ON TRANSFERRING MODEL THEORETIC THEOREMS OF $\mathcal{L}_{\infty,\omega}$ IN THE CATEGORY OF SETS TO A FIXED GROTHENDIECK TOPOS

NATHANAEL LEEDOM ACKERMAN

Abstract. Working in a fixed Grothendieck topos $\mathrm{Sh}(C, J_C)$ we generalize $\mathcal{L}_{\infty,\omega}$ to allow our languages and formulas to make explicit reference to $\mathrm{Sh}(C, J_C)$. We likewise generalize the notion of model. We then show how to encode these generalized structures by models of a related sentence of $\mathcal{L}_{\infty,\omega}$ in the category of sets and functions. Using this encoding we prove analogs of several results concerning $\mathcal{L}_{\infty,\omega}$, such as the downward Löwenheim-Skolem theorem, the completeness theorem and Barwise compactness.

1. Introduction

A remarkable fact about $\mathcal{L}_{\omega_1,\omega}(\mathcal{L})$ is that several important theorems which are true of first order model theory have analogs for the model theory of countable fragments of $\mathcal{L}_{\omega_1,\omega}(\mathcal{L})$. Examples of such theorems include (what we call) the directed embedding theorem, i.e. if all maps in a directed system preserves a fragment of formulas then so do the maps in the limit, the downward Löwenheim-Skolem theorem, the completeness theorem and Barwise's compactness theorem.

One of the most significant discoveries of categorical logic is that the operations of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$ can be described categorically. This observation allows us to study models of sentences of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$ in categories other than the category of sets and functions. One class of categories which are especially well suited to interpret sentences of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$ are Grothendieck toposes. However, while it make sense to study the model theory of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$ in a Grothendieck topos, this model theory can behave very differently than model theory in the category of sets and functions. For example, in general it will be intuitionistic and need not satisfy the law of excluded middle.

A natural question to ask is: "If we fix a Grothendieck topos, which results about the model theory of $\mathcal{L}_{\omega_1,\omega}(\mathcal{L})$ in the category of sets and functions have analogs for the model theory of $\mathcal{L}_{\omega_1,\omega}(\mathfrak{L})$ in our fixed Grothendieck topos?" In this paper we provide a partial answer to this question by proving analogs of each of the theorems mentioned so far. Further, as we are fixing the Grothendieck topos in which we are working, we will be able to prove our theorems for a wider class of formulas and sentences than just $\mathcal{L}_{\omega_1,\omega}(\mathfrak{L})$. Specifically we will be able to prove our analogs for formulas and sentences which makes explicit use of our chosen Grothendieck topos in their definitions. We call the more general formulas "sheaf formulas", the more general sentences "sheaf sentences" and we call the resulting structures "sheaf

1991 *Mathematics Subject Classification.* 03G30, 18C99, 03C90, 03C75, 03C30.

Key words and phrases. Model Theory, Infinitary Logic, Grothendieck Topos, Löwenheim-Skolem , Completeness, Barwise Compactness.

models". As we will see in Section 2.3 our concept of a sheaf formula will not only subsume the notion of a formula of $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$, but will also subsume Kripke-Joyal semantics (for models in a Grothendieck topos).

Our collection of sheaf formulas will generalize $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ in two ways. First, as we have fixed the category our models will live it, our sheaf languages will be able to make explicit reference to objects in this category. For example, our language will be allowed to have *generalized constants*, i.e. constants which are interpreted by generalized elements in any $\mathfrak{L}$-structure.

The second way in which sheaf formulas will be more general than formulas of $\mathcal{L}_{\infty,\omega}(\mathfrak{L})$ has to do with connectives. We can then think of the subobject classifier, Ω, of our fixed Grothendieck topos as a sheaf of truth values. An interpretation of a sheaf formula (in an $\mathfrak{L}$ structure) will then take one of two forms. Either it will be a map between (the interpretation of) two sorts or it will be a map from (the interpretation of) a sort to Ω. We will then be allowed to build new sheaf formulas of the later type from a finite number of other sheaf formulas of the later type using a *connective*, i.e. a map from Ω^n to Ω. This is done in a similar manner to how in continuous logic (see [5]) we consider $[0,1]$ as a metric space of truth values and connectives are maps between $[0,1]^n$ and $[0,1]$.

We will prove the analogs of the previously mentioned theorems of $\mathcal{L}_{\omega_1,\omega}(\mathfrak{L})$ by first constructing an encoding of sheaf $\mathfrak{L}$-structures by models in *Set* of a theory in $\mathcal{L}_{\infty,\omega}(\mathcal{L})$ (where $\mathcal{L}$ is constructed from $\mathfrak{L}$). We will then define an interpretation of sheaf formulas and sheaf sentences by sentences of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$ in such a way that a sheaf model satisfies a sheaf sentence if and only if the corresponding encoded $\mathcal{L}$-structure satisfies the corresponding encoded sentence of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$. With this encoding in hand we will proceed to show how various theorems about $\mathcal{L}_{\infty,\omega}(\mathcal{L})$ can be translated into theorems about sheaf $\mathfrak{L}$-structures, sheaf formulas and sheaf sentences.

One of the main difficulties which we will have to over come in order to define our encodings is the non-first order nature of sheaves. We will avoid this difficulty by working in a category, $\mathrm{Sh}^*(C, J_C)$, which is an absolute version of the category of sheaves on the (weak) site (C, J_C). As we will see, the class of objects and the class of morphisms in $\mathrm{Sh}^*(C, J_C)$ can be defined by sentences of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$. There is however one subtlety in dealing with $\mathrm{Sh}^*(C, J_C)$ that is worth mentioning. While the morphisms can be described by a sentence of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$, that sentence will need to have access to a linear order of order type $|J_C|^+$. For some of our results this poses a problem as on its face it will prohibit us from directly using results which hold for $\mathcal{L}_{\omega_1,\omega}(\mathcal{L})$ but not for $\mathcal{L}_{\omega_2,\omega}(\mathcal{L})$ (in the category of sets). As such it will be important to keep track of the order type of this linear ordering and showing that in many cases it suffices for this linear ordering to be countable.

1.1. **Outline.** We begin this paper in Section 2.1 with a review of the relationship between weak sites and sites as well as a review of the definition of $\mathrm{Sh}^*(C, J_C)$. In particular we show how sites can be constructed form weak sites and how this construction can be mirrored in the construction of the sheafification of a separated presheaf by iterating a particular functor associated to a weak site. We also introduce the important notion of the closure of a subpresheaf in another presheaf. In Section 2.2 we then introduce the notions of sheaf languages and sheaf models before, in Section 2.3, introducing the notions of sheaf formulas and sheaf sentences.

In Section 3 we give our encodings. These encodings will be constructed by means of what we call *components*. A component consists of a language and a Π_2-theory in that language. In Section 3.1 we give the basic components, i.e. those from which the other components will be built. In Section 3.2 we then, using the basic components, define components which encode the elements of sheaf languages and sheaf models. Finally, in Section 3.3 we define the components which capture the notion of when a formula is equivalent to a function/relation in our sheaf language and the components which allow us to build arbitrary sheaf sentences from simpler ones. With these in hand we prove the desired relationship between encoded formulas, sentences and encoded models.

Once we have finished defining our encodings we will prove, in Section 4, our main results. In Section 4.1 we will show that sheaf formulas and sheaf sentences are preserved under directed limits, in Section 4.2 we will prove a downward Löwenheim-Skolem theorem, in Secion 4.3 we will prove a completeness theorem and in Section 4.4 we will prove an analog of Barwise's compactness theorem.

We then end in Section 5 with a list of open questions.

1.2. **Background.** In this paper we will use Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC) as our ambient theory and we will assume all results take place in a fixed model of ZFC which we refer to as Set. We also abuse notation and use Set for the category of sets and functions in this ambient model of ZFC as well. By an admissible set relative to a language $\langle \in, \cdots \rangle$ we mean a transitive set V such that $(V, \in, \dots)$ is a model of Kripke-Platek set theory (KP) relative to a $\langle \in, \dots \rangle$. If φ is in the language of set theory and V is an admissible set then by φ^V we mean the formula where all quantifiers in φ are bound by V. When $V_0, V_1 \vDash \text{KP}$, we will also use $V_0 \prec_n V_1$ to signify that V_0 is an Σ_n elementary substructure of V_1. If X is a set we denote its transitive closure by $tc(X)$.

If C is a category we will denote its collection of objects by $\text{obj}(C)$, its collection of morphisms by $\text{mor}(C)$, the collection of morphisms from c to d by $C[c, d]$ and the collection of morphisms with codomain d by $C[-, d]$. We will assume all categories have distinguished finite limits. For $c \in \text{obj}(C)$, $!_c : c \to 1$ will be the unique map from c to the terminal object. All categories will be locally small.

If C is a small category we let $\text{Presh}(C)$ be the category of presheaves on C. If A is a presheaf on C we let $x \in A$ be shorthand for the statement $x \in \bigcup_{c \in \text{obj}(C)} A(c)$ and we let $\text{dom}(x) \in \text{obj}(C)$ be such that $x \in A(\text{dom}(x))$. We will also assume for every presheaf A that if $c, d \in \text{obj}(C)$ with $c \neq d$ then $A(c) \cap A(d) = \varnothing$. We loose no generality by this assumption, but it will simplify the presentation. If A, B are presheaves we let $A \subseteq B$ mean the obvious thing and if $f : A \to B$ is a map of presheaves we let $\text{ran}(f) = \{y \in B : (\exists x \in A) f(x) = y\}$. We let $|A| \coloneqq |\bigcup_{c \in C} A(c)|$

If (C, J_C) is a site let $\text{Sep}(C, J_C)$ be the category of separated presheaves on (C, J_C) and $\text{Sheaf}(C, J_C)$ be the category of sheaves on (C, J_C). We let $\mathbf{i} : \text{Sheaf}(C, J_C) \to \text{Sep}(C, J_C)$ be the inclusion map and $\mathbf{a} : \text{Sheaf}(C, J_C) \to \text{Sep}(C, J_C)$ be the sheafification functor. Whenever $A, B \in \text{obj}(\text{Sep}(C, J_C))$ with $A \subseteq B$, we will assume $\mathbf{a}(A) \subseteq \mathbf{a}(B)$. While we may not be able to do this simultaneously for all objects of $\text{Sep}(C, J_C)$, for any set of separated presheaves we can choose a specific sheafification which makes this true. As such there is no loss of generality in this assumption, and we make it as it will greatly simplify the presentation.

By a **first order language** we mean a language with sorts, relations on the sorts, and functions between the sorts which are intended to be interpreted in Set.

We will assume the collection of sorts of a first order language is closed under taking finite strings (where $\langle S_1, \dots, S_n\rangle$ will be interpreted as the product of the interpretations of $S_1, \dots, S_n$). If $\mathcal{L}_0 \subseteq \mathcal{L}_1$ are first order languages and $\mathcal{M}$ is an $\mathcal{L}_1$ structure, $\mathcal{M}|_{\mathcal{L}_0}$ is the structure obtained from $\mathcal{M}$ by restricting to the language $\mathcal{L}_0$.

If $T \in \mathcal{L}_{\infty,\omega}(\mathcal{L})$ is a sentence then we let $\vdash T$ denote the statement that there exists an (infinitary) proof of T. We define the **complexity** of a formula $\varphi \in \mathcal{L}_{\infty,\omega}(\mathcal{L})$ as the least κ such that $\varphi \in \mathcal{L}_{\kappa^+,\omega}(\mathcal{L})$.

Suppose $\mathcal{L}$ is a first order language with $S \in \mathcal{L}$ a sort and $E \in \mathcal{L}$ a relation of type S. Further suppose $\varphi, \psi \in \mathcal{L}_{\infty,\omega}(\mathcal{L})$ with $\psi(\cdot)$ a formula with a free variable of type S and which doesn't have $E(\cdot)$ as a subformula. We then we denote the result of replacing all occurrences $E(\cdot)$ in φ by $\psi(\cdot)$ by $\varphi[\psi(\cdot)/E(\cdot)]$. Similarly if $\mathcal{L}_0, \mathcal{L}_1$ are copies of the same language and $\mathcal{L}_0 \subseteq \mathcal{L}^*$, then we denote the language obtained by replacing $\mathcal{L}_0$ by $\mathcal{L}_1$ in $\mathcal{L}^*$ by $\mathcal{L}^*[\mathcal{L}_1/\mathcal{L}_0]$.

For more information on background definitions or results not mentioned here the reader is referred to such standard texts as [4] of [7] for set theory, [9] for category theory, [10] for the theory of sheaves, and [6] or [8] for model theory.

2. Sheaf Models

In this section we will introduce our notion of a *sheaf language*, a *sheaf model*, a *sheaf formula* and a *sheaf sentence*. These are the analogs of first order languages, models in *Set*, and formulas and sentences in $\mathcal{L}_{\infty,\omega}(\mathcal{L})$, except that they will take into account the fact that we are working in a fixed background Grothendieck topos.

2.1. Grothendieck Toposes.

Before we begin it is worth recalling some definitions which will be important later.

Definition 2.1. *A* **weak site** *is a pair (C, J_C) where C is a small category and J_C is a function which takes an object c of C and returns a collection of sieves on c such that:*

- *(Identity) $C[-, c] \in J_C(c)$.*
- *(Base Change) If $I \in J_C(c)$ and $f : d \to c$ then $f^*I := \{g : f \circ g \in I\} \in J_C(d)$.*

We call $J_C(c)$ the **covering sieves** *of c and we let $|J_C| = |\bigcup_{c \in obj(C)} J_C(c)|$ be the* **size** *of J_C.*

For the rest of the paper we will work with a fixed weak site (C, J_C).

Definition 2.2. *A* **site** *is a weak site (C, J_C) satisfying*

- *(Local Character) Whenever $I \in J_C(c)$ and K is any sieve on c, if we have $(\forall d \in obj(C))(\forall f \in I(d))\ f^*K \in J_C(d)$ then $K \in J_C(c)$.*

The relationship between a weak site and a site is similar to the relationship between a basis for a topological space and a topological space. Being a basis for a topological space is an absolute property while being a topological space is not absolute. Similarly, being a weak site is an absolute property while being a site is not absolute. Further, just as for each basis there is a smallest topological space which contains it, for each weak site (C, J_C) there is a smallest site which contains it. The smallest site containing (C, J_C) can be built as the fixed point of an inductive definition.

Definition 2.3. *If (C, J_C) is a weak site then for each $c \in obj(C)$ we define the collection of sieves $J_C^\alpha(c)$ by induction on α:*

- $J_C^1(c) = \{I : (\exists I' \in J_C(c)) I' \subseteq I\}$.
- $J_C^{\omega\cdot\gamma}(c) = \bigcup_{\beta<\omega\cdot\gamma} J_C^\beta(c)$.
- $J_C^{\alpha+1}(c) = \{I : (\exists I' \in J_C(c))(\forall f \in I') f^* I \in J_C^\alpha(dom(f))\}$.

As J_C^α is non-decreasing in α, there is some ordinal such that $J_C^\alpha = J_C^{\alpha+1}$. We let J_C^{ORD} be such a J_C^α.

It is easily checked that (C, J_C^{ORD}) is the smallest site containing (C, J_C) (see [2] for details). If $I \in J_C^{\mathrm{ORD}}(c)$ then we define the **level** of I to be the least α such that $I \in J_C^\alpha(c)$. This fine grained analysis of the construction of the smallest site containing a weak site is important as it will allow us to give a fine grained analysis of the sheafification functor.

Definition 2.4. *Let $F : C^{op} \to Set$ be a presheaf on C. If $c \in obj(C)$ and $I \in J_C(c)$, a* **compatible collection of elements** *(for I) is a collection $\langle (b_i, i) : i \in I\rangle$ such that*

- *For each $d \in obj(C)$, $(\forall i \in I(d))$ $b_i \in F(d)$.*
- *For each $d, d' \in obj(C)$, $(\forall i \in I(d))(\forall i' \in C[d', d]) b_{i\circ i'} = F(i')(b_i)$*

If there is an $b \in F(d)$ such that $F(i)(b) = b_i$ for all $i \in I$ then we say $\langle (b_i, i) : i \in I\rangle$ **covers** *b.*

Definition 2.5. *Let $F : C^{op} \to Set$ be a presheaf. We say F is* **separated** *for (C, J_C) if every compatible collection of elements of F covers at most one element of F. We say F is a* **sheaf** *for (C, J_C) if every compatible collection of elements covers exactly one element of F. We let Sep(C, J_C) be the full subcategory of Presh(C) whose objects are separated presheaves for (C, J_C) and Sheaf(C, J_C) be the full subcategory of Presh(C) whose objects are sheaves for (C, J_C).*

It is not hard to show (see [2]) that $\mathrm{Sep}(C, J_C) = \mathrm{Sep}(C, J_C^{\mathrm{ORD}})$ and that $\mathrm{Sheaf}(C, J_C) = \mathrm{Sheaf}(C, J_C^{\mathrm{ORD}})$. We have chosen to deal with weak sites instead of sites as there are weak sites which are of size κ but for which the minimal site containing them, in any model of set theory, is of size 2^κ. This distinction between sites and weak sites will be important when we want to define the notion of the size of a structure.

If we start with a separated presheaf F for (C, J_C) we can build its sheafification $\mathbf{a}(F)$ in stages that mirror the way in which (C, J_C^{ORD}) was built from (C, J_C).

Definition 2.6. *We define the functors $\mathbf{a}^\alpha : Sep(C, J_C) \to Sep(C, J_C)$ by induction on α as follows:*

- $\mathbf{a}^0 = id$.
- $\mathbf{a}^1(A) = \{b \in \mathbf{a}(A) : (\exists I \in J_C(dom(b)))(\forall f \in I)\ \mathbf{a}(A)(f)(b) \in A(dom(f))\}$.
- $\mathbf{a}^{\omega\cdot\gamma}(A) = \bigcup_{\beta<\omega\cdot\gamma} \mathbf{a}^\beta(A)$.
- $\mathbf{a}^{\alpha+1}(A) = \mathbf{a}^1(\mathbf{a}^\alpha(A))$.

For a map $f : A \to B$ we let $\mathbf{a}^\alpha(f) : \mathbf{a}^\alpha(A) \to \mathbf{a}^\alpha(B)$ be the unique map of presheaves such that $(\forall x \in A) f(x) = \mathbf{a}^\alpha(f)(x)$.

For any separated presheaf A we have $\mathbf{a}^\alpha(A) \subseteq \mathbf{a}^\beta(A)$ whenever $\alpha < \beta$. Further, if $\mathbf{a}^\alpha(A) = \mathbf{a}^{\alpha+1}(A)$ then $\mathbf{a}^\alpha(A) = \mathbf{a}(A)$, although the first α for which this will happen depends on A.

The presheaves $\mathbf{a}^\alpha(A)$ can be thought of as building $\mathbf{a}(A)$ by adding, one layer at a time, all compatible collections of elements for all $I \in J_C(c)$, $c \in \text{obj}(C)$. In particular it is easy to check that $b \in \mathbf{a}^\alpha(A)(c)$ if and only if $\{g : \mathbf{a}(A)(g)(b) \in A\} \in J_C^\alpha(c)$.

One of the difficulties with working with categories of sheaves is that the property of being a sheaf is a second order property (and in particular is not absolute). Our first step towards dealing with this issue is to define a stand in for sheafification of subobjects.

Definition 2.7. *Suppose $A \subseteq B$ are separated presheaves for (C, J_C). We define the* **closure** *of A in B to be $\mathbf{a}(A) \cap B$. We say that A is* **closed** *in B if $\mathbf{a}(A) \cap B = A$. A particularly important case will be where $\mathbf{a}(A) \cap B = B$ in which case we say that A* **covers** *B.*

The intuition is that a subpresheaf A is closed in B if whenever a compatible collection of elements of A covers an element in B that element is already in A. Note that $\mathbf{a}(A) \cap B$ captures this notion because by our convention, if $A \subseteq B$ then $\mathbf{a}(A) \subseteq \mathbf{a}(B)$. The following is also immediate.

Lemma 2.8. *If $A \subseteq B$ then A is closed in B if and only if for all $c \in obj(C)$, all $I \in J_C(c)$ and all $b \in B(c)$, $[\bigwedge_{f \in I} B(f)(b) \in A(c)] \to b \in A(c)$. i.e. if and only if $\mathbf{a}^1(A) \cap B = \mathbf{a}^0(A) \cap B = A$.*

We define the **level** of A in B to be the smallest α such that $\mathbf{a}^\alpha(A) \cap B = \mathbf{a}^{\alpha+1} \cap B$. The level of a subpresheaf A in B can be thought the number of times we need to iteratively add in elements of B which come from compatible collections of elements of A before we stabilize. We can think of closure of A in B as a stand in for the sheafification of A which doesn't require us to add compatible collections of elements which we don't already have in front of us (and hence the closure of A in B isn't second order).

Lemma 2.9. *Suppose $B' \subseteq B$ and $A' = A \cap B'$. Then $[\mathbf{a}^\alpha(A) \cap B] \cap B' = \mathbf{a}^\alpha(A') \cap B'$.*

Proof. This is because $[\mathbf{a}^\alpha(A) \cap B] \cap B' = \mathbf{a}^\alpha(A) \cap B' = \mathbf{a}^\alpha(A \cap B') \cap B' = \mathbf{a}^\alpha(A') \cap B'$. □

This tells us that taking the closure of a subpresheaf A in B is a local property. We now give two results which show that, in some sense, the level of A in B can't be too large.

Proposition 2.10. *Suppose V is an admissible set (with respect to some language) with $(C, J_C) \in V$ and suppose $A, B \in obj(Sep(C, J_C))^V$ with $A \subseteq B$. Then*

1. *For every $\alpha \in ORD(V)$, $\mathbf{a}^\alpha(A) \cap B \in V$.*
2. *The function F_V which takes an $\alpha \in ORD(V)$ and returns $\mathbf{a}^\alpha(A) \cap B \in V$ is uniformly Δ_1-definable over V.*
3. *$\mathbf{a}^{ORD(V)}(A) \cap B = \mathbf{a}(A) \cap B$.*

Proof. (1) follows immediately from (2) and the fact that V is admissible. To see that (2) holds define the function function $G(x, Y)$ as follows. If either $x \notin \text{ORD}(V)$ or Y is not a function with domain x then $G(x, Y) := \emptyset$. Otherwise, if $x = \emptyset$, $G(x, Y) := A$, if x is a limit ordinal then $G(x, Y) := \bigcup_{z \in x} Y(z)$ and if $x = \alpha + 1$ then

$$G(x, Y) := \{b \in \bigcup_{c \in \text{obj}(C)} B(c) : (\exists I \in J_C(c))(\forall f \in I)\ B(f)(b) \in Y(\alpha)\}.$$

It is then easily checked that G is Δ_1 definable over V and that F_V is obtained from G using transfinite recursion. Hence as V is admissible F_V is Δ_1 over V.

To show (3) holds it suffices to show that for all $x \in \mathbf{a}^{\mathrm{ORD}(V)+1}(A) \cap B$ that $x \in \mathbf{a}^{\mathrm{ORD}(V)}(A) \cap B$, or equivalently that there is an $\alpha(x) \in \mathrm{ORD}(V)$ such that $x \in \mathbf{a}^{\alpha(x)}(A) \cap B$.

Let $z \in \mathbf{a}^{\mathrm{ORD}(V)+1}(A) \cap B$. We then have there is some $I \in J_C(\mathrm{dom}(z))$ such that for all $g \in I$, $B(g)(z) \in \mathbf{a}^{\mathrm{ORD}(V)}(A) \cap B$. Then, as $\mathbf{a}^{\mathrm{ORD}(V)}(A) \cap B = \bigcup_{\alpha \in \mathrm{ORD}(V)} \mathbf{a}^{\alpha}(A) \cap B$ the following holds:

$$(V, \in) \vDash \bigvee_{I \in J_C(c)} \bigwedge_{g \in I} (\exists \alpha)\ B(g)(x) \in \mathbf{a}^{\alpha}(A)(X) \cap B.$$

This is a Σ_1-formula and hence, as V satisfies Σ_1 reflection, there is a $V^* \in V$ with $(C, J_C) \in V^*$ such that

$$(V^*, \in) \vDash \bigvee_{I \in J_C(c)} \bigwedge_{g \in I} (\exists \alpha)\ B(g)(x) \in \mathbf{a}^{\alpha}(A)(X) \cap B.$$

But then there is an $I \in J_C(X)$ such that $\bigwedge_{g \in I} (\exists \beta < \mathrm{ORD}(V^*)) B(g)(x) \in \mathbf{a}^{\beta}(A) \cap B$. Hence if $\alpha(x) = \mathrm{ORD}(V^*) + 1$ we have $z \in \mathbf{a}^{\alpha(x)}(A) \cap B$ with $\alpha(x) \in \mathrm{ORD}(V)$.

□

Proposition 2.11. *Suppose V is an admissible set (with respect to some language) and $V \vDash$ "There is a Σ_1-definable well-ordering". Further suppose $(C, J_C) \in V$ and $V \vDash |\kappa| > |J_C|$. If $A, B \in obj(Sep(C, J_C)) \cap V$ then $\mathbf{a}^{\kappa}(A) \cap B = \mathbf{a}(A) \cap B$.*

Proof. Let $x \in \mathbf{a}(A) \cap B$ and let $V_0 \prec_1 V$ with $V_0 \in V$ a Σ_1-elementary substructure such that $\{x, A, B, tc(\{C, J_C\})\} \in V_0$ and $V \vDash |V_0| = |J_C|$. Note that we can find such a substructure as V has a Σ_1-definable well-ordering. Now let $i : V_0 \to V_0^*$ be the transitive collapsing map. Then $\{i, V_0^*\} \in V$.

We have by Proposition 2.10 that there is some $\alpha \in \mathrm{ORD}(V)$ such that $x \in \mathbf{a}^{\alpha}(A) \cap B$. Hence there must be some $\alpha^* \in \mathrm{ORD}(V_0^*)$ such that $i(x) \in \mathbf{a}^{\alpha^*}(i(A)) \cap i(B)$. But we also have that the inverse of the transitive collapse gives injections $i_A : i(A) \to A$ and $i_B : i(B) \to B$. Let A' be the image of $i(A)$ under i_A and let B' be the image of $i(B)$ under i_B. We then have $x \in \mathbf{a}^{\alpha^*}(A') \cap B'$. By Lemma 2.9 we also have $\mathbf{a}^{\alpha^*}(A') \cap B' = [\mathbf{a}^{\alpha^*}(A) \cap B] \cap B'$ and so $x \in \mathbf{a}^{\alpha^*}(A) \cap B$.

But by construction we have $\alpha^* \in V_0^*$ and $V \vDash |V_0^*| = |J_C|$ and so $\alpha^* < \kappa$. Hence for all $x \in \mathbf{a}(A) \cap B$ there is some $\alpha(x) < \kappa$ with $x \in \mathbf{a}^{\alpha(x)}(A) \cap B$. But this then implies $\mathbf{a}^{\kappa}(A) \cap B = \mathbf{a}^{\kappa+1}(A) \cap B$ and so $\mathbf{a}^{\kappa}(A) \cap B = \mathbf{a}(A) \cap B$.

□

Proposition 2.10 and Proposition 2.11 give us a sense of how many times we need to repeatedly apply $\mathbf{a}^1$ before things stabilize. These will be important when we want to encode the notion of the closure of a subpresheaf in another presheaf, and in particular these lemmas will provide limits on how complex that encoding needs to be.

We now turn to the definition of the category in which we will work. As being a sheaf is a second order property, we will want to use a category which is equivalent to Sheaf(C, J_C), but where the objects and morphisms can be described in a first order manner.

Definition 2.12. *Let $Sh^*(C, J_C)$ be such that:*

(a) *The objects of* $Sh^*(C, J_C)$ *are the separated presheaves for* (C, J_C).
(b) *The morphisms of* $Sh^*(C, J_C)[D, R]$ *are the pairs* $\langle f, d\rangle$ *where:*
   – $d \subseteq D$ *and* d *covers* D.
   – $f : d \Rightarrow R$ *is a map of separated presheaves.*
(c) $\langle g, d_*\rangle \circ \langle f, d\rangle = \langle g \circ f, f^{-1}(d_*)\rangle$ *when* $\langle f, d\rangle : D \to D_*$, *and* $\langle g, d_*\rangle : D_* \to R$.
(d) *If* $X \in obj(Sh^*(C, J_C))$ *then the identity on* X *is* $id_X = \langle X, X, id_X\rangle$.

It is worth pointing out that while we will treat $\mathrm{Sh}^*(C, J_C)$ as if it was a category, composition may not be associative and so $\mathrm{Sh}^*(C, J_C)$ may not be a category. However, there is a category closely related to $\mathrm{Sh}^*(C, J_C)$ which is in fact equivalent to $\mathrm{Sheaf}(C, J_C)$.

Definition 2.13. *For all* $\langle f, d_f\rangle, \langle g, d_g\rangle \in Sh^*(C, J_C)[D, R]$ *we say* $\langle f, d_f\rangle$ *is* **equivalent** *to* $\langle g, d_g\rangle$, *which we write as* $\langle f, d_f\rangle \equiv \langle g, d_g\rangle$, *if* $(\forall x \in d_f \cap d_g) f(x) = g(x)$. *We define* $Sh(C, J_C) := Sh^*(C, J_C)/\equiv$ *and we let* $\overline{q} : Sh^*(C, J_C) \to Sh(C, J_C)$ *be the quotient map.*

In what follows we will often prefer to work with $\mathrm{Sh}^*(C, J_C)$ instead of $\mathrm{Sh}(C, J_C)$ as it will allow us to avoid having to use equivalence classes of morphisms. As such we will abuse notation and refer a structure in $\mathrm{Sh}^*(C, J_C)$ as having a property when its image under $\overline{q}$ has that property in $\mathrm{Sh}(C, J_C)$. For example, we define a product in $\mathrm{Sh}^*(C, J_C)$ to be a diagram whose image under $\overline{q}$ is a product in $\mathrm{Sh}(C, J_C)$.

We say a map $\langle f, d\rangle : D \to R$ is **total** if $d = D$, i.e. if f is actually a map of presheaves between D and R. Note that it is not the case that every map is equivalent to a total one. However, there is an inclusion of categories $\iota : \mathrm{Sep}(C, J_C) \to \mathrm{Sh}^*(C, J_C)$ where $\iota(A) = A$ for all separated presheaves and $\iota(f) = \langle f, D\rangle$ for all map $f \in \mathrm{Sep}(C, J_C)[D, R]$. Notice that a map is total if and only if it is in the image of ι.

Lemma 2.14. *There is an equivalence of categories:* $\mathbf{j} : Sh(C, J_C) \to Sheaf(C, J_C)$ *where* $\mathbf{j} \circ \overline{q} \circ \iota = \mathbf{a}$.

Proof. See [2]. □

As $\mathbf{j} \circ \overline{q} \circ \iota = \mathbf{a}$ and as sheafification preserves finite limits, we can assume without loss of generality that in $\mathrm{Sh}^*(C, J_C)$ the limit of any finite diagram consisting of total maps also consists of total maps. In particular we can assume that the distinguished product of any finite collection of objects consists of total maps.

Now that we have our category $\mathrm{Sh}(C, J_C)$ we can make precise the sense in which the closed subpresheaves represent the subobjects of the separated presheaf (in $\mathrm{Sh}(C, J_C)$).

Lemma 2.15. *Suppose* $B \in obj(Sh^*(C, J_C))$, $A \subseteq B$ *and* X *is the subobject of* B *in* $Sh^*(C, J_C)$ *containing* $\langle in_A, A\rangle : A \to B$. *Then the following are equivalent:*

(1) A *is closed in* B.
(2) *If* $\langle f, A'\rangle \in X$ *then* $f : A' \to B$ *factors through* in_A *(as a map of separated presheaves).*

Further every subobject contains a (necessarily) unique map of the form $\langle in_A, A\rangle$ *with* A *closed in* B.

Proof. First notice that since any such f in (2) must be a monic (in the category of separated presheaves) we can assume without loss of generality that f is actually an inclusion and that $A' \subseteq B$.

(1) implies (2): Suppose A is closed in B. Then, as $\langle in_{A'}, A'\rangle$ and $\langle in_A, A\rangle$ are in the same subobject of $\mathrm{Sh}^*(C, J_C)$, we must have $\mathbf{a}(A') = \mathbf{a}(A) \subseteq \mathbf{a}(B)$. Hence as A is closed, we have $A' \subseteq \mathbf{a}(A') \cap B = \mathbf{a}(A) \cap B = A$.

$\neg$ (1) implies $\neg$ (2): Let $A' = \mathbf{a}(A) \cap B$ and hence $A \subsetneq A'$. Then $\mathbf{j} \circ \overline{q}(A) = \mathbf{a}(A) = \mathbf{a}(A') = \mathbf{j} \circ \overline{q}(A')$ and so $\langle in_A, A\rangle$ and $\langle in_{A'}, A'\rangle$ are in the same subobject of B. But $\langle in_{A'}, A'\rangle$ does not factor through $\langle in_A, A\rangle$ and so (2) does not hold. □

We will end this subsection with a discussion of what we mean when we say an object of $\mathrm{Sh}^*(C, J_C)$ has size κ. It turns out that there are several different notions of what it means to be of size κ. We consider two of these notions here. These two notions, along with two others relating to the natural number object, are studied in [3] and we refer the interested reader to [3] for a more thorough discussion of the virtues and problems surrounding each of these notions of size.

Definition 2.16. *We say that $A \in obj(Sh^*(C, J_C))$ is of* **pure size** κ *if* $|\mathbf{j} \circ \overline{q}(A)| = |\mathbf{a}(A)| = \kappa$ *(i.e. the sheafification of A has size κ, as a presheaf).*

From a set theoretic point of view the notion of the pure size of a sheaf is a natural notion. One major drawback though of using pure size is that there are separated presheaves which have size κ but whose sheafification has pure size 2^κ (in any model of set theory).

Definition 2.17. *We say that $A \in obj(Sh^*(C, J_C))$ is* κ**-generated** *if there is an $A^* \in obj(Sh^*(C, J_C))$ such that $\langle A^*, A, in\rangle \equiv \langle A, A, id\rangle$ in $Sh^*(C, J_C)$ and $|A^*| \leq \kappa$.*

An object $A \in \mathrm{obj}(\mathrm{Sh}^*(C, J_C))$ is κ-generated if it can be covered by a sub-presheaf of size at most κ. We have the following relationship between the generated size and pure size of an object of $\mathrm{Sh}^*(C, J_C)$.

Lemma 2.18. *Suppose $A \in obj(Sh^*(C, J_C))$ is κ-generated and $|mor(C)| = \gamma$. Then there is a ζ with $\zeta \leq \kappa^\gamma$ such that A is of pure size ζ.*

Proof. Without loss of generality we can assume $|A| \leq \kappa$. Now for every $x \in \mathbf{a}(A)$ let $x^* : |\mathrm{mor}(C)| \to A \cup \{*\}$ be such that $x^*(f) = \mathbf{a}(A)(f)(x)$ if this is well-defined and in A and $*$ otherwise. As $\mathbf{a}(A)$ is separated we have $x^* = y^*$ if and only if $x = y$. Hence $|\mathbf{a}(A)| \leq \kappa^\gamma$. □

Lemma 2.18, in general, cannot be improved upon.

Example 2.19. *Let C be the category with two objects c, d and let the only non-identity maps be $\{f_i : i \in \gamma\} \subseteq C[c, d]$. Let the only non-total sieve in J_C be $\{f_i : i \in \gamma\} \in J_C(d)$. It is then immediate that if $A \in Sh^*(C, J_C)$ and $|A(c)| = \kappa$ then the pure size of A is κ^γ.*

2.2. **Sheaf Languages and Sheaf Models.** Now that we have defined $\mathrm{Sh}^*(C, J_C)$ we can define our notion of sheaf languages and sheaf models. As with first order languages we will have sorts, functions and relations. However, because we have fixed a Grothendieck topos in which our models will live, we are able to expand the language to take this into account. Specifically in usual first order languages if S is a sort then a constant of type S is realized as an element of the realization of S, i.e. as a map from the terminal object into the realization of S. In our sheaf languages though we will be able to include *generalized constants* of type S which are functions whose interpretations are generalized elements of the realization of S, i.e. maps from an arbitrary fixed objects in our Grothendieck topos into the realization of S. We will also allow sorts which are, essentially, arbitrary finite combinations of sorts from our language and objects in our Grothendieck topos.

Before we give our definition of a sheaf language though, we will fix for the rest of the paper a single distinguished copy of the subobject classifier of $\mathrm{Sh}^*(C, J_C)$ which we denote by Ω.

Definition 2.20. *A* **sheaf language** $\mathfrak{L}$ *consists of the following:*

- *A collection of* **sorts**, $\mathtt{S}_{\mathfrak{L}}$, *which will always be closed under taking finite sequences.*
- *A collection of* **object sorts**, $\mathtt{O}_{\mathfrak{L}}$, *along with a function* $r_{\mathfrak{L}} : \mathtt{O}_{\mathfrak{L}} \to obj(Sh^*(C, J_C)) - \Omega$. *We assume that no sort in* $\mathtt{O}_{\mathfrak{L}}$ *is a sequence of other sorts.*
- *A collection of* **function symbols**, $\mathtt{F}_{\mathfrak{L}}$, *each of which has a domain and codomain which is a sort. We assume* $\mathtt{F}_{\mathfrak{L}}$ *has, for each collection of sorts* $S_1, \dots, S_n$, **projection functions** $\pi_j : \langle S_1, \dots, S_n \rangle \to S_j$.
- *A collection of* **relation symbols**, $\mathtt{R}_{\mathfrak{L}}$, *where to each relation symbol we associate a sort which is its* **type**. *We will often say the domain of a relation is its type and its codomain is* Ω. *We assume* $\mathtt{R}_{\mathfrak{L}}$ *has, for each sort* S, *a relation* $=_S$ *of type* $\langle S, S \rangle$.

From now on $\mathfrak{L}$ and its variants will always represent sheaf languages.

Each sort $A \in \mathtt{O}_{\mathfrak{L}}$ will be interpreted in all $\mathfrak{L}$-structures by the object $r_{\mathfrak{L}}(A)$. In the case where the Grothendieck topos is *Set*, and hence where every object is the colimit of terminal objects, a map $f : A \times S \to B \times T$ with $A, B \in \mathtt{O}_{\mathfrak{L}}$ can be interpreted as a sequence of maps $\langle f_a : S \to T \text{ s.t. } a \in r_{\mathfrak{L}}(A) \rangle$ and $\langle f_a^* : S \to r_{\mathfrak{L}}(B) \text{ s.t. } a \in r_{\mathfrak{L}}(A) \rangle$. Hence in the special case where our Grothendieck topos is *Set*, our notion of a sheaf language is equivalent to the classical notion of a language with the added ability for maps to take values in a fixed set as opposed to a sort.

Definition 2.21. *We say a sheaf language* $\mathfrak{L}$ *is* κ-**generated** *if*

- $\mathtt{S}_{\mathfrak{L}}$ *is of size at most* κ.
- *Each sort* $A \in \mathtt{O}_{\mathfrak{L}}$, $r_{\mathfrak{L}}(A)$ *is* κ*-generated.*

We define the **pure size** *of* $\mathfrak{L}$ *similarly. We let* $|\mathfrak{L}|$ *denote least* κ *such that* $\mathfrak{L}$ *is* κ*-generated.*

Note that in *Set* a language $\mathcal{L}$ is κ-generated if and only if it is of pure size κ if and only if every model for which no sort is empty has size at least κ.

We now give our notion of an $\mathfrak{L}$-structure.

Definition 2.22. *An* $\mathfrak{L}$**-structure** $\mathcal{M}$ *consists of the following:*

- *For each* $S \in \mathtt{S}_{\mathfrak{L}}$ *an object* $S^{\mathcal{M}}$ *of* $Sh^*(C, J_C)$ *such that:*
  - *If* $S = \langle S_1, \dots, S_n \rangle$ *then* $S^{\mathcal{M}} = \prod_{i \leq n} S_i^{\mathcal{M}}$ *is the product of the sequence as presheaves.*
  - *If* $S \in \mathtt{O}_{\mathfrak{L}}$ *then* $S^{\mathcal{M}} = r_{\mathfrak{L}}(S)$.
- *For each function symbol* $f : S \to T$ *a map* $f^{\mathcal{M}} : S^{\mathcal{M}} \to T^{\mathcal{M}}$ *in* $Sh^*(C, J_C)$ *such that:*
  - *If* π_j *is a projection function then* $\pi_j^{\mathcal{M}} : \prod_{i \leq n} S_i^{\mathcal{M}} \to S_j^{\mathcal{M}}$ *is image under* ι *of the corresponding projection map in* $Sep(C, J_C)$.

- *For each relation symbol R of sort S a pair $(R_s^{\mathcal{M}}, R^{\mathcal{M}})$ where:*
  - $R^{\mathcal{M}} : S^{\mathcal{M}} \to \Omega$.
  - $R_s^{\mathcal{M}} \subseteq S^{\mathcal{M}}$ *where $R_s^{\mathcal{M}}$ is closed in $S^{\mathcal{M}}$.*
  - *For all $x \in S^{\mathcal{M}}$, $R^{\mathcal{M}}(x) := \{f \in C[-, dom(x)] : S^{\mathcal{M}}(f)(x) \in R_s^{\mathcal{M}}\}$.*

  and
  - *For each relation symbol $=_S$, $(=_S)_s^{\mathcal{M}} := \{(x,x) : x \in S^{\mathcal{M}}\} \subseteq S^{\mathcal{M}} \times S^{\mathcal{M}}$.*

Note if $\mathcal{M}$ is an $\mathfrak{L}$-structure and if $S = \langle S_1, \dots, S_n\rangle \in \mathtt{S}_{\mathfrak{L}}$ then $\langle S^{\mathcal{M}}, \langle \pi_i^{\mathcal{M}} : i \leq n\rangle\rangle$ is the distinguished product of $S_1^{\mathcal{M}}, \dots, S_n^{\mathcal{M}}$ in $\mathrm{Sh}^*(C, J_C)$ (as we have assumed all distinguished products are total). Also notice by Lemma 2.15 if $R \in \mathtt{R}_{\mathfrak{L}}$ is of type S then either element of the pair $(R_s^{\mathcal{M}}, R^{\mathcal{M}})$ determines the other uniquely (and determines a unique subobject of $S^{\mathcal{M}}$). While it will often be easier to deal with $R_s^{\mathcal{M}}$ there will be situations, such as when dealing with connectives, when we will need $R^{\mathcal{M}}$. As such we have required the realization of a relation symbol to contain both.

There is a subtle point worth mentioning, even though it will not play an important role in what follows. As our language determines, for some sorts, the objects which interpret those sorts, there are languages for which there are no $\mathfrak{L}$-structures. For example, suppose $A, B \in \mathrm{obj}(\mathrm{Sh}^*(C, J_C))$ are such that $\mathrm{Sh}^*(C, J_C)[A, B] = \varnothing$, e.g. if B is a proper closed subset of A (and hence is an element of a proper subobject). If our language requires there to be a function whose domain is A and whose codomain is B, then for that language there would be no $\mathfrak{L}$-structures.

One way in which we could avoid this dilemma would be to only allow the sorts in $\mathtt{O}_{\mathfrak{L}}$ to be in the domain of function symbols. However we have chosen not to do this as it limits the languages which we can consider and none of our results are harmed by the possibility that our language might not admit any structures.

We now give three important definitions.

Definition 2.23. *Suppose $\mathfrak{L}_0 \subseteq \mathfrak{L}_1$ are sheaf languages and $\mathcal{M}$ is an $\mathfrak{L}_1$-structure. We define the* **restriction** *of $\mathcal{M}$ to $\mathfrak{L}_0$, written $\mathcal{M}|_{\mathfrak{L}_0}$, to be the unique $\mathfrak{L}_0$-structure such that:*

- $S^{\mathcal{M}} = S^{\mathcal{M}|_{\mathfrak{L}_0}}$ *for all $S \in \mathtt{S}_{\mathfrak{L}_0}$.*
- $f^{\mathcal{M}} = f^{\mathcal{M}|_{\mathfrak{L}_0}}$ *for all $f \in \mathtt{F}_{\mathfrak{L}_0}$.*
- $R_s^{\mathcal{M}} = R_s^{\mathcal{M}|_{\mathfrak{L}_0}}$ *for all $R \in \mathtt{R}_{\mathfrak{L}_0}$.*

We say that an $\mathfrak{L}_1$-structure $\mathcal{M}$ is an **expansion** *of an $\mathfrak{L}_0$-structure $\mathcal{N}$ (to $\mathfrak{L}_1$) if $\mathcal{N} = \mathcal{M}|_{\mathfrak{L}_0}$.*

Definition 2.24. *We say an $\mathfrak{L}$-structure is κ-***generated** *if*

- *$\mathfrak{L}$ is κ-generated.*
- *Each sort $S^{\mathcal{M}}$ is κ-generated.*

We define the **pure size** *of an $\mathfrak{L}$-structure similarly.*

Definition 2.25. *Suppose $\mathcal{M}, \mathcal{N}$ are $\mathfrak{L}$-structures. An* **$\mathfrak{L}$-homomorphism** *α from $\mathcal{M}$ to $\mathcal{N}$, $\alpha : \mathcal{M} \to \mathcal{N}$, is a collection of maps $\langle \alpha_S : S \in \mathtt{S}_{\mathfrak{L}}\rangle$ in $Sh^*(C, J_C)$ such that*

- *For each $S \in \mathtt{S}_{\mathfrak{L}}$, $\alpha_S : S^{\mathcal{M}} \to S^{\mathcal{N}}$.*
- *For each $S \in \mathtt{O}_{\mathfrak{L}}$, $\alpha_S = id_{S^{\mathcal{M}}}$.*
- *For each $f : S \to T$ in $\mathtt{F}_{\mathfrak{L}}$ we have $\alpha_T \circ f^{\mathcal{M}} \equiv f^{\mathcal{N}} \circ \alpha_S$.*
- *For each $R \in \mathtt{R}_{\mathfrak{L}}$ of type S, we have $R^{\mathcal{M}} \equiv R^{\mathcal{N}} \circ \alpha_S$.*

We say two $\mathfrak{L}$-homomorphisms $\alpha^0, \alpha^1 : \mathcal{M} \to \mathcal{N}$ are **equivalent**, *written $\alpha^0 \equiv \alpha^1$, if for each sort $S \in \mathtt{S}_{\mathfrak{L}}$, $\alpha_S^0 \equiv \alpha_S^1$.*

We call an $\mathfrak{L}$-homomorphism from $\mathcal{M}$ to $\mathcal{N}$ an **inclusion** if each component is an inclusion. We say that $\mathcal{M}$ is an **$\mathfrak{L}$-substructure** of $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$, if there is an inclusion from $\mathcal{M}$ to $\mathcal{N}$, i.e. the inclusion maps $\langle in_S : S^{\mathcal{M}} \to S^{\mathcal{N}} \text{ s.t. } S \in \mathtt{S}_{\mathfrak{L}}\rangle$ form an $\mathfrak{L}$-homomorphism.

We define composition of $\mathfrak{L}$-homomorphisms in the obvious way (i.e. component wise). We define the identity $\mathfrak{L}$-homomorphisms, $id_{\mathcal{N}}$, on an $\mathfrak{L}$-structure $\mathcal{N}$ to be the homomorphism which is the identity in each component. We also say $\alpha : \mathcal{M} \to \mathcal{N}$ and $\beta : \mathcal{M} \to \mathcal{N}$ are **inverse $\mathfrak{L}$-isomorphisms** if $\alpha \circ \beta \equiv id_{\mathcal{N}}$ and $\beta \circ \alpha \equiv id_{\mathcal{M}}$. In other words α and β are inverse $\mathfrak{L}$-isomorphisms if for each sort $S \in \mathtt{S}_{\mathfrak{L}}$, $\overline{q}(\alpha_S)$ and $\overline{q}(\beta_S)$ are inverses in $\mathrm{Sh}(C, J_C)$.

An important property of $\mathfrak{L}$-homomorphisms is that they are absolute.

Lemma 2.26. *Suppose $\mathcal{M}$,$\mathcal{N}$ are $\mathfrak{L}$-structures and for each $S \in S_{\mathfrak{L}}$, $\alpha_S : S^{\mathcal{M}} \to S^{\mathcal{N}}$. Further suppose V is a admissible set with $\{(C, J_C), \mathcal{M}, \mathcal{N}, \mathfrak{L}, \langle \alpha_S : S \in S_{\mathfrak{L}}\rangle\} \in V$. Then $V \vDash$"α is an $\mathfrak{L}$-homomorphism" if and only if $Set \vDash$"α is an $\mathfrak{L}$-homomorphism".*

Proof. First observe that by Proposition 2.10 we have that if $\{\langle f, d_f\rangle, D, R\} \in V$ with $\langle f, d_f\rangle \in \mathrm{Sh}^*(C, J_C)[D, R]$ (in Set) then $\langle f, d_f\rangle \in \mathrm{Sh}^*(C, J_C)[D, R]^V$ as $\{x \in D : (\exists\alpha)\{f \in \mathrm{mor}(C) : D^f(x) \in d_f\} \in J_C^{\alpha}\}^V = D$. Hence we have $\mathcal{M}$ and $\mathcal{N}$ are $\mathfrak{L}$-structures in V and $\alpha_S : S^{\mathcal{M}} \to S^{\mathcal{N}}$ for each $S \in \mathtt{S}_{\mathfrak{L}}$.

The result then follows from the fact that composition in $\mathrm{Sh}^*(C, J_C)$ is absolute and $\Omega^V \subseteq \Omega$. □

We now show how to transform any directed diagram into one where all maps are total and the size of the resulting structures don't change too much. We can extend the notion of a total map to models by saying that an $\mathfrak{L}$-structure $\mathcal{M}$ is **total** if every function symbol is interpreted as a total map, or equivalently, if every sort and function symbol are in the image of ι. Similarly we say that a directed system of $\mathfrak{L}$-homomorphisms is total if all components are, or if equivalently it is the image under ι of a directed system in $\mathrm{Sep}(C, J_C)$.

Proposition 2.27. *Suppose $\langle I, \leq\rangle$ is a partial order such that every pair of elements has an upper bound. Further suppose $\mathfrak{D} := \langle\{\mathcal{M}_i : i \in I\}, \{\alpha^{i,j} : \mathcal{M}_i \to \mathcal{M}_j, i \leq j\}\rangle$ is a directed system of $\mathfrak{L}$-structures and $\mathfrak{L}$-homomorphisms. Then there is a directed system $\mathfrak{D}_0 = \langle\{\mathcal{N}_i : i \in I\}, \{\beta^{i,j} : \mathcal{N}_i \to \mathcal{N}_j, i \leq j\}\rangle$ such that:*

1. *For each $i \in I$, $\mathcal{M}_i$ is an $\mathfrak{L}$-substructure of $\mathcal{N}_i$ with inclusion maps $in^i : \mathcal{M}_i \to \mathcal{N}_i$ which are isomorphisms (in $Sh^*(C, J_C)$).*
2. *For all $i, j \in I$ with $i \leq j$, $\beta^{i,j} \circ in^i = in^j \circ \alpha^{i,j}$. In other words $\mathfrak{D}_0$ is isomorphic, as a directed system, to $\mathfrak{D}$.*
3. *$\mathfrak{D}_0$ is total, i.e. there is a directed system $\mathfrak{D}_0^* = \langle\{\mathcal{N}_i^* : i \in I\}, \{\beta^{i,j,*} : \mathcal{N}_i^* \to \mathcal{N}_j^*, i \leq j\}\rangle$ in $Sep(C, J_C)$ with $\mathfrak{D}_0 = \iota[\mathfrak{D}_0^*]$.*
4. *If $\langle\mathcal{N}_+^*, \langle\beta^{i,*} : \mathcal{N}_i^* \to \mathcal{N}_+^*, i \in I\rangle\rangle$ is the directed limit of $\mathfrak{D}_0^*$ in $Sep(C, J_C)$ then $\langle\iota(\mathcal{N}_+^*), \langle\iota(\beta^{i,*}) : i \in I\rangle\rangle$ is a directed limit of $\mathfrak{D}_0$ in $Sh^*(C, J_C)$. Let $\mathcal{N}_+ = \iota(\mathcal{N}_+^*)$*
5. *$\bigcup_{S \in S_{\mathfrak{L}}} |S^{\mathcal{N}_+}| + |\mathfrak{L}| = \bigcup_{S \in S_{\mathfrak{L}}} \bigcup_{i \in I} |S^{\mathcal{M}_i}| + |\mathfrak{L}|$.*

Proof. We will first show that it suffices to restrict our attention to when $\alpha^{i,j}$ and all interpretations of functions are total.

Let V_S be such that $V_S \prec_n Set$ (for some sufficiently large n), $tc(\{\mathfrak{D}, \mathfrak{L}\}) \in V_S$ and $|V_S| = \bigcup_{S \in \mathtt{S}_{\mathfrak{L}}} \bigcup_{i \in I} |S^{\mathcal{M}_i}|$. Let $t : V_S \to V_S^c$ be the transitive collapse of V_S. Working inside V_S^c we can let $\mathfrak{D}_0^*$ be the result of applying $\mathbf{i} \circ \mathbf{j} \circ \overline{q}$ to each component of $\mathfrak{D}$, i.e. of first mapping each component to Sheaf(C, J_C) and then mapping the result to Sep(C, J_C) via the inclusion of categories, $\mathbf{i}$. We then let $\mathfrak{D}_0 = \iota(\mathfrak{D}_0^*)$ (and in particular (3) follows by definition)

As $\mathbf{j} \circ \overline{q} \circ \iota(A) = \mathbf{a}(A)$ for any separated presheaf, we then have inclusion maps $in^i : \mathcal{M}_i \to \mathcal{N}_i$ for each $i \in I$. Further, by Lemma 2.14 we have $(\overline{q} \circ \iota) \circ \mathbf{i} \circ \mathbf{j} : \mathrm{Sh}(C, J_C) \to \mathrm{Sh}(C, J_C)$ is isomorphic to the identity functor. Hence the inclusion maps form an isomorphism of directed systems and (2) follows.

As $\mathbf{j} \circ \overline{q} \circ \iota = \mathbf{a}$, $\mathbf{a}$ preserves limits and $\mathbf{j} \circ \overline{q}$ is an equivalence of categories, we have that ι preserves limits as well. Hence (4) holds.

Lastly we have that $\bigcup_{S \in \mathbf{S}_{\mathfrak{L}}} |S^{\mathcal{N}_+}| + |\mathfrak{L}| = \bigcup_{S \in \mathbf{S}_{\mathfrak{L}}} \bigcup_{i \in I} |S^{\mathcal{N}_i}| + |\mathfrak{L}|$ which further equals $\bigcup_{S \in \mathbf{S}_{\mathfrak{L}}} \bigcup_{i \in I} |\mathbf{a}(S^{\mathcal{M}_i})^{V_S^c}| + |\mathfrak{L}|$. But as $|V_S^c| = \bigcup_{S \in \mathbf{S}_{\mathfrak{L}}} \bigcup_{i \in I} |S^{\mathcal{M}_i}| + |\mathfrak{L}|$ we have that (5) holds as well.

□

The method used in the proof of Proposition 2.27 of working inside a transitive collapse and then observing that the result has the properties we want and isn't to large (as the transitive collapse is of fixed size) is one which we will use several times in Section 4.

It is worth pointing out that if α as an $\mathfrak{L}$-homomorphism then α_S must preserve $=_S$ for each sort S and hence α_S must be a monic. This observation, along with Proposition 2.27 shows that we can replace any $\mathfrak{L}$-homomorphism by an isomorphic one (in the obvious sense) which in an inclusion.

2.3. **Sheaf Formulas and Sheaf Sentences.** Now that we have our notion of a sheaf language and a sheaf structure we can define our notion of a sheaf formula. Each sheaf formula will be interpreted in a structure as either a map between the realization of two sorts, or a map from the realization of a sort to the subobject classifier. We want the latter collection of sheaf formulas to be closed under the basic logical operations of $\forall$, $\exists$, as well as the infinitary $\bigvee$ and $\bigwedge$. In addition to these operations we will also want our sheaf formulas to be closed under all finitary connectives from our fixed Grothendieck topos. Such a finitary connective is a map from some finite power of the subobject classifier to the subobject classifier.

In the category *Set*, all connectives between $\{\top, \bot\}^n$ and $\{\top, \bot\}$ can be built from the standard connectives $\wedge, \vee, \neg$. Hence in the category of *Set*, there is no difference between requiring the collection of formulas to be closed under $\{\wedge, \vee, \neg\}$ and requiring the collection of formulas to be closed under all connectives between $\{\top, \bot\}^n$ and $\{\top, \bot\}$ for all finite n. This however is a peculiarity of the category *Set* and in a general Grothendiek topos it need not be the case that all maps from Ω^n to Ω can be generated from the maps $\{\wedge, \vee, \neg\}$.

Definition 2.28. *We define the collection $For_{\kappa^+,\omega}(\mathfrak{L})$ of* **partial sheaf formulas** *over $\mathfrak{L}$ (of complexity at most κ) to be the smallest collection where:*

- (L) $\mathtt{F}_{\mathfrak{L}} \cup \mathtt{R}_{\mathfrak{L}} \subseteq For_{\kappa^+,\omega}(\mathfrak{L})$.
- (0) *If $A, B \in \mathtt{O}_{\mathcal{L}}$ and $\alpha : r_{\mathfrak{L}}(A) \to r_{\mathfrak{L}}(B)$ then $\langle 0, \alpha \rangle \in For_{\kappa^+,\omega}(\mathfrak{L})$ with domain A and codomain B.*
- (1) *If $f, g \in For_{\kappa^+,\omega}(\mathfrak{L})$ and $cod(f) = dom(g)$ then $\langle 1, g, f \rangle \in For_{\kappa^+,\omega}(\mathfrak{L})$, with domain $dom(f)$ and codomain $cod(g)$. We abbreviate $\langle 1, g, f \rangle$ by $g \circ f$.*

(2) *If* $\{f_i : i \leq n\} \subseteq For_{\kappa^+,\omega}(\mathfrak{L})$ *all with the same domain and such that* $\{cod(f_i) : i \leq n\} \subseteq \mathcal{S}_{\mathfrak{L}}$*, then* $\langle 2, f_i : i \leq n\rangle \in For_{\kappa^+,\omega}(\mathfrak{L})$ *with domain* $dom(f_i)$ *and codomain* $\langle cod(f_1), \dots, cod(f_n)\rangle$*. We abbreviate* $\langle 2, f_i : i \leq n\rangle$ *as* $\prod_{i\leq n} f_i$*.*
(3) *If* $\{f_i : i \leq n\} \subseteq For_{\kappa^+,\omega}(\mathfrak{L})$ *each with the same domain and each with codomain* Ω*, and if* $X \subseteq \Omega$ *with* $\beta : X^n \to X$ *then* $\langle 3, \beta, X, f_i : i \leq n\rangle \in For_{\kappa^+,\omega}(\mathfrak{L})$*, with domain* $dom(f_i)$ *and codomain* Ω*. We abbreviate* $\langle 3, \beta, X, f_i : i \leq n\rangle$ *as* $\beta \circ_X \prod_{i\leq n} f_i$ *and we call* X *the* **confines** *of* $\beta \circ_X \prod_{i\leq n} f_i$*.*
(4,5) *If* $|K| \leq \kappa$ *and* $\{f_i : i \in K\} \subseteq For_{\kappa^+,\omega}(\mathfrak{L})$ *all with the same domain and all with codomain* Ω *then* $\langle 4, f_i : i \in K\rangle, \langle 5, f_i : i \in K\rangle \in For_{\kappa^+,\omega}(\mathfrak{L})$ *with domain* $dom(f_i)$ *and codomain* Ω*. We abbreviate* $\langle 4, f_i : i \in K\rangle$ *as* $\bigvee_{i\in K} f_i$ *and* $\langle 5, f_i : i \in K\rangle$ *as* $\bigwedge_{i\in K} f_i$*.*
(6,7) *If* $f, g \in For_{\kappa^+,\omega}(\mathfrak{L})$ *with same domain,* $cod(f) = \Omega$ *and the* $cod(g) \in \mathcal{S}_{\mathfrak{L}}$ *then* $\langle 6, f, g\rangle, \langle 7, f, g\rangle \in For_{\kappa^+,\omega}(\mathfrak{L})$ *with domain* $dom(f)$ *and codomain* Ω*. We abbreviated* $\langle 6, f, g\rangle$ *as* $(\forall_g) f$ *and* $\langle 7, f, g\rangle$ *as* $(\exists_g) f$*.*

We let $For_{\infty,\omega}(\mathfrak{L}) = \bigcup_{\kappa\in ORD} For_{\kappa^+,\omega}(\mathfrak{L})$

Definition 2.29. *For* $\psi_0, \psi_1 \in For_{\kappa^+,\omega}(\mathfrak{L})$ *we say* ψ_0 *is a* **subformula** *of* ψ_1*, written* $\psi_0 \leq \psi_1$*, if* $\psi_0 \in tc(\psi_1)$*. We will also abuse notation and say* $\langle X, \beta\rangle \leq \psi$ *if* $\beta \circ_X \prod_{i\leq n} f_i \leq \psi$ *for some* $\langle f_i : i \leq n\rangle$*. Note that* $\leq$ *is well-founded.*

With the exception of (3), each item from Definition 2.28 has a self explanatory interpretation. We call a map $X^n \to X$ where $X \subseteq \Omega$ a **partial connective** and formulas from Definition 2.28 (3) are meant to interpret partial connectives. Note that as Ω is injective, every partial connective is the restriction of a connective to its confines.

Notice the domain of every formula is a sort and the codomain is either a sort or Ω. In particular connectives are not themselves formulas. Given a sheaf $\mathfrak{L}$-structure and a sheaf formula φ, we will want to be able to expand our sheaf structure so that φ is *named*, i.e. so that φ is equivalent to a function or relation in the language. In order to do this we will want to simultaneously name every subformula of φ. However, if we had allowed connectives to be formulas, in any formula which contains a connective we would need a sort isomorphic to Ω. This would pose a problem though as there are weak sites of size κ for which the subobject classifier Ω is not of generated size less than 2^κ. In this situation, if we allowed connectives to be formulas, any structure which had a named connective would itself have to be of generated size at least 2^κ (even though the weak site itself was only of size κ). We solve this problem by dealing with partial connectives instead of with connectives. The cost however is that not all (partial) sheaf formulas will be interpretable in all sheaf models.

Definition 2.30. *We define* $\sqsubseteq$ *to be the smallest partial order on* $For_{\infty,\omega}(\mathfrak{L})$ *such that:*

- $\langle 3, \beta_0, X, f_i : i \leq n\rangle \sqsubseteq \langle 3, \beta_1, Y, g_i : i \leq n\rangle$ *if and only if* $X \subseteq Y$*,* $\beta_0 = \beta_1|_X$ *and* $f_i \sqsubseteq g_i$ *for each* $i \leq n$*.*
- *Otherwise* $\langle a_0, b^0_i : i \leq \zeta_0\rangle \sqsubseteq \langle a_1, b^1_i : i \leq \zeta_1\rangle$ *if and only if* $a_0 = a_1$*,* $\zeta_0 = \zeta_1$ *and for each* $i \leq \zeta_0$*,* $b^0_i \sqsubseteq b^1_i$*.*

We say a formula is **total** *if it is maximal in* $\sqsubseteq$*.*

Intuitively $\varphi_0 \sqsubseteq \varphi_1$ if φ_0 and φ_1 are built from simpler formulas in exactly the same way, except whenever there is a partial connective the domain of that connective in φ_1 contains the domain in φ_0 and both connectives take the same values when they are both defined.

The following is then immediate

Lemma 2.31. *For every ψ there is a total ψ' with $\psi \sqsubseteq \psi'$. Further a formula ψ is total if and only if whenever $\langle 3, \beta, X, \langle f_i : i \leq n \rangle\rangle \preceq \psi$, $X = \Omega$*

Proof. This follows immediately from the fact that Ω is injective and hence any partial map $X^n \to X \subseteq \Omega$ can be extended to a map $\Omega^n \to \Omega$. □

Now that we have our notion of sheaf formula we want to describe how to interpret a partial sheaf formula in a sheaf model. Unfortunately though not all partial sheaf formulas will be interpretable in all sheaf models. In particular if we try and compose a partial connective with a formula which takes values outside of the confines of the partial connective, then we run into problems. We will deal with this issue by allowing some interpretations to take the special value of $\Uparrow$.

Definition 2.32. *Suppose $\mathcal{M}$ is an $\mathfrak{L}$-structure. For each $\varphi \in For_{\kappa^+,\omega}(\mathfrak{L})$ we define $\varphi^{\mathcal{M}} : dom(\varphi)^{\mathcal{M}} \to cod(\varphi)^{\mathcal{M}}$ by induction along $\preceq$. Notice, that as a base case we have already defined $\varphi^{\mathcal{M}}$ when $\varphi \in R_{\mathfrak{L}} \cup F_{\mathfrak{L}}$.*

Next, if there is a $\psi \preceq \varphi$ such that $\psi^{\mathcal{M}} = \Uparrow$ then $\varphi^{\mathcal{M}} = \Uparrow$. Otherwise we have the following:

- *If $\langle 0, \alpha\rangle \in For_{\kappa^+,\omega}(\mathfrak{L})$ then $\langle 0, \alpha\rangle^{\mathcal{M}} = \alpha$.*
- *If $f \circ g \in For_{\kappa^+,\omega}(\mathfrak{L})$ then $(f \circ g)^{\mathcal{M}} = g^{\mathcal{M}} \circ f^{\mathcal{M}}$.*
- *If $\prod_{i \leq n} f_i \in For_{\kappa^+,\omega}(\mathfrak{L})$ then $[\prod_{i\leq n} f_i]^{\mathcal{M}}$ is a morphism g from $dom(f_i)^{\mathcal{M}}$ to $\prod_{i\leq n} cod(f_i)^{\mathcal{M}}$ such that $\pi_j^{\mathcal{M}} \circ g = f_j^{\mathcal{M}}$ for each $j \leq n$.*
- *If $\beta \circ_X \prod_{i\leq n} f_i \in For_{\kappa^+,\omega}(\mathfrak{L})$ then $[\beta \circ_X \prod_{i\leq n} f_i]^{\mathcal{M}} = \beta \circ \prod_{i\leq n} f_i^{\mathcal{M}}$ if $ran(f_i^{\mathcal{M}}) \subseteq X$ for each $i \leq n$ and $\Uparrow$ otherwise.*
- *If $\{f_i : i \in I\} \subseteq For_{\kappa^+,\omega}(\mathfrak{L})$ then $[\bigvee_{i\in I} f_i]^{\mathcal{M}} = \bigvee_{i\in I} f_i^{\mathcal{M}}$ and $[\bigwedge_{i\in I} f_i]^{\mathcal{M}} = \bigwedge_{i\in I} f_i^{\mathcal{M}}$.*
- *If $(\forall_g)f, (\exists_g)f \in For_{\kappa^+,\omega}(L)$ then $[(\forall_g)f]^{\mathcal{M}} = (\forall_{g^{\mathcal{M}}})f^{\mathcal{M}}$ and $[(\exists_g)f]^{\mathcal{M}} = (\exists_{g^{\mathcal{M}}})f^{\mathcal{M}}$.*

We say φ is **legal** *for $\mathcal{M}$ if $\varphi^{\mathcal{M}} \neq \Uparrow$.*

We then have the following relationship between $\sqsubseteq$ and being legal.

Lemma 2.33. *Let $\mathcal{M}$ be a sheaf model and let $\varphi_0 \sqsubseteq \varphi_1$. Then*

(a) *If φ_0 is legal for $\mathcal{M}$ then φ_1 is legal for $\mathcal{M}$.*
(b) *If $\varphi_0^{\mathcal{M}}$ is legal for $\mathcal{M}$ then $\varphi_0^{\mathcal{M}} = \varphi_1^{\mathcal{M}}$.*
(c) *If φ_0 is total then it is legal for all $\mathfrak{L}$-structures.*

Proof. (a), (b) follow by an easy induction on $\preceq$. For (c) notice by Lemma 2.31 that if φ_0 is total then every connective has confines Ω. □

It will often be useful to have a formula be equivalent to a function or relation in our language.

Definition 2.34. *Suppose $\varphi \in For_{\kappa^+,\omega}(\mathfrak{L})$ and $H_\varphi \in F_{\mathfrak{L}} \cup R_{\mathfrak{L}}$ with $dom(\varphi) = dom(H_\varphi)$ and $cod(\varphi) = cod(H_\varphi)$. If φ is legal for an $\mathfrak{L}$-structure $\mathcal{M}$ and $\varphi^{\mathcal{M}} \equiv H_\varphi^{\mathcal{M}}$ then we say H_φ is a* **name** *for φ (in $\mathcal{M}$).*

We then have the following easy connection between names for formulas and homomorphisms which preserve formulas.

Lemma 2.35. *Suppose* $A \subseteq For_{\kappa^+,\omega}(\mathfrak{L})$, $S_{\mathfrak{L}} = S_{\mathfrak{L}_A}$ *and* $[F_{\mathfrak{L}_A} - F_{\mathfrak{L}}] \cup [R_{\mathfrak{L}_A} - R_{\mathfrak{L}}] = \{H_\varphi : \varphi \in A\}$ *where* $dom(H_\varphi) = dom(\varphi)$ *and* $cod(H_\varphi) = cod(\varphi)$.

Then for any $\mathfrak{L}$*-structure* $\mathcal{M}$ *for which each* $\varphi \in A$ *is legal, there is a unique expansion* $\mathcal{M}_A$ *to* $\mathfrak{L}_A$ *where* H_φ *names* φ *for each* $\varphi \in A$.

Definition 2.36. *Suppose* $\mathcal{M}, \mathcal{N}$ *are* $\mathfrak{L}$*-structures,* $\alpha : \mathcal{M} \to \mathcal{N}$ *and* $\varphi : S \to T$ *is legal for both* $\mathcal{M}$ *and* $\mathcal{N}$. *We say that* α **preserves** φ *if:*

- $\alpha_T \circ \varphi^{\mathcal{M}} \equiv \varphi^{\mathcal{N}} \circ \alpha_S$ *if* $T \in S_{\mathfrak{L}}$.
- $\varphi^{\mathcal{M}} \equiv \varphi^{\mathcal{N}} \circ \alpha_S$ *if* $T = \Omega$.

The following is then immediate.

Lemma 2.37. *Suppose* $\mathcal{M}, \mathcal{N}$ *are* $\mathfrak{L}$*-structures and* $A \subseteq For_{\kappa^+,\omega}(\mathfrak{L})$ *with each* $\varphi \in A$ *legal for both* $\mathcal{M}$ *and* $\mathcal{N}$. *Then an* $\mathfrak{L}$*-homomorphism* $\alpha : \mathcal{M} \to \mathcal{N}$ *preserves every formula in* A *if and only if* $\alpha : \mathcal{M}_A \to \mathcal{N}_A$ *is also an* $\mathfrak{L}_A$*-homomorphism.*

We have shown how to interpret sheaf formulas in $\mathfrak{L}$-structures. However, these interpretations don't provide us with any statements whose truth value we can externally evaluate (i.e. in *Set* and not in the $\mathrm{Sh}(C, J_C)$). The notion of a sheaf sentence will provide us with statements about $\mathfrak{L}$-structures which will either be true or false (in our model *Set* of ZFC).

Definition 2.38. *We let* $Sen_{\kappa^+,\omega}(\mathfrak{L})$ *be the smallest collection such that:*

- *If* $f, g \in For_{\kappa^+,\omega}(\mathfrak{L})$ *with* $dom(f) = dom(g)$ *and* $cod(f) = cod(g)$ *then* $\langle 9, f, g\rangle \in Sen_{\kappa^+,\omega}(\mathfrak{L})$. *We abbreviate* $\langle 9, f, g\rangle$ *as* $f \equiv g$. *We call these* **basic sentences**
- *If* $T \in Sen_{\kappa^+,\omega}(\mathfrak{L})$ *then so is* $\langle 10, T\rangle$. *We abbreviate* $\langle 10, T\rangle$ *as* $\check{\neg} T$.
- *If* $|K| \leq \kappa$ *and* $\{T_i : i \in K\} \subseteq Sen_{\kappa^+,\omega}(\mathfrak{L})$ *then* $\langle 11, T_i : i \in K\rangle, \langle 12, \langle T_i : i \in K\rangle\rangle \in Sen_{\kappa^+,\omega}(\mathfrak{L})$. *We abbreviate* $\langle 11, \langle T_i : i \in K\rangle\rangle$ *by* $\check{\bigvee}_{i\in K} T_i$ *and* $\langle 12, \langle T_i : i \in K\rangle\rangle$ *by* $\check{\bigwedge}_{i\in K} T_i$.

We let $Sen_{\infty,\omega}(\mathfrak{L}) = \bigcup_{\kappa \in ORD} Sen_{\kappa^+,\omega}(\mathfrak{L})$.

The intuition is that a basic sentence determines whether or not two formulas are interpreted by equivalent maps. Arbitrary sentences are then boolean combinations of basic ones.

Definition 2.39. *If* $T_0 \in Sen_{\kappa^+,\omega}(\mathfrak{L}) \cup For_{\kappa^+,\omega}(\mathfrak{L})$ *and* $T_1 \in Sen_{\kappa^+,\omega}(\mathfrak{L})$ *we say* $T_0 \preceq T_1$ *if* $T_0 \in tc(T_1)$. *In this case we say that* T_0 *is a* **subsentence** *or* **subformula** *of* T_1 *(as appropriate). We also define a* **fragment** *to be a subset of* $For_{\infty,\omega}(\mathfrak{L}) \cup Sen_{\infty,\omega}(\mathfrak{L})$ *which is closed under* $\preceq$.

Now that we have our collection of sentences we want to define when an $\mathfrak{L}$-structure satisfies a sentence. We define this by induction.

Definition 2.40. *Suppose* $T \in Sen_{\kappa^+,\omega}(\mathfrak{L})$. *If there is a formulas* $\varphi \preceq T$ *such that* φ *is not legal for* $\mathcal{M}$ *then* T *is not legal for* $\mathcal{M}$ *and* $\mathcal{M} \nvDash T$. *If however* T *is legal for* $\mathcal{M}$ *then we define* $\mathcal{M} \vDash T$ *by induction as follows:*

- $\mathcal{M} \vDash f \equiv g$ *if and only if* $f^{\mathcal{M}} \equiv g^{\mathcal{M}}$.
- $\mathcal{M} \vDash \check{\neg} T$ *if and only if* $\mathcal{M} \nvDash T$ *(and* T *is legal for* $\mathcal{M}$*).*
- $\mathcal{M} \vDash \check{\bigvee}_{i\in I} T_i$ *if there is some* $i \in I$ *such that* $\mathcal{M} \vDash T_i$
- $\mathcal{M} \vDash \check{\bigwedge}_{i\in I} T_i$ *if* $\mathcal{M} \vDash T_i$ *for each* $i \in I$.

It is worth taking a second to discuss the difference between $\bigwedge$ and $\check{\bigwedge}$, and $\bigvee$ and $\check{\bigvee}$. We can think of formulas with codomain Ω as functions which takes a structure and returns an *internal* subsets of given sort, i.e. subobjects of the sort. Then $\bigwedge$ and $\bigvee$ are the (internal) operations inherited from the corresponding lattice operations on Ω. In a similar vein we can think of a sentence as a function from structures to $\{\top, \bot\}$, the subobject classifier of *Set*. In this way a sentence represents an *external* subset of structures. Then $\check{\bigwedge}$ and $\check{\bigvee}$ are then the (external) operations inherited from the corresponding lattice operations on $\{\top, \bot\}$.

There is a particular class of sentences which will play an important role in Section 4. We say a sentence $T \in \mathrm{Sen}_{\kappa^+,\omega}(\mathfrak{L})$ is **simple** if for each $\varphi \in \mathrm{For}_{\kappa^+,\omega}(\mathfrak{L})$, $\varphi \preceq T$ implies $\varphi \in \mathtt{F}_{\mathfrak{L}} \cup \mathtt{R}_{\mathfrak{L}}$. In other words a sentence is simple if it does not make any mention of any of the operations used in constructing the formulas of the language. Note that simple sentences are legal for all $\mathfrak{L}$-structures.

Lemma 2.41. *For each $T \in Sen_{\kappa^+,\omega}(\mathfrak{L})$ let $P(T) = \{\varphi \in For_{\kappa^+,\omega}(\mathfrak{L}) : \varphi \preceq T\}$. Then for each $N_T = \langle H_\varphi : \varphi \in P(T)\rangle$ with $N_T \cap P(T) = \varnothing$ there is a basic sentence T_{N_T} such that if N_T names all elements of $P(T)$ in $\mathcal{M}$ then $\mathcal{M} \vDash T$ if and only if $\mathcal{M} \vDash T_{N_T}$.*

Proof. T_{N_T} is obtained from T by replacing all occurrences of $(f \equiv g) \preceq T$ with $H_f \equiv H_g$ for any $f, g \in P(T)$. □

Lemma 2.41 tells us that we can reduce the satisfaction relation for sentences to the satisfaction relation for simple sentences when all subformulas are named. This will be very important when we want to apply our encodings in Section 4.

We now end this section by considering how our sheaf languages, sheaf formulas and sheaf models relate to the Kripke-Joyal semantics for the Mitchell-Bénabou language (see [10]). First recall that if $\varphi \in \mathcal{L}_{\infty,\omega}(\mathcal{L})$ is of type A (where A is an object of $\mathrm{Sh}(C, J_C)$) then the *Mitchel-Bénabou language* allows us to associate to φ a subobject $\{x : \varphi(x)\} \subseteq A$. If $\alpha : U \to A$ is then a generalized element of A the *Kripke-Joyal semantics* says that U *forces* $\varphi(\alpha)$, $U \Vdash \varphi(\alpha)$, if and only if $\mathrm{ran}(\alpha) \subseteq \{x : \varphi(x)\}$.

In particular we have $U \Vdash \varphi(\alpha)$ if and only if α factors through the subobject $\{x : \varphi(x)\}$. But if $\varphi^* : A \to \Omega$ is the map corresponding to the subobject $\{x : \varphi(x)\}$, then $U \Vdash \varphi(\alpha)$ if and only if $\varphi^* \circ \alpha$ factors through $\top : 1 \to \Omega$, i.e. $\varphi^* \circ \alpha \equiv \top \circ !_U$.

Now suppose $S \in \mathtt{S}_{\mathfrak{L}}$, $U^* \in \mathtt{O}_{\mathfrak{L}}$ with $r_{\mathfrak{L}}(U^*) = U$, and $\mathcal{M}$ is an $\mathfrak{L}$-structure such that $S^{\mathcal{M}} = A$. Next let $\overline{\varphi}(x)$ be the formula in $\mathrm{For}_{\infty,\omega}(\mathfrak{L})$ with domain S and codomain Ω which is constructed in the same fashion as $\varphi(x)$. Then $\overline{\varphi}^{\mathcal{M}}(x)$ is a map from A to Ω which has the same interpretation as the formulas $\varphi(x)$ (from the Mitchell-Bénabou language of $\mathrm{Sh}(C, J_C)$). Further, if $\alpha : U^* \to S$ is any function symbol in $\mathfrak{L}$ then $U \Vdash \varphi(\alpha^{\mathcal{M}})$ if and only if $\mathcal{M} \vDash \overline{\varphi} \circ \alpha \equiv \top \circ !_U$.

In this way we see that the (analog of) Kripke-Joyal semantics for the Mitchell-Bénabou language is subsumed by our notion of a sheaf formula.

3. Representations and Components

In Section 4 we will prove analogs of the directed embedding theorem, the downward Löwenheim-Skolem theorem, a completeness theorem as well as an analog of Barwise's compactness theorem. We will do this by showing that each of these theorems can be reduced to the corresponding theorem on structures in the category of sets. In order make this reduction we will need to do three things.

(1) For each sheaf language $\mathfrak{L}$ we need to find an encoding of $\mathfrak{L}$ by a first order language $\text{Enc}(\mathfrak{L})$ and a for each sheaf $\mathfrak{L}$-structure $\mathcal{M}$ an encoding of $\mathcal{M}$ by a $\text{Enc}(\mathfrak{L})$-structure $\text{Enc}(\mathcal{M})$.
(2) For each fragment A of $\mathfrak{L}$ formulas and each $\{H_\varphi \in \mathtt{F}_{\mathfrak{L}} \cup \mathtt{R}_{\mathfrak{L}} : \varphi \in A\}$ we will need a sentence of $\mathcal{L}_{\infty,\omega}(\text{Enc}(\mathfrak{L}))$ which holds of $\text{Enc}(\mathcal{M})$ if and only if for each $\varphi \in A$, H_φ is a name for φ in $\mathcal{M}$.
(3) For each simple sentence T we will need an encoding of T by a sentence $\langle\!\langle T \rangle\!\rangle \in \mathcal{L}_{\infty,\omega}(\text{Enc}(\mathfrak{L}))$ where $\text{Enc}(\mathcal{M}) \vDash \langle\!\langle T \rangle\!\rangle$ if and only if $\mathcal{M} \vDash T$.

We will accomplish these three goals by defining *components* which can be realized in a *Set*-model.

A component is a pair consisting of a language along with a Π_2-theory in the language, which is required to be satisfied for the component to be realized. We will combine these components to build the necessary encodings. Note the fact that all theories are Π_2 will be important for proving the directed embeddings theorem (Theorem 4.1).

Definition 3.1. *A* **component***,* $\mathfrak{C}(P)$*, consists of a pair* $\langle \textit{Lan}[\mathfrak{C}(P)], \textit{Th}[\mathfrak{C}(P)]\rangle$ *where*

- $\textit{Lan}[\mathfrak{C}(P)]$ *is a language.*
- $\textit{Th}[\mathfrak{C}(P)] \subseteq \mathcal{L}_{\infty,\omega}(\textit{Lan}[\mathfrak{C}(P)])$ *is a* Π_2 *theory.*

If $\mathcal{M}$ is an $\mathcal{L}$-structure we say $\mathfrak{C}(P)$ is **realized** in $\mathcal{M}$ if $\text{Lan}[\mathfrak{C}(P)] \subseteq \mathcal{L}$ and $\mathcal{M} \vDash \text{Th}[\mathfrak{C}(P)]$. We also say a component $\mathfrak{C}(P_0)$ is **contained** in a component $\mathfrak{C}(P_1)$, written $\mathfrak{C}(P_0) \subseteq \mathfrak{C}(P_1)$ if $\text{Lan}[\mathfrak{C}(P_0)] \subseteq \text{Lan}[\mathfrak{C}(P_1)]$ and $\vdash \text{Th}[\mathfrak{C}(P_1)] \rightarrow \text{Th}[\mathfrak{C}(P_0)]$.

We will often abuse notation and use $\mathfrak{C}(P)$ to refer to both $\text{Lan}[\mathfrak{C}(P)]$ and $\text{Th}[\mathfrak{C}(P)]$ when no confusion will arise. For example, if $\mathcal{L}_0, \mathcal{L}_1$ are two copies of a language and $\mathcal{L}_0 \subseteq \mathcal{L}_1$ we will write $\mathfrak{C}(P)[\mathcal{L}_0/\mathcal{L}_1]$ for the component $\langle(\text{Lan}[\mathfrak{C}(P)] - \mathcal{L}_0) \cup \mathcal{L}_1, \text{Th}[\mathfrak{C}(P)][\mathcal{L}_0/\mathcal{L}_1]\rangle$. We also will write $\mathcal{M} \vDash \mathfrak{C}(P)$ for $\mathcal{M} \vDash \text{Th}[\mathfrak{C}(P)]$, and $X \in \mathfrak{C}(P)$ for $X \in \text{Lan}[\mathfrak{C}(P)]$, etc.

Each component which we introduce will be intended to *encode* some part of a sheaf language, sheaf model, sheaf formula, or sheaf sentence. As we introduce these components we will also explain how they are related to what they are intended to encode. This relationship will often take the form of a Δ_0-definable surjection or bijection. When this is the case we will abuse notation and refer to the map which takes models of a component and returns what it represents by Rep. We will likewise abuse notation and refer to its inverse, i.e. the map which takes some part of our sheaf structure and returns a model of a component which encodes it, by Enc. In this case we say that $\text{Rep}(\mathcal{M})$ is the **representation** of $\mathcal{M}$ and $\text{Enc}(A)$ is the **encoding** of A.

While we will always state explicitly which component a symbol represents, we will find that by the end of the paper the notation can get a little unwieldy. To help visually signal what is going on we will use the following convention. If some part of the language of the component is not contained in any other component then we will place two dots, as in $\ddot{[]}$, over the name. Usually fundamental components, from which others components will be built, will be of this form.

If the component consists purely of other components which collectively satisfy some extra sentences and if the component has an explicit name describing it, then we place four dots, as in $\ddot{\ddot{[]}}$, over the name of the component. If however the component consists purely of other components which collectively satisfy some extra sentences and the purpose of the component is to express the relationship between these other components then we place the description of the components within two angled brackets like ⟪⟫.

We break this section into three parts. In Section 3.1 we define our basic components. These are the components from which everything else will be built. In Section 3.2 we use our basic components to define encodings of $\mathfrak{L}$-structures. Then in Section 3.3 we define the encodings used for expressing the fact that a fragment is named and for encoding simple sentences.

3.1. **Basic Components.** We break our basic components into three groups. In Section 3.1.1 we define the components which encode pieces of the category of separated presheaves. In Section 3.1.2 we define the components needed to encode when one subpresheaf is the closure of another. We will accomplish this by defining a component which allows us to iterate the operation of $\mathbf{a}^1(\cdot) \cap B$ until it stabilizes. In order to do this iteration we will need to define a sort which contains *enough* ordinals. In Section 3.1.3 we define components which represent maps from a sort to the subobject classifier. Defining these maps will require some care as we don't want our encoded models to have to encode all of the subobject classifier. To accomplish this we will use the fact that each element of Ω is a subset of the morphisms C. We will then define a map from S to Ω as a relation I on $S \times \mathrm{mor}(C)$ where x gets mapped to $\{f \in \mathrm{mor}(C) : I(x, f)\}$.

We end this section on basic components in Section 3.1.4 where we define structures which are not components (but will be part of a component in Section 3.3.1). Specifically we define the structure which will allow us to encode partial connectives. This structure is not a component as it is not something which can be realized in a *Set*-structure. Rather this structure will be a collection of conditions on formulas which allow us to encode the partial connective, given that our encoding is treating elements of Ω as subsets of $\mathrm{mor}(C)$.

3.1.1. *Sorts, Subpresheaves and Functions.* In this section we will define components which are related to separated presheaves.

Definition 3.2. *We say $\ddot{\mathcal{S}}$ is an* **encoded sort** *if it is a component which contains:*
- *For each $c \in obj(C)$ a (unique) sort S^c.*
- *For each $f \in C[c,d]$ a (unique) function $S^f : S^d \to S^c$.*

and which says for each $c \in obj(C)$:
- $(\forall x : S^c) \bigwedge_{f,g,h \in mor(C), h = g \circ f} S^f \circ S^g(x) = S^h(x)$.
- $(\forall x : S^c) S^{id_c}(x) = x$.
- $(\forall x, y : S^c) \bigwedge_{I \in J(c)} [\bigwedge_{f \in I} S^f(x) = S^f(y)] \to x = y$.

Let $\mathrm{Rep} : \mathrm{Mod}(\ddot{\mathcal{S}}) \to \mathrm{obj}(\mathrm{Sep}(C, J_C))$ be such that $\mathrm{Rep}(\mathcal{M})(c) = (S^c)^{\mathcal{M}}$ for $c \in \mathrm{obj}(C)$ and $\mathrm{Rep}(\mathcal{M})(f) = (S^f)^{\mathcal{M}}$ for $f \in \mathrm{mor}(C)$. It is then immediate that Rep is a Δ_0-definable bijection, and we let Enc be its inverse. In particular encoded sorts are exactly the structures which capture separated presheaves on (C, J_C).

We will use the shorthand $\ddot{\mathcal{S}}^{\mathcal{M}}$ for $\mathrm{Rep}(\mathcal{M}|_{\ddot{\mathcal{S}}})$ when $\ddot{\mathcal{S}}$ is an encoded sort in $\mathcal{M}$. In what follows $\ddot{\mathcal{S}}$ and its variants will be encoded sorts. Note that encodings and representations preserve size.

Lemma 3.3. *Suppose $A \in obj(Sep(C, J_C))$. Then $|A| = |Enc(A)|$.*

Note the following is immediate.

Lemma 3.4. *If $\ddot{\mathcal{S}}_1, \dots, \ddot{\mathcal{S}}_n$ are encoded sorts and $\ddot{\mathcal{S}}_*$ is such that $S^c_* = \langle S^c_1, \dots, S^c_n \rangle$ and $S^f_* = \langle S^f_1, \dots, S^f_n \rangle$ then $\ddot{\mathcal{S}}_*$ is an encoded sort, which we denote by $\ddot{\mathcal{S}}_1 \times \cdots \times \ddot{\mathcal{S}}_n$. Further, if $\ddot{\mathcal{S}}_1, \dots, \ddot{\mathcal{S}}_n$ are encoded sorts in $\mathcal{M}$ then so is $\ddot{\mathcal{S}}_*$ and $\ddot{\mathcal{S}}^{\mathcal{M}}_* = \ddot{\mathcal{S}}^{\mathcal{M}}_1 \times \cdots \times \ddot{\mathcal{S}}^{\mathcal{M}}_n$.*

Our next component captures the notion of being a subpresheaf. Before we give this component though we will need a related definition.

Definition 3.5. *Suppose $\ddot{\mathcal{S}} \subseteq \mathcal{L}$ is an encoded sort and suppose $\vec{\varphi} = \langle \varphi^c : c \in obj(C) \rangle$ where for each $c \in obj(C)$, $\varphi^c \in \mathcal{L}_{\infty,\omega}(\mathcal{L})$ is a formula whose free variable is of sort S^c. We say $\vec{\varphi}$ is an* **encoded formula** *(of sort $\ddot{\mathcal{S}}$) in a structure, if the structure satisfies the theory $Th_{For}(\vec{\varphi})$ which says:*

- *$\ddot{\mathcal{S}}$ is an encoded sort.*
- $\bigwedge_{c,d \in obj(C)} \bigwedge_{f \in C[c,d]} (\forall x : S^d) \varphi^c(x) \to \varphi^d(S^f(x))$.

An encoded formula is a collection of formulas which cohere in a way so as to describe a subpresheaf of our encoded sort. If $\vec{\varphi}$ is an encoded formula of sort $\ddot{\mathcal{S}}$ in a structure $\mathcal{M}$, then we let $\vec{\varphi}^{\mathcal{M}}$ be the presheaf where for any $c \in \text{obj}(C)$, $\vec{\varphi}^{\mathcal{M}}(c) = \{x \in (S^c)^{\mathcal{M}} : \mathcal{M} \vDash \varphi(x)\}$. It is then clear that $\vec{\varphi}^{\mathcal{M}} \subseteq \ddot{\mathcal{S}}^{\mathcal{M}}$.

Definition 3.6. *We say $\ddot{E}$ is an* **encoded subset** *of sort $\ddot{\mathcal{S}}$ if it is a component which contains:*

- *The encoded sort $\ddot{\mathcal{S}}$.*
- *$\langle E^c : c \in obj(C) \rangle$ where for each $c \in obj(C)$, E^c is a relation of type S^c.*

and which proves $Th_{For}(\langle E^c : c \in obj(C) \rangle)$.

If $\ddot{E}$ is realized as an encoded subset in $\mathcal{M}$ we use the shorthand $\ddot{E}^{\mathcal{M}}$ for $\langle E^c : c \in \text{obj}(C) \rangle^{\mathcal{M}}$.

Let SSep be the collection of pairs, $\langle A_0, A_1 \rangle$, of objects of $\text{Sep}(C, J_C)$ with $A_0 \subseteq A_1$. Now if we let $\text{Rep} : \text{Mod}(\ddot{E}) \to \text{SSep}$ be such that $\text{Rep}(\mathcal{M}) = \langle \ddot{E}^{\mathcal{M}}, \ddot{\mathcal{S}}^{\mathcal{M}} \rangle$ then Rep is a Δ_0-definable bijection. We call its inverse Enc. In this way we see that $\ddot{E}$ captures the notion of being a subpresheaf.

Our next component will capture being a morphism of presheves. Before we give this component though, we will give the notion of an encoded term.

Definition 3.7. *Suppose $\ddot{\mathcal{S}} \cup \ddot{\mathcal{T}} \subseteq \mathcal{L}$ and suppose $\vec{t} = \langle t^c : c \in obj(C) \rangle$ where for each $c \in obj(C)$, $t^c \in \mathcal{L}_{\infty,\omega}(\mathcal{L})$ is a term of type $S^c \to T^c$. We say $\vec{t}$ is an* **encoded term** *(of type $\ddot{\mathcal{S}} \to \ddot{\mathcal{T}}$) in an structure if the structure satisfies the theory $Th_{Ter}(\vec{t})$ which says:*

- *$\ddot{\mathcal{S}}$, $\ddot{\mathcal{T}}$ are encoded sorts.*
- $\bigwedge_{c \in obj(C)} (\forall x : S^c) \bigwedge_{g \in C[d,c]} t^d \circ S^g(x) = T^g \circ t^c(x)$.

If $\vec{t}$ is an encoded term of type $\ddot{\mathcal{S}} \to \ddot{\mathcal{T}}$ in a structure $\mathcal{M}$, then we let $\vec{t}^{\mathcal{M}}$ be the morphism of presheaves where for any $c \in \text{obj}(C)$, and $x \in \ddot{\mathcal{S}}^{\mathcal{M}}$, $\vec{t}^{\mathcal{M}}(x) = y$ if and only if $\mathcal{M} \vDash t^c(x) = y$. It is then easily checked that $\vec{t}^{\mathcal{M}} : \ddot{\mathcal{S}}^{\mathcal{M}} \to \ddot{\mathcal{T}}^{\mathcal{M}}$ is a map of presheaves.

Definition 3.8. *We say $\ddot{f}$ is an* **encoded function** *with domain $\ddot{\mathcal{S}}$ and codomain $\ddot{\mathcal{T}}$ if it is a component which contains:*

- *Encoded sorts $\ddot{\mathcal{S}}$ and $\ddot{\mathcal{T}}$.*
- *$\langle f^c : c \in obj(C) \rangle$ where each f^c is a function symbol of type $S^c \to T^c$.*

and which proves $Th_{Ter}(\langle f^c : c \in obj(C) \rangle)$.

If $\ddot{f}$ is realized as an encoded function in $\mathcal{M}$ we use the shorthand $\ddot{f}^{\mathcal{M}}$ for $\langle f^c : c \in \text{obj}(C)\rangle^{\mathcal{M}}$.

If $\text{Rep} : \text{Mod}(\ddot{f}) \to \text{mor}(\text{Sep}(C, J_C))$ be such that $\text{Rep}(\mathcal{M}) = \ddot{f}^{\mathcal{M}}$ then Rep is a Δ_0-definable bijection. We call its inverse Enc.

Now that we have defined three of the basic components, we will introduce a shorthand which will greatly simplify our presentation. Suppose ψ is a sentence of $\mathcal{L}_{\infty,\omega}(\mathcal{L})$. Let ψ^c be the result of replacing in ψ each occurrence of a sort S with a sort S^c, each occurrence of a relation symbols E by a relation E^c and each occurrence of a function symbol f by a function symbol f^c. Further let $\widehat{\psi}$ be the result of (formally) replacing each occurrence of a sort S with the encoded sort $\ddot{\mathcal{S}}$, each occurrence of a relation E by an encoded subset $\ddot{E}$ and each occurrence of a function symbol f by an encoded function $\ddot{f}$. We then use $\widehat{\psi}$ as a shorthand for $\bigwedge_{c \in \text{obj}(C)} \psi^c$.

3.1.2. *Covers.* Suppose $\ddot{\mathcal{S}}$ is an encoded sort and $\ddot{E}_0$ and $\ddot{E}_1$ are encoded subsets of $\ddot{\mathcal{S}}$. In this section we will define the component which say that $\ddot{E}_1$ is the closure of $\ddot{E}_0$ in $\ddot{\mathcal{S}}$. We will do this by adding an (encoding of an) initial segment of the ordinals to our theory and then adding structure which allows us to iterate $\mathbf{a}^{\alpha}(\ddot{E}_0) \cap \ddot{\mathcal{S}}$ through these ordinals. In order to do this we will (in general) need our structure to contain all ordinals less than or equal to $|J_C|^+ + 1$, and this can only be expressed in $\mathcal{L}_{|J_C|^{++},\omega}$ and not in $\mathcal{L}_{|J_C|^+,\omega}$. As such it will important to define the ordinals in such a way that if we happen to have $\mathbf{a}^{\alpha}(\ddot{E}_0) \cap \ddot{\mathcal{S}} = \ddot{E}_1$ with $\alpha < |J_C|^+$, then our encoded models will not be saddled with unnecessary, overly complex, structure coming from lager ordinals than are necessary.

We first define (our encoding of) the ordinals.

Definition 3.9. *We say $\ddot{\mathcal{O}}_\gamma$ is an* **encoding of ordinals** *(up to $\gamma + 2$) if it is a component which contains:*

- *A sort O.*
- *Constants $\{\widehat{i} : i \le \gamma + 1\} \cup \{\widehat{\infty}, \widehat{\infty_{-1}}\}$ of sort O*
- *A relation $\le$ of type $\langle O, O\rangle$.*

and which proves:

- *$\le$ is a linear order.*
- *$\widehat{\infty_{-1}}$ is the predecessor of $\widehat{\infty}$.*
- *$(\forall x : O) \bigvee_{i \le \gamma+1} x = \widehat{i}$.*
- *If $i \le j \le \gamma + 1$ then $\widehat{i} \le \widehat{j}$.*
- *If $i \le j \le k \le \gamma + 1$ then $\widehat{i} = \widehat{j} \to \widehat{i} = \widehat{k}$.*

We will abuse notation in what follows and treat O as an encoded sort where $O^c = O$ for all $c \in \text{obj}(C)$ and $O^f = id_O$ for all $f \in C[c, d]$. Unlike other components of which a model may have many different copies, we will require that any structure which realizes this component realizes it only once. Further we will assume that all such structures realizes it with the (exact) same sort O and the same relation $\le$ (although they may realize it with different constants).

The following lemma is then easily checked.

Lemma 3.10. *If $\gamma_0 < \gamma_1$ then*

- *$Lan[\ddot{\mathcal{O}}_{\gamma_0}] \subseteq Lan[\ddot{\mathcal{O}}_{\gamma_1}]$*
- *$\vdash Th[\ddot{\mathcal{O}}_{\gamma_0}] \to Th[\ddot{\mathcal{O}}_{\gamma_1}]$.*

- *Every model of $\ddot{\mathcal{O}}_{\gamma_0}$ has a unique expansion to a model of $\ddot{\mathcal{O}}_{\gamma_1}$.*

Note the expansion of a model of $\ddot{\mathcal{O}}_{\gamma_0}$ to $\ddot{\mathcal{O}}_{\gamma_1}$ simply sets $\widehat{i} = \widehat{\gamma_0 + 1}$ for all $\gamma_0 < i$.

Definition 3.11. *Suppose $\mathcal{M} \vDash \ddot{\mathcal{O}}_\gamma$. We define the* **height** *of $\mathcal{M}$ to be the order type of $(O^{\mathcal{M}}, \leq^{\mathcal{M}})$.*

Let $\mathrm{Lim}(x) \coloneqq (\forall \beta : O)\beta < x \to (\exists \gamma : O)\beta < \gamma < x$. $\mathrm{Lim}(x)$ is a Σ_1 formula of type O which holds if and only if there is no largest element less than x. The purpose of having the ordinals is to allow us to give the following (inductive) definition.

Definition 3.12. *We say $\ddddot{Cov}_\gamma(\ddot{E}_0, \ddot{E}_1)$ is a (γ-)***witness** *to $\vec{E}_0$ covering $\vec{E}_1$ if it is a component which contains:*

- *An encoding of ordinals up to γ, $\ddot{\mathcal{O}}_\gamma$.*
- *Encoded subsets $\ddot{E}_0, \ddot{E}_1$ of type $\ddot{\mathcal{S}}$.*
- *An encoded subset $\ddot{W}$ of type $\ddot{\mathcal{S}} \times O$.*

and which proves

1. $(\forall x : \ddot{\mathcal{S}})\ddot{E}_0(x) \leftrightarrow \ddot{W}(x, \widehat{0})$.
2. $(\forall x : \ddot{\mathcal{S}})\ddot{E}_1(x) \leftrightarrow \ddot{W}(x, \widehat{\infty})$.
3. $(\forall x : \ddot{\mathcal{S}})(\forall \alpha : O)Lim(\alpha) \to [\ddot{W}(x, \alpha) \leftrightarrow (\exists \beta : O)\beta < \alpha \wedge \ddot{W}(x, \beta)]$.
4. $\bigwedge_{c \in obj(C)}(\forall x : S^c)(\forall \alpha : O)\neg Lim(\alpha) \to [W^c(x, \alpha) \leftrightarrow \bigvee_{I \in J_C(c)} \bigwedge_{g \in I(c)}(\exists \beta : O)\beta < \alpha \wedge W^{dom(g)}(S^g(x), \beta)$.
5. $(\forall x : \ddot{\mathcal{S}})\ddot{W}(x, \widehat{\infty}) \leftrightarrow \ddot{W}(x, \widehat{\infty_{-1}})$.

Now an important point to realize is that the witnesses are, more or less, absolute.

Lemma 3.13. *Suppose $\mathcal{M}$ is an $Lan[\ddddot{Cov}_\gamma(\ddot{E}_0, \ddot{E}_1)]$-structure. Then the following are equivalent:*

- *$\ddddot{Cov}_\gamma(\ddot{E}_0, \ddot{E}_1)$ is a γ-witness to $\ddot{E}_0$ covering $\ddot{E}_1$.*
- *For each $\beta \leq \gamma + 1$, $\{x : (x, \widehat{\beta}) \in \ddot{W}^{\mathcal{M}}\} = \mathbf{a}^\beta(\ddot{E}_0^{\mathcal{M}}) \cap \ddot{\mathcal{S}}^{\mathcal{M}}$.*

Proof. This is an easy induction on β, given Definition 3.12 (1), (3) and (4). □

Further we have

Lemma 3.14. *Suppose $\mathcal{M}$ realizes $\ddddot{Cov}_\gamma(\ddot{E}_0, \ddot{E}_1)$ is a γ-witness to $\ddot{E}_0$ covering $\ddot{E}_1$. Then ${\ddot{E}_1}^{\mathcal{M}} = \mathbf{a}({\ddot{E}_0}^{\mathcal{M}}) \cap \ddot{\mathcal{S}}^{\mathcal{M}}$.*

Proof. Suppose $\mathcal{M} \vDash \widehat{\infty_{-1}} = \widehat{\alpha}$. We have by Definition 3.12 (5) and Lemma 3.13 that $\mathbf{a}^\alpha({\ddot{E}_0}^{\mathcal{M}}) \cap \ddot{\mathcal{S}}^{\mathcal{M}} = \mathbf{a}^{\alpha+1}({\ddot{E}_0}^{\mathcal{M}}) \cap \ddot{\mathcal{S}}^{\mathcal{M}}$. The result follows from Definition 3.12 (2).

□

In this way having $\ddddot{\mathrm{Cov}}_\gamma(\ddot{E}_0, \ddot{E}_1)$ be a (γ)-witness to $\ddot{E}_0$ covering $\ddot{E}_1$ does in fact capture the fact that $\ddot{E}_0$ covers $\ddot{E}_1$ in $\ddot{\mathcal{S}}$. Further Lemma 3.13 shows that in this case $\ddot{W}$ is completely determined and the exact nature of the ordinals is unimportant, so long as there are enough of them. In particular the following corollary follows immediately from Lemma 3.13.

Corollary 3.15. *Suppose*

- *$\gamma_0 < \gamma_1$.*
- *$\ddddot{Cov}_{\gamma_0}(\ddot{E}_0, \ddot{E}_1)$ is a γ_0-witness that $\ddot{E}_0$ covers $\ddot{E}_1$ that is realized in $\mathcal{M}_0$.*

- $\ddddot{Cov}_{\gamma_1}(\ddot{E}_0, \ddot{E}_1)$ *is a* γ_1*-witness that* $\ddot{E}_0$ *covers* $\ddot{E}_1$ *that is realized in* $\mathcal{M}_1$*.*
- $\ddot{\mathcal{S}}^{\mathcal{M}_0} = \ddot{\mathcal{S}}^{\mathcal{M}_1}$ *and* $\ddot{E}_0^{\mathcal{M}_0} = \ddot{E}_0^{\mathcal{M}_1}$*.*

Then

- *For all* $x \in \ddot{\mathcal{S}}^{\mathcal{M}_0}$ *and all* $\beta \le \gamma_0+1$*,* $(x, \widehat{\beta}) \in \ddot{W}^{\mathcal{M}_0}$ *if and only if* $(x, \widehat{\beta}) \in \ddot{W}^{\mathcal{M}_1}$*.*
- *For all* $\gamma_0 \le \beta \le \gamma_1 + 1$*,* $\mathcal{M}_1 \vDash (\forall x : \ddot{\mathcal{S}})\ddot{W}(x, \widehat{\beta}) \leftrightarrow \ddot{E}_1(x)$*.*

However not only is $\ddot{W}$-completely determined, but the following two corollaries show that it is provably completely determined.

Corollary 3.16. *Suppose*

$$T_0 \coloneqq \ddddot{Cov}_\gamma(\ddot{E}_0, \ddot{E}_1)[\ddot{W}_0/\ddot{W}] \text{ and } T_1 \coloneqq \ddddot{Cov}_\gamma(\ddot{E}_0, \ddot{E}_1)[\ddot{W}_1/\ddot{W}],$$

i.e. T_0, T_1 *are* $\ddddot{Cov}_\gamma(\ddot{E}_0, \ddot{E}_1)$ *with* $\ddot{W}_0, \ddot{W}_1$ *substituted in for* $\ddot{W}$ *(respectively). We then have*

$$(*) \quad \vdash T_0 \wedge T_1 \to [(\forall x : \ddot{\mathcal{S}})(\forall \alpha : O)\ddot{W}_0(x, \alpha) \leftrightarrow \ddot{W}_1(x, \alpha)]$$

Proof. First notice that if γ is countable and $|J_C| = \omega$, then $(*) \in \mathcal{L}_{\omega_1,\omega}(\mathcal{L})$ (for an appropriate language $\mathcal{L}$). But by Lemma 3.13 $(\forall x : \ddot{\mathcal{S}})(\forall \alpha : O)\ddot{W}_0(x, \alpha) \leftrightarrow \ddot{W}_1(x, \alpha)$ is true in all structures which satisfy $T_0 \wedge T_1$. Hence, by the completeness theorem for $\mathcal{L}_{\omega_1,\omega}(\mathcal{L})$ we have that there is a proof of $(*)$.

Now if we have $(*) \notin \mathcal{L}_{\omega_1,\omega}(\mathcal{L})$, then there is some forcing extension $Set[G]$ of Set where $(T_0 \wedge T_1 \in \mathcal{L}_{\omega_1,\omega}(\mathcal{L}))^{Set[G]}$. But being a γ-witness that $\ddot{E}_0$ covers $\ddot{E}_1$ is absolute and so by the previous paragraph we have there is a proof of $(*)$ in $Set[G]$. But the existence of a proof is absolute and hence there must be a proof of $(*)$ in Set.

□

Corollary 3.16 tells us that the witness predicate $\ddot{W}$ is provably completely determined by $\ddot{E}_0$, $\ddot{\mathcal{S}}$ and the ordinals. In particular this gives justification for not mentioning $\ddot{W}$ as a parameter in the component $\ddddot{\mathrm{Cov}}_\gamma(\ddot{E}_0, \ddot{E}_1)$

Corollary 3.17. *If* $\gamma_0 < \gamma_1$ *then* $\vdash \ddddot{Cov}_{\gamma_0}(\ddot{E}_0, \ddot{E}_1) \to \ddddot{Cov}_{\gamma_1}(\ddot{E}_0, \ddot{E}_1)$

Proof. This follows immediately from Lemma 3.10. □

As the exact nature of the ordinals in our structures will be unimportant we will often want to talk about when two structures minus their ordinals are the same. We therefore have the following definition.

Definition 3.18. *Suppose* $\ddot{\mathcal{O}}_\gamma \subseteq \mathcal{L}$*. Let* $\mathcal{L}'$ *be the language where* $S_{\mathcal{L}'} = S_{\mathcal{L}} - \{S : O \in tc(\{S\})\}$*,* $F_{\mathcal{L}'} = \{f \in F_{\mathcal{L}} : dom(f), cod(f) \in S_{\mathcal{L}'}\}$*, and* $R_{\mathcal{L}'} = \{R \in R_{\mathcal{L}} : dom(R) \in S_{\mathcal{L}'}\}$*. We then say two* $\mathcal{L}$*-structures* $\mathcal{M}, \mathcal{N}$ *are* **equivalent without ordinals***, written* $\mathcal{M} \risingdotseq \mathcal{N}$*, if* $\mathcal{M}|_{\mathcal{L}'} = \mathcal{N}|_{\mathcal{L}'}$*.*

It is worth mentioning that the only components which will make use of the ordinals are covers (and components which use covers).

As a consequence we have that we can find an extension which has a γ-witness to $\ddot{E}_0$ covering $\ddot{E}_1$ if and only if $\ddot{E}_0$ actually covers $\ddot{E}_1$.

Corollary 3.19. *Suppose* $\ddot{\mathcal{S}}, \ddot{E}_0, \ddot{E}_1 \subseteq \mathcal{L}$ *and* $O \notin \mathcal{L}$*. We then have the following are equivalent for an* $\mathcal{L}$*-structure* $\mathcal{M}$*:*

(1) $\mathbf{a}(\ddot{E}_0^{\mathcal{M}}) \cap \ddot{\mathcal{S}}^{\mathcal{M}} = \ddot{E}_1^{\mathcal{M}}$.

(2) *For some* γ *there is an extension* $\mathcal{M}_\gamma$ *of* $\mathcal{M}$ *to an* $\mathcal{L} \cup Lan[\ddot{\ddot{Cov}}_\gamma(\ddot{E}_0, \ddot{E}_1)]$*-structure such that* $\mathcal{M}_\gamma \models \ddot{\ddot{Cov}}_\gamma(\ddot{E}_0, \ddot{E}_1)$.
(3) *Up to isomorphism there is a unique extension* $\mathcal{M}_{|J_C|^+}$ *of* $\mathcal{M}$ *to* $\mathcal{L} \cup Lan[\ddot{\ddot{Cov}}_{|J_C|^+}(\ddot{E}_0, \ddot{E}_1)]$*-structure such that* $\mathcal{M}_{|J_C|^+} \models \ddot{\ddot{Cov}}_{|J_C|^+}(\ddot{E}_0, \ddot{E}_1)$.

Proof. The equivalence of (1) and (2) follows immediately from Lemma 3.14 and the equivalence of (2) and (3) follows from Proposition 2.11 and Lemma 3.13. □

Another easy but important consequence of Lemma 3.13 is the following.

Corollary 3.20. *Suppose*

- $\mathcal{M} \subseteq \mathcal{N}$ *is a substructure.*
- $\mathcal{N} \models \ddot{\ddot{Cov}}_\gamma(\ddot{E}_0, \ddot{E}_1)$.

Then for all $\alpha \leq \gamma + 1$, $\{x : \mathcal{M} \models \ddot{W}(x, \widehat{\alpha})\} = \{x : \mathcal{N} \models \ddot{W}(x, \widehat{\alpha})\} \cap \ddot{\mathcal{S}}^{\mathcal{M}}$.

Proof. By Lemma 3.13 we have $\{x : (x, \widehat{\beta}) \in \ddot{W}^{\mathcal{M}}\} = \mathbf{a}^\beta(\ddot{E}_0^{\mathcal{M}}) \cap \ddot{\mathcal{S}}^{\mathcal{M}}$ and $\{x : (x, \widehat{\beta}) \in \ddot{W}^{\mathcal{N}}\} = \mathbf{a}^\beta(\ddot{E}_0^{\mathcal{N}}) \cap \ddot{\mathcal{S}}^{\mathcal{N}}$. The result then follows from Lemma 2.9 and the fact that $\ddot{E}_0^{\mathcal{M}} = \ddot{E}_0^{\mathcal{N}} \cap \ddot{\mathcal{S}}^{\mathcal{M}}$.

□

3.1.3. *Sieves and Subobjects.* In this section we show how to encode maps from an encoded sort $\ddot{\mathcal{S}}$ to the subobject classifier. Our method will be first to define an encoded sort $\ddot{\mathcal{C}}$ with a constant $\widehat{f}$ of sort $C^{\mathrm{dom}(f)}$ for every $f \in \mathrm{mor}(C)$. We then define an encoded sieve on $c \in \mathrm{obj}(C)$ to be a relation of type C^c which satisfies a specific theory. An encoded subobject will then be an encoded subset $\ddot{R}$ of $\ddot{\mathcal{S}} \times \ddot{\mathcal{C}}$ where for all $x \in \ddot{\mathcal{S}}$, $\{f \in \ddot{\mathcal{C}} : \ddot{R}(x, f)\}$ is a closed sieve and the map $x \mapsto \{f \in \ddot{\mathcal{C}} : \ddot{R}(x, f)\}$ is the desired map from $\ddot{\mathcal{S}}$ to Ω which is encoded by $\ddot{R}$.

Definition 3.21. *We say* $\ddot{\mathbb{C}}$ *is an* **encoding of the morphisms of** C *if it is a component which contains:*

- *An encoded sort* $\ddot{\mathcal{C}}$.
- *For each* $c \in obj(C)$, *a set* $\{\widehat{g} : g \in C[-, c]\}$ *of constants of sort* C^c.

and which proves $Th_{CSi(c)}(C^c)$ *which says*

- *For each* $c \in obj(C)$ *and all* $g_0, g_1 \in C[-, c]$ *with* $g_0 \neq g_1$ *we have* $\widehat{g_0} \neq \widehat{g_1}$.
- *For each* $c \in obj(C)$, $(\forall x : C^c) \bigvee_{g \in C[-,c]} x = \widehat{g}$.
- *If* $g_1 \in C[c, d_0], h_1 \in C[c, d_1]$ *is the pullback of* $g_0 \in C[d_0, e], h_0 \in C[d_1, e]$ *then* $C^{g_0}(\widehat{h_0}) = \widehat{g_1}$.

Like the encoding of ordinals, while the encoding of morphisms of C can be realized in any structure, we require that it be realized at most once and that when it is realized it is always realized by the same language (in all structures).

Definition 3.22. *We say* $\ddot{I}_c$ *is an* **encoded closed sieve** *on* c *if it is a component which contains:*

- $\ddot{\mathbb{C}}$, *the encoding of morphisms of* C.
- *A relation* I *of sort* C^c.

and which proves:

(1) $\bigwedge_{h \in C[d,c], g \in C[e,d]} I(\widehat{h}) \rightarrow I(\widehat{h \circ g})$.
(2) $\bigwedge_{h \in C[d,c]} [\bigvee_{K \in J_C(d)} \bigwedge_{g \in K} I(\widehat{g \circ h})] \rightarrow I(\widehat{h})$.

Suppose $\ddot{\mathbb{C}} \subseteq \mathcal{L}$ and $\psi(\cdot) \in \mathcal{L}_{\infty,\omega}(\mathcal{L})$ is a formula of sort C^c. If $\mathcal{M}$ is an $\mathcal{L}$-structure then we define $\wr\psi(\cdot)\wr^{\mathcal{M}} := \{g : \mathcal{M} \vDash \psi(\widehat{g})\} \subseteq C[-,c]$.

Lemma 3.23. *Suppose $\mathcal{M}$ is an $Lan[\ddot{I}_c]$-structure. Then the following are equivalent*

1. $\mathcal{M} \vDash \ddot{I}_c$.
2. $\wr I(\cdot)\wr^{\mathcal{M}}$ *is a closed sieve on c.*

Proof. That (2) implies (1) is immediate from Definition 3.22. Further notice that by Definition 3.22 (1), if $\mathcal{M} \vDash \ddot{I}_c$ then $\wr I(\cdot)\wr^{\mathcal{M}}$ is a sieve.

Now to get a contradiction assume there is an $\mathcal{M}$ such that $\mathcal{M} \vDash \ddot{I}_c$ but $\wr I(\cdot)\wr^{\mathcal{M}}$ is not closed, i.e. for some $g \in C[d,c]$, $g^*(\wr I(\cdot)\wr^{\mathcal{M}}) \in J_C^{\mathrm{ORD}}(d)$ but $g \notin \wr I(\cdot)\wr^{\mathcal{M}}$. Let α be the level of $g^*(\wr I(\cdot)\wr^{\mathcal{M}})$. Without loss of generality we can assume that α is minimal such that these conditions hold.

Now if the level of $g^*(\wr I(\cdot)\wr^{\mathcal{M}})$ is 0, then by Definition 3.22 (2) we have $g \in \wr I(\cdot)\wr^{\mathcal{M}}$ and hence we can assume $\alpha > 0$. In particular, we can assume that there is a sieve $K \in J_C(d)$ such that for every $f \in K$, $f^*(g^*(\wr I(\cdot)\wr^{\mathcal{M}})) \in J_C^{\mathrm{ORD}}(\mathrm{dom}(f))$ and has level strictly less than α. But by our inductive assumption this implies for each $f \in K$ that $g \circ f \in \wr I(\cdot)\wr^{\mathcal{M}}$. Then by Definition 3.22 (2) we have $g \in \wr I(\cdot)\wr^{\mathcal{M}}$ contradicting our assumption.

Hence whenever $\mathcal{M} \vDash \ddot{I}_c$ we have $\wr I(\cdot)\wr^{\mathcal{M}}$ is closed and (1) implies (2).

□

Lemma 3.23 shows that $\ddot{I}_c$ captures what we mean by a closed sieve on c.

Definition 3.24. *We say $\ddddot{\theta}$ is an* **encoded subobject** *of sort $\ddot{\mathcal{S}}$ if it is a component which contains*

- $\ddot{\mathbb{C}}$, *an encoding of the morphisms of C.*
- *An encoded sort $\ddot{\mathcal{S}}$.*
- *An encoded subset $\ddot{\theta}$ of type $\ddot{\mathcal{S}} \times \ddot{\mathcal{C}}$.*

and which proves for each $c \in obj(C)$:

1. $(\forall y : S^c)\, Th[\ddot{I}_c][\theta^c(y,x)/I(x)]$.
2. $\bigwedge_{f \in C[d,c]} (\forall x : S^c) \bigwedge_{g \in C[e,d]} \theta^c(x, \widehat{f \circ g}) \leftrightarrow \theta^d(S^f(x), \widehat{g})$.

We now want to show that being an encoded subobject captures the notion of being a map to the subobject classifier. Suppose $\mathcal{M}$ is an Lan$[\ddddot{\theta}]$-structure and $\ddddot{\theta}$ is an encoded subobject realized in $\mathcal{M}$. Then let Rep$(\mathcal{M})$ be such that for $x \in (S^c)^{\mathcal{M}}$, Rep$(\mathcal{M})(x) = \wr\theta(x,\cdot)\wr^{\mathcal{M}}$.

Lemma 3.25. *The following are equivalent for a $Lan[\ddddot{\theta}]$-structure $\mathcal{M}$:*

1. $\ddddot{\theta}$ *is an encoded subobject of sort $\ddot{\mathcal{S}}$ realized in $\mathcal{M}$.*
2. *$Rep(\mathcal{M})$ is a map of presheaves from $\ddot{\mathcal{S}}^{\mathcal{M}}$ to Ω.*

Proof. That (2) implies (1) is immediate from the definition. To see (1) implies (2) notice that if $\ddddot{\theta}$ is an encoded subobject of sort $\ddot{\mathcal{S}}$ then Lemma 3.23 tells us that Rep$(\mathcal{M})$ is a function from $\bigcup_{c \in \mathrm{obj}(C)} \ddot{\mathcal{S}}^{\mathcal{M}}(c)$ to $\bigcup_{c \in \mathrm{obj}(C)} \Omega(c)$. But Definition 3.24 (2) is satisfied if and only if for each $f \in C[c,d]$ and $x \in S^c$ we have Rep$(\mathcal{M})(S^f(x)) = f^*(\mathrm{Rep}(\mathcal{M})(x))$, i.e. if Rep$(\mathcal{M})$ is a map of presheaves.

□

Now notice that as Ω is a sheaf, the map ι restricts to a bijection between the categories $\mathrm{Sep}(C, J_C)[-,\Omega]$ and $\mathrm{Sh}(C, J_C)[-,\Omega]$. Hence by Lemma 3.25 we have that if $\mathrm{Rep} : \mathrm{Mod}(\ddot{\theta}) \to \mathrm{Sep}(C, J_C)[-,\Omega]$ then $\overline{q} \circ \iota \circ \mathrm{Rep}$ is a Δ_0-definable bijection between $\mathrm{Mod}(\ddot{\theta})$ and $\mathrm{Sh}(C, J_C)[-,\Omega]$. We call the inverse to $\overline{q} \circ \iota \circ \mathrm{Rep}$, Enc. We will use $\ddot{\theta}^{\mathcal{M}}$ as a shorthand for $\iota \circ \mathrm{Rep}(\mathcal{M}|_{\ddot{\theta}})$.

3.1.4. *Partial Connectives: A Non-Component.* In this section we introduce the one piece which is not a component, i.e. which will not itself be an explicit subset of our encoded structures. Specifically we discuss what it means for a collection of formulas to encode a subset of Ω and for a collection of formulas to encode a partial connective.

Definition 3.26. *Let Lan_{SO} be the language which contains:*

- *$\ddot{\mathbb{C}}$, an encoding of morphisms of C.*
- *For each $c \in obj(C)$ a relation X^c of sort C^c.*

Suppose $\dot{SO}(\varphi) = \langle \varphi^c : c \in obj(C)\rangle$ where for each $c \in obj(C)$, $\varphi^c \in \mathcal{L}_{\kappa^+,\omega}(\{X^c\})$ is a quantifier free sentence. We say $\dot{SO}(\varphi)$ is a **definable subset** *of Ω (of complexity κ) if the following holds:*

(1) $\vdash \varphi^c \to [X^c$ *is an encoded closed sieve*$]$.
(2) $\vdash \bigwedge_{f\in C[d,c]} \left[\bigwedge_{g\in C[e,d]} X^d(\widehat{g}) \leftrightarrow X^c(\widehat{f\circ g})\right] \to [\varphi^c \to \varphi^d]$.

We say that $\dot{SO}(\varphi)^*$ **defines** *the function $\dot{SO}(\varphi)^*(c) \coloneqq \{\wr X^c(\cdot)\wr^{\mathcal{M}} : \mathcal{M} \vDash \varphi^c\}$.*

Definition 3.26 (2) says if $\mathcal{M}$ is an $\{X^c, X^d\}\cup\ddot{\mathbb{C}}$-structure $\mathcal{M}$ and $f^*(\wr X^c(\cdot)\wr^{\mathcal{M}}) = \wr X^d(\cdot)\wr^{\mathcal{M}}$ then $\wr X^d(\cdot)\wr^{\mathcal{M}}$ is in our definable subset whenever $\wr X^c(\cdot)\wr^{\mathcal{M}}$ is.

Lemma 3.27. *If $\dot{SO}(\varphi)$ is a definable subset of Ω then $\dot{SO}(\varphi)^*$ is a subpresheaf of Ω.*

Proof. That $\dot{\mathrm{SO}}(\varphi)^*(c) \subseteq \Omega(c)$ for each $c \in \mathrm{obj}(C)$ follows immediately from Definition 3.26 (1) that $\dot{\mathrm{SO}}(\varphi)^*$ is a subpresheaf follows immediately from Definition 3.26 (2). □

It turns out that every subset of Ω is definable with some complexity.

Lemma 3.28. *For every $Z \subseteq \Omega$ there is a definable subset $\dot{SO}(\varphi)$ of complexity at most $2^{|mor(C)|}$ with $\dot{SO}(\varphi)^* = Z$.*

Proof. For I a closed sieve on c, let $\eta_I \coloneqq \bigwedge_{f\in I} X^c(\widehat{f}) \wedge \bigwedge_{f\notin I} \neg X^c(\widehat{f})$. Then let $\varphi^c \coloneqq \bigvee_{I\in Z(c)} \eta_I \wedge \bigwedge_{I\notin Z(c)} \neg\eta_I$.

□

An example of a definable subset of Ω of complexity $|J_C|$ is $\mathrm{Th}_{\mathrm{CSi}} \coloneqq \langle \mathrm{Th}_{\mathrm{CSi}(c)}(X^c) : c \in \mathrm{obj}(C)\rangle$. It is clear that $\mathrm{Th}^*_{\mathrm{CSi}} = \Omega$ and we say that a definable subset $\dot{\mathrm{SO}}(\varphi)$ of Ω is **total** if $\vdash \bigwedge_{c\in\mathrm{obj}(C)} \varphi^c \leftrightarrow \mathrm{Th}_{\mathrm{CSi}(c)}(X^c)$.

Note that being a definable subset of Ω is an absolute property (i.e. is true in all models of set theory). However, having $\dot{\mathrm{SO}}(\varphi)^* = \Omega$ is not in general absolute. We can think of being total as an absolute analog of having $\dot{\mathrm{SO}}(\varphi)^* = \Omega$.

The following lemma is also immediate.

Lemma 3.29. *If $V_0 \subseteq V_1$ are models of ZFC and $\dot{SO}(\varphi)$ is a definable subset of Ω, then $(\dot{SO}(\varphi)^*)^{V_0} = (\dot{SO}(\varphi)^*)^{V_1} \cap \Omega^{V_0}$.*

We now show how to define partial connectives.

Definition 3.30. *Let $Lan^c_{Con(n)}$ be the language which contains the sort C^c and for each $c \in obj(C)$ relations $Y^c_1, \dots, Y^c_n$ of sort C^c.*

Suppose $\dot{SO}(\varphi)$ is a definable subset of Ω (of complexity κ) and $\dot{Con}(\psi) = \langle \psi^c : c \in obj(C)\rangle$ where for each $c \in obj(C)$, $\psi^c(y) \in \mathcal{L}_{\kappa^+,\omega}(Lan^c_{Con(n)})$ is a quantifier free formula of sort C^c. We say $\langle \dot{SO}(\varphi), \dot{Con}(\psi)\rangle$ is a **definable partial connective** *(of complexity κ) if the following holds:*

1. *For each $c \in obj(C)$, $\vdash \bigwedge_{i\leq n} \varphi^c[Y^c_i(x)/X^c(x)] \to \varphi^c[\psi^c(x)/X^c(x)]$.*
2. *For each $c, d \in obj(C)$ and $f \in C[d,c]$,*

$$\vdash \left[\bigwedge_{i\leq n}\left[\bigwedge_{g\in C[e,d]} Y^d_i(\widehat{g}) \leftrightarrow Y^c_i(\widehat{f\circ g})\right]\right] \to \left[\bigwedge_{g\in C[e,d]} \psi^d(\widehat{g}) \leftrightarrow \psi^c(\widehat{f\circ g})\right].$$

Let $\dot{\mathrm{Con}}(\psi)^* : (\dot{\mathrm{SO}}(\varphi)^*)^n \to \dot{\mathrm{SO}}(\varphi)^*$ be such that $\dot{\mathrm{Con}}(\psi)^*(I_1,\dots,I_n) = I_*$ if and only if (for some $c \in \mathrm{obj}(C)$) there is a $\mathrm{Lan}^c_{\mathrm{Con}(n)}$-structure $\mathcal{M}$ with $\wr Y_i(\cdot)\wr^{\mathcal{M}} = I_i$ for each $i \leq n$ and $\wr\psi^c(\cdot)\wr^{\mathcal{M}} = I_*$. We say that $\langle \dot{\mathrm{SO}}(\varphi), \dot{\mathrm{Con}}(\psi)\rangle$ **defines** the pair $\langle \dot{\mathrm{SO}}(\varphi)^*, \dot{\mathrm{Con}}(\psi)^*\rangle$.

Definition 3.30 (2) says that for $f \in C[d,f]$ and an $\mathrm{Lan}^c_{\mathrm{Con}(n)} \cup \mathrm{Lan}^d_{\mathrm{Con}(n)}$-structure $\mathcal{M}$, if for each $i \leq n$, $f^*(\wr Y^c_i(\cdot)\wr^{\mathcal{M}}) = \wr Y^d_i(\cdot)\wr^{\mathcal{M}}$ then we also have $f^*(\wr\psi^c(\cdot)\wr^{\mathcal{M}}) = \wr\psi^d(\cdot)\wr^{\mathcal{M}}$.

Lemma 3.31. *If $\langle \dot{SO}(\varphi), \dot{Con}(\psi)\rangle$ is a definable partial connective then $\dot{Con}(\psi)^*$ restricts to a map of presheaves from $[\dot{SO}(\varphi)^*]^n$ to $\dot{SO}(\varphi)^*$.*

Proof. That the image of any tuple from $\dot{\mathrm{SO}}(\varphi)^*$ is in $\dot{\mathrm{SO}}(\varphi)^*$ follows from Definition 3.30 (1). That $\dot{\mathrm{Con}}(\psi)^*$ is a function follows from the fact that $\psi^c \in \mathcal{L}_{\infty,\omega}(\mathrm{Lan}^c_{\mathrm{Con}(n)})$ and hence if we have two structures $\mathcal{M},\mathcal{N}$ which agree on $\mathrm{Lan}^c_{\mathrm{Con}(n)}$ then they must agree on ψ^c. Finally, that $\dot{\mathrm{Con}}(\psi)^*$ is a map of presheaves follows from Definition 3.30 (2). □

We call a definable partial connective $\langle \dot{\mathrm{SO}}(\varphi), \dot{\mathrm{Con}}(\psi)\rangle$ **total** if $\dot{\mathrm{SO}}(\varphi)$ is total. We will sometimes refer to total definable partial connectives simply as *definable connectives*. Definable connectives encode maps from Ω^n to Ω (in any model of set theory).

Lemma 3.32. *Suppose $X \subseteq \Omega$ and $\beta : X^n \to X$ is a partial connective. Then there is a definable partial connective $\langle \dot{SO}(\varphi), \dot{Con}(\psi)\rangle$ of complexity at most $|X| \leq 2^{|mor(C)|}$ such that:*

- *$\dot{SO}(\varphi)^* = X$.*
- *$\dot{Con}(\psi)^*$ restricted to X^n equals β.*

Proof. First let $\dot{\mathrm{SO}}(\varphi)$ be as in Lemma 3.28. Next, for I a closed sieve on c, let $\eta^i_I := \bigwedge_{f\in I} Y^c_i(\widehat{f}) \wedge \bigwedge_{f\notin I} \neg Y^c_i(\widehat{f})$. We then let $\psi^c(x) := \bigvee_{I_1,\dots,I_n\in X}[[\bigwedge_{i\leq n}\eta^i_{I_i}] \to \bigvee_{f\in\beta(I_1,\dots,I_n)} x = \widehat{f}]$. □

The following lemma is also immediate.

Lemma 3.33. *If $V_0 \subseteq V_1$ are models of ZFC and $\langle \dot{SO}(\varphi), \dot{Con}(\psi)\rangle$ is a definable partial connective. Then for any $(x \in [\dot{SO}(\varphi)^n]^*)^{V_0}$, $(\dot{Con}(\psi)^*(x))^{V_0} = (\dot{Con}(\psi)^*(x))^{V_1}$.*

We now end with a few important example of definable connective of complexity $|\text{mor}(C)|$.

Example 3.34. *First recall that if A_1, A_2 are subpresheaves of a sheaf A then we define $A_1 \Rightarrow A_2$ to be the subpresheaf where, for $c \in obj(C)$, $e \in A(c)$ if and only if $\bigwedge_{f \in C[d,c]} A(f)(e) \in A_1(d)$ implies $A(f)(e) \in A_2(d)$.*

Now if we interpret A_i as maps $a_i : A \to \Omega$ and $[a_1 \Rightarrow a_2] \in Sh^*(C, J_C)[A, \Omega]$ then for $e \in A(c)$, $[a_1 \Rightarrow a_2](e) = \{f \in C[-,c] : (\forall g \in C[-, dom(f)]) f \circ g \in A_1(c) \to f \circ g \in A_2(c)\}$. Now let

$$\widehat{\Rightarrow}^c(x) := \bigwedge_{f \in C[-,c]} \left[x = \widehat{f} \to \bigwedge_{g \in C[-,dom(f)]} Y_1^c(\widehat{f \circ g}) \to Y_2^c(\widehat{f \circ g}) \right].$$

It is then immediate that $\langle Th_{CSi}, \widehat{\Rightarrow} \rangle$ is a $|mor(C)|$-definable connective which defines the operation $\Rightarrow : \Omega^2 \to \Omega$.

Example 3.35. *Let*

$$\widehat{=}^c(x) := \bigwedge_{f \in C[-,c]} \left[x = \widehat{f} \to \bigwedge_{g \in C[-,dom(f)]} Y_1(\widehat{f \circ g}) \leftrightarrow Y_2(\widehat{f \circ g}) \right].$$

It is then immediate that $\langle Th_{CSi}, \widehat{=} \rangle$ is a $|mor(C)|$-definable connective which defines the operation $=_\Omega : \Omega^2 \to \Omega$.

Example 3.36. *Suppose $a : 1 \to \Omega$. Then we can define the connective*

$$\widehat{a}^c(x) = \bigwedge_{f \in C[-,c]} \bigvee_{f \in a(1)(c)} x = \widehat{f}.$$

It is then immediate that $\langle Th_{CSi}, \widehat{a} \rangle$ is a $|mor(C)|$-definable connective which defines the operation $a : 1 \to \Omega$.

3.2. **Model Components.** In this section we show how to combine basic components to encode sheaf models. We break this into four subsections. In Section 3.2.1 we deal with components associated to sorts, in Section 3.2.2 we deal with components associated to functions and, in Section 3.2.3 we deal with components associated to relations. Then, once we have defined all of these components we combine them to define our encoding of sheaf models.

3.2.1. *Sorts.* First we give a component which pins down when a separated presheaf is isomorphic (as a separated presheaf) to a given fixed separated presheaf.

Definition 3.37. *Suppose A is a separated presheaf. We say $\ddot{Con}_A(\ddot{\mathcal{S}}_A)$* **encodes** *$A$ if it is a component which contains:*

- *An encoded sort $\ddot{\mathcal{S}}_A$.*
- *For each $c \in obj(C)$ and $a \in A(c)$ a constant $\widehat{a}$ of sort S_A^c.*

and which proves:

- *For each $c \in obj(C)$, $(\forall x : S_A^c) \bigvee_{a \in A(c)} x = \widehat{a}$.*
- *For each $c \in obj(C)$, $\bigwedge_{a,a' \in A(c)} \widehat{a} \neq \widehat{a'}$.*
- *For each $g \in mor(C)$, $\bigwedge_{a = A(g)(a')} S_A^g(\widehat{a'}) = \widehat{a}$.*

The following lemma is immediate.

Lemma 3.38. *If $\ddot{\mathcal{S}}_A$ is an encoded sort in a structure $\mathcal{M}$, then $\mathcal{M}$ has an expansion which satisfies $\ddot{Con}_A(\ddot{\mathcal{S}}_A)$ if and only if $\ddot{\mathcal{S}}_A^{\mathcal{M}}$ is isomorphic to A (in $Sep(C, J_C)$).*

We next define the component which encodes products in the category $\mathrm{Sh}^*(C, J_C)$.

Definition 3.39. *We say* $\ddddot{\mathrm{Prod}}(\ddot{\mathcal{S}}_i : i \leq n)$ *is an* **encoded product** *of the encoded sorts* $\ddot{\mathcal{S}}_0, \ldots, \ddot{\mathcal{S}}_n$ *if it is a component which contains:*

- *Encoded sorts* $\ddot{\mathcal{S}}_i$ *for* $i \leq n$.
- *An encoded sort* $\ddot{\mathcal{S}}_*$.
- *Encoded functions* $\ddot{\pi}_i : \ddot{\mathcal{S}}_* \to \ddot{\mathcal{S}}_i$ *for each* $i \leq n$.

and which proves:

- $(\forall x, y : \ddot{\mathcal{S}}_*)[\bigwedge_{i \leq n} \ddot{\pi}_i(x) = \ddot{\pi}_i(y)] \to x = y$.
- $(\forall x_1 : \ddot{\mathcal{S}}_1)\cdots(\forall x_n : \ddot{\mathcal{S}}_n)(\exists x : \ddot{\mathcal{S}}_*) \bigwedge_{i \leq n} \ddot{\pi}_i(x) = x_i$.

It is easy to see that $\ddddot{\mathrm{Prod}}(\ddot{\mathcal{S}}_i : i \leq n)$ is an encoded product in a structure $\mathcal{M}$ if and only if $\langle \ddot{\mathcal{S}}_*^{\mathcal{M}}, \langle \ddot{\pi}_i^{\mathcal{M}} : i \leq n \rangle \rangle$ is a product of $\ddot{\mathcal{S}}_0^{\mathcal{M}}, \ldots, \ddot{\mathcal{S}}_n^{\mathcal{M}}$ in $\mathrm{Sep}(C, J_C)$ if and only if $\langle \ddot{\mathcal{S}}_*^{\mathcal{M}}, \langle \iota(\ddot{\pi}_i^{\mathcal{M}}) : i \leq n \rangle \rangle$ is the distinguished product of $\ddot{\mathcal{S}}_0^{\mathcal{M}}, \ldots, \ddot{\mathcal{S}}_n^{\mathcal{M}}$ in $\mathrm{Sh}^*(C, J_C)$.

Definition 3.40. *We say* $\ddot{=}_{\ddot{\mathcal{S}}}$ *is an* **encoding of equality** *on* $\ddot{\mathcal{S}}$ *if it is a component which contains:*

- *An encoded subset* $\ddot{=}_{\ddot{\mathcal{S}}}$ *of type* $\ddot{\mathcal{S}} \times \ddot{\mathcal{S}}$.

and which proves:

- $(\forall x, y : \ddot{\mathcal{S}}) \ddot{=}_{\ddot{\mathcal{S}}}(x, y) \leftrightarrow x = y$.

3.2.2. *Functions.*

Definition 3.41. *We say* ${}_{\gamma}\ddddot{f}$ *is an* **encoded morphism** *(of height* γ*) with domain* $\ddot{\mathcal{S}}$ *and codomain* $\ddot{\mathcal{T}}$ *if it is a component which contains*

- *Encoded sorts* $\ddot{\mathcal{S}}$ *and* $\ddot{\mathcal{T}}$.
- *Encoded subsets* $\ddot{D}_f$, $\ddot{D}_1$ *of type* $\ddot{\mathcal{S}}$.
- *A (*γ*-)***witness***,* $\ddddot{Cov}_{\gamma}(\ddot{D}_f, \ddot{D}_1)$ *to* $\ddot{D}_f$ *covering* $\ddot{D}_1$.
- *An encoded subset* $\ddot{f}$ *of type* $\ddot{\mathcal{S}} \times \ddot{\mathcal{T}}$.

and which proves

- $(\forall x : \ddot{\mathcal{S}}) \ddot{D}_1(x)$.
- $(\forall x : \ddot{\mathcal{S}}) \ddot{D}_f(x) \leftrightarrow (\exists y : \ddot{\mathcal{T}}) \ddot{f}(x, y)$.
- $(\forall x : \ddot{\mathcal{S}})(\forall y, y' : \ddot{\mathcal{T}}) \ddot{f}(x, y) \wedge \ddot{f}(x, y') \to y = y'$.

Let $\mathrm{Rep} : \bigcup_{\gamma \in \mathrm{ORD}} \mathrm{Mod}({}_{\gamma}\ddddot{f}) \to \mathrm{mor}(\mathrm{Sh}^*(C, J_C))$ be such that when $\mathrm{Rep}(\mathcal{M}) = \langle f, d_f \rangle$ then $\mathrm{dom}(\langle f, d_f \rangle) = \ddot{\mathcal{S}}^{\mathcal{M}} = \ddot{D}_1^{\mathcal{M}}$, $\mathrm{cod}(\langle f, d_f \rangle) = \ddot{\mathcal{T}}^{\mathcal{M}}$, $d_f = \ddot{D}_f^{\mathcal{M}}$ and $f(x) = y$ if and only if $\mathcal{M} \models \ddot{f}(x, y)$. We then immediately have the following lemma.

Lemma 3.42. *Rep is a* Δ_0*-definable surjection with Rep*$(\mathcal{M}_0) =$ *Rep*$(\mathcal{M}_1)$ *if and only if* $\mathcal{M}_0 \simeq \mathcal{M}_1$.

We will use ${}_{\gamma}\ddddot{f}^{\mathcal{M}}$ as a short hand for $\mathrm{Rep}(\mathcal{M}|_{{}_{\gamma}\ddddot{f}})$. We will also omit the subscript representing the ordinals when it is clear from context. In particular if $\ddddot{f}$ is an encoded morphism, the corresponding encoded set which represents the graph of $\ddddot{f}$ will be $\ddot{f}$.

Lemma 3.43. *For every* $\gamma \in$ *ORD and* $\mathcal{M} \in$ *Mod*$({}_{\gamma}\ddddot{f})$ *there is an* $\mathcal{M}^* \in$ *Mod*$({}_{|J_C|^+}\ddddot{f})$ *with* ${}_{\gamma}\ddddot{f}^{\mathcal{M}} = {}_{|J_C|^+}\ddddot{f}^{\mathcal{M}^*}$.

Proof. This follow immediately from Corollary 3.19 and the fact that $\text{Rep}(\mathcal{M}) = \text{Rep}(\mathcal{M}^*)$ if and only if $\mathcal{M} \approx \mathcal{M}^*$. ☐

In this way we see that $_{|J_C|^+}\ddddot{f}$ really does capture the notion of being a morphism in $\text{Sh}^*(C, J_C)$.

3.2.3. *Relations.*

Definition 3.44. *Suppose $\ddot{\mathcal{S}}$ is an encoded sort and suppose $\vec{\varphi}$ is an encoded formula of type $\ddot{\mathcal{S}}$. Let $Th_{CFor}(\vec{\varphi})$ be the sentence which says:*

- $\bigwedge_{c \in obj(C)} (\forall x : S^c) \bigwedge_{I \in J_C(c)} [\bigwedge_{g \in I} \varphi^{dom(g)}(S^g(x)) \to \varphi^c(x)]$.

It is then immediate that if $\ddot{E}$ is an encoded subset of type $\ddot{\mathcal{S}}$ in $\mathcal{M}$ then $\mathcal{M} \vDash \text{Th}_{\text{CFor}}(\ddot{E})$ if and only if $\mathbf{a}^1(\ddot{E}^{\mathcal{M}}) \cap \ddot{\mathcal{S}}^{\mathcal{M}} = \ddot{E}^{\mathcal{M}} \cap \ddot{\mathcal{S}}$, i.e $\ddot{E}^{\mathcal{M}}$ is closed in $\ddot{\mathcal{S}}^{\mathcal{M}}$. If an encoded subset, $\ddot{E}$ of $\ddot{\mathcal{S}}$, satisfies $\text{Th}_{\text{CFor}}(\ddot{E})$ then we say $\ddot{E}$ is a encoded **closed** subset.

Definition 3.45. *We say $\ddddot{Rel}(\ddot{R}_s, \dddot{R})$ is an* **encoded relation** *of type $\ddot{\mathcal{S}}$ if it is a component which contains:*

- *An encoded closed subset $\ddot{R}_s$ of type $\ddot{\mathcal{S}}$.*
- *An encoded subobject $\dddot{R}$ of type $\ddot{\mathcal{S}}$.*

and which proves:

- $\bigwedge_{c,d \in obj(C)} \bigwedge_{g \in C[d,c]} (\forall x : S^c) R^c(x, \widehat{g}) \leftrightarrow R_s^d(S^g(x))$.

The following is then immediate.

Lemma 3.46. *Suppose in a structure $\mathcal{M}$, $\ddot{\mathcal{S}}$ is an encoded sort, $\ddot{R}_s$ is an encoded subset of $\ddot{\mathcal{S}}$ and $\dddot{R}$ is an encoded subobject of type $\ddot{\mathcal{S}}$. Then $\mathcal{M} \vDash \ddddot{Rel}(\ddot{R}_s, \dddot{R})$ if and only if $\ddot{R}_s^{\mathcal{M}} \subseteq \ddot{\mathcal{S}}^{\mathcal{M}}$ is a pullback of $\top : 1 \to \Omega$ along $\dddot{R}^{\mathcal{M}} : \ddot{\mathcal{S}} \to \Omega$ (in $Sh^*(C,, J_C)$).*

In particular the following is immediate from Lemma 2.15 and Lemma 3.25.

Corollary 3.47. *Suppose $\ddot{\mathcal{S}}$ is an encoded sort. Then*

- *For every structure $\mathcal{M}$ which realizes $\dddot{R}$ as an encoded subobject of type $\ddot{\mathcal{S}}$ there is a unique expansion of $\mathcal{M}$ to an $Lan[\ddddot{Rel}(\ddot{R}_s, \dddot{R})]$-structure $\mathcal{M}^*$ where $\mathcal{M}^* \vDash \ddddot{Rel}(\ddot{R}_s, \dddot{R})$.*
- *For every structure $\mathcal{M}$ which realizes $\ddot{R}_s$ as an encoded closed subset of type $\ddot{\mathcal{S}}$ there is a unique expansion of $\mathcal{M}$ to an $Lan[\ddddot{Rel}(\ddot{R}_s, \dddot{R})]$-structure $\mathcal{M}^*$ where $\mathcal{M}^* \vDash \ddddot{Rel}(\ddot{R}_s, \dddot{R})$.*

3.2.4. *Models.* We are finally ready to define an encoding of a sheaf model.

Definition 3.48. *Suppose $\mathfrak{L}$ is a sheaf language. We say $\ddddot{Lan}_\gamma(\mathfrak{L})$ is an* **encoding of sheaf $\mathfrak{L}$-structures** *(of height γ) if it is a component which contains*

- *For every $S \in \mathcal{S}_{\mathfrak{L}}$ an encoded sort $\ddot{\mathcal{S}}$.*
- *For every $f \in \mathcal{F}_{\mathfrak{L}}$ with domain S and codomain T an encoded morphism $_\gamma\ddddot{f}$ (of height γ) with domain $\ddot{\mathcal{S}}$ and codomain $\ddot{\mathcal{T}}$.*
- *For every $R \in \mathcal{R}_{\mathfrak{L}}$ of type S an encoded relation $\ddddot{Rel}(\ddot{R}_s, \dddot{R})$ of type $\ddot{\mathcal{S}}$.*
- *For each $S \in \mathcal{S}_{\mathfrak{L}}$ an encoding of equality $\ddot{=}_{\ddot{\mathcal{S}}}$ on $\ddot{\mathcal{S}}$.*
- *For each $S_p = \langle S_1, \ldots, S_n \rangle \in \mathcal{S}_{\mathfrak{L}}$ an encoded product $\ddddot{Prod}(\ddot{\mathcal{S}}_i : i \le n)[\ddot{\mathcal{S}}_p / \ddot{\mathcal{S}}_*]$ of $\ddot{\mathcal{S}}_1, \ldots, \ddot{\mathcal{S}}_n$.*

- *For each $S \in \mathtt{O}_{\mathfrak{L}}$ an encoding of $r_{\mathfrak{L}}(S)$, $\ddot{Con}_{r_{\mathfrak{L}}}(\ddot{\mathcal{S}})$.*

Let Rep : $\bigcup_{\gamma \in \mathrm{ORD}} \mathrm{Mod}(\ddot{\ddot{\mathrm{Lan}}}_\gamma(\mathfrak{L})) \to \mathfrak{L}$-Structures be such that when $\mathrm{Rep}(\mathcal{M}) = \mathcal{N}$ then for all $S \in \mathtt{S}_{\mathfrak{L}}$, $\ddot{\mathcal{S}}^{\mathcal{M}} = S^{\mathcal{N}}$, for all $f \in \mathtt{F}_{\mathfrak{L}}$, ${}_\gamma\ddot{\ddot{f}}^{\mathcal{M}} = f^{\mathcal{N}}$ and for all $R \in \mathtt{R}_{\mathfrak{L}}$, $\langle \ddot{R}_s^{\mathcal{M}}, \ddot{\ddot{R}}^{\mathcal{M}} \rangle = \langle R_s^{\mathcal{N}}, R^{\mathcal{N}} \rangle$. We then have

Lemma 3.49. *The following hold:*

(1) *Rep is a Δ_0-surjection.*
(2) *$Rep(\mathcal{M}_0) = Rep(\mathcal{M}_1)$ if and only if $\mathcal{M}_0 \simeq \mathcal{M}_1$.*
(3) *For each $\gamma \in ORD$ and each $\mathcal{M} \in Mod(\ddot{\ddot{Lan}}_\gamma(\mathfrak{L}))$ there is (up to isomorphism) a unique $\mathcal{M}^* \in Mod(\ddot{\ddot{Lan}}_{|J_C|^+}(\mathfrak{L}))$ with $Rep(\mathcal{M}) = Rep(\mathcal{M}^*)$.*

Proof. (1) follows immediately from the analogous results for each component. (2) follows from Lemma 3.42. (3) follows from (2) and Corollary 3.19. □

If $\mathcal{N}$ is an $\mathfrak{L}$-structure with $\mathrm{Rep}(\mathcal{M}) = \mathcal{N}$ then we denote by $\mathrm{Enc}(\mathcal{N}) \in \mathrm{Mod}(\ddot{\ddot{\mathrm{Lan}}}_{|J_C|^+}(\mathfrak{L}))$ the structure from Lemma 3.49 (3).

The next lemma follows immediately from Lemma 3.10.

Lemma 3.50. *Suppose $\gamma_0 < \gamma_1$. Then*

- *$Lan[\ddot{\ddot{Lan}}_{\gamma_0}(\mathfrak{L})] \subseteq Lan[\ddot{\ddot{Lan}}_{\gamma_1}(\mathfrak{L})]$.*
- *$\vdash Th[\ddot{\ddot{Lan}}_{\gamma_0}(\mathfrak{L})] \to Th[\ddot{\ddot{Lan}}_{\gamma_1}(\mathfrak{L})]$.*
- *$\ddot{\ddot{Lan}}_{\gamma_0}(\mathfrak{L})$ has complexity $\max\{|J_C|, |\gamma_0|, |\mathfrak{L}|\}$.*

3.3. **Formula and Sentence Components.** In this section we show how to encode sentences. We do this by first showing in Section 3.3.1 how to encode when a formula is named in a structure. Next in Section 3.3.2 we show how to encode simple sentences. Lemma 2.41 then tell us that this is enough to encode arbitrary sentences.

3.3.1. *Formula Components.* We begin showing how to characterize a map in $\mathrm{Sh}^*(C, J_C)$ as well as showing how to characterize various operations on morphisms.

Definition 3.51. *Suppose A, B are separated presheaves and $\alpha = \langle \alpha_f, d_\alpha \rangle \in Sh^*(C, J_C)[A, B]$. We say $\langle\!\langle \ddot{\ddot{g}} :=_\gamma \alpha \rangle\!\rangle$* **defines** *$\alpha$ (with height γ) if it is a component which contains:*

- *An encoding of A, $\ddot{Con}_A(\ddot{\mathcal{S}}_A)$ and an encoding of B, $\ddot{Con}_B(\ddot{\mathcal{S}}_B)$.*
- *An encoded morphism $\ddot{\ddot{g}}$ (of height γ) with domain $\ddot{\mathcal{S}}_A$ and codomain $\ddot{\mathcal{S}}_B$*

and which proves:

- *$\bigwedge_{c \in obj(C)} \bigwedge_{a \in d_\alpha(c)} D_g^c(\widehat{a}) \wedge \bigwedge_{a \notin d_\alpha(c)} \neg D_g^c(\widehat{a})$.*
- *$\bigwedge_{c \in obj(C)} \bigwedge_{a \in A(c), \alpha(a) = b} g^c(\widehat{a}, \widehat{b})$.*

The following lemma is then immediate.

Lemma 3.52. *If $\mathcal{M}$ is a $Lan[\langle\!\langle \ddot{\ddot{g}} :=_\gamma \alpha \rangle\!\rangle]$-structure with $\widehat{a}^{\mathcal{M}} = a$ for all $a \in A$ and $\widehat{b}^{\mathcal{M}} = b$ for all $b \in B$ then $\mathcal{M} \vDash \langle\!\langle \ddot{\ddot{g}} :=_\gamma \alpha \rangle\!\rangle$ if and only $\ddot{\ddot{g}}^{\mathcal{M}} = \alpha$.*

In this way we have encoded the morphism $\alpha \in \mathrm{Sh}^*(C, J_C)$.

Definition 3.53. *We say $\langle\!\langle \ddot{\ddot{g}} :=_\gamma \ddot{\ddot{f}}_1 \circ \ddot{\ddot{f}}_0 \rangle\!\rangle$* **defines the composition** *(of height γ) of $\ddot{\ddot{f}}_1$ with $\ddot{\ddot{f}}_0$ if it is a component which contains:*

- *Encoded morphism $\ddot{\ddot{f}}_0 : \ddot{\mathcal{S}} \to \ddot{\mathcal{T}}$, $\ddot{\ddot{f}}_1 : \ddot{\mathcal{T}} \to \ddot{\mathcal{U}}$ and $\ddot{\ddot{g}} : \ddot{\mathcal{S}} \to \ddot{\mathcal{U}}$ (of height γ).*

and which proves:

- $(\forall x : \ddot{\mathcal{S}})\ddot{D}_g(x) \leftrightarrow (\exists y : \ddot{\mathcal{T}})\ddot{D}_{f_1}(y) \wedge \ddot{f}_0(x, y)$.
- $(\forall x : \ddot{\mathcal{S}})(\forall z : \ddot{\mathcal{U}})\ddot{g}(x, z) \leftrightarrow (\exists y : \ddot{\mathcal{T}})\ddot{f}_0(x, y) \wedge \ddot{f}_1(y, z)$.

The following lemma is then immediate.

Lemma 3.54. *Suppose* $\ddddot{f}_0 : \ddot{\mathcal{S}} \to \ddot{\mathcal{T}}$, $\ddddot{f}_1 : \ddot{\mathcal{T}} \to \ddot{\mathcal{U}}$ *and* $\ddddot{g} : \ddot{\mathcal{S}} \to \ddot{\mathcal{U}}$ *are encoded morphisms (of height* γ*) in* $\mathcal{M}$*. Then* $\mathcal{M} \vDash \langle\!\langle \ddddot{g} :=_\gamma \ddddot{f}_1 \circ \ddddot{f}_1 \rangle\!\rangle$ *if and only* $\ddddot{g}^{\mathcal{M}} = \ddddot{f}_1^{\mathcal{M}} \circ \ddddot{f}_0^{\mathcal{M}}$.

In this way $\langle\!\langle \ddddot{g} :=_\gamma \ddddot{f}_1 \circ \ddddot{f}_0 \rangle\!\rangle$ captures composition of morphisms.

Definition 3.55. *We say* $\langle\!\langle \ddot{E}_1 :=_\gamma \ddddot{f}^{-1}[\ddot{F}] \rangle\!\rangle$ **defines the inverse image** *of* $\ddot{F}$ *by* $\ddddot{f}^{-1}$ *if it is a component which contains:*

- *An encoded morphism* $\ddddot{f}$ *(of height* γ*) with domain* $\ddot{\mathcal{S}}$ *and codomain* $\ddot{\mathcal{T}}$.
- *Encoded subsets* $\ddot{E}_0, \ddot{E}_1$ *of sort* $\ddot{\mathcal{S}}$.
- *A closed encoded subset* $\ddot{F}$ *of sort* $\ddot{\mathcal{T}}$.
- $\ddddot{Cov}_\gamma(\ddot{E}_0, \ddot{E}_1)$*, a* γ-**witness** *to* $\vec{E}_0$ *covering* $\vec{E}_1$.

and which proves:

- $(\forall x : \ddot{\mathcal{S}})\ddot{E}_0(x) \leftrightarrow \ddot{D}_f(x) \wedge (\exists y : \ddot{\mathcal{T}})\ddot{f}(x, y) \wedge \ddot{F}(y)$.

Lemma 3.56. *If* $\langle\!\langle \ddot{E}_1 :=_\gamma \ddddot{f}^{-1}[\ddot{F}] \rangle\!\rangle$ *defines the inverse image of* $\ddot{F}$ *by* $\ddddot{f}^{-1}$ *in* $\mathcal{M}$ *then*

(1) $\ddot{E}_1^{\mathcal{M}}$ *is in the subobject of* $\ddot{\mathcal{S}}^{\mathcal{M}}$ *corresponding to the pullback of* $\ddot{F}^{\mathcal{M}}$ *along* $\ddddot{f}^{\mathcal{M}}$ *in* $Sh^*(C, J_C)$.

(2) *If* $\mathcal{M} \vDash \ddddot{Rel}(\ddot{E}_1, \ddddot{I_E})$ *and* $\mathcal{M} \vDash \ddddot{Rel}(\ddot{F}, \ddddot{I_F},)$ *then* $\ddddot{I_E}^{\mathcal{M}} = \ddddot{f}^{\mathcal{M}} \circ \ddddot{I_F}^{\mathcal{M}}$.

Proof. To see (1) holds note $\ddot{E}_0^{\mathcal{M}}$ is the pullback of $\ddot{F}^{\mathcal{M}} \cap \operatorname{ran}(f^{\mathcal{M}})$ along $f^{\mathcal{M}}$ in $\mathrm{Sep}(C, J_C)$. Then applying $\mathbf{j} \circ \overline{q}$ to all maps and subobjects we see that $\mathbf{j} \circ \overline{q}(\ddot{E}_0^{\mathcal{M}})$ is in the subobject $(\mathbf{j} \circ \overline{q}(\ddddot{f}^{\mathcal{M}}))^{-1}[\mathbf{j} \circ \overline{q}(\ddot{E}_0^{\mathcal{M}})]$, as $\mathbf{j} \circ \overline{q}$ preserves pullbacks. But we then also have by Lemma 3.14 that $\mathbf{j} \circ \overline{q}(\ddot{E}_0^{\mathcal{M}}) = \mathbf{j} \circ \overline{q}(\ddot{E}_1^{\mathcal{M}})$.

Finally (2) then follows immediately from (1) and Lemma 3.46. □

Definition 3.57. *We say* $\langle\!\langle \ddddot{g} :=_\gamma \prod_{i \leq n} \ddddot{f}_i \rangle\!\rangle$ **defines the product** *of* $\langle \ddddot{f}_i : i \leq n \rangle$ *if it is a component which contains:*

- *An encoded product* $\ddddot{Prod}(\ddot{\mathcal{S}}_i : i \leq n)$ *(of height* γ*).*
- *Encoded morphisms* $\ddddot{f}_i : \ddot{\mathcal{T}} \to \ddot{\mathcal{S}}_i$ *(of height* γ*) for* $i \leq n$.
- *An encoded morphism* $\ddddot{g} : \ddot{\mathcal{T}} \to \ddot{\mathcal{S}}_*$ *(of height* γ*).*

and which proves:

- $(\forall x : \ddot{\mathcal{T}})\ddot{D}_g(x) \leftrightarrow \bigwedge_{i \leq n} \ddot{D}_{f_i}(x)$.
- $(\forall x : \ddot{\mathcal{S}})\ddot{g}(x, y) \leftrightarrow \bigwedge_{i \leq n} \ddot{f}_i(x, \pi_i(y))$.

We then easily have the following lemma.

Lemma 3.58. *Suppose* $\ddddot{g} : \ddot{\mathcal{T}} \to \ddot{\mathcal{S}}_*$ *and* $\ddddot{f}_i : \ddot{\mathcal{T}} \to \ddot{\mathcal{S}}_i$ $(i \leq n)$ *are encoded morphisms (of height* γ*).*

(1) *If* $\mathcal{M} \vDash \langle\!\langle \ddddot{g} :=_\gamma \prod_{i \leq n} \ddddot{f}_i \rangle\!\rangle$ *then* $\ddddot{g}^{\mathcal{M}}$ *is a product of* $\langle \ddddot{f}_i^{\mathcal{M}} : i \leq n \rangle$ *(in* $Sh^*(C, J_C)$*).*

(2) *If* $\ddddot{{}_{|J_C|^+} f_i} : \ddot{\mathcal{T}} \to \ddot{\mathcal{S}}_i$ *are encoded morphisms realized in* $\mathcal{M}$*, then there is a unique expansion of* $\mathcal{M}$ *to* $\mathcal{M}^*$ *which contain a new encoded morphism* $\ddddot{{}_{|J_C|^+} g}$ *such that* $\mathcal{M} \vDash \langle\!\langle \ddddot{g} :=_{|J_C|^+} \prod_{i \leq n} \ddddot{f}_i \rangle\!\rangle$.

Proof. (1) Follows immediately from the definition of products. (2) follows from Corollary 3.19. ☐

We now show how to compose encoded subobjects with partial connectives.

Definition 3.59. *Suppose $\langle X, \beta\rangle = \langle \dot{SO}(\varphi)_\beta, \dot{Con}(\psi)_\beta\rangle$ is a definable partial connective in $Lan_{Con(n)}$ and let $\psi^c_*(x) = \psi^c_\beta[F^c_i(x,y)/Y^c_i(y)]$, $\varphi^c_i(x) = \varphi^c_\beta[F^c_i(x,y)/X^c(y)]$ i.e. the result of substituting $F^c_i(x,y)$ in for $Y^c_i(y)$ and $X^c(y)$ everywhere.*

We say $\langle\!\langle \ddot{\ddot{G}} :=_X \beta \circ \langle \ddot{\ddot{F}}_i : i \leq n\rangle\rangle\!\rangle$ **defines composition with** *β if it is a component which contains:*

- *An encoded sort $\ddot{\mathcal{S}}$.*
- *Encoded subobjects $\ddot{\ddot{G}}$, $\{\ddot{\ddot{F}}_i : i \leq n\}$ of type $\ddot{\mathcal{S}}$.*

and which proves:

1. *For all $c \in obj(C)$, $\bigwedge_{i\leq n}(\forall x : S^c)\varphi^c_i(x)$.*
2. *For all $c \in obj(C)$, $(\forall x : S^c)\bigwedge_{g\in C[-,c]} \ddot{\ddot{G}}(x, \widehat{g}) \leftrightarrow \psi^c_*(x, \widehat{g})$.*

Lemma 3.60. *Suppose $\mathcal{M}$ realizes encoded subobjects $\ddot{\ddot{G}}$, $\{\ddot{\ddot{F}}_i : i \leq n\}$. Then the following are equivalent:*

(a) $\mathcal{M} \vDash \langle\!\langle \ddot{\ddot{G}} :=_X \beta \circ \langle \ddot{\ddot{F}}_i : i \leq n\rangle\rangle\!\rangle$.

(b) *Both*

  (i) *For each $i \leq n$, $ran(\ddot{\ddot{F}}_i^{\mathcal{M}}) \subseteq \dot{SO}(\varphi)^*_\beta$.*

  (ii) $\ddot{\ddot{G}}^{\mathcal{M}} = \dot{Con}(\psi)^*_\beta \circ [\prod_{i\leq n} \ddot{\ddot{F}}_i^{\mathcal{M}}]$

Proof. By Lemma 3.27 Definition 3.59 (1) and (b)(i) are equivalent in any encoded model, and by Lemma 3.31 Definition 3.59 (2) and (b)(ii) are equivalent in any encoded model.

☐

Lemma 3.61. *The complexity of $Th[\langle\!\langle \ddot{\ddot{G}} := \beta_X \circ \langle \ddot{\ddot{F}}_i : i \leq n\rangle\rangle\!\rangle]$ is* max*{complexity of $\dot{SO}(\varphi)_\beta$, complexity of $\dot{Con}(\psi)_\beta, |J_C|\}$.*

Now we show how to define quantifiers.

Definition 3.62. *Let $\ddot{\ddot{\mathfrak{Q}}}_\gamma$ be the component which is the union of the following:*

- *An encoded subset $\ddot{E}$ of sort $\ddot{\mathcal{S}}$.*
- *An encoded morphism $\ddot{\ddot{f}}$ from $\ddot{\mathcal{S}}$ to $\ddot{\mathcal{T}}$ (of height γ).*
- *Encoded sets $\ddot{F}_0, \ddot{F}$ of sort $\ddot{\mathcal{T}}$.*
- *A γ-witness $\ddot{\dot{Cov}}_\gamma(\ddot{F}_0, \ddot{F})$*

We say $\langle\!\langle \ddot{F} :=_\gamma (\forall \ddot{\ddot{f}})\ddot{E}\rangle\!\rangle$ **defines universal quantification** *if it the component which contains $\ddot{\ddot{\mathfrak{Q}}}_\gamma$ and further proves:*

- $(\forall y : \ddot{\mathcal{T}})\ddot{F}_0(y) \leftrightarrow [(\forall x : \ddot{\mathcal{S}})\ddot{f}(x,y) \rightarrow \ddot{E}(y)]$.

We say $\langle\!\langle \ddot{F} :=_\gamma (\exists \ddot{\ddot{f}})\ddot{E}\rangle\!\rangle$ **defines existential quantification** *if it is the component which contains $\ddot{\ddot{\mathfrak{Q}}}_\gamma$ and further proves:*

- $(\forall y : \ddot{\mathcal{T}})\ddot{F}_0(y) \leftrightarrow [(\exists x : \ddot{\mathcal{S}})\ddot{f}(x,y) \wedge \ddot{E}(y)]$.

The following lemma is then immediate.

Lemma 3.63. *We have*

(1a) *If $\mathcal{M} \vDash \langle\!\langle \ddot{F} :=_\gamma (\exists \ddot{\ddot{f}})\ddot{E}\rangle\!\rangle$ then $\ddot{F}^{\mathcal{M}}$ is closed in $\ddot{\mathcal{S}}^{\mathcal{M}}$ and in the same subobject as $(\exists \ddot{\ddot{f}}^{\mathcal{M}})\ddot{E}^{\mathcal{M}}$ (in $Sh^*(C, J_C)$).*

(1b) *If* $\mathcal{M} \vDash \langle\!\langle \ddot{F} :=_{\gamma} (\forall \dddot{f})\ddot{E} \rangle\!\rangle$ *then* $\ddot{F}^{\mathcal{M}}$ *is closed and in the same subobject as* $(\forall \dddot{f}^{\mathcal{M}})\ddot{E}^{\mathcal{M}}$ *(in* $Sh^*(C, J_C)$*).*

(2) *Suppose in* $\mathcal{M}$ *realizes* $\ddot{E}$ *as an encoded closed subset of* $\ddot{\mathcal{S}}$ *and* $\dddot{f} : \ddot{\mathcal{S}} \to \ddot{\mathcal{T}}$ *is an encoded morphism (of some height). Then there is an* $\mathcal{M}_0$ *such that* $\mathcal{M}_0 \approx \mathcal{M}$*, the height of* $\mathcal{M}_0$ *is* $|J_C|^+$ *and* $\mathcal{M}_0$ *has an expansion* $\mathcal{M}_0^*$ *which realizes* $\langle\!\langle \ddot{F} :=_{|J_C|^+} (\exists \dddot{f})\ddot{E} \rangle\!\rangle$ *and* $\langle\!\langle \ddot{F} :=_{|J_C|^+} (\forall \dddot{f})\ddot{E} \rangle\!\rangle$*.*

Proof. That (1a) and (1b) hold for any encoded model follows from the definition of quantification and Lemma 3.14. That (2) holds follows from Corollary 3.19. □

Lemma 3.64. *We have the following for* $\gamma_0 < \gamma_1$*.*

- $\vdash Th[\langle\!\langle \ddot{F} :=_{\gamma_0} (\forall \dddot{f})\ddot{E} \rangle\!\rangle] \to Th[\langle\!\langle \ddot{F} :=_{\gamma_1} (\forall \dddot{f})\ddot{E} \rangle\!\rangle]$
- $\vdash Th[\langle\!\langle \ddot{F} :=_{\gamma_0} (\exists \dddot{f})\ddot{E} \rangle\!\rangle] \to Th[\langle\!\langle \ddot{F} :=_{\gamma_1} (\exists \dddot{f})\ddot{E} \rangle\!\rangle]$

Proof. This follows immediately from Corollary 3.17 and the fact that the only mention of the constants $\widehat{\zeta}$ for ordinals ζ are in $\mathrm{Th}[\ddot{\mathcal{O}}_{\gamma_i}]$. . □

We now show how to name conjunctions and disjunctions.

Definition 3.65. *Let* $\dddot{\mathfrak{B}}$ *be the component which is the union of the following:*

- *An encoded sort* $\ddot{\mathcal{S}}$*.*
- *Encoded closed subsets,* $\ddot{F}$ *and* $\ddot{E}_i$*,* $i \in I$*, of sort* $\ddot{\mathcal{S}}$*.*

We let $\langle\!\langle \ddot{F} := \bigwedge_{i \in I} \ddot{E}_i \rangle\!\rangle$ *be the sort which contains* $\dddot{\mathfrak{B}}$ *and further proves:*

- $(\forall x : \ddot{\mathcal{S}})\ddot{F}(x) \leftrightarrow \bigwedge_{i \in I} \ddot{E}_i(x)$

We let $\langle\!\langle \ddot{F} :=_{\gamma} \bigvee_{i \in I} \ddot{E}_i \rangle\!\rangle$ *be the sort which contains* $\dddot{\mathfrak{B}}$ *as well as:*

- *An encoded set* $\ddot{F}_0$ *of sort* $\ddot{\mathcal{S}}$*.*
- *A* γ*-witness that* $\ddot{F}_0$ *covers* $\ddot{F}$*,* $\dddot{Cov}_{\gamma}(\ddot{F}_0, \ddot{F})$*.*

and which proves:

- $(\forall x : \ddot{\mathcal{S}})\ddot{F}_0(x) \leftrightarrow \bigvee_{i \in I} \ddot{E}_i(x)$

The following lemma follows easily from the definition of infinite conjunctions and disjunctions in $\mathrm{Sh}^*(C, J_C)$.

Lemma 3.66. *We have:*

(1a) *If* $\mathcal{M} \vDash \langle\!\langle \ddot{F} := \bigwedge_{i \in I} \ddot{E}_i \rangle\!\rangle$ *then* $\ddot{F}^{\mathcal{M}}$ *is in the same subobject as* $\bigwedge_{i \in I} \ddot{E}^{\mathcal{M}}$ *(in* $Sh^*(C, J_C)$*).*

(1b) *If* $\mathcal{M} \vDash \langle\!\langle \ddot{F}_1 :=_{\gamma} \bigvee_{i \in I} \ddot{E}_i \rangle\!\rangle$ *then* $\ddot{F}^{\mathcal{M}}$ *is in the same subobject as* $\bigvee_{i \in I} \ddot{E}^{\mathcal{M}}$ *(in* $Sh^*(C, J_C)$*).*

(2) *Suppose* $\mathcal{M}$ *realizes* $\ddot{E}_i$ *are encoded closed subsets of* $\ddot{\mathcal{S}}$ *(for* $i \in I$*) and* $\mathcal{M}$ *does not contain any encoded ordinals. Then there is an expansion* $\mathcal{M}^*$ *of* $\mathcal{M}$ *which realizes* $\langle\!\langle \ddot{F} := \bigwedge_{i \in I} \ddot{E}_i \rangle\!\rangle$ *and* $\langle\!\langle \ddot{F} :=_{|J_C|^+} \bigvee_{i \in I} \ddot{E}_i \rangle\!\rangle$*.*

Proof. That (1a) and (1b) hold for any encoded model follows from the definition of infinite conjunctions and disjunctions of subobjects in $\mathrm{Sh}^*(C, J_C)$ and Lemma 3.14. That (2) holds follows from Corollary 3.19. □

Lemma 3.67. *We have the following for* $\gamma_0 < \gamma_1$*.*

- $\vdash Th[\langle\langle \ddot{F}_1 :=_{\gamma_0} \bigvee_{i\in I} \ddot{E}_i \rangle\rangle] \to Th[\langle\langle \ddot{F}_1 :=_{\gamma_1} \bigvee_{i\in I} \ddot{E}_i \rangle\rangle]$

Proof. This follows immediately from Corollary 3.17. □

Now that we have all of these sentences which name formulas, we can say when a fragment is named.

Definition 3.68. *Suppose A is a fragment and $N_A = \langle H_\varphi : \varphi \in A\rangle \subseteq F_{\mathcal{L}} \cup R_{\mathcal{L}} - A$ and let $Q_A = \langle Q_\beta : \beta \circ_{X_\beta} \prod_{i\leq n} f_i \in A\rangle$ where $Q_\beta = \langle \varphi_\beta, \psi_\beta\rangle$ is a definable partial connectives which defines $\langle X_\beta, \beta\rangle$. Let $\ddot{\ddot{Nam}}_\gamma(A, N_A, Q_A)$ be the component which contains:*

(0) *For each $A, B \in O_{\mathcal{L}}$ and $\alpha : r_{\mathcal{L}}(A) \to r_{\mathcal{L}}(B)$ in A, $\langle\langle \ddot{\ddot{H}}_\alpha :=_\gamma \alpha\rangle\rangle$.*
(1a) *For each $g \circ f \in A$ with $cod(g) \in S_L$, $\langle\langle \ddot{\ddot{H}}_{g\circ f} :=_\gamma \ddot{\ddot{H}}_g \circ \ddot{\ddot{H}}_f \rangle\rangle$.*
(1b) *For each $g \circ f \in A$ with $cod(g) = \Omega$ then $\langle\langle (\ddot{H}_{g\circ f})_s :=_\gamma \ddot{\ddot{f}}^{-1}[(\ddot{H}_f)_s]\rangle\rangle$*
(2) *For each $\{f_i : i \leq n\}$ all with codomain in $S_{\mathcal{L}}$ and with $\prod_{i\leq n} f_i \in A$, $\langle\langle \ddot{\ddot{H}}_{\prod_{i\leq n} f_i} :=_\gamma \prod_{i\leq n} \ddot{\ddot{H}}_{f_i}\rangle\rangle$.*
(3) *For each $\beta \circ_{X_\beta} \prod_{i\leq n} f_i \in A$, $\langle\langle \ddot{\ddot{H}}_{\beta\circ_{X_\beta} \prod_{i\leq n} f_i} := Q_\beta \circ \langle \ddot{\ddot{H}}_{f_i} : i \leq n\rangle\rangle\rangle$.*
(4) *For each $\bigvee_{i\in I} E_i \in A$, $\langle\langle (\ddot{H}_{\bigvee_{i\in I} E_i})_s :=_\gamma \bigvee_{i\in I} (\ddot{H}_{E_i})_s\rangle\rangle$.*
(5) *For each $\bigwedge_{i\in I} E_i \in A$, $\langle\langle (\ddot{H}_{\bigwedge_{i\in I} E_i})_s := \bigwedge_{i\in I} (\ddot{H}_{E_i})_s\rangle\rangle$*
(6) *For each $(\forall_f)E \in A$, $\langle\langle (\ddot{H}_{(\forall_f)E})_s :=_\gamma (\forall_{\ddot{\ddot{H}}_f})(\ddot{H}_E)_s\rangle\rangle$.*
(7) *For each $(\exists_f)E \in A$, $\langle\langle (\ddot{H}_{(\exists_f)E})_s :=_\gamma (\exists_{\ddot{\ddot{H}}_f})(\ddot{H}_E)_s\rangle\rangle$.*

We then have the following theorem which sums up the the results of this section.

Theorem 3.69. *Suppose $A \subseteq For_{\kappa^+,\omega}(\mathcal{L})$ is a fragment each of whose formulas is legal for a sheaf model $\mathcal{M}^*$, $\mathcal{M}^* = Rep(\mathcal{M})$, i.e. $\mathcal{M}^*$ is a representation of $\mathcal{M}$ and that N_A and Q_A are as in Definition 3.68. Then the following hold:*

(1) *If $\mathcal{M} \vDash \ddot{\ddot{Nam}}_\gamma(A, N_A, Q_A)$ then $\{H_\varphi : \varphi \in A\}$ are names for A in $\mathcal{M}^*$.*
(2) *If $\{H_\varphi : \varphi \in A\}$ are names for A in $\mathcal{M}^*$ then there is an $\mathcal{M}_0$ such that $\mathcal{M}_0 \simeq \mathcal{M}$ and $\mathcal{M}_0 \vDash \ddot{\ddot{Nam}}_{|J_C|^+}(A, N_A, Q_A)$.*
(3) *$\vdash Th[\ddot{\ddot{Nam}}_{\gamma_0}(A, N_A, Q_A)] \to Th[\ddot{\ddot{Nam}}_{\gamma_1}(A, N_A, Q_A)]$ if $\gamma_0 < \gamma_1$.*
(4) *The complexity of $\ddot{\ddot{Nam}}_\gamma(A, N_A, Q_A)$ is at most the supremum of the set $\{|A|, |\gamma|, |J_C|$, the complexity of all Q_β in $Q_A\}$.*

Proof. (1) and (2) follow immediately from Lemma 3.52, Lemma 3.54, Lemma 3.56, Lemma 3.58, Lemma 3.58, Lemma 3.63, Lemma 3.66. (3) follows from Corollary 3.17 and the fact that the only difference between $\ddot{\ddot{\mathrm{Nam}}}_{\gamma_0}(X)$ and $\ddot{\ddot{\mathrm{Nam}}}_{\gamma_1}(X)$ occur on components which witness one encoded subset covering another. □

Theorem 3.69 is the most important of result of Section 3.3.1. It shows how we can collect all of our encodings together to get names for all formulas in a fragment.

3.3.2. *Sentence Components.* We now show how to encode basic sentences. This, along with the encoding of names, will allow us to encode arbitrary sheaf sentences.

Definition 3.70. *We say $\langle\langle \ddot{\ddot{f}}_0 \equiv_\gamma \ddot{\ddot{f}}_1\rangle\rangle$* **defines the equivalence** *of $\ddot{\ddot{f}}_0$ and $\ddot{\ddot{f}}_1$ if it is a component which contains:*

- *Encoded morphisms $\ddot{\ddot{f}}_0$ and $\ddot{\ddot{f}}_0$ (of height γ) both of which have domain $\ddot{\mathcal{S}}$ and codomain $\ddot{\mathcal{T}}$.*

and which proves:

- $(\forall x : \ddot{\mathcal{S}})(\forall y : \ddot{\mathcal{T}})\ddot{D}_{f_0}(x) \wedge \ddot{D}_{f_1}(x) \to [\ddot{f}_0(x,y) \leftrightarrow \ddot{f}_1(x,y)]$.

The following lemma is immediate.

Lemma 3.71. *Suppose $\ddddot{f}_0$ and $\ddddot{f}_1$ are encoded morphisms (of height γ) from $\ddot{\mathcal{S}}$ to $\ddot{\mathcal{T}}$ realized in $\mathcal{M}$. Then $\mathcal{M} \models \langle\!\langle \ddddot{f}_0 \equiv_\gamma \ddddot{f}_1 \rangle\!\rangle$ if and only if $\ddddot{f}_0^{\mathcal{M}} \equiv \ddddot{f}_1^{\mathcal{M}}$ (as morphisms in $Sh^*(C, J_C)$).*

Definition 3.72. *We say $\langle\!\langle \ddddot{Rel}(\ddot{R}^0_s, \ddddot{R}^0) \equiv \ddddot{Rel}(\ddot{R}^1_s, \ddddot{R}^1) \rangle\!\rangle$* **defines the equivalence** *of $\ddddot{Rel}(\ddot{R}^0_s, \ddddot{R}^0)$ and $\ddddot{Rel}(\ddot{R}^1_s, \ddddot{R}^1)$ if it is a component which contains:*

- *Encoded relations $\ddddot{Rel}(\ddot{R}^0_s, \ddddot{R}^0)$ and $\ddddot{Rel}(\ddot{R}^1_s, \ddddot{R}^1)$ of type $\ddot{\mathcal{S}}$.*

and which proves:

- $(\forall x : \ddot{\mathcal{S}})\ddot{R}^0_s(x) \leftrightarrow \ddot{R}^1_s(x)$.

The following lemma is immediate.

Lemma 3.73. *Suppose $\ddddot{Rel}(\ddot{R}^0_s, \ddot{R}^0)$ and $\ddddot{Rel}(\ddot{R}^1_s, \ddot{R}^1)$ are encoded relations of type $\ddot{\mathcal{S}}$ realized in $\mathcal{M}$. Then $\mathcal{M} \models \langle\!\langle \ddddot{Rel}(\ddot{R}^0_s, \ddddot{R}^0) \equiv \ddddot{Rel}(\ddot{R}^1_s, \ddddot{R}^1) \rangle\!\rangle$ if and only if $\ddddot{R}^{0\,\mathcal{M}} \equiv \ddddot{R}^{1\,\mathcal{M}}$ (as morphisms in $Sh^*(C, J_C)$).*

Definition 3.74. *We then define the following by induction on simple sentences:*

- *For $T \in Sen_{\infty,\omega}(\mathfrak{L})$ let $\langle\!\langle \check{\neg} T \rangle\!\rangle$ be the component which contains $\langle\!\langle T \rangle\!\rangle$ and which proves $\neg Th[\langle\!\langle T \rangle\!\rangle]$.*
- *For $\{T_i : i \in K\} \subseteq Sen_{\infty,\omega}(\mathfrak{L})$ let $\langle\!\langle \check{\bigvee}_{i\in K} T_i \rangle\!\rangle$ and $\langle\!\langle \check{\bigwedge}_{i\in K} T_i \rangle\!\rangle$ be components which contain each $\langle\!\langle T_i \rangle\!\rangle$ and where:*
  - *$\langle\!\langle \check{\bigvee}_{i\in K} T_i \rangle\!\rangle$ proves $\bigvee_{i\in K} Th[\langle\!\langle T_i \rangle\!\rangle]$.*
  - *$\langle\!\langle \check{\bigwedge}_{i\in K} T_i \rangle\!\rangle$ proves $\bigwedge_{i\in K} Th[\langle\!\langle T_i \rangle\!\rangle]$.*

Lemma 3.75. *If T is a simple sentence and $\mathcal{M}^*$ is a sheaf structure with $M^* = Rep(M)$, then $\mathcal{M}^* \models T$ if and only if $\mathcal{M} \models \langle\!\langle T \rangle\!\rangle$.*

Proof. This is immediate from the Definition 3.74, Lemma 3.71 and Lemma 3.73. ☐

Note that if T is not legal for $\mathcal{M}$ then we have $\mathcal{M} \not\models T$ and $\mathcal{M} \not\models \neg T$. In particular we have restricted our attention here to simple sentences as simple sentences are legal in all structures.

Definition 3.76. *Suppose $T \in Sen_{\kappa^+,\omega}(\mathfrak{L})$ and we have the notation from Lemma 2.41 and Definition 3.68. Let $\ddddot{Sen}_\gamma(T)$ be the component which is the union of the following components:*

- *$\ddddot{Nam}_\gamma(P(T), N_T, Q_{P(T)})$.*
- *$\langle\!\langle T_{N_T} \rangle\!\rangle$.*

Theorem 3.77. *Suppose $T \in Sen_{\kappa^+,\omega}(\mathfrak{L})$ is legal for $\mathcal{M}^*$. Then the following hold:*

1. *If $Rep(\mathcal{M}) = \mathcal{M}^*$ and $\mathcal{M} \models \ddddot{Sen}_\gamma(T)$ then $\mathcal{M}^* \models T$.*
2. *If $\mathcal{M}^* \models T$ then there is an expansion $\mathcal{M}^*_0$ of $\mathcal{M}^*$ with everything in $P(T)$ and an $\mathcal{M}_0$ such that $Rep(\mathcal{M}_0) = \mathcal{M}^*_0$ and $\mathcal{M}^*_0 \models \ddddot{Sen}_{|J_C|^+}(T)$.*
3. *$\vdash Th[\ddddot{Sen}_{\gamma_0}(T)] \to Th[\ddddot{Sen}_{\gamma_1}(T)]$ if $\gamma_0 < \gamma_1$.*

(4) *The complexity of* $\ddot{\overline{Sen}}_\gamma(T)$ *is at most the supremum of* $\{tc(T), |\gamma|, |J_C|$, *the complexity of all* $Q_\beta \in Q_{P(T)}\}$.

Proof. (1) and (2) follows from Lemma 2.41 and Corollary 3.19, Theorem 3.69 and Lemma 3.75. (3) and (4) then follow from Theorem 3.69 (3) and (4). □

4. Applications

In this section we will use our knowledge about models of $\mathcal{L}_{\infty,\omega}$ in *Set* along with the encodings from Section 3 to deduce facts about sheaf models and sheaf sentences.

4.1. **Elementary Chains.** As an example of the strength of our encoding we provide a proof of the directed embedding theorem. We will deduce this from the corresponding result for *Set*-structures.

Theorem 4.1 (Directed Embedding Theorem). *Let* $A \subseteq For_{\kappa^+,\omega}(\mathfrak{L})$ *be a fragment and suppose* $\langle I, \leq\rangle$ *is a partial order such that every pair of elements has an upper bound. Further suppose* $\mathfrak{D} = \langle\{\mathcal{M}_i : i \in I\}, \{\alpha^{i,j} : \mathcal{M}_i \to \mathcal{M}_j, i \leq j\}\rangle$ *is a directed system of sheaf models such that each formula in* A *is valid for each* $\mathcal{M}_i$ *and each* $\alpha^{i,j}$ *preserves all formulas in* A. *Then* $\mathfrak{D}$ *has a directed limit* $\langle\mathcal{M}_+, \langle\alpha^i : \mathcal{M}_i \to \mathcal{M}_+, i \in I\rangle\rangle$ *where:*

(1) *Each each* $\varphi \in A$ *is valid for* $\mathcal{M}_+$ *and each* α^i *preserves all* $\varphi \in A$.
(2) *Suppose* $T \in Sen_{\infty,\omega}(\mathfrak{L})$ *is such that* $\mathcal{M}_i \vDash T$ *for all* $i \in I$ *and* $P(T) \subseteq A$. *Then* $\mathcal{M}_+ \vDash T$.
(3) $\bigcup_{S\in S_{\mathfrak{L}}} |S^{\mathcal{M}_+}| + |\mathfrak{L}| = \bigcup_{i\in I} \bigcup_{S\in S_{\mathcal{L}}} |S^{\mathcal{M}_i}| + |\mathfrak{L}|$

Proof. First note that by Proposition 2.27 and the fact that (1) and (2) are closed under isomorphisms of directed diagrams it suffices to restrict attention to total directed systems with total directed limits.

To see (1) holds note from Lemma 2.37 that if $\mathcal{M}_i^A$ is an expansion of $\mathcal{M}_i$ by (only) adding names for all formulas in A (in the same way) then $\mathfrak{D}^A = \langle\{\mathcal{M}_i^A : i \in I\}, \{\alpha^{i,j} : i \leq j\}\rangle$ is a directed system as well. Let $\mathcal{M}_+^*$ be the directed limit of this system. We then have by Lemma 2.37 that if $\mathcal{M}_+^* = \mathcal{M}_+^A$ than each α^i preserves all formulas in A.

Notice though that as we can assume all maps $\alpha^{i,j}$ are total we also have that $\langle\{\mathrm{Enc}(\mathcal{M}_i^A) : i \in I\}, \{\mathrm{Enc}(\alpha^{i,j}) : i \leq j\}\rangle$ is a directed system of $\ddot{\overline{\mathrm{Lan}}}_\gamma(\mathfrak{L})$-structures and $\mathrm{Enc}(\mathcal{M}_+^*)$ is its directed limit.

But by Theorem 3.69 we have that $\mathrm{Enc}(\mathcal{M}_i^A) \vDash \mathrm{Th}[\ddot{\overline{\mathrm{Nam}}}_{|J_C|^+}(A)]$ for each $i \in I$ and that $\mathrm{Th}[\ddot{\overline{\mathrm{Nam}}}_{|J_C|^+}(A)]$ is Π_2. Hence because in *Set*-structures Π_2-sentences are preserved by directed limits, $\mathrm{Enc}(\mathcal{M}_+^*) \vDash \mathrm{Th}[\ddot{\overline{\mathrm{Nam}}}_{|J_C|^+}(A)]$ and so by Theorem 3.69 that $\mathcal{M}_+^* = \mathcal{M}_+^A$ (i.e. $\mathcal{M}_+^*$ has names for each formula in A which corresponds to the names in each $\mathcal{M}_i$).

To see that (2) holds it suffices to show, by Lemma 2.41, Lemma 2.37 and the previous paragraph, that (2) holds for simple sentences in $\mathfrak{D}^A$. But if T is a simple sentence then $\mathrm{Th}[\langle\!\langle T\rangle\!\rangle]$ is Π_2 (by Definition 3.70, Definition 3.72 and Definition 3.74). □

4.2. **Downward Löwenheim-Skolem Theorem.** We now prove an analog of the downward Löwenheim-Skolem Theorem.

Theorem 4.2 (Generated Downward Löwenheim-Skolem Theorem). *Suppose $A \subseteq For_{\kappa^+,\omega}(\mathfrak{L}) \cup Sen_{\kappa^+,\omega}(\mathfrak{L})$ is a fragment and Q_A is as in Definition 3.68. Further suppose $|A|, |J_C|, |\mathfrak{L}| \leq \kappa$ and all definable partial connectives in Q_A have complexity at most κ.*

If $\mathcal{M}$ is an $\mathfrak{L}$-structure and $Y \subseteq \bigcup_{S \in \mathcal{S}_{\mathfrak{L}}} S^{\mathcal{M}}$ is of size $\leq \kappa$ then there is an $\mathfrak{L}$-structure $\mathcal{N}_Y$ such that:

1. *$Y \subseteq \mathcal{N}_Y \subseteq \mathcal{M}$ and $\mathcal{N}_Y$ is at most κ-generated,.*
2. *The inclusion map $in : \mathcal{N}_Y \to \mathcal{M}$ preserves all formulas in A.*
3. *For any sentence $T \in A$ valid for $\mathcal{M}$, $\mathcal{M} \vDash T$ if and only if $\mathcal{N}_Y \vDash T$.*

Proof. By Lemma 2.37 and Lemma 2.41 in order to show (2) and (3) it suffices to assume all sentences of A are named in $\mathcal{M}$ and then to find $\mathcal{N}_Y$ for which the theorem holds for simple sentences and where for each $\varphi \in A$ if H_φ is a name for φ in $\mathcal{M}$ it is also a name for φ in $\mathcal{N}_Y$.

Let $\mathcal{M}^*$ be an encoding of $\mathcal{M}$ and let $V \prec_n Set$ be a substructure such that $\{tc(\{Y, \mathfrak{L}, A\}), \mathcal{M}, \mathcal{M}^*\} \in V$ and $|V| = \kappa$. Let $i : V \to V_0$ be the transitive collapse of V and let $\mathcal{N}^* = i(\mathcal{M}^*)$. We then have $i_*^{-1} : i(\mathcal{N}^*) \to \mathcal{M}^*$ is a homomorhpism and we can let $\mathcal{N}_Y = \mathrm{Rep}(\mathrm{ran}(i_*^{-1}))$.

Note that if $T \in A$ then as V_0 is a (Σ_n-)elementary substructure of Set we have $\mathcal{M} \vDash T$ if and only if $i(\mathcal{M}) \vDash T$ (as $i(T) = T$) if and only if $i(\mathcal{M}^*) \vDash \langle\!\langle T \rangle\!\rangle$ if and only if $\mathcal{N}_Y \vDash T$.

Next note that, by our assumption in the first paragraph, we have $\mathcal{M}^* \vDash \ddot{\mathrm{Nam}}_{|J_C|^+}(A)$. Hence as $tc(A) \in V$ we also have $\mathcal{N}^* \vDash i(\ddot{\mathrm{Nam}}_{|J_C|^+}(A))$. But $i(\ddot{\mathrm{Nam}}_{|J_C|^+}(A)) = \ddot{\mathrm{Nam}}_{i(|J_C|^+)}(A)$ and so by Theorem 3.69 (3) we have $\mathcal{N}^* \vDash \ddot{\mathrm{Nam}}_{|J_C|^+}(A)$ as well (as $i(|J_C|^+) \leq |J_C|^+$). Hence if H_φ is a name for φ in $\mathcal{M}$, H_φ is also a name for φ in $\mathcal{N}_Y$ and so the inclusion map preserves φ.

All that is left is to show (1). But clearly $Y \subseteq \mathcal{N}_Y$ and, as $|V| = \kappa$ we have $\mathcal{N}_Y$ must be κ-generated.

□

We now have a similar result for pure size, provided we have some condition on the cardinality.

Corollary 4.3 (Pure Downward Löwenheim-Skolem Theorem). *If $\kappa^{|mor(C)|} = \kappa$ then we can assume $\mathcal{N}_Y$ in Theorem 4.2 has pure size at most κ.*

Proof. This follows immediately from Lemma 2.18.

□

Note that in general we cannot do away with the assumption in Corollary 4.3. For example if (C, J_C) is as in Example 2.19 and $\mathfrak{L}$ has a single sort S, if $Y \subseteq S^{\mathcal{M}}(c)$ with $|Y| = \kappa$ then any substructure $\mathcal{N}_Y \subseteq \mathcal{M}$ has pure size at least $\kappa^{|\mathrm{mor}(C)|}$.

4.3. **Completeness.** We now turn our attention to countable weak sites and sentences of $\mathcal{L}_{\omega_1,\omega}$. In particular we show that there is a completeness theorem in this context.

Definition 4.4. *We say a sentence T is κ-**valid** if whenever T is legal for $\mathcal{M}$ and $\mathcal{M}$ has height at most κ, then $\mathcal{M} \vDash T$. We say T is **valid** if it is κ-valid for all κ.*

For the rest of this section suppose $V_0 \subseteq V_1$ are transitive models of set theory with the same ordinals such that $A \in V_0$ is a fragment.

Definition 4.5. *Suppose $T \in Sen_{\infty,\omega}(\mathfrak{L})$. We define a* **proof up to** α *of T to be a proof of:*

- $Pr_\alpha(T) \coloneqq [\ddot{\ddot{Lan}}_\alpha(\mathfrak{L}) \wedge \ddot{\ddot{Nam}}_\alpha(P(T), N_T, Q_{P(T)})] \to \langle\!\langle T_{N_T} \rangle\!\rangle$.

In what follows it will be useful to have a notion of how complicated a sentence is to express. We define the **complexity** of $T \in \mathrm{Sen}_{\kappa^+,\omega}(\mathfrak{L})$ to be the complexity of $Pr_{|J_C|^+}(T)$.

We then have the following

Lemma 4.6. *If $T \in Sen_{\infty,\omega}(\mathfrak{L})$ then the following are equivalent:*

(0) *T has a proof up to α for all $\alpha \in ORD$.*
(1) *T has a proof up to α for all $\alpha <$ (complexity of $T)^+$.*
(2) *T has a proof up to α for some $\alpha \geq$ (complexity of $T)^+$.*

Proof. Notice that (0) immediately implies (1). Next suppose (0) doesn't hold. In particular suppose that there is no proof of T up to α for some $\alpha \in$ ORD. Now for any particular α having a proof up to α (of T) is absolute (as it is just a matter of having a proof of a sentence of $\mathcal{L}_{\infty,\omega}$, which is an absolute property of a sentence).

Let $V \prec_1 Set$ with $\{\alpha, tc(\{T, P(T), (C, J_C), \mathfrak{L}\})\} \in V$ and $|V|$ = (complexity of T). Let $i : V \to V_0$ be the transitive collapsing map. Then in V_0, $i(T) = T$ does not have a proof up to $i(\alpha) = \alpha'$. Hence T does not have a proof up to α' in Set. But $|V|$ =(complexity T) and so $\alpha' <$ (complexity of $T)^+$, contradicting (1).

In particular we have shown that (0) and (1) are equivalent. But (0) easily implies (2) and (2) implies (1) because Theorem 3.77 (3) implies any model of $\neg Pr_\alpha(T)$ is also a model of $\neg Pr_{\alpha'}(T)$ for all $\alpha' > \alpha$. Hence we are done. ∎

Now as a consequence of Lemma 4.6 we have the following.

Lemma 4.7. *If $V_0 \vDash$ "(complexity $T) = \omega_1$" then the following are equivalent:*

(0) *T is valid in V_1.*
(1) *T has a proof up to α for all $\alpha < \omega_1^{V_0}$ in V_0.*
(2) *In V_1: All sheaf models for which T is legal satisfy T.*
(3) *In V_0: All countably generated sheaf models for which T is legal satisfy T.*

Proof. First note that because $Pr_\alpha(T) \in V_0$ for all $\alpha \in \mathrm{ORD}(V_0)$ and if Pr_α has a proof in V_1 it must also have a proof in V_0 the equivalence of (0) and (1) follows from Lemma 4.6.

Next note that (2) easily implies (3) as being legal, as well as the satisfaction relation between $\mathfrak{L}$-structures and sentences of $\mathrm{Sen}_{\infty,\omega}(\mathfrak{L})$, is absolute.

Next assume (3) holds. If $\alpha < \omega_1^{V_0}$ then working in V_0, $Pr_\alpha(T) \in \mathcal{L}_{\omega_1,\omega}$ is true in all countable models and hence (by the downward Löwenheim-Skolem theorem) is true in all models. But then $Pr_\alpha(T)$ is valid (by the completeness theorem for $\mathcal{L}_{\omega_1,\omega}$ in Set) and so we have $\vdash Pr_\alpha(T)$. (1) follows as α was arbitrary $< \omega_1^{V_0}$.

Finally to show that $\neg$(2) implies $\neg$(1) (and hence (1) implies (2)), notice that if there is some sheaf model in V_1 which doesn't satisfy T then there is some κ such that T doesn't have a proof in V_1 up to κ. But then T doesn't have a proof in V_0 up to κ either (as $Pr_\kappa(T) \in V_0$ and $\mathrm{ORD}(V_0) = \mathrm{ORD}(V_1)$). Hence by Lemma 4.6, there is some $\alpha < \omega_1^{V_0}$ such that T doesn't have a proof up α.

□

Theorem 4.8 (Completeness Theorem)**.** *If $\mathfrak{L}$ is countably generated then the collection of sentences of $Sen_{\omega_1,\omega}(\mathfrak{L})$ which have countable complexity and are valid is $\Sigma_1(\omega_1)$.*

Proof. First notice that the collection of sentences T such that if $\langle X,\beta\rangle \leq T$ then $\langle X,\beta\rangle$ is definable with countable complexity is a (uniformly) Σ_1 collection of sentences. Let Tot be this collection. The $\Sigma_1(\omega_1)$ definition is then $\{T \in Tot : (\forall\alpha < \omega_1) \vdash Pr_\alpha(T)\}$. □

Note that this does not mean that the collection is uniformly Σ_1. The reason is this definition uses ω_1, which is not absolute, as a parameter.

Lemma 4.9. *If $\mathfrak{L}$ is countably generated then the collection of sentences of $Sen_{\omega_1,\omega}(\mathfrak{L})$ which have countable complexity and are valid is Π_2 over the hereditarily countable sets (HC).*

Proof. First notice that any such sentence is in HC. Also notice the the collection $Tot \cap HC$ (from the proof of Theorem 4.8) is Σ_1 over HC. The Π_2 definition is then $\{T \in Tot : (\forall\alpha) \vdash Pr_\alpha(T)\}$. □

The previous completeness theorem only worked for sentences with connectives that were definable by formulas with countable complexity. We now turn to the general case.

In what follows suppose $X \subseteq \Omega$ is countably generated and $T \in \mathrm{Sen}_{\omega_1,\omega}(\mathfrak{L})$ is a $\sqsubseteq$-maximal sentence. Let T_X be a sentence such that all partial connectives have a domain containing X and $T_X \sqsubseteq T$.

Theorem 4.10. *T is valid if and only if T_X is ω_1-valid for every countable X.*

Proof. Left implies Right: Any $\mathcal{M}$-structure which satisfies T and for which T_X is legal also satisfies T_X by Lemma 2.33. Further, if T is valid, then as it is $\sqsubseteq$-maximal, it hold in all sheaf models.

¬Left implies ¬Right: Suppose there is some sheaf model $\mathcal{M}$ such that $\mathcal{M} \not\models T$. Then, as T is $\sqsubseteq$-maximal we must have $\mathcal{M} \models \neg T$. Now let $V \prec_n Set$ be countable with $\mathcal{M} \in V$ and let $i : V \to V_0$ be the transitive collapse. Then $i(\mathcal{M}) \models i(T)$. But as $i(\Omega) \subseteq \Omega$, $i(T)$ is legal for $i(\mathcal{M})$ and $i(\mathcal{M}) \models \neg T_{i(\mathcal{M})}$. Therefore $T_{i(\Omega)}$ is not ω_1-valid (as $i(\mathcal{M})$ must have height $< \omega_1$).

□

Corollary 4.11. *Let $\mathfrak{P}_\omega(\Omega)$ be the collection of countable subpresheaves of Ω. Then the collection of valid sheaf sentences in $Sen_{\omega_1,\omega}(\mathfrak{L})$ is $\Sigma_1(\omega_1, \mathfrak{P}_\omega(\Omega))$.*

Proof. This follows immediately from the Theorem 4.8 and Theorem 4.10. □

4.4. **Barwise Compactness.** In this section we show that for certain admissible sets a version of Barwise's compactness theorem holds.

Theorem 4.12. *Suppose V is a countable Σ_1-admissible set (with respect to some language) such that*

- *$V \models$ "There exists a Σ_1-definable well-ordering".*
- *$\{(C, J_C), \mathfrak{L}\} \in V$.*

- $V \vDash (\exists\kappa)|\kappa| > |mor(C)|$.

Further suppose $T \subseteq V \cap Sen_{\omega_1,\omega}(\mathfrak{L})$ is Σ_1 over V and the collection of definable partial connectives $\langle Q_{\langle X,\beta\rangle} : \langle X,\beta\rangle \leq \check{\bigwedge} T\rangle$ is also Σ_1 over V (where $Q_{\langle X,\beta\rangle}$ defines $\langle X,\beta\rangle$). We then have that if every V-finite subset[1] of T has a model in V then T also has a model.

Proof. Let $\alpha = (|J_C|^+)^V$. By our assumption on T and the definable partial connectives, the collection $T^* := \{\ddot{\ddot{\mathrm{Sen}}}_\alpha(U) : U \in T\}$ is also Σ_1 over V.

Now if $F \subseteq T$ is V-finite we have by assumption that F has a model in V. But then by Proposition 2.11, Lemma 3.13 and the fact that the only components which use encoded ordinals are encoded witnesses to covers, we have $\{\ddot{\ddot{\mathrm{Sen}}}_\alpha(U) : U \in F\}$ also has a model.

By Barwise compactness we then have that T^* has a model $\mathcal{M}^*$. But then by Theorem 3.77 (1) we have $\mathrm{Rep}(\mathcal{M}^*) \vDash T$.

□

Just as with Barwise compactness we can't assume that the resulting model is actually in V. However unlike with Barwise compactness our proof makes fundamental use of the fact that the models realizing the V-finite subsets of T are themselves V-finite.

5. Open Questions

We now discuss some open questions for future research.

Completeness

(1a) In Theorem 4.8 is the parameter ω_1-necessary? Or more concretely, is there a weak site (C, J_C) such that the collection valid sentences of $\mathrm{Sen}_{\omega_1,\omega}(\mathfrak{L})$ in $\mathrm{Sh}(C, J_C)$ is not Σ_1-definable?

(1b) What are some criteria on a weak site (C, J_C) which ensure that the collection of valid sentences of $\mathrm{Sen}_{\omega_1,\omega}(\mathfrak{L})$ in $\mathrm{Sh}(C, J_C)$ is Σ_1-definable with a Σ_1-definition which is independent of the model of set theory? For example the trivial weak site is an instance of such a (C, J_C).

Barwise Compactness

(2a) In Theorem 4.12 can the conditions on V, beyond admissibility, be weakened?

(2b) What are some conditions on a weak site (C, J_C) which will ensure that Barwise Compactness holds for all admissible sets V? For example the trivial weak site is an instance of such a (C, J_C).

Hanf Number

Define the **Hanf number of $\mathrm{Sen}_{\omega_1,\omega}(\mathfrak{L})$ for** (C, J_C) to be the smallest κ such that for any $\varphi \in \mathrm{Sen}_{\omega_1,\omega}(\mathfrak{L})$ if there is a sheaf model $\mathcal{M}$ such that $\mathcal{M} \vDash \varphi$ and $\mathcal{M}$ is not γ-generated for any $\gamma < \kappa$ then for all κ^* there is a sheaf model $\mathcal{M}^*$ such that $\mathcal{M}^* \vDash \varphi$ and $\mathcal{M}^*$ is not γ^*-generated for any $\gamma^* < \kappa^*$.

[1]Recall a set is "V-finite" if it is an element of V.

If (C, J_C) is the trivial site, so $\mathrm{Sh}(C, J_C) = Set$, then it known that the Hanf number of $\mathrm{Sen}_{\omega_1,\omega}(\mathfrak{L})$ for (C, J_C) is $\beth_{\omega_1}$.

(3) Is it the case that for any countable weak site (C, J_C) the Hanf number of $\mathrm{Sen}_{\omega_1,\omega}(\mathfrak{L})$ for (C, J_C) is $\beth_{\omega_1}$?

Generalized Ultrametric Spaces

For a complete lattice Γ a Γ-ultrametric metric space is an ultrametric space whose distances take values in Γ. The category of Γ-ultrametric spaces is equivalent to the category of flabby separated presheaves on Γ^{op} (see [1] for more on this). As such the results of this paper should generalize in a straight forward way to models in the category of Γ-ultrametric spaces (i.e. sheaf models all of whose sorts are flabby). This suggests two collections of questions.

(4a) If we look only at models in the category of Γ-ultrametric spaces (for a fixed Γ), does the collection of valid $\mathrm{Sen}_{\omega_1,\omega}(\mathfrak{L})$ in $\mathrm{Sh}(C, J_C)$ have a Σ_1 definition? If so can the definition be made uniform in Γ? Does Barwise compactness hold for all admissible sets V?

(4b) Do the results of this paper generalize to continuous model theory, i.e. model theory where the sorts are interpreted as complete metric spaces (as in [5])?

References

[1] N. L. Ackerman. Completeness in generalized ultrametric spaces. *p-Adic Numbers Ultrametric Anal. Appl.*, 5(2):89–105, 2013.

[2] Nathanael Leedom Ackerman. Relativized Grothendieck topoi. *Ann. Pure Appl. Logic*, 161(10):1299–1312, 2010.

[3] Nathanael Leedom Ackerman. The number of countable models in categories of sheaves. In *Models, logics, and higher-dimensional categories*, volume 53 of *CRM Proc. Lecture Notes*, pages 1–27. Amer. Math. Soc., Providence, RI, 2011.

[4] Jon Barwise. *Admissible sets and structures*. Springer-Verlag, Berlin, 1975. An approach to definability theory, Perspectives in Mathematical Logic.

[5] Ita¨ı Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov. Model theory for metric structures. In *Model theory with applications to algebra and analysis. Vol. 2*, volume 350 of *London Math. Soc. Lecture Note Ser.*, pages 315–427. Cambridge Univ. Press, Cambridge, 2008.

[6] Wilfrid Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[7] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[8] H. Jerome Keisler. *Model theory for infinitary logic. Logic with countable conjunctions and finite quantifiers*. North-Holland Publishing Co., Amsterdam, 1971. Studies in Logic and the Foundations of Mathematics, Vol. 62.

[9] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[10] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

Department of Mathematics, Harvard University, One Oxford Street, Cambridge, MA 02138

E-mail address: nate@math.harvard.edu