Geometric Invariant Theory

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Introduction

A theorem from differential topology

**Theorem**

Let $G$ be a Lie group and $X$ a smooth manifold. Suppose $G$ acts on $X$ smoothly, freely, and properly. Then the orbit space $X/G$ is a topological manifold of dimension $\dim X - \dim G$, and admits a unique smooth structure such that the quotient map $\pi: X \to X/G$ is a smooth submersion.

- The crux of the theorem is finding a smooth structure on $X/G$.
- The hypotheses on the group action are very strong.
- Relaxing some of these easily leads to singularities in $X/G$ or even worse.
Why are we interested in taking quotients?

- Group actions are an ubiquitous and important theoretical construction in many areas of mathematics. It is natural to ask for an object behaving like an orbit space.
- Classification of objects can sometimes be reduced to the computation of a quotient.
- In geometric language, moduli spaces can often be realized as quotients of spaces of marked objects. This approach has been widely used in algebraic geometry to construct moduli spaces parametrizing curves, vector bundles, and others.
Before we begin, let us make a few remarks about the content of this talk.

- Unless otherwise specified, from now on we will work with algebro-geometric objects.
- Assume an algebraically closed base field $k$. 
Let $G$ be an algebraic group $G$ and $X$ a (projective) variety. An action of $G$ on $X$ (denoted $G \acts X$) is a morphism $G \times X \to X$ satisfying some additional constraints. There are at least two meanings one could invest in a potential “quotient object”.

- Considering the orbit space $X/G$ as a candidate for the quotient is very natural. This is the approach we take for manifolds. Unfortunately, the space $X/G$ rarely has the structure of a variety (often not Hausdorff).

- There always exists a dense open $G$-invariant $U \subset X$ such that $U/G$ is a variety [Rosenlicht]. On the other hand, we lose compactness.

- The notion of orbit space depends on viewing the variety $X$ as a set of points. The concept of a point in modern algebraic geometry is very subtle. This makes it harder to express this idea formally.
Categorical quotients

Definition

Call $\varphi: X \to Y$ a **categorical quotient** if for all $f: X \to Z$ constant on orbits it factors through $Y$.

- $\varphi: X \to Y$ is uniquely determined (up to canonical isomorphism) and has good functorial properties.
- Unfortunately, $Y$ may not have the desired geometric properties.
- Developing this idea leads us to study **stacks**, a much bigger undertaking.
Example 1

Consider \( k^\times = k \setminus \{0\} \) acting on \( k^2 \) by

\[
t \cdot (x, y) = (tx, t^{-1}y).
\]

The orbits are

- for all all \( \alpha \in k^\times \), \( \{(x, y) \in k^2 \mid xy = \alpha\} \),
- punctured \( x \)-axis,
- punctured \( y \)-axis,
- the origin.

Remarks:

- The orbit space is not Hausdorff since there are no unique limits.
- To get a Hausdorff quotient we need to combine the last 3 orbits. After doing that we get a line with coordinate \( \alpha \).
- This is a categorical quotient.
Example 2

Consider $k^\times$ acting on $k^n$ (for $n \geq 2$) by

$$t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n).$$

- The orbits are punctured lines through 0 and $\{0\}$.
- The origin lies in the closure of every orbit, so any morphism constant on orbits is constant.
- The topology on the orbit space is undesirable. There is a decent candidate for a quotient space – a point.
Consider \( k^\times \) acting on \( k^n \setminus \{0\} \) (for \( n \geq 2 \)) by

\[
t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n).
\]

- There exists a quotient \( \mathbb{P}^{n-1} \) which is both an orbit space and a categorical quotient.
- Note that removing only one point had an immense effect on the resulting quotient.
- To ask for a reasonable quotient we should allow ourselves, within limits, to glue orbits and throw away points. This strategy is realized by GIT.
Basic algebraic geometry

Before getting into details, let us recall a few basic principles from algebraic geometry.

- To each variety $X$ we associate an algebra, denoted $A(X)$, consisting of all morphisms $X \rightarrow k$. The algebra structure is induced by the algebra structure on the codomain $k$. In other words, we add and multiply maps $X \rightarrow k$ pointwise.

- There is a bijective correspondence:

\[
\begin{align*}
\left\{ \begin{array}{c}
\text{affine varieties} / k \\
\text{varieties} / k
\end{array} \right\} & \xrightarrow{A(-)} \left\{ \begin{array}{c}
f.g. \text{nilpotent-free} \\
algebras / k
\end{array} \right\} \\
\text{Spec}(-) & \xleftarrow{\text{Spec}(-)}
\end{align*}
\]

One of the directions is called **Hilbert’s Nullstellensatz**.

- There is a similar correspondence between projective varieties and certain graded algebras. The analogue of $\text{Spec}(-)$ is called $\text{Proj}(-)$.
Let $G$ be an algebraic group and $X$ an affine variety receiving an action from $G$.

- There is an induced action of $G$ on $A(X)$ given by $(g \cdot f)(x) = f(g^{-1} \cdot x)$.
- One idea for construing $X/G$ is to consider the following procedure.
  
  $X \text{ affine} \xrightarrow{\sim} A(X) \xrightarrow{\sim} A(X)^G \xrightarrow{\sim} \text{Spec } A(X)^G$

- This leads us to study the algebra $A(X)^G$. 

The algebra $A(X)^G$ II

- We are ultimately interested in $\text{Spec } A(X)^G$, so it is natural to ask whether $A(X)^G$ is finitely generated and nilpotent-free (see the correspondence above).
- Lack of nilpotent elements comes for free since $A(X)^G$ is a subalgebra of $A(X)$.
- Finite generation is a much harder question. It is also known as Hilbert’s 14th problem.
- Nagata constructed a counterexample in positive characteristic [Nagata1, Nagata2].
- Even more remarkable, he showed that actions of reductive groups $G$ always behave well. In fact, many groups of interest are reductive, including $\text{SL}(n), \text{GL}(n), \text{PGL}(n)$. 
The GIT strategy I

- Restrict considerations to reductive algebraic groups $G$. (There is much recent progress towards working with non-reductive groups [DoranKirwan, Kirwan].)
- Assume the action $G \curvearrowright X \subset \mathbb{P}^n$ is linear, that is, it acts through a morphism $G \to \text{GL}(n + 1)$.
- For projective varieties $X \subset \mathbb{P}^n$, carry out the following:
  1. select a $G$-invariant open $X^{ss} \subset X$ by throwing away undesirable points;
  2. cover $X^{ss}$ with $G$-invariant affine opens $U \subset X^{ss}$;
  3. for each such open $U$, construct its quotient $\text{Spec} \ A(U)^G$;
  4. glue the quotients together.
The GIT construction

The GIT strategy II

It turns out this is equivalent to a more direct construction.

\[
\begin{align*}
\text{projective variety } & X \subset \mathbb{P}^n \\
\downarrow & \\
\text{affine cone } & \hat{X} \subset k^{n+1} \\
\downarrow & \\
A(\hat{X}) & \\
\downarrow & \\
A(\hat{X})^G & \\
\downarrow & \\
\text{Proj } A(\hat{X})^G
\end{align*}
\]

The difficulty remains in computing \( A(\hat{X})^G \). We need a more direct way to analyze \( X^{ss} \) and the points of \( \text{Proj } A(\hat{X})^G \).
Stability I

Assume we have an algebraic group $G$ acting on a projective variety $X \subset \mathbb{P}^n$.

- GIT splits the points of $X$ into three categories:
  - stable,
  - strictly semistable (semistable),
  - unstable.

- We denote these by $X^s \subset X^{ss} \subset X$, where both $X^s$ and $X^{ss}$ are $G$-invariant open subsets in $X$. 
Despite the fact we omitted the technical definitions of stability and semistability, here are a few useful facts to note.

- **Stable points** are the best behaved ones. Their quotient is an honest orbit space.
- **(Strictly) semistable points** are still decently behaved retaining categorical properties. Their orbits may need to be identified in the quotient.
- **Unstable points** need to be discarded in order to form a quotient with reasonable geometric properties.
The good quotient I

As before, $G \acts X$.

Definition
A morphism of varieties $\varphi: X \to Y$ is called a **good quotient** of $X$ by $G$ if:

- $\varphi$ is constant on orbits, surjective and affine;
- if $U \subset Y$ is open, then $\varphi^*: A(U) \xrightarrow{\cong} A(\varphi^{-1}(U))^G$;
- for any closed $G$-invariant $W \subset X$, its image $\varphi(W) \subset Y$ is closed;
- for any disjoint closed $G$-invariant $W_1, W_2 \subset X$, their images are also disjoint.

If furthermore $Y$ is an orbit space for this action, we call it a **geometric quotient** of $X$ by $G$. 
The good quotient II

- We denote good and geometric quotients respectively by
  \[ Y = X \!//\! G, \quad Y = X / G. \]
- Good quotients are categorical in the sense we described earlier.
  \[ \varphi(x_1) = \varphi(x_2) \text{ iff } G \cdot x_1 \cap G \cdot x_2 \neq \emptyset. \]
- If \( Y = X \!//\! G \) is a good quotient and \( G \) acts on \( X \) with closed orbits, then \( Y \) is a geometric quotient.
The first major result

**Theorem**

Let $G$ be a reductive algebraic group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Then:

- There exists a good quotient $\varphi : X^{ss} \to Y$ and $Y = X^{ss} // G$ is projective.
- There exists an open $Y^s \subset Y$ such that $\varphi^{-1}(Y^s) = X^s$ and $Y^s = X^s / G$.
- If $\varphi(x_1) = \varphi(x_2)$ for $x_1, x_2 \in X^{ss}$, then $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset$.
- A point $x \in X^{ss}$ is stable iff $x$ has finite stabilizer and $G \cdot x \subset X^{ss}$ is closed.
1-parameter subgroups

- Even with the conditions listed in the previous theorem in practice it is fairly hard to determine which points are stable, strictly stable, and unstable.

- To each Lie group one can associate a Lie algebra whose non-zero elements correspond to 1-parameter subgroups. In a similar manner, one can study a group action on variety by restricting the action to 1-parameter subgroups.

**Definition**

A **1-parameter subgroup** (1-PS) \( \lambda \) of a group \( G \) is a non-constant homomorphism of algebraic groups

\[
\lambda: k^\times \rightarrow G.
\]
Let $\lambda : k^\times \rightarrow G$ be a 1-PS.

- The action of $k^\times$ through $\lambda$ may be diagonalized.
- In other words, since the action is linear we have a composition of morphisms

$$k^\times \xrightarrow{\lambda} G \xrightarrow{\psi} \text{GL}(n+1).$$

Then there exists a basis $e_0, \ldots, e_n$ of $k^{n+1}$ such that

$$\psi \circ \lambda(t) = \begin{pmatrix} t^{r_0} \\ \vdots \\ t^{r_n} \end{pmatrix}.$$
Suppose we are given a 1-PS \( \lambda: k^x \to G \) and a point \( x \in X \subset \mathbb{P}^n \). Let \( \hat{x} = (\hat{x}_0, \ldots, \hat{x}_n) \in k^{n+1} \) be a lift of \( x \in \mathbb{P}^n \). Set

\[
\mu(x, \lambda) = \max\{-r_i \mid \hat{x}_i \neq 0\}.
\]

One can verify that \( \mu(x, \lambda) \) is independent of the choices \( \hat{x}, \{e_i\} \). We call \( \mu(x, \lambda) \) the \( \lambda \)-weight of \( x \).

The positivity of \( \mu \) is closely related to limits of points, hence to closure of orbits. More precisely:

\[
\begin{align*}
\mu(x, \lambda) > 0 & \iff \lim_{t \to 0} t \cdot \hat{x} \text{ does not exist;} \\
\mu(x, \lambda) = 0 & \iff \lim_{t \to 0} t \cdot \hat{x} \neq 0 \text{ exists;} \\
\mu(x, \lambda) < 0 & \iff \lim_{t \to 0} t \cdot \hat{x} = 0 \text{ exists.}
\end{align*}
\]
Note that

\[ \mu(g \cdot x, \lambda) = \mu(x, g^{-1} \lambda g). \]

- As we will see in a moment, we are interested in the integers \( \mu(x, \lambda) \)
  for all \( x \in X \) and \( \lambda \) 1-PS in \( G \).
- We can further restrict our attention 1-PS up to conjugation.
The Hilbert-Mumford criterion I

Following the definitions of stability and semistability, it is not hard to show the following:

- $x$ is semistable $\implies \mu(x, \lambda) \geq 0$ for all 1-PS $\lambda$;
- $x$ is stable $\implies \mu(x, \lambda) > 0$ for all 1-PS $\lambda$.

Theorem (Hilbert-Mumford criterion)

For a reductive group $G$ acting linearly on a projective variety $X \subset \mathbb{P}^n$, both arrows $\implies$ may be replaced by $\iff$. 
For $G = \text{SL}(p)$, it is well-known that each 1-PS subgroup is conjugate to out of the form

$$
\lambda(t) = \begin{pmatrix}
    t^{r_1} \\
    \vdots \\
    t^{r_p}
\end{pmatrix},
$$

where $\sum r_i = 0$, $r_1 \geq \cdots \geq r_p$, and not all $r_i = 0$.

**Corollary**

*Suppose $\text{SL}(p)$ acts linearly on a projective variety $X \subset \mathbb{P}^n$. Then $x \in X$ is stable (semistable) iff*

$$
\mu(g \cdot x, \lambda) > 0 \ (\geq 0) \quad \forall g \in \text{SL}(p), \text{1-PS } \lambda \text{ of the above form}.
$$
Example 1

- For $n \geq 2$, consider the vector space $V_n$ of homogeneous degree $n$ polynomials in two variables, namely,

$$V_n = \left\{ f = \sum_{i=0}^{n} a_i x^i y^{n-i} \mid a_i \in k \right\}.$$

A **binary form of degree** $n$ is an element (point) of $\mathbb{P}V_n$.

- The zeros of $f \in V_n$ determine $n$ points in $\mathbb{P}^1$, counted with multiplicity. Another way to view this is by factoring $f$ into linear forms. This demonstrates that

$$\text{Sym}^n \mathbb{P}^1 \cong \mathbb{P}V_n \cong \mathbb{P}^n.$$
Example 1 (continued)

- There is a natural action of SL(2) on $\mathbb{P}^1$, and this lifts to an action on $\mathbb{P}V_n$.
- Any 1-PS of SL(2) is conjugate to one of the form
  \[
  \lambda_r(t) = \begin{pmatrix} t^r \\ t^{-r} \end{pmatrix}, \quad r \geq 1.
  \]
- All $\lambda_r$ acts diagonally with respect to the basis $\{x^i y^{n-i} \mid 0 \leq i \leq n\}$ of $V_n$, namely,
  \[
  \lambda_r(t) \cdot \left( \sum a_i x^i y^{n-i} \right) = \sum t^{r(n-2i)} a_i x^i y^{n-i}.
  \]

Then
  \[
  \mu \left( \sum a_i x^i y^{n-i}, \lambda_r \right) = r(n - 2i_0)
  \]

where $i_0$ is the smallest $i$ such that $a_i \neq 0$. 
Example 1 (continued)

Therefore

\[ \mu \geq 0(>0) \iff i_0 \leq n/2 (< n/2) \text{ for a given form } [f] \]
\[ \iff \text{the point } [1:0] \text{ occurs with multiplicity } \leq n/2 \]
\[ (< n/2) \text{ for a given form.} \]

Taking into account the action of SL(2), we conclude an unordered \( n \)-tuple is:

- stable iff no point of \( \mathbb{P}^1 \) occurs with multiplicity \( \geq n/2 \);
- semistable iff no point of \( \mathbb{P}^1 \) occurs with multiplicity \( > n/2 \).

For odd \( n \), stable is equivalent to semistable. For even \( n \), there exist strictly semistable points so the geometric quotient is not projective.
Example 2

- A degree $d$ curve in $\mathbb{P}^2$ is given by a single degree $d$ homogeneous form
  \[ f = \sum_{i+j+k=d} a_{ijk} x^i y^j z^k \]
  up to scaling (formally, a point in $\mathbb{P}(\text{Sym}^d(\mathbb{C}^3)^\vee)$).
- We can imagine the coefficients $a_{ijk}$ forming an equilateral triangle with side-length $d + 1$. The barycentric coordinates of $a_{ijk}$ are given by the triple $(i, j, k)$. 

\[
\begin{tikzpicture}
  \node (a00d) at (0,0) {$a_{00d}$};
  \node (ad00) at (-2,-2) {$a_{d00}$};
  \node (a0d0) at (2,-2) {$a_{0d0}$};
  \draw (a00d) -- (ad00); \draw (a00d) -- (a0d0); \draw (ad00) -- (a0d0);
\end{tikzpicture}
\]
Example 2 (continued)

- Since $\text{SL}(3)$ acts naturally on $\mathbb{P}^2$, this induces an action on the space of degree $d$ forms.

- For integers $a, b, c$ satisfying $a + b + c = 0$, consider the 1-PS $\lambda_{abc}$ of $\text{SL}(3)$ given by

$$
\lambda_{abc}(t) = \begin{pmatrix}
  t^a \\
  t^b \\
  t^c 
\end{pmatrix}.
$$

We get

$$
\lambda_{abc}(t) \cdot \left( \sum_{i+j+k=0} a_{ijk} x^i y^j z^k \right) = \sum_{i+j+k=0} t^{ai+bj+ck} a_{ijk} x^i y^j z^k.
$$
Example 2 (continued)

- Consider the following subspaces given in barycentric coordinates:

\[ L_{abc} = \{ ai + bj + ck = 0 \}, \]
\[ L^+_{abc} = \{ ai + bj + ck > 0 \}, \]
\[ L^-_{abc} = \{ ai + bj + ck < 0 \}. \]

- It is not hard to see that \( \lambda_{abc} \) acts on the monomials lying in \( L_{abc} \) with weight 0, on the monomials in \( L^+_{abc} \) with positive weight, and on the monomials in \( L^-_{abc} \) with negative weight.

- A degree \( d \) curve is stable (semistable) if each of its translates has non-zero coefficients \( a_{ijk} \) lying in \( L^+_{abc} \) and \( L^-_{abc} \) \((L^+_{abc} \cup L_{abc} \text{ and } L^-_{abc} \cup L_{abc})\) for all triples \((a, b, c)\) satisfying \( a + b + c = 0 \) and \( a \geq b \geq c \).
Example 2 (continued)

\[ a_{00d} \]

\[ (d/3, d/3, d/3) \]

\[ a_{d00} \]

\[ a_{0d0} \]

\[ L_{abc}^+ \]

\[ L_{abc}^- \]
For the sake of specificity, let us consider $d = 3$, namely the cubic curves in $\mathbb{P}^2$. For convenience we will denote this space by $\mathbb{P}^9$. This is closely related to the moduli space $\overline{M}_1$.

\[
\begin{array}{cccc}
a_{003} & & & \\
a_{102} & a_{012} & & \\
a_{201} & a_{111} & a_{021} & \\
a_{300} & a_{210} & a_{120} & a_{030}
\end{array}
\]
The line of attack here is twofold.

- Analyze what stability and semistability means in terms of vanishing of the coefficients $a_{ijk}$.
- Analyze what vanishing means geometrically, that is, what curves satisfy the relevant conditions.

Both include multiple cases, and the second stage could get particularly tedious with algebraic computations. We restrict ourselves to providing a summary of these computations.
Example 2 (continued)

- $C$ is stable
  $\Leftrightarrow$
  $C$ is smooth

- $C$ is semistable
  $\Leftrightarrow$
  $C$ has at worst ordinary double points

- $C$ is unstable
  $\Leftrightarrow$
  $C$ has a cusp or worse singularity
We conclude

\[(\mathbb{P}^9)^{ss} // SL(3) \cong \mathbb{P}^1,\]
\[(\mathbb{P}^9)^s / SL(3) \cong k.\]

The three semistable orbits are identified. A coordinate on \(k\) is given by the \(j\)-invariant.
Conclusion

- GIT provides a useful yet simple tool for constructing quotients in algebraic geometry. The ideas pioneered by Mumford and Hilbert have also found application to other areas of mathematics.
- The Hilbert-Mumford criterion could be used to suggest which objects are suitable to include in a moduli space and which are not.
- Many moduli spaces have been given GIT constructions. All resulting objects are projective varieties.
- Prominent examples include moduli spaces of curves, polarized curves, vector bundles, and others.
- GIT has also been useful in analyzing the birational geometry of some of these spaces.
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Towards non-reductive geometric invariant theory.

Joe Harris and Ian Morrison.

Frances Kirwan.
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D. Mumford, J. Fogarty, and F. Kirwan.

David Mumford.
Stability of projective varieties.

Masayoshi Nagata.
On the 14-th problem of Hilbert.
Masayoshi Nagata.
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Maxwell Rosenlicht.
A remark on quotient spaces.
Thank you!