Open Covers and Compactness

Suppose \((X, d)\) is a metric space.

**Definition**
Let \(E \subseteq X\). An *open cover* of \(E\) is a collection \(\{G_\alpha : \alpha \in I\}\) of open subsets of \(X\) such that \(E \subseteq \bigcup_{\alpha \in I} G_\alpha\).

**Definition**
A subset \(K\) of \(X\) is *compact* if every open cover contains a *finite* subcover.

In other words if \(\{G_\alpha : \alpha \in I\}\) is a collection of open subsets of \(X\) with \(K \subseteq \bigcup_{\alpha \in I} G_\alpha\) then there is a finite set \(\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq I\) such that

\[
K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}
\]
Examples of Compact Sets:
- Every finite set is compact.
- Any closed interval \([a, b]\) in \(\mathbb{R}^1\).

Examples of Non-Compact Sets:
- \(\mathbb{Z}\) in \(\mathbb{R}^1\).
- Any open interval \((a, b)\) in \(\mathbb{R}^1\).
- \(\mathbb{R}^1\) as a subset of \(\mathbb{R}^1\).
Relative Compactness

**Theorem**
Suppose $(X, d)$ is a metric space and $K \subseteq Y \subseteq X$. Then $K$ is a compact subset of $(X, d)$ if and only if $K$ is a compact subset of $(Y, d)$.

So unlike with closed and open sets, a set is “compact relative a subset $Y$” if and only if it is compact relative to the whole space.
Compact Subsets are Closed

Theorem

*Compact subsets of a metric space are closed.*
Closed Subsets and Compactness

**Theorem**
*Closed subsets of compact sets are compact.*

**Corollary**
*If $F$ is closed and $K$ is compact then $F \cap K$ is compact.*
Intersection of Compact Sets

**Theorem**
If \( \{ K_\alpha : \alpha \in I \} \) is a collection of compact subsets of a metric space \( X \) such that the intersection of every finite subcollection of \( \{ K_\alpha : \alpha \in I \} \) is non-empty then \( \bigcap_{\alpha \in I} K_\alpha \) is nonempty.

**Corollary**
If \( \{ K_n : n \in \mathbb{N} \} \) is a sequence of nonempty compact sets such that \( K_n \supseteq K_{n+1} \) (for \( n = 1, 2, 3, \ldots \)) then \( \bigcap_{1}^{\infty} K_n \) is not empty.
Theorem

Every infinite subset of a compact set $K$ has a limit point in $K$. 
Intersection of $k$-cells

**Theorem**
If $\{I_n : n \in \mathbb{N}\}$ is a sequence of nonempty intervals in $\mathbb{R}^1$ such that $I_n \supseteq I_{n+1}$ (for $n = 1, 2, 3, \ldots$) then $\bigcap_{1}^{\infty} I_n$ is not empty.

**Theorem**
Let $k$ be a positive integer. If $\{I_n : n \in \mathbb{N}\}$ is a sequence of nonempty $k$-cells such that $I_n \supseteq I_{n+1}$ (for $n = 1, 2, 3, \ldots$) then $\bigcap_{1}^{\infty} I_n$ is not empty.
$k$-Cells are Compact

**Theorem**

*Every $k$-cell is compact.*
Closed and Bounded Subsets of $\mathbb{R}^k$

**Theorem**

If $E \subseteq \mathbb{R}^k$ then the following are equivalent:

(a) $E$ is closed and bounded.

(b) $E$ is compact.

(c) Every infinite subset of $E$ has a limit point in $E$.

**Corollary (Weierstrass)**

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$. 
Define two subsets $A, B$ of a metric space $X$ are said to be separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty. I.e. if no point of $A$ lies in the closure of $B$ and no point of $B$ lies in the closure of $A$.

A set $E \subseteq X$ is said to be connected if $E$ is not the union of two nonempty separated sets.

Note that while any two separated sets are disjoint, not all disjoint sets are separate.

Consider $[0, 1]$ and $(1, 2)$. $[0, 1] \cap (1, 2) = \emptyset$ but $1 \in [0, 1]$ and 1 is a limit point of $(1, 2)$.
Connected Subsets of $\mathbb{R}^1$

**Theorem**

A subset $E$ of the real line $\mathbb{R}^1$ is connected if and only if it has the following property: If $x \in E$, $y \in E$ and $x < z < y$ then $z \in E$. 