# Atomic models higher up

J. Millar and G. E. Sacks
Brown University and Harvard University

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**Abstract**

There exists a countable structure $\mathcal{M}$ of Scott rank $\omega_1^{CK}$ where $\omega_1^{\mathcal{M}} = \omega_1^{CK}$ and where the $\mathfrak{L}_{\omega_1^{CK},\omega}$-theory of $\mathcal{M}$ is not $\omega$-categorical. The Scott rank of a model is the least ordinal $\beta$ where the model is prime in its $\mathfrak{L}_{\omega\beta,\omega}$-theory. Most well-known models with unbounded atoms below $\omega_1^{CK}$ also realize a non-principal $\mathfrak{L}_{\omega_1^{CK},\omega}$-type; such a model that preserves the $\Sigma_1$-admissibility of $\omega_1^{CK}$ will have Scott rank $\omega_1^{CK}+1$. Makkai ([4]) produces a hyperarithmetical model of Scott rank $\omega_1^{CK}$ whose $\mathfrak{L}_{\omega_1^{CK},\omega}$-theory is $\omega$-categorical. A computable variant of Makkai's example is produced in [2].

## Introduction

The structure of the paper is as follows. In the introduction we discuss Scott rank. The majority of the paper is then spent constructing the theory $th^{\omega_1^{CK}}$ by reverse engineering a Scott analysis. Finally we apply Barwise compactness to the theory developed earlier in order to construct our desired model $\mathcal{M}$.

Our method is something like a Henkin construction, in that we first build a theory and then build a model satisfying the theory. We give a brief outline of the general argument. The theory $th^{\omega_1^{CK}}$ has complexity $\Delta_1^{\omega_1^{CK}}$ and is also complete and consistent in a countable language fragment of $\mathfrak{L}_{\omega_1^{CK},\omega}$. Additionally, $th^{\omega_1^{CK}}$ contains some, but only countably many, non-principal types, all of which fail to be $\Sigma_1^{\omega_1^{CK}}$. The theory in fact has a unique completion in $\mathfrak{L}_{\omega_1^{CK},\omega}$ and the non-principal types continue to be non-principal in $\mathfrak{L}_{\omega_1^{CK},\omega}$.

Having lost compactness as we venture outside of first order logic, we regain weaker version of compactness for certain special fragments, and in particular for $\mathfrak{L}_{\omega_1^{CK},\omega}$. Indeed we consider the language $\mathfrak{L}_{\omega_1^{CK},\omega}$ as a definable class inside the initial fragment $L(\omega_1^{CK})$ of the constructible universe, which is in particular a universe of (KP) set theory. Using Barwise compactness we omit $\omega_1^{CK}$ from a universe of (KP) set theory, resulting in a non-standard end extension of $L(\omega_1^{CK})$, and build inside this universe a model of $th^{\omega_1^{CK}}$ that omits all the non-principal $\mathfrak{L}_{\omega_1^{CK},\omega}$-types. The resulting model respects the $\Sigma_1$-admissibility of $\omega_1^{CK}$, due to the omission of $\omega_1^{CK}$ in the ambient universe, and has Scott rank $\omega_1^{CK}$ due to the omission of the non-principal types. As claimed, this results in a model of Scott rank $\omega_1^{CK}$ that doesn't violate the $\Sigma_1$-admissibility of $\omega_1^{CK}$, and whose $\mathfrak{L}_{\omega_1^{CK},\omega}$-theory is not $\omega$-categorical.

Since the intended audience is model theorists we include basic definitions and facts for the relevant higher recursion theory and set theory.

**Definition 1.** The ordinal $\omega_1^{CK}$ is the least non-recursive ordinal: that is, no recursive well order $A, <$ has order type $\omega_1^{CK}$ and for every $\beta < \omega_1^{CK}$ there is a recursive well order of order type $\beta$. For an arbitrary set $A$ the ordinal $\omega_1^A$ is the least non-recursive ordinal using $A$ as an oracle.

**Definition 2.** A $\Delta_0$ formula of set theory is a formula in the language of set theory with only bounded quantifiers. One can then define the usual hierarchy of $\Sigma_n$ and $\Pi_n$ formulas. One can also define a generalized Kleene predicate, and therefore functions of various $\omega_1^{CK}$-complexity. Intuitively an $\omega_1^{CK}$-recursive function will resolve on any input using a computation with fewer than $\omega_1^{CK}$ many steps. As usual, a superscript refers to an oracle: a formula is $\Sigma_1^{\omega_1^{CK}}$ if it is $\Sigma_1$ relative to oracle calls to $\omega_1^{CK}$.

**Definition 3.** A subset of $L(\beta)$ with a $\Delta_n$ definition is called $\Delta_n(L(\beta))$, or $\Delta_n(\beta)$ for short. Similarly for subsets with $\Sigma_n$ and $\Pi_n$ definitions.

**Fact.** A set is $\Delta_1(\omega)$ exactly if it can be encoded as a computable subset of $\omega$. This is because of the highly effective $\omega$ enumeration of the universe $L(\omega)$. A set is $\Delta_1(\omega_1^{CK})$, that is, it is a subset of $L(\omega_1^{CK})$ with a $\Delta_1$ definition, exactly if it can be encoded as a *hyperarithmetical* subset of $\omega$. Likewise, a set is $\Sigma_1(\omega_1^{CK})$ exactly if it can be encoded as a $\Pi_1^1$ subset of $\omega$. The essential ingredient is the set of ordinal notations, a $\Pi_1^1$-complete subset of the natural numbers encoding $\omega_1^{CK}$. It is natural to refer to $\Delta_1(\omega_1^{CK})$ as $\Delta_1^{\omega_1^{CK}}$. One is then free to choose one of two interpretations: the definable set belongs to

$L(\omega_1^{CK})$ and the formula naturally has oracle calls to the set of ordinals, OR the definable set belongs to the natural numbers and the formula has oracle calls to an encoding of the set of ordinals. (See Sacks [7] for a more thorough explanation.)

Perhaps the most important example of an $\omega_1^{CK}$-recursive function is the projectum map, which is an injection $\pi : \omega_1^{CK} \to \omega$. The projectum map comes from the study of ordinal notations. If $n$ is the ordinal notation for $\beta$ then the partial function $\pi^{-1}$ applied to $n$ returns $\beta$ in $\beta$ steps, essentially the length required to step down the transfinite recursion. In addition for $n \notin \pi(\omega_1^{CK})$ there is no way of determining in fewer than $\omega_1^{CK}$ steps that $\pi^{-1}(n)$ is not defined. Therefore $\pi^{-1}$ is $\omega_1^{CK}$-recursively enumerable but not $\omega_1^{CK}$-recursive. In this paper the projectum map $\pi$ makes an appearance in the high level priority argument. Also the restrictions $\pi_\beta = \pi|_\beta$ are used in the combinatorics of the Scott analysis. Since $\pi_\beta^{-1}$ finishes all its computations in fewer than $\beta$ steps, $\pi_\beta^{-1}$ is $\omega_1^{CK}$-recursive, in contrast to $\pi^{-1}$.

**Barwise Compactness**. Let $\Theta$ be a $\Sigma_1^{\omega_1^{CK}}$ set of sentences. If every $\Delta_1^{\omega_1^{CK}}$ subset of $\Theta$ has a model, then $\Theta$ also has a model.

**Fact.** The initial fragment of the constructible universe of height $\omega_1^{CK}$, or $L(\omega_1^{CK})$, satisfies the axioms of Kripke Platek set theory (KP). Comparing (KP) to the standard (ZF) axioms of set theory, one sees that the power set axiom is omitted and the axiom of replacement is replaced by the weaker $\Sigma_1$ replacement: for all $\Sigma_1$ formulas of set theory $\varphi(x, y)$:

$$\forall X \exists Y (\forall x \in X \exists! y \varphi(x, y)) \to (\forall x \in X \exists! y \in Y \varphi(x, y)).$$

**Definition 4.** The language $\mathfrak{L}_{\beta,\omega}$ is all formulas with conjunction depth $< \beta$ and quantifier depth $< \omega$. This language is a $\Delta_0$ class of $L(\beta)$.

In the definition of Scott rank we consider *infinite* conjunction depth. For example, first order logic $\mathfrak{L}_{\omega,\omega}$ is all formulas with infinite conjunction depth of 0 and $\mathfrak{L}_{\omega 2,\omega}$ is all formulas with infinite conjunction depth of 1. For large limit ordinals of course there is no difference; for example $\mathfrak{L}_{\omega_1^{CK},\omega}$ is all formulas with infinite conjunction depth of $\omega_1^{CK}$.

**Definition 5.** We say that $X$ respects the $\Sigma_1$-admissibility of $\omega_1^{CK}$ iff the the initial fragment of height $\omega_1^{CK}$ of the constructible universe built over $X$, or $L(X, \omega_1^{CK})$, still satisfies (KP).

Given a set $A$ the ordinal $\omega_1^A$ is the least $\beta$ so that $L(A, \beta)$ satisfies (KP). The set $X$ respects the $\Sigma_1$-admissibility of $\omega_1^{CK}$ iff $\omega_1^X = \omega_1^{CK}$, that is, $\omega_1^{CK}$ remains the first non-computable ordinal with $X$ as oracle.

**Definition 6.** The Scott rank of a countable model $\mathcal{M}$ is the minimal $\beta$ so that $\mathcal{M}$ is the prime model of its $\mathfrak{L}_{\omega\beta,\omega}$-theory, which is called the Scott theory of the model $\mathcal{M}$.

This is a top-down definition of Scott rank. Historically Scott rank was based on a variant of the following important algorithmic analysis [9].

**Definition 7.** The Scott analysis of a countable model $\mathcal{M}$ with a countable theory is the following iterative construction of theories and Scott languages. Starting with the first order language $\mathfrak{L}^1(\mathcal{M})$ and first order theory $th^1(\mathcal{M})$ iteratively build $th^{\beta+1}(\mathcal{M})$ and $\mathfrak{L}^{\beta+1}(\mathcal{M})$ by first augmenting the language $\mathfrak{L}^\beta(\mathcal{M})$ with the non-principal types of the theory $th^\beta(\mathcal{M})$, that is,

$$\mathfrak{L}^{\beta+1}(\mathcal{M}) =_{def} \mathfrak{L}^\beta(\mathcal{M}) \cup \{\wedge p(\bar{x}) \mid p \text{ a non-principal type of } th^\beta(\mathcal{M})\}$$

closed under negation, conjunction and quantification, and then extending the theory $th^\beta(\mathcal{M})$ to

$$th^{\beta+1}(\mathcal{M}) =_{def} \mathfrak{L}^{\beta+1}(\mathcal{M}) \text{ theory of } \mathcal{M}.$$

At limits take unions

$$\mathfrak{L}^\gamma(\mathcal{M}) =_{def} \bigcup_{\beta<\gamma} \mathfrak{L}^\beta(\mathcal{M})$$

$$th^\gamma(\mathcal{M}) =_{def} \bigcup_{\beta<\gamma} th^\beta(\mathcal{M}).$$

For an $n$-tuple $\bar{a} \in \mathcal{M}$ and ordinal $\beta$ let $tp^\beta[\bar{a}](\bar{x})$ be the conjunction of the formulas in $n$ free variables in $\mathfrak{L}^\beta(\mathcal{M})$ satisfied by $\bar{a}$; call this the $\mathfrak{L}^\beta(\mathcal{M})$ type of $\bar{a}$. The least ordinal $\beta$ at which the type $tp^\beta[\bar{a}](\bar{x})$ is an orbit under automorphism is called the Scott rank of $\bar{a}$. The least ordinal at which all the types have become orbits under automorphism is called the length of the Scott analysis.

It is common knowledge that if the Scott rank of $\mathcal{M}$ is $\beta$ then the length of the Scott analysis is at most $\omega\beta$. The proof runs as follows. If two elements $a, b \in \mathcal{M}$ are distinguished by a formula $\varphi \in \mathfrak{L}_{\omega\beta,\omega}$ then one can find a formula $\theta_{\varphi,a,b} \in \mathfrak{L}^{\omega\beta}(\mathcal{M})$ that also distinguishes $a$ and $b$. This is done by inductively replacing quantifiers with infinite conjunctions and arguing in the base case that if a partial type distinguishes two elements then indeed a complete type distinguishes two elements. Therefore, if $\overline{tp}^{\beta}[a] \neq \overline{tp}^{\beta}[b]$ then $tp^{\omega\beta}[a] \neq tp^{\omega\beta}[b]$. This is sufficient to show that (length of analysis) $\leq \omega \cdot SR(\mathcal{M})$. A detailed proof is provided in a preprint [5]. As a corollary we have the following.

**Lemma 0.1.** Say $\omega\beta = \beta$ (for example $\beta = \omega_1^{CK}$) and the length of the Scott analysis of $\mathcal{M}$ is $\beta$. If non-principal $\mathfrak{L}^{\beta}(\mathcal{M})$ types are realized in $\mathcal{M}$ then the Scott rank of $\mathcal{M}$ is $\beta + 1$, and otherwise the Scott rank of $\mathcal{M}$ is $\beta$.

Since $\beta = \omega\beta$ we have for all $\bar{a}, \bar{b} \in \mathcal{M}$ that $\overline{tp}^{\beta}[\bar{a}] = \overline{tp}^{\beta}[\bar{b}]$ if and only if $tp^{\beta]}[\bar{a}] = tp^{\beta}[\bar{b}]$. A standard back and forth argument shows that the $\mathfrak{L}^{\beta}(\mathcal{M})$-types are orbits under automorphism exactly when $\mathcal{M}$ is homogeneous in the language $\mathfrak{L}^{\beta}(\mathcal{M})$, that is, for every $\bar{a}_1, \bar{a}_2, \bar{b}_1 \in \mathcal{M}$ with $tp^{\beta}[\bar{a}_1](\bar{b}_1)$ there exists $\bar{b}_2$ with $tp^{\beta}[\bar{a}_1, \bar{a}_2](\bar{b}_1, \bar{b}_2)$. Pick $\gamma < \beta$. Since $\omega\gamma < \beta$ the model $\mathcal{M}$ is not homogeneous in its $\mathfrak{L}^{\omega\gamma}(\mathcal{M})$-theory. By the first part of the lemma this is equivalent to stating that $\mathcal{M}$ is not homogeneous in its $\mathfrak{L}_{\omega\gamma,\omega}$-theory. Therefore $\mathcal{M}$ is not the prime model of its $\mathfrak{L}_{\omega\gamma,\omega}$-theory for any $\gamma < \beta$ and $SR(\mathcal{M}) \geq \beta$. Now, if $\mathcal{M}$ realizes only principal types of $\mathfrak{L}_{\omega\beta,\omega}$ then indeed $\mathcal{M}$ is the prime model of its $\mathfrak{L}_{\omega\beta,\omega}$-theory. If $\mathcal{M}$ realizes non-principal $\mathfrak{L}_{\omega\beta,\omega}$-types then by the homogeneity of the $\mathfrak{L}_{\omega\beta,\omega}$-types each type $\overline{tp}^{\beta+1}[\bar{a}]$ is isolated by the $\mathfrak{L}_{\omega(\beta+1),\omega}$-formula $\vee\overline{tp}^{\beta}[\bar{a}](\bar{x})$ and $\mathcal{M}$ is the prime (and only countable) model of its $\mathfrak{L}_{\omega(\beta+1),\omega}$-theory.

$\square$

**Remark 0.2.** The back and forth argument mentioned above also shows that the Scott analysis produces a **strictly monotone refinement** of the partition into types of $\cup_{n<\omega}\mathcal{M}^n$. That is, for every $\beta$ there must be some $\mathfrak{L}^{\beta}(\mathcal{M})$-type that splits into more than one $\mathfrak{L}^{\beta+1}(\mathcal{M})$ type realized in $\mathcal{M}$. The refinement terminates when the types have become orbits under automorphism.

**Remark 0.3.** If the definition of the Scott analysis is modified by only adding those types realized in the model then the above theorem no longer holds. For example if there is a first order type $p^1(x)$ that is omitted from $\mathcal{M}$ then the $\mathfrak{L}^2(\mathcal{M})$-theory will continue to have $p^1(x)$ as a non-principal type while the larger $\mathfrak{L}_{\omega 2,\omega}$-theory will include $\forall x\ \neg p^1(x)$. However, the lemma still holds, since for all $\beta$ and for all $\bar{a} \in \mathcal{M}$, the sets $tp^{\beta}[\bar{a}]^{\mathcal{M}}$ and $\overline{tp}^{\beta}[\bar{a}]^{\mathcal{M}}$ are equal. Although not considered in this paper, this modification could be a useful observation for making sure the languages remain countable.

Having established the relationship between Scott rank and the length of the Scott analysis, we turn to bounds on Scott rank.

**Lemma 0.4.** (Nadel, [6]) The Scott analysis of a countable model $\mathcal{M}$ has length $\leq \omega_1^{\mathcal{M}}$.

The essential data used in Nadel's bound is the strictly monotone refinement of the types. The cruder observation that $SR(\mathcal{M}) < \omega_1$ for any countable $\mathcal{M}$ also relies on the strictly monotone refinement of types. Nadel's bound can be thought of as the analogous result in the effective realm. The analogy is not precise since the bound of $\omega_1^{\mathcal{M}}$ is obtainable.

**Definition 8.** A model $\mathcal{M}$ has high rank if the length of its Scott analysis is $\omega_1^{\mathcal{M}}$.

In this paper we are concerned with models in the class $\{X \subset \omega \mid \omega_1^X = \omega_1^{CK}\}$, where high rank means Scott rank $\omega_1^{CK}$ or $\omega_1^{CK}+1$. The most easily constructed models of high rank have non-principal types in their $\mathfrak{L}_{\omega_1^{CK},\omega}$ theory. Such models have Scott rank $\omega_1^{CK}+1$ and in particular the Scott theory $th^{\omega_1^{CK}+1}(\mathcal{M})$ is $\aleph_0$-categorical. Makkai's model has Scott rank $\omega_1^{CK}$, and in fact the Scott theory $th^{\omega_1^{CK}}(\mathcal{M})$ is also $\aleph_0$-categorical. The result of this paper can be stated as follows: here is a countable model $\mathcal{M}$ of high rank whose Scott theory is not $\aleph_0$-categorical.

# Section 1

## The first order case

We construct a complete, consistent, and computable first order theory with countably many types, including some non-principal types, where no non-principal type is computably enumerable. This construction is well known in computable model theory and employs a finite injury argument, which naively uses $\Sigma_2$-replacement. Happily, finite injury arguments can generally be finessed to work with $\Sigma_1$-replacement ([7]).

**Lemma 0.5.** ***(well known)*** There is a computable tree $T \subset 2^{<\omega}$ with a non-isolated branch that is not computably enumerable and where all other branches are isolated.

*Proof.* We give the proof in detail to set the stage for the construction in the next section. Requirement $R_i,\ i < \omega$, states that the non-isolated branch is not the characteristic function for the $i^{th}$ recursively enumerable set $S_i$. We effectively decide whether elements of $2^{<\omega}$ belong to our tree $T$ according to an enumeration of $2^{<\omega}$ with parents enumerated before children, keeping in mind a primary candidate $\hat{p}$ and a set of potential candidates $\hat{q}_i$ for the non-isolated branch. As time passes, the primary candidate stabilizes on longer and longer initial segments and after $\omega$ stages becomes a well defined branch in $T$. A *witness* is a pair $W = (w, v) \in \omega \times 2$, where the $w^{th}$ bit of the primary candidate takes value $v$. A witness $W_i$ is used to establish requirement $R_i$. At any stage a requirement is in one of three disjoint states: *unhappy*, *addressed*, or *finitely satisfied*. Additionally, if the requirement is addressed or finitely satisfied it is happy. All transitions between these states are possible except for the transition from finitely satisfied to addressed. Transiitons from happy states to the unhappy state correspond to injuries.

**Property** $*(n)$ means the following:

The set $T$ is a finite initial subtree of $2^{<\omega}$, and membership in $T$ has been determined for the first $n$ nodes in the enumeration of $2^{<\omega}$. The candidate branches – primary and potential – end in terminal nodes $p, \{q_i\}$ of $T$. Some initial segment $R_1, \ldots, R_k, k \leq n$, of the requirements have been either addressed or finitely satisfied and for these requirements there are corresponding witnesses $W_1 = (w_1, v_1), \ldots, W_k = (w_k, v_k)$. The remaining requirements are unhappy. The witnesses occur in order, that is, $w_1 < w_2 < \ldots < w_k$. Also, for $i \leq k$ we have $i < w_i$. Requirement $R_i$ is satisfied if and only if $w_i$ enters the $i^{th}$ r.e. set $S_i$ in fewer than $n$ steps, and in this case $v_i = 0$, that is, the $w_i^{th}$ bit of the primary candidate $\hat{p}$ has chosen the opposite value 0. If $R_i$ is addressed then $w_i$ does not enter $S_i$ in fewer than $n$ steps, and also $v_i = 1$. For each $i$ with $R_i$ addressed there is a potential candidate $q_i$ which agrees with the primary candidate $p$ on the first $w_i - 1$ bits, and takes the opposite value 0 on the bit $w_i$. Finally, the potential candidate $q_i$ is isolated above bit $w_i$.

**Stage** 1 **of the construction.** Initially the tree $T$ is empty and all requirements are unhappy. Check if 1 enters the first r.e. set $S_1$ in 1 step. If it does then set the primary candidate $p$ to the string 0 and set the state of requirement 1 to *finitely satisfied.* If it does not then make the following adjustments: the primary candidate $p$ is the string 1, the first potential candidate $q_1$ is the string 0, the witness $W_1$ is $(1, 1)$ and the state of requirement 1 is *addressed.* The tree now satisfies $*(1)$.

**Stage $n + 1$ of the construction.** Inductively assume $*(n)$.

1. Consider the $(n + 1)^{st}$ node in the enumeration of the tree $2^{<\omega}$. If its membership in the subtree $T$ is undetermined, decide that the node does not belong to $T$ unless its parent belongs to $T$ and its sibling does not belong to $T$, in which case decide the node does belong to $T$. If the parent of the node was a candidate and the node has been added then update that candidate to the node.

2. Let $I$ be the set of indices of addressed requirements. For each $i \in I$ check if $w_i$ enters the $i^{th}$ r.e. set $S_i$ in $\leq n+1$ steps. If not, proceed to 3. Let $j \in I$ be the least index where $w_j$ enters $S_j$ in $\leq n$ steps. If no such $j$ exists proceed to 3. Otherwise the injury begins. For $i > j$ update $R_i$ to unhappy, remove the potential candidate $q_i$ and remove the witness $W_i$. Update the primary candidate $p$ to the potential candidate $q_j$, update $R_j$ to finitely satisfied, and update $W_j = (w_j, 0)$.

3. Let $i$ be the least index where $R_i$ is unhappy. Add a new potential candidate $q_i = p \wedge 0$. Update the primary candidate $p = p \wedge 1$. Define the witness $W_i$ to be $(|p|, 1)$.

We leave it to the reader to verify that the tree now satisfies property $*(n + 1)$.

We claim that for each $n$ the state of requirement $R_n$ eventually stabilizes, either to addressed or finitely satisfied. Notice that requirements do not change from finitely satisfied to addressed, and that if requirements $1, \ldots, n$ have stabilized, then requirement $n + 1$ does not remain unhappy. Therefore the changes of $R_n$ from happy to unhappy (that is, the injuries to requirement $n$) are interleaved with state changes of higher priority requirements (i.e., $R_1, \ldots, R_{n-1}$). It follows inductively that requirement $R_n$ changes states at most a finite number of times, and ends either as addressed or finitely satisfied. This is the classical finite injury argument.

We also claim that the primary candidate $\hat{p}$ is a well defined branch through the tree. Every change in the first $n$ bits of $\hat{p}$ is accompanied by a change of state for a requirement $R_k$ where $k < n$. Recall $k < v_k < n$ where the witness $W_k = (w_k, v_k)$. The requirements $R_1, \ldots, R_n$ eventually stabilize, and therefore so do the first $n$ bits of $\hat{p}$.

As required, we have produced a recursive tree with a unique non-isolated branch $p$. Since all the requirements are satisfied, $p : \omega \to 2$ cannot be the characteristic function for any recursively enumerable set. $\dashv$

**Corollary 0.6.** *There is a complete, consistent, computable theory with countably many types including some non-principal types, where no non-principal type is computably enumerable.*

The signature is countably many unary predicates $\{U_i | i < \omega\}$. For every string $s \in 2^{\leq\omega}$ define

$$\theta_s(x) =_{def} \bigwedge_{i<|s|} \neg^{s_i+1} U_i(x)$$

where $(\neg)^1 U(x) = \neg U(x)$ and $(\neg)^2 U(x) = \neg\neg U(x) = U(x)$. For example, $\theta_{10}(x)$ is $U_1(x) \wedge \neg U_2(x)$. If $t$ is a substring of $s$ then $\theta_s(x)$ implies $\theta_t(x)$. When $|s| < \omega$ the formula $\theta_s(x)$ is first order and when $s$ is a terminal branch, that is, when $|s| = \omega$, we claim the infinite conjunction $\theta_s(x)$ determines a complete one type for any theory in this signature with which it is consistent. For any model $\mathcal{M}$ with this signature there is an associated tree $\{s \in 2^{<\omega} \mid \text{ there exists } a \in M, \theta_s(a)\}$. Conversely for any subtree $T \subset 2^{<\omega}$ we have the model completion $\{\exists^\infty x\ \theta_s(x) \mid s \in T\} \cup \{\forall x\ \neg\theta_s(x) \mid s \notin T\}$ of the universal axioms $\{\forall x\ \neg\theta_s(x) \mid s \notin T\}$. We have just described a bijection between subtrees of $2^{<\omega}$ and a natural subset of theories in this signature. Under this bijection branches of the tree correspond to one-types with no parameters. Equivalently, the type of a single element is completely determined by which unary predicates it satisfies. Isolated branches correspond to principal types: a branch is isolated above node $s$ exactly when $\theta_s(x)$ is a complete formula. Additionally the type of a tuple is determined by the one-types of the elements in the tuple. Therefore if we start with a tree $T \subset 2^{<\omega}$ with a non-isolated, non-recursive branch $p$ where all other branches are isolated, the corresponding theory will have a non-principal, non-recursive one-type $\theta_p(x)$ and all other one-types are principal. An $n$-type will be non-principal and non-recursive exactly if $\theta_p(x)$ is a sub-type, and all other $n$-types are principal. $\dashv$

## The higher order case

**Theorem 0.7.** There is a complete and consistent $\Delta^1(\omega_1^{CK})$ theory $th^{\omega_1^{CK}}$ in a countable fragment $\mathfrak{L}^{\omega_1^{CK}}$, that contains some, but only countably many, non-principal types, all of which fail to be $\Sigma_1(\omega_1^{CK})$. Additionally if $\mathcal{M} \models th^{\omega_1^{CK}}$ then $\mathfrak{L}^{\omega_1^{CK}}(\mathcal{M}) \subset \mathfrak{L}^{\omega_1^{CK}} \subset \mathfrak{L}_{\omega_1^{CK},\omega}$.

We now introduce a variant of the Scott analysis that differs from the original definition because 1) it may include more than complete types at each level and 2) it does not take a structure $\mathcal{M}$ as input.

**Definition 9.** A modified Scott analysis of height $\delta$ is a set of theories and languages $\{\mathfrak{L}^\beta, th^\beta \mid \beta \leq \delta\}$ satisfying the following conditions. The language $\mathfrak{L}^1$ is first order, each $th^\beta$ is a complete and consistent theory in $\mathfrak{L}^\beta$ with only countably many types, for each $\beta < \delta$

$$\mathfrak{L}^{\beta+1}(\mathcal{M}) =_{def} \mathfrak{L}^\beta(\mathcal{M}) \cup P^\beta$$

closed under negation, conjunction and quantification, where $P^\beta$, called the set of non-principal basic partial types, is a countable set of non-principal partial $th^\beta$-types whose closure contains all the complete non-principal $th^\beta$-types. Finally for $\gamma$ a limit ordinal take

$$\mathfrak{L}^\gamma(\mathcal{M}) =_{def} \bigcup_{\beta<\gamma} \mathfrak{L}^\beta(\mathcal{M})$$

$$th^\gamma(\mathcal{M}) =_{def} \bigcup_{\beta<\gamma} th^\beta(\mathcal{M}).$$

Any theory $th^\beta$ belonging to a Scott analysis of height $\beta \leq \delta$ is called a modified Scott theory of height $\delta$

We will need the following variation on lemma 0.1.

**Lemma 0.8.** Say $\omega\beta = \beta$ and that $th^\beta$ is a modified Scott theory of height $\beta$ and that $\mathcal{M}$ is a homogeneous model of $th^\beta$. If non-principal $\mathfrak{L}^\beta$-types are realized in $\mathcal{M}$ then $SR(\mathcal{M}) = \beta + 1$ and otherwise $SR(\mathcal{M}) = \beta$.

The lemma follows from lemma 0.1 and the observation that for $\delta < \beta$ we have $\mathfrak{L}^\delta(\mathcal{M}) \subset \mathfrak{L}^\delta \subset \mathfrak{L}_{\omega\delta,\omega}$. From now on we refer to the modified Scott analysis simply as the Scott analysis. We keep track of the type refinement of the Scott analysis using trees of the kind described below.

**Definition 10.** For this paper a tree is a partially ordered set $T, <$ so that for every $x \in T$ the set $\hat{x} =_{def} \{y \in T \mid y < x\}$ is well ordered by $<$. The height of $x$, $ht(x)$, is the order type of $\hat{x}$. The $\alpha^{th}$ level of $T$ is $T_\alpha =_{def} \{x \in T \mid ht(x) = \alpha\}$. An immediate successor of a node is called a child. A branch refers to a well-ordered cofinal subset of the tree. Our trees will have the following properties:

- for all $\alpha < ht(T)$ the level $T_\alpha$ is countable,
- every element has either 1 or 2 immediate successors,
- for all limit $\alpha$ if $ht(x) = ht(y) = \alpha$ and $\hat{x} = \hat{y}$ then $x = y$,
- and if $ht(x) < \beta$ then $x$ has a successor at level $\beta$.

An element of height $\lambda$ is represented as a string $s : \lambda \to 2$, with $s(\beta)$ the $\beta^{th}$ bit of branch $s$ and $s|_\beta$ the initial segment of $s$ of length $\beta$.

We have defined $P^\delta$ as a set of non-principal basic partial types. We discuss the structure of the non-principal basic partial types in greater detail, and also define basic partial types.

**Definition 11.** We have specific names for the non-principal basic partial types:

$$P^\delta = \{b^\beta_{n,k}(\bar{x}_n) \mid n, k < \omega\} \cup \{q^\beta_{n,\gamma}(\bar{x}_n) \mid n < \omega, \gamma < \omega\beta\} \cup \{Q^\beta_n(\bar{x}_n)\}$$

where $b^\beta_{n,k}(\bar{x}_n)$ is called a regenerating type, $q^\beta_{n,\gamma}(\bar{x}_n)$ is called a priority type, and $Q^\beta_n(\bar{x}_n)$ is called a seed type. These types satisfy the following coherence. Fix $\gamma < \beta$. The restriction of the regenerating type $b^\beta_{n,k}(\bar{x}_n)$ to the language $\mathfrak{L}^\gamma$ is the regenerating type $b^\gamma_{n,k}(\bar{x}_n)$ and likewise the restriction of the seed type $Q^\beta_n(\bar{x}_n)$ to the language $\mathfrak{L}^\gamma$ is the seed type $Q^\gamma_n(\bar{x}_n)$. For $\delta < \omega\gamma$ the restriction of the priority type $q^\beta_{n,\delta}(\bar{x}_n)$ to the language $\mathfrak{L}^\gamma$ is $q^\gamma_{n,\delta}(\bar{x}_n)$. However for $\omega\gamma \leq \delta$ the restriction of $q^\beta_{n,\delta}(\bar{x}_n)$ to $\mathfrak{L}^\gamma$ is the seed type $Q^\gamma_n(\bar{x}_n)$.

**Notational convention:** in the above definition we use the following shorthand: $\bar{x}_n$ stands for the $n$-tuple $x_1, \ldots, x_n$, so for example we write $b^\beta_{n,k}(\bar{x}_n)$ for $b^\beta_{n,k}(x_1, \ldots, x_n)$.

For the next several sections it is unnecessary to understand the point of coherence of the non-principal basic partial types, or for that matter the point of the priority and seed types. We set the stage for the priority argument by using the regenerating types to define the important pseudo-predicates, as follows.

**Definition 12.** The formula

$$U_{n,\omega\gamma+2k}(\bar{x}_n) =_{def} \exists y_{n+1}\ b^{\gamma}_{n+1,k}(\bar{x}_n, y_{n+1})$$

is called a pseudo-predicate, where $b^{\gamma}_{n+1,k}(\bar{x}_n, y_{n+1}) \in P^{\beta}$ is a regenerating type. Likewise the formula

$$U_{n,\omega\gamma+2k+1}(\bar{x}_n)$$

is called a priority pseudo-predicate, where $q^{\gamma}_{n+1,k}(\bar{x}_n, y_{n+1})$ is a priority type.

Again the distinction between priority pseudo-predicates and regenerating pseudo-predicates is not initially important.

**Definition 13.** For a string $s \in 2^{\omega_1^{CK}}$ define

$$\theta_{n,s}(\bar{x}_n) =_{def} \bigwedge_{i<|s|} \neg^{s_i+1} U_{n,s_i}(\bar{x}_n).$$

When the string $s$ has limit ordinal length $|s| = \omega\delta$ then $\theta_{s,n}$ is a candidate for a basic partial type.

**Definition 14.** Associated to our Scott analysis is a sequence of nested trees $\{T^{\beta}_n \mid \beta \leq \omega_1^{CK}\}$ , $ht(T^{\beta}_n) = \omega\beta$. A basic partial type of $\mathfrak{L}^{\beta}$ is a formula $\theta_{n,s}(\bar{x}_n)$ where $s$ is a branch of tree $T^{\beta}_n$. The trees are called the trees of partial types.

For example, if a basic partial type of $th^{\gamma}$ refines into countably many basic partial types of $th^{\gamma+1}$ then the corresponding branch of $T^{\gamma}_n$ will extend to countably many branches of $T^{\gamma+1}_n$.

**Fact 1.** The Scott analysis is constructed in such a way that the set $P^{\delta}$ is exactly the set of basic partial types where the corresponding branch is non-isolated. As one would expect, therefore, a non-principal basic partial type is exactly a basic partial type whose corresponding branch is non-isolated.

We have the following bootstrapping effect. The regenerating 2-types of height $\beta$ generate the unary pseudo-predicates of height $\beta + 1$, which taken together generate the non-principal basic partial 1-types of height $\beta + 1$. Likewise, the regenerating 3-types at level $\beta$ generate the basic partial 2-types of height $\beta + 1$. Since we have basic partial $n$-types for all $n < \omega$ the bootstrapping effect continues indefinitely. See Figure 1.

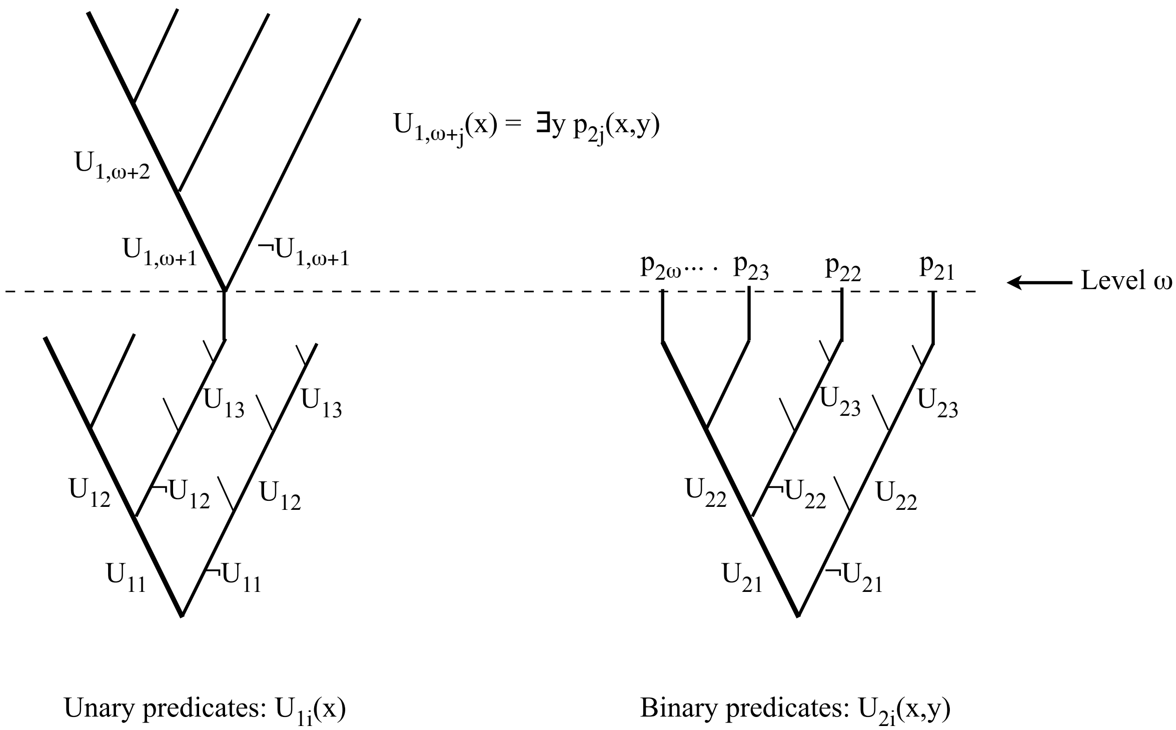


Figure 1: Regenerating types

**The first order theory $th^1$**

We begin by fixing the trees $T^1_n$ used to build the basic partial types. These trees can be chosen with a great deal of flexibility, modulo the following constraints. We insist that each tree $T^1_n$ is computable, that there is a countably infinite number of non-isolated branches, that the string $jk =_{def} 000$ has a single extension, and in fact that there is a computable enumeration of all the branches. Additionally we assume the non-isolated branches of $T^1_n$ are computably partitioned into an $\omega$-sequence of regenerating branches $\{b^1_{n,k} \mid k < \omega\}$, an $\omega$-sequence of priority branches $\{q^1_{n,k} \mid k < \omega\}$ and a seed branch $\{Q^1_n\}$. For example we can define $T^1_n$, identical for all $n < \omega$, as follows. A string $s \in 2^{<\omega}$ belongs to $T^1_n$ if it either has at most two zeroes or is the string of all zeroes. In this example the non-isolated branches are those with at most one 0, with the branch of all 1's the unique limit of non-isolated branches. Finally, the seed branch could be the branch of all ones, the priority branches could be the remaining even non-isolated branches (according to the lexicographic order) and the regenerating branches could be the odd non-isolated branches.

Our signature consists of $n$-ary predicates $\{U_{n,k}(\bar{x}_n) \mid 0 < n, k < \omega\}$, unary predicates $\{S_n \mid 0 < n < \omega\}$ and a unary function $f$. The predicate $U_{n,k}$ corresponds to the $k^{th}$ level of tree $T^1_n$ and the string $s \in 2^{\leq\omega}$ corresponds to the conjunction

$$\theta_{n,s}(\bar{x}_n) =_{def} \bigwedge_{k<|s|} (\neg)^{1+s_k} U_{n,k}(\bar{x}_n).$$

The language $\mathfrak{L}^1$ is the first order language of this signature.
Note that when $|s| < \omega$ the formula $\theta_{ns}$ is first order. When $|s| = \omega$ the formula $\theta_{ns}$ is a first order *basic partial types*. We conflate the name of the non-principal basic partial types and their associated non-isolated branches, so that for example $b^1_{n,k}(\bar{x}_n)$ is the $k^{th}$ regenerating $n$-type as well as the $k^{th}$ regenerating branch of $T^1_n$. For a while we write $\theta_{n,b^1_{nk}}(\bar{x}_n)$ and not $b^1_{n,k}(\bar{x}_n)$ in order to emphasize the fact that the regenerating types are particular instances of the basic partial types $\theta_{n,s}(\bar{x})$.

**Definition 15.** The set $P^1$ of non-principal basic partial types is

$$P^1 = \{b^1_{n,k}(\bar{x}_n) \mid n, k < \omega\} \cup \{q^1_{n,k}(\bar{x}_n) \mid n, k < \omega\} \cup \{Q^1_n(\bar{x}_n)\}.$$

**Axioms for $th^1$**

- **UB**, or universal bootstrap axioms. The type of an n-tuple $\bar{x}$ only encodes information about the tree $T^1_n$ if for each $i \leq n$ we have $f(x_{i+1}) = x_i$ and additionally $x_1$ belongs to the "bottom sort" $S_1$. The predicates $S_n$ are introduced to eliminate quantifiers, with the function $f$ mapping from $S_{n+1}$ to $S_n$. For $0 < n, k < \omega$:
  - $\forall x_1 \ldots x_n \ \neg\theta_{n,jk}(\bar{x}) \leftrightarrow (S_1(x_1) \wedge \bigwedge_{i<n} f(x_{i+1}) = x_i)$
  - $\forall x \ \neg S_1(x) \rightarrow f^n(x) \neq x$
  - $\forall x \ (f(x) = y \rightarrow (S_{n+1}(x) \leftrightarrow S_n(y)))$
  - $\forall x \ S_1(x) \rightarrow f(x) = x$
  - $\forall x \ S_n(x) \rightarrow \neg S_k(x), k \neq n$

- **UT**, or universal tree axioms. The only branches of $2^{<\omega}$ encoded by $n$-types are those belonging to $T^1_n$. For all $n$ and for all finite strings $s \notin T^1_n$:

$$\forall \overline{x} \ \neg\theta_{ns}(\overline{x})$$

- **EC**, or existential closure axioms.
  - For all $t \neq jk$ with $t \in T^1_{n+1}$

$$\forall \bar{x}_n \ \ \neg\theta_{n,jk}(\bar{x}_n) \rightarrow \exists^{>k} y_{n+1} \ \theta_{n+1,t}(\bar{x}_n, y_{n+1})$$

  - For all $k$

$$\forall x \exists^{>k} y \ f(y) = x.$$

We begin with a description of several one-types, without proof, to give a rough picture of what is going on. Say $a$ is an element of a model of $th^1$ – what can the type of $a$ be? If $S_1(a)$ then the complete type of $a$ is determined by its basic partial one-type, that is $\theta_{1,s}(a)$ for some string $s \in T_1^1$. If $S_2(a)$ then the complete type of $a$ is determined by

$$\theta_{1,s_1}(f(a)) \wedge \theta_{2,s_2}(a, f(a))$$

where $s_1 \in T_1^1$ and $s_2 \in T_2^1$. The basic partial one type of $a$ itself is the junk partial one type, namely $\theta_{1,jk}(a)$. The entire purpose of $a$ is to establish that $f(a)$ satisfies certain higher order pseudo-predicates. Specifically, if $p_{2,n}^{\beta}(a, f(a))$ then $U_{1,\omega\beta+n}(f(a))$. One can view the role of the non-principal basic partial two-types as introducing non-homogeneity in the complete one-types satisfying $S_1$. With countably many independent non-homogeneities there can be new non-principal $S_1$ one-types at the next level of language, and the Scott analysis perpetuates. Likewise, the role of the basic partial three-types is to introduce non-homogeneity in the complete one-types of elements satisfying $S_2$ and so on.

**Lemma 0.9.** The first order theory $th^1$ is complete and consistent in $\mathfrak{L}^1$.

*Proof.* We assume completeness and postpone the details of the proof until a later section. We prove consistency by building a model $\mathcal{M} \models th^1$. For each $n$ let $M_n$ be an infinite-to-one cover of the non-zero branches $T_n^1$, witnessed by the map

$$s : M_n \to \{\text{branches of } T_n^1\}.$$

Take the universe of $\mathcal{M}$ to be

$$M =_{def} \bigcup_{n<\omega} \prod_{i \leq n} M_i$$

Consider an element $(\bar{a}_n) \in \mathcal{M}$. That is, $(\bar{a}_n)$ is the finite product $(a_1, \ldots, a_n)$, where $s(a_i)$ is a branch in $T_i^1$. Set $f((\bar{a}_n)) = (\bar{a}_{n-1})$, set $S_n((\bar{a}_n))$ and set $\neg S_k((\bar{a}_n))$ for $k \neq n$. For the $n$-tuple $(\bar{a}_1), (\bar{a}_2), \ldots, (\bar{a}_n)$ set

$$\theta_{n,s(a_n)}((\bar{a}_1), (\bar{a}_2), \ldots, (\bar{a}_n)).$$

For all $n$-tuples $\bar{b}$ not of this form and for all $k$, define $\neg U_{nk}(\bar{b})$. Note that this implies $\theta_{n,jk}(\bar{b})$. By inspection $\mathcal{M}$ is a model of $th^1$. ⊣

As an exercise the reader may consider the reduct of $th^1$ to the language

$$S_1, S_2, \{U_{1k}(x) \mid k < \omega\}, \{U_{2k}(x_1, x_2) \mid k < \omega\}, f$$

and convince themselves that the maximal Scott rank of any model $\mathcal{M}$ of this reduct is two.

### The theory $th^2$

The language $\mathfrak{L}^2$ is the closure of $\mathfrak{L}^1 \cup P^1$ under negation, conjunction and quantification. By the completeness of $th^1$ all the complete types of $th^1$ are finite conjunctions of basic partial types, therefore $P^1$ satisfies the requirement for a Scott analysis. For $n, k < \omega$ define the $n$-ary pseudo-predicate $U_{n,\omega+k}(\bar{x})$ as

$$U_{n,\omega+k}(\overline{x}) =_{def} \exists y\ \theta_{n+1,b^1_{n+1,k}}(\overline{x}, y).$$

For every string $s : \lambda \to 2$ (with $\lambda \leq \omega 2$) define

$$\theta_{ns}(\overline{x}) =_{def} \bigwedge_{\beta<\lambda} (\neg)^{1+s_\beta} U_{n\beta}(\overline{x}).$$

As in the first order case we fix the trees $T^2_n$ used to build basic partial types of $th^2$. The tree $T^2_n$ is a subtree of $2^{<\omega 2}$ with $T^1_n$ as the initial subtree of height $\omega$. The non-isolated branches can be computably partitioned as

$$\{b^2_{n,k} \mid n, k < \omega\} \cup \{q^2_{n,\gamma} \mid n < \omega, \gamma < \omega 2\} \cup \{Q^2_n\}$$

with the following restrictions. For each $n, k < \omega$ the regenerating branch $b^2_{n,k}$ extends the regenerating branch $b^1_{n,k}$: that is, $b^2_{n,k}|_\omega = b^1_{n,k}$. Likewise $q^2_{n,k}|_\omega = q^1_{n,k}$ , $Q^2_n|_\omega = Q^1_n$ and $q^2_{n,\omega+k}|_\omega = Q^1_n$ for $n, k < \omega$. We postpone the specific choice for the trees $T^2_n$ until the the next section.

Note that the seed branch $Q^1_n \in T^1_n$ has countably many non-isolated extensions (it grows an extra $\omega$-sequence of fresh priority branches) and the other non-isolated branches of $T^1_n$ have a single non-isolated extension. Every isolated branch of $T^1_n$ continues to be isolated.

**Axioms for $th^2$**

- **(UB)**, or universal bootstrap axioms. The type of an n-tuple $\bar{x}$ only encodes information about the tree $T_n$ if for each $i \leq n$ we have $f(x_{i+1}) = x_i$ and additionally $x_1$ belongs to the "bottom sort" $S_1$. The predicates $S_n$ are introduced to eliminate quantifiers, with the function $f$ mapping from $S_{n+1}$ to $S_n$. For all $n$ and $k$:
  - $\forall x_1 \dots x_n \ \neg\theta_{n,jk}(\bar{x}) \leftrightarrow (S_1(x_1) \wedge \bigwedge_{i<n} f(x_{i+1}) = x_i)$
  - $\forall x \ \neg S_1(x) \rightarrow f^n(x) \neq x$
  - $\forall x \ (f(x) = y \rightarrow (S_{n+1}(x) \leftrightarrow S_n(y)))$
  - $\forall x \ S_1(x) \rightarrow f(x) = x$
  - $\forall x \ S_n(x) \rightarrow \neg S_k(x)$
  - $\forall x \bigvee_n S_n(x)$ (**new axiom removes $\mathcal{M}^{\neg S}$**).

- **(UT)**, or universal tree axioms. The only branches of $2^{<\omega 2}$ encoded by $n$-types are those belonging to $T_n^2$. For all $n$ and for all strings $s \in 2^{<\omega 2} \backslash T_n^2$:
$$\forall \bar{x}_n \ \neg\theta_{n,s}(\bar{x}_n).$$

- **(EC)**, or existential closure axioms.

  1. For all $t \neq jk$ with $t \in T_{n+1}^1$
  $$\forall \bar{x}_n \ \ \neg\theta_{n,jk}(\bar{x}_n) \rightarrow \exists^{>k} y \ \theta_{n+1,t}(\bar{x}_n, y).$$
  2. Take $s, t \neq jk$ with $s \in T_n^2$ and $t \in T_{n+1}^2$ excluding those pairs where $t|_\omega = b_{n+1,k}^1$ and $s(\omega + k) = 0$. Then for all $k$
  $$\forall \bar{x}_n \ \ \theta_{n,s}(\bar{x}) \rightarrow \exists^{>k} y \ \theta_{n+1,t}(\bar{x}_n, y).$$
  3. For $t \in T_{n+1}^2$ and $n < \omega$
  $$(\exists y \theta_{n+1,t|_\omega}(\bar{x}_n, y)) \rightarrow (\exists z \ \theta_{n+1,t}(\bar{x}_n, z)).$$
  4. For $n < \omega$
  $$\forall \bar{x}_n \exists y \ \neg\theta_{n,jk}(\bar{x}_n) \rightarrow Q_{n+1}^1(\bar{x}_n, y).$$

**Lemma 0.10.** The theory $th^2$ is complete and consistent in the language $\mathfrak{L}^2$.

*Proof.* Again we postpone the proof of completeness. We prove consistency by building a model $\mathcal{M} \models th^2$. For each $n$ let $M_n$ be an infinite cover of the non-zero branches of $T_n^2$ witnessed by the map

$$s : M_n \to \{\text{branches of } T_n^2\}$$

and set

$$M =_{def} \bigcup_{n<\omega} \prod_{i\leq n} M_i.$$

An element of $M$ is a finite product $(a_1, \ldots, a_n)$ with each $a_i \in M_i$. Denote this element by $(\bar{a}_n)$. Denote the value of $s$ with input $a \in M_n$ as the branch $s[a] \in T_n^2$, and denote the $\gamma^{th}$ bit of $s[a] : \omega 2 \to 2$ as $s[a](\gamma)$. We now remove a subset of $M$. For every $a \in M_n$ define $\text{Neg}(a) =_{def} \{k \mid s[a](\omega + k) = 0\}$ and define

$$\text{NEG} =_{def} \{\bar{a}_n \in M \mid \exists\, i < n, \exists k < \omega \text{ where } a_{i+1}|_\omega = b^1_{i+1,k} \text{ and } k \in \text{Neg}(a_i)\}.$$

Set the universe of $\mathcal{M}$ to be $M \backslash \text{NEG}$. For all $(\bar{a}_n) \in \mathcal{M}$ decide the atomic diagram of $(\bar{a}_n)$ as follows. Set $f((\bar{a}_n)) = (\bar{a}_{n-1})$; if $\bar{a}_n \in \mathcal{M}$ then $\bar{a}_{n-1} \in \mathcal{M}$ and therefore the function $f$ is well defined. Set $S_n((\bar{a}_n))$ and set $\neg S_k((\neg \bar{a}_n))$ for $k \neq n$. Set the basic partial type of the $n$-tuple $f^{n-1}(\bar{a}_n), f^{n-2}(\bar{a}_n), \ldots, (\bar{a}_n)$, or equivalently the $n$-tuple $(\bar{a}_1), (\bar{a}_n), \ldots, (\bar{a}_n)$, to be

$$\theta_{n,s(a_n)|_\omega}((\bar{a}_1), (\bar{a}_2), \ldots, (\bar{a}_n)).$$

**Claim 1:** Given $(\bar{a}_n) = (a_1 \cdot a_2 \cdot \ldots \cdot a_n) \in \mathcal{M}$. If $s[a_n](\omega + k) = 1$ then $U_{n,\omega+k}((\bar{a}_1) \ldots (\bar{a}_n))$ and if $s[a_n](\omega + k) = 0$ then $\neg U_{n,\omega+k}((\bar{a}_1) \ldots (\bar{a}_n))$. In other words the branch $s[a_n]$ encodes the basic partial $n$-type of

$$(\bar{a}_1), (\bar{a}_2), \ldots, (\bar{a}_n) = (a_1), (a_1 \cdot a_2), \ldots, (a_1 \cdot a_2 \cdot \ldots \cdot a_n) \in \mathcal{M}^n$$

in the language $\mathfrak{L}^2$.

First assume bit $\omega + k$ of $s(a_n)$ is 1. Pick $a_{n+1} \in M^{n+1}$ with $s(a_{n+1}|_\omega) = b^1_{n+1,k}$, the $k^{th}$ non-isolated branch of $T^1_{n+1}$. Since $s[a_n](\omega + k) = 1$ and $\bar{a}_n \in \mathcal{M}$ it follows that $\bar{a}_{n+1} \notin \text{NEG}$ and therefore $\bar{a}_{n+1} \in \mathcal{M}$ as well. The first order partial $n+1$-type of $(\bar{a}_1), \ldots, (\bar{a}_{n+1})$ was chosen to be

$$\theta_{n+1,a_{n+1,k}|_\omega}((\bar{a}_1) \ldots (\bar{a}_{n+1})) = \theta_{n+1,b_{n+1,k}}((\bar{a}_1) \ldots (\bar{a}_{n+1}))$$

and therefore $\exists y\, \theta_{n+1,b^1_{n+1,k}}((\bar{a}_1),\ldots,(\bar{a}_n),y)$. As required we have shown that $U_{n,\omega+k}((\bar{a}_1),\ldots,(\bar{a}_n))$.

Next assume $s[a_n](\omega+k)=0$ and suppose for contradiction there exists $b\in\mathcal{M}$ where $\theta_{n+1,b^1_{n+1,k}}((\bar{a}_1),\ldots,(\bar{a}_n),b)$. There must be some $a_{n+1}\in M_{n+1}$ where $b=(\bar{a}_{n+1})=(a_1\cdot\ldots\cdot a_n\cdot a_{n+1})\in\mathcal{M}$. Additionally, we must have $s[a_{n+1}]|_\omega = b^1_{n+1,k}$. Since $k\in \mathrm{Neg}(a_n)$ it follows that $\bar{a}_{n+1}\in \mathrm{NEG}$ and $\bar{a}_{n+1}\notin\mathcal{M}$. Contradiction. ⊣

**Claim 2:** The axioms (UB) and (UT) are satisfied.

The axioms (UB) are satisfied by trivial inspection. For (UT) it is sufficient to show that $\neg\theta_{n,t}((\bar{a}_1),\ldots,(\bar{a}_n))$ for all $a_i\in M_i$ with each $(\bar{a}_i)\in\mathcal{M}, i\le n$ and for all $t\notin T^2_n,\ |t|<\omega 2$; all other tuples $\bar{b}\in\mathcal{M}$ satisfy the junk type $\theta_{n,jk}(\bar{b})$. Take $\lambda<\omega 2$ where $s[a_n](\lambda)\ne t(\lambda)$. By the previous claim we have $U^{s[a_n](\lambda)+1}_{n,\lambda}((\bar{a}_1),\ldots,(\bar{a}_n))$ and therefore $\neg\theta_{n,t}((\bar{a}_1),\ldots,(\bar{a}_n))$. ⊣

**Claim 3:** The axioms (EC) are satisfied.
(EC), axiom 1: take $t\in T^2_{n+1}$ with $|t|<\omega$ and $d_1,\ldots,d_n\in\mathcal{M}$ where $\neg\theta_{n,jk}(d_1,\ldots,d_n)$. We must find infinitely many $c_k\in\mathcal{M}, k<\omega$ where $\theta_{n+1,t}(d_1,\ldots,d_n,c_k)$. By (UT) there exists $s\in T^2_n$ with $|s|=\omega 2$ with $\theta_{n,s}(d_1,\ldots,d_n)$. By construction there must be $a_i\in M_i, i\le n$ with $d_i=(a_1\cdot\ldots\cdot a_i)$ and where $s[a_n]=s$. Now, choose a branch $t'^2_{n+1}$ extending $t$ which does not extend any of the non-isolated branches $b^1_{n+1,1}, b^1_{n+1,2},\ldots$, and choose $a_{n+1,k}\in M_{n+1}, k<\omega$ so that $s[a_{n,k}]=t'$. Then for each $k$, $c_k =_{def} (a_1\cdot\ldots\cdot a_n\cdot a_{n+1,k})\in M\backslash\mathrm{NEG}=\mathcal{M}$ satisfies $\theta_{n+1,t}(d_1,\ldots,d_n,c_k)$.

(EC), axiom 2: take $s\in T^2_n$ and $t\in T^2_{n+1}$ excluding those pairs where $t|_\omega=b^1_{n+1,k}$ and $s(\omega+k)=0$. The same argument as above works, choosing an arbitrary branch $t'^2_{n+1}$ of length $\omega 2$ extending $t$.
(EC), axioms 3 and 4 follow similarly. This ends the proof of consistency. ⊣

## The theory $th^\beta$

We define an inductive property $TP(\beta)$ on trees $\{T^\beta_n \mid n<\omega\}$, and prove that given trees $\{T^\beta_n \mid n<\omega\}$ satisfying property $TP(\beta)$ we can construct a Scott theory $th^\beta$ of height $\beta$ so that the trees $\{T^\beta_n \mid n<\omega\}$ are the trees of partial types. This sets the framework for the priority argument of the next section.

We become concerned with the combinatorics of the trees, and clarify the roles of the priority types $q^\beta_{n,\gamma}(\bar{x}_n)$ and seed types $Q^\beta_n(\bar{x}_n)$. The non-isolated branches of tree $T^{\beta}_{n+1}$ fulfill three distinct functions. First, they approximate the final non-isolated branches of the tree $T^{\omega_1^{CK}}_{n+1}$. Second, they provide a source of pseudo-predicates for $T^{\beta+1}_n$. Finally, they provide choice for the priority argument. In order to maintain the priority argument, the set of non-isolated branches must increase with $\beta$. This is because a priority requirement induces a multitude of negative choices and negative choices have the potential to kill off the boot-strapping process. To illustrate the problem of negative choices recall that the regenerating branch $b^2_{n+1,1} \in T^2_{n+1}$ extends the regenerating branch $b^1_{n+1,1} \in T^1_{n+1}$, that is, $b^1_{n+1,1} = b^2_{n+1,1}|_{\omega}$. Therefore the pseudo-predicate $U_{n,\omega 2+1}(\bar{x}) = \exists y \ \theta_{n+1,b^2_{n+1,1}}(\bar{x}, y)$ of $\mathfrak{L}^3$ and the pseudo-predicate $U_{n,\omega+1}(\bar{x}) = \exists y \ \theta_{n+1,b^1_{n+1,1}}(\bar{x}, y)$ of $\mathfrak{L}^2$ are related as follows: $\neg U_{n,\omega+1}(\bar{x})$ implies $\neg U_{n,\omega 2+1}(\bar{x})$. It is inconsistent to have $s \in T^3_n$ with $s(\omega + 1) = 0$ and $s(\omega 2 + 1) = 1$. At the extreme, if $s(\omega + n) = 0$ for all $n < \omega$ and all the non-isolated branches of $T^3_{n+1}$ extend non-isolated branches of $T^2_{n+1}$ then $s(\beta) = 0$ for all $\omega 2 < \beta$ and in particular $s$ cannot be a non-isolated branch. We solve the problem of negative choices by separating the functions of regeneration and providing choice for the priority argument.

**Inductive property** $TP(\delta)$. We have a set of trees $\{T^{\beta}_n \mid n < \omega\}$ of height $\omega\beta$ for every $\beta \leq \delta$. The trees cohere. The non-isolated branches of height $\beta$ are partitoned into three sets: an $\omega$-sequence of *regenerating branches* $\{b^{\beta}_{n,k} \mid k < \omega\}$, a $\omega\beta$-sequence of *priority branches* $\{q^{\beta}_{n,\gamma} \mid \gamma < \omega\beta\}$, and a a *seed branch* $Q^{\beta}_n$. The regenerating branches are responsible for perpetuating the Scott analysis. The priority branches are responsible for the priority argument. The seed branches are responsible for generating fresh priority branches; specifically the branches $q^{\beta+1}_{n,\beta+1}$ extend the seed branch $Q^{\beta}_n$. Additionally there are sets of *potentially non-isolated branches* $\{Q^{\beta}_n[\gamma] \mid \gamma < \beta\}$, $\{b^{\beta}[\gamma]_{n,k} \mid \gamma < \beta\}$ and $\{q^{\beta}[\gamma]_{n,\alpha} \mid \gamma < \beta\}$ approximating the respective non-isolated branches. All of these branches are continuously defined.

**Definition:** Say that a set of named branches $\{b^{\gamma}_{n,k}$ branch of $T^{\gamma}_n \mid \gamma < \delta\}$ is *continuously defined* if for every $\omega\beta < \delta$ the direct limit $\underline{lim}_{\gamma<\omega\beta}\ b^{\gamma}_{n,k}$ exists and equals $b^{\omega\beta}_{n,k}$.

Additionally, the restriction of a primary candidate of height $\beta$ to a language $\mathfrak{L}^{\gamma}$ for $\gamma < \beta$ results in either the same primary candidate of height $\gamma$, or in a potential candidate of the same kind, also of height $\gamma$. For example, $b^{\beta}_{n,k}|_{\gamma} = b^{\gamma}_{n,k}$ or $b^{\beta}_{n,k}|_{\gamma} = b^{\gamma}_{n,k}[\delta]$ for some $\delta$. The exception to this rule is $q^{\beta}_{n,\delta}$ when $\delta < \gamma$, in which case $q^{\beta}_{n,\delta}|_{\gamma} = Q_{n,\gamma}$ as described earlier.

We restrict the negative choices as follows. We want all of the non-isolated branches, and all of the potential non-isolated branches, to make positive choices for the regenerating pseudo-predicates. Looking ahead, the regenerating branches correspond to even pseudo-predicates and the priority branches correspond to odd pseudo-predicates. Therefore all non-isolated branches and all potentially non-isolated branches take the value of one at all even heights. The exception to this rule is the types $q^\beta_{n,\gamma}$ and their approximations, which take the value of 0 at heights $\omega\beta+2$ for $\gamma \leq \beta$. Additionally for every $\gamma < \beta$ and $k < \omega$ there are infinitely many branches $s$ of $T^\beta_k$ with $s(\gamma) = 1$. For each branch $s \in T^\beta_n$ define

$$\mathrm{NegB}^\delta(s) =_{def} \{k < \omega \mid s(\omega\delta + 2k) = 0\}$$

and

$$\mathrm{NegQ}^\delta(s) =_{def} \{k < \omega \mid s(\omega\delta + 2k + 1) = 0\}.$$

For every branch $s \in T^\beta_n$ and $\gamma < \delta \leq \beta$ we require that $\mathrm{NegB}^\gamma(s) \subset \mathrm{NegB}^\delta(s)$. This is because $b^\delta_{n+1,k}|_\gamma = b^\gamma_{n+1,k}$ and so we expect

$$th^\beta \vdash \forall \bar{x}\ (\neg U_{n,\omega\gamma+2k}(\bar{x}) \rightarrow \neg U_{n,\omega\delta+2k}(\bar{x})).$$

Similarly we require that $\pi^{-1}_\gamma(\mathrm{NegQ}^\gamma(s)) \subset \pi^{-1}_\delta(\mathrm{NegQ}^\delta(s))$, since $q^\delta_{n+1,\alpha}|_\gamma = q^\gamma_{n+1,\alpha}$. This finishes the description of property $TP(\delta)$.

Assume we are given trees $\{T^\delta_n \mid n < \omega\}$ satisfying property $TP(\delta)$. We define languages $\mathfrak{L}^\beta$ and theories $th^\beta$ for $\beta \leq \delta$. The regenerating branches of $\{T^\delta_n \mid n < \omega\}$ correspond to the even pseudo-predicates:

$$U_{n,\omega\beta+2k}(\bar{x}) =_{def} \exists y\ b^\beta_{n+1,k}(\bar{x}, y).$$

The priority branches correspond to the odd pseudo-predicates, using the inverse $\pi^{-1}_\beta$ of the restriction of the projectum map:

$$U_{n,\omega\beta+2k+1}(\bar{x}) =_{def} \exists y\ q^\beta_{n+1,\pi^{-1}_\beta(k)}(\bar{x}, y).$$

The seed branches do not correspond to pseudo-predicates; all relevant tuples $\bar{x}$ will make the positive choice $\exists y\ Q^\beta_{n+1}(\bar{x}, y)$. The language $\mathfrak{L}^{\beta+1}$ is the closure of $\mathfrak{L}^\beta \cup \{U_{n,\omega\beta+k} \mid k < \omega\}$ under conjunction, negation and quantification. Associated to each tuple $\bar{x}$ and branch $s$ of $T^\delta_n$ is the formula

$$\theta_{n,s}(\bar{x}) =_{def} \bigwedge_{\beta<|s|} (\neg)^{1+s(\beta)} U_{n,\beta}(\bar{x}).$$

**Axioms for $th^\beta$**

**UB**, or universal bootstrap axioms. The type of an $n$-tuple $\bar{x}$ only encodes information about the tree $T_n^\beta$ if for each $i \leq n$ we have $f(x_{i+1}) = x_i$ and additionally $x_1$ belongs to the sort $S_1$. The predicates $S_n$ are introduced to eliminate quantifiers, with the function $f$ mapping from $S_{n+1}$ to $S_n$. For all $n$ and $k$ with $n \neq k$:

- – $\forall x_1 \dots x_n \ \neg\theta_{n,jk}(\bar{x}) \leftrightarrow (S_1(x_1) \wedge \bigwedge_{i<n} f(x_{i+1}) = x_i)$
  – $\forall x \ \neg S_1(x) \rightarrow f^n(x) \neq x$
  – $\forall x \ (f(x) = y \rightarrow (S_{n+1}(x) \leftrightarrow S_n(y)))$
  – $\forall x \ S_1(x) \rightarrow f(x) = x$
  – $\forall x \ S_n(x) \rightarrow \neg S_k(x)$
  – $\forall x \bigvee_n S_n(x)$
- **UT**, or universal tree axioms. The only branches of $2^{<\omega\beta}$ encoded by $n$-types are those belonging to $T_n^\beta$. For all $n$ and for all strings $s \in 2^{<\omega\beta} \backslash T_n^\beta$:
$$\forall \bar{x} \ \neg\theta_{n,s}(\bar{x})$$
- **EC**, or existential closure axioms.
  1. For all $t \neq jk$ with $t \in T_{n+1}^1$
  $$\forall \bar{x} \ \ \neg\theta_{n,jk}(\bar{x}) \rightarrow \exists^{>k} y \ \theta_{n+1,t}(\bar{x}, y)$$
  2. Take $s, t \neq jk$ with $s \in T_n^\beta$ and $t \in T_{n+1}^\beta$ with the following restriction. If $t|_{\omega\gamma} = b_{n+1,k}^\gamma$ then $s(\omega\gamma + 2k) = 1$ and if $t|_{\omega\gamma} = q_{n+1,\alpha}^\gamma$ then $s(\omega\gamma + 2\pi_\delta(\alpha) + 1) = 1$, for all $\gamma < \beta$. Then for all $n, k$
  $$\forall \bar{x} \ \ \theta_{n,s}(\bar{x}) \rightarrow \exists^{>k} y \ \theta_{n+1,t}(\bar{x}, y)$$
  3. For $t \in T_{n+1}^\beta$
  $$(\exists y \theta_{n+1,t|_{\omega\beta}}(\bar{x}_n, y)) \rightarrow (\exists z \ \theta_{n+1,t}(\bar{x}_n, z))$$
  4. For $n < \omega$
  $$\forall \bar{x}_n \exists y \neg\theta_{n,jk}(\bar{x}_n) \rightarrow Q_{n+1}^\gamma(\bar{x}_n, y)$$

**Theorem 0.11.** Given trees $\{T_n^\delta \mid n < \omega\}$ satisfying $TP(\delta)$ the theory $th^\delta$ described above is a complete and consistent Scott theory of rank $\delta$.

*Proof.* In order to prove completeness, we describe a normal form for formulas of $\mathfrak{L}^\beta$.

**Definition 16.** For each successor ordinal $\beta + 1$ the language of pseudo-quantifier free rank $\beta + 1$ formulas is defined to be the closure of

$$\mathfrak{L}^\beta \cup \{U_{n,\omega\beta+k}(\bar{x}_n) \mid n, k < \omega\} \cup \{\exists x \bigwedge_n \neg S_n(x)\}$$

under negation and conjunction. For a limit ordinal $\omega\beta$ the pseudo-quantifier free formulas of rank $\omega\beta$ are all the formulas.

Fix a language $\mathfrak{L}^\beta$. Let $\sigma =_{def} \sigma(1) \dots \sigma(|\sigma|) = n$ be a finite increasing sequence of finite numbers and define the sequence of strings $\bar{s}_\sigma$ as $s_{\sigma(1)} s_{\sigma(2)} \dots s_{\sigma(|\sigma|)}$ where $s_{\sigma(i)}$ is a branch through $T^\beta_{\sigma(i)}$ for each $i, 1 \leq i \leq |\sigma|$. For all $\gamma < \omega\beta$ define $\bar{s}_\sigma|_\gamma$ to be the sequence $s_{\sigma(1)}|_\gamma s_{\sigma(2)}|_\gamma \dots s_{\sigma(|\sigma|)}|_\gamma$ of initial segments of the $s_{\sigma(i)}$ branches of length $\gamma$. Define

$$\Delta^\gamma[\bar{s}_\sigma](x) =_{def} \bigwedge_{i \leq |\sigma|} \theta_{i, s_{\sigma(i)}|_\gamma}(f^{n-1}(x) \dots f^{n-\sigma(i)}(x)).$$

Now let $G$ be a directed graph on the set of numbers $\{1, \dots, n\}$ with no loops and where every vertex has out degree one. Define

$$\varphi_G(\bar{x}_{|G|}) =_{def} \bigwedge_{i \to j \in \text{Edge}(G)} f(x_i) = x_j$$

and define $\Delta^0_{G,S}(\bar{x}_n) = \varphi_G(\bar{x}_n) \wedge \varphi_S(\bar{x}_n)$ where $\varphi_S$ is a conjunction in the language $\{S_i \mid i < \omega\}$. Define the *sources* of $G$ as those vertices with in degree 0.

**Definition 17.** A basic formula of rank $\gamma$ is any formula

$$\Delta^0_{G,S}(\bar{x}_n) \wedge \bigwedge_{b \text{ source vtx of } G} \Delta^\gamma[\bar{s}_{\sigma_b}](x_b).$$

Note that the basic formulas of rank $\gamma < \omega\beta$ belong to the language $\mathfrak{L}^\beta$.

We claim without proof that every pseudo-quantifier free formula of $\mathfrak{L}^\beta$ is provably equivalent via $th^\beta$ to a finite disjunction of basic formulas.

**Lemma 0.12.** For every pseudo-quantifier free $\varphi(\bar{x}, y)$ of $\mathfrak{L}^\beta, \beta \leq \delta$ there is a pseudo-quantifier free $\xi_\varphi(\bar{x})$ of $\mathfrak{L}^\beta$ so that $th^\beta \vdash \forall \bar{x}((\exists y \varphi(\bar{x}, y)) \leftrightarrow \xi_\varphi(\bar{x}))$.

*Proof.* The case of interest is the successor case $\beta + 1 < \delta$, where we inductively assume the lemma holds for $\gamma \leq \beta$. We assume without loss of generality that $\varphi(\bar{x}, y)$ is a basic formula of rank $\gamma < \omega(\beta + 1)$ and also that both $\bar{x}$ and $\bar{x}, y$ are $f$-closed. Therefore $y$ is a source vertex and $\varphi(\bar{x}, y)$ is either of the form

$$\theta_{1,s}(y) \wedge \xi(\bar{x})$$

or of the form

$$\Delta^\gamma[\bar{s}_\sigma](y) \wedge \ (f(y) = x) \ \wedge \xi(\bar{x})$$

where in both cases $\xi(\bar{x})$ is a basic formula not involving $y$. We consider the first case. If $s \notin T_1^\beta$ then by the UT axioms we can choose $\xi_\varphi(\bar{x})$ to be false. On the other hand if $s \in T_1^\beta$ then by the EC axioms $th^\beta \vdash \exists^{>m} z \ \theta_{1,s}(z)$ for all $m$. Rewrite $\xi_\varphi(\bar{x})$ as $\bigwedge_{i \in I} \theta_{1,s}(x_i) \ \wedge \ \Delta^\gamma(\bar{x})$ where $\Delta^\gamma$ makes no claims involving $\theta_{1,s}$. Choosing $m > |I|$ for the EC axiom we have

$$th^{\beta+1} \vdash \forall \bar{x} \ ((\bigwedge_{i \in I} \theta_{1,s}(x_i)) \leftrightarrow (\exists y \ \theta_{1,s}(y) \wedge (\bigwedge_{i \in I} \theta_{1,s}(x_i))))$$

and therefore as desired $th^{\beta+1} \vdash \forall \bar{x}((\exists y \varphi(\bar{x}, y)) \leftrightarrow \xi_\varphi(\bar{x}))$.

We now consider the case where $\varphi(\bar{x}, y)$ is of the form

$$\Delta^\gamma[\bar{s}_\sigma](y) \wedge \ (f(y) = x) \ \wedge \xi(\bar{x}).$$

Say $\sigma(|\sigma|) = n$. Then $(f^{n-1}(y), \ldots, f^{n-\sigma(i)}(y)) = (f^{n-2}(x), \ldots, f^{n-\sigma(i)-1}(x))$ except when $i = |\sigma|$, in which case $(f^{n-1}(y), \ldots, f^{n-\sigma(i)}(y)) = (f^{n-2}(x), \ldots, x, y)$. Therefore

$$\Delta^\gamma[\bar{s}_\sigma](y) = \theta_{n,s_n|_\gamma}(f^{n-2}(x), \ldots, x, y) \wedge \bigwedge_{i < |\sigma|} \theta_{\sigma(i), s_{\sigma(i)}|_\gamma}(f^{n-2}(x) \ldots f^{n-\sigma(i)-1}(x)).$$

Since $th^\beta \vdash \theta_{n,s_n|_\gamma}(f^{n-2}(x), \ldots, x, y) \rightarrow \ (f(y) = x)$ we can reduce $\varphi(\bar{x}, y)$ to

$$\theta_{n,t}(f^{n-2}(x), \ldots, x, y) \wedge \Delta^\gamma[\bar{s}_\sigma](x) \wedge \xi_1(\bar{x}),$$

where $\xi_1(\bar{x})$ is a pseudo-quantifier free formula not involving $y$, where $t \in T^\beta_n$ is a branch of length $\gamma < \omega(\beta+1)$, where $s_{\sigma(1)}, \ldots, s_{\sigma(j)}$ are branches in $T^\beta_{\sigma(1)}, \ldots, T^\beta_{\sigma(j)}$ respectively, and where $\sigma(1) < \ldots < \sigma(j) < n$.
**Case 1**. Say the branch $t \in T^\beta_n$ is isolated by level $\alpha < \omega\beta$. Then

$$th^{\beta+1} \vdash \forall \bar{x}, y(\theta_{n,t}(f^{n-2}(x), \ldots, x, y) \leftrightarrow \theta_{n,t|_\alpha}(f^{n-2}(x), \ldots, x, y)).$$

By induction there exists $\xi_2(x)$ with

$$th^\beta \vdash \forall x((\exists y\ \theta_{n,t|_\alpha}(f^{n-2}(x), \ldots, x, y) \wedge \Delta^\alpha[\bar{s}_\sigma](x)) \leftrightarrow \xi_2(x)).$$

Setting $\xi_\varphi(\bar{x}) = \xi_2(x) \wedge \xi_1(\bar{x})$ we therefore have

$$th^{\beta+1} \vdash \forall \bar{x}((\exists y \varphi(\bar{x}, y)) \leftrightarrow \xi_\varphi(\bar{x})),$$

as desired.
**Case 2**. Say the branch $t \in T^\beta_n$ is not isolated. Then set $\xi_\varphi(\bar{x})$ to be

$$\xi_2(x) \wedge \Delta^\gamma[\bar{s}_\sigma](x) \wedge \xi_1(\bar{x}),$$

where $\xi_2(x)$ is chosen as follows. If $t|_{\omega\beta} = b^\beta_{n,k}$ then $\xi_2(x) = U_{n-1,\omega\beta+2k}(x)$, if $t|_{\omega\beta} = q^\beta_{n,k}$ then $\xi_2(x) = U_{n-1,\omega\beta+2k+1}(x)$ and if $t|_{\omega\beta} = Q^\beta_n$ then $\xi_2(x)$ is null. Now by EC axioms 3 and 4, we have

$$th^{\beta+1} \vdash (\forall x\ ((\exists y\ \theta_{n,t}(f^{n-2}(x), \ldots, x, y)) \leftrightarrow \xi_2(x)))$$

in all three of the situations. It follows that

$$th^{\beta+1} \vdash ((\exists y \varphi(\bar{x}, y)) \leftrightarrow \xi_\varphi(\bar{x})).$$

That ends the proof for the successor case $\beta+1$. The limit case is clear. ⊣

The above lemma establishes that $th^\beta$ eliminates quantifiers down to pseudo-quantifier free rank $\beta$ formulas. The truth value of all pseudo-quantifier free rank $\beta$ sentences is established by $th^\beta$, and therefore $th^\beta$ is complete in the language $\mathfrak{L}^\beta$.

**Lemma 0.13.** For all $\beta \leq \delta$ the theory $th^\beta$ is consistent.

*Proof.* As before we build a model $\mathcal{M} \models th^\beta$. For each $n$ let $M_n$ be an infinite cover of the non-zero branches of $T^\beta_n$ witnessed by the map $s$, and set

$$M =_{def} \bigcup_{n<\omega} \prod_{i \leq n} M_i.$$

For every $a \in M_n$ and $\delta < \beta$ recall that $\mathrm{NegB}^\delta(s[a]) = \{k \mid s[a](\omega\beta+2k) = 0\}$ and that $\mathrm{NegQ}^\delta(a) =_{def} \{(k \mid s[a](\beta\omega + 2k + 1) = 0\}$. Now define:

$$\begin{aligned}\mathrm{NEGB} =_{def} \{\bar{a}_n \in M \mid \exists\, i < n, k < \omega, \delta < \beta, \quad & a_{i+1}|_{\omega\delta} = b^\delta_{i+1,k} \\ \text{and } k \in \mathrm{NegB}^\delta(s[a_i])\},\end{aligned}$$

and

$$\begin{aligned}\mathrm{NEGQ} =_{def} \{\bar{a}_n \in M \mid \exists\, i < n, k < \omega, \delta < \beta, \quad & a_{i+1}|_{\omega\delta} = q^\delta_{i+1,\pi_\delta^{-1}(k)} \\ \text{and } k \in \mathrm{NegQ}^\delta(s[a_i])\}.\end{aligned}$$

Set the universe of $\mathcal{M}$ to be $M \backslash \mathrm{NEGB} \backslash \mathrm{NEGQ}$. For all $(\bar{a}_n) \in \mathcal{M}$ decide the atomic diagram of $(\bar{a}_n)$ as follows. Set $f((\bar{a}_n)) = (\bar{a}_{n-1})$; if $(\bar{a}_n) \in \mathcal{M}$ then $(\bar{a}_{n-1}) \in \mathcal{M}$ and therefore the function $f$ is well defined. Set $S_n((\bar{a}_n))$ and set $\neg S_k((\bar{a}_n))$ for $k \neq n$. Set the basic partial type of the $n$-tuple $f^{n-1}(\bar{a}_n), f^{n-2}(\bar{a}_n), \ldots, (\bar{a}_n)$, or equivalently the $n$-tuple $(\bar{a}_1), (\bar{a}_2), \ldots, (\bar{a}_n)$, to be

$$\theta_{n,s(a_n)|_\omega}((\bar{a}_1), (\bar{a}_2), \ldots, (\bar{a}_n)).$$

**Claim:** Given $(\bar{a}_n) = (a_1 \cdot a_2 \cdot \ldots \cdot a_n) \in \mathcal{M}$. For $\gamma \leq \beta$, if $s[a_n](\omega\gamma + k) = 1$ then $U_{n,\omega\gamma+k}((\bar{a}_1) \ldots (\bar{a}_n))$ and if $s[a_n](\omega\gamma+k) = 0$ then $\neg U_{n,\omega\gamma+k}((\bar{a}_1), \ldots, (\bar{a}_n))$. In other words the branch $s[a_n]$ encodes the basic partial $n$-type of

$$(\bar{a}_1), (\bar{a}_2), \ldots, (\bar{a}_n) = (a_1), (a_1 \cdot a_2), \ldots, (a_1 \cdot a_2 \cdot \ldots \cdot a_n) \in \mathcal{M}^n$$

in the language $\mathfrak{L}^{\beta+1}$.

*Proof.* For $\omega\gamma + k = k$ we have $U_{n,k}((\bar{a}_1), \ldots, (\bar{a}_n))$ iff $s[a_n](k) = 1$ by the choice of atomic diagram. For $\gamma \geq 1$ the cases of even and odd $k$ are slightly different; we provide a proof for $\omega\delta + 2k$ and leave the odd case to the reader. First assume $s[a_n](\omega\gamma + 2k) = 1$ and pick $a_{n+1} \in M^{n+1}$ with $s(a_{n+1}|_{\omega\gamma}) = b^\delta_{n+1,k}$, the $k^{th}$ regenerating branch of $T^\gamma_{n+1}$. Note that for all $\alpha < \delta$ we have $b^\delta_{n+1,k}|_{\omega\alpha} = b^\gamma_{n+1,k}$ or $b^\delta_{n+1,k}|_{\omega\gamma} = b^\gamma_{n+1,k}[\rho]$ for some potential candidate $b^\alpha_{n+1,k}[\rho]$ (which is isolated in the tree $T^\alpha_{n+1}$). In particular $b^\gamma_{n+1,k}|_{\omega\alpha} \neq q^\alpha_{n+1,\epsilon}$ for any $\epsilon$, and therefore $(\bar{a}_{n+1}) \notin \mathrm{NEGQ}$. We rule out the possibility that $(\bar{a}_{n+1}) \in \mathrm{NEGB}$ by examining the branch $s[a_n]$. By the requirement on the tree $T^\beta_n$ we must have $\mathrm{NegB}^\alpha(s[a_n]) \subset \mathrm{NegB}^\gamma(s[a_n])$, and so $s[a_n](\omega\alpha+2k) = 1$ for all $\alpha < \gamma$. Therefore $(\bar{a}_{n+1}) \notin \mathrm{NEGB}$ and $(\bar{a}_{n+1}) \in \mathcal{M}$. It follows that $\theta_{n+1,b^\gamma_{n+1,k}}((\bar{a}_1), \ldots, (\bar{a}_n), (\bar{a}_{n+1}))$ and thus $U_{n,\omega\gamma+2k}((\bar{a}_1), \ldots, (\bar{a}_n))$. Next assume $s[a_n](\omega\gamma+2k)=0$ and suppose for contradiction there exists $c \in \mathcal{M}$ where $\theta_{n+1,b^{\gamma}_{n+1,k}}((\bar{a}_1),\ldots,(\bar{a}_n),c)$. There must be some $a_{n+1} \in M_{n+1}$ where $c = (\bar{a}_{n+1}) = (a_1 \cdot \ldots \cdot a_n \cdot a_{n+1}) \in \mathcal{M}$. Additionally, we must have $s[a_{n+1}]|_{\omega\alpha} = b^{\gamma}_{n+1,k}$. Since $2k \in \mathrm{Neg}(a_n)$ it follows that $(\bar{a}_{n+1}) \in$ NEGB and $\bar{a}_{n+1} \notin \mathcal{M}$. Contradiction. The remaining proof that $\mathcal{M}$ satisfies the axioms of $th^{\beta}$ is a simple variant of the consistency proof of $th^2$, and is left to the reader. ⊓⊔

We have established that given trees $\{T^{\delta}_n \mid n < \omega\}$ satisfying $TP(\delta)$ the corresponding theory $th^{\beta}$ is complete in $\mathfrak{L}^{\beta}$ and consistent, for all $\beta \leq \delta$. ⊓⊔

**Corollary 0.14.** Suppose $th^{\omega_1^{CK}}$ is a theory as above where the corresponding trees $\{T_n^{\omega_1^{CK}} \mid n < \omega\}$ satisfy $TP(\omega_1^{CK})$. Then $th^{\beta}$ has countably many non-principal types, and the complexity of these types is bounded from below by the least complexity of a non-isolated branch of the trees $T_n^{\omega_1^{CK}}$, $n < \omega$.

*Proof.* We have quantifier elimination to a normal form of pseudo-quantifier free formulas. It follows that a complete $n$-type of $th^{\omega_1^{CK}}$ is of the form

$$\Delta^0_G(\bar{x}_n) \wedge \bigwedge_{\text{b source vtx of G}} \Delta^{\omega_1^{CK}}[\bar{s}(b)_{n_b}](x_b)$$

where each $\bar{s}(b)_{n_b} = s(b)_1, s(b)_2, \ldots, s(b)_{n_b}$ and each $s(b)_i$ is a branch through the tree $T_i^{\omega_1^{CK}}$. The complexity of the type is the complexity of the set of branches $\{s(b)_i \mid b \text{ source vertex of G}, i \leq n_b\}$. The type is non-principal exactly when one of these branches is non-isolated, and therefore the complexity of a non-principal type is bounded below by the complexity of a non-isolated branch. It is also clear that there are only countably many non-principal types, since the trees have only countably many branches according to $TP(\omega_1^{CK})$.

# The priority argument

Kreisel and Sacks [3] show that finite injury priority arguments can be lifted to $\omega_1^{CK}$. The proof relies on the projectum map $\pi : \omega_1^{CK} \to \omega$ and the existence of a simultaneous enumeration of all the $\Sigma_1^{\omega_1^{CK}}$ sets. The $\Sigma_1^{\omega_1^{CK}}$ sets are enumerated as increasing $\omega_1^{CK}$-finite sets, where a set is $\omega_1^{CK}$-finite if it is $\Delta_1^{\omega_1^{CK}}$ and bounded. One first picks a canonical enumeration of the

$\omega_1^{CK}$-finite sets as follows: choose an $\omega_1^{CK}$-recursive function $k(\gamma,\nu)$ where if $k(\gamma,\nu)=0$ then $\gamma<\nu$ and if $K$ is $\omega_1^{CK}$-finite then $K=\{\gamma \mid k(\gamma,\nu)=0\}$ for exactly one $\nu<\omega_1^{CK}$. Write $K=K_\nu$, so $\{K_\nu \mid \nu<\omega_1^{CK}\}$ is the enumeration of $\omega_1^{CK}$-finite sets. Now there is another $\Sigma_1(\omega_1^{CK})$ function $r(\sigma,\epsilon)$ where $K_{r(\sigma,\epsilon)}\subset K_{r(\tau,\epsilon)}$ if $\sigma\leq\tau$ and if $B$ is $\Sigma_1(\omega_1^{CK})$ then $B=\cup\{K_{r(\sigma,\epsilon)} \mid \sigma<\omega_1^{CK}\}$ for some $\epsilon<\omega_1^{CK}$. Define $S_\epsilon=\cup\{K_{r(\sigma,\epsilon)} \mid \sigma<\omega_1^{CK}\}$ and fix $\{S_\epsilon \mid \epsilon<\omega_1^{CK}\}$ as the canonical enumeration of the $\omega_1^{CK}$-recursively enumerable sets.

In the priority argument there will naturally be $\omega_1^{CK}$ many requirements, essentially following the enumeration $\{S_\epsilon \mid \epsilon<\omega_1^{CK}\}$ of $\omega_1^{CK}$-recursively enumerable sets. Say they are given a recursive enumeration $\{R_\sigma \mid \sigma<\omega_1^{CK}\}$. Use the projectum map $\pi$ to reorder the priority of the requirements as follows. At stage $\beta<\omega_1^{CK}$ of the argument define requirement $\mathrm{Req}(n)$ for $n<\omega$ as $R_{\pi^{-1})(n)}$, with the caveat that $\pi^{-1}(n)$ must be computed before the standard actions of the requirement are carried out. Recall that when it converges, $\pi^{-1}(n)$ takes $|\pi^{-1}(n)|$ steps to converge. The priority of the requirements is $\mathrm{Req}(1)>\mathrm{Req}(2)>\mathrm{Req}(3),\ldots$. For requirement $\mathrm{Req}(n)$ let $\alpha_{n,1},\ldots,\alpha_{n,k_n}$ enumerate the stages at which an injury occurs to that requirement. As in the standard finite injury argument this sequence is finite, indeed $k_n\leq 2^n$. In other words the original combinatorics carry over once the priorities have been rearranged. For $\Sigma_1$-admissible ordinals with larger projecta new arguments must be used. They are present in Sacks and Simpson [8] but are not needed here since we only consider $\alpha=\omega_1^{CK}$.

**Theorem 0.15.** There exists a $\Delta_1(\omega_1^{CK})$ set of trees $\{T_n^{\omega_1^{CK}} \mid n<\omega\}$ satisfying $TP(\omega_1^{CK})$ where additionally none of the non-isolated branches are $\omega_1^{CK}$-recursively enumerable.

Without requiring property $TP(\omega_1^{CK})$ the theorem would follow immediately from the Kreisel/Sacks argument above and the Lemma 0.5 presented at the beginning. We describe the tension arising from the addition of requirement $TP(\omega_1^{CK})$. As in the first order case there will be primary candidates and potential candidates for the non-principle types. The potential candidates will always differ from their associated primary candidate by a negative choice: for some priority branch $q$ the potential candidate will fail to have an extension to the corresponding pseudo-predicate. As the argument progresses, the primary candidate may switch over to one of the potential candidates. This will add a negative choice to the primary candidate; one might worry that at some point all the choices for priority pseudo-predicates become exhausted. To allay this fear, note that the final axiom of (EC) guarantees that every $n$-type has an extension to the seed partial $(n+1)$-type, $\exists y\ Q^{\gamma}_{n+1}(\bar{x}, y)$, and that at every successor stage $\beta + 1$ countably many new priority branches $q^{\beta+1}_{n,\beta+k}$ are introduced, all of which extend $Q^{\beta}_{n}$. Therefore the priority argument may indeed stall at level $\beta$, but a new crop of choices becomes available at level $\beta + 1$. While it is acceptable for the priority argument to stall occasionally, the regeneration of the Scott analysis must remain strictly monotone. This is the reason for the separate collection of regeneration branches and pseudo-predicates for which all primary and potential candidates make positive choices.

*Proof of theorem.* We run the priority argument simultaneously for all the non-isolated branches of the trees. For every named branch $s \in \{b_{n,k} \mid n, k < \omega\} \cup \{q_{n,\delta} \mid n < \omega, \delta < \omega_1^{CK}\} \cup \{Q_n \mid n < \omega\}$ and for every $\delta < \omega_1^{CK}$ the requirement $Rs(n)$ states that the branch $s$ is not the characteristic function for the set $S_{\pi^{-1}(n)}$. As such, requirement $Rs(n)$ first attempts to compute $\pi^{-1}(n)$ and then, if it is successful, begins to check if A *witness* for the non-isolated branch $s$ is a pair $Ws = (w, v) \in \omega_1^{CK} \times 2$ where the $w^{th}$ bit of the primary candidate for branch $s$ takes the value $v$. If $Ws = (w, v)$ then refer to $w$ as $Ws[w]$ and $v$ as $Ws[v]$. Witness $Ws_n$ is used to establish requirement $Rs(n)$. At any stage a requirement is in one of three disjoint states: *unhappy, addressed, or* $\omega_1^{CK}$*-finitely satisfied.* Additionally, if the requirement is addressed or $\omega_1^{CK}$-finitely satisfied call it a *happy* requirement. All transitions between these states are possible except for the transition from $\omega_1^{CK}$-finitely satisfied to addressed. Transitions from a happy state to the unhappy state correspond to injuries to the requirement.

**Property** $*TP(\delta)$ means the following.

The trees $\{T^{\delta}_{n} \mid n < \omega\}$ satisfy the tree property $TP(\delta)$ and have complexity $\Delta_1(\delta)$. For each branch $s \in \{q^{\delta}_{n,\beta} \mid \beta \leq \omega\delta\} \cup \{b^{\delta}_{n,k} \mid n, k < \omega\} \cup \{Q^{\delta}_{n} \mid n < \omega\}$ some initial segment $Rs(1), \ldots, Rs(k_{s,\delta})$, for $k_{s,\delta} \leq \omega$ of the requirements are in a happy state. Additionally there are witnesses $Ws_n$ for each of these happy requirements, and associated computations $Cs_n$. The computation $Cs_n$ first attempts to compute $\pi^{-1}(n)$, and if it is successful starts to check if the ordinal $Ws_n[w]$ belongs to the $\omega_1^{CK}$-recursively enumerable set $S_{\pi^{-1}(n)}$. If $Cs_n$ verifies that $Ws_n[w]$ enters the set $S_{\pi^{-1}(n)}$ in fewer than $\delta$ steps then $Rs(k)$ is addressed, and otherwise $Rs(k)$ is $\omega_1^{CK}$-finitely satisfied. If $Rs[n]$ is addressed then $Ws_n[v] = 0$, and if $Rs[n]$ is $\omega_1^{CK}$-finitely satisfied then $Ws_k[n] = 1$. For each $n$ with $Rs[n]$ addressed there is a branch sharing the initial segment of length $Ws_n[v]$ with the primary candidate $s$, and isolated in the tree $T_n^\delta$ by taking the value 0 on the bit $Ws_n[v]$. If the value of $\pi^{-1}(k)$ is determined then this branch becomes the $\pi^{-1}(k)^{th}$ *potential candidate* for $s$. Note that there are exactly $\delta$ many potential candidates for $s$, in agreement with the property $TP(\delta)$. Finally, the witness heights $\{Ws_1[w] \dots Ws_{k_{s,\beta}}[w]\}$ occur in order of priority, that is, if $\pi(\gamma) < \pi(\beta)$ then $Ws_\gamma[w] < Ws_\beta[w]$. Additionally, $\pi^{-1}(k) < Ws_k[w]$ for every $k$ in the range of $\pi$.

**Stages 1 of the construction.** We begin with trees $\{T_n^1 \mid n < \omega\}$ where $T_n^1 \subset 2^{<\omega}$ and a finite string $s$ belongs to $T_n^1$ if $s(k) = 0$ at most twice. Fix $n$. The lexicographic order type of the non-isolated branches of $T_n^1$ is an $\omega$-sequences followed by a final limit. The $\omega$-sequence is the set of regeneration branches $\{b_{n,k}^1 \mid k < \omega\}$ and the final limit – that is the branch of all ones – is the seed branch $Q_n^1 \in T_n^1$. There are no priority branches at this level so $TP(0)$ is not quite satisfied.

**Stage $\gamma$, where $1 < \gamma$ is finite or $\gamma = \omega\delta + 1$.** At this stage we stall on the priority argument and simply build trees $\{T_n^\gamma \mid n < \omega\}$ satisfying $TP(\gamma)$. Extend every terminal branch $s \in T_n^{\gamma-1}$ as positively as possible, modulo the constraints that $\mathrm{NegQ}^{\gamma-1}(s) \subset \mathrm{NegQ}^\gamma(s)$ and $\pi_{\gamma-1}^{-1}(\mathrm{NegP}^{\omega\delta}(s)) \subset \pi_\gamma^{-1}(\mathrm{NegP}^\gamma(s))$. For those branches extending non-isolated branches in $T_n^{\gamma-1}$ add countably many approximating branches that differ at a single even height $\omega(\gamma) + 2k$ from the original branch. Take the lexicographically least extension of the seed branch $Q_n^{\gamma-1}$, that is, the branch $s$ that differs from $Q_n^\gamma$ by $s(\omega(\gamma-1)+2) = 0$. Let this branch be $q_\gamma^\gamma$. Add countably many branches that approximate $q_\gamma^\gamma$, each differing at a single even height $\omega(\gamma) + 2k$, for $k > 1$.

**Stage $\delta + 2$ of the construction.** Inductively assume $*TP(\delta + 1)$. For each branch $s^{\delta+1} \in \{q_{n,\gamma}^{\delta+1} \mid \gamma \leq \delta\} \cup \{b_{n,k}^{\delta+1} \mid n, k < \omega\} \cup \{Q_n^{\delta+1} \mid n < \omega\}$ do the following two steps.

- Step 1. Let $k < \omega$ be the least number where $Rs(k)$ is addressed, and where according to the computation $Cs_k$ the witness $Ws_k(v)$ has entered the set $S_{\pi^{-1}(k)}$ at step $\delta + 2$. If no such $k$ exists then define $s^{\delta+2}|_{\omega(\delta+1)} =_{def} s^{\delta+1}$ and proceed to the next step. Otherwise, since the computation $Cs_k$ has established the identity of the ordinal $\pi^{-1}(k)$ it follows that $\pi^{-1}(k) < \delta + 2$ and there is a potential candidate $s^{\delta+1}[\pi^{-1}(k)]$ to branch $s^{\delta+1}$. We do not change $s^{\delta+1}$, since it will likely be used to create a pseudo-predicate. Instead, we define

$s^{\delta+2}|_{\omega(\delta+1)} =_{def} s^{\delta+1}[\pi^{-1}(k)]$. This is an injury to branch $s$. For all requirements $Rs(m)$ of lower priority $n < m$ change the state of $Rs(m)$ to unhappy.

- Step 2. Let $m < \omega$ be the least number where $Rs(m)$ is unhappy, if it exists (it may be that all requirements are currently happy). We use the priority branch $q^{\delta+1}_{n+1,\delta+1}$ to create a witness $Ws_m$ for this requirement, as follows. Since $\delta+1 \notin \pi_\delta^{-1}(\mathrm{NegQ}^\delta(s))$ we are free to choose positively or negatively at the height corresponding to $q^{\delta+1}_{n+1,\delta+1}$. Therefore if computation $Cs_m$ establishes in $\leq \delta{+}2$ steps that $\omega(\delta{+}1){+}2\pi_{\delta+2}(\delta{+}1) = 0$, then set $Ws_m[v] = 0$ and set the requirement to $\omega_1^{CK}$-finitely satisfied. Otherwise set $Ws_m[v] = 1$ and set the requirement to addressed.

The construction of the trees $\{T^{\delta+2}_n \mid n < \omega\}$ is the same as in the previous case, except that each non-isolated branch $s^{\delta+2}$ is built to extend $s^{\delta+2}|_{\omega(\delta+1)}$ as defined in step 1 above. We leave it to the reader to verify that the trees satisfy property $*TP(\delta + 1)$.

**Stage $\omega\delta$ of the construction.** At limit stages take the unions of the trees: $T^{\omega\delta}_n =_{def} \cup_{\gamma<\omega\delta} T^\gamma_n$.

The proofs that each requirement is eventually met, and that each named branch coalesces into a well-defined non-isolated branch, are the same as the finite injury proof presented in Lemma 0.5. ⊣

# Main result

Combining the previous two theorems we have the following result:

**Corollary 0.16.** There exists a complete and consistent Scott theory $th^{\omega_1^{CK}}$ in the language $\mathfrak{L}_{\omega_1^{CK},\omega}$ of complexity $\Delta_1^{\omega_1^{CK}}$ with some, but only countably many, non-principle types, all of which fail to be $\Sigma_1^{\omega_1^{CK}}$.

**Theorem.** There exists a countable structure $\mathcal{M}$ such that:

- $\omega_1^{\mathcal{M}} = \omega_1^{CK}$,
- the $\mathfrak{L}_{\omega_1^{CK},\omega}$ theory of $\mathcal{M}$ is $\Delta_1^{L(\omega_1^{CK})}$,

- the Scott rank of $\mathcal{M}$ is $\omega_1^{CK}$,
- $\mathcal{M}$ is an atomic model of its $\mathfrak{L}_{\omega_1^{CK},\omega}$ theory, and
- the $\mathfrak{L}_{\omega_1^{CK},\omega}$ theory of $\mathcal{M}$ is not $\omega$-categorical.

*Proof.* Let $th^{\omega_1^{CK}}$ be the theory of Corollary 0.17 above. The desired $\mathcal{M}$ will be a countable model of $th^{\omega_1^{CK}}$ constructed via Barwise compactness and type omitting.

Let (Z) be the following set of sentences.

- (Z1) The atomic diagram of $L(\omega_1^{CK})$ in the sense of $\mathfrak{L}_{\omega_1^{CK},\omega}$. For example, each $\beta < \omega_1^{CK}$ is named by $\underline{\beta}$ and is characterized by the sentence

$$\forall x(x < \underline{\beta} \leftrightarrow \bigvee_{\gamma<\beta} x = \underline{\gamma}).$$

  It follows that every model of (Z1) is an end extension of $L(\omega_1^{CK})$.
- (Z2) The axioms of $\Sigma_1$ admissibility.
- (Z3) Let $d$ be a constant denoting an ordinal and not occurring in (Z1). For each $\beta < \omega_1^{CK}$:

$$\underline{\beta} < d.$$

- (Z4) Let $\underline{\mathcal{M}}$ denote a countable structure whose underlying language is that of $th^{\omega_1^{CK}}$. For each sentence $\mathfrak{F} \in th^{\omega_1^{CK}}$:

$$\underline{\mathcal{M}} \models \mathfrak{F}.$$

Note that (Z) is $\Delta_1^{L(\omega_1^{CK})}$ because $th^{\omega_1^{CK}}$ is. Hence (Z) has a countable model $\mathcal{N}$ by Barwise compactness. Let $\mathcal{M}^{\mathcal{N}}$ be the structure denoted by $\underline{\mathcal{M}}$ in $\mathcal{N}$. By (Z4) $\mathcal{M}^{\mathcal{N}}$ is a model of $th^{\omega_1^{CK}}$ and has Scott rank $\geq \omega_1^{CK}$ according to lemma 0.9 of Section 1. The proof of Barwise compactness is a Henkin argument that needs two modifications to assure:

- (i) $\omega_1^{CK} \notin \mathcal{N}$,
- (ii) $\mathcal{M}^{\mathcal{N}}$ realizes no non-principal types of $th^{\omega_1^{CK}}$.

Condition (i) implies $\omega_1^{\mathcal{M}^{\mathcal{N}}} = \omega_1^{CK}$ because any ordinal computable from a real in $\mathcal{N}$ belongs to $\mathcal{N}$. Condition (ii) implies $\mathcal{M}$ has Scott rank $\leq \omega_1^{CK}$.

**Establishing (i).** Suppose $e$ is a constant that denotes an ordinal and the Henkin construction at stage $n$ imples that $\beta < e$ for all $\beta < \omega_1^{CK}$. Introduce a new constant $e'$ and add to (Z) the axioms

$$\{\beta < e' < e \mid \beta < \omega_1^{CK}\}.$$

If the addition were contradictory, then for some $\beta_0 < \omega_1^{CK}$, the axioms

$$\{\beta < e' < e \mid \beta < \beta_0\}$$

would be contradictory. This last follows from the fact that a consequence of a $\Sigma_1^{\omega_1^{CK}}$ set $K$ of sentences is a consequence of some $K_0 \in L(\omega_1^{CK}) \cap 2^K$. But then

$$\beta < \beta_0 < e$$

for all $\beta < \beta_0$ would be contradictory, which it is not. On the other hand if $(e = \beta)$ is consistent at stage $n$ for some $\beta < \omega_1^{CK}$ then add one such sentence to the Henkin construction at stage $n$.

**Establishing (ii).** Let $p$ be a non-principal $m$-type of theory $th^{\omega_1^{CK}}$ and let $\bar{b}$ be a constant that denotes an $m$-tuple of $\mathcal{M}^{\mathcal{N}}$. Suppose for contradiction that the Henkin construction at stage $n$ implies $\mathfrak{F}(\bar{b})$ for all $\mathfrak{F} \in p$. Then $p$ is $\Sigma_1^{\omega_1^{CK}}$, hence $\Delta_1^{\omega_1^{CK}}$ since the type is complete. But $th^{\omega_1^{CK}}$ has no $\Delta^{\omega_1^{CK}}$ non-principle types and we have established the contradiction.

Hence, for every non-principal $m$-type $p$ and $m$-tuple $\bar{b} \in \mathcal{M}^{\mathcal{N}}$ there must be some $\mathfrak{F}(\bar{x}) \in p$ such that $\neg\mathfrak{F}(\bar{b})$ is consistent with the Henkin construction at stage $n$. Since $th^{\omega_1^{CK}}$ has only countably many non-principle types it is possible to address each such pair $p, \bar{b}$ in the countably many steps of the Henkin construction.

Finally, the $\mathfrak{L}_{\omega_1^{CK},\omega}$ theory of $\mathcal{M}$ is $th^{\omega_1^{CK}}$ which has a non-principal type and so is not $\omega$-categorical. ⊣

# Conclusion

We end with two open questions, one about reducing the complexity of the model and the other about increasing the Scott rank.

**Open question 1**. Does the main result hold if one insists on a computable model?

**Open question 2**. Does the main result hold when $\omega_1^{CK}$ is replaced by an arbitrary countable admissible ordinal?[1]

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[1] Answered in the affirmative by Cameron Freer in his Ph.D Thesis, Harvard University, 2008.