ON THE ABSENCE OF TIME IN MATHEMATICS

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Journalists write under the sign of the *time-hook*. Most words that appear in a newspaper must have a hook to the present moment. Journalists should offer good answers to the two questions an editor will ask about their text:

> Why couldn't this have been printed yesterday?
> Why shouldn't it be printed tomorrow?
>
> Newsprint yellows fast.

Lawyers and judges are slaves to time procedure. Moses himself learned this the hard way. Before Moses received the substance of the Ten Commandments, his father-in-law Jethro set him straight about judicial pacing [1]. Even when laws are given by divinity, procedure precedes substance.

The Athenian law courts – according to Plato's *Apology* – had water-clocks: if you couldn't prove your case before the water-clock stopped dripping, you couldn't prove your case. A friend once pointed out to me that Plato thumbed his nose at those legal water-clocks by writing the dialogue *The Theaetetus*, which can be read as a deposition answering the charges brought against Socrates, but some thirty years after the fact, making the case that philosophy, at least, is not ruled by time restrictions. I want to make the same case, only more so, for mathematics.

In my book, *Imagining numbers (particularly the square root of minus fifteen)* [2] I have a very brief section (pp. 149-151), with the same title as this article, which was used as a quotation in FLM **23**(3) (p. 4). Here I wish to elaborate on the absence of time, on the timelessness, of mathematics. Now "timeless" seems to mean *outside time* as well as *rich with a superfluity of time*. I propose to follow both strands:

- the content and arguments of mathematics have no time-hook, as discussed above, even though they incorporate within their imagined world fictionalised, temporal sequences of arguments, constructions and procedures

- the arcs of mathematical thought weave through millenia; this influences the pacing of the great projects in mathematics, and the sense of enduring responsibility that its practitioners have for their work.

The "absence" I refer to is not that the concept of time is missing as an object of thought in mathematics. We have, after all, a clean, Zeno-paradox-avoiding time-model (the real number line) about which we can say profound things. But that's just a model. When we make a computation on a computer, we are, of course, very interested in how long it will take to run. We want some computations to be performed quickly; we want others to be guaranteed to take a long time: the marvellous enterprise of public key encryption depends for its security on producing integers that come along with some assurance that the process of factorizing them into products of primes is unfeasible with current technology.

Many beautiful algorithms in modern number theory come along with *a priori* running-time estimates – the proof of these delicate estimates being as interesting and as innovative as the establishment of the algorithm in the first place. The use these algorithms are put to, and sometimes the possibility, and attractiveness, of some direction of research, depends upon questions of time-efficiency of programs.

That a modern topologist, for example, can draw a knot (with the mouse) on the computer screen, and get an instantaneous record of all the vital statistics of the knot, such as its *fundamental group* and its *volume* if it is hyperbolic, opens up research paths that were unavailable to earlier generations.

We have an entire field of research, *computer science*, that deals with issues such as whether a given type of problem can be algorithmically solved in *polynomial time* (in terms of the data), in *exponential time*, or in fine-grained distinctions between the two. These issues, expressed in *ad infinitum* time vocabulary, have, to be sure, *finite time* implications.

The question, "How rapidly does it grow?", is a natural one in mathematics. In mathematical logic, *the Ackermann hierarchy* provides a vocabulary to ask this question – the interest being that if a certain function grows *faster* than can be constructed within the frame of a given formal system, and if that function is the solution to a certain question, why then that question is *unsolvable* in that formal system.

Teleological concerns are embedded in much mathematics: they are front and center in stability problems in the theory of dynamical systems and qualitative questions regarding the classical three-body problem. The "whole of eternity" is invoked in Bernoulli's discussion of what we call the 'law of large numbers'.

With all this, it would appear that mathematics is utterly time-drenched, in contrast to the implication of the title of this essay [5].

It is rather, first, that *our* time, *our lived time*, seems to be irrelevant to the enterprise (no time-hooks!); and, second, when one does mathematics one is often inspired by the sense of taking but a tiny part in a sustained conversation through the ages that shows every sign of providing new understandings, new intuitions, and broader viewpoints for as long as humanity continues.

History, of course, has time-hooks. I chose, in my book [2], a random sentence from Churchill's writings ("He then proceeded to state that the guarantee he had given Czechosolvakia no longer in his opinion had validity." (p. 149)) just to recognize that such sentences abound in time-points

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crucial for the understanding of the thought expressed.

Poetry, also, deals with the urgency of our lived time in a way that mathematics does not. Is there a poet who never once has butted against the puzzles, the heartaches, presented by time? In *Imaginary numbers* I work with a few lines of Shakespeare's sonnets that, among many other things, do exactly that. Ezra Pound's dictum [3] "Literature is news that stays news," points to the combined timelessness and timeliness of poetry.

Outside of time

The 'time words' in our mathematical discussions are often indicators of possible time-sequences of our coming to know things (If we know X then we also know Y, because …). We formulate various epistemological time-lines in our demonstrations. The most famous (and most poured over) cable of strands of 'epistemological time-lines' is the woven sequences of interdependent propositions in Euclid.

Let us make a repertoire of the kinds of things we can do with our fictional epistemological time. We can roll it forwards or backwards ('reduction to a prior case'), or more strikingly, we can go both ways in a symmetrical phalanx, as in the classical combination of *analysis* (going backwards from the result one wants) and *synthesis* (going forwards from the hypotheses one has). And, as we shall discuss later, we can pluck infinite sequences of events entirely out of our fictionalized time-line and deal with them as a single thing.

Even in ancient times we were well practiced in the art of going epistemologically backwards to trace back to the prime movers, the *archai* of thought, called by the various names *principles, axioms, postulates, hypotheses, common notions*. Nowadays, with Hilbert's formulation of *formal systems*, we understand that we can start anywhere, provided – of course – that we don't end up with a contradiction. This shift of emphasis is curious: the ancients worried so much about beginnings, the moderns about endings.

Any reader of the last two books of Aristotle's *Metaphysics* is painfully aware how intensely the ancients dote on beginnings and priority. Aristotle even plays with a tentative genealogy of mathematical concepts:

> the incomposite is prior to composite
>
> the planes which exist independently must be prior to those which are present in the immovable solids.

Certainly, any construction proceeds in a step-by-step way. Our mathematical writings are filled with time-sequences of 'coming to construct things' (first extend the line AB until it intersects with CD, *then* …). This, of course, is similar to the *sequentiality* of instructions for recipes in cookbooks, and rules of instruction: algorithms are sequential, if they are not parallel, and even if they are parallel there are sequences of operations in them. All this has no hook to any particular 'time zero'. Today is as good as tomorrow, or next millenium; a procrastinator's paradise, you might say.

It gets a bit more amusing when the mathematical recipes involve infinite, or indefinite, repetition, a distinct 'no no' for, say, *The joy of cooking*. In the simplest instances, as in the determination of limits of infinite sequences, although we might actually *say,*

> add 1/2 to 1/4 to 1/8, and so on, to get 1,

we simply *mean* our traditional definition of infinite sum, which avoids the problematic "and so on." As is amply written about, our traditional definition of the limit of a sequence replaces a head-on, probably meaningless, and certainly fruitless, encounter with *actual infinity* by an indefinitely protracted imaginary interchange: "if you give me an ε I'll give you a δ". What remains as a problem is the potential indefiniteness of this hypothetical conversation: this imaginary dialogue might be doomed to go on forever. But all is well if the "I" in this dialogue comes up with a general finite rule for a δ to offer, when faced with any ε. The surprise and delight, of course, is that this finite strategy for response can often be made.

We tend to replace "and so on" in the various sequentially infinite recipes that we come up with by possibly protracted, but not necessarily sequential, imaginary interchanges that sometimes can be dealt with all at once. The axiom of mathematical induction is the most sterling example of this: instead of dealing with an infinite row of dominoes each knocking down the next to endless repetition, you can concern yourself with a general proposition. Such is the pliability of epistemological time.

Richness of time

Mathematics is not special in treating itself to the generosity of unbounded time. Keats might well generalize his famous observation to include all objects of art as foster-children "of Silence and Slow Time" [4]. All the arts are media of 'conversation through the ages,' modern poets engaging with Homer, with Sappho, and modern painters with the masters of the cave in Lascaux. Nevertheless, that the tyranny of time has no sway on mathematics is strongly felt by its practitioners and is sometimes cited as one of the inspiring elements of the activity. This could be the reason for some of the good qualities, and some of the unusual qualities, of the culture of mathematics.

I think, for example, of the leisurely custom of expansive letter-writing that practicing mathematicians engage in, even in this epoch. I think of two specific letters written by one contemporary mathematician to another, on the same mathematical subject, one directly continuing the theme of the other, in a style which might lead a reader of these letters to think they were sent off in sequential mail deliveries, if not for the dates on them which show a hiatus of seventeen years. (I hope these letters will eventually be published.)

This is not to say that mathematicians are not servants to time (or patience, or cash). It is rather that despite the changing fads of interest in this theory or that, the favoring of this approach or that, despite the changes in notation or notions of rigor from epoch to epoch, despite new tools, new unifications, new aims, there is an underlying timelessness in the basic *conversation that is mathematics*.

Mathematics has had, for example, thousand-year-long projects, where the mathematicians at work in these projects maintain, throughout, a fairly stable sense of what constitutes important ideas, and important progress. An example of this is the search for positive integral solutions (X,Y) of the diophantine equations:

$Y^2 - 2X^2 = 1,$

$Y^2 - 3X^2 = 1,$

…

$Y^2 - dX^2 = 1,$

where d is a positive integer and not a perfect square.

Why one might be fascinated by this any of our young students of number theory would already know. These equations play a central role in many of the fundamental questions about numbers, their solutions (X, Y) yield the rational numbers Y/X that "best approximates" the quadratic surd $\sqrt{d}$ ("best" in terms of the size of the denominator), the size of these solutions tell much about the subtle behaviour of the continued fraction expansion of these quadratic surds, and on top of it all the equations themselves were the crucial lever that Matjasevic used – some 40 years ago – when he disproved Hilbert's 10th Problem and showed that the question of whether or not the general polynomial equation in many variables with integer coefficients has a solution in integers is *algorithmically unsolvable*.

Given the importance, then, of these equations, it is amazing, and irritating, how little we still know about the erratic sizes of their (smallest) solutions. The desire to come to a better understanding of these solutions animates a search already vigorously engaged in by the Indian mathematicians a millenium ago, by European mathematicians beginning in the 17th century, and continues, with no slacking of intensity, today. Pierre de Fermat mentioned in a letter that the smallest solution to:

$Y^2 - 109X^2 = 1$, d = 109, is:
$Y = 158070671986249$
$X = 15140424455100$.

If Fermat came back to life today, we could tell him that his example is respectably close to the currently known theoretical upper bound, and that we still don't know how crude the asymptotics of this theoretical upper bound is (*very crude*, if a certain still unresolved problem raised by Gauss has an affirmative answer).

We might also tell Fermat that this question is linked to the all-important issue of unique factorization in the rings of integers of quadratic number fields. Of course, at this point we would have to explain a viewpoint that would be a bit new to him – but he would catch on, and, more to the point, all this would genuinely be about a single, and ongoing, topic.

Mathematics, then, is accustomed to long conversations, and I mean really long.

In summary, mathematics makes fluent use of something that might be called "epistemological time," has a span of attention for the themes of interest to it that lasts for millennia and presents itself to many of its practitioners as possessing the quality of *timelessness*. And all this deserves close study.

Notes

[1] Exodus 18.
[2] Mazur, B. (2003) *Imagining numbers (particularly the square root of minus fifteen)*, St. Ives, UK, Allen Lane, the Penguin Press.
[3] Pound, E. (1960, first published 1934) *ABC of reading*, New York, NY, New Directions Publishing Corporation.
[4] John Keats in *Ode on a Grecian urn*.
[5] The quotation below is from Edith Sylla's forthcoming translation of Jacob Bernoulli's *Ars Conjectandi*, published posthumously in 1713. I am thankful to her for providing it for me.

[If] the observations of all events were continued for the whole of eternity (with the probability finally transformed into perfect certainty) then everything in the world would be observed to happen in fixed ratios and with a constant law of alternation. Thus in even the most accidental and fortuitous we would be bound to acknowledge a certain quasi necessity and, so to speak, fatality. I do not know whether or not Plato already wished to assert this result in his dogma of the universal return of things to their former positions [*apocatastasis*], in which he predicted that after the unrolling of innumerable centuries everything would return to its original state.

(From Jacob Bernoulli's *Ars Conjectandi* [5])