0.1 Hermitian Forms

Now that we have defined what a Euclidean space is we want to consider the analog for complex vector spaces. Specifically we want to look at what the dot product should be if our space is a complex vector space instead of a real one.

Well if we have \( \langle x_1, \ldots, x_n \rangle \in \mathbb{C}^n \) then its length as an element of \( \mathbb{R}^{2n} \) is

\[
\sqrt{(a_1^2 + b_1^2) + \cdots + (a_n^2 + b_n^2)} = \sqrt{x_1 \bar{x}_1 + \cdots + x_n \bar{x}_n}
\]

Where \( \bar{x} \) is the complex conjugation of \( x \).

**Standard Hermitian Product**

**Definition 0.1.0.1.** This suggests that the right gener-
alization for dot product on complex vector spaces is the Standard Hermitian product

\[ \langle X, Y \rangle = \overline{X}^t Y = \overline{x_1}y_1 + \cdots + \overline{x_n}y_n \]

**Lemma 0.1.0.2.** Now this product has the following nice properties

- \( \langle X, Y \rangle \) agrees with the dot product on the reals
- \( (\forall X \neq 0) \langle X, X \rangle \) is a positive real.

(\text{Linearity in the second variable}) \( \langle X, cY \rangle = c \langle X, Y \rangle \) and \( \langle X, Y_1 + Y_2 \rangle = \langle X, Y_1 \rangle + \langle X, Y_2 \rangle \)

(\text{Conjugate Linearity in the first variable}) \( \langle cX, Y \rangle = \overline{c} \langle X, Y \rangle \) and \( \langle X_1 + X_2, Y \rangle = \langle X_1, Y \rangle + \langle X_2, Y \rangle \)

(\text{Hermitian Symmetry}) \( \langle X, Y \rangle = \overline{\langle Y, X \rangle} \)

**Proof.** Immediate. \( \square \)

Now we can define the generalization of a bilinear form to complex vector spaces. **Hermitian Form**
**Definition 0.1.0.3.** We define a Hermitian Form on a complex vector space $V$ to be a function

$$\langle \cdot, \cdot \rangle : V \times V \to V$$

such that $\langle \cdot, \cdot \rangle$ satisfies

- Linearity in the second variable
- Conjugate linearity in the first variable
- Hermitian symmetry

Just as in the real case we can define **Matrix Associated to**

**Definition 0.1.0.4.** Let $\langle \cdot, \cdot \rangle$ be a Hermitian form on a complex finite dimensional vector space $V$. Further let $B = \{v_1, \cdots, v_n\}$ be a basis for $V$. We then define the matrix $A$ associated to the form $\langle \cdot, \cdot \rangle$ as

$$A = (a_{ij}) \text{ where } a_{ij} = \langle v_i, v_j \rangle$$

Just as in the case of bilinear forms we have
Lemma 0.1.0.5. Let $\langle , \rangle$ be a Hermitian form on a complex finite dimensional vector space $V$. Further let $B = \{v_1, \cdots, v_n\}$ be a basis for $V$ and let $v, w \in V$ be such that $v = BX$ and $w = BY$ then

$$\langle v, w \rangle = X^t A Y$$

Proof. This is exactly the same as in the case of bilinear forms.

There is one difference though between the Hermitian form and the bilinear form case. While in a bilinear form any matrix gave rise to a bilinear form, in the case of a Hermitian form we know that

$$a_{ij} = \langle v_i, v_j \rangle = \overline{\langle v_j, v_i \rangle} = \overline{a_{ji}}$$

which leads us to the following definition

**Adjoint of a matrix**

Definition 0.1.0.6. Let $A$ be a complex matrix then
we define the adjoint of $A$ to be 

$$A^* = A^t$$

**Lemma 0.1.0.7.** We then have the following properties of adjoints

$$(A + B)^* = A^* + B^*$$

$$(AB)^* = B^*A^*$$

$$(A^*)^{-1} = (A^{-1})^*$$

$$A^{**} = A$$

**Proof.** Immediate

**Definition 0.1.0.8.** A matrix is **self-adjoint** or **Hermitian** if

$$A^* = A$$
Theorem 0.1.0.9. Let $\langle , \rangle$ be a Hermitian form on a complex finite dimensional vector space $V$. Let $A$ be the matrix associated to $\langle , \rangle$ relative to a basis. Then $A$ is a Hermitian Matrix and
\[
\langle X, Y \rangle = X^* A Y
\]

Further, if $A$ is Hermitian then $X^* A Y$ is a Hermitian form

Proof. The proof is essentially identical to the one for real symmetric bilinear forms. The only thing which is left to check is Hermitian symmetry and I will leave it to you to check that. $\square$

Lemma 0.1.0.10. The real hermitian matrixes are the real symmetric matrixes

Proof. Immediate. $\square$
0.2  Change of Base

**Change of Base for Hermitian Form**

**Theorem 0.2.0.11.** Let $A$ be the matrix associated to a Hermitian form with respect to a basis $B$. Then the matrixes which represent the same form with respect to different basis are those of the form

$$A' = QAQ^*$$

for some invertible matrix $Q \in GL_n(\mathbb{C})$

**Proof.** Let $P \in GL_n(\mathbb{C})$ be the matrix from $B$ to $B'$. Let $X, Y$ represent $v, w$ with respect to $B$ and $X', Y'$ represent $v, w$ with respect to $B'$. Further let $A'$ be the matrix associated with $\langle , \rangle$ relative to the basis $B'$. Then we have

$$PX = X'$$
$$PY = Y'$$
and

\[ X^*AY = \langle v, w \rangle = (X')^*A'Y' = (PX)^*A'(PY) = X^*(P^*A'P)Y \]

and so we have \( A = P^*A'P \) or \( A' = QAQ^* \) where \( Q = (P^*)^{-1} \]

Unitary Matrices

Definition 0.2.0.12. For Hermitian forms, the analog of orthogonal matrixes are called Unitary Matrixes. A matrix is unitary if

\[ P^*P = I \text{ or } P^* = P^{-1} \]

For example

\[ \frac{1}{\sqrt{2}} \begin{bmatrix} c & 1 \\ 1 & -i \end{bmatrix} \]

is unitary.

Lemma 0.2.0.13. The unitary matrixes for a group

\[ U_n = \{ P | P^*P = I \} \]
Proof. Immediate \qed

Change of base and Standard Hermitian Product

Corollary 0.2.0.14. A change of base preserves the standard hermitian product (i.e. \( X^*Y = X'^*Y'^* \)) if and only if the change of base matrix is unitary.

Proof. Immediate \qed

0.3 Carry over from bilinear forms

Orthogonal, Positive Definite definitions

Definition 0.3.0.15. Let \( \langle \cdot, \cdot \rangle \) be a Hermitian form. We can say that two vectors \( v, w \) are orthogonal if and only if

\[ \langle v, w \rangle = 0 \]
Similarly we can define $\langle ., \rangle$ to be positive definite if and only if
\[
\langle v, v \rangle \text{ is a positive real number if } v \neq 0
\]

\[\textbf{Hermitian Space}\]

**Definition 0.3.0.16.** We say a complex vector space $V$ with a positive definite hermitian form is a **Hermitian Space**

\[\textbf{Carry over from bilinear forms.}\]

**Theorem 0.3.0.17.** We then have Sylvester's theorem carries over to the case of hermitian forms.

A Hermitian form has a orthonormal basis if and only if it is positive definite

If $W \subseteq V$ and $\langle , \rangle$ restricted to $W$ is non-degenerate then
\[
V = W \oplus W^\perp
\]
Proof. This proof is identical to the one for real vector spaces.

\[ \square \]

0.4 Spectral Theorem

Notice that now we have two seemingly different interpretation of an \( n \times n \) matrix. Such a matrix can be associated to

- A bilinear form \( \langle \cdot, \cdot \rangle : V \times V \rightarrow F \)
- A linear transformation \( T : V \rightarrow V \).

Now we are going to combine these two uses of matrixes.

0.4.1 Theorems

Recall from last semester

**Linear Operator change of base**

**Theorem 0.4.1.1.** Let \( T : V \rightarrow V \) be a linear operator on \( V \) and let \( M \) be the matrix associated to it with respect to a basis \( B = (v_1, \ldots, v_n) \).

Now let \( P \in GL_n(F) \) be the matrix associated with a change of base from \( B \)
to $B' = (w_1, \ldots, w_n)$. Then the matrix associated to $T$ with respect to the basis $B'$ is

$$ M' = PMP^{-1} $$

Proof. Let $v = \sum a_i v_i = \sum b_i w_i$. So in particular we know that

1. $Tv$ expressed with basis $B'$

$$ = M'(v \text{ expressed with basis } B') $$

$$ = P \circ M(v \text{ expressed with basis } B) $$

$$ = P \circ MP^{-1}(v \text{ expressed with basis } B') $$

2. $\langle v, w \rangle$

Theorem 0.4.1.2. Let $T : V \to V$ be a linear operator on a hermitian space $V$. Further let $M$ be the matrix associated to $T$ with respect to an orthonormal basis. Then

(a) The matrix $M$ is hermitian if and only if $\langle v, Tw \rangle = \langle Tv, w \rangle$ for all $v, w \in V$. In this case we say $T$ is a Hermitian Operator.

(b) The matrix $M$ is unitary if and only if $\langle v, w \rangle = \langle Tv, Tw \rangle$ for all $v, w \in V$. In this case we say $T$ is a Unitary Operator.

Proof. Let $X, Y$ be the coordinate vectors so that $v = BX, w = BY$ and hence $\langle v, w \rangle = X^*Y$ and $Tv = BMX$. 

Hermitian Operators, Unitary Operators

Hermitian Operators, Unitary Operators
Part (a):

We have \( \langle v, Tw \rangle = X^*(MY) \) and \( \langle Tv, w \rangle = (MX)^*Y = X^*M^*Y \).

In particular we therefore have \((\forall v, w \in V)\langle v, Tw \rangle = \langle Tv, w \rangle\) if and only if \(M = M^*\) or \(M\) is hermitian. Part (b):

Similarly we have \(\langle Tv, Tw \rangle = (MX)^*(MY) = X^*(M^*M)Y\)

So, in particular we therefore have \((\forall v, w \in V)\langle Tv, Tw \rangle = \langle v, w \rangle\) if and only if \(M^*M = I\), or \(M\) is unitary. \(\square\)

Spectral Theorem

**Theorem 0.4.1.3** (Spectral Theorem).  

(a) Let \(T\) be a hermitian operator on a hermitian vector space \(V\). Then there is an orthonormal basis for \(V\) consisting of eigenvectors of \(T\).

(b) Matrix form Let \(M\) be a hermitian matrix. There is a unitary matrix \(P\) such that \(PMP^*\) is a real diagonal matrix.

**Proof.** WE WILL PROVE THIS NEXT TIME. \(\square\)

0.5 TODO

- Go through Lang’s book on the same topics.