Before Utility

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When we say that something—some type of 'goods,' or some set-up— is *useful,* we invoke—perhaps implicitly—the existence of some agent or agents (that can make use of it), of some type of situation (in which it can be put to use), of some mode of operation (i.e., way of using it) and some goal (the reason why it is useful).

1 Usefulness of goods/ friendship/ justice

At times Aristotle focuses his attention on 'goods' in the context of commerce, as in the Nichomachean Ethics Book V.5 ([1]):

> All goods must therefore be measured by some one thing... this unit is in truth *demand*, which holds all things together (for if men did not need one another's goods at all, or did not need them equally, there would be either no exchange or not the same exchange); but money has become by convention a sort of representative of demand.

but in Book VII of The *Politics* ([2]) the word has broader personal significance. The goal—writes Aristotle— of "external goods, goods of the body, and goods of the soul" is, in the end, happiness, and:

> happiness whether consisting in pleasure or virtue, or both, is more often found with those who are most highly cultivated in their mind and in their character, and have only a moderate share of external goods, than among those who possess external goods to a useless extent but are deficient in higher qualities; and this is not only matter of experience, but, if reflected upon, will easily appear to be in accordance with reason. *For, whereas external goods have a limit, like any other instrument, and all things useful are of such a nature that where there is too much of them they must either do harm, or at any rate be of no use, to their possessors, every good of the soul, the greater it is, is also of greater use, if the epithet useful as well as noble is appropriate to such subjects.*

In the last quoted sentence one already sees what one might call the principle of concavity of the relationship between sheer quantity and assessment of usefulness—as will be a theme of later discussions[1].

'Usefulness' itself has its limits in human interactions, as in love and friendship (Nichomachean Ethics Book VIII; [3]):

> ... friends whose affection is based on utility do not love each other in themselves, but in so far as some benefit accrues to them from each other. And similarly with those whose friendship is based on pleasure: for instance, we enjoy the society of witty people not because of what they are in themselves, but because they are agreeable to us. Hence in a friendship based on utility or on pleasure men love their friend for their own good or their own pleasure, and not as being the person loved, but as useful or agreeable. And therefore these friendships are based on an accident, since the friend is not loved for being what he is, but as affording some benefit or pleasure as the case may be. ... And utility is not a permanent quality; it differs at different times. Hence when the motive of the friendship has passed away, the friendship itself is dissolved, having existed merely as a means to that end.

In the view of Epicurus, the essential human pursuit is happiness or some Epicurian version of *eudaimonia*, so the notion of 'usefulness' in his writings is pointed toward that primary human goal, happiness. It is interesting to try to interpret, then, his treatment *Justice,*[2] as being so strongly linked to 'usefulness' in his thought ([4] at least as formulated by Diogenes Laertius in the Principal Doctrines: *Kuriai Doxia*).

- (31) Natural justice is a symbol or expression of usefulness, to prevent one person from harming or being harmed by another.
- (36) Taken generally, justice is the same for all, to wit, something found useful in mutual association; but in its application to particular cases of locality or conditions of whatever kind, it varies under different circumstances.
- (37) Among the things accounted just by conventional law, whatever in the needs of mutual association is attested to be useful, is thereby stamped as just, whether or not it be the same for all; and in case any law is made and does not prove suitable to the usefulness of mutual association, then this is no longer just. And should the usefulness which is expressed by the law vary and only for a time correspond with the prior conception, nevertheless for the time being it was just, so long as we do not trouble ourselves about empty words, but look simply at the facts.

Curious definition of just: "useful for mutual association" (38) below simply repeats:

[1] The Greek word for useful *ophelimos* was 'nominalized into English' in the form of *ophelimity* (by the economist Arthur Cecil Pigou; 1877-1959), meaning the capacity to satisfy a need, desire, or want.

[2] which Epicurus takes to be principally the promotion of social arrangements where no one is 'harming or is being harmed by others'

- (38) Where without any change in circumstances the conventional laws, when judged by their consequences, were seen not to correspond with the notion of justice, such laws were not really just; but wherever the laws have ceased to be useful in consequence of a change in circumstances, in that case the laws were for the time being just when they were useful for the mutual association of the citizens, and subsequently ceased to be just when they ceased to be useful.

2 Bernoulli

The innocent act of nominalizing this adjective *useful*, i.e., turning it into a noun (*usefulness, utility*)—a move that was made in quite early days of economic thought—has the effect of increasing the complexity and subtlety of the questions that you can ask about this concept. For example, can "it" be measured? And if so, what species of object can one measure it against? (*'Itself'* turns out to be a possible answer.)

Utility, the curious (possibly measurable, but certainly subjective) concept, relates to the notions of *Value*, *Price* and *Money*. In the hands of Daniel Bernoulli (∼1738) it makes specific connection with *Risk* and *Expectation* as well. The essay *Exposition of a New Theory on the Measurement of Risk* [5] radiates with the intensity of original ideas, and it also has the delicious crudeness of fresh thinking—the rough edges are not smoothed out in any way. In [5] Bernoulli dives into a qualitative description of the measurement—not only of someone's—a pauper's or millionaire's—sense of the utility of money, per se, but also—as the title indicates—of the person's assessment of expected outcome—an admittedly subjective assessment—of some venture that will be entered into with incomplete knowledge; i.e., in the context of *risk*. For example, in sections 3-5 he considers the prospects of

> a very poor fellow [who] obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble.
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> . . .
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> [T]he determination of the *value* of an item must not be based on its *price*, but rather on the *utility* it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount.

At this point in Bernoulli's essay the notion 'utility' has a qualitative—but not quantitative—status. The viewpoint changes abruptly though:

> [L]et us use this as a fundamental rule: If the utility of each possible profit expectation is multiplied by the number of ways which it can occur, and we then divide the sum of these products by the total number of possible cases, a **mean utility** [moral expectation] will be obtained, and the profit which corresponds to this utility will equal the value of the risk in question. Thus it becomes evident that no valid measurement of the value of a risk can be obtained without consideration being given to its utility, that is to say, the utility of whatever gain accrues to the individual or, conversely, how much profit is required to yield a given utility.

> The utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed. Considering the nature of man, it seems to me that the foregoing hypothesis is apt to be valid for many people to whom this sort of comparison, can be applied. Only a few do not spend their entire yearly incomes. But, if among these, one has a fortune worth a hundred thousand ducats and another a fortune worth the same number of semi-ducats and if the former receives from it a yearly income of, five thousand ducats while the latter obtains the same number of semi-ducats it is quite clear that to the former a ducat has exactly the same significance as a semi-ducat to the latter, and that, therefore, the gain of one ducat will have to the former no higher value than the gain of a semi-ducat to the latter. Accordingly, if each makes a gain of one ducat the latter receives twice as much utility from it, having been enriched by two semi- ducats.

Bernoulli then retreats from precise quantitative formulations:

> However it hardly seems plausible to make any precise generalizations since the utility of an item may change with circumstances. Thus, though a poor man generally obtains more utility than does a rich man from an equal gain, it is nevertheless conceivable, for example, that a rich prisoner who possesses two thousand ducats but needs two thousand ducats more to repurchase his freedom, will place a higher value on a gain of two thousand ducats than does another man who has less money than he.

And then returns to quantitative precision (with *logarithmic curves* measuring utility against more objective markers).

> ... in order to perceive the problem more correctly we shall assume that there is an imperceptibly small growth in the individual's wealth which proceeds continuously by infinitesimal increments. Now it is highly probable that any increase in wealth, no matter how insignificant, *will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed.*

In plain English, the moral Bernoulli wants to draw is this. Let us be given that there is a measure of utility $u(t)$ that depends on $t :=$ the amount of goods, say, —or money—that you have. (That there is a meaningful such function measuring the level of utilty of the goods in your possession is, of course, already the big assumption.) And that the curve $u(t)$ representing utility as a function of t is *concave*. In a fair game, if you have x amount (of money) and you are willing to invest some of it, say y with a 50% probability of losing it or gaining $2y$, then—if you play the game—your mean utility will decrease, even if you are in this 'fair game.'

Her's why: you have begun the game with 'utility' $u(x)$. If you lose, you'd have utility $u(x-y)$; if you win it would be $u(x+y)$. Since it's 50/50 (if you play the game) your expected 'mean' utility—following Bernoulli—is $\frac{u(x-y)+u(x+y)}{2}$, which by the concavity of the function $u(t)$ is strictly less than $u(x)$, your starting utility level.

He then works with somewhat shocking explicitness, in effect inverting his utility function—specifically: *turning his logarithmic utility function into what one might call a 'multiplicative' function that describes perceived value*—to get a 'mean' expected value in various situations. He explains why there are occasions, therefore, where everyone—e.g., agents working on opposite sides of a deal should eagerly participate—given that different agents will have different amounts of wealth—hence the utility of a gain or loss will be significantly different. He applies this to *Insurance* and to dividing risk, spreading it over a number of situations (should you have all your cargo shipped in one ship, or spread it over a few ships?)

The basic arithmetic is simple enough. First the pre-Bernoulli analysis: if you have \$100 and invest in a project where you have an equal chance of losing \$50 or gaining \$50, the straightforward evaluation is that it is a totally fair (neutral, in a sense) game, in that you will end up with either a bank account of \$50 or \$150 with equal probability, so your 'Expected wealth' after the game is

$$\frac{1}{2}50 + \frac{1}{2}150 = 100 \tag{1}$$

dollars, exactly what it was before the game.

Now, given

- Bernoulli's hypothesis that expected utility has a logarithmic relationship to wealth[3], and
- that the Expected utility of an event that will have one of a range of outcomes, each with specific probabilities, is the sum, over the range of outcomes, of the product of the Utility of the particular outcome times the probability that it will occur,

Bernoulli reckons your 'Expected wealth', after the game above, to be:

$$\$\sqrt{50 \cdot 150} = \$86.60 \tag{2}$$

So, an 'Expected loss' of \$13:40. He writes:

[3]which is a big assumption; far too explicit to be meaningful!

> We must strongly emphasize this truth, although it be self evident: the imprudence of a gambler will be the greater the larger the part of his fortune which he exposes to a game of chance.

E.g., if—for example—the gambler has \$10000, the 'Expected wealth', after the game would be:

$$\$\sqrt{9950 \cdot 10050} = \$9999.87 \tag{3}$$

So, an 'Expected loss' of 13 cents.

Let's do same analysis with 'insurance': imagine that the probability of a certain event happening is 1/4 and if it happens, it will cost you \$100; if it doesn't, you have as much money as you had before—say \$1000. The insurance company, is worth \$100,000. Following the recipe $(900)^{1/4}1000^{3/4}$, your expected wealth after the event is \$973, so if your insurance premium is less than \$27, that sounds like it's worth it. But for the insurance company the recipe is

$$(99,900)^{1/4}100,000^{3/4} = 99984.28.$$

So if the insurance company charges more than $100,000 - 99,984.28 = \$15.72$, it may expect a profit. Any premium, then between \$16 and \$27 would—given Bernoulli's analysis(!) —be a reasonable risk to take for either the insurer or the insuree.

Bernoulli ends his essay discussing (what is now known as) the St. Petersburg Paradox: how much would you be willing to pay to be a participant of the game of the following sort. In this game you can only win, not lose. A coin is thrown, and if it is heads the first throw, you get \$1 (and the game ends). If it is tails, it gets thrown again and if—then—it lands on heads you get \$2 (and the game ends). ... If it is tails the first n times and is heads the $(n+1)$st time, you get $\$2^n$ dollars (and the game ends).

How much money might you naively expect to get on average if you play this game in long runs, if–for example—this game is scheduled to be cut off, finished or not, after n (or fewer, if it is finished at fewer) times? And what if it is slated to go on indefinitely? *There's an underlying empirical/psychological question here, and it pays to think about it on that level as well, before closer analysis, worth taking some time to discuss.*

The issue here is that if you apply the naive notion of "Expectation" to this problem— you arrive at one—unbelievable—answer, but if you work with "Expected utility" you get an answer that (you might argue) conforms—at least qualitatively— more closely to what people would actually pay to get into this game.

A naive computation of "expectation" of the money you might expect to win is

$$\sum_{k=1}^{\infty} \text{Prob}(k) \cdot \text{Payoff}(k),$$

where Prob(k) is the probability that you'll win at the k-th stage, and Payoff(k) is the payoff if you win at the k-th stage. So,

$$\sum_{k=1}^{\infty} \text{Prob}(k) \cdot \text{Payoff}(k) \quad = \quad \sum_{k=1}^{\infty} \frac{1}{2^k} 2^{k-1} \quad = \quad \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \tag{4}$$

which seems to recommend that you should be willing to pay absolutely any amount to get into that game. But (apparently) none of us would pay a very high figure to get into this game—so what's wrong? Are we (somewhat irrationally) risk averse? Or is there already something simply wrong or paradoxical with this naive notion of Expectation? Or perhaps, does the utterly unreal nature of this strange game spook us? A neat discussion ensues, along with remarks about the parallel work of Gabriel Cramer (1728; in a letter to Nicholas Bernoulli).

To quote Bernoulli quoting Cramer:

> The "paradox" consists in the infinite sum which calculation yields ... This seems absurd since no reasonable man would be willing to pay 20 ducats as equivalent. You ask for an explanation of the discrepancy between the mathematical calculation and the vulgar evaluation. I believe that it results from the fact that, in their theory, mathematicians evaluate money in proportion to its quantity while, in practice, people with common sense evaluate money in proportion to the utility they can obtain from it.

Cramer then explains how how unravels the paradox—not terribly different from the way Bernoulli does. Here's how Bernoulli deals with it:

If your wealth is α and the game actually can plausibly be played for n times, then—as Bernoulli theorizes!—the value of the game *for you* is

$$V_n(\alpha) := \prod_{k=1}^{n} (\alpha + 2^{k-1})^{\frac{1}{2^k}} - \alpha.$$

And this converges for any given α (as $n \to \infty$). E.g., if you own nothing, i.e., if $\alpha = 0$, its value is in the limit as $n \to \infty$ is:

$$V_\infty(0) = \prod_{k=1}^{\infty} (2^{k-1})^{\frac{1}{2^k}} = \prod_{k=2}^{\infty} (2^{k-1})^{\frac{1}{2^k}}$$

$$= 2^{\sum_{j=1}^{\infty} \frac{j}{2^{j+1}}} = 2^1 = 2.$$

This is a weird conclusion, *and an interesting discussion might be: does any of this make sense?*

Bernoulli ends his essay with an engaging discussion regarding the fact—paradox, in essence—that (as computed by Bernoulli) if your wealth α gets larger and larger, your expectation of gain gets larger as well. It's left quite a bit unresolved, but see [6] and [7]. People offer specific (low) amounts to enter such a game so this becomes a behavioral issue, or curiosity: what does their behavior tell us about personal assessments of risk?

We could discuss this. Various suggestions are in the literature: a prospective player of this game might be aware that the casino has finite resources so the game, as presented is untenable, or bogus/ also things that happen with miniscule odds are neglectable even if a naive computation of Expectation seems to get us forget this/ etc.

For an experimental study, see [8] where, among other things the authors test people's willingness to participate in various versions of the game and offer bids; and also in inversions of the game where—if you enter, you can lose as well as win:

> Recent experimental research has returned to the questions originally posed by Bernoulli: Is human choice behaviour in St. Petersburg lotteries (a) inconsistent with expected value theory and (b) consistent with risk aversion? Recent experiments (Cox, Sadiraj and Vogt 2009; Neugebauer 2010) have used real money payoffs and finite games in order, respectively, to provide the experiment subjects with economic motivation and the experimenters offers of the lottery with credibility. Data from these experiments are inconsistent with risk neutrality but consistent with risk aversion, in this way appearing to provide support for Bernoullis general conclusion about risk aversion (but not his specific conclusion about log utility).

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