QUANTIFIER RANK SPECTRA OF SCATTERED SENTENCES OF $\mathcal{L}_{\omega_1,\omega}$

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Abstract. In this paper we show that, assuming the existence of theories with a nice metalanguage, there is a set Z, unbounded in ω_1, such that for every $\alpha \in Z$ there is a scattered sentence $S_\alpha \in \mathcal{L}_{\omega_1,\omega}$ where the quantifier rank of S_α is less than or equal to ω but the supremum of the quantifier ranks of countable models of S_α is α.

1. Introduction

In this paper we study the possible quantifier rank spectra of sentences of $\mathcal{L}_{\omega_1,\omega}$ under the assumption that a nice metalanguage exists. We show that under these assumptions there is a set Z of ordinals unbounded in ω_1 such that for each $\alpha \in Z$ there exists a scattered sentence $S_\alpha \in \mathcal{L}_{\omega_1,\omega}$ with quantifier rank less than or equal to ω but with quantifier rank spectrum unbounded in α.

We begin in Section 2 by introducing the theory of infinitely branching trees. In Section 3 we then discuss some more elaborate tree structures. These will include structures which allows us to compare the heights of trees as well as what we call full trees.

In Section 4 we introduce the concept of base predicates and of archetypes. We also discuss the fundamental properties which our structures satisfy. We then describe a collection of archetypes on the theory of full trees.

In Section 5 we continue our discussion of archetypes on our theory of a single tree and introduce the notion of a definable collection of archetypes. We then prove various results concerning theories which have a definable collection of archetypes and show that the collection of archetypes for the theory of full trees satisfies all but two of the conditions needed to be a definable collection. In this section we also discuss the connection between our theories and a counterexample to Vaught's Conjecture (this part will require much stronger assumption than we make for the rest of the paper)

While the theory of trees is a very rich theory, in order to obtain our results we need to "glue" two trees together. In Section 6 we introduce this method of gluing and discuss pairs of archetypes and the conditions we want these pairs of archetypes to satisfy.

In Section 7 we introduce the notion of a definable collection of pair of archetypes and provide a method of constructing models of any theory which has such a collection. In Section 8 we provide a method for bounding the quantifier rank of theories which have a definable collection of archetypes.

Finally in Section 9 we introduce our scattered sentences S_α and prove the quantifier rank spectrum of S_α is cofinal in α.

1.1. **Notation.** We use lowercase letters $\{x, y, z, \dots\}$ to represent elements of a model or variables and lowercase bold letters $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots\}$ to represent finite tuples of elements of a model or of variables. We also use lightface letters to represent elements of their bold counterparts. For example x_i would be an element of $\mathbf{x}$.

We will use commas to break up tuples into disjoint, possibly empty, subtuples. Further, when a formula is expressed with arguments which are variables it is assumed that all free variables of the formula are among those shown. For example in $\varphi(\mathbf{x}, \mathbf{y})$ it is assumed that $\mathbf{x} \cap \mathbf{y} = \emptyset$ and all free variables of φ are among $\mathbf{x} \cup \mathbf{y}$. We use $(\exists^n \mathbf{x})\varphi(\mathbf{x}, \mathbf{y})$ as a shorthand for $(\exists \mathbf{x}_1, \dots, \mathbf{x}_n) \bigwedge_{i \leq n} \varphi(\mathbf{x}_i, \mathbf{y}) \wedge \bigwedge_{i \neq j} \mathbf{x}_i \cap \mathbf{x}_j = \emptyset$. We use $(\exists^\infty \mathbf{x})\varphi(\mathbf{x}, \mathbf{y})$ as a shorthand for $\bigwedge_{i<\omega}(\exists^i \mathbf{x})\varphi(\mathbf{x}, \mathbf{y})$.

If $L \subseteq L'$ are languages and M is an L'-structure then we define $M|_L$ to be the restriction of M to an L structure. If $U \in L'$ is a unary relation we define $(\forall^U x)\varphi(x)$ as $(\forall x)U(x) \rightarrow \varphi(x)$ and $(\exists^U x)\varphi(x)$ as $(\exists x)U(x) \wedge \varphi(\mathbf{x})$. If $\varphi \in \mathcal{L}_{\infty,\omega}(L - \{U\})$, φ^U is obtained by uniformly replacing $(\forall x)$ with $(\forall^U x)$ and $(\exists x)$ with $(\exists^U x)$. We say $U^M \models \varphi \Leftrightarrow M \models \varphi^U$.

We define the *quantifier rank* of a formula $\varphi \in \mathcal{L}_{\infty,\omega}(L')$ as the height of the tree representing φ and we define the quantifier rank of a model as the least quantifier rank of a complete formulas in $\mathcal{L}_{\infty,\omega}$ which it satisfies (see [2]). The *quantifier rank spectrum* of a formula $\varphi \in \mathcal{L}_{\infty,\omega}(L)$ is the set of quantifier ranks of its countable models. We assume that all models mentioned in this paper are countable.

2. Theory of Trees

2.1. **Background Trees.** As with any structure which involves a partial ordering, a choice must be made at the outset with regards to the direction of the ordering symbol, "$<_1$". We have chosen $a <_1 b$ to mean a is further from the root than b.

2.1.1. *Theory T_T.*

Definition 2.1. Let $L_T = \{r, <_1\} \cup \{\text{level}_n : n \in \omega\}$ where r is a constant, level_n are unary relations, and $<_1$ is a binary relation. We will write $x <_1 y$ instead of $<_1 (x, y)$.

Definition 2.2. $T_T \in \mathcal{L}_{\omega_1,\omega}(L_T)$, *the theory of infinitely branching trees*, consists of the conjunction of the following L_T sentences

- $(\forall x)\text{level}_0(x) \leftrightarrow x = r$
- $(\forall x, y)x <_1 y \rightarrow \bigvee_{i \in \omega} \text{level}_{i+1}(x) \wedge \text{level}_i(y)$
- $(\forall x, y, z)(x <_1 y) \wedge (x <_1 z) \rightarrow y = z$

- $(\forall x)x \neq r \rightarrow (\exists y)x <_1 y$
- $(\forall y)(\exists^\infty x)x <_1 y$

For the rest of this paper all theories will contain T_T.

Notice T_T implies that $\text{level}_n(x)$ is equivalent to "there is a $<_1$-chain of length n from x to r" and hence is equivalent to "x is on the nth level of the tree".

Definition 2.3. Let $\leq$ be the transitive closure of $<_1$.

$$x \leq y \Leftrightarrow \bigvee_{n\in\omega} (\exists z_0, z_1, \cdots, z_n)(z_0 = y) \wedge (z_n = x) \bigwedge_{0\leq i<n} z_{i+1} <_1 z_i$$

Also define $x < y \Leftrightarrow x \leq y \wedge x \neq y$.

Definition 2.4. Let $M \models T_T$. We call r^M the *root* of M. If $M \models \text{level}_n(x)$ we say x is on *level* n ($\text{level}(x) = n$).

2.1.2. *Closed Tuples.*

Definition 2.5. Let $M \models T_T$. We say $\mathbf{x} \subset M$ is *closed* if $M \models (\forall a \in \mathbf{x})(\forall b)a <_1 b \rightarrow b \in \mathbf{x}$. We say $M \models \text{Closed}(\mathbf{x})$ if $\mathbf{x}$ is closed.

Closed tuples are important because given any tuple we can find its closure in a definable way. As such whenever we compare two tuples we are really comparing their closures.

Definition 2.6. We define the *closure* of $\{a_1, \ldots, a_n\}$ to be the smallest closed set containing $\{a_1, \ldots, a_n\}$ and we denote it $\overline{\{a_1, \ldots, a_n\}}$.

2.2. **Quantifier Rank.**

Definition 2.7. Let $L_T \subseteq L_K$ be a language and let M, N be L_K structures such that $M, N \models T_T$. Then $p : M \rightarrow N$ is a *Tree Partial Isomorphism* if p is a partial isomorphism from M to N and $M \models \text{Closed}(\text{dom}(p))$.

Definition 2.8. We say $\langle I_\gamma : \gamma < \alpha\rangle$ is a *Sequence of Tree Partial Isomorphisms* from M to N if

- $(\forall \gamma < \beta < \alpha)I_\beta \subseteq I_\gamma$.
- $(\forall \gamma < \alpha)(\forall p \in I_\gamma)$ p is a tree partial isomorphism from M to N.
- If $\beta + 1 < \alpha$, $p \in I_{\beta+1}$, $a \in M$ and $\text{dom}(p) \cup \{a\}$ is closed then there is a $b \in N$ such that $p \cup (a, b) \in I_\beta$
- If $\beta + 1 < \alpha$, $p \in I_{\beta+1}$, $b \in N$ and $\text{range}(p) \cup \{b\}$ is closed then there is an $a \in M$ such that $p \cup (a, b) \in I_\beta$

We call the last two conditions are *tree back and forth property* and if such a sequence exists we say $M \equiv^T_\alpha N$.

Theorem 2.9. *Let $\mathcal{I} = \langle I_i : i < \omega * \alpha\rangle$ be a sequence of tree partial isomorphisms from M and N. Then there is a sequence $\mathcal{J} = \langle J_i : i < \alpha\rangle$ of partial isomorphisms from M to N*

Proof. Let $s \in J_\beta$ if and only $(\exists p \in I_{\omega*\beta}) s \subseteq p$. Then for each $\gamma \leq \beta < \alpha$, $J_\beta \subseteq J_\gamma$ and if $s \in J_\beta$, s is a partial isomorphism.

Next suppose we have $s \in J_{\beta+1}$, $s \subseteq p \in I_{\omega*(\beta+1)}$, and $a \in M$. If $\text{level}_n(a)$ then $\overline{a} = \{a = a_n, a_{n-1}, \ldots, a_0 = r\}$ where $\text{level}_i(a_i)$ and $a_{i+1} <_1 a_i$. Because $p \in I_{\omega*\beta+n+1}$, we can find $b \in N$ such that $\overline{b} = \{b = b_n, b_{n-1}, \ldots, b_0 = r\}$ and $p' = p \cup \{(a_i, b_i) : i \leq n\} \in I_{\omega*\beta}$. So $s \cup (a, b) \in J_\beta$.

We get the other direction (were we are given a $b \in N$ and find an $a \in M$) in exactly the same way. Hence $\mathcal{J}$ is a sequence of partial isomorphisms. □

Corollary 2.10. *If $M \equiv^T_{\omega*\alpha} N$ then $M \equiv_\alpha N$.*

3. Other Trees

3.1. **Subtrees.**

Definition 3.1. Let $L_P = L_T \cup \{P\}$ where P is a unary relation.

Definition 3.2. Let $T_{sub}(P) \in \mathcal{L}_{\omega_1,\omega}(L_P)$ be the conjunction of

- T_T
- $(\forall x, y) P(y) \wedge y <_1 x \rightarrow P(x)$
- $(\forall x)(\exists^\infty y) y <_1 x \wedge \neg P(y)$

Definition 3.3. If $M \models T_{sub}(P)$ let $P^M = \{x \in M : M \models P(x)\}$

Theorem 3.4. *If $(Tr, \preceq)$ is a countable tree (see [3] for a definition) then there is a unique countable model $M \models T_{sub}(P)$ such that $(P^M, \leq) \cong (Tr, \preceq)$.*

Proof. Because any element of M has infinitely many elements extending it which don't satisfy P. □

Having an infinite branching tree in the background doesn't add any structure to our models of trees. However having the background tree will make the presentation of our results much nicer.

Definition 3.5. Let $M \models T_{sub}(P)$ and $b \in M$. We define the *P-Height of* b ($\text{height}_P(b)$) recursively as follows.

- If $M \models \neg P(b)$ then $\text{height}_P(b) = -\infty$
- If $M \models P(b)$ and $(\forall x <_1 b) M \models \neg P(x)$ then $\text{height}_P(b) = 0$
- $\text{height}_P(b) = \sup\{\text{height}_P(x) + 1 : x <_1 b \wedge P(x)\}$ if it is defined.
- $\text{height}_P(b) = \infty$ otherwise.

Definition 3.6. Let $M \models T_{sub}(P)$. We define the *P-spectrum of* M to be $\text{Spec}_P(M) = \{\text{height}_P(b) : b \in M\}$

Definition 3.7. Suppose $M \models T_{sub}(P)$ and suppose $\mathbf{b} = (b_1, \cdots, b_n) \in M$ is a tuple. We define *P-Height Type* of $\mathbf{b}$ to be the function $\text{htype}_P(\mathbf{b}) : (x_1, \ldots, x_n) \rightarrow \omega_1 \cup \{-\infty, \infty\}$ such that $(\forall i \leq n) M \models \text{htype}_P(\mathbf{b})(x_i) = \text{height}_P(b_i)$

There is one more concept we need before we go on. This is the idea of a slant line. In the trees we will be interested in the $\mathcal{L}_{\omega_1,\omega}$-type of an element will be "approximately" determined by its P-height and level. Slant lines are a tool which allow us to compare the complexity of elements which are on different levels of a tree.

Definition 3.8. A function $f : \omega \to \omega_1 \cup \{-\infty, \infty\}$ is a *slant line* if for all $i \in \omega$

- $f(i) > f(i+1)$
- If $f(i)$ is a successor ordinal then $f(i) = f(i+1) + 1$

Here we consider $-\infty > -\infty$ and $\infty > \infty$ and we let $sl_{-\infty}, sl_{\infty}$ be slant lines which have the constant value of $-\infty, \infty$ respectively.

If f is a slant line then we say $f < \alpha$ (for an ordinal α) if $f(i) < \alpha$ for all $i \in \omega$

Definition 3.9. Let $M \models T_{sub}(P) \wedge \text{level}(a) = \text{level}(b) = n$. Let f be a slant line. We say that *a and b have the same P-height up to f* ($\text{height}_P(a)|f = \text{height}_P(b)|f$) if either

- $\text{height}_P(a) = \text{height}_P(b)$ or
- $\text{height}_P(a) \geq f(n)$ and $\text{height}_P(b) \geq f(n)$

Definition 3.10. Let $M, N \models T_{sub}(P)$, $\mathbf{a} = (a_0, \cdots, a_n) \in M$, $\mathbf{b} = (b_0, \cdots, b_n) \in N$ be closed tuples such that $M \models a_i <_1 a_j \Leftrightarrow N \models b_i <_1 b_j$ for all $i, j \leq n$ and let f be a slant line. We say that $\mathbf{a}, \mathbf{b}$ have the same *P-height type up to f* ($\text{htype}_P(\mathbf{a})|f = \text{htype}_P(\mathbf{b})|f$) if

$$(\forall 0 \leq i \leq n)\ \text{height}_P(a_i)|f = \text{height}_P(b_i)|f$$

Theorem 3.11. *For all $\alpha \in \omega_1$ there is a formula $\vartheta^{\leq}_{\alpha}(x) \in \mathcal{L}_{\omega_1,\omega}(L_P)$ such that*

$$T_{sub}(P) \vdash (\forall x)\vartheta^{\leq}_{\alpha}(x) \leftrightarrow \alpha \leq \mathit{height}_P(x)$$

and quantifier rank$(\vartheta_\alpha(x)) = \alpha$

Proof. Let $\vartheta_0(x) = P(x)$ and let $\vartheta^{\leq}_{\beta}(x) \Leftrightarrow \bigvee_{\gamma<\beta} (\exists y <_1 x)\vartheta^{\leq}_{\gamma}(y)$ □

Definition 3.12. Let $L_P \subseteq L_K$ and let $T_K \vdash T_{sub}(P)$ and let $X \subseteq \{-\infty, \infty\} \cup \omega_1$. We define $\mathcal{M}_X(T_K) = \{M : M \models T_K, |M| = \omega, \text{Spec}_P(M) \subseteq X\}$. We will omit T_K when it is clear from context.

3.2. **Comparing Height.**

Definition 3.13. Let $L_H = L_P \cup \{H_{\leq}\}$ where $H_{\leq}$ is a binary relation. We define as shorthand $H_=(a, b) \Leftrightarrow H_{\leq}(a, b) \wedge H_{\leq}(b, a)$ and $H_<(a, b) \Leftrightarrow H_{\leq}(a, b) \wedge \neg H_=(a, b)$.

Definition 3.14. The theory $T_{Height} \in \mathcal{L}_{\omega_1,\omega}(L_H)$ is the conjunction of the following:

Background Trees	$T_{sub}(P)$
Linearity	$(\forall x, y) H_{\leq}(x, y) \vee H_{\leq}(y, x)$
Transitivity	$(\forall x, y, z)[H_{\leq}(x, y) \wedge H_{\leq}(y, z)] \rightarrow H_{\leq}(x, z)$
Tree Ordering	$(\forall x, y, a)[a < x \wedge P(x) \wedge H_{\leq}(x, y)] \rightarrow H_{<}(a, y)$
Base Case	$(\forall x, y) \neg P(x) \rightarrow H_{\leq}(x, y)$

Theorem 3.15. *If $M \models T_{Height}$ then*

- $M \models (\forall a, b) H_{\leq}(a, b) \Rightarrow height_P(a) \leq height_P(b)$
- $M \models (\forall a, b)\ height_P(a) < height_P(b) \wedge height_P(a) \neq \infty \rightarrow H_{<}(a, b).$

Proof. This is by induction on the height of a. □

Theorem 3.16. *For each α there is a formula $\varphi_\alpha(x, y) \in \mathcal{L}_{\omega_1,\omega}(L_P)$ such that whenever $M \models T_{Height}$ and $Spec_P(M) \subseteq \{-\infty\} \cup \alpha$, $M \models \varphi_\alpha(x, y) \leftrightarrow H_{\leq}(x, y)$*

Proof. This follows from Theorem 3.11 and Theorem 3.15. □

3.3. **Full Trees.** In order to construct a scattered theory of trees we need to limit the number of models of our theory with a given spectrum. The best possible limitation we could hope for would be to construct a theory where a model is uniquely determined by its spectrum

In this section we define a theory of trees whose well-founded models are determined by their spectrum. This will provide an example of a theory which contains much of the structure we are studying (for single trees).

Definition 3.17. $T_{Full} \in \mathcal{L}_{\omega_1,\omega}(L_H)$, *the theory of full trees*, is the conjunction of the following:

Comparing Heights	T_{Height}
Fullness	$(\forall x, y) H_{<}(x, y) \rightarrow (\exists^\infty z) H_{=}(x, z) \wedge z <_1 y$

If $M \models T_{Full}$ we say $M|_{L_P}$ is a *full* tree.

In full trees every element is extended by infinitely many elements of every possible height.

Theorem 3.18. *Suppose $M \models T_{Full}$ with $a \in M$. If $\beta < height_P(a)$ then $M \models (\exists^\infty b) b <_1 a$ and $height_P(b) = \beta$.*

Proof. If $\beta < \infty$ this follows from Theorem 3.15. If $\beta = \infty$ this follows from the fact that ($\text{height}_P(a) = \infty$) $\Leftrightarrow$ (the tree extending a in P^M is ill-founded) $\Leftrightarrow$ (($\exists^\infty b) b <_1 a$ and the tree extending b in P^M is ill-founded) □

Theorem 3.19. *If $M, N \models T_{Full}(P)$ and $Spec_P(M) = Spec_P(N)$ then $M|_{L_P} \equiv_\infty N|_{L_P}$.*

Proof. Let $I = \{p : p$ is a tree partial isomorphism in L_P and p preserves height$\}$. I then satisfies the tree back and forth property because both M and N are full. □

Notice even if $\text{Spec}_P(M) = \text{Spec}_P(N)$ and $M, N \models T_{Full}$ we do not in general have $M \equiv_\infty N$. The reason is that on ill-founded branches $H_\leq(x,y)$ does not describe heights accurately. Rather on ill-founded branches $H_\leq(x,y)$ introduces an arbitrary linear order. So, for any countable model $M \models T_{Full}$ with $\infty \in \text{Spec}_P(M)$, there is a set of 2^ω many countable models M_i where $M_i \models T_{Full}$, $M_i \not\cong M_j$ if $i \neq j$ but $M_i|_{L_P} \cong M|_{L_P}$.

Corollary 3.20. *If $M, N \models T_{Full}$ and $Spec_P(M) = Spec_P(N) = \{-\infty\} \cup \alpha$ then $M \equiv_\infty N$.*

Proof. We know that $M|_{L_P} \equiv_\infty N|_{L_P}$ by Theorem 3.19. But by Theorem 3.16 we know that there is a formula $\varphi_\alpha \in \mathcal{L}_{\omega_1,\omega}(L_P)$ which defines $H_\leq$ in M and N. □

4. Collections

Before we introduce the structure we want to place on our trees it is worth discussing "abstract structure" in general terms. In this section we will discuss two types of "abstract structure", collections of base predicates and collections of archetypes.

4.1. Collections Base Predicates.

The first type of abstract structure we will want to discuss are collections of base predicates. A collection of base predicates is a collection of relations which satisfy very basic homogeneity and completeness conditions. These collections are important because eventually the structure we want our theory to satisfy can be framed in terms of the existence of a collection of base predicates with some extra properties.

Definition 4.1. Let $L_T \subset L_K$ be a language and $T_K \in \mathcal{L}_{\omega_1,\omega}(L_K)$. We say $\text{BP}_K \subseteq L_K - L_T$ is a *collection of base predicates for T_K* if T_K satisfies the following. (We omit mention of T_K when it is clear from context)

(Truth on Base Predicates)
For all $A \in \text{BP}_K$ there is a complete quantifier free formula $\varphi_A \in \mathcal{L}_{\omega_1,\omega}(\text{BP}_K \cup L_T)$ such that

$$T_K \vdash (\forall \mathbf{x}) A(\mathbf{x}) \to \varphi_A(\mathbf{x})$$

(Uniqueness of Base Predicate)
For all $A, A' \in \text{BP}_K$, $T_K \vdash (\forall \mathbf{x}) A(\mathbf{x}) \wedge A'(\mathbf{x})$ or $T_K \vdash (\forall \mathbf{x})(\neg A(\mathbf{x}) \vee \neg A'(\mathbf{x}))$

(Closed Domains)
For all $A \in \text{BP}_K$, $T_K \vdash (\forall \mathbf{x}) A(\mathbf{x}) \to \text{Closed}(\mathbf{x})$

(Uniqueness of Domains)
For all $A \in \text{BP}_K$, $T_K \vdash (\forall x_1, \dots, x_n) A(x_1, \dots, x_n) \to \bigwedge_{i \neq j} x_i \neq x_j$

(Permutation of Domains)
For all $A \in \text{BP}_K$ with $arity(A) = n$ and $\sigma : n \to n$ a bijection there is an

$A_\sigma \in \mathrm{BP}_K$ such that $T_K \vdash (\forall x_1, \ldots, x_n)A(x_1, \ldots, x_n) \leftrightarrow A_\sigma(x_{\sigma(n)}, \ldots, x_{\sigma(n)})$

<u>(Completeness for Base Predicates)</u>
$T_K \vdash (\forall \mathbf{x}) \bigvee_{n\in\omega}(\exists y_1, \ldots, y_n) \bigvee_{A\in \mathrm{BP}_K, arity(A)=n+|\mathbf{x}|} A(\mathbf{x}, y_1, \ldots, y_n)$

<u>(Amalgamation for Base Predicates)</u>
For all $A, B, C \in \mathrm{BP}_K$ if $T_K \vdash (\forall \mathbf{x}, \mathbf{y})A(\mathbf{x}, \mathbf{y}) \rightarrow C(\mathbf{y})$ and $T_K \vdash (\forall \mathbf{y}, \mathbf{z})B(\mathbf{y}, \mathbf{z}) \rightarrow C(\mathbf{y})$ then there exists a $D \in \mathrm{BP}_K$ such that $T_K \vdash (\forall \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})D(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) \rightarrow (A(\mathbf{x}, \mathbf{y}) \wedge B(\mathbf{y}, \mathbf{z}))$

<u>(Homogeneity for Base Predicates)</u>
If $A \in \mathrm{BP}_K$ and $T_K \vdash (\forall \mathbf{x}, \mathbf{y})A(\mathbf{x}, \mathbf{y}) \rightarrow B(\mathbf{x})$ then $T_K \vdash (\forall \mathbf{x})[B(\mathbf{x}) \rightarrow (\exists^\infty \mathbf{y})A(\mathbf{x}, \mathbf{y})]$

Notice that (Truth on Base Predicate), (Completeness for Base Predicates) and (Homogeneity for Base Predicates) ensure that for any $M, N \models T_K$, $M|_{\mathrm{BP}_K \cup L_T} \equiv_\infty N|_{\mathrm{BP}_K \cup L_T}$. So while a collection of base predicates does ensure a certain amount of structure is present, that structure is in some sense uniquely defined (although how that structure interacts with the rest the language L_K is not).

Definition 4.2. If BP_K is a collection of base predicates for $T_K \in \mathcal{L}_{\omega_1,\omega}(L)$ then we define $Th(\mathrm{BP}_K, T_K) \in \mathcal{L}_{\omega_1,\omega}(\mathrm{BP}_K \cup L_T)$ to be the conjunction of

- $(\forall \mathbf{x}) \bigvee_{n\in\omega}(\exists y_1, \ldots, y_n) \bigvee_{A\in \mathrm{BP}_K, arity(A)=n+|\mathbf{x}|} A(\mathbf{x}, y_1, \ldots, y_n)$
- $(\forall \mathbf{x})[B(\mathbf{x}) \rightarrow (\exists^\infty \mathbf{y})A(\mathbf{x}, \mathbf{y})]$ where $A, B \in \mathrm{BP}_K$ and $T_K \vdash (\forall \mathbf{x}, \mathbf{y})A(\mathbf{x}, \mathbf{y}) \rightarrow B(\mathbf{x})$
- $(\forall \mathbf{x})A(\mathbf{x}) \leftrightarrow \varphi_A(\mathbf{x})$ for all $A \in \mathrm{BP}_K$

Theorem 4.3. *Suppose BP is a collection of base predicates for T. If $T^* \in \mathcal{L}_{\omega_1,\omega}(L^*)$ where $BP \cup L_T \subseteq L^*$ and $T^* \vdash Th(BP, T)$ then $Th(BP, T) = Th(BP, T^*)$ and BP is a collection of base predicates for T^*.*

Proof. This is immediate from the definition of $Th(\mathrm{BP}, T)$. □

4.1.1. *Examples.*

Definition 4.4. Let $\mathrm{BP}_T = \{A_\varphi(\mathbf{x}) : \varphi(\mathbf{x}) \in \mathcal{L}_{\omega_1,\omega}(L_T)$ is a complete quantifier free formula and $\vdash (\forall \mathbf{x})\varphi(\mathbf{x}) \rightarrow \mathrm{Closed}(\mathbf{x})\}$

Definition 4.5. Let $T_T^{\mathrm{BP}} \in \mathcal{L}_{\omega_1,\omega}(L_T \cup \mathrm{BP}_T)$ be the conjunction of

- T_T
- $(\forall \mathbf{x})A_\varphi(\mathbf{x}) \leftrightarrow \varphi(\mathbf{x})$ for each $A \in \mathrm{BP}_T$

Theorem 4.6. *BP_T is a collection of base predicates for T_T^{BP}.*

Proof. This is immediate from the definitions. □

4.2. **Collection of Archetypes.** We now introduce our second "abstract structure", collections of archetypes. An archetype is nothing more than an "abstract property" and so before we begin discussing characteristics we want our archetypes to have, we must first introduce a method of talking about them.

Definition 4.7. If $\mathcal{M} \subseteq \{M \models T_K : |M| \leq \omega\}$ then define $S(\mathcal{M}) = \bigcup\{\mathrm{Spec}_P(M) : M \in \mathcal{M}\}$

Definition 4.8. For the rest of this paper we fix a language $L_T \subseteq L_K$ and a theory $T_K \in \mathcal{L}_{\infty,\omega}(L_K)$, and a collection of base predicates $\mathrm{BP}_K \subseteq L_K - L_T$ for T_K. We also fix $\mathcal{M}_K \subseteq \{M \models T_K : |M| \leq \omega\}$.

Definition 4.9. Suppose $\mathcal{M} \subseteq \{M \models T_K : |M| \leq \omega\}$ and $\varphi(\mathbf{x}, \mathbf{y}), \psi(\mathbf{y}, \mathbf{z})$ are "abstract properties" of elements of $\mathcal{M}$. We say that $\varphi(\mathbf{x}, \mathbf{y})$ *Forces*$_{\mathcal{M}}$ $\psi(\mathbf{y}, \mathbf{z})$ ($\varphi(\mathbf{x}, \mathbf{y}) \Vdash_{\mathcal{M}_K} \psi(\mathbf{y}, \mathbf{z})$) if

$$(\forall M \in \mathcal{M}_K)(\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in M) M \models \varphi(\mathbf{a}, \mathbf{b}) \Rightarrow M \models \psi(\mathbf{b}, \mathbf{c})$$

We say that the *domain* of $\varphi(\mathbf{x}, \mathbf{y})$ is $\mathbf{x}, \mathbf{y}$

Definition 4.10. We will use $(\varphi(\mathbf{x}, \mathbf{y}) \Vdash_K \psi(\mathbf{y}, \mathbf{z}))$ as a shorthand for $(\varphi(\mathbf{x}, \mathbf{y}) \Vdash_{\mathcal{M}_K} \psi(\mathbf{y}, \mathbf{z}))$. When $X \subseteq \{-\infty, \infty\} \cup \omega_1$ we use $\Vdash_X$ as a shorthand for $\Vdash_{\mathcal{M}_X(T_K)}$ and if $X = \{-\infty\} \cup \alpha$ we will use $\Vdash_\alpha$ as a shorthand for $\Vdash_X$.

Definition 4.11. Let $\mathrm{AT}_K = \mathrm{AT}(T_K, \mathrm{BP}_K, \mathcal{M}_K)$ be a collection of "abstract properties" of elements of models of T_K. We say that AT_K is a *Collection of Archetypes (for T_K, BP_K and $\mathcal{M}_K$)* if it satisfies the following. (We omit mention of T_K, BP_K and $\mathcal{M}_K$ when they are clear from the context)

<u>(Truth on Atomic Formulas)</u>
For all $\phi \in \mathrm{AT}_K$, $\phi(\mathbf{x}), \phi(\mathbf{y}) \Vdash_K (\forall$ quantifier free formula $\theta \in \mathcal{L}_{\omega_1,\omega}(L_K))[\theta(\mathbf{x}) \leftrightarrow \theta(\mathbf{y})]$

<u>(Completeness of Archetypes)</u>
For all $\sigma, \tau \in \mathrm{AT}_K$, if $(\exists M \in \mathcal{M}_K) M \models (\exists \mathbf{a}, \mathbf{b}) \sigma(\mathbf{a}) \wedge \tau(\mathbf{a}, \mathbf{b})$ then $\tau(\mathbf{x}, \mathbf{y}) \Vdash_K \sigma(\mathbf{x})$

<u>(Existence of Empty Archetypes)</u>
There exists $\varphi \in \mathrm{AT}_K$ such that for all $M \in \mathcal{M}_K$, $M \models \varphi$ and $\mathrm{dom}(\varphi) = \emptyset$

<u>(Trivial Amalgamation)</u>
For each $\sigma, \tau \in \mathrm{AT}_K$ there exists a $\mathrm{Trivial}_{\sigma,\tau}(r, \mathbf{x}, \mathbf{y}) \in \mathrm{AT}_K$ such that

- $\sigma(r, \mathbf{x}) \wedge \tau(r, \mathbf{y}) \Vdash_K (\exists^\infty \mathbf{z}) \mathrm{Trivial}_{\sigma,\tau}(r, \mathbf{x}, \mathbf{y}, \mathbf{z})$

(recall that r is the root of the tree)

(Base Predicates Imply Archetypes)
For all $A \in \mathrm{BP}_K$, $A(\mathbf{x}) \Vdash_K (\exists \sigma \in \mathrm{AT}_K)\sigma(\mathbf{x})$

(Archetypes imply Base Predicates)
For all $\sigma \in \mathrm{AT}_K$ there is an $A \in \mathrm{BP}_K$ such that $\sigma(\mathbf{x}) \Vdash_K A(\mathbf{x})$

Lemma 4.12. *For all $\phi \in AT$*

$$\phi(\boldsymbol{x}) \Vdash_K (\forall \boldsymbol{y}) \bigvee_{n \in \omega} (\exists z_1, \cdots, z_n) \bigvee_{\psi(\boldsymbol{x},\boldsymbol{y},z_1,\ldots,z_n) \Vdash_K \phi(\boldsymbol{x})} \psi(\boldsymbol{x}, \boldsymbol{y}, z_0, \cdots, z_n)$$

Proof. This follows from (Extension to Base Predicates) and (Base Predicates Imply Archetypes). □

We eventually want our archetypes to be generalizations of actual $\mathcal{L}_{\infty,\omega}$ types so we will want an archetype to completely determine all necessary properties of its domain. However, while working with as much generality as we are in this section it is hard to pin down exactly what this would mean. As such the requirements on a collection of archetypes are there to capture as much as can be said without knowing substantially more about our theory.

Definition 4.13. For the rest of this paper we will fix a collection of archetypes AT_K for T_K, BP_K and $\mathcal{M}_K$.

4.3. **Full Trees.**

Definition 4.14. Lets consider a collections of archetypes on $T_{Full} \cup T_T^{\mathrm{BP}} = T'_{Full}$.

Definition 4.15. Let $\mathrm{AT}_{Full} = \{(f, A_\varphi)(\mathbf{x}) : (\exists M \models T'_{Full}) M \models (\exists \mathbf{a})(f = \mathrm{htype}_P(\mathbf{a})) \wedge \varphi(\mathbf{a}) \wedge \mathrm{Closed}(\mathbf{a})$ (where $A_\varphi \in \mathrm{BP}_T)\}$. If $\psi(\mathbf{x}) = (f, A_\varphi) \in \mathrm{AT}_{Full}$ then

- When $B \in \mathcal{L}_{\omega_1,\omega}(L_H \cup \mathrm{BP}_T)$ is quantifier free, $\psi(\mathbf{x}) \Vdash_{\{-\infty,\infty\}\cup\omega_1} B(\mathbf{x})$ if and only if $\vdash (\forall \mathbf{x})\varphi(\mathbf{x}) \rightarrow B(\mathbf{x})$
- $\psi(\mathbf{x}) \Vdash_{\{-\infty,\infty\}\cup\omega_1} \mathrm{htype}(\mathbf{x}) = f$

It is not hard to see that this is a collection of archetypes for T_{full}, BP_T and $M_{\omega_1 \cup \{-\infty,\infty\}}(T_{Full})$.

5. Definable Collection of Archetypes

Having a collection of archetypes is only a start. We need some way to relate our archetypes to our language and, more specifically, to our base predicates. First though we need another piece of structure which we call "Extra Information". This is a way of getting out of an archetypes more information than just the levels and heights of the elements of its domains.

5.1. **Extra Information.**

Definition 5.1. Let Y be a countable set not previously mentioned. We say $\mathcal{EI} = \langle (\mathrm{EI}_K, \prec_{EI}), \mathrm{ei}_K, \varphi^{\mathrm{EI}} : \varphi \in \mathrm{AT}_K \rangle$ is *Extra Information* (For $\mathrm{AT}_K, \mathrm{BP}_K$, and $\mathcal{M}_K$) if

- $\mathrm{EI}_K \subseteq S(\mathcal{M}_K) \times Y \cup \{\emptyset\}$
- $\prec_K$ is a partial quasi-order on EI_K
- $\mathrm{ei}_K : \mathcal{M}_K \cup \mathrm{AT}_K \to \mathrm{Powerset}(\mathrm{EI}_K)$
- For all $\varphi \in \mathrm{AT}_K$, $\varphi^{\mathrm{EI}} : \mathrm{dom}(\varphi) \to \mathrm{EI}_K$

and $\mathcal{EI}_K$ satisfy the following properties

Archetypes

- For all $\varphi \in \mathrm{AT}_K$, $\mathrm{ei}_K(\varphi) = \mathrm{range}(\varphi^{\mathrm{EI}})$
- $(\forall \varphi \in \mathrm{AT}_K)\varphi(\mathbf{x}) \Vdash_K \mathrm{height}_P(x) = \alpha \Leftrightarrow (\exists y \in Y)\varphi^{\mathrm{EI}}(x) = (\alpha, y)$
- If $M \in \mathcal{M}_K$, $\phi \in \mathrm{AT}_K$, then $M \models (\exists \mathbf{x})\phi(\mathbf{x}) \Leftrightarrow \mathrm{ei}_K(\phi) \subseteq \mathrm{ei}_K(M)$.

Models

- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(M)$ is linearly quasi-ordered by $\prec_{\mathrm{EI}}$
- $(\forall M \in \mathcal{M}_K)(\forall x, y \in \mathrm{EI}_K) x \prec_{\mathrm{EI}} y \wedge y \in \mathrm{ei}_K(M) \to x \in \mathrm{ei}_K(M)$
- $(\forall M \in \mathcal{M}_K)(\exists (\alpha, y) \in \mathrm{EI}_K)\mathrm{ei}_K(M) = \{x : x \prec_{\mathrm{EI}} (\alpha, y)\}$
- $(\forall (\alpha, y) \in \mathrm{EI}_K)(\exists M \in \mathcal{M}_K)\mathrm{ei}_K(M) = \{x : x \prec_{\mathrm{EI}} (\alpha, y)\}$

Order

- If $(\omega * \alpha + n, x), (\omega * \beta + m, y) \prec_{\mathrm{EI}} z$ and $\beta < \alpha$ then $(\omega * \beta + m, y) \prec_{\mathrm{EI}} (\omega * \alpha + n, x)$ and $(\omega * \alpha + n, x) \not\prec_{\mathrm{EI}} (\omega * \beta + m, y)$

Up until now the only properties of elements which we have discussed are their height and their level. However, as our theories become increasingly complicated it is not always the case that the height and level of an element completely determine its $\mathcal{L}_{\infty,\omega}$-type. Extra Information gives us a way to talk about other properties elements of our models might have.

Definition 5.2. Let $\sigma, \tau \in \mathrm{AT}_K$. We say that σ and τ are *compatible* $(\sigma \| \tau)$ if $(\exists M \in \mathcal{M}_K)\mathrm{ei}_K(\sigma) \cup \mathrm{ei}_K(\tau) \subseteq \mathrm{ei}_K(M)$.

In what follows we discuss many different ways in which archetypes can be combined. However, it will not always make sense for two archetypes to exist at the same time. As such, when considering different ways to combine archetypes, we only want to assume facts about sets of archetypes which can all co-exist in the same model.

Definition 5.3. Let $\sigma(x_1, \ldots, x_n), \tau(x_1, \ldots, x_n) \in \mathrm{AT}$ and let sl be a slant line. We say that σ and τ are the same up to a slant line sl $(\sigma|sl = \tau|sl)$ if

- $\sigma(x_1, \ldots, x_n), \tau(y_1, \ldots, y_n) \Vdash_K A(x_1, \ldots, x_n) \leftrightarrow A(y_1, \ldots, y_n)$ for all quantifier free formulas $A \in \mathcal{L}_{\omega_1,\omega}(L_K)$.

and for all $i \leq n$ either

- $\sigma^{\mathrm{EI}}(x_i) = \tau^{\mathrm{EI}}(x_i)$ or

- $\sigma(x_1, \ldots, x_n), \tau(y_1, \ldots, y_n) \Vdash_K \text{height}_P(x_i) \geq sl(\text{level}(x_i))$ and $\text{height}_P(y_i) \geq sl(\text{level}(y_i))$

We define $\text{ei}_K(\sigma|sl) = \{(\alpha, y) : (\exists x \in \text{dom}(\sigma))\sigma^{\text{EI}}(x) = (\alpha, y)$ and $\sigma(\mathbf{x}) \Vdash_K \alpha = \text{height}_P(x) < sl(\text{level}(x))\}$

Intuitively two elements which are on different levels of a tree contain the same information about the spectrum of their model if they are on the same slant line. So two archetypes σ and τ are the same up to a slant line sl if they give the same answers to questions which only require information below sl.

Theorem 5.4. *For the rest of this paper we fix Extra Information $\mathcal{EI}_K = \langle (EI_K, \prec_{EI}), ei_K, \varphi^{EI} : \varphi \in AT_K\rangle$ for AT_K, BP_K and $\mathcal{M}_K$.*

5.2. **Definable Collection of Archetypes.**

Definition 5.5. We say AT_K is a *Definable Collection of Archetypes (with respect to $T_K, BP_K, \mathcal{EI}_K$ and $\mathcal{M}_K$)* if it satisfies

<u>(Prediction)</u>
For all $\sigma, \tau \in \text{AT}_K$ with $\tau(\mathbf{x}, \mathbf{y}) \Vdash_K \sigma(\mathbf{x})$ there is an $\eta_\tau(\mathbf{a})$ such that

- $(\forall M \in \mathcal{M}_K)\ M \models (\exists \mathbf{x}, \mathbf{y})\tau(\mathbf{x}, \mathbf{y}) \Rightarrow M \models (\exists \mathbf{a})\eta(\mathbf{a})$

and there is a $A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \in \text{BP}_K$ such that

- $T_K \vdash (\forall \mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a})A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \rightarrow A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ (where $\text{Trivial}_{\eta_\tau,\sigma}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \Vdash_K A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ and $A_\eta \in \text{BP}_K$)
- $\text{Trivial}_{\eta_\tau,\sigma}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \wedge A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \Vdash_K \tau(\mathbf{x}, \mathbf{y})$

<u>(Prediction Up To A Slant Line)</u>
For all $\sigma, \sigma', \tau \in \text{AT}_K$ such that $\tau(\mathbf{x}, \mathbf{y}) \Vdash_K \sigma(\mathbf{x})$, $\sigma' \| \tau$ and $\sigma|sl = \sigma'|sl$ there is an $\eta_{\tau|sl}(\mathbf{a})$ where

- $(\forall M \in \mathcal{M}_K)\text{ei}_K(\tau|sl) \cup \text{ei}_K(\sigma') \subseteq \text{ei}_K(M) \Rightarrow \text{ei}_K(\eta_{\tau|sl}) \subseteq \text{ei}_K(M)$

and there is a base predicate $A_{\sigma|sl,\tau|sl}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a})$ such that

- $T_K \vdash A_{\sigma|sl,\tau|sl}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \rightarrow A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ (where $\text{Trivial}_{\eta_{\tau|sl},\sigma'}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \Vdash_K A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ and $A_\eta \in \text{BP}_K$)
- $\text{Trivial}_{\eta_{\tau|sl},\sigma'}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \wedge A_{\sigma|sl,\tau|sl}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \Vdash_K (\exists \tau' \in \text{AT}_K)\tau'(\mathbf{x}, \mathbf{y}) \wedge \tau'|sl = \tau|sl$.

<u>(Truth on Height)</u>
For all $\phi \in \text{AT}_K$, $\phi(\mathbf{x}), \phi(\mathbf{y}) \Vdash_K \text{htype}(\mathbf{x}) = \text{htype}(\mathbf{y})$

(Prediction) and (Prediction Up To A Slant Line) are the most important conditions considered in this section (and the only two conditions not satisfied by AT_{Full}). These conditions are what connect our archetypes to our language. They allow us to pass information from the archetypes down to the base predicates.

We don't want our base predicates to be able to tell us anything explicitly about the height of our elements (because we want every base predicate to be realized in every model). But, the base predicates can give us information about the relative heights of two elements (in some way). So in the statement of prediction we can think of η_τ as some archetype disjoint from σ but with elements whose heights are the same as those in τ. Then $A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a})$ is the base predicate which says "look at the tuple $\mathbf{a}$ and use it as a guide for what the heights of the tuple $\mathbf{y}$ should be".

As we will see, (Prediction) allow us to show that a model is determined by the archetypes it realizes. (Prediction Up To A Slant Line) though, while similar to (Prediction), will be used to show that any two models which realize similar archetypes are in fact similar. This will allow us to get a lower bound on the quantifier rank of our models.

Definition 5.6. For the rest of this paper we will assume that AT_K is a definable collection of archetypes (for $\mathrm{BP}_K, \mathcal{EI}_K$ and $\mathcal{M}_K$).

5.3. **Results.**

Theorem 5.7. *Suppose* $\sigma(\boldsymbol{x}), \tau(\boldsymbol{x}, \boldsymbol{y}) \in AT_K$, $\tau(\boldsymbol{x}, \boldsymbol{y}) \Vdash_K \sigma(\boldsymbol{x})$ *and* $ei_K(\tau) \subseteq ei_K(M)$. *Then* $M \models (\forall \boldsymbol{x})\sigma(\boldsymbol{x}) \rightarrow (\exists^\infty \boldsymbol{y})\tau(\boldsymbol{x}, \boldsymbol{y})$

Proof. This follows immediately from (Prediction) and (Homogeneity of Base Predicates) □

Definition 5.8. Define $\mathrm{ATYPE}(M) = \{\phi \in \mathrm{AT}_K : \mathrm{ei}_K(\phi) \subseteq \mathrm{ei}_K(M)\}$

If $M, N \in \mathcal{M}_K$ and sl is a slant line, we say $\mathrm{ATYPE}(M)|sl = \mathrm{ATYPE}(N)|sl$ if

$$(\forall \phi \in \mathrm{ATYPE}(M))(\exists \psi \in \mathrm{ATYPE}(N))(\phi|sl = \psi|sl)$$

and

$$(\forall \psi \in \mathrm{ATYPE}(N))(\exists \phi \in \mathrm{ATYPE}(M))(\phi|sl = \psi|sl)$$

Lemma 5.9. $(\forall \phi \in AT)\phi \in ATYPE(M) \Leftrightarrow M \models (\exists \boldsymbol{x})\phi(\boldsymbol{x})$

Proof. This follows from our definition of $\mathcal{EI}_K$. □

Theorem 5.10. *If* $M, N \in \mathcal{M}_K$ *and* $ATYPE(M) = ATYPE(N)$ *then* $M \cong N$.

Proof. First notice that as M and N are countable it suffices to show that $M \equiv_\infty N$.

Definition 5.11. Let $I(M, N) = \{f : M \rightarrow N$ s.t.

- f is a tree partial isomorphism
- There exists $\sigma_f \in \mathrm{AT}_K$ and $\mathbf{a} \in M, \mathbf{b} \in N$ such that $M \models \sigma_f(\mathrm{dom}(f), \mathbf{a})$, $N \models \sigma_f(\mathrm{range}(f), \mathbf{b})$

We need to show that $I(M, N) \subseteq I(M, N)$ satisfies the tree back and forth condition. Let $f \in I(M, N)$, $a \in M$ such that $\mathrm{dom}(f) \cup \{a\}$ is closed.

Let $\sigma' \in \mathrm{AT}$ be such that $M \models \sigma'(\mathrm{dom}(f), a, \mathbf{a}')$ with $\mathbf{a} \subseteq \mathbf{a}'$ (which we know must exist by Lemma 4.12). Then we have by (Completeness of Archetypes) that $\sigma'(\mathbf{x}, a, \mathbf{a}') \Vdash_K \sigma_f(\mathbf{x}, \mathbf{a})$. So by Theorem 5.7 $N \models (\exists b, \mathbf{b}')\sigma'(\text{range}(f), b, \mathbf{b}')$ and hence $f \cup (a, b) \in I(M, N)$.

The case where we are given a $b \in N$ and we need to find an $a \in M$ is done analogously. Hence $I(M, N)$ has the tree back and forth property and $M \equiv^T_\infty N$. So $M \equiv_\infty N$. □

Theorem 5.10 shows that models in $\mathcal{M}_K$ are determined by the "types of archetypes" they realize.

Corollary 5.12. *If $M, N \in \mathcal{M}_K$ and $ei_K(M) = ei_K(N)$ then $M \cong N$.*

Theorem 5.13. $|S(\mathcal{M}_K)| \leq |\mathcal{M}_K| \leq \max\{\omega, |S(\mathcal{M}_K)|\}$

Proof. First notice that $|\{\text{ei}_K(M) : M \in \mathcal{M}_K\}| = |\text{EI}_K|$ by the conditions on extra information. Further, by Corollary 5.12 if $\text{ei}_K(M) = \text{ei}_K(N)$ then $M \cong N$, so $|\mathcal{M}_K| = |\text{EI}_K|$. Because $|Y| \leq \omega$ (in the definition of Extra Information) we know that $|S(\mathcal{M}_K)| \leq |\text{EI}_K| \leq |S(\mathcal{M}_K) \times Y| \leq \max\{\omega, |S(\mathcal{M}_K)|\}$. □

Theorem 5.14. *If $ei_K(M) \subseteq ei_K(N)$ and sl is a slant line with $sl < \omega * \gamma \subset Spec_P(M)$ then $ATYPE(M)|sl = ATYPE(N)|sl$*

Proof. First notice that $\text{ATYPE}(M) \subseteq \text{ATYPE}(N)$. So it suffices to show that for every $\varphi \in \text{ATYPE}(N)$ there is a $\psi \in \text{ATYPE}(M)$ such that $\psi|sl = \varphi|sl$.

Let $\hat{\emptyset} \in \text{AT}_K$ be such that $\text{dom}(\hat{\emptyset}) = \emptyset$ and $\text{ei}_K(\hat{\emptyset}) \subseteq \text{ei}_K(M)$. Then $\varphi(\mathbf{x}) \Vdash_K \hat{\emptyset}$. Now by the Other conditions on extra information we know that for all $\phi \in \text{ATYPE}(N)$ $\text{ei}_K(\phi|sl) \subseteq \text{ei}_K(M)$. So, by (Prediction up to a Slant Line) we then know there is a ψ satisfying $\text{ei}_K(\psi) \subseteq \text{ei}_K(M)$ and $\psi|sl = \varphi|sl$. □

Theorem 5.15. *If $M, N \in \mathcal{M}_K$, $ei_K(M) \subseteq ei_K(N)$ and $\omega * \gamma \subset Spec_P(M) \subseteq Spec_P(N)$ then $M \equiv^T_{\omega*\gamma} N$*

Proof. We need to define a sequence of partial tree isomorphisms from M to N of length at least $\omega * \gamma$.

Definition 5.16. Define $I_\zeta(M, N) = I_\zeta$ as follows:
$I_{\omega*\zeta+n} = \{f : M \to N s.t.$

- f is a tree partial isomorphism
- There exists a slant line sl, $\sigma_f, \tau_f \in \text{AT}_K$ and $\mathbf{a} \in M, \mathbf{b} \in N$ such that
  - $sl < \omega * (\zeta + 1)$ and $sl(|\text{dom}(f)| + n) \geq \omega * \zeta\}$
  - $M \models \sigma_f(\text{dom}(f), \mathbf{a})$, $N \models \tau_f(\text{range}(f), \mathbf{b})$
  - $\sigma_f|sl = \tau_f|sl$

We need to show that $\langle I_\mu : \mu < \omega * \gamma \rangle$ satisfies the tree back and forth property. Let $\omega * \zeta + n + 1 < \omega * \gamma$ and let $f \in I_{\omega*\zeta+n+1}$, $b \in N$ such that $\text{range}(f) \cup \{b\}$ is closed.

Let $\sigma' \in \text{AT}$ be such that $N \models \tau'(\text{range}(f), b, \mathbf{b}')$ with $\mathbf{b} \subseteq \mathbf{b}'$ and

$\tau'(\mathbf{x}, a, \mathbf{a}') \Vdash_K \tau_f(\mathbf{x}, \mathbf{a})$. We know $\tau' || \sigma_f$ by assumption so (Prediction up to a Slant Line) tells us there is an archetype $\eta_{\tau'|sl}$ and a base predicate $A_{\tau|sl,\tau'|sl}$ such that whenever

$$(*) \quad M \models \text{Trivial}_{\eta_{\tau'|sl},\sigma_f}(\mathbf{c}, \text{dom}(f), \mathbf{a}, \mathbf{d}) \wedge A_{\tau|sl,\tau'|sl}(\text{dom}(f), a, \mathbf{a}', \mathbf{c}, \mathbf{d}, \mathbf{e})$$

then $M \models \sigma'(\text{dom}(f), a, \mathbf{a}')$ for some $\sigma' \in \text{AT}_K$ where $\mathbf{a} \subseteq \mathbf{a}'$ and $\tau'|sl = \sigma'|sl$.

But we also know $\text{ei}_K(\tau'|sl) \cup \text{ei}_K(\sigma_f) \subseteq M$ by Theorem 5.14 and so $M \models (\exists^\infty \mathbf{c})\eta_{\tau'|sl}(\mathbf{c})$. Hence $M \models (\exists \mathbf{c}, \mathbf{d})\text{Trivial}_{\eta_{\tau'|sl}(\mathbf{c}),\sigma_f}(\mathbf{c}, \text{dom}(f), \mathbf{a}, \mathbf{d})$ and by (Homogeneity of Base Predicates) M satisfies $(*)$ for some $a, \mathbf{a}', \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$. So if we let a be as above then $g = f \cup (a, b) \in I_{\omega*\eta+n}$ (because $sl(|\text{dom}(f)| + n + 1) = sl(|\text{dom}(g)| + n)$.

We do the case where we are given $a \in M$ and we find $b \in N$ analogously. Hence we have proved that $\langle I_\mu : \mu < \omega * \gamma\rangle$ has the tree back and forth property and $M \equiv^T_{\omega*\gamma} N$. □

Theorem 5.17. *If $\omega * \gamma \in Spec_P(M)$ then the quantifier rank of M is at least γ ($qr(M) \geq \gamma$)*

Proof. For all $\beta < \gamma$ there is a $\sigma_\beta \in \text{AT}_K$ such that

- $\sigma_\beta(\mathbf{x}) \Vdash_K (\exists x)\omega * \beta \leq \text{height}_P(x) < \omega * (\beta + 1)$
- $M \models (\exists \mathbf{x})\sigma_\beta(\mathbf{x})$

By the definition of Extra Information there must be a $(\omega * \beta + n, y) \in \text{ei}_K(M)$. Let M_β be the model such that $\text{ei}_K(M_\beta) = \{i \in \text{EI}_K : i \prec_K (\omega * \beta + n, y)\}$. Then by Theorem 5.15 $M_\beta \equiv^T_{\omega*\beta} M$ so $M \equiv_\beta M_\beta$ and $\text{qr}(M) > \beta$ (as $M_\beta \neq M$). But as β was arbitrary we have $\text{qr}(M) \geq \gamma$. □

5.4. **Existence of Definable Collections of Archetypes and Vaught's Conjecture.** Notice that if $X \subseteq X' \subseteq \omega_1 \cup \{-\infty, \infty\}$ then any definable collection of archetypes up to $\mathcal{M}_{X'}(T)$ yields a definable collection of archetypes up to $\mathcal{M}_X(T)$ by restricting $(\text{EI}_K, \prec_{\text{EI}})$ to $\{(\alpha, y) : \alpha \in X\}$. So, as X gets larger, the statement "there exists a definable collection of archetypes for $\mathcal{M}_X(T)$" becomes stronger. Hence the strongest statement we can make of this form is "there exists a definable collection of archetypes for $M_{\omega_1 \cup \{-\infty,\infty\}}(T)$ (for some T)". This statement is significantly stronger what we will need for two reasons.

First, it assumes that there is a single definable collection of archetypes which works for all $\{-\infty\} \cup \alpha$. But as we will see, to construct our scattered sentence S_α we will only need a theory with a definable collection of archetypes up to $\{-\infty\} \cup \alpha$ and we will not care what the models look like that have larger spectra. Hence we do not require a single collection of archetypes for all $\{-\infty\} \cup \alpha$.

Second, the existence of a definable collection of archetypes for an $\mathcal{M}_X$ where $\infty \in X$ is a very strong assumption. Much of the work in studying trees as they relate to Vaught's conjecture comes from trying to deal with the ill-founded branches.

In what follows we will develop a method that will allow us to ignore ill-founded branches all together. However the cost will be a strict upper bound on our quantifier rank.

Theorem 5.18. *Suppose T has a definable collection of archetypes for $\mathcal{M}_{\{-\infty,\infty\}\cup\omega_1}(T)$. Then T has ω_1 many countable models.*

Proof. We know that every countable model M must have $\mathrm{Spec}_P(M) \subseteq \{-\infty, \infty\} \cup \omega_1$ and hence be in $\mathcal{M}_{\{-\infty,\infty\}\cup\omega_1}(T)$. By Theorem 5.13 we know that T has $|\{-\infty, \infty\} \cup \omega_1| = \omega_1$ many countable models. □

The central argument of the proposed counterexample of Robin Knight [4] is (essentially) a construction of a definable collection of archetypes for $\{-\infty, \infty\} \cup \omega_1$ (that these condition follow from slight modifications to the theory Θ in [4] is proved in [1]).

5.5. **Full Trees.**

5.5.1. *Extra Information.*

Definition 5.19. Define $\mathcal{EI}_{Full} = \langle(\mathrm{EI}_{Full}, \prec_{Full}), \mathrm{ei}_{Full}, \varphi^{\mathrm{EI}} : \varphi \in \mathrm{AT}_{Full}\rangle$ as follows

- $\mathrm{EI}_{Full} = \{(\alpha, n) : \alpha \in \{-\infty\} \cup \omega_1, n \in \omega\}$
- $(\beta, m) \prec_{\mathrm{EI}} (\alpha, n) \Leftrightarrow \beta + m \leq \alpha + n$ (where $\infty + n = \infty$ and $-\infty + n = -\infty$ for all $n \in \omega$).
- $\varphi^{\mathrm{EI}}(x) = (\alpha, n)$ such that $\varphi(\mathbf{x}) \Vdash_K \mathrm{height}_P(x) = \alpha$ and $\mathrm{level}(x) = n$.
- $\mathrm{ei}_K(M) = \{(\alpha, n) : (\exists a \in M)\mathrm{height}_P(a) = \alpha, \mathrm{level}(a) = n\}$.

Theorem 5.20. *$\mathcal{EI}_{Full}$ is extra information for $T_{Full}(P)$ for $\mathcal{M}_{\{-\infty\}\cup\omega_1}(T_{Full})$*

Proof. The only conditions which aren't self evident are those in Models. To see that these are satisfied notice that if $M \models (\exists a)\mathrm{height}_P(a) = \alpha$ and $\mathrm{level}(a) = n$ then for all $\beta < \alpha$ such that $\beta + m \leq \alpha + n$ (with $m > n$), $M \models (\exists b < a)\mathrm{height}_P(b) = \beta$ and $\mathrm{level}(b) = m$. □

Notice here, for the first time, we actually need that our trees are full (and not just that we can compare heights). Also notice that $\mathcal{EI}_{Full}$ is not extra information up to $M_{\{-\infty,\infty\}\cup\omega_1}(T_{Full})$. This is because $\mathrm{ei}_K(M)$ only determines $M|_{L_P}$, and $M|_{L_P}$ only determines M if $\infty \notin \mathrm{Spec}_P(M)$.

5.5.2. *Results.*

Theorem 5.21. *If $\tau, \sigma \in AT_{Full}$ and $\tau(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{\omega_1} \sigma(\boldsymbol{x})$ then $\sigma(\boldsymbol{x}) \Vdash_{\omega_1} (\exists^{\infty}\boldsymbol{y})\tau(\boldsymbol{x}, \boldsymbol{y})$*

Proof. This follows immediately from the definition of AT_{Full}. □

Theorem 5.22. *For all $M, N \in \mathcal{M}_{\{-\infty\}\cup\omega_1}(T_{Full})$ such that $|M| = |N| = \omega$,*

$$ei_K(M) \subseteq ei_K(N) \Leftrightarrow Spec_P(M) \subseteq Spec_P(N)$$

Proof. This follows immediately from the definition of $\mathcal{EI}_{Full}$. □

Theorem 5.23. *AT_{Full} satisfies (Truth on Height)*

Proof. Immediate. □

6. Pairs of Archetypes

Now that we have introduced our theory of a single tree we want to consider a theory of two trees. We will do this in very much the same way.

6.1. Gluing Copies of $T_{sub}(P)$.

Definition 6.1. Let $L_P(2) = \{P_0, P_1\} \cup L_T$ where P_0 and P_1 are unary relations.

Definition 6.2. Let $T_{sub}(P_0, P_1) \in \mathcal{L}_{\infty,\omega}(L_P(2))$ be the theory containing

- $T_{sub}(P_i)$ for each $i \in \{0, 1\}$
- $(\forall x)P_0(x) \rightarrow P_1(x)$

So if $M \models T_{sub}(P_0, P_1)$ then $P_0^M \subseteq P_1^M$.

Theorem 6.3. *If $M \models T_{sub}(P_0, P_1)$ then*

$$(\forall a \in M)height_{P_0}(a) \leq height_{P_1}(a)$$

Proof. This follows from the fact that P_0^M is a subtree of P_1^M. □

Corollary 6.4. *If $M \models T_{sub}(P_0, P_1)$ and $\infty \notin Spec_{P_1}(M)$ then $Spec_{P_0}(M) \subseteq Spec_{P_1}(M)$*

Proof. This follows immediately from Theorem 6.3. □

Definition 6.5. Let $L_K(2) = L_T \cup \{(K, i) : i \in \{0, 1\}, K \in L_K - L_T\}$ and let $[L_K]_i = L_T \cup \{(K, i) : K \in L_K - L_T\}$ for $i \in \{0, 1\}$. If M is a model of $L_K(2)$ define $[M]_i = M|_{[L_K]^i}$. We say $[M]_i$ is the model at level i. We will also write K_i for (K, i) when no confusion can arise.

Definition 6.6. Let $T_K(2) \in \mathcal{L}_{\omega_1,\omega}(L_K(2))$ be the conjunction of the following

- $T_{sub}(P_0, P_1)$
- T_K in $[L_K]_0$ and T_K in $[L_K]_1$

6.2. Collection of Pairs of Archetypes.

When we move from the context of a single tree to a pair of trees we also want to move from a single archetypes to a pair of archetypes. However to do this we need another piece of structure. We need a way to tell when one pair of archetypes is valid as an extension of another.

Specifically we may have two pairs $\langle \tau_0(\mathbf{x}, \mathbf{y}), \tau_1(\mathbf{x}, \mathbf{y})\rangle$ and $\langle \sigma_0(\mathbf{x}), \sigma_1(\mathbf{x})\rangle$ where $\tau_i(\mathbf{x}, \mathbf{y}) \Vdash_K \sigma_i(\mathbf{x})$ for $i \in \{0, 1\}$, but $\langle \tau_0(\mathbf{x}, \mathbf{y}), \tau_1(\mathbf{x}, \mathbf{y})\rangle$ is not a valid extension of $\langle \sigma_0(\mathbf{x}), \sigma_1(\mathbf{x})\rangle$. We will denote the fact that $\langle \tau_0(\mathbf{x}, \mathbf{y}), \tau_1(\mathbf{x}, \mathbf{y})\rangle$ is a valid extension of $\langle \sigma_0(\mathbf{x}), \sigma_1(\mathbf{x})\rangle$ by writing $\langle \tau_0(\mathbf{x}, \mathbf{y}), \tau_1(\mathbf{x}, \mathbf{y})\rangle \lhd \langle \sigma_0(\mathbf{x}), \sigma_1(\mathbf{x})\rangle$. For notational convenience we will denote the pair $\langle \sigma_0(\mathbf{x}), \sigma_1(\mathbf{x})\rangle$ by $\langle \sigma_i\rangle(\mathbf{x})$.

Definition 6.7. We say $(2-\mathrm{AT}_K, \lhd)$ is a *Collection of Pairs of Archetypes (for AT_K)* if the following hold.

(Pairs of Archetypes)
If $\langle \sigma_i \rangle \in 2-\mathrm{AT}$ then $\sigma_i \in \mathrm{AT}$ for $i \in \{0,1\}$.

(Restriction of Arity for Consistent Pairs of Archetypes)
If $\langle \tau_i \rangle, \langle \sigma_i \rangle \in 2-\mathrm{AT}_K$, $\langle \tau_i \rangle(\mathbf{x},\mathbf{y},\mathbf{z}) \lhd \langle \sigma_i \rangle(\mathbf{x})$ and for $i \in \{0,1\}$ $\tau_i(\mathbf{x},\mathbf{y},\mathbf{z}) \Vdash_K \eta_i(\mathbf{x},\mathbf{y})$ then

- $\langle \eta_i \rangle \in 2-\mathrm{AT}_K$
- $\langle \tau_i \rangle(\mathbf{x},\mathbf{y},\mathbf{z}) \lhd \langle \eta_i \rangle(\mathbf{x},\mathbf{y})$ and $\langle \eta_i \rangle(\mathbf{x},\mathbf{y}) \lhd \langle \sigma_i \rangle(\mathbf{x})$

(Consistency of $\lhd$)
If $\langle \phi_i \rangle, \langle \psi_i \rangle \in 2-\mathrm{AT}_K$ and $\langle \phi_i \rangle(\mathbf{x},\mathbf{y}) \lhd \langle \psi_i \rangle(\mathbf{x})$ then for for $i \in \{0,1\}$ $\phi_i(\mathbf{x},\mathbf{y}) \Vdash_K \psi_i(\mathbf{x})$.

(Transitivity of $\lhd$)
If $\langle \phi_i \rangle, \langle \psi_i \rangle, \langle \zeta_i \rangle \in 2-\mathrm{AT}_K$, $\langle \zeta_i \rangle(\mathbf{x},\mathbf{y},\mathbf{z}) \lhd \langle \phi_i \rangle(\mathbf{x},\mathbf{y})$ and $\langle \phi_i \rangle(\mathbf{x},\mathbf{y}) \lhd \langle \psi_i \rangle(\mathbf{x})$ then $\langle \zeta_i \rangle(\mathbf{x},\mathbf{y},\mathbf{z}) \lhd \langle \psi_i \rangle(\mathbf{x})$

(Consistency of Height)
If $\langle \phi_i \rangle \in 2-\mathrm{AT}_K$ then

$$\phi_0(x_1,\dots,x_n), \phi_1(y_1,\dots,y_n) \Vdash_K \bigwedge_{i \le n} \mathrm{height}_P(x_i) \le \mathrm{height}_P(y_i)$$

(Empty Archetypes)
If $\hat{\emptyset} \in \mathrm{AT}_K$ is an archetype with $\mathrm{dom}(\hat{\emptyset}) = \emptyset$ then $\langle \hat{\emptyset}_i \rangle = \langle \hat{\emptyset}, \hat{\emptyset} \rangle \in 2-\mathrm{AT}_K$ and for all $\langle \sigma_i \rangle \in 2-\mathrm{AT}_K$, $\langle \sigma_i \rangle(\mathbf{x}) \lhd \langle \hat{\emptyset}_i \rangle$

(Extension of 0-Archetypes)
Suppose $\langle \sigma_i \rangle \in 2-\mathrm{AT}_K$ and $\tau_1(\mathbf{x},\mathbf{y}) \Vdash_K \sigma_1(\mathbf{x})$. Then there exists $\langle \eta_i \rangle \in 2-\mathrm{AT}$ such that

- $\eta_1(\mathbf{x},\mathbf{y},\mathbf{z}) \Vdash_K \tau_1(\mathbf{x},\mathbf{y})$
- $\langle \eta_i \rangle(\mathbf{x},\mathbf{y},\mathbf{z}) \lhd \langle \sigma_i \rangle(\mathbf{x})$
- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(\tau_1) \subseteq M \Leftrightarrow \mathrm{ei}_K(\eta_1) \subseteq \mathrm{ei}_K(M)$
- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(\sigma_0) \subseteq M \Leftrightarrow \mathrm{ei}_K(\eta_0) \subseteq \mathrm{ei}_K(M)$

(Extension of 1-Archetypes)
Suppose $\langle \sigma_i \rangle \in 2-\mathrm{AT}_K$ and $\tau_0(\mathbf{x},\mathbf{y}) \Vdash_K \sigma_0(\mathbf{x})$. Then there exists $\langle \eta_i \rangle \in 2-\mathrm{AT}$ such that

- $\eta_0(\mathbf{x},\mathbf{y},\mathbf{z}) \Vdash_K \tau_0(\mathbf{x},\mathbf{y})$
- $\langle \eta_i \rangle(\mathbf{x},\mathbf{y},\mathbf{z}) \lhd \langle \sigma_i \rangle(\mathbf{x})$
- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(\tau_0) \subseteq M \Leftrightarrow \mathrm{ei}_K(\eta_0) \subseteq \mathrm{ei}_K(M)$
- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(\tau_0) \cup \mathrm{ei}_K(\sigma_1) \subseteq M \Rightarrow \mathrm{ei}_K(\eta_1) \subseteq \mathrm{ei}_K(M)$

Definition 6.8. If M is a model of $L_K(2)$ and $\langle \sigma_i \rangle \in 2 - \mathrm{AT}$ we say $M \models \langle \sigma_i \rangle(\mathbf{x})$ if $[M]_0 \models \sigma_0(\mathbf{x})$ and $[M]_1 \models \sigma_1(\mathbf{x})$.

The conditions on a collection of pairs of archetypes guarantee that $\lhd$ is a partial order on the pairs of archetypes which respect height. They also (almost) guarantee that given any pair of archetypes and an extension of one archetype in the pair we can find an extension of the pair which also extends the other archetype and is realized in every model the original pair was realized in.

The one exception to this is (Extension of 1-Heights). This condition only guarantees an extension if there is a single model M which realizes all archetypes involved. This will be a common feature of several of our assumptions when dealing with multiple trees. This is because in order to glue two trees together with the properties we need there has to be some form of compatibility between their elements.

Definition 6.9. For the rest of the paper let $(2 - \mathrm{AT}_K, \lhd)$ be a collection of pairs of archetypes for T_K.

6.3. **Examples.**

Definition 6.10. Let $2 - \mathrm{AT}_{Full} = \{(\sigma_0, \sigma_1) : \sigma_0, \sigma_1 \in \mathrm{AT}_{Full}, |\mathrm{dom}(\sigma_0)| = |\mathrm{dom}(\sigma_1)|$ and $\sigma_0(\mathbf{x}), \sigma_1(\mathbf{y}) \Vdash_{\omega_1} \mathrm{height}_P(x_i) \leq \mathrm{height}_P(y_i)\}$.

Further let $(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y}) \lhd (\sigma_0, \sigma_1)(\mathbf{x})$ if and only if $\tau_0(\mathbf{x}, \mathbf{y}) \Vdash_K \sigma_0(\mathbf{x})$ and $\tau_1(\mathbf{x}, \mathbf{y}) \Vdash_K \sigma_1(\mathbf{x})$

Theorem 6.11. *$(2 - AT_{Full}, \lhd)$ is a collection of pairs of archetypes (for AT_{Full})*

Proof. This is immediate from the definition. □

Notice that this doesn't use anything about T_{Full} and AT_{Full} except that AT_{Full} was a collection of archetypes for T_{Full} and that $(\forall \sigma, \tau \in \mathrm{AT}_{Full})\, \sigma \| \tau$.

7. Definable Collection of Pairs of Archetypes

Just as we needed a way to relate our archetypes to our language we need a way to relate our pairs of archetypes to our language.

7.1. **Definable Collections of Pairs of Archetypes.**

Definition 7.1. For the rest of this paper let $L_K(2) \subseteq L_K^2$ be a language with $\mathrm{BP}_K^2 \subseteq L_K^2 - L_K(2)$ and let $T_K^2 \in \mathcal{L}_{\omega_1,\omega}(L_K^2)$ be such that

- $T_K^2 \vdash T_K(2)$
- BP_K^2 is a collection of base predicates for T_K^2

Definition 7.2. Let $\mathcal{M}_K^2 = \{M \models T_K^2 : [M]_0, [M]_1 \in \mathcal{M}_K\}$. We will use $\varphi(\mathbf{x}, \mathbf{y}) \Vdash_K^2 \psi(\mathbf{y}, \mathbf{x})$ as a shorthand for $\varphi(\mathbf{x}, \mathbf{y}) \Vdash_{\mathcal{M}_K^2} \psi(\mathbf{y}, \mathbf{x})$.

Definition 7.3. Let $\mathrm{AT}^2_K = \{[\langle\sigma_i\rangle, C] : (\exists M \in \mathcal{M}^2_K) M \models (\exists \mathbf{x})\langle\sigma_i\rangle(\mathbf{x}) \wedge C(\mathbf{x})$, where $\langle\sigma_i\rangle \in 2 - \mathrm{AT}$ and $C \in \mathrm{BP}^2_K\}$.

If $[\langle\sigma_i\rangle, C], [\langle\tau_i\rangle, D] \in \mathrm{AT}^2_K$ we say $[\langle\tau_i\rangle, D](\mathbf{x}, \mathbf{y}) \blacktriangleleft [\langle\sigma_i\rangle, C](\mathbf{x})$ if $\langle\tau_i\rangle(\mathbf{x}, \mathbf{y}) \lhd \langle\sigma_i\rangle(\mathbf{x})$ and $T^2_K \vdash (\forall \mathbf{x}, \mathbf{y}) D(\mathbf{x}, \mathbf{y}) \to C(\mathbf{x})$.

If $M \in \mathcal{M}^2_K$ then $M \models [\langle\sigma_i\rangle, C](\mathbf{a}) \Leftrightarrow M \models \langle\sigma_i\rangle(\mathbf{a}) \wedge C(\mathbf{a})$

Definition 7.4. We say AT^2_K is a *Definable Collection of Pairs of Archetypes (with respect to $BP^2_K, \mathcal{EI}_K, \mathcal{M}_K$ and $(2 - AT_K, \lhd)$) for T^2_K* if the following hold.

(Collection of Archetypes)
AT^2_K is a collection of archetypes with respect to BP^2_K (see Definition 4.11).

(Minimality)
If $[\langle\sigma_i\rangle, C] \in \mathrm{AT}^2_K$ then $(\exists M \in \mathcal{M}^2_K) M \models (\exists \mathbf{a})[\langle\sigma_i\rangle, C](\mathbf{a})$

(Extra Information on Levels)
For all $M \in \mathcal{M}^2_K$, if $\mathrm{ei}_K(\sigma_i) \subseteq EI([M]_i)$ $i \in \{0, 1\}$ and $[\langle\sigma_i\rangle, C] \in \mathrm{AT}^2_K$ then $M \models (\exists \mathbf{a})[\langle\sigma_i\rangle, C](\mathbf{a})$

(Amalgamation for Pairs of Archetypes)
For each $[\langle\phi_i\rangle, A], [\langle\psi_i\rangle, B], [\langle\zeta_i\rangle, C] \in \mathrm{AT}^2_K$ if

- $[\langle\phi_i\rangle, A](\mathbf{x}, \mathbf{y}) \blacktriangleleft [\langle\zeta_i\rangle, C](\mathbf{y})$
- $[\langle\psi_i\rangle, B](\mathbf{y}, \mathbf{z}) \blacktriangleleft [\langle\zeta_i\rangle, C](\mathbf{y})$
- $(\exists M \in \mathcal{M}_K)\mathrm{ei}_K(\psi_0) \cup \mathrm{ei}_K(\psi_1) \cup \mathrm{ei}_K(\phi_0) \cup \mathrm{ei}_K(\phi_1) \subseteq \mathrm{ei}_K(M)$

then there is a $[\langle\eta_i\rangle, D] \in \mathrm{AT}^2_K$ such that

- $[\langle\eta_i\rangle, E](\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) \blacktriangleleft [\langle\phi_i\rangle, A](\mathbf{x}, \mathbf{y})$
- $[\langle\eta_i\rangle, E](\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) \blacktriangleleft [\langle\psi_i\rangle, B](\mathbf{y}, \mathbf{z})$
- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(\phi_0) \cup \mathrm{ei}_K(\psi_0) \subset \mathrm{ei}_K(M) \Rightarrow \mathrm{ei}_K(\eta_0) \subseteq \mathrm{ei}_K(M)$
- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(\phi_1) \cup \mathrm{ei}_K(\psi_1) \subset \mathrm{ei}_K(M) \Rightarrow \mathrm{ei}_K(\eta_1) \subseteq \mathrm{ei}_K(M)$

(Extension of Base Predicates)
Let $[\langle\sigma_i\rangle, C] \in \mathrm{AT}^2_K$ and suppose $\langle\tau_i\rangle(\mathbf{x}, \mathbf{y}) \lhd \langle\sigma_i\rangle(\mathbf{x})$. Then there is a $D \in \mathrm{BP}^2_K$ such that $[\langle\tau_i\rangle, D] \in \mathrm{AT}^2_K$ and $[\langle\tau_i\rangle, D] \blacktriangleleft [\langle\sigma_i\rangle, C]$

(Prediction For Pairs)
For all $[\langle\sigma_i\rangle, C], [\langle\tau_i\rangle, D] \in \mathrm{AT}^2_K$ such that $[\langle\tau_i\rangle, D](\mathbf{x}, \mathbf{y}) \blacktriangleleft [\langle\sigma_i\rangle, C](\mathbf{x})$ there is an $[\langle\eta_i\rangle, E] \in \mathrm{AT}^2_K$ such that

- $(\exists \mathbf{x}, \mathbf{y})[\langle\tau_i\rangle, D](\mathbf{x}, \mathbf{y}) \Vdash^2_K (\exists \mathbf{a})[\langle\eta_i\rangle, E](\mathbf{a})$

and there is a base predicate $A_{[\langle\sigma_i\rangle, C], [\langle\tau_i\rangle, D]}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \in \mathrm{BP}^2_K$ such that

- $T^2_K \vdash (\forall \mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) A_{[\langle\sigma_i\rangle, C],[\langle\tau_i\rangle, D]}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \rightarrow A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ (where $\text{Trivial}_{[\langle\eta_i\rangle, E],[\langle\sigma_i\rangle, C]}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \Vdash^2_K A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ and $A_\eta \in \text{BP}^2_K$)
- $\text{Trivial}_{[\langle\eta_i\rangle, E],[\langle\sigma_i\rangle, C]}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \wedge A_{[\langle\sigma_i\rangle, C],[\langle\tau_i\rangle, D]}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \Vdash^2_K [\langle\tau_i\rangle, D](\mathbf{x}, \mathbf{y})$

<u>(Prediction Up To A Slant Line For Pairs)</u>
For all $[\langle\sigma_i\rangle, C], [\langle\sigma'_i\rangle, C], [\langle\tau_i\rangle, D] \in \text{AT}^2_K$ such that

- $[\langle\tau_i\rangle, D](\mathbf{x}, \mathbf{y}) \blacktriangleleft [\langle\sigma_i\rangle, C](\mathbf{x})$
- $\sigma_1(\mathbf{x}) = \sigma'_1(\mathbf{x})$
- $\sigma_0|sl = \sigma'_0|sl$
- $(\exists S \in \mathcal{M}_K)\text{ei}_K(\sigma_0)\cup\text{ei}_K(\sigma_1)\cup\text{ei}_K(\sigma'_0)\cup\text{ei}_K(\sigma'_1)\cup\text{ei}_K(\tau_0)\cup\text{ei}_K(\tau_1)\subseteq S$

and for all $M \in \mathcal{M}^2_K$ there is an $[\langle\eta^M_i\rangle, E^M]$ such that

- $(\forall S \in \mathcal{M}_K)\text{ei}_K(\tau_1) \subseteq \text{ei}_K(S) \Rightarrow \text{ei}_K(\eta_1) \subseteq \text{ei}_K(S)$
- $(\forall S \in \mathcal{M}_K)\text{ei}_K(\tau_0|sl) \cup \text{ei}_K(\sigma'_0) \subseteq \text{ei}_K(S) \Rightarrow \text{ei}_K(\eta_0) \subseteq \text{ei}_K(S)$

and a base predicate $A^M_{[\langle\sigma_i\rangle, C],[\langle\tau_i\rangle, D],sl}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \in \text{BP}^2_K$ such that

- $T^2_K \vdash (\forall \mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) A^M_{[\langle\sigma_i\rangle, C],[\langle\tau_i\rangle, D],sl}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \rightarrow A_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ (where $\text{Trivial}_{[\langle\eta^M_i\rangle, E^M],[\langle\sigma'_i\rangle, C]}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \Vdash^2_K A^M_\eta(\mathbf{a}, \mathbf{x}, \mathbf{z})$ and $A^M_\eta \in \text{BP}^2_K$)
- $M \models (\forall \mathbf{a}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z})\text{Trivial}_{[\langle\eta^M_i\rangle, E^M],[\langle\sigma_i\rangle, C]}(\mathbf{a}, \mathbf{x}, \mathbf{z}) \wedge$
  $A^M_{(\langle\sigma_i\rangle, C),(\langle\tau_i\rangle, D)}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{a}) \rightarrow (\exists \tau'_0 \in \text{AT}_K)[\langle\tau'_0, \tau_1\rangle, D](\mathbf{x}, \mathbf{y})$
  and $\tau'_0|sl = \tau_0|sl$

<u>(Local Truth)</u>
Suppose M is a model of L^2_K such that

- $M \models T_K(2)$
- $M \models Th(\text{BP}^2_K, T^2_K)$
- $\text{ei}_K([M]_0) \subseteq \text{ei}_K([M]_1)$
- There exists $f : M^{<\omega} \rightarrow \text{AT}^2_K \cup \{\emptyset\}$ such that
  - $(\forall \mathbf{x})(\exists \mathbf{y}) f(\mathbf{x}, \mathbf{y}) \neq \emptyset$
  - If $f(\mathbf{x}) = [\langle\sigma_i\rangle, C]$ then
    - $M \models C(\mathbf{x})$
    - $\sigma_i(\mathbf{x}) \Vdash_K \text{height}_P(x) = \alpha \Leftrightarrow [M]_i \models \text{height}_P(x) = \alpha$
  - If $f(\mathbf{x}) \neq \emptyset$ and $f(\mathbf{x}, \mathbf{y}) \neq \emptyset$ then $f(\mathbf{x}, \mathbf{y})(\mathbf{x}, \mathbf{y}) \blacktriangleleft f(\mathbf{x})(\mathbf{x})$
  - If $[\langle\tau_i\rangle, D](\mathbf{x}, \mathbf{y}) \blacktriangleleft f(x)(\mathbf{x})$ and $\text{ei}_K(\tau_i) \subseteq \text{ei}_K([M]_i)$ for $i \in \{0, 1\}$ then $(\exists^\infty \mathbf{y}) f(\mathbf{x}, \mathbf{y}) = [\langle\tau_i\rangle, D]$

Then $M \models T^2_K$ and $(\forall \mathbf{x}) f(\mathbf{x}) \neq \emptyset \Rightarrow M \models f(\mathbf{x})(\mathbf{x})$

Just as in the case of a single tree the role of (Prediction For Pairs) will ensure that two models with the same basic properties (in this case having their component trees being isomorphic) are in fact isomorphic.

In a similar manner to the case of a single tree (Prediction Up To A Slant Line For Pairs) will let us tell when two models are equivalent up to a given quantifier rank. However, for reasons which will become obvious in Section 8, we only want to look at restrictions to slant lines on the level 0 model.

But, just as in the case of a single tree, in order to actually use (Prediction Up To A Slant Line For Pairs) to get a lower bound on the quantifier ranks of a model, we need there to be enough models which are similar to it. In the case of a single tree the property of extra information which said "given any $(\alpha, y) \in \mathrm{EI}$ there is a model M such that $\mathrm{ei}_K(M) = \{x : x \prec_K (\alpha, y)\}$" is what guarantees the existence of the models we need. In the case of pairs of trees we need (Local Truth), (Extension of Base Predicates), (Extension of Pairs of Archetypes), and (Amalgamation).

Of these (Local Truth) is the condition which is least clear on first reading. We can think of (Local Truth) as saying that if we build a "term model" assigning archetypes to tuples and using the $\blacktriangleleft$ relation to tell what the extensions are, then we get an actual model of our theory T^2_K. In Section 7.2 this is exactly what we will do.

7.2. **Construction of Models.** In this section we use (Local Truth) to explicitly construct models of T^2_K. These models will be used in Section 7.3 to give a lower bound on the quantifier rank of the models we are interested in. First though we need a definition.

Definition 7.5. If $[\langle\mu_i\rangle, B], [\langle\sigma_i\rangle, C], [\langle\tau_i\rangle, D], [\langle\eta_i\rangle, E] \in \mathrm{AT}^2_K$ are such that

- $[\langle\eta_i\rangle, E](\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) \blacktriangleleft [\langle\sigma_i\rangle, C](\mathbf{x}, \mathbf{y}) \blacktriangleleft [\langle\mu_i\rangle, B](\mathbf{y})$
- $[\langle\eta_i\rangle, E](\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) \blacktriangleleft [\langle\tau_i\rangle, C](\mathbf{y}, \mathbf{z}) \blacktriangleleft [\langle\mu_i\rangle, B](\mathbf{y})$
- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(\sigma_0) \cup \mathrm{ei}_K(\tau_0) \subseteq \mathrm{ei}_K(M) \Leftrightarrow \mathrm{ei}_K(\eta_0) \subseteq \mathrm{ei}_K(M)$
- $(\forall M \in \mathcal{M}_K)\mathrm{ei}_K(\sigma_1) \cup \mathrm{ei}_K(\tau_1) \subseteq \mathrm{ei}_K(M) \Leftrightarrow \mathrm{ei}_K(\eta_1) \subseteq \mathrm{ei}_K(M)$

then we say $[\langle\eta_i\rangle, E]$ is an *amalgamation of* $[\langle\sigma_i\rangle, C]$ *and* $[\langle\tau_i\rangle, D]$ *around* $[\langle\mu_i\rangle, B]$

Theorem 7.6. *If $M_0, M_1 \subseteq \mathcal{M}_K$ with $ei_K(M_0) \subseteq ei_K(M_1)$ then there exists a model $M^* \models T^2_K$ such that $[M^*]_i \cong M_i$.*

Proof. We construct our model M^* in stages. At stage n we construct sequences $\mathbf{a}_n \in M_0$ and $\mathbf{b}_n \in M_1$, bijections $f_n : \mathbf{x}_n \to \mathbf{a}_n$, $g_n : \mathbf{x}_n \to \mathbf{b}_n$ for some $\mathbf{x}_n \in \omega$, and $[\langle\sigma^n_i\rangle, C_n] \in \mathrm{AT}^2_K$ such that

- $f_{n-1} \subset f_n$ and $g_{n-1} \subset g_n$
- $\bigcup_{n\in\omega} \mathbf{a}_n = M_0$ and $\bigcup_{n\in\omega} \mathbf{b}_n = M_1$
- $M_0 \models \sigma^n_0(\mathbf{a}_n)$ and $M_1 \models \sigma^n_1(\mathbf{b}_n)$
- $[\langle\sigma^n_i, C_n\rangle](\mathbf{x}, \mathbf{y}) \blacktriangleleft [\langle\sigma^{n-1}_i\rangle, C_{n-1}](\mathbf{x})$

Further we do this in a way such that for all $[\langle\tau_i\rangle, D], [\langle\zeta_i\rangle, E] \in \mathrm{AT}^2_K$ with

- $[\langle\tau_i\rangle, D](\mathbf{x}, \mathbf{y}) \blacktriangleleft [\langle\zeta_i\rangle, E](\mathbf{x})$
- $\mathrm{ei}_K(\tau_0) \subseteq \mathrm{ei}_K(M_0)$
- $\mathrm{ei}_K(\tau_1) \subseteq \mathrm{ei}_K(M_1)$

$M^* \models (\forall \mathbf{a})[\langle\eta_i\rangle, E](\mathbf{a}) \to (\exists^\infty \mathbf{b})[\langle\tau_i\rangle, D](\mathbf{a}, \mathbf{b})$

This will give us bijections $f : \omega \to M_0$ and $g : \omega \to M_1$. We then define our model M^* as follows

- The underlying set of $M^* = \omega$

- For all $\tau \in \mathrm{AT}_K$ and $\mathbf{x} \subset \omega$, $M^* \models \tau(\mathbf{x})$ in $[L_K]_0$ if and only if $M_0 \models \tau(f[\mathbf{x}])$
- For all $\tau \in \mathrm{AT}_K$ and $\mathbf{x} \subset \omega$, $M^* \models \tau(\mathbf{x})$ in $[L_K]_1$ if and only if $M_1 \models \tau(g[\mathbf{x}])$
- For all $B \in \mathrm{BP}_K$ and $\mathbf{x} \subset \omega$, $M^* \models B(\mathbf{x})$ if and only if there is an $n \in \omega$ such that $\mathbf{x} \subseteq \mathbf{x}_n$ and $T^2_K \vdash (\forall \mathbf{x}_n) C_n(\mathbf{x}_n) \rightarrow B(\mathbf{x})$

A model M^* built this way will satisfy the conditions of (Local Truth) with the function $h(\mathbf{x}_n) = [\langle \sigma^n_i \rangle, C_n]$ and $h(\mathbf{a}) = \emptyset$ if $\mathbf{a} \notin \{\mathbf{x}_i : i \in \omega\}$. Hence M^* will be a model of T^2_K.

All that is left is to construct our f_n, g_n and $[\langle \sigma^n_i \rangle, C_n]$. First let $\langle m^0_i : i \in \omega \rangle$ be an enumeration of M_0, and let $\langle m^1_i : i \in \omega \rangle$ be an enumeration of M_1. Let $\Upsilon = \langle [\langle \tau^j_i \rangle, D_j] : j \in \omega \rangle$ be an enumeration of elements of AT^2_K such that $\mathrm{ei}_K(\tau^j_0) \subset \mathrm{ei}_K(M_0)$ and $\mathrm{ei}_K(\tau^j_1) \subset \mathrm{ei}_K(M_1)$.

Definition 7.7. Let $\langle (A_{j,k}, U_{j,k}, S_{j,k}) : j < \omega \rangle$ be a (countably redundant) enumeration of all triples such that

- $S_{j,k}, U_{j,k}, A_{j,k} \in \Upsilon$
- $U_{j,k}(\mathbf{x}, \mathbf{y}) \blacktriangleleft S_{j,k}(\mathbf{y})$
- $[\langle \tau^k_i \rangle, D_k](\mathbf{y}, \mathbf{z}) \blacktriangleleft S_{j,k}(\mathbf{y})$
- $A_{j,k}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$ is an amalgamation of $[\langle \tau^k_i \rangle, D_k]$ and $U_{j,k}$ around $S_{i,j}$

Notice that by (Amalgamation of Pairs of Archetypes) and (Extension of Base Predicates) this is non-empty for every $k \in \omega$, and because $|\mathrm{AT}| \cup |\mathrm{BP}^2_K| = \omega$ it is countable.

<u>Stage -1:</u>
Let $f_{-1} = g_{-1} = \emptyset$. And let $\mathrm{dom}(\langle \sigma^{-1}_i \rangle) = \emptyset$. Let $\mathbf{x}_{-1} = \mathbf{y}_{-1} = \emptyset$

<u>Stage 3n:</u>
Let $[\langle \sigma^{3n-1}_i \rangle, C_{3n-1} \rangle] = [\langle \tau^k_i \rangle, D_k]$. So $A_{n,k}$ is an amalgamation of $[\langle \sigma^{3n-1}_i \rangle, C_{3n-1} \rangle]$ with $U_{n,k}$ around $S_{n,k}$. Let $[\langle \sigma^{3n}_i \rangle, C_{3n} \rangle] = A_{n,k}$.

We then know by Theorem 5.7 and the fact that $M_0 \models \sigma^{3n-1}_0(\mathbf{a}_{3n-1})$ and $M_1 \models \sigma^{3n-1}_1(\mathbf{b}_{3n-1})$ that there are $\mathbf{a}_{3n}, \mathbf{b}_{3n}$ such that $\mathbf{a}_{3n-1} \subseteq \mathbf{a}_{3n}$, $\mathbf{b}_{3n-1} \subseteq \mathbf{b}_{3n}$, $M_0 \models \sigma^{3n}_0(\mathbf{a}_{3n})$ and $M_1 \models \sigma^{3n}_1(\mathbf{b}_{3n})$. Let f_{3n} and g_{3n} be any maps from $|\mathbf{a}_{3n}| = |\mathbf{b}_{3n}|$ agreeing with f_{3n-1} and g_{3n-1} and such that $M_0 \models \sigma^{3n}_0(f_{3n}[\mathbf{a}_{3n}])$ and $M_1 \models \sigma^{3n}_1(g_{3n}[\mathbf{b}_{3n}])$.

<u>Stage 3n+1:</u>
Let s be the least such that $m^0_s \notin \mathbf{a}_{3n}$. Let $\eta \in \mathrm{AT}$ be such that $M_0 \models \eta(\mathbf{a}_{3n}, m^0_s, \mathbf{c})$. We know by (Extensions to 1-Height) that there is a $\langle \sigma^{3n+1}_i \rangle(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}) \triangleleft \langle \sigma^{3n}_i \rangle(\mathbf{x})$ with $\mathrm{ei}_K(\sigma^{3n+1}_0) \subseteq \mathrm{ei}_K(M_0), \mathrm{ei}_K(\sigma^{3n+1}_1) \subseteq \mathrm{ei}_K(M_1)$ and $\sigma^{3n+1}_0(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}) \Vdash_K \eta(\mathbf{x}, \mathbf{y}, \mathbf{w})$. Now let $\mathbf{e}$ be such that $M_0 \models \sigma^{3n+1}_1(\mathbf{a}_{3n}, m^0_s, \mathbf{c}, \mathbf{e})$ and let $b, \mathbf{d}, \mathbf{f}$ be such that $M_1 \models \sigma^{3n+1}_1(\mathbf{b}_{3n}, b, \mathbf{d}, \mathbf{f})$ (which exist by Theorem 5.7).

Let $\mathbf{a}_{3n+1} = \mathbf{a}_{3n} m_s^0 \mathbf{ce}$, $b_{3n+1} = \mathbf{b}_{3n} b\mathbf{df}$ and let f_{3n+1} and g_{3n+1} be bijections from $|\mathbf{a}_{3n+1}| = |\mathbf{b}_{3n+1}|$ extending f_{3n} and g_{3n} such that $M_0 \models \sigma_0^{3n+1}(f_{3n+1}[\mathbf{a}_{3n+1}])$ and $M_1 \models \sigma_1^{3n+1}(g_{3n+1}[\mathbf{b}_{3n+1}])$. Finally let $C_{3n+1} \in \mathrm{BP}_K^2$ be such that $[\langle\sigma_i^{3n+1}\rangle, C_{3n+1}](\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}) \blacktriangleleft [\langle\sigma_i^{3n}\rangle, C_{3n}](\mathbf{x})$ (which we know must exist by (Extension of Base Predicates)).

Stage 3n+2:
Let s be the least such that $m_s^1 \notin \mathbf{b}_{3n+1}$. Let $\eta \in \mathrm{AT}$ be such that $M_1 \models \eta(\mathbf{b}_{3n+1}, m_s^1, \mathbf{d})$. We know by (Extensions to 0-Height) that there is a $\langle\sigma_i^{3n+2}\rangle(\mathbf{x}, y, \mathbf{w}, \mathbf{z}) \lhd \langle\sigma_i^{3n+1}\rangle(\mathbf{x})$ with $\mathrm{ei}_K(\sigma_0^{3n+2}) \subseteq \mathrm{ei}_K(M_0), \mathrm{ei}_K(\sigma_1^{3n+2}) \subseteq \mathrm{ei}_K(M_1)$ and $\sigma_1^{3n+1}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}) \Vdash_K \eta(\mathbf{x}, \mathbf{y}, \mathbf{w})$. Now let $\mathbf{f}$ be such that $M_1 \models \sigma_1^{3n+2}(\mathbf{b}_{3n+1}, m_s^1, \mathbf{d}, \mathbf{f})$ and let $a, \mathbf{c}, \mathbf{e}$ be such that $M_0 \models \sigma_0^{3n+2}(\mathbf{a}_{3n+1}, a, \mathbf{c}, \mathbf{e})$ (which exist by Theorem 5.7). Let $\mathbf{a}_{3n+2} = \mathbf{a}_{3n+1} a\mathbf{ce}$, $\mathbf{b}_{3n+2} = \mathbf{b}_{3n+1} m_s^1 \mathbf{df}$ and let f_{3n+2} and g_{3n+2} be bijections from $|\mathbf{a}_{3n+2}| = |\mathbf{b}_{3n+2}|$ extending f_{3n+1} and g_{3n+1} such that $M_0 \models \sigma_0^{3n+2}(f_{3n+2}[\mathbf{a}_{3n+2}])$ and $M_1 \models \sigma_1^{3n+2}(g_{3n+2}[\mathbf{b}_{3n+2}])$. Finally let $C_{3n+2} \in \mathrm{BP}_K^2$ be such that $[\langle\sigma_i^{3n+2}\rangle, C_{3n+2}](\mathbf{x}, y, \mathbf{w}, \mathbf{z}) \blacktriangleleft [\langle\sigma_i^{3n+1}\rangle, C_{3n+1}](\mathbf{x})$ (which we know must exist by (Extension of Base Predicates)).

So our construction is done! f and g are bijections by steps $3n+1$ and $3n+2$ and whenever $[\langle\tau_i\rangle, D](\mathbf{x}, \mathbf{y}) \blacktriangleleft [\langle\eta_i\rangle, E](\mathbf{x})$, $\mathrm{ei}_K(\tau_0) \subseteq EI(M_0)$ and $\mathrm{ei}_K(\tau_1) \subseteq EI(M_1)$, then $M^* \models (\forall \mathbf{a})[\langle\eta_i\rangle, E](\mathbf{a}) \to (\exists^\infty \mathbf{b})[\langle\tau_i\rangle, D](\mathbf{a}, \mathbf{b})$ (this was taken care of in Steps $3n$).

□

7.3. **Results.**

Theorem 7.8. *Suppose*

- $[\langle\sigma_i\rangle, C](\boldsymbol{x}), [\langle\tau_i\rangle, D](\boldsymbol{x}, \boldsymbol{y}) \in AT_K^2$
- $[\langle\tau_i\rangle, D](\boldsymbol{x}, \boldsymbol{y}) \blacktriangleleft [\langle\sigma_i\rangle, C](\boldsymbol{x})$
- $ei_K(\tau_i) \subseteq ei_K([M]_i)$

Then $M \models (\forall \boldsymbol{x})[\langle\sigma_i\rangle, C](x) \to (\exists^\infty \boldsymbol{y})[\langle\tau_i\rangle, D](\boldsymbol{x}, \boldsymbol{y})$.

Proof. We know by assumption and (Extra Information on Levels) that $M \models (\exists \mathbf{a}, \mathbf{b})[\langle\tau_i\rangle, D](\mathbf{a}, \mathbf{b})$. The result then follows from (Prediction For Pairs) and (Homogeneity of Base Predicates) □

Theorem 7.9. *If* $M, N \in \mathcal{M}_K^2$ *and* $M_i \cong N_i$ *then* $M \cong N$.

Proof. First notice that as M and N are countable it suffices to show that $M \equiv_\infty N$.

Definition 7.10. Define $I(M, N) = \{f : M \to N$ s.t.

- f is a tree partial isomorphism
- There exists $[\langle\sigma_i^f\rangle, C] \in \mathrm{AT}_K^2$ and $\mathbf{a} \in M, \mathbf{b} \in N$ such that $M \models [\langle\sigma_i^f\rangle, C](\mathrm{dom}(f), \mathbf{a})$, $N \models [\langle\sigma_i^f\rangle, C](\mathrm{range}(f), \mathbf{b})\}$

We need to show that $I(M,N) \subseteq I(M,N)$ has the tree back and forth property. Let $f \in I(M,N)$, $a \in M$ such that $\mathrm{dom}(f) \cup \{a\}$ is closed. Let $[\langle \sigma_i' \rangle, D] \in \mathrm{AT}$ be such that $M \models [\langle \sigma_i' \rangle, D](\mathrm{dom}(f), a, \mathbf{a}')$ with $\mathbf{a} \subseteq \mathbf{a}'$ (we know such exists because AT_K^2 is a collection of archetypes). We therefore have by (Completeness for Archetypes) (for AT_K^2) and (Restriction of Arity for Consistent Pairs of Archetypes) $[\langle \sigma_i' \rangle, D](\mathbf{x}, y, \mathbf{y}') \blacktriangleleft [\langle \sigma_i^f \rangle, C](\mathbf{x}, \mathbf{y})$ (where $\mathbf{y} \subseteq \mathbf{y}'$). We know by Theorem 7.8 that $N \models (\exists b, \mathbf{b}')[\langle \sigma_i' \rangle, D](\mathrm{range}(f), b, \mathbf{b}')$. So $f \cup (a,b) \in I(M,N)$.

We do the case where we are given a $b \in N$ and we find an $a \in M$ analogously and hence have proved that $I(M,N)$ has tree the back and forth property. So $M \equiv_\infty^T N$ and hence $M \equiv_\infty N$. □

This theorem will give an us upper bound on the number of models of our theory.

Corollary 7.11. $|\mathcal{M}_K^2| \leq |\mathcal{M}_K \times \mathcal{M}_K| \leq \max\{\omega, |S(\mathcal{M}_K)|\}$

Theorem 7.12. *Suppose $M, N \in \mathcal{M}_K^2$ such that*

- $M_1 \cong N_1$
- $ei_K(M_0) \cup ei_K(N_0) \subseteq ei_K(M_1)$
- $ATYPE(M_0)|sl = ATYPE(N_0)|sl$

Then $M \equiv_{\omega*\gamma}^T N$.

Proof. We need to define a sequence of partial tree isomorphisms from M to N of length at least $\omega * \gamma$.

Definition 7.13. Define $I_\zeta(M,N) = I_\zeta$ as follows:
$I_{\omega*\zeta+n} = \{f : M \to N$ s.t.

- f is a tree partial isomorphism
- There exists a slant line sl, $\sigma_f, \tau_f \in \mathrm{AT}_K$ and $\mathbf{a} \in M, \mathbf{b} \in N$ such that
  - $sl < \omega * (\zeta + 1)$ and $sl(|\mathrm{dom}(f)| + n) \geq \omega * \zeta\}$
  - $M \models [\langle \sigma_i^f \rangle, C](\mathrm{dom}(f), \mathbf{a})$, $N \models [\langle \tau_i^f \rangle, C](\mathrm{range}(f), \mathbf{b})$
  - $\sigma_1^f = \tau_1^f$
  - $\sigma_0^f|sl = \tau_0^f|sl$

We want to show that $\langle I_\mu : \mu < \omega * \gamma \rangle$ satisfies the tree back and forth property. Let $\omega * \zeta + n + 1 < \omega * \gamma$ and let $f \in I_{\omega*\zeta+n+1}$, $a \in M$ such that $\mathrm{dom}(f) \cup \{a\}$ is closed.

Let $[\langle \sigma_i' \rangle, C'] \in \mathrm{AT}$ be such that $M \models [\langle \sigma_i' \rangle, C'](\mathrm{dom}(f), a, \mathbf{a}')$ with $\mathbf{a} \subseteq \mathbf{a}'$ and $[\langle \sigma_i' \rangle, C'](\mathbf{x}, a, \mathbf{a}') \blacktriangleleft [\langle \sigma_i^f \rangle, C](\mathbf{x}, \mathbf{a})$. We know by (Prediction Up To A Slant Line For Pairs) that there is an $[\langle \eta_i^N \rangle, E^N] \in \mathrm{AT}_K^2$ and a base predicate $A^N_{[\langle \sigma_i \rangle, C], [\langle \tau_i \rangle, D], sl}$ such that

$$(*)\ N \models \mathrm{Trivial}_{[\langle \eta_i^N \rangle, E^N], [\langle \tau_i^f \rangle, C]}(\mathbf{c}, \mathrm{range}(f), \mathbf{b}, \mathbf{d}) \wedge A^N_{[\langle \sigma_i \rangle, C], [\langle \tau_i \rangle, D], sl}(\mathrm{range}(f), \mathbf{b}, b, \mathbf{c}, \mathbf{d}, \mathbf{e})$$

then $N \models [\langle \tau_i' \rangle, C'](\mathrm{range}(f), b, \mathbf{b}')$ for some element $[\langle \tau_i \rangle, D] \in \mathrm{AT}_K^2$ such that $\tau_1' = \sigma_1'$ and $\tau_0'|sl = \sigma_0'|sl$.

But we also know $N \models (\exists \mathbf{c})[\langle \eta_i^N \rangle, E^N](\mathbf{c})$ by our assumptions on M and N and (Extra Information on Levels). So $N \models (\exists \mathbf{c}, \mathbf{d})\text{Trivial}_{[\langle \eta_i^M \rangle, E^M], [\langle \tau_i^f \rangle, C]}(\mathbf{c}, \text{range}(f), \mathbf{b}, \mathbf{d})$. Hence by (Homogeneity of Base Predicates) N satisfies $(*)$. So if we let b be as above then $g = f \cup (a, b) \in I_{\omega * \eta + n}$ (because $sl(|\text{dom}(f)| + n + 1) = sl(|\text{dom}(g)| + n))$.

We do the case where we are given $b \in N$ and we need to find $a \in M$ analogously and hence have proved that $\langle I_\mu : \mu < \omega * \gamma \rangle$ has the tree back and forth property and $M \equiv^T_{\omega * \gamma} N$. □

Notice that $\langle I_\mu : \mu < \omega * \gamma \rangle$ has the added property that if $p \in \bigcup_{\mu \in \omega * \gamma} I_\mu$ then $(\forall a \in \text{dom}(p))\text{height}_{P_1}(a) = \text{height}_{P_1}(p(a))$. This will be important in Section 8.2.

Corollary 7.14. *If $M \in \mathcal{M}^2_K$, $ei_K([M]_0) \subseteq ei_K([M]_1)$ and $Spec_{P_0}(M) = \{-\infty\} \cup \omega * \alpha + n$ then $qr(M) \geq \alpha$.*

Proof. This is immediate from Theorem 7.12 and Theorem 7.6. □

7.4. **Full Trees.**

Definition 7.15. Let $L^2_{Full} = L_{Full}(2) \cup \{H^{i,j}_{\leq} : i, j \in \{0, 1\}\}$ where $H^{i,j}_{\leq}$ are binary relations. We define as shorthand $H^{i,j}_{=}(a, b) \Leftrightarrow H^{i,j}_{\leq}(a, b) \wedge H^{j,i}_{\leq}(b, a)$ and $H^{i,j}_{<}(a, b) \Leftrightarrow H^{i,j}_{\leq}(a, b) \wedge \neg H^{i,j}_{=}(a, b)$.

Definition 7.16. $T^2_{Full} \in \mathcal{L}_{\omega_1,\omega}(L^2_{Full})$, *the theory of 2-full trees* is the conjunction of the following (for $i, j, k \in \{0, 1\}$):

Full Components	$T_{Full}(2)$
Agreement With T_{Height}	$(\forall x, y)H^{i,i}_{\leq}(x, y) \leftrightarrow (H_{\leq}, i)(x, y)$ (i.e. $H_{\leq}$ in $[L_K]_i$)
	$(\forall x, y)H^{i,j}_{\leq}(x, y) \leftrightarrow [(\exists b)H^{j,j}_{\leq}(b, y) \wedge H^{i,j}_{=}(x, b)]$
Linearity	$(\forall x, y)H^{i,j}_{\leq}(x, y) \vee H^{j,i}_{\leq}(y, x)$
Transitivity	$(\forall x, y, z)[H^{i,j}_{\leq}(x, y) \wedge H^{j,k}_{\leq}(y, z)] \rightarrow H^{i,k}_{\leq}(x, z)$
Tree Ordering	$(\forall x, y, a)[a < x \wedge P_i(x) \wedge H^{i,j}_{\leq}(x, y)] \rightarrow H^{i,j}_{<}(a, y)$
Base Case	$(\forall x, y)\neg P_i(x) \rightarrow H^{i,j}_{\leq}(x, y)$
2-Fullness	$(\forall x, y_0, y_1) \bigwedge_{i \in \{0,1\}} H^{i,i}_{<}(y_i, x) \wedge H^{0,1}_{\leq}(y_0, y_1) \rightarrow$ $(\exists^\infty z) \bigwedge_{i \in \{0,1\}} H^{i,i}_{=}(y_i, z) \wedge z <_1 x$

Theorem 7.17. *If $M \models T^2_{Full}$ then for all $i, j \in \{0, 1\}$*

- *$M \models (\forall a, b)H^{i,j}_{\leq}(a, b) \Rightarrow height_{P_i}(a) \leq height_{P_j}(b)$*
- *$M \models (\forall a, b)\ height_{P_i}(a) \leq height_{P_j}(b) \wedge height_{P_i}(a) \neq \infty \rightarrow H^{i,j}_{\leq}(a, b)$.*

Proof. This is by an induction on $\text{height}_{P_i}(a)$ (in a similar way to the proof of Theorem 3.15) □

Theorem 7.18. *Suppose $M \models T^2_{Full}$ with $a \in M$. If $\beta_0 < height_{P_0}(a)$, $\beta_1 < height_{P_1}(a)$ and $\beta_0 \leq \beta_1$ then $M \models (\exists^\infty b) b <_1 a$, $height_{P_0}(b) = \beta_0$ and $height_{P_1}(b) = \beta_1$*

Proof. If $\beta_0, \beta_1 < \infty$ then this follows from Theorem 7.17. If $\beta_0 = \infty$ or $\beta_1 = \infty$ this follows from the fact that $(\forall\beta)\beta < \infty$ and Theorem 7.17. □

Definition 7.19. Let $\mathrm{AT}^2_{Full} = \{[\langle\sigma_i\rangle, C] : \langle\sigma_i\rangle \in 2 - \mathrm{AT}_{Full}, C \in BP_T$ and $(\exists M \models T^2_{Full}, |M| = \omega) M \models (\exists\mathbf{a})[\langle\sigma_i\rangle, C](\mathbf{a})\}$.

Theorem 7.20. *AT^2_{Full} satisfies all the conditions of a definable collection of pairs of archetypes (with respect to $BP_T, \mathcal{EI}_{Full}$, and $(2 - AT_{Full}, \lhd)$) except for (Prediction For Pairs) and (Prediction Up To A Slant Line For Pairs).*

Proof. All the conditions but (Local Truth) are obvious. (Local Truth) follows from the fact that $\langle\tau_i\rangle(\mathbf{x}, \mathbf{y}) \lhd \langle\sigma_i\rangle(\mathbf{x})$ whenever $\tau_0(\mathbf{x}, \mathbf{y}) \Vdash_{\omega_1} \sigma_0(\mathbf{x})$, $\tau_1(\mathbf{x}, \mathbf{y}) \Vdash_{\omega_1} \sigma_1(\mathbf{x})$, and $\tau_0(a_1, \dots, a_n), \tau_1(b_1, \dots, b_n) \Vdash_{\omega_1} \bigwedge_{i \leq n} \mathrm{height}_P(a_i) \leq \mathrm{height}_P(b_i)$. So, any function $f : M^{<\omega} \to \mathrm{AT}^2_{Full}$ with the properties of (Local Truth) must actually be an assignment of pairs of archetypes. □

Theorem 7.21. *Let $M, N \models T^2_{Full}$ such that*

- *$\infty \notin Spec_{P_0}(M) \cup Spec_{P_1}(M) \cup Spec_{P_0}(N) \cup Spec_{P_1}(N)$.*
- *$[M]_0 \equiv_\infty [N]_0$ and $[M]_1 \equiv_\infty [N]_1$*

Then $M \equiv_\infty N$

Proof. Let $I(M, N) = \{p : p$ is a tree partial isomorphism from M to N which preserves height in $[L_P]_0$ and $[L_P]_1\}$. We see that $I(M, N) \subseteq I(M, N)$ has the tree back and forth property by Theorem 7.18 and the fact that there is a single formula $\varphi_{i,j}(x, y) \in \mathcal{L}_{\omega_1,\omega}(L^2_{Full})$ which defines $H^{i,j}_{\leq}$ in M and N (see Theorem 3.16). □

The difficulty with using T^2_{Full} as the basis for out scattered theories doesn't come from the fact that there are too few or too many models with the appropriate spectra, but rather that our method of gluing models together will collapse their quantifier ranks.

8. Scattered Theories

We are now ready to introduce our method of bounding the height of our models and of eliminating models with ill-founded branches from our theories. We do this by gluing two trees together in such a way as to fix the spectrum of the larger tree.

8.1. **Theory.**

Definition 8.1. Let

- $\mathcal{M} \in \mathcal{M}_K$ such that $\infty \notin \mathrm{Spec}_P(\mathcal{M})$.
- $L_Q(\mathcal{M}) = \{\langle c_i : i \in \mathcal{M}\rangle, Q_{tree}, Q_{\mathcal{M}}\}$ where the c_i are constants and Q_{tree}, Q_M are unary relations.
- $L_K(\mathcal{M}) = L^2_K \cup L_Q(\mathcal{M}) \cup \{(H_\leq, 1)\}$. (We will write $H_\leq$ for $(H_\leq, 1)$).

Definition 8.2. Let $T^2_K(\mathcal{M})$ be the conjunction of the following $L_K(\mathcal{M})$ sentences:

$\underline{Q_M}$:

- $(\forall x)Q_{\mathcal{M}}(x) \leftrightarrow \neg Q_{tree}(x) \vee x = r$
- $(\forall x)Q_{\mathcal{M}}(x) \leftrightarrow \bigvee_{a \in \mathcal{M}} x = c_a$
- For all $\phi \in \mathcal{L}_{\infty,\omega}([L_H]_1)$, $Q_{\mathcal{M}} \models \phi(c_{a_1}, \dots c_{a_n})$ iff $\mathcal{M} \models \phi(a_1, \dots a_n)$

Disjointness:

- $(\forall \mathbf{x}, \mathbf{y})Q_{tree}(\mathbf{x}) \wedge Q_M(\mathbf{y}) \wedge |\mathbf{y}| > 0 \wedge r \notin \mathbf{y} \rightarrow \neg U(\mathbf{x}, \mathbf{y})$ where U is any relation not in $[L_H]_1$.

$\underline{L_H}$:

- $(\forall x)(\exists c)Q_{tree}(x) \rightarrow Q_{\mathcal{M}}(c) \wedge H_{\leq}(x, c)$
- $(\forall c)(\exists x)Q_{\mathcal{M}}(c) \rightarrow Q_{tree}(x) \wedge H_{\leq}(c, x)$

Other Axioms:

- $Q_{tree} \models T^2_K$

In Section 6.1 we saw how to combine two trees in such a way that the smaller one is a subtree of the larger one. However, we still have the problem that our archetypes only work with models with certain spectra and so we need some way to limit the models we are considering. The theory $T^2_K(\mathcal{M})$ is there to fix this problem. Specifically, in $T^2_K(\mathcal{M})$ we fix the spectrum of the larger tree and thereby also bounding the spectrum of the smaller subtree. By doing this we find that we no longer have to worry about models of T^2_K with spectra we don't want to consider (like those with ∞ in their spectra.)

Definition 8.3. If $M \models T^2_K(\mathcal{M})$ we define $M^Q = \{x : M \models Q_{tree}(x)\}|_{L^2_K}$

8.2. **Results.**

Theorem 8.4. *Suppose $M \models T^2_K$ and $Spec_P([M]_1) = Spec(\mathcal{M})$. Then there is a unique model $M^* \models T^2_K(\mathcal{M})$ such that $(M^*)^Q \cong M$.*

Proof. This follows immediately from the fact that there is a unique expansion of $[M^Q]_1$ to L_H, that $\{x : M^* \models Q_M(x)\} \cong \mathcal{M}|_{L_H}$, and the Disjointness axiom. □

Theorem 8.5. *Suppose*

- $M, N \models T^2_K(\mathcal{M})$
- *$\langle I_\gamma : \gamma < \alpha\rangle$ is a sequence of tree partial isomorphisms from M^Q to N^Q*
- $(\forall p \in \bigcup_{\gamma<\alpha} I_\gamma)(\forall x \in dom(p))height_{P_1}(x) = height_{P_1}(p(x))$

then $M \equiv^T_\alpha N$

Proof. Let I^M be the set of all tree partial isomorphisms from $(Q_{\mathcal{M}})^M$ to $(Q_{\mathcal{M}})^N$. $I^M \subseteq I^M$ has the tree back and forth property since $T^2_K(\mathcal{M}) \vdash (\forall x)Q_{\mathcal{M}}(x) \rightarrow$ there is a constant c such that $c = x$.

Let $I_\gamma = \{q : (\exists p \in I_\gamma)(\exists p' \in I^M)q = p \cup p'\}$. $\langle I_\gamma : \gamma < \alpha\rangle$ has the tree back and forth property because $\langle I_\gamma : \gamma < \alpha\rangle$ and $I^M \subseteq I^M$ do. Further each element of $\bigcup_{\gamma<\alpha} I$ is also a tree partial isomorphism on $L^2_K(\mathcal{M})$ by the Disjointness axiom and our conditions on $\langle I_\gamma : \gamma < \alpha\rangle$. □

Note that this theorem works for any $T^2_K \vdash T_{sup}(P_0, P_1)$ and doesn't use any assumptions on archetypes.

Corollary 8.6. *If $M, N \models T^2_K(\mathcal{M})$ and $[M^Q]_1 \cong [N^Q]_1$ and $\omega * \gamma \subset Spec_P([M^Q]_0) \cap Spec_P([N^Q]_0)$ then $M \equiv^T_{\omega*\gamma} N$*

Proof. By Theorem 7.12 we know $M^Q \equiv^T_{\omega*\gamma} N^Q$ and further the sequence of tree partial isomorphisms used to witness this satisfies the conditions in Theorem 8.5. So by Theorem 8.5, $M \equiv^T_{\omega*\gamma} N$ □

Corollary 8.7. *If $M \models T^2_K(\mathcal{M})$, $ei_K([M^Q]_0) \subseteq ei_K([M^Q]_1)$ and $\omega * \gamma \subset Spec_P([M^Q]_0)$ then $qr(M) \geq \gamma$.*

Proof. This follows from Corollary 8.6 and the construction in Theorem 7.6. □

Theorem 8.8. *If $M, N \models T^2_K(\mathcal{M})$ and $[M^Q]_i \equiv_\infty [N^Q]_i$ for $i \in \{0, 1\}$ then then $M \equiv_\infty N$*

Proof. We know by Theorem 7.9 that $M^Q \equiv_\infty N^Q$. So this follows from Theorem 8.4. □

Corollary 8.9. *If $M \models T^K_2(\mathcal{M})$ then the $qr(M) \leq \max\{qr([M]_0), qr([M]_1)\}$.*

8.3. **Full Trees.**

Theorem 8.10. *Let $\mathcal{M} \in \mathcal{M}_{\omega_1}$ and suppose $M, N \models T^2_{Full}(\mathcal{M})$. If $M \equiv^T_\omega N$ then $M \cong N$.*

Proof. Suppose that $\langle I_n : n \in \omega\rangle$ is a sequence of tree partial isomorphisms from M to N. Let $\{-\infty\} \cup \alpha = \text{Spec}_{P_0}(M^Q)$ and let $\{-\infty\} \cup \beta = \text{Spec}_{P_0}(N^Q)$. For each $\gamma \in \alpha$ let $x_\gamma \in \mathcal{M}$ be such that $\mathcal{M} \models \text{height}(x_\gamma) = \gamma$. By 2-fullness there is an $a \in M$ such that $M \models H^{0,1}_{=}(a, a) \wedge H^{1,1}_{=}(a, c_{x_\gamma})$. Hence by assumption there must be a $b \in N$ such that $N \models H^{0,1}_{=}(b, b) \wedge H^{1,1}_{=}(b, c_{x_\gamma})$. But by Theorem 7.17 this implies that $\text{height}_{P_0}(b) = \gamma$ and so, as γ was arbitrary we get $\text{Spec}_{P_0}(N^Q) \subseteq \text{Spec}_{P_0}(M^Q)$. We get the other inclusion in the same manner and so $\text{Spec}_{P_0}(M^Q) = \text{Spec}_{P_0}(N^Q)$.

We also have $\text{Spec}_{P_1}(M^Q) = \text{Spec}_{P_1}(N^Q) = \text{Spec}_P(\mathcal{M})$, and so by Corollary 3.20 we have $[M^Q]_i \cong [N^Q]_i$ for $i \in \{0, 1\}$. But this implies $M^Q \cong N^Q$ by 7.21 and the fact that M and N are countable.

So, by Theorem 8.5 applied to T^2_{Full} we have $M \cong N$. □

This shows that in the theory of 2-full trees our method of gluing collapses the quantifier ranks of our models.

9. Main Results

We are now ready to put all of these results together. Our main assumption will be the following.

Assumption

- For all $\alpha \in \omega_1$ there is a theory T_α such that
  - T_α has a definable collection of pairs archetypes AT_α for $\mathcal{M}_\alpha(T_\alpha)$
  - $(\exists M \in \mathcal{M}_\alpha(T_\alpha))\alpha \subseteq \mathrm{Spec}_P(M)$
- For all $\alpha \in \omega_1$ there is a theory T^2_α where
  - $T^2_\alpha \vdash T_\alpha(2)$
  - T^2_α has a definable collection of archetypes with respect to AT_α
  - $\mathrm{qr}(T^2_\alpha) \leq \omega$

Under these assumption we have the following theorem

Theorem 9.1. *There is a set $Z \subseteq \omega_1$ such that*

- *Z is unbounded in ω_1*
- *For all $\alpha \in Z$ there is an $S_\alpha \in \mathcal{L}_{\omega_1,\omega}$ such that*
  - *$qr(S_\alpha) \leq \omega$*
  - *The supremum of the quantifier rank spectrum of S_α is α.*
  - *S_α is scattered with countably may models.*

Proof. Let $Z^* = \{(\alpha, T^2_\beta(\mathcal{M})) : \mathcal{M} \in \mathcal{M}_\beta(T_\beta)$ and the supremum of the quantifier rank spectrum of $T^2_\beta(\mathcal{M}) = \alpha\}$. We then let $Z = \{\alpha : (\exists T \in \mathcal{L}_{\omega_1,\omega})(\alpha, T) \in Z^*\}$ and S_α be such that $(\alpha, S_\alpha) \in Z^*$.

For each $\alpha \in Z$ we know by Corollary 7.11 that S_α is scattered with countably many models and further, by our assumptions and Definition 8.2, it is clear that $\mathrm{qr}(S_\alpha) \leq \omega$ for each α.

Let $\omega * \beta \in \omega_1$. We then know that there is a model $M \in \mathcal{M}_{\omega*\beta}(T_{\omega*\beta})$ such that $\omega * \beta \subseteq \mathrm{Spec}_P(M)$. Now we know that there is a model M^2 of $T^2_{\omega*\beta}$ such that $[M^2]_0 \cong [M^2]_1 \cong M$ by the construction in Theorem 7.6. Let N be the unique expansion of M^2 to a model of $T^2_{\omega*\beta}(M)$. Then by Corollary 8.7 $\mathrm{qr}(N) \geq \beta$. So the supremum of the quantifier rank spectrum of $T^2_{\omega*\beta}(M)$ is greater than or equal to β. Hence Z is unbounded in ω_1. □

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