p, \beta, c_\beta\})). \tag{7.8}$$

But then $A_{\{p,\beta,c_\beta\}} \models \neg\mathcal{F}(\underline{c_\beta}, \underline{\delta})$ for all $\delta < \alpha$, a contradiction since $\delta_{p,\beta,c_\beta} \in A_{\{p,\beta,c_\beta\}}$.

Thus Z is inconsistent.

To remove assumption (7.7), generically extend the universe V to V' so that Z is countable in V'. Then Z is inconsistent in V', hence in V by the absoluteness of provability in the sense of $\mathcal{L}_{\infty,\omega}$.

Since Z is $\Sigma_1^{A_{\{p,\beta\}}}$, there must be an inconsistent $W \subseteq Z$ such that $W \in A_{\{p,\beta\}}$. The set W consists of ($W1$), ($W2$) and ($W3$):

($W1$) is ($Z1$) & ($Z4$) above.

($W2$) Some $A_0 \in A_{\{p,\beta\}}$ such that $A_0 \subseteq$ set of sentences of ($Z2$).

$(W3)$ For some $\delta_1 < \alpha$, the sentence $\neg\mathcal{G}(\underline{c_\beta}, \underline{\delta}, \underline{r})$ for all $\delta < \delta_1$ and every $\underline{r}$ of $(Z2)$ that names an element of $L(\delta_1, tc(\{p, \beta, c_\beta\}))$.
Then there is a deduction $D \in A_{\{p,\beta\}}$ from $(W1)$ & $(W2)$ of

$$\vee\{\mathcal{F}(\underline{c_\beta}, \delta) \mid \delta < \delta_1\}. \tag{7.9}$$

Let ρ_0 be the least ρ such that there is such a $D \in L(\rho, tc(\{p, \beta\}))$; let $\delta_{\{p,\beta\}}$ be the least δ_1 associated with any such $D \in L(\rho_0, tc(\{p, \beta\}))$. Then

$$\delta_{p,\beta,c} \leq \delta_{p,\beta} \tag{7.10}$$

for any c such that $B(c)$ holds. The Σ_1^{ZF} formula $\mathcal{H}$ that defines $\delta_{p,\beta}$ as a partial function of p, β uniformly owes its existence to the effective nature of deducibility in $\mathcal{L}_{\omega_1,\omega}$. The formula $\mathcal{H}$ singles out a deduction in $A_{\{p,\beta\}}$ that establishes the value of $\delta_{p,\beta}$, and can be formulated to succeed in every Σ_1 admissible A, because $p, \beta \in A$ implies $A_{\{p,\beta\}}$ is a Σ_1^A definable (uniformly) subclass of A. □

8. Enumeration of Models under Weak Scattering

Let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ for some countable first order language $\mathcal{L}$, and $T \subseteq \mathcal{L}_0$ a theory with a model. Assume T is weakly scattered as defined in Section 1. For convenience assume T mentions all formulas of $\mathcal{L}_0$; thus $\mathcal{L}_0$ and $\mathcal{L}$ are recoverable from T. Since T need not be scattered, there is no hope of enumerating theories in $L(\omega_1, T)$ whose atomic models are exactly the countable models of T. But some useful vestiges of the constructive features of scattering carry over to weak scattering, and $L(\omega_1, T)$ manages to say a great deal about the countable models of T.

First consider $\mathcal{RH}(T)$, the **raw hierarchy** for the countable models of T. On level 0 of $\mathcal{RH}(T)$, put every T_0 such that $T \subseteq T_0$ and T_0 is a finitarily consistent, ω-complete theory of $\mathcal{L}_0$. (If needed, see the beginning of Section 4 for a review.)

Suppose T_δ is on level δ of $\mathcal{RH}(T)$. Define

$$\delta- = \begin{cases} \delta - 1 \text{ if } \delta \text{ is a successor,} \\ \\ \delta \text{ if } \delta \text{ is not a successor.} \end{cases} \tag{8.1}$$

Let $\mathcal{L}_0(T_{0-})$ be $\mathcal{L}_0$. Assume T_δ extends a unique $T_{\delta-}$ on level $\delta-$ and $\mathcal{L}_\delta(T_{\delta-})$ is countable. If all n-types $(n \geq 1)$ of T_δ are principal, then $\mathcal{L}_{\delta+1}(T_\delta)$ is undefined and T_δ has no extensions on level $\delta + 1$. Otherwise let $\mathcal{L}_{\delta+1}(T_\delta)$ be the least fragment of $\mathcal{L}_{\omega_1,\omega}$ extending $\mathcal{L}_\delta(T_{\delta-})$ and having as a member the conjunction

$$\wedge\{\mathcal{F}(\overrightarrow{x}) \mid \mathcal{F}(\overrightarrow{x}) \in p(\overrightarrow{x})\} \tag{8.2}$$

for every non-principal n-type $p(\overrightarrow{x})$ of T_δ $(n \geq 1)$. Since T is weakly scattered, the set $\mathcal{L}_{\delta+1}(T_\delta)$ is countable.

On level $\delta+1$ of $\mathcal{RH}(T)$ put every $T_{\delta+1}$ that extends T_δ and is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\delta+1}(T_\delta)$.

Put T_λ on level λ if there is a sequence $T_\delta(\delta < \lambda)$ such that:

(a) T_δ is on level δ;
(b)$T_\beta \subseteq T_\gamma$ if $\beta \leq \gamma$; and
(c) $T_\lambda = \cup\{T_\delta \mid \delta < \lambda\}$.

Define $\mathcal{L}_\lambda(T_\lambda)$ to be $\cup\{\mathcal{L}_\delta(T_{\delta-}) \mid \delta < \lambda\}$.

It is straightforward to verify that $\mathcal{A}$ is a countable model of T iff $\mathcal{A}$ is the atomic model of T_δ for some countable δ. Define the **raw tree rank** of $\mathcal{A}$ by

$$rtr(\mathcal{A}) = \text{ (least } \delta)[\mathcal{A} \text{ is the atomic model of some } T_\delta]. \tag{8.3}$$

Propositions 4.5 and 4.6 hold when tr is replaced by rtr. Thus

$$rtr(\mathcal{A}) \leq sr(\mathcal{A}), \tag{8.4}$$

and if $L(\alpha, \langle T, \mathcal{A}\rangle)$ is Σ_1 admissible, then

$$rtr(\mathcal{A}) < \alpha \longrightarrow sr(\mathcal{A}) < \alpha. \tag{8.5}$$

What matters more is what can be expressed inside $L(\alpha, T)$ when $\alpha \leq \omega_1$ and $L(\alpha, T)$ is Σ_1 admissible. Let A_δ be the set of all T_δ's on level δ of $\mathcal{RH}(T)$. The set A_δ will be defined by a β-bounded Δ_0^{ZF} formula (7.1), and its definition as such, denoted by $\ulcorner A_\delta \urcorner$, will belong to $L(\alpha, T)$ when $\delta < \alpha$. The fragment $\mathcal{L}_\delta(T_{\delta-})$ will be constructible from $T_{\delta-}$ via an ordinal $\rho_\delta < \alpha$ for all $T_{\delta-} \in A_{\delta-}$. The pair $\ulcorner A_\delta \urcorner$ and ρ_δ will be defined by a simultaneous $\Sigma_1^{L(\alpha,T)}$ recursion uniformly in α, i.e. the same Σ_1 formula will work for all $\alpha \leq \omega_1$ such that $L(\alpha, T)$ is Σ_1 admissible.

Consider an arbitrary T_δ on level δ of $\mathcal{RH}(T)$. There exists a natural **recovery process** that can be applied to T_δ to recover the unique sequence T_γ $(\gamma < \delta)$ such that

$$\begin{aligned} &T_\gamma \text{ is on level } \gamma, \\ &\gamma_1 \leq \gamma_2 \longrightarrow T_{\gamma_1} \subseteq T_{\gamma_2}, \text{ and} \\ &T_\lambda = \cup\{T_\gamma \mid \gamma < \lambda\} \text{ for all limit } \lambda \leq \delta. \end{aligned} \tag{8.6}$$

The recovery proceeds as follows. It begins with: T_0 is $T_\delta \cap \mathcal{L}_0$. If γ is a successor, then

$$T_\gamma = T_\delta \cap \mathcal{L}_\gamma(T_{\gamma-}). \tag{8.7}$$

If γ is a limit, then $T_\gamma = \cup\{T_\beta \mid \beta < \lambda\}$.

The recovery process can be used to decide whether or not an arbitrary set c is a theory on level δ of $\mathcal{RH}(T)$. The answer is yes iff c passes the following tests at all levels $\gamma \leq \delta$.

Level 0: let c_0 be $c \cap \mathcal{L}_0$; c_0 is an extension of T and a finitarily consistent, ω-complete theory of $\mathcal{L}_0$.

Level $\gamma + 1 \leq \delta$: let $\mathcal{L}_{\gamma+1}(c_\gamma)$ be the least fragment extending $\mathcal{L}_\gamma(c_{\gamma-})$ and having as a member the conjunction

$$\wedge\{\mathcal{F}(\overrightarrow{x}) \mid \mathcal{F}(\overrightarrow{x}) \in p(\overrightarrow{x})\} \tag{8.8}$$

for every non-principal n-type $p(\overrightarrow{x})$ of $c_{\gamma-}$. Let $c_{\gamma+1}$ be $c \cap \mathcal{L}_{\gamma+1}(c_\gamma)$. Then $c_{\gamma+1}$ extends c_γ and is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\gamma+1}(c_\gamma)$.

Level λ (limit) $\leq \delta$: Let c_λ be $\cup\{c_\gamma \mid \gamma < \lambda\}$; let $\mathcal{L}_\lambda(c_\lambda)$ be $\cup\{\mathcal{L}_\gamma(c_{\gamma-}) \mid \gamma < \lambda\}$.

In short c is a theory on level δ of $\mathcal{RH}(T)$ iff c satisfies the recovery process on all levels $\gamma \leq \delta$ and $c = c_\delta$. It will follow below that A_δ is β-bounded Δ_0^{ZF} definable (7.1), where β is large enough to define the recovery process.

An effective version of the recovery process is woven into the $\Sigma_1^{L(\alpha,T)}$ recursive definitions of ρ_δ and $\ulcorner A_\delta \urcorner$ for $0 < \delta < \alpha$. The set $\mathcal{L}_\delta(T_{\delta-})$ is constructible from $T_{\delta-}$ via the ordinal ρ_δ for all $T_{\delta-} \in A_{\delta-}$, and $\ulcorner A_\delta \urcorner$ is a β-bounded Δ_0^{ZF}definition of A_δ. The definition $\ulcorner A_\delta \urcorner$ specifies the value of β, and the Δ_0^{ZF} formula.

Stage 0: $\mathcal{L}_0(T_{0-})$ is $\mathcal{L}_0$; A_0 is the set of all finitarily consistent, ω-complete theories of $\mathcal{L}_0$ extending T. Since $\mathcal{L}_0$ is recoverable from T, the set A_0 is $\beta-$bounded Δ_0^{ZF} definable with $\beta = 0$ and parameter T.

Stage $\delta + 1$. Assume the recursion has produced sequences

$$\{\rho_\gamma \mid \gamma \leq \delta\},\ \ \{\ulcorner A_\gamma \urcorner \mid \gamma \leq \delta\} \in L(\alpha, T) \tag{8.9}$$

such that $\ulcorner A_\gamma \urcorner$ is a β-bounded Δ_0^{ZF}definition of A_γ, and $\mathcal{L}_\gamma(T_{\gamma-})$ $(\gamma \leq \delta)$ is first order definable over

$$L[\rho_\gamma, \mathcal{L}_0; T_{\gamma-}]. \tag{8.10}$$

(The definition of (8.10) follows (7.1).) Consider an arbitrary $T_\delta \in A_\delta$ $(\delta > 0)$. Use the recovery process to construct the unique $T_{\delta-} \in A_{\delta-}$ such that

$$T_{\delta-} \subseteq T_\delta \subseteq \mathcal{L}_\delta(T_{\delta-}). \tag{8.11}$$

The recovery is effective thanks to the sequence ρ_γ $(\gamma \leq \delta)$. Now $\mathcal{L}_{\delta+1}(T_\delta)$ can be defined as above (8.2) but with an effective twist. Let ST_δ be the set of all n-types $(n \geqslant 1)$ of T_δ. Since T is weakly scattered, Corollary 3.2 implies

$$ST_\delta \in L(\omega_1^{T_\delta}, T_\delta), \tag{8.12}$$

the least Σ_1 admissible set with T_δ as a member. Let

$$\gamma_{T_\delta} = (least\ \gamma)[ST_\delta \in L(\gamma, T_\delta)]. \tag{8.13}$$

By Theorem 3.3, the ordinal γ_{T_δ}, as a function of T_δ, is uniformly Σ_1; the same Σ_1^{ZF} formula singles out γ_{T_δ} in $L(\omega_1^{T_\delta}, T_\delta)$ for every $T_\delta \in A_\delta$ and for all δ. By Theorem 7.1(i), there is a γ_δ such that

$$(\forall T_\delta \in A_\delta)[\gamma_{T_\delta} \leq \gamma_\delta < \alpha]. \tag{8.14}$$

Hence $ST_\delta \in L(\gamma_\delta, T_\delta)$ for all $T_\delta \in A_\delta$. Theorem 7.1(ii) implies that γ_δ, as a function of δ, has a uniform Σ_1 definition utilizing the parameters occurring in $\ulcorner A_\delta \urcorner$ and the uniform Σ_1 definition of γ_{T_δ}. Any n-type $p(\overrightarrow{x}) \in ST_\delta$ for any $T_\delta \in A_\delta$ is constructible from T_δ via some ordinal less than γ_δ.

A set $\mathcal{P}_\delta$ of first order definitions can be assembled at level γ_δ of $L(\alpha, T)$ as follows. Let

$$\{p_j^{T_\delta} \mid j \in \mathcal{J}_\delta\} \tag{8.15}$$

be the set of all first order definitions over $L(\gamma, T)$ for all $\gamma < \gamma_\delta$ with parameter $\mathcal{T}_\delta$. For each $T_\delta \in A_\delta$, the object $p_j(T_\delta)$ is the set defined by $p_j(\mathcal{T}_\delta)$ when the parameter $\mathcal{T}_\delta$ is assigned the value T_δ. The set (8.15) has a natural wellordering W_δ definable at level γ_δ, since each $p_j^{\mathcal{T}_\delta}$ is specified by its level $\gamma < \gamma_\delta$ and its Gödel number $e < \omega$ as a formula of ZF. The type $d_\delta(\mathcal{T}_\delta)$, the **default type for** $\mathcal{T}_\delta$, is defined by its action on $T_\delta \in A_\delta$:

$$j(T_\delta) = (\text{least } j \text{ in sense of } W_\delta)[p_j(T_\delta) \text{ is an } n\text{-type of } T_\delta]; \tag{8.16}$$

$$d_\delta(T_\delta) = p_{j(T_\delta)}(T_\delta). \tag{8.17}$$

The formula $p_j^{\mathcal{T}_\delta}$ is a slight variant of $p_j(\mathcal{T}_\delta)$ and is defined by its action on $T_\delta \in A_\delta$.

$$p_j^{T_\delta} = \begin{cases} p_j(T_\delta) \text{ if } p_j(T_\delta) \text{ is an } n\text{-type of } T_\delta, \\ \\ d_\delta(T_\delta), \text{ the default type, otherwise.} \end{cases} \tag{8.18}$$

Let $\mathcal{P}_\delta = \{p_j^{\mathcal{T}_\delta} \mid j \in \mathcal{J}_\delta\}$. Then

(1) For all $T_\delta \in A_\delta$ and $p(\overrightarrow{x}) \in ST_\delta$, there is a $j \in \mathcal{J}_\delta$ such that $p_j^{T_\delta}$ defines $p(\overrightarrow{x})$ at level γ_δ of $L(\alpha, T)$,

(2) and $p_j^{T_\delta} \in ST_\delta$ for all $T_\delta \in A_\delta$ and all $j \in \mathcal{J}_\delta$.

It can happen for some $T_\delta \in A_\delta$ and $j, k \in \mathcal{J}_\delta$ that $j \neq k$ but $p_j^{T_\delta} = p_k^{T_\delta}$. Such repetitions are the price paid to have $\mathcal{P}_\delta \in L(\gamma_\delta + 1, T)$.

The ordinal $\rho_{\delta+1} < \alpha$ is chosen just large enough to develop the sequence ρ_γ $(\gamma \leq \delta)$ needed for the recovery of $T_{\delta-}$ from T_δ $(\delta > 0)$, and the ordinal γ_δ needed to assemble $\mathcal{P}_\delta$. The set $\mathcal{L}_{\delta+1}(T_\delta)$ is first order definable over $L[\rho_{\delta+1}, \mathcal{L}_0; T_\delta]$; its definition begins with $\mathcal{L}_\delta(T_{\delta-})$, adds the conjunction of all formulas in $p_j^{\mathcal{T}_\delta}$ for each $p_j^{\mathcal{T}_\delta} \in \mathcal{P}_\delta$, and closes under the finitary operations that generate a fragment of $\mathcal{L}_{\omega_1,\omega}$.

To complete stage $\delta + 1$, construe $A_{\delta+1}$ to be the set of all x such that the effective version of the recovery process applied to x reports that x is a theory on level $\delta + 1$ of $\mathcal{RH}(T)$. The effective version uses the sequence ρ_γ $(0 < \gamma \leq \delta + 1)$ to define $\mathcal{L}_\gamma(T_{\gamma-})$ from $T_{\gamma-}$ for all $T_{\gamma-} \in A_{\gamma-}$. Thus $A_{\delta+1}$ is β-bounded Δ_0^{ZF} definable with β equal to $\rho_{\delta+1}$, and $\ulcorner A_{\delta+1} \urcorner \in L(\alpha, T)$. The parameter specified by $\ulcorner A_{\delta+1} \urcorner$ is T.

Stage λ (limit). Assume for $0 < \gamma < \lambda$ that $\mathcal{L}_\gamma(T_{\gamma-})$ is constructible from $T_{\gamma-}$ via ρ_γ for all $T_{\gamma-} \in A_{\gamma-}$. Use the effective version of the recovery process to define A_λ as a β-bounded Δ_0^{ZF} class. For $T_\gamma \in A_\lambda$, effectively recover the unique sequence T_γ $(\gamma < \lambda)$ such that T_λ is $\cup\{T_\gamma \mid \gamma < \lambda\}$, and then define $\mathcal{L}_\lambda(T_\lambda)$ to be $\cup\{\mathcal{L}_\gamma(T_{\gamma-}) \mid 0 < \gamma < \lambda\}$.

Makkai[8] showed: if T is a counterexample to Vaught's Conjecture, then T has a model of cardinality ω_1 that is $\mathcal{L}_{\infty,\omega}$ equivalent to a countable model. The following are variants of his results.

Suppose A is a countable Σ_1 admissible set and $T \in A$. Assume $T \subseteq \mathcal{L}_0$, $\mathcal{L}_0$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}$, and $\mathcal{L}$ is a countable first order language.

Also assume every symbol of $\mathcal{L}$ is mentioned in T so that $\mathcal{L}$ is recoverable from T. Let $\mathcal{L}'$ denote an arbitrary fragment of $\mathcal{L}_{\omega_1,\omega}$ that extends $\mathcal{L}$, and T' an arbitrary finitarily consistent, ω-complete theory contained in $\mathcal{L}'$ and extending T. Call T **weakly scattered in** A iff $ST' \in A$ for all $T' \in A$. Theorem 3.3 implies the next result.

Theorem 8.1. *Suppose $\mathcal{A}$ is a countable model of T. Assume T is weakly scattered in $L(\omega_1^{T,\mathcal{A}}, \langle T, \mathcal{A}\rangle)$, and*

$$sr(\mathcal{A}) \geq \omega_1^{T,\mathcal{A}}.$$

Then $\mathcal{A}$ is $\mathcal{L}_{\infty,\omega}$ equivalent to a model of T of cardinality ω_1.

Proof. Let $\alpha = \omega_1^{T,\mathcal{A}}$. Thus $\omega_1^{\mathcal{A}} = \alpha$, since $\omega_1^{\mathcal{A}} + 1 \geq sr(\mathcal{A})$. Let $T_\beta^{\mathcal{A}}$ ($\beta \leq sr(\mathcal{A})$) be the Scott analysis of $\mathcal{A}$ as defined in Section 2. By Theorem 3.3 $ST_\beta^{\mathcal{A}} \in L(\alpha, \langle T, \mathcal{A}\rangle)$ (and so $T_\beta^{\mathcal{A}}$ has a countable atomic model) for all β such that $\beta + 1 < sr(\mathcal{A})$. The set Z is $\Sigma_1^{L(\alpha,\langle T,\mathcal{A}\rangle)}$ and consists of the following sentences:

(Z1) the atomic diagram (in the sense of $\mathcal{L}_{\omega_1,\omega}$) of $L(\alpha, \langle T, \mathcal{A}\rangle)$.

(Z2) $\underline{d}$ is a countable ordinal and $\underline{d} \geq \delta$ (all $\delta < \omega_1^{T,\mathcal{A}}$).

(Z3) $\forall y[y < \underline{d} \rightarrow T_y^{\mathcal{A}}$ has a countable atomic model].

(Z3). axioms of Σ_1 admissibility.

The set Z is consistent since it can be modeled by V (the real world). Every model of Z is an end extension of $L(\alpha, \langle T, \mathcal{A}\rangle)$. Let M be a model of Z that omits α. Thus M has non-standard ordinals greater than every ordinal less than α. Hence $sr(\mathcal{A}) \geq \alpha$ in V and $\alpha \notin M$, so $sr(\mathcal{A}) \geq \gamma$ for some non-standard $\gamma \in M$.

Now work inside M. Let $T_\delta^{\mathcal{A}}$ ($\delta \leq \gamma$) be the Scott analysis of $\mathcal{A}$ up to level γ. Choose a non-standard $\beta < \gamma$. Then $T_\beta^{\mathcal{A}}$ has a countable atomic model $\mathcal{A}_\beta$. There is a map

$$i_{\beta\gamma} : \mathcal{A}_\beta \rightarrow \mathcal{A} \tag{8.19}$$

that is elementary with respect to all formulas of $\mathcal{L}_\beta^{\mathcal{A}}$ (defined in Section 2). Note that $i_{b\gamma}$ is not onto, since $\mathcal{A}_\beta$ is not isomorphic to $\mathcal{A}$ in M.

But $\mathcal{A}_\beta$ is isomorphic to $\mathcal{A}$ in V. Now $\omega_1^{\mathcal{A}_\beta} \leq \alpha$ since $\alpha \notin M$; also $sr(\mathcal{A}_\beta) \geq \delta$ for all $\delta < \alpha$, hence $sr(\mathcal{A}_\beta) \geq \alpha$, and so $\omega_1^{\mathcal{A}_\beta} \geq \alpha$. Thus both $\mathcal{A}_\beta$ and $\mathcal{A}$ are homogeneous models of $T_\alpha^{\mathcal{A}}$ by (2.6). To see they realize the same types of $T_\alpha^{\mathcal{A}}$, choose $p_\alpha \in ST_\alpha^{\mathcal{A}}$ and first suppose $\mathcal{A}_\beta \models p_\alpha(\bar{b})$. In M, note that $\mathcal{A}_\beta \models p_\beta(\bar{b})$ for some type p_β of $T_\beta^{\mathcal{A}}$, and that $\mathcal{A} \models p_\gamma(i_{\beta\gamma}(\bar{b}))$ for some type p_γ of $T_\gamma^{\mathcal{A}}$. Then

$$p_\alpha \subseteq p_\beta \subseteq p_\gamma \tag{8.20}$$

since $i_{\beta\gamma}$ is $\mathcal{L}_\beta^{\mathcal{A}}$ elementary. Hence $\mathcal{A} \models p_\alpha(i_{\beta\gamma}(\bar{b}))$. It follows that

$$i_{\beta\gamma} \text{ is } \mathcal{L}_{\omega_1,\omega} \text{ elementary,} \tag{8.21}$$

since the types of $T_\alpha^{\mathcal{A}}$ realized in $\mathcal{A}_\beta$ are atoms of $\mathcal{L}_{\omega_1,\omega}$.

Now suppose $\mathcal{A} \models p_\alpha(\overline{a})$. In M, the tuple $\overline{a}$ realizes p_γ in $\mathcal{A}$, a type of $T_\gamma^{\mathcal{A}}$. Choose a non-standard $\delta < \beta$. Let p_β be the restriction of p_γ to $\mathcal{L}_\beta^{A}$, and let p_δ be the restriction to $\mathcal{L}_\delta^{\mathcal{A}}$. Then $p_\alpha \subseteq p_\delta \subseteq p_\beta \subseteq p_\gamma$. So

$$\mathcal{A} \models \exists \overline{x} p_\delta(\overline{x}). \tag{8.22}$$

But then $\exists \overline{x} p_\delta(\overline{x}) \in T_{\delta+1} \subseteq T_\beta$, so p_δ, hence p_α, is realized in $\mathcal{A}_\beta$.

Thanks to the above there exist structures $\mathcal{B}_0$ and $\mathcal{B}_1$, both isomorphic to $\mathcal{A}$, such that $\mathcal{B}_0 \subsetneqq \mathcal{B}_1$ and the inclusion map i is $\mathcal{L}_{\omega_1,\omega}$ elementary. A strictly expanding $\mathcal{L}_{\omega_1,\omega}$ elementary chain $\mathcal{B}_\delta$ $(\delta \leq \omega_1)$ is defined by iterating i.

For $\delta < \omega_1$, assume $\mathcal{B}_\delta$ is isomorphic to $\mathcal{A}$. Then enlarge $\mathcal{B}_\delta$ to $\mathcal{B}_{\delta+1}$, another copy of $\mathcal{A}$.

For limit $\lambda \leq \omega_1$, let $\mathcal{B}_\lambda$ be the union of the $\mathcal{B}_\delta$'s $(\delta < \lambda)$.

$\mathcal{B}_{\omega_1}$ is an $\mathcal{L}_{\omega_1,\omega}$ elementary extension of $\mathcal{B}_0$, hence $\mathcal{L}_{\omega_1,\omega}$- equivalent to $\mathcal{A}$, consequently $\mathcal{L}_{\infty,\omega}$-equivalent to $\mathcal{A}$. □

Corollary 8.2. *Suppose T is weakly scattered. For each $\beta < \omega_1^T$, assume T has a model of Scott rank $\geq \beta$. Then T has a countable model $\mathcal{A}$ such that*

$$sr(\mathcal{A}) \geq \omega_1^{T,\mathcal{A}} = \omega_1^T,$$

and every such $\mathcal{A}$ is $\mathcal{L}_{\infty,\omega}$ equivalent to a model of T of cardinality ω_1.

9. Bounds on Weakly Scattered Theories

Once again let $\mathcal{L}_0$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ for some countable first order language $\mathcal{L}$, and $T \subseteq \mathcal{L}_0$ a weakly scattered theory with a model. Assume $L(\alpha, T)$ is Σ_1 admissible. Consider B_α, a $\Delta_1^{L(\alpha,T)}$ set of sentences designed so that every model of B_α constitutes a node on level α of $\mathcal{RH}(T)$, the raw hierarchy for T. The axioms of B_α are:

(1) $T \subseteq T_0$ and T_0 is a finitarily consistent, ω-complete theory of $\mathcal{L}_0$.
(2) T_δ has a non-principal n-type for some n (all $\delta < \alpha$).
(3) $T_\delta \subseteq T_{\delta+1}$ and $T_{\delta+1}$ is a finitarily consistent, ω-complete theory of $\mathcal{L}_{\delta+1}(T_\delta)$ (all $\delta < \alpha$).
(4) $T_\lambda = \cup\{T_\delta \mid \delta < \lambda\}$ and $\mathcal{L}_\lambda(T_\lambda) = \cup\{\mathcal{L}_\delta(T_{\delta-}) \mid \delta < \lambda\}$ (all limit $\lambda < \alpha$).

Then B_α is $\Delta_1^{L(\alpha,T)}$ because Section 8 shows how to construct $\mathcal{L}_\delta(T_{\delta-})$ from $T_{\delta-}$ via the ordinal ρ_δ defined by a $\Sigma_1^{L(\alpha,T)}$ recursion on $\delta < \alpha$.

Sets $\mathcal{P}_\delta$ and $\mathcal{J}_\delta$ were defined just after (8.14). Define ***p* is on level *δ*** by

$$p = p_j^{\mathcal{T}_\delta} \text{ for some } j \in \mathcal{J}_\delta. \tag{9.1}$$

A **split at level *δ*** is a sentence of the form: p is on level δ, and there exist r and r' on level $\delta + 1$ such that $r \neq r'$ and both r and r' extend p. The sentence in abbreviated form is $< p, r, r' >$. A split is a sentence of $\mathcal{L}_{\omega_1,\omega} \cap L(\alpha, T)$, because $\mathcal{P}_\delta,\ \mathcal{P}_{\delta+1} \in L(\alpha, T)$. The triple $< p, r, r' >$

is a ***k*−split** if p has arity k. Let K be a set of k-splits**.** The set K is **unbounded** iff

$$\forall\beta < \alpha(\exists\delta > \beta)[K \text{ has a } k\text{-split on level } \delta]. \tag{9.2}$$

K has the **predecessor property** iff there is a partial function $f(p,\gamma)$ such that: if $\gamma < \delta$ and $< p, r, r' >\in K$ and asserts p splits at level δ , then $f(p,\gamma)$ is defined and belongs to $\mathcal{J}_\gamma$, and

$$B_\alpha \vdash [< p, r, r' >\longrightarrow (p^{\mathcal{T}_\gamma}_{\boldsymbol{f(p,\gamma)}} \text{ is extended by } p)]. \tag{9.3}$$

If such an f exists, then there is one that is $\Sigma_1^{L(\alpha,T)}$ definable, since the $\Delta_1^{L(\alpha,T)}$ definability of B_α implies the deduction claimed by (9.3) can be found in $L(\alpha, T)$.

The **effective** k**-splitting hypothesis** holds for T at α iff there exists an unbounded $\Delta_1^{L(\alpha,T)}$ set K of k-splits such that K has the predecessor property and $B_\alpha \cup K$ is consistent (in the sense of $\mathcal{L}_{\omega_1,\omega}$ restricted to $L(\alpha,T)$) if B_α is. Consider Makkai's example [7] (also [5]) mentioned in Section 1. It can be formulated as a fragment $\mathcal{L}_0$ and a theory $T_M \subseteq \mathcal{L}_0$, both arithmetically definable, with the following properties:

(1) T_M is not weakly scattered.

(2) Every countable model $\mathcal{A}$ of T_M has Scott rank at most $\omega_1^{\mathcal{A}}$.

(3) For every countable Σ_1 admissible $L(\alpha)$, there is a countable model $\mathcal{A}$ of T_M such that $\omega_1^{\mathcal{A}} = \alpha = sr(\mathcal{A})$.

Despite (1) it is possible to develop a crude hierarchy for T_M with a superficial resemblance to the raw hierarchy $\mathcal{RH}(T)$ of Section 8. For $\delta < \omega_1$ put theory $T' \supseteq T_M$ on level δ if there exists a countable model $\mathcal{A}$ of T_M such that $sr(\mathcal{A}) = \delta$ and $T' = T^{\mathcal{A}}_{sr(\mathcal{A})}$ (as defined in Section 2). Since T_M is not weakly scattered, it is not possible to give a bounded description of all types associated with all theories on level δ, as was done with $\mathcal{P}_\delta$ in Section 8. Nonetheless some of the types on level δ have properties that lend credence to the effective k-splitting hypothesis. The model $\mathcal{A}$ of (3) above is a tree with ω many levels and infinite paths. Some nodes of $\mathcal{A}$ have foundation rank $(fr) < \infty$. Foundation rank $\omega\delta + m$ corresponds to atoms of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ of rank δ. Associated with level δ of $\mathcal{CH}(T_M)$, the crude hierarchy for T_M, are types of the form

$$x \text{ is on level } \delta \text{ of } \mathcal{A} \text{ and } fr(x) \geq \omega\delta + m \tag{9.4}$$

that split on level $\delta + 1$ of $\mathcal{CH}(T)$. On level $\gamma < \delta$ (9.4) has a predecessor similar to(9.4) with δ replaced by γ.

Theorem 9.1. *Suppose T is weakly scattered and $L(\alpha,T)$ is countable and Σ_2 admissible. For each $\beta < \alpha$, suppose T has a model of Scott rank at least β. If for some k, the effective k-splitting hypothesis holds for T at α, then T has a countable model $\mathcal{A}$ such that*

$$\omega_1^{\mathcal{A}} = \alpha \text{ and } sr(\mathcal{A}) = \alpha + 1.$$

Proof. By Barwise Compactness, T has a model $\mathcal{A}$ such that $L(\alpha, < T, \mathcal{A} >)$ is Σ_1 admissible and $sr(\mathcal{A}) \geq \alpha$. Then $rtr(\mathcal{A}) \geq \alpha$ by (8.5) and so B_α is consistent. Let K be an unbounded $\Delta_1^{L(\alpha,T)}$ set of k-splits with a $\Sigma_1^{L(\alpha,T)}$ predecessor function $f(\gamma, p)$. A model of $B_\alpha \cup K$ is constructed so that T_α has a non-principal type q_α and the structure

$$L[\alpha, T; T_\alpha, q_\alpha] \tag{9.5}$$

is Σ_1 admissible with respect to Σ_1 formulas that include T_α and q_α as atomic predicates. Then, as in the type omitting proof of Theorem 6.1, T has a model $\mathcal{A}_1$ realizing q_α and such that $\omega_1^{\mathcal{A}_1} = \alpha$. The universe of (9.5) is the result of iterating first order definability through the ordinals less than α starting with T and with T_α, q_α as additional atomic predicates. The construction of (9.5) is Henkinesque and gradually decides all sentences of rank less than α in a standard language $\mathcal{L}_{\alpha,T} \in \Delta_1^{L(\alpha,T)}$ that names all elements of (9.5) and is able to express how each one is defined from those of lower definability rank. The language $\mathcal{L}_{\alpha,T}$ does not have symbols T_α or q_α but does have symbols T_β and q_β for all $\beta < \alpha$. There is one twist. The Σ_1 admissibility of (9.5) is not obtained by an effective type omitting argument that omits α as in the proof of Theorem 6.1, but by direct manipulation of ranked sentences of $\mathcal{L}_{\alpha,T}$. The twist avoids Henkin constants.

Let S_n be the set of sentences chosen by the end of stage n. The set S_n will be $\Sigma_2^{L(\alpha,T)}$ definable. The definition of S_0 requires some preparation. Consider $p_j^{\mathcal{T}_\gamma}$ for some $j \in \mathcal{J}_\gamma$. $p_j^{\mathcal{T}_\gamma}$ is said to be K-**unbounded** if the set of all δ such that

$$\exists < p, r, r' > [< p, r, r' > \in K, p \text{ is on level } \delta, f(p, \gamma) = p_j^{\mathcal{T}_\gamma}] \tag{9.6}$$

is unbounded in α. Thus $B_\alpha \cup K$ implies $p_j^{\mathcal{T}_\gamma}$ has unboundedly many extensions that split in K. K-unboundedness is a $\Pi_2^{L(\alpha,T)}$ property. K-**bounded** means: not K-unbounded.

$$\text{Claim: For all } \gamma \text{ there is a } K\text{-unbounded type on level } \gamma. \tag{9.7}$$

Suppose not. Then for each $j \in \mathcal{J}_\gamma$, there is a least β_j such that for all $\delta \geq \beta_j$, (9.6) is false. The ordinal β_j, as a function of j, is $\Sigma_2^{L(\alpha,T)}$, hence bounded by some $\beta_\infty < \alpha$. But then K is bounded by β_∞. A set $U \subseteq K$ is said to be **bounded** if

$$\exists \beta < \alpha (\forall \delta > \beta)[U \text{ does not have a } k\text{-split on level } \delta].$$

Definition of S_0. Start with $B_\alpha \cup K$. Add:

(1) sentences of $\mathcal{L}_{\alpha,T}$ that express how each element of (9.5) is defined from elements of lower rank;
(2) q_β is a type on level β $(\beta < \alpha)$;
(3) q_β is extended by q_γ $(\beta < \gamma < \alpha)$;
(4) $q_\beta \neq p$ ($\beta < \alpha$ and p is K-bounded).

Note that "q_β is a type on level β" is a ranked sentence, in particular a disjunction, by the remarks following (8.14).

S_0 is $\Sigma_2^{L(\alpha,T)}$ definable since K-boundedness is $\Sigma_2^{L(\alpha,T)}$. To check the consistency of S_0, let M be a model of $B_\alpha \cup K$ that specifies the structure of $L(\alpha, T; T_\alpha)$ but says nothing about q_γ for any $\gamma < \alpha$. Fix $\tau < \alpha$. Suppose $\gamma < \tau$; then M can be interpreted as a model of those sentences in S_0 that mention q_γ only for $\gamma < \tau$. Choose a K-unbounded p_τ on level τ with the aid of 9.7. Define

$$U_\tau \quad = \quad \{s \mid \exists t, t'[< s, t, t' >\in K] \text{ and } f(s,\tau) = p_\tau\}, \tag{9.8}$$
$$U_\gamma^r \quad = \quad \{s \mid s \in U_\tau \wedge f(s,\gamma) = r\} \ (\gamma < \tau). \tag{9.9}$$

Fix $\gamma < \tau$. There must be a K-unbounded r on level γ. Suppose not. Then U_γ^r is bounded for every r on level γ. But

$$U_\tau = \cup\{U_\gamma^r \mid r \text{ is on level } \gamma\}. \tag{9.10}$$

Hence U_τ is bounded by the Σ_2 admissibility argument used to prove (9.7), and so p_τ is K-bounded.

For each $\gamma < \tau$, choose a K-unbounded r_γ on level γ. To see that for each $\gamma < \tau$,

$$B_\alpha \cup K \vdash r_\gamma \text{ is extended by } p_\tau, \tag{9.11}$$

let $s \in U_\gamma^{r_\gamma}$. Then $s \in U_\tau$. Assume $B_\alpha \cup K$. Then s extends $f(s,\tau) = p_\tau$ and s extends $f(s,\gamma) = r_\gamma$. Hence p_τ extends r_γ.

It follows from (9.11) that

$$B_\alpha \cup K \vdash r_{\gamma_1} \text{ is extended by } r_{\gamma_2} \tag{9.12}$$

when $\gamma_1 < \gamma_2 < \tau$. Now M, as promised above, can be interpreted as a model of that part of S_0 that mentions q_γ only for $\gamma < \tau$ by setting the interpretation of q_γ in M equal to that of r_γ.

Definition of S_{n+1}. Assume S_n is consistent and $\Sigma_2^{L(\alpha,T)}$. There are two cases.

Case a. Suppose $\mathcal{F} = \vee\{\mathcal{F}_i \mid i \in I\}$ is a ranked sentence such that $S_n \cup \{\mathcal{F}\}$ is consistent. S_{n+1} is $S_n \cup \{\mathcal{F}_{i'}\}$ for some $i' \in I$ such that $S_n \cup \{\mathcal{F}_{i'}\}$ is consistent.

Case b. The purpose of this case is to establish Δ_0 bounding, hence Σ_1 replacement, for (9.5). Let $\mathcal{D}(x,y)$ be a Δ_0^{ZF} formula with constants naming elements of (9.5). Fix $\rho < \alpha$, and regard $\mathcal{D}(x,y)$ as possibly defining a many-valued function $d(x)$ from ρ into α that is Δ_0 in the sense of (9.5) For each $\delta < \rho$, define

$$H_\delta = \{\neg D(\delta,\gamma) \mid \gamma < \alpha\}. \tag{9.13}$$

*Subcase b*1. Suppose there is a $\delta < \rho$ such that $S_n \cup H_\delta$ is consistent. Let δ' be such a δ, and put S_{n+1} equal to $S_n \cup H_{\delta'}$. Then $d(\delta')$ will be undefined.

*Subcase b*2. Suppose $b1$ fails. Then for each $\delta < \rho$:

$$S_n \vdash \vee\{D(\delta,\gamma) \mid \gamma < \alpha\}; \tag{9.14}$$

so by Barwise Compactness there is a $c(\delta) < \alpha$ such that

$$S_n \vdash \vee\{D(\delta,\gamma) \mid \gamma < c(\delta)\}. \tag{9.15}$$

$c(\delta)$ can be defined via deductions from S_n as a $\Sigma_2^{L(\alpha,T)}$ function of δ. Let c be $\sup\{c(\delta) \mid \delta < \rho\}$. Then $c < \alpha$ and $d(\delta)$ $(\delta < \rho)$ will be bounded by c.

Define $S = \cup\{S_n \mid n < \omega\}$. By Case a, S specifies (9.5). q_α is a non-principal type of T_α, because for every $\beta < \alpha$, S_0 and (9.7) compel q_β to be K-unbounded and consequently to split. (An instance of *Case a* results in the choice of a K-unbounded p such that $(q_\beta = p)$ belongs to S.) By *Case b*, (9.5) is Σ_1 admissible. It follows, as in the proof of Theorem 6.1, that T has a model $\mathcal{A}_1$ that realizes q_α and such that $\omega_1^{\mathcal{A}_1} = \alpha$. Hence $sr(\mathcal{A}) = \alpha + 1$. □

Corollary 9.2. *(Bounding) Suppose T is weakly scattered and for some k satisfies the effective k-splitting hypothesis at α. If $L(\alpha,T)$ is Σ_2 admissible and*

$$(\forall \text{ countable } \mathcal{A})\left[\mathcal{A} \models T \longrightarrow sr(\mathcal{A}) \leq \omega_1^{\mathcal{A}}\right], \tag{9.16}$$

then

$$(\exists\beta < \alpha)(\forall\mathcal{A})\left[\mathcal{A} \models T \longrightarrow sr(\mathcal{A}) < \beta\right]. \tag{9.17}$$

10. Further Results and Open Questions

Weakening the assumption of effective k-splitting in Section 9 is under study. At this writing it appears likely that the predecessor (9.3) property can be dropped from the assumption: all that is needed is an unbounded $\Delta_1^{L(\alpha,T)}$ set of k-splits consistent with B_α; then the existence of a predecessor function can be proved. There is a price to pay: the type structure $p_j^{\mathcal{T}_\delta}$ $(\delta < \alpha)$ of a weakly scattered theory T has to be treated with greater delicacy. A further weakening, less likely but more than plausible, is to rule out the existence of RN-models of T. Call $\mathcal{A}$ an **RN-model** of T iff (i) $sr(\mathcal{A}) = \omega_1^{\mathcal{A}}$, (ii) $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ is ω-categorical, and (iii) for each n there is a $\beta < \omega_1^{\mathcal{A}}$ such that each principal n-type of $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ of arity n is generated by a formula of rank less than β. (The theory $T^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ is defined in Section 2.) Makkai[7] produces an $\mathcal{A}$ that satisfies (i) and (ii) but not (iii).

It appears that iterated forcing has a role to play above and also in the construction of an α-saturated model of T when T is weakly scattered and has countable models of unbounded Scott rank. But that is another story.

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