Research statement

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Overview

My main research interests center around algebraic and arithmetic geometry and number theory. They particularly concern questions of modularity, moduli and fields of definition. I found excellent objects of study in Calabi-Yau varieties and in elliptic surfaces. Recently I have started to extend the techniques to study other projective varieties, mainly different classes of surfaces, in particular Enriques surfaces.

This note sketches current and future research activities with references to the relevant previous achievements.

Contents

1 Singular K3 surfaces

Two-dimensional Calabi-Yau varieties are called K3 surfaces. They have not only great relevance in algebraic and arithmetic geometry, but also in string theory. My research mainly explores the deep connections between arithmetic and geometry.

A complex K3 surface S is called *singular* (in the sense of exceptional, but not non-smooth) if it has the maximal Picard number $\rho(S) = 20$. Hence the transcendental lattice T_S has rank two and gives rise to a two-dimensional Galois representation. If S is defined over $\mathbb{Q}$, modularity follows from a result by Livné [L]. My interest in singular K3 surfaces was originally motivated by the inverse problem, as raised independently by B. Mazur and D. van Straten.

Theorem 1 (Elkies, Schütt)
Every known newform of weight 3 with rational coefficients is associated to a singular K3 surface over $\mathbb{Q}$.

Theorem 1 gives the first uniform answer to the problem of realising modular forms of weight greater than two geometrically in a single class of varieties.

As a first step towards its proof, I proved in [S5] that the associated newform is essentially determined by the CM-field K of class group exponent 1 or 2. There are 65 such fields known, and by a result of Weinberger [We], there is at most one more (which would be ruled out by the extended Riemann hypothesis). Extremal elliptic K3 surfaces and Kummer surfaces of Weil restrictions of CM-elliptic curves give singular K3 surfaces over $\mathbb{Q}$ for 36 of these fields. Together with N. D. Elkies, I determined singular K3 surfaces over $\mathbb{Q}$ for the remaining newforms [ES1]. To achieve this, we searched in families of elliptic K3 surfaces of high rank for suitable specialisations – a topic of independent interest that we are currently pursuing further (cf. 2.1).

My ultimate aim in this area is the classification of all singular K3 surfaces over number fields of fixed degree. In analogy to elliptic curves with CM, the number of these varieties is finite by a result of Šafarevič [Š].

Goal
Classify all singular K3 surface over a given number field, in particular over $\mathbb{Q}$, up to isomorphism.

In [S4], I derived upper and lower bounds for the minimal degree of the field of definition. The proof uses classical results of CM and the basic theory of singular abelian and K3 surfaces, developed by Shioda with Mitani [SM] resp. Inose [SI]. For many cases, this gives already a complete solution. For the general case, I derived the following additional criterion in [S7]:

Theorem 2
Let S be a singular K3 surface over some number field L. Assume that $NS(S)$ is generated by divisors defined over L. Let $d = disc(T_S) < 0$. Then the extension $L(\sqrt{d})$ contains the ring class field $H(d)$.

The proof of this result uses the known shape of the ζ-function over some extension, class group theory and the Artin-Tate conjecture. Independently the theorem was proved by Elkies for the special case $L = \mathbb{Q}$ by different techniques [E]. Theorem 2 also provides a direct proof of Šafarevič's finiteness result for singular K3 surfaces over number fields.

2 Arithmetic aspects of algebraic surfaces

Next to the continuation of the study of singular K3 surfaces, I have started to extend the circle of techniques and ideas to other classes of algebraic surfaces. My research currently centers on the following three themes.

2.1 K3 surfaces with high Picard number or extra structure

As a side-project, originating from [ES1], N. Elkies and I consider families of K3 surfaces of high Picard number. The motivating example is the Dwork pencil, consisting of one parameter-deformations of the Fermat quartic surface

$$X_\lambda : \quad x_0^4 + x_1^4 + x_2^4 + x_3^4 = \lambda\, x_0 x_1 x_2 x_3.$$

We investigate this family from a purely algebraic point of view. Working with an isogenous family of K3 surface Y_λ (the "mirror quotient" by a $(\mathbb{Z}/4\mathbb{Z})^2$ subgroup of the automorphism group), we reprove that both families have Picard rank at least 19. This was originally shown by Dwork [D] using p-adic techniques. We determine the parametrising modular curve and thus find all specialisations with $\lambda^4 \in \mathbb{Q}$ and Picard rank 20. Using Niemeier lattices, we also determine all jacobian elliptic fibrations on Y_λ. Similar techniques apply to other families of K3 surfaces of high Picard number.

As an application of our techniques, we want to compute the Néron-Severi lattice of (a general surface) X_λ. This is generally a very difficult task, but the new methods developed with T. Shioda and R. van Luijk in [SSvL] might enable us to achieve it in the present situation.

In a different direction, another project with N. D. Elkies concerns the supersingular K3 surface in characteristic two with Artin invariant one [ES2]. We classify all jacobian elliptic fibrations on this surface and relate them to each other. Such classifications have previously only been achieved in characteristic zero.

A project with R. Livné and N. Yui investigates K3 surfaces with non-symplectic automorphisms which operate trivially on the Néron-Severi group [LSY]. With minimal transcendental lattice, these surfaces have been classified by Kondō [K]. We prove their modularity and address mirror symmetry on the arithmetic side. Another paper [S6] concerns those K3 surfaces S which were not fully classified by Kondō: with an automorphism of order $m = 2^k$ acting trivially on $\mathrm{NS}(S)$ while T_S has rank m. I classify these K3 surfaces completely and investigate their arithmetic As a new feature, I compare arithmetic properties with mirror symmetry.

2.2 Surfaces with given Picard number

It was a longstanding open problem to exhibit an explicit K3 surface over $\mathbb{Q}$ with Picard number $\rho = 1$. This was achieved by R. van Luijk in [vL] by fairly sophisticated means. In particular, his method required to determine the characteristic polynomial of Frobenius over finite fields $\mathbb{F}_p$ which was computed through point counting over extensions up to $\mathbb{F}_q$ for $q = p^{11}$.

As opposed to van Luijk's method, I propose a systematic approach that replaces almost random K3 surfaces by fairly well understood K3 surfaces over $\mathbb{Q}$. Here we have good control over the Picard number under reduction mod p. Instead of counting points, it suffices to analyse the Galois action on the Néron-Severi group (which could actually be very big!).

As an extension of these ideas, I plan to investigate algebraic surfaces with given Picard number within a fixed class of surfaces of general type. As first object, a project with R. van Luijk targets quintic surfaces [vL]. Only for very few Picard numbers, there were quintic surfaces known previously (cf. [Sh]). Recently I filled several gaps by exhibiting quintic surface of Delsarte type with different Picard numbers [S8], but there are still many Picard numbers missing. This case could also give a hint whether it might be possible at all to pursue similar ideas for other classes of surfaces of general type. In the next paragraph I will comment on the specific case of maximal Picard number.

2.3 Surfaces with maximal Picard number and Shimura varieties

Surfaces with maximal Picard number have appeared in several of my papers. In characteristic zero, the Picard number $\rho(X)$ is bounded by $h^{1,1}(X)$. The question arises whether the bound is attained within a fixed class of surfaces.

In the K3 case, complex surfaces with maximal Picard number are called singular (cf. Sect. 1). They lie dense in the moduli space and have interesting arithmetic structures. This is illustrated by Thm. 2 and also explains our interest in 2.1.

The general problem of maximal Picard number, however, is wide open. For instance, consider surfaces in $\mathbb{P}^3$ of degree $d > 4$. So far, the only known surface with maximal Picard number was the Fermat sextic (Beauville). Then a quintic surface with maximal Picard number was exhibited in [S8]. The surface admits an automorphism of order 15 which also induces a rich arithmetic structure.

I believe more generally that automorphisms of higher order will help us to produce surfaces of higher degree with maximal Picard number, again reflecting the deep interplay between geometry and arithmetic. In order to study this, I started an extensive project with B. van Geemen that investigates the Shimura varieties governing the variations of Hodge structures of certain algebraic surfaces. Here the concept of half twists is very powerful, since it provides a direct connection with abelian varieties by [vG]. The abelian varieties can be used to study the degeneration locus on the Shimura variety.

These methods can also be applied to K3 surfaces. Namely, we plan to extend the study of the families of K3 surfaces in [S6] from the point of view of Hodge structures and Shimura curves.

2.4 Arithmetic and geometric aspects of Enriques surfaces

Together with K. Hulek, I have launched a program to investigate arithmetic aspects of Enriques surfaces. The main problems concern the question of fields of definition and the Galois action on the Néron-Severi group. We have found proof that the rich theory of K3 surface gives rise to interesting phenomena on the arithmetic side of Enriques surfaces.

At first, our project led us to geometric considerations involving elliptic fibrations. We start with a rational elliptic surface and consider quadratic base changes (which generally are K3 surfaces). In [HS1] we found a canonical way to endow the base change with Enriques involutions. Geometrically this makes a recent result of Ohashi explit that K3 surfaces may admit arbitrarily many Enriques involutions [O]. This approach has many interesting application on the geometric and arithmetic side.

Specifically, the technique from [HS1] applies to singular K3 surfaces. On every possible singular K3 surface (with at most 62 exceptions) we can construct Enriques involutions explicitly in a geometric way. The novel point about this construction is that it goes beyond the abstract results using lattice theory.

In [HS2], we show how arithmetic results about singular K3 surfaces as in section 1 can be descended to the Enriques quotients. Overall we obtain a picture that is very similar to section 1:

Theorem 3 (Hulek, Schütt)
Let Y denote a complex Enriques surface that is covered by a singular K3 surface S. Let d denote the discriminant of $NS(S)$. Then Y admits a model over the ring class field $H(d)$.

We aim to investigate this uniform result further. Possible extensions direct towards Thm. 2 and the classification of Enriques surfaces as above over fixed number fields.

Another interesting aspect in this context concerns Brauer groups. A complex Enriques surface Y has $\mathrm{Br}(Y) = \mathbb{Z}/2\mathbb{Z}$. One can ask how $\mathrm{Br}(Y)$ behaves under the pull-back by the universal cover $\pi : X \to Y$. In [B], Beauville showed that generically $\pi^* \mathrm{Br}(Y) = \mathbb{Z}/2\mathbb{Z} \subset \mathrm{Br}(X)$. He posed the problem to exhibit an explicit Enriques surface (over some number field, or say, over $\mathbb{Q}$) where $\pi^* \mathrm{Br}(Y) = \{0\}$.

Such an example has been given in [HS1] based on the geometric base change construction. A project with A. Garbagnati extends this example to families of K3 surfaces. Our first results are related to Kummer surfaces of product type. They possess an interesting underlying structure that also relates to the problems discussed in 2.1.

3 Further projects

3.1 Modularity and non-liftability of Calabi-Yau varieties

In [S1] und [S2] I found several modular Calabi-Yau threefolds over $\mathbb{Q}$. Following a construction of Schoen [Sc], these varieties arose as fibre products of semi-stable rational elliptic sufaces with section. Recently Cynk and van Straten showed how modularity gives rise to non-liftable threefolds at the bad primes [CvS]. This cannot happen for Calabi-Yau varieties in dimension one and two by a result of Deligne.

In a new project with S. Cynk, we have extended these ideas to other fibre products [CS]. We allow the rational elliptic surfaces to involve additive singular fibres in such a way that the fibre product still admits a small resolution in the category of algebraic spaces. The results are innovative in two respects: on the one hand introducing new modular varieties, on the other hand showing non-liftability at new primes. As an illustration, we mention the following implications of our work:

Theorem 4

(i) There is a non-liftable three-dimensional Calabi-Yau space in any characteristic less than 100 *except possibly for* $53, 83$.

(ii) There is a non-liftable three-dimensional Calabi-Yau space in characteristic 521201.

3.2 Singular fibres of elliptic surfaces

The singular fibres of elliptic surfaces have been classified by Kodaira and Tate. The question arises what the maximal singular fibres within a fixed class of elliptic surfaces are. Here it is convenient to consider elliptic surfaces over $\mathbb{P}^1$ with section and fix the Euler number. Naturally, the situation differs in characteristic zero and $p > 0$: The latter features supersingular surfaces with $\rho = b_2$ while over $\mathbb{C}$, Lefschetz' theorem gives $\rho \leq h^{1,1}$. The papers [S3] and [SS1] (together with A. Schweizer) derive a complete solution when the maximal fibres mod p exceed the restriction from Lefschetz' bound in characteristic zero. In [SS2], we also proved uniqueness in the K3 case.

We want to continue these investigations in a new project concerning elliptic surfaces over an elliptic base curve with only one or two singular fibres. Over the ground field $\mathbb{C}$, these have been completely classified. In positive characteristic, we aim to establish essentially the same classification; only small characteristics come with some extra features due to bad reduction and wild ramification.

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