

Interior Regularity of Solutions to the Isotropically Constrained Plateau Problem

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Abstract

In this paper, we study the regularity of isotropically area-minimizing surfaces. We prove a partial regularity theorem which says that if an $W^{1,2}$ isotropic map from a two-dimensional disk into \mathbb{R}^{2n} minimizes area relative to its boundary among isotropic competitors and is close enough in $W^{1,2}$ norm to a linear holomorphic isotropic map, then it is smooth in the interior. Furthermore, we prove that the solution to the isotropically constrained Plateau problem exists and has a smooth interior with possibly isolated singularities.

1 Introduction

In [S-W], Schoen and Wolfson used variational approach to study the existence of special Lagrangian surfaces. They pointed out that if a *smooth* closed Lagrangian submanifold in a Kähler-Einstein manifold is stationary for the volume functional among all Lagrangian variations, then it is in fact minimal (hence special Lagrangian). Thus they suggested that minimizing area among all Lagrangian cycles in a homology class will lead to the production of a closed special Lagrangian submanifold *if the minimizer is smooth*. However, the smoothness of the area minimizers is not so clear. Schoen and Wolfson studied this regularity problem in two-dimensional case. They proved that a Lagrangian map from a 2-dimensional disk to a 4-dimensional Kähler manifold which minimizes area among all Lagrangian maps in the same homology class is in fact a branched immersion away from a finite set of points. As a consequence, they produced branched immersed 2-dimensional Lagrangian cycles with finite number of singularities which minimizes area among its Lagrangian homology class.

In this present paper, using the similar approach of Schoen and Wolfson, we extend their regularity result to higher CO-dimensions. We study the constrained area minimizing problem among the class of isotropic surfaces. In particular we concentrate on the following two-dimensional isotropically constrained Plateau problem in \mathbb{R}^{2n} .

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Let $\omega = \sum_{k=1}^n dx^k \wedge dy^k$ be the standard symplectic form on \mathbb{R}^{2n} . A surface Σ is called isotropic if $\omega|_{\Sigma} = 0$. Let D_1 be the unit disk in \mathbb{R}^2 . Let Γ be a closed piecewise C^1 Jordan curve in \mathbb{R}^{2n} which bounds some isotropic disk. It is natural to ask whether Γ bounds an isotropic disk which has the least area among all isotropic disks bounded by Γ . More precisely, define

$$\mathcal{X}_{\Gamma, I} = \{l \in W^{1,2}(D_1, \mathbb{R}^{2n}) : l^* \omega = 0, \\ l|_{\partial D_1} \text{ is continuous and monotone onto } \Gamma\}$$

The isotropically constrained Plateau problem then asks two questions:

- (1) Does there exist an area minimizer in this class?
- (2) What is the regularity of this area minimizer if it exists?

It is fairly easy to get a positive answer to the first question. We shall show in section 2 that the set of weakly isotropic maps is weakly closed. The existence of area minimizer in this class then follows from standard arguments. We will call it a weak solution to the isotropically constrained Plateau problem (Minicozzi in his Ph.D thesis [Mi] gave the similar weak existence of the solution to Legendrianly constrained Plateau problem).

The regularity of this solution turns out to be a much harder problem. We shall first extend Schoen and Wolfson's monotonicity formula for Lagrangian stationary surfaces in \mathbb{R}^4 ([S-W]) to higher co-dimensional isotropically stationary surfaces in \mathbb{R}^{2n} . Consequently we deduce a partial regularity results which says that if an $W^{1,2}$ isotropic map from a two-dimensional disk into \mathbb{R}^{2n} minimizes area relative to its boundary among isotropic competitors and is close enough in $W^{1,2}$ norm to a linear holomorphic isotropic map, then it is smooth interiorly (Theorem 5.10). By studying the stability of isotropically stationary tangent cones and by using the dimension reduction argument, we obtain the interior regularity theorem which concludes that the solution to the isotropically constrained Plateau problem is smooth away from a finite set (Theorem 7.4).

The organization of this paper is as follows:

First we introduce the weakly isotropic maps and prove the weak existence of the solution to the isotropically constrained Plateau problem (section 2). In section 3 we present the monotonicity for isotropically stationary surfaces. In section 4 we prove the excess decay lemma and the consequent $C^{1,\mu}$ partial regularity theorem. The higher regularity issue (from $C^{1,\mu}$ to C^∞) is of a subtle nature and will be addressed in section 5. In section 6 we classify all the isotropically stationary tangent cones. Then in the last section we give the interior regularity result.

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2 Existence of Weak solutions to the Isotropically Constrained Plateau Problem

In this section we will pose the isotropically constrained Plateau problem and prove the weak existence result.

Let $\omega = \sum_{k=1}^n dx^k \wedge dy^k$ be the standard symplectic form in \mathbb{R}^{2n} . A surface $\Sigma^2 \subset \mathbb{R}^{2n}$ is called *isotropic* if $\omega|_{\Sigma} = 0$.

Definition 2.1. *Let $\Sigma^2 \subset \mathbb{R}^{2n}$ be an isotropic surface. A deformation F_s on \mathbb{R}^{2n} is called isotropic with respect to Σ if $F_s(\Sigma)$ are isotropic for all small s .*

Σ is called isotropically stationary if

$$\left. \frac{d(\text{Area}(F_s(\Sigma_s)))}{ds} \right|_{s=0} = 0$$

for any isotropic deformation F_s which is compactly supported on Σ

We will use $\eta = \sum_{k=1}^n x^k dy^k - y^k dx^k$ to denote a primitive of ω in \mathbb{R}^{2n} , use $D_r(t)$ to denote the disk in \mathbb{R}^2 centered at t with radius r , and use $C_r(t)$ to denote $\partial D_r(t)$.

Definition 2.2 (Weakly isotropic maps). *Let $\Omega \subset \mathbb{R}^2$ be a domain. $l \in W^{1,2}(\Omega, \mathbb{R}^{2n})$ is called weakly isotropic if $l^*\omega = 0$ for a.e. $t \in \Omega$. We will use $W_I^{1,2}(\Omega, \mathbb{R}^{2n})$ to denote the set of all weakly isotropic maps.*

Lemma 2.3. *The set*

$$\Gamma_{I,M} = \{l \in W_I^{1,2}(D_1, \mathbb{R}^{2n}) : E(l) \leq M\}$$

is weakly closed, where $E(l) = \int_{D_1} |\nabla l|^2 dt$ is the energy of l .

Proof. Let $\{l_i\} \subset \Gamma_{I,M}$. Assume it converges weakly to l . Then by the lower-semi-continuity of energy, $E(l) \leq M$. So it remains to show that l is weakly isotropic.

Let $\eta = xdy - ydx$ be a primitive of ω in \mathbb{R}^{2n} . It is easy to see that a map h is weakly isotropic if and only if $\int_{C_r(t)} h^*\eta = 0$ for a.e. $t \in D_1$ and $r \in (0, 1 - |t|)$, where $C_r(t)$ is the circle centered at t with radius r .

By Rellich lemma, since $E(l_i) \leq M$, we might assume that (by passing to a subsequence) $\{l_i\}$ converges to l strongly in L^2 . Co-area formula then implies that for a.e. $C_r(t) \subset D_1$, $l_i|_{C_r(t)}$ converges to $l|_{C_r(t)}$ strongly in L^2 and weakly in $W^{1,2}$. Hence,

$$\lim_{i \rightarrow \infty} \int_{C_r(t)} l_i^* \eta = \int_{C_r(t)} l^* \eta,$$

which shows that $\int_{C_r(t)} l^* \eta = 0$. □

Isoperimetric inequality in the setting of isotropic surfaces is very important to construct comparison surfaces. It is proven by Gromov[Gr] and Allcock[Al].

Proposition 2.4 (Gromov, Allcock). *Given any $W^{1,2}$ map ϕ from the unit circle C into \mathbb{R}^{2n} satisfying $\int_C \phi^*(xdy - ydx) = 0$, there exists a weakly isotropic map $l \in W^{1,2}(D, \mathbb{R}^{2n})$, such that $l = \phi$ on C and*

$$Area(L(D)) \leq cLength(l(C))^2.$$

Using the isoperimetric inequality, we are able to prove the Hölder continuity of isotropically area minimizing maps.

Proposition 2.5. *If $l \in W^{1,2}(D_1, \mathbb{R}^{2n})$ is weakly isotropic, weakly conformal and minimizes area among all isotropic maps with the same boundary value. Then there is a $\mu \in (0, 1)$ and an absolute constant c such that $l \in C^{0,\mu}(D_{1/2})$. Moreover*

$$|l(P) - l(Q)| \leq cA(l(D_1))^{1/2}|P - Q|^\mu, \text{ for any } P, Q \in D_{1/2}.$$

Proof. By scaling, we may assume $Area(D_1) = 1$. Given any $p \in D_{1/2}$, let $f(r) = E(l|_{D_r(p)})$. The conformality of l implies $f(r) = 2Area(l|_{D_r(p)})$. Since l is isotropic, we have $\int_{C_r(p)} l^*\eta = 0$. Therefore Proposition 2.4 in addition to the fact that l is isotropically area minimizing, gives that

$$Area(l|_{D_r(p)}) \leq cLength(l|_{C_r(p)})^2.$$

Using Co-area formula and Cauchy-Schwartz inequality, it is easy to obtain

$$Length(l|_{C_r(p)})^2 \leq 2\pi r f'(r).$$

Therefore $f(r) \leq cr f'(r)$, where $c > 0$ is an absolute constant. Integrating, we get

$$f(r) \leq (f(r_1)r_1^{-c})r^c,$$

for any $r \leq r_1$ such that $D_{r_1}(p) \subset D_1$. Since $p \in D_{1/2}$, we could let $r_1 = 1/3$. Note $f(1/3) \leq Area(D_1) = 1$. So we have

$$\int_{D_r(p)} |\nabla l|^2 dt \leq c_1 r^c,$$

where c_1 and c are absolute positive constant. Morrey's estimate(see [G-T]) then implies that l is $C^{0,\mu}$ and

$$|l(P) - l(Q)| \leq c_2|P - Q|^\mu,$$

for any P, Q in $D_{1/2}$, where c_2 is an absolute constant. □

Definition 2.6. *A weakly isotropic map $l : D_1 \rightarrow \mathbb{R}^{2n}$ is called exact if there is a $W^{1,2}$ function ϕ on D_1 such that $d\phi = l^*(xdy - ydx)$ (we will call ϕ a lifting function of l).*

If an isotropic map is exact then we could use the function ϕ to lift the map into the contact map in \mathbb{R}^{2n+1} with the standard contact structure $\alpha = d\phi - (xdy - ydx)$. More precisely, for a weakly exact isotropic map $l : D_1 \rightarrow \mathbb{R}^{2n}$, we can define a map $\tilde{l} : D_1 \rightarrow \mathbb{R}^{2n+1}$ by

$$\tilde{l}(t) = (l(t), \phi(t)) \in \mathbb{R}^{2n+1}, \quad (1)$$

where ϕ is the lifting function. Then clearly $\tilde{l}^*\alpha = 0$.

Let $l : D_1 \rightarrow \mathbb{R}^{2n}$ be a weakly isotropic, weakly conformal and isotropically area-minimizing map. By Proposition 2.5, l is continuous. Therefore by locally integrating, we see the local existence of a lifting function ϕ .

Proposition 2.7. *If $l \in W^{1,2}(D_1, \mathbb{R}^{2n})$ is weakly isotropic, weakly conformal and minimizes area among all isotropic maps with the same boundary value. Then l is locally exact.*

Now we consider the isotropically constrained Plateau problem. Let Γ be a piecewise C^1 closed Jordan curve such that $\int_{\Gamma} \eta = 0$. Define

$$\mathcal{X}_{\Gamma, I} = \{l \in W_I^{1,2}(D_1, \mathbb{R}^{2n}) : l|_{\partial D_1} \text{ is continuous and is a monotone map onto } \Gamma\} \quad (2)$$

By Proposition 2.4, Γ bounds some isotropic disk, so $\mathcal{X}_{\Gamma, I}$ is not empty.

Let $A_{\Gamma, I} = \inf_{l \in \mathcal{X}_{\Gamma, I}} \text{Area}(l)$ and let $E_{\Gamma, I} = \inf_{l \in \mathcal{X}_{\Gamma, I}} E(l)$. Classical result in Plateau problem gives:

Lemma 2.8. *$A_{\Gamma, I} = \frac{1}{2}E_{\Gamma, I}$. Moreover if $E(l_0) = E_{\Gamma, I}$, then $A(l_0) = A_{\Gamma, I}$.*

The above lemma enables us to minimize energy instead of area. To quotient out the conformal group of the disk, we consider a subclass. Let p_1, p_2 and p_3 be distinct points in ∂D_1 and q_1, q_2 and q_3 be distinct points on Γ . Define

$$\mathcal{X}'_{\Gamma, I} = \{l \in \mathcal{X}_{\Gamma, I} : l|_{\partial D_1}(p_i) = q_i, i = 1, 2, 3\}. \quad (3)$$

We now have the following existence theorem of the isotropically constrained Plateau problem.

Theorem 2.9. *There is a weakly conformal, weakly isotropic map $l_0 \in \mathcal{X}'_{\Gamma, I}$ such that $A(l_0) = A_{\Gamma, I}$. Moreover l_0 is Hölder continuous in the interior of D_1 .*

Proof. Since energy is conformally invariant, we see that

$$\inf_{l \in \mathcal{X}'_{\Gamma, I}} E(l) = E_{\Gamma, I}. \quad (4)$$

It suffices to show that there is an energy minimizer in $\mathcal{X}'_{\Gamma, I}$. Let $\{l_i\} \subset \mathcal{X}'_{\Gamma, I}$ such that $\lim_{i \rightarrow \infty} E(l_i) = E_{\Gamma, I}$. Assume $\{l_i\}$ converges weakly to l_0 in $W^{1,2}$ sense and strongly in L^2 sense. The lower-semi-continuity of energy gives $E(l_0) \leq E_{\Gamma, I}$. It suffices now to show that l_0 is in the class of $\mathcal{X}'_{\Gamma, I}$. Lemma 2.3

shows that l_0 is weakly isotropic. Classical Courant-Lebesgue lemma(see [La]) gives that the class $\{l|_{\partial D_1} : l \in \mathcal{X}'_{\Gamma,I}\}$ is equicontinuous. Therefore by Arzela-Ascoli lemma we might assume(by passing to a subsequence) $\{l_i|_{\partial D_1}\}$ converges in C^0 sense to a continuous function $\varphi : \partial D_1 \rightarrow \mathbb{R}^{2n}$. Clearly φ is monotone onto Γ and $\varphi(p_i) = q_i$ for $i = 1, 2, 3$. We know that the trace map is continuous from $W^{1,2}(D_1, \mathbb{R}^{2n})$ to $W^{1/2,2}(\partial D_1, \mathbb{R}^{2n})$, and $W^{1/2,2}(\partial D_1, \mathbb{R}^{2n})$ is compactly embedded in $L^2(\partial D_1, \mathbb{R}^{2n})$. Therefore the weak convergence of $\{l_i\}$ to l_0 in $W^{1,2}(D_1, \mathbb{R}^{2n})$ implies the strong convergence of $\{l_i|_{\partial D_1}\}$ to $l_0|_{\partial D_1}$ in $L^2(\partial D_1, \mathbb{R}^{2n})$. Hence we $l_0|_{\partial D_1} = \varphi$. So $l_0|_{\partial D_1}$ is continuous, monotone onto Γ and $l_0|_{\partial D_1}(p_i) = q_i$. Hence $l_0 \in \mathcal{X}'_{\Gamma,I}$ and $E(l) = E_{\Gamma,I}$. By Lemma 2.8, l_0 is the area minimizer. The Hölder continuity follows from Proposition 2.5. \square

3 Extension of Schoen-Wolfson's Monotonicity to Higher Co-dimensions

Monotonicity for two-dimensional Lagrangian stationary surfaces was obtained by Schoen and Wolfson in [S-W]. With a slight modification we can extend their result to higher co-dimensions.

Instead of deriving the monotonicity directly in the Lagrangian setting, Schoen and Wolfson consider the lifting of a Lagrangian surfaces into a contact space and derive the monotonicity upstairs. Let $(x^1, \dots, x^n, y^1, \dots, y^n, \varphi)$ be the coordinate on \mathbb{R}^{2n+1} . And let $\alpha = d\varphi - \eta$ be the standard contact form on \mathbb{R}^{2n+1} , where $\eta = xdy - ydx$. A submanifold N^k is called contact if $\alpha|_N = 0$. A vector field X in \mathbb{R}^{2n+1} is called a contact vector field if its flow preserves the contact structure, i.e. $L_X\alpha = f\alpha$ for some function f . Just like in the symplectic case, any ambient function in \mathbb{R}^{2n+1} generates a contact vector field.

Lemma 3.1. *Let $h : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$ be any smooth function. Define a vector field*

$$\begin{aligned} X_h = & h_x \frac{\partial}{\partial y} - h_y \frac{\partial}{\partial x} - h_\varphi (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) \\ & + (-2h + xh_x + yh_y) \frac{\partial}{\partial \varphi}. \end{aligned}$$

Then X_h is a contact vector field(and will be called the Hamiltonian vector field for h).

Define $\Pi : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n}$ to be the projection along φ direction. For vectors Z_1, Z_2 in \mathbb{R}^{2n+1} , define

$$\langle Z_1, Z_2 \rangle_d = \langle \Pi_* Z_1, \Pi_* Z_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean metric on \mathbb{R}^{2n} . Note that $\langle \cdot, \cdot \rangle_d$ is not a Riemannian metric on \mathbb{R}^{2n+1} since it degenerates. But it is not difficult to see that the restriction of $\langle \cdot, \cdot \rangle_d$ to any contact linear subspace is non-degenerate.

Let $l : D \rightarrow \mathbb{R}^{2n}$ be an isotropic map, where D is a 2-dimensional disk. Assume l is exact and isotropically stationary (in the sense that the first variation of area is zero for compactly supported deformations of l through isotropic maps). Then as described in section 2, the exactness of l gives a lifting map $\tilde{l} : D \rightarrow \mathbb{R}^{2n+1}$ via $\tilde{l} = (l, \varphi)$. Now let Z be any contact vector field in \mathbb{R}^{2n+1} such that Z vanishes along $\tilde{l}(\partial D)$ and let F_t be the flow generated by Z . Define $\tilde{l}_t := F_t \circ \tilde{l}$ and define an isotropic deformation l_t of l by

$$l_t := \Pi \circ \tilde{l}_t.$$

Then first variation of area gives

Proposition 3.2. *Let $l : D \rightarrow \mathbb{R}^{2n}$ be an isotropic map which is exact and isotropically stationary. Let $\tilde{l} \rightarrow \mathbb{R}^{2n+1}$ be the lifting of l . If Z is a contact vector field in \mathbb{R}^{2n+1} such that Z vanishes along $\tilde{l}(\partial D)$, then*

$$\int_{\Sigma} \operatorname{div}_{\Sigma} Z \, d\mu_{\Sigma} = 0. \quad (5)$$

Here $\operatorname{div}_{\Sigma} Z := \sum_{k=1}^2 \langle \nabla_{e_k}^0 Z, e_k \rangle_d$, where ∇^0 denotes the Euclidean connection on \mathbb{R}^{2n+1} and $\{e_i\}_{i=1}^2$ is a set of orthonormal tangent vector of $\Sigma := \tilde{l}(D)$ with respect to the metric $\langle \cdot, \cdot \rangle_d$.

The idea to get the monotonicity is to find a proper Hamiltonian function h and apply Proposition 3.2 to the contact vector field X_h .

Let $s = 2^{-1}(x^2 + y^2)$ and

$$r = \sqrt{2}(s^2 + \varphi^2)^{1/4}. \quad (6)$$

r plays the same role as the distance function and will be called the modified distance function in \mathbb{R}^{2n+1} . Define

$$\tilde{B}_{\sigma}(0) = \{P \in \mathbb{R}^{2n+1} : r(P) < \sigma\}. \quad (7)$$

Also define a change of variables,

$$t = \log(\sqrt{s^2 + \varphi^2}), \quad \theta = \arctan \frac{\varphi}{s}.$$

Since $l(D) \subset \mathbb{R}^{2n}$ is isotropic, we have the orthogonal decomposition of the tangent space of \mathbb{R}^{2n} :

$$T\mathbb{R}^{2n} = Tl(D) \oplus J Tl(D) \oplus Q \quad (8)$$

where J is the standard complex structure of \mathbb{R}^{2n} . Let $\pi : T\mathbb{R}^{2n} \rightarrow Q$ be the orthogonal projection.

Proposition 3.3. *Let $h(t, \theta) : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$ be a smooth function. Then*

$$\begin{aligned} & \operatorname{div}_{\Sigma} X_{h(t, \theta)} \\ = & -2h_t e^{-t} \sin \theta - 2h_{\theta t} e^{-t} \cos \theta \\ & + (h_{\theta t} - h_{\theta}) (2 \|\nabla^{\Sigma} \theta\|_d^2 + e^{-2t} \|\pi(\vec{P})\|^2) \\ & + \{-2h_t + h_{tt} - h_{\theta\theta}\} \langle \nabla^{\Sigma} \theta, \nabla^{\Sigma} t \rangle_d, \end{aligned}$$

where \vec{P} is the position vector of l in \mathbb{R}^{2n} .

We will prove this proposition in Appendix A.

Once Proposition 3.3 is established, we could use the exactly same argument in Schoen-Wolfson's paper to derive the monotonicity. For the convenience of future use, we include the argument here.

Fix $a > 0$, let $\rho_a(t)$ to be a smooth function of t such that

$$\rho_a(t) = \begin{cases} 1 - 2a^{-2}e^t & \text{if } t \leq \log(a^2/2) - 1, \\ 0 & \text{if } t \geq \log(a^2/2). \end{cases}$$

Let

$$\Gamma_1 = \{(t, \theta) : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \theta \leq -t - \log(\frac{a^2}{2}) - 1, \theta \geq t - \log(\frac{a^2}{2}) - 1\},$$

$$\Gamma_2 = \{(t, \theta) : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \theta \geq -t - \log(\frac{a^2}{2}), \theta \leq t - \log(\frac{a^2}{2})\},$$

$$\Gamma_3 = \{(t, \theta) : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\} \setminus (\Gamma_1 \cup \Gamma_2).$$

Consider the following wave equation:

$$\begin{cases} h_{tt} - h_{\theta\theta} - 2h_t = 0 \\ h(t, 0) = 0 \\ h_\theta(t, 0) = \rho_a(t) \end{cases} \quad (9)$$

Clearly we see that the solution of equation(9) is given by

$$h^{(a)} = \begin{cases} \theta - 2a^{-2}e^{-t} \sin \theta & \text{if } (t, \theta) \in \Gamma_1, \\ ** & \text{if } (t, \theta) \in \Gamma_3, \\ 0 & \text{if } (t, \theta) \in \Gamma_2, \end{cases} \quad (10)$$

where ** is some smooth function. Define

$$\begin{aligned} G^{(a)} &= h_\theta^{(a)} - h_{\theta t}^{(a)}, \\ F^{(a)} &= \frac{1}{2}a^2 e^{-t} (h_{\theta t}^{(a)} \cos \theta + h_t^{(a)} \sin \theta). \end{aligned}$$

Clearly $G^{(a)} = 1$ and $F^{(a)} = 1$ in Γ_1 . By scaling,

$$\begin{aligned} F^{(a)}(t, \theta) &= F^{(1)}(t - \log(a^2/2), \theta), \\ G^{(a)}(t, \theta) &= G^{(1)}(t - \log(a^2/2), \theta). \end{aligned}$$

Schoen and Wolfson [S-W] proved the following proposition concerning $F^{(a)}$ and $G^{(a)}$:

Proposition 3.4 (Schoen-Wolfson). *The function $F^{(a)}$ is non-negative. And G satisfies $0 \leq G^{(a)} \leq 1$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Furthermore, there is a fixed constant θ_0 such that $G^{(a)} - G^{(b)} \geq 0$ for $0 \leq b \leq \theta_0 a$.*

Now by Proposition 3.3, we have

$$\operatorname{div}_\Sigma X_{h^{(a)}} = 4a^{-2}F^{(a)} - (2\|\nabla^\Sigma\theta\|_d^2 + e^{-2t}\|\pi(\vec{P})\|^2)G^{(a)}. \quad (11)$$

Clearly the support of $h^{(a)}$ is contained in

$$\{(t, \theta) : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), t \leq \log(a^2/2) + \pi/2\}$$

which is in fact $\tilde{B}_{\sqrt{2}e^{\pi/4}a}(0)$ in \mathbb{R}^{2n+1} . Assume $0 < b < a$, then $h^{(a)} - h^{(b)}$ is a smooth function compactly supported in $\tilde{B}_{\sqrt{2}e^{\pi/4}a}(0)$. Therefore Proposition 3.2 and (11) gave,

$$\begin{aligned} & a^{-2} \int_\Sigma F^{(a)} d\mu_\Sigma - b^{-2} \int_\Sigma F^{(b)} d\mu_\Sigma \\ &= 2^{-1} \int_\Sigma (G^{(a)} - G^{(b)}) (\|\nabla^\Sigma\theta\|_d^2 + 2^{-1}e^{-2t}\|\pi(\vec{P})\|^2) d\mu_\Sigma. \end{aligned} \quad (12)$$

Combine Proposition 3.4 and (12), we have

Theorem 3.5 (Monotonicity). *Let $l : D \rightarrow \mathbb{R}^{2n}$ be a weakly isotropic, weakly conformal, exact, and isotropically stationary map with a contact lifting $\tilde{l} : D \rightarrow \mathbb{R}^{2n}$. Let $\Sigma = \tilde{l}(D)$. Assume $\tilde{l}(0) = 0$ and $\tilde{l}(\partial D)$ is outside $\tilde{B}_r(0)$ (defined in (7)). Then there are constants c_1 and c_2 which only depend on $\operatorname{Area}(\Sigma)$ and r such that*

$$c_1 \leq \sigma^{-2} \operatorname{Area}(\Sigma \cap \tilde{B}_\sigma(0)) \leq c_2,$$

for any $\sigma \in (0, r)$.

4 $C^{1,\mu}$ ε -Regularity for mapping problem

Through out this section we will assume $l \in W^{1,2}(D_2, \mathbb{R}^{2n})$ is weakly isotropic, weakly conformal and area-minimizing among all isotropic maps with the same boundary value. By Proposition 2.7, l is locally exact. Therefore monotonicity applies. Also proposition 2.5 tells us that l is actually $C^{0,\mu}$.

We shall use a version of excess decay result to get the $C^{1,\mu}$ regularity of l .

First quote a reversed Poincaré type inequality which is proven by Schoen and Wolfson [S-W].

Proposition 4.1 (Schoen-Wolfson). *Assume that $A(l(D_2)) \leq c_1$, and l_0 is a linear holomorphic map into a isotropic plane in \mathbb{R}^{2n} . For any $\varepsilon > 0$, there exists an absolute constant c and a constant $\delta = \delta(\varepsilon)$ such that if*

$$\int_{D_2} \{|\nabla(l - l_0)|^2 + |l - l_0|^2\} dt \leq \delta,$$

then

$$\int_{D_1} |\nabla(l - l_0)|^2 dt \leq \varepsilon \int_{D_2} |\nabla(l - l_0)|^2 dt + c \int_{D_2} |l - l_0|^2 dt.$$

Now we define the excess type quantity

$$E(l, l_0, r) = r^{-2} \int_{D_r} |\nabla(l - l_0)|^2 dt, \quad (13)$$

where l_0 is a linear holomorphic map into an isotropic plane in \mathbb{R}^{2n} .

Lemma 4.2 (Excess Decay Lemma). *Assume $A(l(D_1)) \leq c_1$. There exist constants $\varepsilon > 0$ and $\bar{\theta} \in (0, 1)$ depending only on c_1 and an absolute constant c such that if*

$$E(l, l_0, 1) + \int_{D_1} |l - l_0|^2 dt \leq \varepsilon, \quad (14)$$

then there exists a linear holomorphic map l_1 into an isotropic plane such that

$$\begin{aligned} E(l, l_1, \bar{\theta}) &\leq \frac{1}{2} E(l, l_0, 1), \\ \bar{\theta}^{-4} \int_{D_{\bar{\theta}}} |l - l_1|^2 dt &\leq c E(l, l_0, 1). \end{aligned} \quad (15)$$

Proof. Without loss of generality, assume $l_0(t^1, t^2) = (t^1, t^2, 0, 0, \dots, 0)$ is the identity map of D_1 into the $x^1 x^2$ -plane. Assume the lemma is not true, then we have a sequence of maps $l^{(j)} = (x^{(j)1}, \dots, x^{(j)n}, y^{(j)1}, \dots, y^{(j)n})$ such that

$$E(l^{(j)}, l_0, 1) + \int_{D_1} |l^{(j)} - l_0|^2 dt = \varepsilon_j, \quad (16)$$

where $\lim_{j \rightarrow \infty} \varepsilon_j = 0$, but none of them satisfies (15). Let $E_j = E(l^{(j)}, l_0, 1)$. Define

$$\begin{aligned} u^{(j)1} &= \frac{1}{\sqrt{E_j}} (x^{(j)1} - t^1) - \alpha^{(j)1}, \\ u^{(j)2} &= \frac{1}{\sqrt{E_j}} (x^{(j)2} - t^2) - \alpha^{(j)2}, \\ w^{(j)k} &= \frac{1}{\sqrt{E_j}} x^{(j)k} - \gamma^{(j)k}, \quad k = 3, \dots, n, \\ v^{(j)1} &= \frac{1}{\sqrt{E_j}} y^{(j)1} - \beta^{(j)1}, \\ v^{(j)2} &= \frac{1}{\sqrt{E_j}} y^{(j)2} - \beta^{(j)2}, \\ q^{(j)k} &= \frac{1}{\sqrt{E_j}} y^{(j)k} - \mu^{(j)k}, \quad k = 3, \dots, n, \end{aligned} \quad (17)$$

where α, β, γ , and μ are constants to make $\int_{D_1} u^{(j)} = 0$, $\int_{D_1} v^{(j)} = 0$, $\int_{D_1} w^{(j)} = 0$, and $\int_{D_1} q^{(j)} = 0$. Then clearly

$$\int_{D_1} |\nabla u^{(j)}|^2 dt + \int_{D_1} |\nabla v^{(j)}|^2 dt + \int_{D_1} |\nabla w^{(j)}|^2 dt + \int_{D_1} |\nabla q^{(j)}|^2 dt = 1. \quad (18)$$

Standard Poincaré inequality gives that $u^{(j)}, v^{(j)}, w^{(j)}$ and $q^{(j)}$ are bounded in $W^{1,2}$, so by Alaoglu Theorem and Rellich's Theorem we can assume (by passing to a subsequence) that they converge to u, v, w, q respectively weakly in $W^{1,2}$ and strongly in L^2 . The lower-semi-continuity of energy gives

$$\int_{D_1} |\nabla u|^2 dt + \int_{D_1} |\nabla v|^2 dt + \int_{D_1} |\nabla w|^2 dt + \int_{D_1} |\nabla q|^2 dt \leq 1. \quad (19)$$

On the other hand, from (16), we know that $x^{(j)1}(t^1, t^2)$ converges strongly to t^1 in L^2 , $x^{(j)2}(t^1, t^2)$ converges strongly to t^2 in L^2 and $x^{(j)k}(t^1, t^2), y^{(j)k}(t^1, t^2)$ converge strongly to 0 in L^2 sense.

Claim 1. u is a holomorphic from D_1 to $x^1 x^2$ -plane.

Proof. Let $t = t^1 + \sqrt{-1}t^2$, $x = x^1 + \sqrt{-1}x^2$, $y = y^1 + \sqrt{-1}y^2$, and $z^k = x^k + \sqrt{-1}y^k$ for $k = 3, \dots, n$. Then the conformality of $l^{(j)}$ gives us

$$\frac{\partial x^{(j)}}{\partial t} \frac{\partial \bar{x}^{(j)}}{\partial t} + \frac{\partial y^{(j)}}{\partial t} \frac{\partial \bar{y}^{(j)}}{\partial t} + \sum_{k=3}^n \frac{\partial z^{(j)k}}{\partial t} \frac{\partial \bar{z}^{(j)k}}{\partial t} = 0.$$

Cauchy -Schwartz inequality implies

$$\int_{D_1} \left| \frac{\partial x^{(j)}}{\partial t} \frac{\partial \bar{x}^{(j)}}{\partial t} \right| dt \leq \sum_{\alpha=1}^2 \int_{D_1} |\nabla y^{(j)\alpha}|^2 dt + \sum_{k=3}^n \int_{D_1} (|\nabla y^{(j)k}|^2 + |\nabla x^{(j)k}|^2) dt \quad (20)$$

Using the fact that $\frac{\partial \bar{x}^{(j)}}{\partial t} = \frac{\partial x^{(j)}}{\partial t} \frac{\partial \bar{x}^{(j)}}{\partial t} - \frac{\partial(x^{(j)}-t)}{\partial t} \frac{\partial(\bar{x}^{(j)}-\bar{t})}{\partial t}$, we get from (20) that

$$\int_{D_1} \left| \frac{\partial \bar{x}^{(j)}}{\partial t} \right| dt \leq \int_{D_1} |\nabla(l - l_0)|^2 dt = E_j.$$

Thus

$$\int_{D_1} \left| \frac{\partial \bar{u}^{(j)}}{\partial t} \right| dt \leq \sqrt{E_j}.$$

Let j go to infinity, we see that u is holomorphic. \square

Claim 2. *There is a biharmonic function $f : D_1 \rightarrow \mathbb{R}$, such that $(v^1, v^2) = \nabla f$. And w^k, q^k are harmonic functions on D_1 for $k = 3, \dots, n$.*

Proof. Let $\eta, \tilde{x}^k, \tilde{y}^k, k = 3, 4, \dots, n$ be smooth functions of x^1, x^2 with compact support in $\tilde{D}_1 = \{(x^1, x^2) : (x^1)^2 + (x^2)^2 \leq 1\}$. Define \tilde{y}^1, \tilde{y}^2 to be functions of $x^1, x^2, x^k, \dots, y^k, k = 3, \dots, n$ by

$$\tilde{y}^\alpha = \sum_{k=3}^n \left(x^k \frac{\partial \tilde{y}^k}{\partial x^\alpha} - y^k \frac{\partial \tilde{x}^k}{\partial x^\alpha} \right) + \frac{\partial \eta}{\partial x^\alpha}, \alpha = 1, 2. \quad (21)$$

Now let X be a vector field in the cylinder $\tilde{D}_1 \times \mathbb{R}^{2n-2}$ defined by

$$X := \sum_{k=3}^n (\tilde{x}^k \frac{\partial}{\partial x^k} + \tilde{y}^k \frac{\partial}{\partial y^k}) + \sum_{\alpha=1}^2 \tilde{y}^\alpha \frac{\partial}{\partial y^\alpha}. \quad (22)$$

It is easy to check that $d(X \lrcorner \omega) = 0$. Therefore X is a symplectic vector field, i.e. its flow F_s preserves the symplectic form ω (see [M-S]).

Since $l^{(j)}$ is isotropically area minimizing, by Proposition 2.5, it is $C^{0,\mu}$ and has uniform $C^{0,\mu}$ bound. Hence we can assume $l^{(j)} - l_0$ is pointwise small since its L^2 norm is small. Therefore, $F_s \circ l^{(j)}(t^1, t^2)$ has compact support in D_1 for j sufficiently large. Since $l^{(j)}$ minimizes energy, we get $\frac{d}{ds}|_{s=0} \int_{D_1} |\nabla F_s \circ l^{(j)}|^2 dt = 0$. Thus we have

$$\begin{aligned} & \sum_{k=3}^n \int_{D_1} \nabla x^{(j)k} \cdot \nabla \tilde{x}^k(x^{(j)1}, x^{(j)2}) dt \\ & + \sum_{k=3}^n \int_{D_1} \nabla y^{(j)k} \cdot \nabla \tilde{y}^k(x^{(j)1}, x^{(j)2}) dt \\ & + \sum_{\alpha=1}^2 \int_{D_1} \nabla y^{(j)\alpha} \cdot \nabla \tilde{y}^\alpha(x^{(j)1}, x^{(j)2}, \dots, x^{(j)n}, y^{(j)3}, \dots, y^{(j)n}) dt, \\ & = 0, \end{aligned} \quad (23)$$

where ∇ is the gradient of t in D_1 . Using the chain rule, for example

$$\nabla \tilde{x}^k(x^{(j)1}(t), x^{(j)2}(t)) = \sum_{\alpha=1}^2 \frac{\partial \tilde{x}^k}{\partial x^\alpha}(x^{(j)1}(t), x^{(j)2}(t)) \nabla x^{(j)\alpha},$$

together with (21) and (23), we get

$$\begin{aligned} & \sum_{k=3}^n \int_{D_1} \sum_{\alpha=1}^2 \frac{\partial \tilde{x}^k}{\partial x^\alpha}(x^{(j)1}(t), x^{(j)2}(t)) \nabla x^{(j)\alpha} \cdot \nabla x^{(j)k} dt \\ & + \sum_{k=3}^n \int_{D_1} \sum_{\alpha=1}^2 \frac{\partial \tilde{y}^k}{\partial x^\alpha}(x^{(j)1}(t), x^{(j)2}(t)) \nabla x^{(j)\alpha} \cdot \nabla y^{(j)k} dt \\ & + \sum_{\alpha=1}^2 \sum_{k=3}^n \int_{D_1} \{ \nabla y^{(j)\alpha} \cdot \nabla [x^{(j)k}(t) \frac{\partial \tilde{y}^k}{\partial x^\alpha}(x^{(j)1}(t), x^{(j)2}(t)) \\ & \quad - y^{(j)k}(t) \frac{\partial \tilde{x}^k}{\partial x^\alpha}(x^{(j)1}(t), x^{(j)2}(t))] \} dt \\ & + \sum_{\alpha=1}^2 \int_{D_1} \nabla y^{(j)\alpha} \cdot \nabla \frac{\partial \eta}{\partial x^\alpha}(x^{(j)1}(t), x^{(j)2}(t)) dt \\ & = 0. \end{aligned}$$

Now note that $\nabla x^{(j)\alpha} = \nabla(x^{(j)\alpha} - t^\alpha) + \nabla t^\alpha$. Therefore

$$\begin{aligned}
& \left| \sum_{k=3}^n \sum_{\gamma=1}^2 \left(\int_{D_1} \frac{\partial x^{(j)k}}{\partial t^\gamma} \frac{\partial \tilde{x}^k}{\partial x^\gamma} dt + \int_{D_1} \frac{\partial y^{(j)k}}{\partial t^\gamma} \frac{\partial \tilde{y}^k}{\partial x^\gamma} dt \right) \right. \\
& \quad \left. + \sum_{\alpha,\gamma=1}^2 \int_{D_1} \frac{\partial y^{(j)\alpha}}{\partial t^\gamma} \frac{\partial^2 \eta}{\partial x^\alpha \partial x^\gamma} dt \right| \\
& \leq \sum_{k=3}^n \sum_{\alpha=1}^2 \left(\tilde{c} \int_{D_1} \|\nabla(x^{(j)\alpha} - t^\alpha)\| \cdot \|\nabla x^{(j)k}\| dt \right. \\
& \quad \left. + \tilde{c} \int_{D_1} \|\nabla(x^{(j)\alpha} - t^\alpha)\| \cdot \|\nabla y^{(j)k}\| dt \right. \\
& \quad \left. + \tilde{c} \int_{D_1} \|\nabla y^{(j)\alpha}\| \cdot \|\nabla x^{(j)k}\| dt + \tilde{c} \int_{D_1} |x^{(j)k}| \cdot \|\nabla y^{(j)\alpha}\| dt \right. \\
& \quad \left. + \tilde{c} \int_{D_1} \|\nabla y^{(j)\alpha}\| \cdot \|\nabla y^{(j)k}\| dt + \tilde{c} \int_{D_1} |y^{(j)k}| \cdot \|\nabla y^{(j)\alpha}\| dt \right)
\end{aligned}$$

where \tilde{c} depends on the C^2 norm of \tilde{x}^k , \tilde{y}^k and η . All \tilde{x}^k , \tilde{y}^k , η and their derivatives above are evaluated at $(x^{(j)1}(t^1, t^2), x^{(j)2}(t^1, t^2))$.

Thus Cauchy-Schwartz inequality implies

$$\begin{aligned}
& \left| \sum_{k=3}^n \sum_{\gamma=1}^2 \left(\int_{D_1} \frac{\partial x^{(j)k}}{\partial t^\gamma} \frac{\partial \tilde{x}^k}{\partial x^\gamma} dt + \int_{D_1} \frac{\partial y^{(j)k}}{\partial t^\gamma} \frac{\partial \tilde{y}^k}{\partial x^\gamma} dt \right) \right. \\
& \quad \left. + \sum_{\alpha,\gamma=1}^2 \int_{D_1} \frac{\partial y^{(j)\alpha}}{\partial t^\gamma} \frac{\partial^2 \eta}{\partial x^\alpha \partial x^\gamma} dt \right| \\
& \leq \tilde{c} E(l^{(j)}, l_0, 1) \\
& \quad + \sum_{k=3}^n \sum_{\alpha=1}^2 \left(\tilde{c} \int_{D_1} |x^{(j)k}| \cdot \|\nabla y^{(j)\alpha}\| dt + \tilde{c} \int_{D_1} |y^{(j)k}| \cdot \|\nabla y^{(j)\alpha}\| dt \right).
\end{aligned} \tag{24}$$

By Cauchy-Schwartz,

$$\int_{D_1} |y^{(j)k}| \cdot \|\nabla y^{(j)\alpha}\| dt \leq \|y^{(j)k}\|_{L^2(D_1)} E^{1/2}(l^{(j)}, l_0, 1). \tag{25}$$

Therefore combine (17), (24) and (25) we get

$$\begin{aligned}
& \left| \sum_{k=3}^n \sum_{\gamma=1}^2 \left(\int_{D_1} \frac{\partial w^{(j)k}}{\partial t^\gamma} \frac{\partial \tilde{x}^k}{\partial x^\gamma} dt + \int_{D_1} \frac{\partial q^{(j)k}}{\partial t^\gamma} \frac{\partial \tilde{y}^k}{\partial x^\gamma} dt \right) \right. \\
& \quad \left. + \sum_{\alpha,\gamma=1}^2 \int_{D_1} \frac{\partial v^{(j)\alpha}}{\partial t^\gamma} \frac{\partial^2 \eta}{\partial x^\alpha \partial x^\gamma} dt \right| \\
& \leq \tilde{c} E^{1/2}(l^{(j)}, l_0, 1) + \tilde{c} \sum_{k=3}^n (\|x^{(j)k}\|_{L^2(D_1)} + \|y^{(j)k}\|_{L^2(D_1)}),
\end{aligned} \tag{26}$$

where $\tilde{x}^k, \tilde{y}^k, \eta$ and their derivatives above are evaluated at $(x^{(j)1}(t^1, t^2), x^{(j)2}(t^1, t^2))$.

Since $x^{(j)\alpha}$ converges to t^α strongly in L^2 for $\alpha = 1, 2$, we conclude that $\frac{\partial \tilde{x}^k}{\partial x^\gamma}(x^{(j)1}(t), x^{(j)2}(t)), \frac{\partial \tilde{y}^k}{\partial x^\gamma}(x^{(j)1}(t), x^{(j)2}(t))$ and $\frac{\partial^2 \eta}{\partial x^\gamma \partial x^\alpha}(x^{(j)1}(t), x^{(j)2}(t))$ converge strongly in L^2 to $\frac{\partial \tilde{x}^k}{\partial x^\gamma}(t^1, t^2), \frac{\partial \tilde{y}^k}{\partial x^\gamma}(t^1, t^2)$ and $\frac{\partial^2 \eta}{\partial x^\gamma \partial x^\alpha}(t^1, t^2)$ respectively. Therefore, in view of (16) and (26), if we view $\tilde{x}^k, \tilde{y}^k, \eta$ as functions defined on (t^1, t^2) , we would get

$$\begin{aligned} & \sum_{k=3}^n \sum_{\gamma=1}^2 \left(\int_{D_1} \frac{\partial w^k}{\partial t^\gamma} \frac{\partial \tilde{x}^k}{\partial t^\gamma} dt + \int_{D_1} \frac{\partial q^k}{\partial t^\gamma} \frac{\partial \tilde{y}^k}{\partial t^\gamma} dt \right) \\ & + \sum_{\alpha, \gamma=1}^2 \int_{D_1} \frac{\partial v^\alpha}{\partial t^\gamma} \frac{\partial^2 u}{\partial t^\alpha \partial t^\gamma} dt \\ & = 0. \end{aligned} \quad (27)$$

Since $\tilde{x}^k, \tilde{y}^k, \eta$ are arbitrary, we see that w^k, q^k are harmonic functions for $k = 3, \dots, n$ and

$$\sum_{\alpha, \gamma=1}^2 \int_{D_1} \frac{\partial v^\alpha}{\partial t^\gamma} \frac{\partial^2 u}{\partial t^\alpha \partial t^\gamma} dt = 0 \quad (28)$$

for any $\eta \in C_c^\infty(D_1)$. To show that (v^1, v^2) is the gradient of some function we notice that the map

$$(t^1, t^2) \mapsto (x^{(j)1}, x^{(j)2}, \dots, x^{(j)n}, v^{(j)1}, v^{(j)2}, q^{(j)3}, \dots, q^{(j)n})$$

is still isotropic for any j . Thus by Lemma 2.3, the weak limit as j goes to infinity is still isotropic. But the weak limit is $(t^1, t^2, 0, \dots, 0, v^1, v^2, q^3, \dots, q^n)$. Therefore

$$\frac{\partial v^1}{\partial t^2} - \frac{\partial v^2}{\partial t^1} = 0. \quad (29)$$

Now combine (28) and (29), we get the existence of a biharmonic function $f : D_1 \rightarrow \mathbb{R}$, such that $(v^1, v^2) = \nabla f$. □

From standard elliptic estimates, we get

$$\begin{aligned} \sup_{D_r} |u^\alpha - u_0^\alpha|^2 & \leq cr^4 \int_{D_1} |\nabla u^\alpha|^2 dt, \\ \sup_{D_r} |v^\alpha - v_0^\alpha|^2 & \leq cr^4 \int_{D_1} |\nabla v^\alpha|^2 dt, \\ \sup_{D_r} |w^k - w_0^k|^2 & \leq cr^4 \int_{D_1} |\nabla w^k|^2 dt, \\ \sup_{D_r} |q^k - q_0^k|^2 & \leq cr^4 \int_{D_1} |\nabla q^k|^2 dt, \end{aligned} \quad (30)$$

where $u_0^\alpha, v_0^\alpha, w_0^k, q_0^k$ are the linear part of the Taylor expansion of each function at 0. Using the strong L^2 convergence of $u^{(j)}, v^{(j)}, q^{(j)}, w^{(j)}$, together with (19), we get for large j

$$\begin{aligned} & \int_{D_\theta} \{|u^{(j)} - u_0|^2 + |v^{(j)} - v_0|^2 + |w^{(j)} - w_0|^2 + |q^{(j)} - q_0|^2\} dt \\ & \leq 2 \int_{D_\theta} \{|u - u_0|^2 + |v - v_0|^2 + |w - w_0|^2 + |q - q_0|^2\} dt \\ & \leq c\theta^6. \end{aligned}$$

Now let

$$\tilde{l}_0^{(j)} = l_0 + \sqrt{E_j}(u_0 + \alpha^{(j)}, w_0 + \gamma^{(j)}, v_0 + \beta^{(j)}, q_0 + \mu^{(j)})$$

where $\alpha^{(j)}, \dots, \mu^{(j)}$ are defined in (17). Then

$$\int_{D_\theta} |l^{(j)} - \tilde{l}_0^{(j)}|^2 dt \leq c\theta^6 E(l^{(j)}, l_0, 1) \quad (31)$$

Claim 3. *we can perturb $\tilde{l}_0^{(j)}$ up to order E_j (in the C^1 norm sense) to make it a linear conformal map $l_1^{(j)}$ into an isotropic plane.*

Proof. Notice that since v_0 is the gradient of a quadratic function in t , simple calculation shows that

$$\omega(\tilde{l}_{0*}^{(j)} \frac{\partial}{\partial t^1}, \tilde{l}_{0*}^{(j)} \frac{\partial}{\partial t^2}) = O(E_j).$$

Therefore by a linear perturbation of order $O(E_j)$ only in y^1 and y^2 direction, we can modify $\tilde{l}_0^{(j)}$ to a linear map whose image is an isotropic plane in \mathbb{R}^{2n} . Also since the map $t + \sqrt{E_j}(u_0 + \alpha^j)$ is holomorphic, we see that the Hopf differential of the perturbed linear map is of order $O(E_j)$ (note that we did the previous perturbation without change the x^1 and x^2 coordinates), therefore a perturbation of order $O(E_j)$ will make the map into a linear holomorphic map. \square

Therefore, we have the L^2 estimate

$$\begin{aligned} & \int_{D_\theta} |l^{(j)} - l_1^{(j)}|^2 dt \quad (32) \\ & \leq \int_{D_\theta} |l^{(j)} - \tilde{l}_0^{(j)}|^2 dt + \int_{D_\theta} |l_1^{(j)} - \tilde{l}_0^{(j)}|^2 dt \\ & \leq c\theta^6 E_j + c\theta^2 E_j^2 \end{aligned}$$

and the energy estimate

$$\begin{aligned}
& \int_{D_\theta} |\nabla(l^{(j)} - l_1^{(j)})|^2 dt & (33) \\
& \leq \int_{D_\theta} |\nabla(l^{(j)} - \tilde{l}_0^{(j)})|^2 dt + \int_{D_\theta} |\nabla(l_1^{(j)} - \tilde{l}_0^{(j)})|^2 dt \\
& \leq cE_j + c\theta^2 E_j^2 \\
& \leq cE_j.
\end{aligned}$$

Note that in (32) we can choose $\theta = \bar{\theta}$ small and δ_1 small such that if $E_j \leq \delta_1$, then

$$\bar{\theta}^{-4} \int_{D_{\bar{\theta}}} |l^{(j)} - l_1^{(j)}|^2 dt \leq cE_j \quad (34)$$

Now apply (a scaled version of) Proposition 4.1 (also using (34)). For any $\epsilon > 0$, there exists an $\delta = \delta(\epsilon) > 0$ such that if $E_j \leq \delta$, we get

$$\int_{D_{\bar{\theta}}} |\nabla(l^{(j)} - l_1^{(j)})|^2 dt \leq \epsilon \int_{D_{2\bar{\theta}}} |\nabla(l^{(j)} - l_1^{(j)})|^2 dt + c\bar{\theta}^{-2} \int_{D_{2\bar{\theta}}} |l^{(j)} - l_1^{(j)}|^2 dt. \quad (35)$$

where c is an absolute constant. Combine (32) and (35) we have

$$\begin{aligned}
& \bar{\theta}^{-2} \int_{D_{\bar{\theta}}} |\nabla(l^{(j)} - l_1^{(j)})|^2 dt \\
& \leq c\epsilon(2\bar{\theta})^{-2} \int_{D_{2\bar{\theta}}} |\nabla(l^{(j)} - l_1^{(j)})|^2 dt + c\bar{\theta}^2 E_j + c\bar{\theta}^{-2} E_j^2.
\end{aligned}$$

Then choose $\bar{\theta}$, ϵ and E_j small we get,

$$\bar{\theta}^{-2} \int_{D_{\bar{\theta}}} |\nabla(l^{(j)} - l_1^{(j)})|^2 dt \leq \frac{E_j}{2}. \quad (36)$$

(34) and (36) give a contradiction to the hypothesis that $l^{(j)}$ does not satisfy (15). This completes the proof of the lemma. \square

Remark 4.3. *Using standard iteration argument, the excess decay lemma gives*

$$r^{-2} \int_{D_r(t)} |\nabla(l - l_{t,r})|^2 dt \leq cr^\mu. \quad (37)$$

for any $t \in D_{1/2}, r \geq 0$, where $l_{t,r}$ is some linear holomorphic function into an isotropic plane. This implies that l is $C^{1,\mu}$ in $D_{1/2}$.

5 Higher Regularity

Unlike in the classical minimal surface case, the higher regularity for a $C^{1,\mu}$ isotropically stationary surface is rather a subtle problem.

In this section first we will apply a variational method to get a *compactly supported* $C^{1,\mu}$ solution to the equation $\operatorname{div}X = f$, where f is a $C^{0,\mu}$ function with compact support in $B_{1/2}$ such that $\int_{B_1} f dt = 0$. Then using this result, we construct a suitable class of $C^{1,\mu}$ isotropic variations for the isotropically stationary surface Σ and get the consequent Euler-Lagrangian equation. Finally using a Schauder type estimate, we prove that Σ is smooth.

In this section we will use $B_r(x)$ (simply B_r if $x = 0$) to denote the disk in \mathbb{R}^2 centered at x with radius r .

5.1 Compactly Supported Solutions to $\operatorname{div}X = f$ Using Weighted Sobolev Spaces

In this subsection, we will discuss the equation $\operatorname{div}X = f$ where f is a $C^{0,\mu}$ function with compact support in $B_{1/2}$ such that $\int_{B_1} f dt = 0$. We would like to construct a solution which is both $C^{1,\mu}$ and compactly supported in B_1 . By adding an additional constraint $\frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1} = 0$, we could get a $C^{1,\mu}$ solution from standard elliptic theory, but this solution does not generally have compact support. So we shall prove the existence by minimizing a certain functional in a weighted Sobolev space with an exponentially decay weight.

Define a weight function $\rho : B_1 \rightarrow \mathbb{R}^+$ by

$$\begin{aligned} \rho(x) &\equiv 1 \text{ for } x \in B_{1/2}, \\ \rho(x) &= e^{-1/d}, \text{ for } x \in B_1 \setminus B_{3/4}, \\ \rho &> 0, \text{ in the interior of } B_1. \end{aligned} \tag{38}$$

where $d(x) = \operatorname{dist}(x, \partial B_1) = 1 - |x|$.

Notice that any derivative of ρ decays exponentially near the boundary and therefore we can extend ρ smoothly by defining ρ to be identically zero outside B_1 .

Define the weighted L^2 norm of a function by

$$\|u\|_{L^2_\rho(B_1)} := \left(\int_{B_1} \rho |u|^2 dx \right)^{1/2}.$$

Define the weighted L^2 space $L^2_\rho(B_1) := \{u : \|u\|_{L^2_\rho(B_1)} < \infty\}$. And define the weighted Sobolev space $H^2_\rho(B_1)$ to be the set of all weakly differentiable functions in B_1 such that both u and ∇u are in $L^2_\rho(B_1)$. $H^2_\rho(B_1)$ is a Hilbert space with the inner product

$$\langle u, v \rangle_\rho := \langle \rho^{1/2} \nabla u, \rho^{1/2} \nabla v \rangle_{L^2} + \langle \rho^{1/2} u, \rho^{1/2} v \rangle_{L^2}.$$

For a detailed discussion of weighted Sobolev spaces, see ([C]).

Now we prove a Poincaré inequality for the weighted Sobolev space.

Lemma 5.1. *Let $u \in H_\rho^2(B_1)$, where ρ is given in (38). Then there is an absolute constant c such that*

$$\inf_{\lambda \in \mathbb{R}} \|u - \lambda\|_{L_\rho^2(B_1)} \leq c \|\nabla u\|_{L_\rho^2(B_1)}.$$

Proof. Choose λ to be a constant such that

$$\min\{\mathcal{L}^2\{x \in B_{1/2} : u(x) \leq \lambda\}, \mathcal{L}^2\{x \in B_{1/2} : u(x) \geq \lambda\}\} \geq 1/3 \mathcal{L}^2(B_{1/2}).$$

where \mathcal{L}^2 is the Lebesgue measure in \mathbb{R}^2 .

Standard Poincaré inequality(see [Si1] page 38) implies that for any $\varepsilon > 0$,

$$\int_{B_{1-\varepsilon}} |u(x) - \lambda|^2 dx \leq c \int_{B_{1-\varepsilon}} |\nabla u(x)|^2 dx, \quad (39)$$

where c is an absolute constant(in particular independent of ε). Define $\check{\rho}(d) = \rho(1-d)$. Since ρ is a function of the distance d to the boundary of B_1 and is monotonically decreasing for $d < 1/8$, $\check{\rho}'(d) \geq 0$ for $d < 1/8$. Integrating along ε (using (39)), we get

$$\int_0^{1/8} \check{\rho}'(\varepsilon) \int_{B_{1-\varepsilon}} |u(x) - \lambda|^2 dx d\varepsilon \leq c \int_0^{1/8} \check{\rho}'(\varepsilon) \int_{B_{1-\varepsilon}} |\nabla u(x)|^2 dx d\varepsilon.$$

Therefore

$$\begin{aligned} & \int_0^{1/8} \check{\rho}'(\varepsilon) \int_{\varepsilon < d < 1/8} |u(x) - \lambda|^2 dx d\varepsilon \\ & \leq c \int_0^{1/8} \check{\rho}'(\varepsilon) \int_{\varepsilon < d < 1/8} |\nabla u(x)|^2 dx d\varepsilon + c \int_{B_{7/8}} |\nabla u(x)|^2 dx. \end{aligned} \quad (40)$$

Now co-area formula gives

$$\begin{aligned} & \int_0^{1/8} \check{\rho}'(\varepsilon) \int_{\varepsilon < d < 1/8} |u(x) - \lambda|^2 dx d\varepsilon \\ & = \int_0^{1/8} \check{\rho}'(\varepsilon) \left(\int_\varepsilon^{1/8} \int_{\partial B_{1-s}} |\nabla d|^{-1} |u(x) - \lambda|^2 d\sigma ds \right) d\varepsilon. \end{aligned}$$

Integration by parts and noticing $|\nabla d| = 1$, we have(since the boundary

terms vanish)

$$\begin{aligned}
& \int_0^{1/8} \tilde{\rho}'(\varepsilon) \int_{\varepsilon < d < 1/8} |u(x) - \lambda|^2 dx d\varepsilon \\
&= \int_0^{1/8} \int_{\partial B_{1-\varepsilon}} \rho |\nabla d|^{-1} |u(x) - \lambda|^2 d\sigma d\varepsilon. \\
&= \int_{d < 1/8} \rho |u(x) - \lambda|^2 dx \\
&= \int_{B_1} \rho |u(x) - \lambda|^2 dx - \int_{B_{7/8}} \rho |u(x) - \lambda|^2 dx \\
&\geq \int_{B_1} \rho |u(x) - \lambda|^2 dx - c \int_{B_{7/8}} |u(x) - \lambda|^2 dx \\
&\geq \int_{B_1} \rho |u(x) - \lambda|^2 dx - c \int_{B_{7/8}} \rho |\nabla u(x)|^2 dx
\end{aligned} \tag{41}$$

Similarly

$$\int_0^{1/8} \tilde{\rho}'(\varepsilon) \int_{\varepsilon < d < 1/8} |\nabla u(x)|^2 dx d\varepsilon \leq \int_{B_1} \rho |\nabla u(x)|^2 dx \tag{42}$$

Finally combine (40), (41) and (42), we get

$$\int_{B_1} \rho |u(x) - \lambda|^2 dx \leq c \int_{B_1} \rho |\nabla u(x)|^2 dx,$$

which completes the proof. \square

We now use a variational approach to construct a compactly supported $C^{1,\mu}$ solution with compact support to the equation $\operatorname{div} X = f$.

For any $f \in C^{0,\mu}(B_1)$ with compact support in $B_{1/2}$ such that $\int_{B_1} f dt = 0$, define a functional $\mathcal{F} : H_\rho^2(B_1) \rightarrow \mathbb{R}$ by

$$\mathcal{F}(u) := \int_{B_1} (2^{-1} \rho |\nabla u|^2 - f \rho u). \tag{43}$$

The Euler-Lagrangian equation of the critical point of \mathcal{F} is

$$\int_{B_1} \rho \langle \nabla u, \nabla \varphi \rangle = \int_{B_1} f \rho \varphi, \tag{44}$$

for any $\varphi \in C_c^\infty(B_1)$.

Standard Schauder theory gives that if u is a weak solution of (44), then u is $C^{2,\mu}$ interiorly (since f is $C^{0,\mu}$). And in fact we have the stronger conclusion:

Lemma 5.2. *Assume $f \in C^{0,\mu}(B_1)$ has compact support in $B_{1/2}$. If $u \in H_\rho^2(B_1)$ is a weak solution of (44) (ρ as defined in (38)), then u is $C^{2,\mu}$ in*

the interior of B_1 . Moreover if $X = \rho \nabla u$, then by extending X to be identically zero outside B_1 , X becomes $C^{1,\mu}$ in B_2 and satisfies

$$\begin{aligned} \operatorname{div} X &= f; \\ \operatorname{support} X &\subset B_1; \\ \|X\|_{C^{1,\mu}(B_2)} &\leq c(\|u\|_{L^2_\rho(B_1)} + \|f\|_{C^{0,\mu}(B_2)}). \end{aligned} \tag{45}$$

Proof. Since ρ is always positive in the interior of B_1 , standard Schauder theory implies that u is $C^{2,\mu}$ in the interior of B_1 . So u is a classical solution of the equation

$$\operatorname{div}(\rho \nabla u) = f$$

in the interior of B_1 . Let $v = \rho^{1/2}u$, then $v \in L^2(B_1)$ and v satisfies

$$\Delta v - \frac{1}{2}(\rho^{-1} \Delta \rho - 2^{-1} \rho^{-2} |\nabla \rho|^2) v = f.$$

Since $\rho(x) = e^{-1/d}$ near the boundary, the $C^{n,\mu}$ norm of $(\rho^{-1} \Delta \rho - 2^{-1} \rho^{-2} |\nabla \rho|^2)$ decays like $d^{-N(n)}$ near the boundary. So standard Schauder theory implies that

$$\|v\|_{C^{2,\mu}(B_{1-\epsilon})} \leq c\epsilon^{-N}(\|v\|_{L^2(B_1)} + \|f\|_{C^{0,\mu}(B_1)}).$$

where N is a fixed positive integer.

Let $X = \rho \nabla u$, then

$$X = \rho^{1/2} \nabla v - 2^{-1} v \rho^{-1/2} \nabla \rho.$$

So we see

$$\|X\|_{C^{1,\mu}(B_{1-\epsilon} \setminus B_{1-2\epsilon})} \leq c e^{-\frac{1}{2\epsilon}} \epsilon^{-N_1} (\|v\|_{L^2(B_1)} + \|f\|_{C^{0,\mu}(B_1)}).$$

Therefore near boundary the $C^{1,\mu}$ norm of X decays like $e^{-1/4d}$. Thus if we extend X trivially outside B_1 , the new X satisfies (45). \square

Now we want to prove the existence of the weak solution to (44). We shall do it by proving the existence of a minimizer of the functional \mathcal{F} in $H^2_\rho(B_1)$.

Lemma 5.3. *Assume $f \in C^{0,\mu}(B_1)$ has compact support in $B_{1/2}$ and $\int_{B_1} f dt = 0$. Then there is a minimizer u of the functional \mathcal{F} (defined in (43)) in $H^2_\rho(B_1)$. Moreover $\|u\|_{L^2_\rho(B_1)} \leq c\|f\|_{L^2(B_1)}$.*

Proof. Let $M_f := \inf\{\mathcal{F}(u) | u \in H^2_\rho(B_1)\}$.

Claim 4. $0 \geq M_f \geq c\|f\|_{L^2(B_1)}^2$.

Proof. By choosing u to be identically zero, we see $M_f \leq 0$. To prove the lower bound, first notice that since $\int_{B_1} f = 0$ and $f\rho = f$, we get $\mathcal{F}(u) = \mathcal{F}(u - u_0)$, where u_0 is any constant. So (by replacing u by $u - u_0$ for a suitable constant

u_0) the Poincaré inequality(Lemma 5.1) then gives $\|u\|_{L^2_\rho(B_1)} \leq c\|\nabla u\|_{L^2_\rho(B_1)}$. Therefore

$$\begin{aligned}\mathcal{F}(u) &\geq c\|\nabla u\|_{L^2_\rho(B_1)}^2 - c\|f\|_{L^2(B_1)} \cdot \|u\|_{L^2_\rho(B_1)} \\ &\geq c\|\nabla u\|_{L^2_\rho(B_1)}^2 - c\|f\|_{L^2(B_1)} \cdot \|\nabla u\|_{L^2_\rho(B_1)}.\end{aligned}$$

The last line in the above inequality is a quadratic function of $\|\nabla u\|_{L^2_\rho(B_1)}$, the minimum of which is $c\|f\|_{L^2(B_1)}^2$. \square

Now assume $\lim_{k \rightarrow \infty} \mathcal{F}(u_k) = M_f$ for a sequence $\{u_k\} \subset H^2_\rho(B_1)$. Claim 4 implies that (by replacing u_k by $u_k - u_{k_0}$ for a suitable constant u_{k_0}) $\|\nabla u_k\|_{L^2_\rho(B_1)} \leq c\|f\|_{L^2(B_1)}$. Then the Poincaré inequality(Lemma 5.1) tell us $\|u_k\|_{H^2_\rho(B_1)} \leq c\|f\|_{L^2(B_1)}$. Hence we may assume that u_k converge to some u weakly in $H^2_\rho(B_1)$ and also weakly in $L^2_\rho(B_1)$. Direct method then gives $\mathcal{F}(u) = M_f$ and $\|\nabla u\|_{L^2_\rho(B_1)} \leq c\|f\|_{L^2(B_1)}$. Therefore u is a minimizer of \mathcal{F} with $\|u\|_{L^2_\rho(B_1)} \leq c\|f\|_{L^2(B_1)}$. \square

The minimizer u is clearly a weak solution of (44). Notice $\|u\|_{L^2_\rho(B_1)} \leq c\|f\|_{L^2(B_1)} \leq c\|f\|_{C^{0,\mu}(B_1)}$. Apply Lemma 5.2, we get the following result:

Proposition 5.4. *Let f be a $C^{0,\mu}$ function in $B_2 \subset \mathbb{R}^2$ such that support $f \subset B_{1/2}$ and $\int_{B_2} f = 0$. Then there exists a $C^{1,\mu}$ vector field X in B_2 such that*

$$\begin{aligned}\operatorname{div} X &= f; \\ \operatorname{support} X &\subset B_1; \\ \|X\|_{C^{1,\mu}(B_2)} &\leq c\|f\|_{C^{0,\mu}(B_2)}.\end{aligned}\tag{46}$$

5.2 Constructing Isotropic Variations

In this subsection we will use proposition 5.4 to construct a class of isotropic variations for an isotropic surface Σ . Notice a graphical surface Σ^* over $x^1 x^2$ -plane given by $y^{*\alpha}$, x^{*k} and y^{*k} ($k = 3, \dots, n$ and $\alpha = 1, 2$) is isotropic if and only if

$$\frac{\partial y^{*1}}{\partial x^2} - \frac{\partial y^{*2}}{\partial x^1} = \sum_{k=3}^n \frac{\partial x^{*k}}{\partial x^1} \frac{\partial y^{*k}}{\partial x^2} - \frac{\partial y^{*k}}{\partial x^1} \frac{\partial x^{*k}}{\partial x^2}\tag{47}$$

Lemma 5.5. *Assume Σ is a $C^{1,\mu}$ isotropic surface which is graphical over B_2 in $x^1 x^2$ -plane. Let $\tilde{x}^3(x^1, x^2), \dots, \tilde{x}^n(x^1, x^2), \tilde{y}^3(x^1, x^2), \dots,$ and $\tilde{y}^n(x^1, x^2)$ be smooth functions with compact support in B_2 and let \tilde{y}^1 and \tilde{y}^2 be $C^{1,\mu}$ functions with compact support in B_2 . Assume*

$$\begin{aligned}\frac{\partial \tilde{y}^1}{\partial x^2} - \frac{\partial \tilde{y}^2}{\partial x^1} &= \sum_{k=3}^n \left(\frac{\partial \tilde{x}^k}{\partial x^1} \frac{\partial y^k}{\partial x^2} - \frac{\partial y^k}{\partial x^1} \frac{\partial \tilde{x}^k}{\partial x^2} \right. \\ &\quad \left. + \frac{\partial x^k}{\partial x^1} \frac{\partial \tilde{y}^k}{\partial x^2} - \frac{\partial \tilde{y}^k}{\partial x^1} \frac{\partial x^k}{\partial x^2} \right).\end{aligned}\tag{48}$$

Then the compactly supported vector field X on Σ defined by

$$X := \sum_{k=3}^n (\tilde{x}^k \frac{\partial}{\partial x^k} + \tilde{y}^k \frac{\partial}{\partial y^k}) + \sum_{\alpha=1}^2 \tilde{y}^\alpha \frac{\partial}{\partial y^\alpha}$$

is isotropically integrable, i.e. there exists a family of isotropic graphical surfaces $\{\Sigma_t\}$ given by $y^\alpha(t, x^1, x^2)$, $x^k(t, x^1, x^2)$ and $y^k(t, x^1, x^2)$ ($k = 3, \dots, n$ and $\alpha = 1, 2$) such that $\Sigma_0 = \Sigma$ and

$$\frac{\partial y^\alpha(t)}{\partial t} \Big|_{t=0} = \tilde{y}^\alpha, \quad \frac{\partial x^k(t)}{\partial t} \Big|_{t=0} = \tilde{x}^k, \quad \frac{\partial y^k(t)}{\partial t} \Big|_{t=0} = \tilde{y}^k, \quad (49)$$

for $\alpha = 1, 2$ and $k = 3, \dots, n$.

Proof. Let $y^\alpha(0) = y^\alpha$, $x^k(0) = x^k$ and $y^k(0) = y^k$. Define

$$x^k(t) = x^k(0) + t\tilde{x}^k, \quad y^k(t) = y^k(0) + t\tilde{y}^k$$

Then the isotropic constraint (47) for $y^\alpha(t)$, $x^k(t)$ and $y^k(t)$ becomes (using the fact that $\Sigma_0 = \Sigma$ is isotropic)

$$\begin{aligned} & \frac{\partial(y^1(t) - y^1(0))}{\partial x^2} - \frac{\partial(y^2(t) - y^2(0))}{\partial x^1} \\ &= t \sum_{k=3}^n \left(\frac{\partial \tilde{x}^k}{\partial x^1} \frac{\partial y^k}{\partial x^2} - \frac{\partial y^k}{\partial x^1} \frac{\partial \tilde{x}^k}{\partial x^2} + \frac{\partial x^k}{\partial x^1} \frac{\partial \tilde{y}^k}{\partial x^2} - \frac{\partial \tilde{y}^k}{\partial x^1} \frac{\partial x^k}{\partial x^2} \right) \\ & \quad + t^2 \sum_{k=3}^n \left(-\frac{\partial \tilde{x}^k}{\partial x^1} \frac{\partial \tilde{y}^k}{\partial x^2} + \frac{\partial \tilde{y}^k}{\partial x^1} \frac{\partial \tilde{x}^k}{\partial x^2} \right). \end{aligned} \quad (50)$$

By Proposition 5.4, there exist $C^{1,\mu}$ functions λ^1 and λ^2 with compact support in B_2 such that

$$\frac{\partial \lambda^1}{\partial x^2} - \frac{\partial \lambda^2}{\partial x^1} = \sum_{k=3}^n \left(-\frac{\partial \tilde{x}^k}{\partial x^1} \frac{\partial \tilde{y}^k}{\partial x^2} + \frac{\partial \tilde{y}^k}{\partial x^1} \frac{\partial \tilde{x}^k}{\partial x^2} \right).$$

Now define

$$y^\alpha(t, x^1, x^2) := t^2 \lambda^\alpha(x^1, x^2) + t \tilde{y}^\alpha(x^1, x^2) + y^\alpha(0, x^1, x^2).$$

It is then easy to see that $x^k(t)$, $y^k(t)$ and $y^\alpha(t)$ satisfies the isotropic constraint equation (50). So the surface Σ_t is isotropic for any t . Moreover by differentiating (50) with respect to t , we see that \tilde{x}^k , \tilde{y}^k and \tilde{y}^α satisfy (49) and (48). Therefore X is isotropically integrable. This completes the proof. \square

Lemma 5.5 immediately implies the following Euler-Lagrangian equation for isotropically stationary surfaces:

Lemma 5.6. *Assume Σ is a $C^{1,\mu}$ isotropic surface which is graphical over B_2 in x^1x^2 -plane. If Σ is isotropically stationary, then*

$$\begin{aligned} & \sum_{k=3}^n \int_{B_2} \nabla^\Sigma x^k \cdot \nabla^\Sigma \tilde{x}^k d\mu_\Sigma + \sum_{k=3}^n \int_{B_2} \nabla^\Sigma y^k \cdot \nabla^\Sigma \tilde{y}^k d\mu_\Sigma \\ & + \sum_{\alpha=1}^2 \int_{B_2} \nabla^\Sigma y^\alpha \cdot \nabla^\Sigma \tilde{y}^\alpha d\mu_\Sigma \\ & = 0, \end{aligned} \quad (51)$$

for any smooth functions $\tilde{x}^3(x^1, x^2), \dots, \tilde{x}^n(x^1, x^2), \tilde{y}^3(x^1, x^2), \dots, \tilde{y}^n(x^1, x^2)$ with compact support in B_2 and any $C^{1,\mu}$ functions $\tilde{y}^1(x^1, x^2)$ and $\tilde{y}^2(x^1, x^2)$ with compact support in B_2 such that (48) holds. Here ∇^Σ is the Riemannian connection of the induced metric on Σ .

Proof. By Lemma 5.5, the vector field on Σ

$$X := \sum_{k=3}^n (\tilde{x}^k \frac{\partial}{\partial x^k} + \tilde{y}^k \frac{\partial}{\partial y^k}) + \sum_{\alpha=1}^2 \tilde{y}^\alpha \frac{\partial}{\partial y^\alpha}$$

is isotropically integrable. Therefore apply the standard first variation formula to X , we get (51). \square

5.3 $C^{1,\mu}$ to C^∞ and the Partial Regularity Theorem

Now we are in the position to prove that any $C^{1,\mu}$ isotropically stationary surface is smooth.

We will use the standard notation $[\cdot]_{\mu,\Omega}$ to denote the semi-norm and $\|\cdot\|_{k,\mu,\Omega}$ (sometimes we also use $\|\cdot\|_{C^{k,\mu}(\Omega)}$) to denote the $C^{k,\mu}$ norm (see [G-T]).

Proposition 5.7. *Assume Σ is a $C^{1,\mu}$ isotropic surface which is graphical over B_2 in x^1x^2 -plane. If Σ is isotropically stationary, then Σ is C^∞ in B_1 .*

Proof. We may assume (by scaling) that

$$\|x^k\|_{C^{1,\mu}(B_2)} \leq \varepsilon, \|y^k\|_{C^{1,\mu}(B_2)} \leq \varepsilon, \|y^\alpha\|_{C^{1,\mu}(B_2)} \leq \varepsilon, \quad (52)$$

for $k = 3, \dots, n$ and $\alpha = 1, 2$.

In lemma 5.6, if we let $\tilde{x}^4(x^1, x^2), \dots, \tilde{x}^n(x^1, x^2), \tilde{y}^3(x^1, x^2), \dots, \tilde{y}^n(x^1, x^2)$ be identically zero, then the Euler-Lagrangian equation for x^3 becomes

$$\int_{B_2} \nabla^\Sigma x^3 \cdot \nabla^\Sigma \tilde{x}^3 d\mu + \sum_{\alpha=1}^2 \int_{B_2} \nabla^\Sigma y^\alpha \cdot \nabla^\Sigma \tilde{y}^\alpha d\mu = 0, \quad (53)$$

for any smooth function $\tilde{x}^3(x^1, x^2)$ and $C^{1,\mu}$ functions $\tilde{y}^1(x^1, x^2)$ and $\tilde{y}^2(x^1, x^2)$ with compact support in B_2 such that

$$\frac{\partial \tilde{y}^1}{\partial x^2} - \frac{\partial \tilde{y}^2}{\partial x^1} = \frac{\partial \tilde{x}^3}{\partial x^1} \frac{\partial y^3}{\partial x^2} - \frac{\partial y^3}{\partial x^1} \frac{\partial \tilde{x}^3}{\partial x^2} \quad (54)$$

For $f(x^1, x^2) \in C^{0,\mu}(B_2)$ with compact support in B_2 such that $\int_{B_2} f dt = 0$, define $\mathcal{G}(f)$ to be the set of all $\Gamma = (\Gamma^1, \Gamma^2)$ such that Γ is $C^{1,\mu}$ vector field on B_2 with compact support in B_2 and $\frac{\partial \Gamma^1}{\partial x^2} - \frac{\partial \Gamma^2}{\partial x^1} = f$. Proposition 5.4 implies that $\mathcal{G}(f)$ is not empty.

Clearly for any $\Gamma = (\Gamma^1, \Gamma^2) \in \mathcal{G}(\frac{\partial \tilde{x}^3}{\partial x^1} \frac{\partial y^3}{\partial x^2} - \frac{\partial y^3}{\partial x^1} \frac{\partial \tilde{x}^3}{\partial x^2})$ and any $\Pi = (\Pi^1, \Pi^2) \in \mathcal{G}(\frac{\partial \tilde{x}^3}{\partial x^1} \frac{\partial \delta^{(h)} y^3}{\partial x^2} - \frac{\partial \delta^{(h)} y^3}{\partial x^1} \frac{\partial \tilde{x}^3}{\partial x^2})$, the vector $\delta^{(h)}\Gamma - \Pi$ is a compactly supported solution to (54) with \tilde{x}^3 replaced by $\delta^{(h)}\tilde{x}^3$, where $\delta^{(h)}$ is the standard difference quotient operator. Therefore (53) tells us (by replacing \tilde{x}^3 by $\delta^{(h)}\tilde{x}^3$)

$$\begin{aligned} & \int_{\Omega} \nabla^{\Sigma} x^3 \cdot \nabla^{\Sigma} \delta^{(h)} \tilde{x}^3 d\mu_{\Sigma} \\ & + \sum_{\alpha=1}^2 \int_{\Omega} \nabla^{\Sigma} y^{\alpha} \cdot \nabla^{\Sigma} \{ \delta^{(h)} \Gamma^{\alpha} - \Pi^{\alpha} \} d\mu_{\Sigma} \\ & = 0, \end{aligned} \quad (55)$$

for any $\Gamma \in \mathcal{G}(\frac{\partial \tilde{x}^3}{\partial x^1} \frac{\partial y^3}{\partial x^2} - \frac{\partial y^3}{\partial x^1} \frac{\partial \tilde{x}^3}{\partial x^2})$ and $\Pi \in \mathcal{G}(\frac{\partial \tilde{x}^3}{\partial x^1} \frac{\partial \delta^{(h)} y^3}{\partial x^2} - \frac{\partial \delta^{(h)} y^3}{\partial x^1} \frac{\partial \tilde{x}^3}{\partial x^2})$.

To get $C^{1,\mu}$ estimates of $\delta^{(h)}x^3$, we shall need the following Schauder type estimate.

Lemma 5.8. *Let $a_{ij}(x)$, $b_i(x)$, $c_i(x)$, $d(x)$, $f_i(x)$ and $g(x)$ be $C^{0,\mu}$ functions on B_2 . Assume $q_1(x), \dots, q_m(x)$ are $C^{1,\mu}$ functions on B_2 and $p_{ikj}(x)$ are $C^{0,\mu}$ on B_2 , for $i, k = 1, 2$ and $j = 1, \dots, m$.*

Assume $u(x) \in C^{1,\mu}(B_2)$ satisfies

$$\begin{aligned} & \sum_{i,j=1}^2 \int_{B_2} a_{ij} D^i u D^j \varphi + \sum_{i=1}^2 \int_{B_2} b_i \varphi D^i u + \sum_{i=1}^2 \int_{B_2} c_i u D^i \varphi \\ & + \int_{B_2} d u \varphi \\ & = \sum_{i=1}^2 \int_{B_2} f_i D^i \varphi d\mu + \int_{B_2} g \varphi d\mu + \sum_{j=1}^m \sum_{i,k=1}^2 \int_{B_2} p_{ikj} D^k \Gamma_j^i d\mu. \end{aligned} \quad (56)$$

for any $\varphi \in C_c^{\infty}(B_2)$, and for any

$$\Gamma_j = (\Gamma_j^1, \Gamma_j^2) \in \mathcal{G}(D^1 q_j D^2 \varphi - D^2 q_j D^1 \varphi), \quad j = 1, \dots, m.$$

We assume the ellipticity condition:

$$\sum_{i,j=1}^2 a_{ij}(x) \xi^i \xi^j \geq \Lambda |\xi|^2, \quad (57)$$

and

$$\begin{aligned} \|a_{ij}\|_{C^{0,\mu}(B_2)} &\leq \Lambda_1, & \|b_i\|_{C^{0,\mu}(B_2)} &\leq \Lambda_1, \\ \|c_i\|_{C^{0,\mu}(B_2)} &\leq \Lambda_1, & \|d\|_{C^{0,\mu}(B_2)} &\leq \Lambda_1. \end{aligned}$$

Then

$$\begin{aligned} \|u\|_{C^{1,\mu}(B_1)} &\leq c(\|u\|_{C^0(B_2)} + \|g\|_{C^{0,\mu}(B_2)} + \sum_{i=1}^2 \|f_i\|_{C^{0,\mu}(B_2)}) \\ &\quad + \sum_{j=1}^k \sum_{i,k=1}^2 \|p_{ikj}\|_{C^{0,\mu}(B_2)} \|Dq_j\|_{C^{0,\mu}(B_2)}, \end{aligned} \quad (58)$$

where $c = c(\Lambda, \Lambda_1)$.

The proof of this lemma is a slight modification of the proof of the standard Schauder theory by scaling in [Si3] and will be left in Appendix B.

Now apply lemma 5.8 to equation (55) (also using the small $C^{1,\mu}$ norm condition (52)), we get

$$\begin{aligned} &\|\delta_i^{(h)} x^3\|_{C^{1,\mu}(B_1)} \\ &\leq c\left\{\sum_{k=3}^n (\|\delta_i^{(h)} x^k\|_{C^0(B_2)} + \|\delta_i^{(h)} y^k\|_{C^0(B_2)}) + \sum_{\alpha=1}^2 \|\delta_i^{(h)} y^\alpha\|_{C^0(B_2)}\right\} \\ &\quad + c\epsilon\left\{\sum_{k=3}^n (\|\delta_i^{(h)} x^k\|_{C^{1,\mu}(B_2)} + \|\delta_i^{(h)} y^k\|_{C^{1,\mu}(B_2)})\right. \\ &\quad \left.+ \sum_{\alpha=1}^2 \|\delta_i^{(h)} y^\alpha\|_{C^{1,\mu}(B_2)}\right\}. \end{aligned} \quad (59)$$

Similarly, the same result is true for $\|\delta_i^{(h)} x^k\|_{C^{1,\mu}(B_1)}$ ($k=4, \dots, n$) and $\|\delta_i^{(h)} y^k\|_{C^{1,\mu}(B_1)}$ ($k=3, \dots, n$).

To get estimates for $\|\delta_i^{(h)} y^\alpha\|_{C^{1,\mu}(B_1)}$, we use a different variation. In the Euler-Lagrangian equation, choose \tilde{x}^k and \tilde{y}^k to be identically zero for $k = 3, \dots, n$. And let

$$\tilde{y}^\alpha = \frac{\partial u}{\partial x^\alpha}, \quad \alpha = 1, 2,$$

where u is any $C^{2,\mu}$ function with compact support in B_1 . Then the Euler Lagrangian equation (51) becomes

$$\sum_{\alpha=1}^2 \int_{\Sigma} \nabla^{\Sigma} y^\alpha \cdot \nabla^{\Sigma} \frac{\partial u}{\partial x^\alpha} d\mu_{\Sigma} = 0. \quad (60)$$

Also we have the isotropic constraint equation:

$$\frac{\partial y^1}{\partial x^2} - \frac{\partial y^2}{\partial x^1} = \sum_{k=3}^n \frac{\partial x^k}{\partial x^1} \frac{\partial y^k}{\partial x^2} - \frac{\partial y^k}{\partial x^1} \frac{\partial x^k}{\partial x^2}. \quad (61)$$

(60) and (61) form a mixed order elliptic system for y^1 and y^2 . Difference quotient and standard Schauder theory for elliptic system (see [D-N]) imply

$$\begin{aligned}
& \|\delta_i^{(h)} y^\alpha\|_{C^{1,\mu}(B_1)} \tag{62} \\
& \leq c \left\{ \sum_{k=3}^n (\|\delta_i^{(h)} x^k\|_{C^0(B_2)} + \|\delta_i^{(h)} y^k\|_{C^0(B_2)}) + \sum_{\alpha=1}^2 \|\delta_i^{(h)} y^\alpha\|_{C^0(B_2)} \right\} \\
& \quad + c\epsilon \left\{ \sum_{k=3}^n (\|\delta_i^{(h)} x^k\|_{C^{1,\mu}(B_2)} + \|\delta_i^{(h)} y^k\|_{C^{1,\mu}(B_2)}) \right. \\
& \quad \left. + \sum_{\alpha=1}^2 \|\delta_i^{(h)} y^\alpha\|_{C^{1,\mu}(B_2)} \right\}
\end{aligned}$$

To proceed we need an abstract lemma which is very useful in dealing with interior estimates (for a proof see [Si2]).

Lemma 5.9. *Let S be a real valued sub-additive function on the class of all balls in $B_R(x_0) \subset \mathbb{R}^n$ (i.e., $S(A) \leq \sum_{i=1}^k S(A_i)$ whenever $A \subset \cup_{i=1}^k A_i$). Suppose $\theta > 0$ and $l \geq 0$ are given. Then there is an $\epsilon_0 = \epsilon_0(\theta, l, n) > 0$ such that if*

$$S(B_{\theta\rho}(x)) \leq \epsilon_0 S(B_\rho(x)) + \gamma\rho^{-l}$$

for all $B_\rho(x) \subset B_R(x_0)$ and for some constant γ (independent of ρ and x), then

$$S(B_{\theta R}(x_0)) \leq C(\theta, l, n)\gamma.$$

A scaled version of (59) and (62) gives

$$\begin{aligned}
& \sum_{k=3}^n ([\delta_i^{(h)} x^k]_{\mu, B_{\rho/2}(x)} + [\delta_i^{(h)} y^k]_{\mu, B_{\rho/2}(x)}) + \sum_{\alpha=1}^2 [\delta_i^{(h)} y^\alpha]_{\mu, B_{\rho/2}(x)} \\
& \leq \epsilon \left\{ \sum_{k=3}^n ([\delta_i^{(h)} x^k]_{\mu, B_\rho(x)} + [\delta_i^{(h)} y^k]_{\mu, B_\rho(x)}) + \sum_{\alpha=1}^2 [\delta_i^{(h)} y^\alpha]_{\mu, B_\rho(x)} \right\} \\
& \quad + C\rho^{-2} \left\{ \sum_{k=3}^n (\|\delta_i^{(h)} x^k\|_{C^0(B_2)} + \|\delta_i^{(h)} y^k\|_{C^0(B_2)}) \right. \\
& \quad \left. + \sum_{\alpha=1}^2 \|\delta_i^{(h)} y^\alpha\|_{C^0(B_2)} \right\}
\end{aligned}$$

for any $B_\rho(x) \subset B_2$.

Therefore applying abstract lemma 5.9 with

$$S(B_\rho(x)) = \sum_{k=3}^n ([\delta_i^{(h)} x^k]_{\mu, B_\rho(x)} + [\delta_i^{(h)} y^k]_{\mu, B_\rho(x)}) + \sum_{\alpha=1}^2 [\delta_i^{(h)} y^\alpha]_{\mu, B_\rho(x)},$$

we get (choose ϵ small)

$$\begin{aligned}
& \sum_{k=3}^n (\|\delta_i^{(h)} x^k\|_{C^{1,\mu}(B_1)} + \|\delta_i^{(h)} y^k\|_{C^{1,\mu}(B_1)}) + \sum_{\alpha=1}^2 \|\delta_i^{(h)} y^\alpha\|_{C^{1,\mu}(B_1)} \\
\leq & c \left\{ \sum_{k=3}^n \|\delta_i^{(h)} x^k\|_{C^0(B_2)} + \|\delta_i^{(h)} y^k\|_{C^0(B_2)} \right. \\
& \left. + \sum_{\alpha=1}^2 \|\delta_i^{(h)} y^\alpha\|_{C^0(B_2)} \right\}.
\end{aligned}$$

Let h go to 0 we see that x^k , y^k and y^α are $C^{2,\mu}$. Iterate, we get that they are C^∞ . \square

Finally combine Proposition 5.7 and Lemma 4.2 (in particular remark 4.3), we have

Theorem 5.10 (Partial Regularity Theorem). *Let $l : D_2 \rightarrow \mathbb{R}^{2n}$ be a weakly conformal, weakly isotropic map which minimizes area among all the isotropic maps with the same boundary value. Let l_0 be a linear holomorphic map into an isotropic plane. Assume $l(0) = l_0(0) = 0$, and $\text{Area}(l(D_1)) \leq c_1$. Then there is a $\varepsilon_0 \geq 0$ depending only on c_1 such that if*

$$\int_{D_1} |\nabla(l - l_0)|^2 dt + \int_{D_1} |l - l_0|^2 dt \leq \varepsilon_0,$$

then l is a smooth immersion in $D_{1/2}$.

6 Tangent Cones

Standard technique of proving interior regularity from partial regularity involves the study of tangent cones and the use of dimension reducing argument. In the first part of this section, we construct tangent cones at each point of an isotropically area minimizing map. In the second part, we classify all the isotropically stationary cones with isolated singularities and study their stability.

Proposition 6.1. *Let l be a weakly conformal, exact, isotropically stationary map from \mathbb{R}^2 to \mathbb{R}^{2n} . Suppose $l(0) = 0$ and l is proper (in the sense that the pre-image of any compact set is compact). Also suppose that $a^{-2} \int_{\mathbb{R}^2} F^{(a)} \circ \tilde{l} dt$ is constant for any $a \geq 0$, where \tilde{l} is a contact lifting of l such that $\tilde{l}(0) = 0$ and $F^{(a)}$ is defined in Section 3. Then l is a cone in the sense that the position vector is in the tangent plane of l . Moreover the contact lifting \tilde{l} in \mathbb{R}^{2n+1} lies in $\{\varphi = 0\}$.*

Proof. First we prove that $\varphi = 0$. Since $a^{-2} \int_{\mathbb{R}^2} F^{(a)} \circ \tilde{l} dt$ is constant for $a \geq 0$, by the monotonicity formula (12), we see that $\theta = \text{constant}$ and $\pi(\vec{P}) = 0$, where \vec{P} is the position vector of l and π is the orthogonal projection onto

Q (see (8)). Therefore $\varphi = \lambda s$, where λ is a constant. We are going to show that $\lambda = 0$. Note the modified distance function(see section 3) $r^2 = 2(1 + \lambda^2)s$. Now fixed a ball $\tilde{B}_{a_0}(0)$ (defined in (6)). Let $\xi : \mathbb{R} \rightarrow \mathbb{R}$ be a concave cutoff function such that:

$$\xi(\theta) = 1, \quad \text{if } \theta \geq a_0^2 2^{-2}(1 + \lambda^2)^{-1},$$

and $\xi(0) = 0$, $\xi'(\theta) > 0$ for $\theta < a_0^2 2^{-2}(1 + \lambda^2)^{-1}$. Then we see that since $r^2 = 2^{-1}(1 + \lambda^2)s$, the Hamiltonian function $\xi \circ s$ is identically 1 outside $\tilde{B}_{a_0}(0)$. Since l is proper, $l^{-1}(B_{a_0}(0))$ is compact in \mathbb{R}^2 . Therefore the function $\xi \circ s \circ l$ is identically 1 outside a compact set in \mathbb{R}^2 . Let $X_{\xi \circ s}$ be the contact vector field defined in Lemma 3.1. Then first variation gives $\int_{\Sigma} \text{div}_{\Sigma} X_{\xi \circ s} d\mu_{\Sigma} = 0$, where $\Sigma = l(\mathbb{R}^2)$. Note that

$$\text{div}_{\Sigma} X_{\xi \circ s} = \xi' \text{div}_{\Sigma} X_s + \nabla(\xi') \cdot X_s = \xi'' \nabla s \cdot X_s,$$

where we use the chain rule of X_h operator and the fact that $\text{div}_{\Sigma} X_s = 0$. Using the fact that \tilde{l} is contact and the fact that $\varphi = \lambda s$, we get that

$$\nabla \varphi \cdot X_s = \lambda \nabla s \cdot X_s = |\nabla \varphi|^2.$$

If $\lambda \neq 0$, then we see that $\int_{\Sigma} \xi'' |\nabla \varphi|^2 d\mu_{\Sigma} = 0$. Since ξ is concave, $\varphi = \text{constant}$. But since $\varphi(0) = 0$ we get that $\varphi \equiv 0$ in $\tilde{l}^{-1}(B_{a_0})$. The arbitrariness of a_0 gives that $\varphi \equiv 0$ in \mathbb{R}^2 .

Now $\varphi \equiv 0$ implies that the position vector \vec{P} is orthogonal to $JTl(D)$ where J is the standard complex structure in \mathbb{R}^{2n} . Note as discussed in section 3, since $l(D)$ is isotropic, the tangent space of \mathbb{R}^{2n} at the origin admits the orthogonal decomposition:

$$T\mathbb{R}^{2n} = Tl(D) \oplus JTl(D) \oplus Q$$

However, we already see that the orthogonal projection of \vec{P} onto Q is zero(since $\pi(\vec{P}) = 0$). Therefore, $\vec{P} \in Tl(D)$, which completes the proof. \square

Now we are able to construct tangent cones for an isotropically area-minimizing map. Let $l : D_1 \rightarrow \mathbb{R}^{2n}$ be weakly conformal, exact, isotropically area minimizing. Let $t_0 \in D_1$. For any sequence $\epsilon_j \rightarrow 0$, define rescaled maps $l_j : D_{\epsilon_j^{-1}} \rightarrow \mathbb{R}^{2n}$ by

$$l_j(\tau) = \delta^{-1} l(\epsilon_j \tau + t_0),$$

where $\delta_j = \sqrt{\text{Area}(l(D_{2\epsilon}(t_0)))}$ is chosen such that $\text{Area}(l_j(D_2)) = 1$. Clearly l_j is still minimizing.

The same argument in Schoen and Wolfson's paper(also using Proposition 6.1) implies that (1) $\{l_j\}_{j=1}^{\infty}$ converges strongly in $W_{loc}^{1,2}(\mathbb{R}^2, \mathbb{R}^{2n})$ to a weakly conformal, exact, isotropically area-minimizing map $l_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^{2n}$; (2) l_0 is actually a cone in the sense that the position vector lying in the tangent plane of l ; and (3) l_0 is a proper map with $l_0^{-1}(\{0\}) = \{0\}$.

We will henceforth call such a limiting map l_0 a *tangent map of l at t_0* (with respect to $\{\varepsilon_j\}$) or a *tangent cone of l at t_0* .

The following lemma studies the case where the image of a tangent map is contained in a plane.

Lemma 6.2. *Let $l_0(\tau) : \mathbb{R}^2 \rightarrow \mathbb{R}^{2n}$ be a tangent map of l at t_0 . If the image of l_0 is contained in an isotropic plane, then (after a unitary coordinate change of \mathbb{R}^{2n} to make the image plane be the x_1x_2 -plane) $l_0(\tau) = a\tau^n$ for some complex constant $a \neq 0$ and some positive integer n . In particular l_0 is a smooth immersion away from the origin and the image of l_0 is the plane.*

Proof. Because every deformation inside x_1x_2 -plane is isotropic, l_0 is area minimizing as a map from \mathbb{R}^2 to \mathbb{R}^2 . Therefore, by standard harmonic function theory, l_0 is a harmonic map, hence smooth in \mathbb{R}^2 . Since l_0 is also conformal, (by possibly reversing the orientation of \mathbb{R}^2) we see that l_0 is holomorphic. Since l_0 is proper and $l_0^{-1}(\{0\}) = \{0\}$, standard complex analysis theory implies l_0 is a polynomial of the form $a\tau^n$. \square

Now we study double blow-ups.

Lemma 6.3. *Let $l_0(\tau)$ be a tangent map of l at t_0 . Let $\tau_0 \in \mathbb{R}^2 \setminus \{0\}$. Let $l_1(\xi)$ be a tangent map of l_0 at τ_0 . Then the image of l_1 is a plane. In particular, by Lemma 6.2, l_1 is a smooth immersion away from the origin.*

Proof. Standard blow up argument shows that the vector $x_0 := l_0(\tau_0)$ is actually in the tangent plane $T_{l_1(\xi)}l_1(\mathbb{R}^2)$ for any non-zero ξ . Therefore the image $l_1(\mathbb{R}^2)$ is invariant under the translation in the direction of x_0 . By a unitary transformation, we may assume the direction of x_0 is the x_1 -axis. Then $l_1(\mathbb{R}^2)$ is a product of x_1 -axis and a one-dimensional curve γ in \mathbb{R}^{2n-1} (which is in fact a one-dimensional cone since $l_1(\mathbb{R}^2)$ is a cone). Since $l_1(\mathbb{R}^2)$ is isotropic, this curve γ must be in \mathbb{R}^{2n-2} spanned by $x_2, \dots, x_n, y_2, \dots, y_n$. It is obvious that any deformation in \mathbb{R}^{2n-2} is isotropic with respect to $l_1(\mathbb{R}^2)$. Therefore, the fact that $l_1(\mathbb{R}^2)$ is isotropically area minimizing in \mathbb{R}^{2n} implies that γ is absolutely length minimizing in \mathbb{R}^{2n-2} . Since the only one-dimensional minimizing cone in \mathbb{R}^{2n-2} is a straight line or a straight ray, we see γ is a line or a ray. Therefore $l_1(\mathbb{R}^2)$ is a plane or a half plane. In any case it is contained in a plane. Then by Lemma 6.2, the image of l_1 is a plane. \square

In the rest of this section we study two-dimensional isotropically stationary cones which are smooth away from origin.

First derive a geometric version of the Euler-Lagrangian equation for isotropically stationary surfaces.

Theorem 6.4. *Suppose $\Sigma^2 \subset \mathbb{R}^{2n}$ is an isotropically stationary surface with mean curvature vector H . Then*

$$\begin{aligned}\delta(H \lrcorner \omega) &= 0, \\ d(H \lrcorner \omega) &= 2\beta, \\ H &\in JT\Sigma,\end{aligned}$$

where β is $(0, 2)$ -tensor on Σ defined by $\beta(X, Y) = \sum_{j=1}^2 \langle \nabla_{e_j}^\perp Y, J \nabla_{e_j}^\perp X \rangle$, with $\{e_j\}_{j=1}^2$ being an orthonormal basis for $T\Sigma$.

Proof. To get the second equation, use Cartan's formula, we have

$$d(H \lrcorner \omega)(X, Y) = \langle \nabla_X H, JY \rangle - \langle \nabla_Y H, JX \rangle.$$

Let $\{e_j\}_{j=1}^2$ is an orthonormal basis for $T\Sigma$. Then $H = \sum_{j=1}^2 \nabla_{e_j}^\perp e_j$. Therefore

$$\langle \nabla_X H, JY \rangle = \sum_{j=1}^2 \langle \nabla_X \nabla_{e_j}^\perp e_j, JY \rangle = \sum_{j=1}^2 \langle \nabla_X^\perp \nabla_{e_j}^\perp e_j, JY \rangle,$$

where we used the fact that JY is in $(T\Sigma)^\perp$ since Σ is isotropic.

Assume the basis $\{e_j\}_{j=1}^2$ is normal at one point. Then using Codazzi equation(see [D]) and the fact that $\langle \nabla_X^\perp Y, JZ \rangle$ is fully symmetric for X, Y, Z in $T\Sigma$, we get

$$d(H \lrcorner \omega)(X, Y) = \sum_{j=1}^k 2 \langle \nabla_{e_j}^\perp Y, J \nabla_{e_j}^\perp X \rangle$$

This completes the proof of the second equation.

To get the third equation, note that since Σ is isotropically stationary, $\int_\Sigma \langle H, X \rangle d\mu_\Sigma = 0$ for any isotropic vector field with compact support. Since any $X \in (JT\Sigma)^\perp$ satisfies $(X \lrcorner \omega)|_\Sigma = 0$, it gives rise to a (local) isotropic variation. This implies that $H \in JT\Sigma$.

First variation also implies $\int_\Sigma \langle H, J \nabla f \rangle d\mu_\Sigma = 0$ for any $f : M \rightarrow \mathbb{R}$ which has compact support in Σ . But since $H \in JT\Sigma$, we get that $\int_\Sigma \langle H, J \nabla^\Sigma f \rangle d\mu_\Sigma = 0$, where $\nabla^\Sigma f$ is the projection of ∇f onto $T\Sigma$. This implies $\int_\Sigma \langle \delta(H \lrcorner \omega), f \rangle d\mu_\Sigma = 0$. The arbitrariness of f implies $\delta(H \lrcorner \omega) = 0$. \square

Remark 6.5. *The theorem above is true for any isotropically stationary submanifold Σ^k in a Kähler manifold (N^{2n}, g, J) , only with the second equation $d(H \lrcorner \omega) = 2\beta$ replaced by $d(H \lrcorner \omega) = 2\beta + 2ric_\Sigma$, where $ric_\Sigma(X, Y) := Ric_\Sigma(X, JY)$ for X, Y in $T\Sigma$.*

Now we can classify all isotropically stationary cones in \mathbb{R}^{2n}

Theorem 6.6. *Let Σ be an isotropic cone over the origin (in the sense that $\eta_\lambda \# \Sigma = \Sigma$ for any $\lambda > 0$, where $\eta_\lambda : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is the homothety map defined by $\eta_\lambda x = x/\lambda$). Assume also that Σ is smooth away from origin and is isotropically stationary in \mathbb{R}^{2n} . Then (up to a unitary transformation), the intersection of Σ with S^{2n-1} is a curve $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^{2n}$ parametrized by (using complex coordinate in \mathbb{R}^{2n})*

$$\begin{aligned} z_1(s) &= \sqrt{\frac{q}{p+q}} e^{\sqrt{-1}ps} \\ z_2(s) &= \sqrt{\frac{p}{p+q}} e^{-\sqrt{-1}qs} \\ z_3(s) &= 0, \dots, z_n(s) = 0. \end{aligned}$$

for some positive integers p and q .

Proof. Let $\gamma(\theta) : [0, L] \rightarrow S^{2n-1}$ be parametrized by arclength(i.e, $\|\gamma'(\theta)\| = 1$ for any θ). Then the cone Σ can be parametrized as

$$\sigma(t, \theta) = t\gamma(\theta), \quad t \in (0, \infty), \theta \in [0, L].$$

Note that an orthonormal basis for tangent planes of Σ is given by $e_1 = \sigma_* \frac{\partial}{\partial t} = \gamma(\theta)$ and $e_2 = \frac{\sigma_* \frac{\partial}{\partial \theta}}{\|\sigma_* \frac{\partial}{\partial \theta}\|} = \dot{\gamma}(\theta)$. Then the mean curvature vector of Σ is given by $H = \nabla_{e_1}^\perp e_1 + \nabla_{e_2}^\perp e_2 = t^{-1}(\ddot{\gamma}(\theta))^\perp$. Calculation using Theorem 6.4 gives

$$\ddot{\gamma} - b_0 J \dot{\gamma} + \gamma = 0, \quad (63)$$

for some real constant b_0 . Let $\gamma(\theta) = (z_1(\theta), \dots, z_n(\theta))$ where (z_1, \dots, z_n) is the complex coordinate in \mathbb{R}^{2n} . Then (63) gives us

$$\ddot{z}_k - \sqrt{-1}b_0 \dot{z}_k + z_k = 0, \quad \text{for } k = 1, \dots, n.$$

Solve it, we get

$$z_k = c_k e^{\sqrt{-1}\lambda_1 \theta} + d_k e^{\sqrt{-1}\lambda_2 \theta},$$

where c_k and d_k are complex constants and

$$\lambda_1 = \frac{b_0 + \sqrt{b_0^2 + 4}}{2}, \quad \lambda_2 = \frac{b_0 - \sqrt{b_0^2 + 4}}{2}.$$

Moreover since $\langle \gamma, \gamma \rangle = 1$, $\langle \dot{\gamma}, \dot{\gamma} \rangle = 1$ and $\langle \dot{\gamma}, J\gamma \rangle = 0$, we have

$$\begin{cases} \sum_{k=1}^n c_k \bar{c}_k = \frac{\lambda_2}{\lambda_2 - \lambda_1}, \\ \sum_{k=1}^n d_k \bar{d}_k = \frac{\lambda_1}{\lambda_1 - \lambda_2}, \\ \sum_{k=1}^n c_k \bar{d}_k = 0. \end{cases}$$

Therefore by a unitary transformation we may assume $c_1 = \sqrt{\frac{\lambda_2}{\lambda_2 - \lambda_1}}$, $c_2 = \dots = c_n = 0$ and $d_1 = 0$, $d_2 = \sqrt{\frac{\lambda_1}{\lambda_1 - \lambda_2}}$, $d_3 = \dots = d_n = 0$. Thus

$$\begin{bmatrix} z_1(\theta) \\ z_2(\theta) \\ \vdots \\ z_n(\theta) \end{bmatrix} = e^{\sqrt{-1}\lambda_1 \theta} \begin{bmatrix} \sqrt{\frac{\lambda_2}{\lambda_2 - \lambda_1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + e^{\sqrt{-1}\lambda_2 \theta} \begin{bmatrix} 0 \\ \sqrt{\frac{\lambda_1}{\lambda_1 - \lambda_2}} \\ \vdots \\ 0 \end{bmatrix}$$

Since γ is closed, we get (note $\lambda_1 > 0$ and $\lambda_2 < 0$)

$$\lambda_1 L = 2\pi p, \quad \lambda_2 L = -2\pi q,$$

for some positive integers q and p , where L is the length of γ . Now use the fact that $\lambda_1 \lambda_2 = -1$ we deduce

$$\lambda_1 = \sqrt{\frac{p}{q}}, \quad \lambda_2 = -\sqrt{\frac{q}{p}}.$$

The theorem then follows by a reparametrization of the curve using $s = 2\pi\theta/L$. \square

We can see that these isotropic cones are exactly those Lagrangian cones described in Schoen and Wolfson's paper [S-W]. By applying the second variational formula, they have proven

Proposition 6.7 (Schoen-Wolfson). *The cones with $|p - q| > 1$ are strictly unstable in \mathbb{R}^4 for any Hamiltonian variation fixing a neighborhood of the cone vertex. And cones with $|p - q| = 1$ are strictly stable in \mathbb{R}^4 for any Hamiltonian variation fixing a neighborhood of the cone vertex.*

And the multiply covered cones are strictly unstable for any p and q .

Therefore those cones with $|p - q| > 1$ are automatically strictly unstable in \mathbb{R}^{2n} . And by a careful study of the second variation, we actually get that cones with $|p - q| = 1$ are also strictly stable in \mathbb{R}^{2n} .

Proposition 6.8. *Those cones with $|p - q| = 1$ are strictly stable in \mathbb{R}^{2n} for any Hamiltonian variation fixing a neighborhood of the cone vertex.*

Proof. Let $f : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be any smooth function which is supported in $B_R(0) \setminus B_\varepsilon(0)$. Let $Z = J\nabla f$ be the Hamiltonian vector field. Therefore $Z|_\Sigma$ is of the form

$$J\nabla h + \sum_{k=3}^n (f_k \frac{\partial}{\partial x^k} + g_k \frac{\partial}{\partial y^k})$$

where h , f_k and g_k are arbitrary function of x^1, x^2, y^1 and y^2 . Then using second variational formula(see [S-W]) we see

$$\delta^2|\Sigma|(V_{J\nabla f}) = \delta^2|\Sigma|(V_{J\nabla h}) + \delta^2|\Sigma|(V_W),$$

where $W = \sum_{k=3}^n f_k \frac{\partial}{\partial x^k} + g_k \frac{\partial}{\partial y^k}$ is in \mathbb{R}^{2n-4} and where $\delta^2|\Sigma|(V_Z)$ denotes the second variation with respect to the vector field Z . To prove that cones with $|p - q| = 1$ are strictly stable, it suffices(since those cones are already strictly stable in \mathbb{R}^4 for any Hamiltonian variation) to show that $\delta^2|\Sigma|(V_W) > 0$ for any vector field W in \mathbb{R}^{2n-4} . A direct computation shows that

$$\begin{aligned} & \delta^2|\Sigma|(V_W) \\ &= \int_{[\varepsilon, R] \times [0, L]} \{ |\frac{\partial W}{\partial t}|^2 + t^{-2} |\frac{\partial W}{\partial \theta}|^2 + b_0 t^{-2} \langle W, J \frac{\partial W}{\partial \theta} \rangle \} t dt d\theta \end{aligned}$$

where we assume $\gamma(\theta)$ is parametrized by arclength and L is the length of the curve.

For notational simplicity, we only prove the case where $n = 3$ (the general case follows from the same argument). Assume $W = f_3 \frac{\partial}{\partial x^3} + g_3 \frac{\partial}{\partial y^3}$. Let $z = x^3 + \sqrt{-1}y^3$.

By the L^2 orthogonal decomposition of W , it suffices to prove the case where $W = f(t)e^{\sqrt{-1}2\pi k\theta/L}$ where k is an integer. Straightforward calculation shows that

$$\begin{aligned} & \delta^2 |\Sigma|(V_W) \\ &= L \int_{\varepsilon}^R t |f'(t)|^2 dt + L \frac{2\pi k}{L} \left(\frac{2\pi k}{L} - b_0 \right) \int_{\varepsilon}^R t^{-2} |f(t)|^2 dt. \end{aligned}$$

Note that (without loss of generality assume $p > q$)

$$b_0 = \sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}, \quad L = 2\pi\sqrt{pq}.$$

Since $p - q = 1$, $\frac{2\pi k}{L} \left(\frac{2\pi k}{L} - b_0 \right) > 0$ for any integer $k \neq 0$. This implies that for any integer k

$$\delta^2 |\Sigma|(V_W) > 0,$$

for $W = f(t)e^{\sqrt{-1}2\pi k\theta/L}$. □

7 Interior Regularity Theorem

In section 2, we showed that the weak solution to the isotropically constrained Plateau problem exists. In this section we will prove that this weak solution is smooth away from a finite set.

For a map $l : \Omega \rightarrow \mathbb{R}^{2n}$. Define regular points to be points in Ω at which l is a smooth immersion. We denote the set of all regular points as $Reg(l)$. And define the singular set $Sing(l) = \Omega \setminus Reg(l)$.

The following proposition says that singularities converge to a singularity.

Proposition 7.1. *Let $l^{(k)} : D_2 \rightarrow \mathbb{R}^{2n}$ be a sequence of weakly conformal, weakly exact and isotropically area minimizing maps. Assume $\{l^{(k)}\}$ converges strongly in $W_{loc}^{1,2}(D_2, \mathbb{R}^{2n})$ to a weakly conformal, exact and isotropically area minimizing map l . Let $t^{(k)} \in Sing(l^{(k)}) \cap D_1$. Suppose $\lim_{j \rightarrow \infty} t^{(k)} = t_0$. Then $t_0 \in Sing(l)$.*

Proof. Without loss of generality, assume $t_0 = 0$. Assume 0 is a regular point of l . Let ε_j be a sequence going to 0. Define the corresponding tangent map φ of l at 0, as the limit of $l_j(\tau) = \delta_j^{-1} l(\varepsilon_j \tau)$. For each $l^{(k)}$ define $l_j^{(k)}(\tau) = \delta_j^{-1} l^{(k)}(\varepsilon_j \tau)$. Now for a fixed j , since $\{l^{(k)}\}$ converges to l , we have an α_j such that

$$\|l_j^{(\alpha_j)} - l_j\|_{W^{1,2}(D_1)} \leq \frac{1}{j},$$

and since $\{t^{(k)}\}$ goes to 0, we might also assume

$$\left| \frac{t^{(\alpha_j)}}{\varepsilon_j} \right| \leq \frac{1}{j}.$$

Therefore, we easily get that

$$\lim_{j \rightarrow \infty} l_j^{(\alpha_j)} = \varphi,$$

in $W^{1,2}(D_1, \mathbb{R}^{2n})$.

Since 0 is regular for l , we see that φ is a linear conformal map into a isotropic plane. Therefore, we can apply our local regularity theorem (Theorem 5.10) to get a constant θ such that for sufficient large j , $l_j^{(\alpha_j)}$ is smooth immersion in D_θ . But note that $\frac{t^{(\alpha_j)}}{\varepsilon_j}$ is a singularity of $l_j^{(\alpha_j)}$, and $\frac{t^{(\alpha_j)}}{\varepsilon_j} \rightarrow 0$. A contradiction. \square

An immediate consequence of the above proposition is the following result.

Proposition 7.2. *Let $l : D_1 \rightarrow \mathbb{R}^{2n}$ be weakly conformal, weakly exact and isotropically area minimizing map. Let $t_0 \in D_1$. If every tangent cone of l at t_0 is a smooth immersion away from the origin, then l is Hölder continuous in D_1 and is a smooth immersion in $D_r(t_0) \setminus \{t_0\}$ for some $r > 0$.*

Proof. Without loss of generality, assume $t_0 = 0$. If l is not smooth immersion in any deleted neighborhood of 0, then we have a sequence of points $\{t_j\} \subset \text{Sing}(l)$ which converges to 0. Let $\varepsilon_j = |t_j|$. Let l_0 be the tangent cone at 0 with respect to $\{\varepsilon_j\}$ with the corresponding rescaled sequence $\{l_j\}$. Clearly $\tau_j = t_j/\varepsilon_j$ is a singularity of l_j . Since $|\tau_j| = 1$, we might assume (by passing to a subsequence) that $\{\tau_j\}$ converges to a point τ in the unit circle S^1 . By Proposition 7.1 τ should be a singularity of the limiting map l_0 . But by hypothesis, every tangent map at 0 is regular away from the origin. A contradiction. \square

Now we are in the position to prove the main regularity theorem.

Theorem 7.3 (Interior Regularity Theorem). *Let $l : D_2 \rightarrow \mathbb{R}^{2n}$ be a weakly conformal, weakly isotropic map which minimizes area among isotropic maps with the same boundary value. There is a finite subset S of D_1 such that l is a smooth immersion on $D_1 \setminus S$.*

Proof. Let Ω be the set of points in D_1 such that the image of every tangent map is a plane. Let $t_0 \in \Omega$. Then Lemma 6.2 says that every tangent cone of t_0 is a smooth immersion away from the origin. Therefore, by Proposition 7.2, l is a smooth immersion in a deleted neighborhood of t_0 .

Let $B = D_1 \setminus \Omega$. For a point $t_1 \in B$, let l_0 be a tangent cone at t_1 whose image is not a plane. Let $\tau_0 \in \mathbb{R}^2$ be any point away from the origin. Lemma 6.3 then says that any tangent cone of l_0 at τ_0 is a smooth immersion away from the origin. Therefore by Proposition 7.2, l_0 is smooth immersion in a deleted neighborhood of any point away from the origin. This implies that the singular

set of l_0 is discrete. Then the fact that l_0 is a geometric cone implies that l_0 is smooth away from the origin. Therefore, by the classification of isotropically stationary cones (see Theorem 6.6) we see that (by Proposition 6.7) the cone $C = l_0(\mathbb{R}^2)$ is conformal equivalent to the complex plane \mathbb{R}^2 via a map $\xi : C \rightarrow \mathbb{R}^2$. Thus $\xi \circ l_0$ is a weakly conformal, isotropically minimizing, proper map with only one zero. Therefore by Lemma 6.2, we see $\xi \circ l_0$ is smooth immersion away from the origin. This implies that l_0 itself is a smooth immersion away from origin. Thus Proposition 7.2 gives that l is a smooth immersion in a deleted neighborhood of t_0 .

So combine the above two arguments, we see that $Sing(l)$ is discrete (hence finite) in D_1 . □

Now finally combine Theorem 2.9 and Theorem 7.3 we have the following regularity result of the solution to the isotropically constrained Plateau problem:

Theorem 7.4. *Let D_1 be the unit disk in \mathbb{R}^{2n} . Let Γ be a closed piecewise C^1 Jordan curve in \mathbb{R}^{2n} such that $\int_{\Gamma}(xdy - ydx) = 0$. Let*

$$\mathcal{X}_{\Gamma, I} = \{l \in W_I^{1,2}(D_1, \mathbb{R}^{2n}) : \\ l|_{\partial D_1} \text{ is continuous and is a monotone map onto } \Gamma\}.$$

Then $\mathcal{X}_{\Gamma, I}$ is not empty and there exists an area minimizer l_0 among $\mathcal{X}_{\Gamma, I}$. Moreover, l_0 is Hölder continuous in D_1 and is smooth immersion in the interior of D_1 with possibly isolated singularities.

APPENDIX A

In this appendix we prove proposition 3.3 in section 3.

Proposition .1. *Let $h(t, \theta) : \mathbb{R}^{2n+1} \rightarrow R$ be a smooth function. Then*

$$\begin{aligned} & div_{\Sigma} X_{h(t, \theta)} \\ &= -2h_t e^{-t} \sin \theta - 2h_{\theta t} e^{-t} \cos \theta \\ &+ (h_{\theta t} - h_{\theta})(2\|\nabla^{\Sigma} \theta\|_d^2 + e^{-2t} \|\pi(\vec{P})\|^2) \\ &+ \{-2h_t + h_{tt} - h_{\theta\theta}\} \langle \nabla^{\Sigma} \theta, \nabla^{\Sigma} t \rangle_d \end{aligned}$$

Proof. Straightforward calculation shows

$$ds = e^t(\cos \theta dt - \sin \theta d\theta), \quad d\varphi = e^t(\sin \theta dt + \cos \theta d\theta), \quad (64)$$

and

$$d\theta = e^{-t}(\cos \theta d\varphi - \sin \theta ds), \quad dt = e^{-t}(\sin \theta d\varphi + \cos \theta ds). \quad (65)$$

Since the degenerate inner product $\langle \cdot, \cdot \rangle_d$ does not involve φ , we see that $\operatorname{div}_\Sigma Z$ does not depend on the $\frac{\partial}{\partial \varphi}$ component. Therefore we shall use $X \approx Y$ to indicate that X and Y have the same $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ components. Then

$$X_s \approx x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad X_\varphi \approx -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

Lemma .2. *Using the same notations as in proposition 3.2, we have*

$$\begin{aligned} \langle e_i, X_s \rangle_d &= e_i(\varphi), & \langle e_i, X_\varphi \rangle_d &= -e_i(s), \\ \operatorname{div}_\Sigma X_s &= 0, & \operatorname{div}_\Sigma X_\varphi &= -2. \end{aligned}$$

Proof. We only prove the last equation.

$$\begin{aligned} \operatorname{div}_\Sigma X_\varphi &= \sum_{k=1}^2 \langle \nabla_{e_k}^0 (-x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}), e_k \rangle_d \\ &= - \sum_{k=1}^2 \langle e_k(x) \frac{\partial}{\partial x} + e_k(y) \frac{\partial}{\partial y}, e_k \rangle_d \\ &= - \sum_{k=1}^2 (e_k(x)^2 + e_k(y)^2) \\ &= -2, \end{aligned}$$

where in the last equation we use the fact that $1 = \|e_k\|_d = e_k(x)^2 + e_k(y)^2$, for each $k = 1, 2$. \square

Since $l(D) \subset \mathbb{R}^{2n}$ is isotropic, we have the orthogonal decomposition of the tangent space of \mathbb{R}^{2n} :

$$T\mathbb{R}^{2n} = Tl(D) \oplus J Tl(D) \oplus Q$$

where J is the standard complex structure of \mathbb{R}^{2n} . Let $\pi : T\mathbb{R}^{2n} \rightarrow Q$ be the orthogonal projection.

Lemma .3. *Using the same notations as in proposition 3.2, we have*

$$\begin{aligned} \operatorname{div}_\Sigma X_t &= -\frac{2 \sin \theta}{e^t} - 2 \langle \nabla^\Sigma \theta, \nabla^\Sigma t \rangle_d, \\ \operatorname{div}_\Sigma X_{-\theta} &= 2 \|\nabla^\Sigma \theta\|_d^2 + e^{-2t} \|\pi(\vec{P})\|^2, \end{aligned}$$

where $\nabla^\Sigma f := \sum_{i=1}^2 e_i(f) e_i$, and where $\vec{P} = (x, y)$ is the position vector of l in \mathbb{R}^{2n} .

Proof. To prove the first equation, we have

$$\begin{aligned} \operatorname{div}_\Sigma X_t &= \operatorname{div}_\Sigma (t_s X_s + t_\varphi X_\varphi) \\ &= \langle X_s, \nabla^\Sigma t_s \rangle + \langle \nabla^\Sigma t_\varphi, X_\varphi \rangle_d + t_s \operatorname{div}_\Sigma X_s + t_\varphi \operatorname{div}_\Sigma X_\varphi \\ &= \langle X_s, \nabla^\Sigma t_s \rangle_d + \langle \nabla^\Sigma t_\varphi, X_\varphi \rangle_d - 2t_\varphi, \end{aligned}$$

where we used the fact that $div_{\Sigma}X_s = 0$ and $div_{\Sigma}X_{\varphi} = -2$.

Using (64), (65) and lemma .2, we have

$$\begin{aligned}\langle \nabla^{\Sigma}t_s, X_s \rangle_d &= \sum_{i=1}^2 \langle e_i(t_s)e_i, X_s \rangle_d = \sum_{i=1}^2 e_i(t_s)e_i(\varphi) \\ &= -\sum_{i=1}^2 [e_i(\theta)e_i(t) + \sin\theta \cos\theta e_i^2(\theta) + \sin\theta \cos\theta e_i^2(t)].\end{aligned}$$

Similarly

$$\langle \nabla^{\Sigma}(t_{\varphi}), X_{\varphi} \rangle_d = -\sum_{i=1}^2 [e_i(\theta)e_i(t) - \sin\theta \cos\theta e_i^2(\theta) - \sin\theta \cos\theta e_i^2(t)]$$

Combine the three equations above, we get

$$\begin{aligned}div_{\Sigma}X_t &= \langle X_s, \nabla^{\Sigma}t_s \rangle_d + \langle \nabla^{\Sigma}t_{\varphi}, X_{\varphi} \rangle_d - 2t_{\varphi} \\ &= -2\sum_{i=1}^2 e_i(\theta)e_i(t) - 2t_{\varphi} \\ &= -2\langle \nabla^{\Sigma}\theta, \nabla^{\Sigma}t \rangle_d - 2e^{-t} \sin\theta,\end{aligned}$$

which gives the first equation.

Similar computations yields

$$div_{\Sigma}X_{\theta} = -2\theta_{\varphi} + \|\nabla^{\Sigma}t\|_d^2 - \|\nabla^{\Sigma}\theta\|_d^2. \quad (66)$$

Next we would like to get a relation between $\|\nabla^{\Sigma}t\|_d^2$ and $\|\nabla^{\Sigma}\theta\|_d^2$. Note a calculation using (64) and (65) gives

$$\begin{aligned}\|\nabla^{\Sigma}t\|_d^2 + \|\nabla^{\Sigma}\theta\|_d^2 &= e^{-2t}(\|\nabla^{\Sigma}\varphi\|_d^2 + \|\nabla^{\Sigma}s\|_d^2) \\ &= e^{-2t}\sum_{i=1}^2 (e_i^2(\varphi) + e_i^2(s)).\end{aligned} \quad (67)$$

Since \tilde{l} is contact,

$$e_i(\varphi) = xe_i(y) - ye_i(x) = \langle \vec{P}, Je_i \rangle,$$

where $\vec{P} = (x, y)$ is the position vector of l in \mathbb{R}^{2n} . Also

$$e_i(s) = xe_i(s) + ye_i(y) = \langle \vec{P}, e_i \rangle$$

Therefore

$$\begin{aligned}\|\nabla^{\Sigma}\varphi\|_d^2 + \|\nabla^{\Sigma}s\|_d^2 &= \sum_{i=1}^2 (\langle \vec{P}, Je_i \rangle^2 + \langle \vec{P}, e_i \rangle^2) \\ &= \|\vec{P}\|^2 - \|\pi(\vec{P})\|^2 \\ &= 2s - \|\pi(\vec{P})\|^2.\end{aligned} \quad (68)$$

Finally combine (67) and (68), we get

$$\|\nabla^\Sigma t\|_d^2 + \|\nabla^\Sigma \theta\|_d^2 = 2 \frac{\cos \theta}{e^t} - \frac{\|\pi(\vec{P})\|^2}{e^{2t}}. \quad (69)$$

Hence from (66), (69) and the fact that $\theta_\varphi = e^{-t} \cos \theta$, we see

$$\operatorname{div}_\Sigma X_\theta = -2\|\nabla^\Sigma \theta\|_d^2 - e^{-2t}\|\pi(\vec{P})\|^2$$

□

Using the chain rule we get

$$\begin{aligned} \operatorname{div}_\Sigma X_h &= \operatorname{div}_\Sigma (h_\theta X_\theta + h_t X_t) \\ &= \langle X_\theta, \nabla^\Sigma h_\theta \rangle_d + \langle X_t, \nabla^\Sigma h_t \rangle_d + h_\theta \operatorname{div}_\Sigma X_\theta + h_t \operatorname{div}_\Sigma X_t. \end{aligned}$$

Then direct calculation using (69), lemma .3 and the fact that $\langle e_i, X_\theta \rangle_d = -e_i(t)$ and $\langle e_i, X_t \rangle_d = e_i(\theta)$ will yield the result. □

APPENDIX B

In this appendix we prove Lemma 5.8 in section 5.3

Lemma .4. *Let $a_{ij}(x)$, $b_i(x)$, $c_i(x)$, $d(x)$, $f_i(x)$ and $g(x)$ be $C^{0,\mu}$ functions on B_2 . Assume $q_1(x), \dots, q_m(x)$ are $C^{1,\mu}$ functions on B_2 and $p_{ikj}(x)$ are $C^{0,\mu}$ on B_2 , for $i, k = 1, 2$ and $j = 1, \dots, m$.*

Assume $u(x) \in C^{1,\mu}(B_2)$ satisfies

$$\begin{aligned} &\sum_{i,j=1}^2 \int_{B_2} a_{ij} D^i u D^j \varphi + \sum_{i=1}^2 \int_{B_2} b_i \varphi D^i u + \sum_{i=1}^2 \int_{B_2} c_i u D^i \varphi \quad (70) \\ &+ \int_{B_2} d u \varphi \\ &= \sum_{i=1}^2 \int_{B_2} f_i D^i \varphi d\mu + \int_{B_2} g \varphi d\mu + \sum_{j=1}^m \sum_{i,k=1}^2 \int_{B_2} p_{ikj} D^k \Gamma_j^i dx. \end{aligned}$$

for any $\varphi \in C_c^\infty(B_2)$, and for any

$$\Gamma_j = (\Gamma_j^1, \Gamma_j^2) \in \mathcal{G}(D^1 q_j D^2 \varphi - D^2 q_j D^1 \varphi),$$

where \mathcal{G} is defined in section 5.3. We assume the ellipticity condition:

$$\sum_{i,j=1}^2 a_{ij}(x) \xi^i \xi^j \geq \Lambda |\xi|^2, \quad (71)$$

and

$$\begin{aligned} \|a_{ij}\|_{C^{0,\mu}(B_2)} &\leq \Lambda_1, \quad \|b_i\|_{C^{0,\mu}(B_2)} \leq \Lambda_1, \\ \|c_i\|_{C^{0,\mu}(B_2)} &\leq \Lambda_1, \quad \|d\|_{C^{0,\mu}(B_2)} \leq \Lambda_1. \end{aligned}$$

Then

$$\begin{aligned} \|u\|_{C^{1,\mu}(B_1)} &\leq c(\|u\|_{C^0(B_2)} + \|g\|_{C^{0,\mu}(B_2)} + \sum_{i=1}^2 \|f_i\|_{C^{0,\mu}(B_2)}) \\ &\quad + \sum_{j=1}^m \sum_{i,k=1}^2 \|p_{ikj}\|_{C^{0,\mu}(B_2)} \|Dq_j\|_{C^{0,\mu}(B_2)}, \end{aligned} \quad (72)$$

where $c = c(\Lambda, \Lambda_1)$.

Proof. We shall use a scaling argument which is a modification of the approach in [Si3].

First consider the constant coefficients case. Assume u satisfies

$$\begin{aligned} &\sum_{i,j=1}^2 \int_{B_2} a_{ij} D^i u D^j \varphi \\ &= \sum_{i=1}^2 \int_{B_2} f_i D^i \varphi d\mu + \sum_{j=1}^m \sum_{i,k=1}^2 \int_{B_2} p_{ikj} D^k \Gamma_j^i dx, \end{aligned} \quad (73)$$

for any $\varphi \in C_c^\infty(B_2)$, and for any

$$\Gamma_j = (\Gamma_j^1, \Gamma_j^2) \in \mathcal{G}(D^1 q_j D^2 \varphi - D^2 q_j D^1 \varphi),$$

where a_{ij} are constants satisfying the elliptic condition (71).

First step is to prove that for any $\epsilon > 0$, there is a $\eta = \eta(\epsilon, \Lambda, \Lambda_1) \in (0, 2)$ such that

$$\begin{aligned} [Du]_{\mu, B_\eta} &\leq \epsilon [Du]_{\mu, B_2} + C \left(\sum_{i=1}^2 [f_i]_{\mu, B_2} \right. \\ &\quad \left. + \sum_{j=1}^m \sum_{i,k=1}^2 \|p_{ikj}\|_{0,\mu,B_2} \|Dq_j\|_{0,\mu,B_2} \right), \end{aligned} \quad (74)$$

where C only depends on Λ and Λ_1 .

Assume (74) is not true, then we have a fixed $\epsilon_0 > 0$, a sequence $\{\eta_l\}_{l=1}^\infty \rightarrow 0$ and sequences $\{a_{ij}^{(l)}\}_{i,j=1}^\infty$, $\{u^{(l)}\}_{l=1}^\infty$, $\{f_i^{(l)}\}_{i=1}^\infty$, $\{p_{ikj}^{(l)}\}_{i,k,j=1}^\infty$ and $\{q_j^{(l)}\}_{j=1}^\infty$ such that they satisfy (73) but

$$\begin{aligned} [Du^{(l)}]_{\mu, B_{\eta_l}} &\geq \epsilon_0 [Du^{(l)}]_{\mu, B_2} + l \left(\sum_{i=1}^2 [f_i^{(l)}]_{\mu, B_2} \right. \\ &\quad \left. + \sum_{j=1}^m \sum_{i,k=1}^2 \|p_{ikj}^{(l)}\|_{0,\mu,B_2} \|Dq_j^{(l)}\|_{0,\mu,B_2} \right) \end{aligned} \quad (75)$$

By definition of $[Du^{(l)}]_{\mu, B_{\eta_l}}$, there are points x_l and y_l in B_{η_l} such that

$$|Du^{(l)}(x_l) - Du^{(l)}(y_l)| \geq (2N)^{-1} [Du^{(l)}]_{\mu, B_{\eta_l}} |x_l - y_l|^\mu, \quad (76)$$

where N is an absolute constant. Let

$$\sigma_l = |x_l - y_l|, \quad \lambda_l = [Du^{(l)}]_{\mu, B_2}.$$

Now scale. Define

$$\begin{aligned} \tilde{u}^{(l)}(x) &= \sigma_l^{-1-\mu} \lambda_l^{-1} u^{(l)}(\sigma_l x + x_l), \quad \tilde{f}_i^{(l)}(x) = \sigma_l^{-\mu} \lambda_l^{-1} f_i^{(l)}(\sigma_l x + x_l), \\ \tilde{p}_{ikj}^{(l)}(x) &= p_{ikj}^{(l)}(\sigma_l x + x_l), \quad \tilde{q}_j^{(l)}(x) = \sigma_l^{-1-\mu} \lambda_l^{-1} q_j^{(l)}(\sigma_l x + x_l) \end{aligned} \quad (77)$$

Then $\tilde{u}^{(l)}$ is a weak solution to (73) with $B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})$, $a_{ij}^{(l)}$, $\tilde{f}_i^{(l)}$, $\tilde{p}_{ikj}^{(l)}$ and $\tilde{q}_j^{(l)}$ in the place of B_2 , a_{ij} , f_i , p_{ikj} and q_j . Moreover

$$[D\tilde{u}^{(l)}]_{\mu, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} = 1.$$

Then (75) and (77) imply

$$\begin{aligned} & \sum_{j=1}^m \sum_{i,k=1}^2 \{ \sigma_l^\mu \|\tilde{p}_{ikj}^{(l)}\|_{0, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} \|D\tilde{q}_j^{(l)}\|_{0, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} \\ & + \|\tilde{p}_{ikj}^{(l)}\|_{0, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} [D\tilde{q}_j^{(l)}]_{\mu, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} \\ & + [\tilde{p}_{ikj}^{(l)}]_{\mu, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} \|D\tilde{q}_j^{(l)}\|_{0, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} \\ & + \sigma_l^{-\mu} [\tilde{p}_{ikj}^{(l)}]_{\mu, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} [D\tilde{q}_j^{(l)}]_{\mu, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} \} \\ & + \sum_{i=1}^2 [f_i^{(l)}]_{\mu, B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})} \\ & \leq l^{-1} \lambda_l^{-1} [Du^{(l)}]_{\mu, B_{\eta_l}} \\ & \leq l^{-1} \end{aligned} \quad (78)$$

Also (75) and (76) give

$$|D\tilde{u}^{(l)}(z_l) - D\tilde{u}^{(l)}(0)| \geq \lambda_l^{-1} [Du^{(l)}]_{\mu, B_{\eta_l}} \geq \epsilon_0, \quad (79)$$

where $z_l = (y_l - x_l) \sigma_l^{-1}$.

Clearly for a fixed $R > 0$, $B_R(0) \subset B_{\sigma_l^{-1}}(-2x_l \sigma_l^{-1})$ for sufficiently large l .

Now let $r^{(l)}(x) := \tilde{u}^{(l)}(0) + D\tilde{u}^{(l)}(0) \cdot x$ be the first degree Taylor polynomial of $\tilde{u}^{(l)}$ at 0. And let $v^{(l)}(x) = \tilde{u}^{(l)} - r^{(l)}$. We see that $v^{(l)}$ is a weak solution of (73) with B_R , $a_{ij}^{(l)}$, $\tilde{f}_i^{(l)} - \tilde{f}_i^{(l)}(0)$, $\tilde{p}_{ikj}^{(l)} - \tilde{p}_{ikj}^{(l)}(0)$ and $\tilde{q}_j^{(l)}$ in the place of B_2 , a_{ij} , f_i , p_{ikj} and q_j respectively.

Also we have

$$[Dv^{(l)}]_{\mu, B_R} \leq 1, \quad |Dv^{(l)}(z_l)| \geq \epsilon_0, \quad Dv^{(l)}(0) = 0, \quad (80)$$

which in turn imply

$$\|Dv^{(l)}\|_{0,B_R} \leq cR^\mu, \quad \sup_{B_R} |v^{(l)}| \leq cR^{1+\mu}. \quad (81)$$

Now fix a function φ with compact support in $B_{R/2}$. Notice that (78) implies that $[\tilde{f}_i^{(l)}]_{\mu,B_R}$ goes to 0 as l goes to infinity. Arzela-Ascoli lemma then gives (by passing to a subsequence) that $\tilde{f}_i^{(l)} - \tilde{f}_i^{(l)}(0)$ converge uniformly to the zero function. Also by Arzela-Ascoli, $v^{(l)}$ converge in $C^{1,\mu}(B_R)$ to some function $v^{(R)}$ such that

$$[Dv^{(R)}]_{\mu,B_R} \leq 1, \quad |Dv^{(R)}(z^*)| \geq \epsilon_0, \quad Dv^{(R)}(0) = 0. \quad (82)$$

where $z^* = \lim_{l \rightarrow \infty} z_l$ (in particular $|z| = 1$).

For this fixed $\varphi \in C_c^\infty(B_R)$, choose $\Gamma_j = (\Gamma_j^1, \Gamma_j^2) \in \mathcal{G}(D^1 \tilde{q}_j^{(l)} D^2 \varphi - D^2 \tilde{q}_j^{(l)} D^1 \varphi)$ such that $\|\Gamma_j\|_{C^{1,\mu}(B_R)} \leq c(R) \|D^1 \tilde{q}_j^{(l)} D^2 \varphi - D^2 \tilde{q}_j^{(l)} D^1 \varphi\|_{C^{0,\mu}(B_R)}$ (the existence of such Γ is given by proposition 5.4). Therefore

$$\begin{aligned} & \|\Gamma_j\|_{1,\mu,B_R} & (83) \\ & \leq c(R) \|D^1 \tilde{q}_j^{(l)} D^2 \varphi - D^2 \tilde{q}_j^{(l)} D^1 \varphi\|_{0,\mu,B_R} \\ & \leq C(R) [D^1 \tilde{q}_j^{(l)} D^2 \varphi - D^2 \tilde{q}_j^{(l)} D^1 \varphi]_{\mu,B_R}, \\ & \leq C(R) \|\varphi\|_{1,\mu,B_R} \|D\tilde{q}_j^{(l)}\|_{0,\mu,B_R} \end{aligned}$$

where we used the fact that $D^1 \tilde{q}_j^{(l)} D^2 \varphi - D^2 \tilde{q}_j^{(l)} D^1 \varphi$ has compact support in B_R . Therefore

$$[(\tilde{p}_{ikj}^{(l)} - \tilde{p}_{ikj}^{(l)}(0)) D^k \Gamma_j^i]_{\mu,B_R} \quad (84)$$

$$\begin{aligned} & \leq \|\tilde{p}_{ikj}^{(l)} - \tilde{p}_{ikj}^{(l)}(0)\|_{0,B_R} [D^k \Gamma_j^i]_{\mu,B_R} + [\tilde{p}_{ikj}^{(l)}]_{\mu,B_R} \|D^k \Gamma_j^i\|_{0,B_R} \\ & \leq C(R) \|\varphi\|_{1,\mu,B_R} [\tilde{p}_{ikj}^{(l)}]_{\mu,B_R} \|D\tilde{q}_j^{(l)}\|_{0,\mu,B_R}. \end{aligned} \quad (85)$$

Combining (78) and (84), we see that

$$[(\tilde{p}_{ikj}^{(l)} - \tilde{p}_{ikj}^{(l)}(0)) D^k \Gamma_j^i]_{\mu,B_R} \leq C(R) l^{-1} \|\varphi\|_{1,\mu,B_R}. \quad (86)$$

Therefore since $(\tilde{p}_{ikj}^{(l)} - \tilde{p}_{ikj}^{(l)}(0)) D^k \Gamma_j^i$ has compact support in B_R , Arzela-Ascoli plus (86) implies that (by passing to a subsequence), they converge (as l goes to ∞) in $C^{0,\mu}(B_R)$ norm to the zero function.

To sum up, by letting l go to ∞ in (73) with B_R , $v^{(l)}$, $a_{ij}^{(l)}$, $\tilde{f}_i^{(l)} - \tilde{f}_i^{(l)}(0)$, $\tilde{p}_{ikj}^{(l)} - \tilde{p}_{ikj}^{(l)}(0)$ and $q_j^{(l)}$ in the place of B_2 , u , a_{ij} , f_i , p_{ikj} and q_j respectively, we have

$$\int_{B_R} a_{ij} D^i v^{(R)} D^j \varphi dx = 0.$$

for any $\varphi \in C_c^\infty(B_R)$, where $a_{ij} = \lim_{l \rightarrow \infty} a_{ij}^{(l)}$. Notice $v^{(R_1)}|_{B_{R_1}} = v^{(R_2)}$ for any $R_1 \geq R_2$, so by letting R go to ∞ we get a function v defined in \mathbb{R}^2 such that

$$[Dv]_{\mu, \mathbb{R}^2} \leq 1, \quad |Dv(z^*)| \geq \epsilon_0, \quad Dv(0) = 0, \quad \|v\|_{0, B_R} \leq cR^{1+\mu}. \quad (87)$$

and

$$\int_{\mathbb{R}^2} a_{ij} D^i v D^j \varphi dx = 0,$$

for any $\varphi \in C_c^\infty(\mathbb{R}^2)$. Standard harmonic function theory implies that v is smooth and

$$\|DDv\|_{0, B_{R/2}} \leq CR^{-2} \|v\|_{0, B_R}.$$

In view of (87), we have

$$\|DDv\|_{0, B_{R/2}} \leq CR^{\mu-1}.$$

Let R go to ∞ we see $DDv \equiv 0$. So v is a polynomial of at most degree one, which is a contradiction to $|Dv(z^*)| \geq \epsilon_0$ and $Dv(0) = 0$ in (87). This completes that proof of (74).

Now we claim that for any $\epsilon > 0$ and for any $\theta \in (0, 2)$, there is a constant $C = C(\epsilon, \theta, \Lambda, \Lambda_1)$ such that

$$\begin{aligned} [Du]_{\mu, B_\theta} &\leq \epsilon [Du]_{\mu, B_2} + C(\|u\|_{0, B_2} + \sum_{i=1}^2 [f_i]_{\mu, B_2}) \\ &\quad + \sum_{j=1}^m \sum_{i, k=1}^2 \|p_{ikj}\|_{0, \mu, B_2} \|Dq_j\|_{0, \mu, B_2} \end{aligned} \quad (88)$$

To prove this, given ϵ and θ , let $y \in B_\theta$. Apply a scaled version of (74) to the ball $B_{2-\theta}(y)$, we get an $\eta = \eta(\epsilon, \Lambda, \Lambda_1)$ such that

$$\begin{aligned} [Du]_{\mu, B_{\eta(2-\theta)}(y)} &\leq \epsilon [Du]_{\mu, B_{2-\theta}(y)} + C(\theta, \Lambda, \Lambda_1) \left(\sum_{i=1}^2 [f_i]_{\mu, B_{2-\theta}(y)} \right) \\ &\quad + \sum_{j=1}^m \sum_{i, k=1}^2 \|p_{ikj}\|_{0, \mu, B_{2-\theta}(y)} \|Dq_j\|_{0, \mu, B_{2-\theta}(y)} \end{aligned} \quad (89)$$

Now for $x \in B_\theta$, consider two cases. First assume $x \in B_{\eta(1-\theta)}(y)$. Then (89) directly gives

$$\begin{aligned} \frac{|Du(x) - Du(y)|}{|x - y|^\mu} &\leq \epsilon [Du]_{\mu, B_2} + C \left(\sum_{i=1}^2 [f_i]_{\mu, B_2} \right) \\ &\quad + \sum_{j=1}^m \sum_{i, k=1}^2 \|p_{ikj}\|_{0, \mu, B_2} \|Dq_j\|_{0, \mu, B_2} \end{aligned} \quad (90)$$

The second case is when $|x - y| \geq \eta(2 - \theta)$. Then interpolation inequality implies that for any $\delta > 0$,

$$\begin{aligned} \frac{|Du(x) - Du(y)|}{|x - y|^\mu} &\leq \eta^{-\mu}(2 - \theta)^{-\mu} \sup_{B_\theta} |Du| \\ &\leq \eta^{-\mu}(2 - \theta)^{-\mu} (\delta \theta^\mu [Du]_{\mu, B_\theta} + C(\delta) \|u\|_{0, B_\theta}) \end{aligned}$$

Choose δ small enough such that $\eta^{-\mu}(2 - \theta)^{-\mu} \delta \leq \epsilon$, we get

$$\frac{|Du(x) - Du(y)|}{|x - y|^\mu} \leq \epsilon [Du]_{\mu, B_2} + C(\epsilon, \theta) \|u\|_{0, B_2}.$$

Combine the two cases, we get (88).

A scaled version of (88)(we fix $\theta = 1$) implies

$$\begin{aligned} [Du]_{\mu, B_{\rho/2}(y)} &\leq \epsilon [Du]_{\mu, B_\rho(y)} + C\rho^{-2} (\|u\|_{0, B_2} + \sum_{i=1}^2 [f_i]_{\mu, B_2}) \\ &\quad + \sum_{j=1}^m \sum_{i, k=1}^2 \|p_{ikj}\|_{0, \mu, B_2} \|Dq_j\|_{0, \mu, B_2}, \end{aligned} \quad (91)$$

for any $B_\rho(y) \subset B_2$

Now apply the abstract Lemma 5.9 to (91) with

$$S(B_\rho(y)) = [Du]_{\mu, B_\rho(y)}$$

together with interpolation inequalities, we get (72), which completes the proof of the lemma in the constant coefficient case for the solution of equation (73). Then by applying the standard technique of freezing coefficients, we prove the lemma in the general case. \square

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