A specific question

Question

Let E be an elliptic curve over $\mathbb{Q}$, and $\chi : G_{\mathbb{Q}} \to \mathbb{C}^{\times}$ a character. How likely is it that $L(E, \chi, 1) = 0$?

A specific question

Question

Let E be an elliptic curve over $\mathbb{Q}$, and $\chi : G_{\mathbb{Q}} \to \mathbb{C}^{\times}$ a character. How likely is it that $L(E, \chi, 1) = 0$?

Is $\{\chi : L(E, \chi, 1) = 0 \text{ and } \chi \text{ has order } d\}$ finite for large d?

Is $\{\chi : L(E, \chi, 1) = 0 \text{ and } \chi \text{ has order } d\}$ empty for large d?

A specific question

Question

Let E be an elliptic curve over $\mathbb{Q}$, and $\chi : G_{\mathbb{Q}} \to \mathbb{C}^{\times}$ a character. How likely is it that $L(E, \chi, 1) = 0$?

Is $\{\chi : L(E, \chi, 1) = 0 \text{ and } \chi \text{ has order } d\}$ finite for large d?

Is $\{\chi : L(E, \chi, 1) = 0 \text{ and } \chi \text{ has order } d\}$ empty for large d?

One can ask similar questions with $\mathbb{Q}$ replaced by any number field and E replaced by a form of weight greater than 2.

David-Fearnley-Kisilevsky conjecture

Conjecture (David-Fearnley-Kisilevsky)

Let $p \geq 7$ be a prime and E an elliptic curve over $\mathbb{Q}$. Then there are only finitely many χ of order p such that $L(E, \chi, 1) = 0$.

They also made a prediction for the growth of the number of such χ (ordered by conductor) when $p = 3$ or 5.

This conjecture was motivated by random matrix statistics. More on this tomorrow.

Conjecture

The following conjecture is (sort of) motivated by the statistics of modular symbols and θ-coefficients.

Conjecture

Suppose E is an elliptic curve over $\mathbb{Q}$. Then there are only finitely many $\chi : G_{\mathbb{Q}} \to \mathbb{C}^{\times}$ such that

- $L(E, \chi, 1) = 0$, *and*
- φ*(order of* χ*)* > 4.

Conjecture

The previous conjecture implies:

Conjecture

Suppose E is an elliptic curve over $\mathbb{Q}$, and $L/\mathbb{Q}$ is an (infinite) abelian extension such that $\mathrm{Gal}(L/\mathbb{Q})$ has only finitely many characters of order 2, 3, and 5. Then $E(L)$ is finitely generated.

Conjecture

The previous conjecture implies:

Conjecture

Suppose E is an elliptic curve over $\mathbb{Q}$, and $L/\mathbb{Q}$ is an (infinite) abelian extension such that $\mathrm{Gal}(L/\mathbb{Q})$ has only finitely many characters of order 2, 3, and 5. Then $E(L)$ is finitely generated.

This is known if $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}_\ell \times$ (finite group) (Kato, Rohrlich).
The hypotheses also apply if:

1. $\mathrm{Gal}(L/\mathbb{Q}) = \hat{\mathbb{Z}}$, or
2. L is the maximal abelian ℓ-extension of $\mathbb{Q}$, with $\ell \geq 7$, or
3. L is the compositum of all such fields (1) and (2).

Vertical line integrals

Let E be an elliptic curve over $\mathbb{Q}$ and

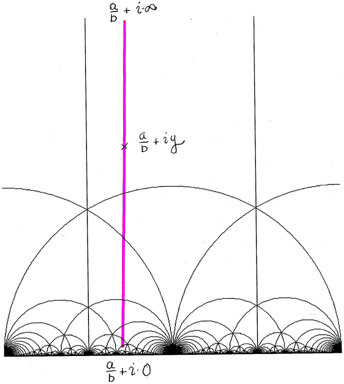
$$f_E(z)dz = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} dz$$

the modular form attached to E, viewed as differential form on the upper-half plane.

For any rational number $r = a/b$, form the integral

$$2\pi i \int_{r+i\cdot 0}^{r+i\cdot\infty} f_E(z)dz.$$

Integrating over vertical lines in the upper half-plane



Raw modular symbols

Symmetrize or anti-symmetrize to define **raw $(\pm)$ modular symbol** attached to the rational number r:

$$\langle r\rangle_E^{\pm} := \pi i\left(\int_{i\infty}^{r} f_E(z)dz \pm \int_{i\infty}^{-r} f_E(z)dz\right)$$

Raw modular symbols

Symmetrize or anti-symmetrize to define **raw $(\pm)$ modular symbol** attached to the rational number r:

$$\langle r\rangle_E^{\pm} := \pi i\left(\int_{i\infty}^{r} f_E(z)dz \pm \int_{i\infty}^{-r} f_E(z)dz\right)$$

The raw modular symbols $\langle r\rangle_E^{\pm}$ take values in the discrete subgroup of $\mathbb{R}$ generated by $\frac{1}{D}\Omega_E^{\pm}$ for some positive D.

In this discussion, for simplicity, we'll consider only the $+$-raw modular symbols:

$$\langle r\rangle := \langle r\rangle_E^{+}.$$

L-functions and modular symbols

Theorem

For every even primitive Dirichlet character χ of conductor m,

$$\sum_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} \chi(a)\langle a/m \rangle = \tau(\chi) L(E, \bar{\chi}, 1).$$

Here $\tau(\chi)$ is the Gauss sum.

Dirichlet characters as Galois characters

For a cyclic extension $L/\mathbb{Q}$ of conductor m we have a canonical surjection

$$\begin{array}{ccccc} (\mathbb{Z}/m\mathbb{Z})^\times & \xrightarrow{\sim} & \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_m)/\mathbb{Q}) & \twoheadrightarrow & \mathrm{Gal}(L/\mathbb{Q}) \\ a & \longmapsto & & & \sigma_{a,L}. \end{array}$$

which allows us to think of Dirichlet characters as Galois characters, and vice-versa.

θ-elements

For a cyclic extension $L/\mathbb{Q}$ the **θ-element**

$$\theta_{L/\mathbb{Q}} := \theta_{E,L/\mathbb{Q}}$$

is the element in the group ring $\mathbb{R}[\mathrm{Gal}(L/\mathbb{Q})]$

$$\theta_{L/\mathbb{Q}} \ := \sum_{a\in(\mathbb{Z}/m\mathbb{Z})^\times} \langle a/m \rangle \cdot \sigma_{a,L} \ = \sum_{g\in\mathrm{Gal}(L/\mathbb{Q})} c_{E,g} \cdot g,$$

where the **θ-coefficients** $c_{E,g}$ are given by

$$c_{E,g} = c_g = \sum_{a\,:\,\sigma_{a,L}=g} \langle a/m \rangle.$$

L-functions and θ-elements

One has

$$L(E,\chi,1)=0 \iff \sum_{a\in(\mathbb{Z}/m\mathbb{Z})^\times} \chi(a)\langle a/m\rangle = 0$$
$$\iff \sum_{g\in \mathrm{Gal}(L/\mathbb{Q})} \chi(g)c_g = 0$$
$$\iff \chi\left(\theta_{L/\mathbb{Q}}\right) = 0.$$

We are interested in the statistics of

- the raw modular symbols $\langle a/m\rangle$,
- the θ-coefficients c_g,

and we want to use computational exploration to suggest how often $L(E,\chi,1)=0$.

Relations satisfied by the modular symbols

Let N be the conductor of E. For every $r \in \mathbb{Q}$, modular symbols satisfy the relations:

Relations satisfied by the modular symbols

Let N be the conductor of E. For every $r \in \mathbb{Q}$, modular symbols satisfy the relations:

- $\boxed{\langle r \rangle = \langle r+1 \rangle}$ since $f_E(z) = f_E(z+1)$

Relations satisfied by the modular symbols

Let N be the conductor of E. For every $r \in \mathbb{Q}$, modular symbols satisfy the relations:

- $\boxed{\langle r \rangle = \langle r+1 \rangle}$ since $f_E(z) = f_E(z+1)$
- $\boxed{\langle r \rangle = \langle -r \rangle}$ by definition

Relations satisfied by the modular symbols

Let N be the conductor of E. For every $r \in \mathbb{Q}$, modular symbols satisfy the relations:

- $\boxed{\langle r \rangle = \langle r+1 \rangle}$ since $f_E(z) = f_E(z+1)$
- $\boxed{\langle r \rangle = \langle -r \rangle}$ by definition
- Atkin-Lehner relation: if w_E is the global root number of E, and $aa'N \equiv 1 \pmod{m}$, then $\boxed{\langle a'/m \rangle = w_E \cdot \langle a/m \rangle}$

Relations satisfied by the modular symbols

Let N be the conductor of E. For every $r \in \mathbb{Q}$, modular symbols satisfy the relations:

- $\boxed{\langle r \rangle = \langle r+1 \rangle}$ since $f_E(z) = f_E(z+1)$
- $\boxed{\langle r \rangle = \langle -r \rangle}$ by definition
- Atkin-Lehner relation: if w_E is the global root number of E, and $aa'N \equiv 1 \pmod{m}$, then $\boxed{\langle a'/m \rangle = w_E \cdot \langle a/m \rangle}$
- Hecke relation: if a prime $\ell \nmid N$ and a_ℓ is the ℓ-th Fourier coefficient of f_E, then $\boxed{a_\ell \cdot \langle r \rangle = \langle \ell r \rangle + \sum_{i=0}^{\ell-1} \langle (r+i)/\ell \rangle}$

Regularities in the modular symbols data

There are some significant *regularities* in the values of modular symbols.

For example, consider the behavior of contiguous sums of the modular symbol:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{a=0}^{\lfloor mx \rfloor} \left\langle \frac{a}{m} \right\rangle \quad = \quad \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{a_n}{n^2} \sin(\pi n x).$$

Regularities in the modular symbols data

There are some significant *regularities* in the values of modular symbols.

For example, consider the behavior of contiguous sums of the modular symbol:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{a=0}^{\lfloor mx \rfloor} \left\langle \frac{a}{m} \right\rangle \quad = \quad \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{a_n}{n^2} \sin(\pi n x).$$

Conjecture of M-R-S recently proved by Kim & Sun.

Random distribution of modular symbols

Theorem (Petridis-Risager)

The distribution determined by the data

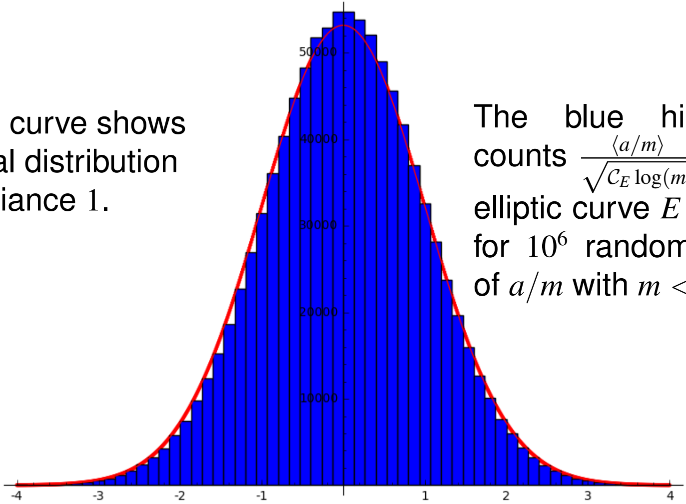
$$\frac{\langle a/m\rangle}{\sqrt{\mathcal{C}_E \log(m)}} \; : \; m \geq 1, a \in (\mathbb{Z}/m\mathbb{Z})^\times$$

is normal with variance 1.

Here $\mathcal{C}_E$ is an explicit constant: if E is semistable then

$$\mathcal{C}_E := \frac{6}{\pi^2} \cdot \prod_{p \mid N} \frac{p}{p+1} \cdot L(\mathrm{Sym}^2(f_E), 2).$$

$E = 11a1$



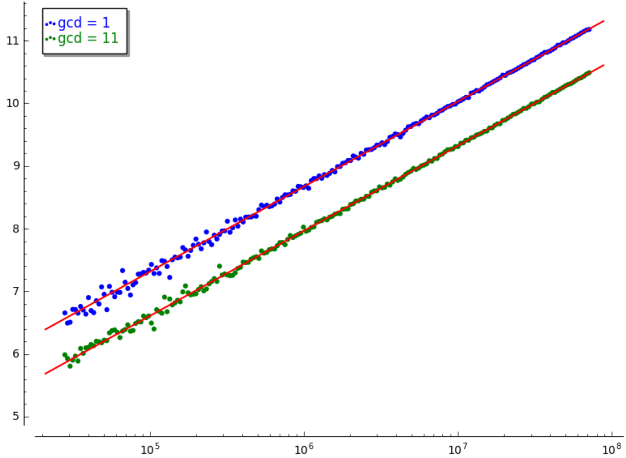
The red curve shows a normal distribution with variance 1.

The blue histogram counts $\frac{\langle a/m \rangle}{\sqrt{\mathcal{C}_E \log(m)}}$ for the elliptic curve $E = 11a1$, for 10^6 random values of a/m with $m < 10^{16}$.

The variance

Let $\mathrm{Var}(E,m)$ denote the variance of $\langle a/m \rangle$, $a \in (\mathbb{Z}/m\mathbb{Z})^{\times}$.

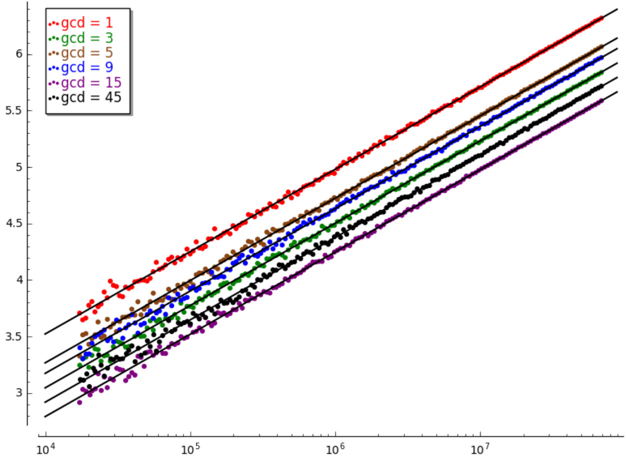
This is a graph of $\mathrm{Var}(E,m)$ versus m for the curve $11a1$.
The two lines correspond to $\gcd(m,11)=1$ and $\gcd(m,11)=11$.



The variance

Let $\mathrm{Var}(E, m)$ denote the variance of $\langle a/m \rangle$, $a \in (\mathbb{Z}/m\mathbb{Z})^\times$.

This is a graph of $\mathrm{Var}(E, m)$ versus m for the curve $45a1$.
The lines correspond to the six possible values of $\gcd(m, 45)$.



The ‘Variance slope’ and ‘Variance shift’

Conjecture (M-R)

For every divisor κ of N_E there is a $\mathcal{D}_{E,\kappa} \in \mathbb{R}$ such that

$$\lim_{\substack{m\to\infty \\ \gcd(m,N)=\kappa}} \mathrm{Var}(E,m) - \mathcal{C}_E \cdot \log m = \mathcal{D}_{E,\kappa}$$

The ‘Variance slope’ and ‘Variance shift’

Conjecture (M-R)

For every divisor κ *of* N_E *there is a* $\mathcal{D}_{E,\kappa} \in \mathbb{R}$ *such that*

$$\lim_{\substack{m\to\infty \\ \gcd(m,N)=\kappa}} \mathrm{Var}(E,m) - \mathcal{C}_E \cdot \log m = \mathcal{D}_{E,\kappa}$$

Petridis & Risager recently announced a proof of an “averaged over m” version of this conjecture, including an explicit formula for $\mathcal{D}_{E,\kappa}$.

Recall θ-coefficients and θ-elements

Suppose $L/\mathbb{Q}$ has conductor m.

$$c_g := \sum_{a\,:\,\sigma_a = g} \langle a/m \rangle \quad \text{for } g \in \mathrm{Gal}(L/\mathbb{Q}),$$

$$\theta_L := \sum_{g \in \mathrm{Gal}(L/\mathbb{Q})} c_g \cdot g \in \mathbb{R}[\mathrm{Gal}(L/\mathbb{Q})].$$

Then for all faithful $\chi : \mathrm{Gal}(L/\mathbb{Q}) \hookrightarrow \mathbb{C}^\times$,

$$\chi(\theta_L) = \tau(\chi) L(E, \bar{\chi}, 1).$$

We want to know how often this vanishes.

Distribution of θ-coefficients

For simplicity suppose that $\ell := [L : K]$ is an odd prime, and suppose χ is a nontrivial character of $\mathrm{Gal}(L/K)$.

1. $\chi(\theta_L) = 0 \iff$ all c_g are equal.
2. The Hecke action shows that

$$\sum_{g \in \mathrm{Gal}(L/K)} c_g = \prod_{p|m} (a_p - 2) L(E, 1).$$

3. Atkin-Lehner duality induces an 'involution' $g \to g'$ such that

$$c_{g'} = w_E \cdot c_g.$$

We call the unique fixed point of this involution the **sensitive** element of $\mathrm{Gal}(L/\mathbb{Q})$.

Distribution of θ-coefficients

Combining these properties, let X be a set of representatives of the $(\ell-1)/2$ orbits $\{g, g'\}$ under the involution. Then

$$\chi(\theta_L) = 0 \iff c_g = \prod_{p|m}(a_p - 2)L(E,1)/\ell \quad \text{for every } g \in X$$

Distribution of θ-coefficients

Combining these properties, let X be a set of representatives of the $(\ell-1)/2$ orbits $\{g, g'\}$ under the involution. Then

$$\chi(\theta_L) = 0 \iff c_g = \prod_{p|m}(a_p - 2)L(E,1)/\ell \quad \text{for every } g \in X$$

Question

How likely is it that $c_g = \prod_{p|m}(a_p - 2)L(E,1)/\ell$?

The distribution of θ-coefficients for fixed d

Fix d odd. For $[L : \mathbb{Q}]$ cyclic of order d and conductor m, each θ-coefficient c_g is a sum of $\varphi(m)/d$ modular symbols.

If this were a *random* sum of modular symbols, we would expect the variance of the c_g to be close to $(\mathcal{C}_E \log(m) + \mathcal{D}_{E,\kappa})\varphi(m)/d$.

The distribution of θ-coefficients for fixed d

Fix d odd. For $[L:\mathbb{Q}]$ cyclic of order d and conductor m, each θ-coefficient c_g is a sum of $\varphi(m)/d$ modular symbols.

If this were a *random* sum of modular symbols, we would expect the variance of the c_g to be close to $(\mathcal{C}_E \log(m) + \mathcal{D}_{E,\kappa})\varphi(m)/d$.

Let $\Lambda_{E,d}(t)$ be the distribution determined by the data

$$(L, g, m) \mapsto \frac{c_g}{\sqrt{(\mathcal{C}_E \log(m) + \mathcal{D}_{E,\kappa}) \cdot \varphi(m)/d}}$$

where (L, g, m) runs through all triples such that:

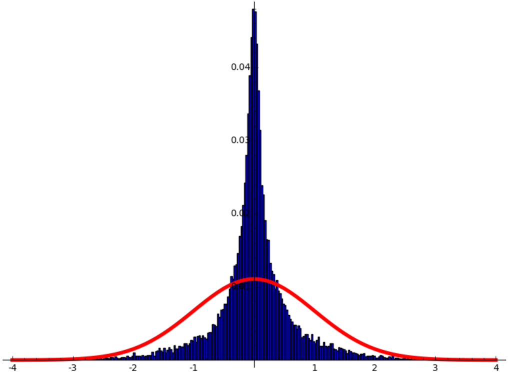
- $L/\mathbb{Q}$ is cyclic of order d,
- $g \in \mathrm{Gal}(L/\mathbb{Q})$ is not the *sensitive element*,
- m is the conductor of $L/\mathbb{Q}$.

The distribution of θ-coefficients for fixed d

We originally expected the $\Lambda_{E,d}(t)$ would be a normal distribution with variance 1.

The distribution of θ-coefficients for fixed d

We originally expected the $\Lambda_{E,d}(t)$ would be a normal distribution with variance 1. However, when $E = 11a1$ and $d = 3$:



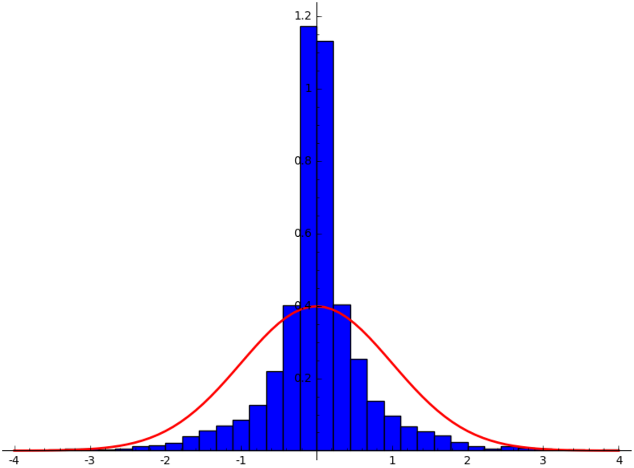
(the red curve is the normal distribution with variance 1). This histogram is typical of other elliptic curves for $d = 3$.

The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod{d}$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L : \mathbb{Q}] = d$,

$d = 3$

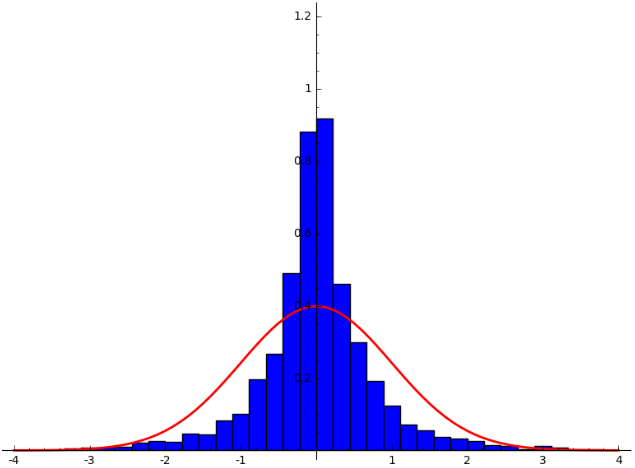


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod{d}$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L : \mathbb{Q}] = d$,

$d = 5$

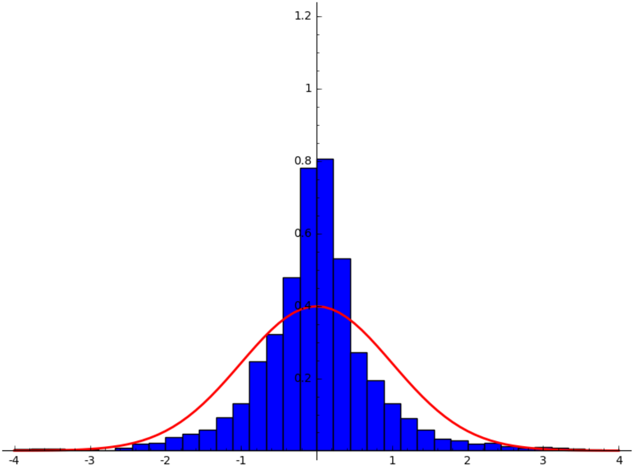


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod d$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L:\mathbb{Q}] = d$,

$d = 7$

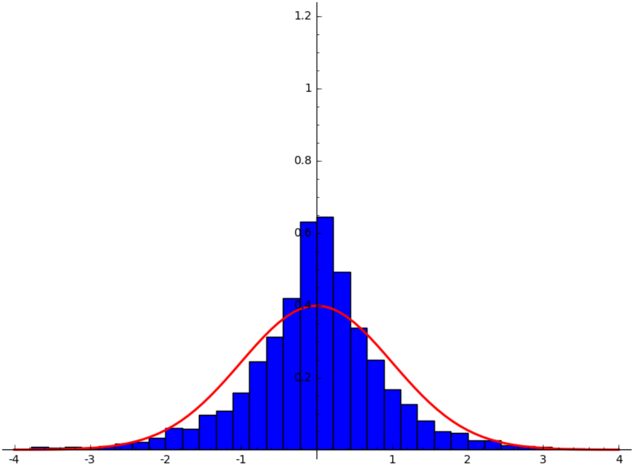


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod{d}$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L : \mathbb{Q}] = d$,

$d = 11$

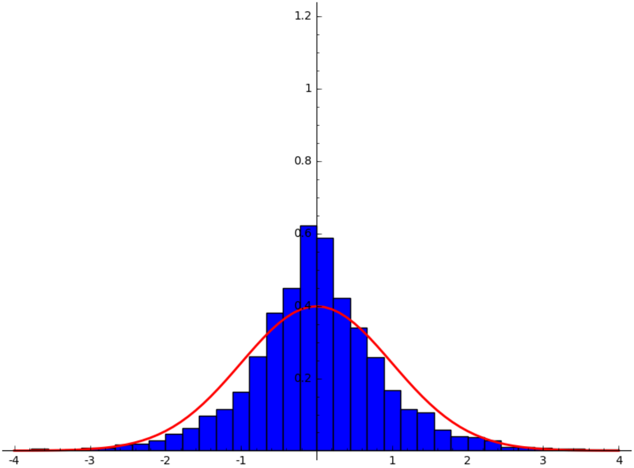


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod{d}$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L : \mathbb{Q}] = d$,

$d = 13$

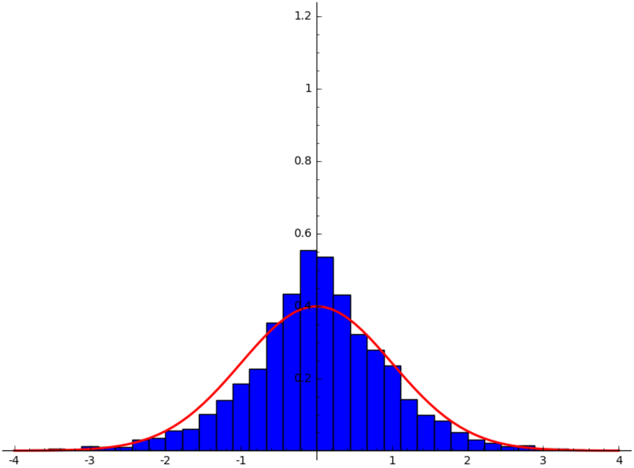


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod{d}$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L : \mathbb{Q}] = d$,

$d = 17$

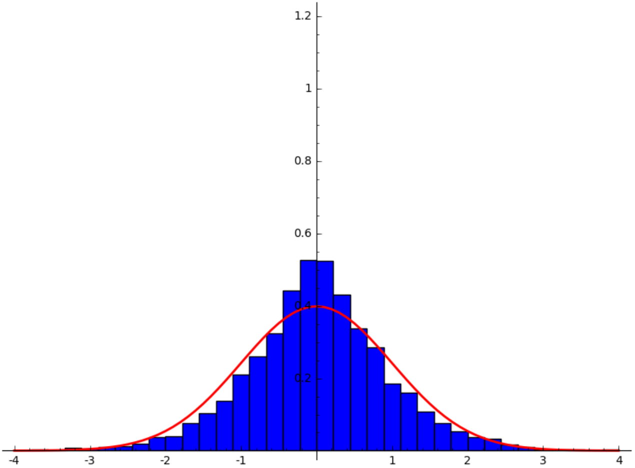


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod d$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L:\mathbb{Q}] = d$,

$d = 23$

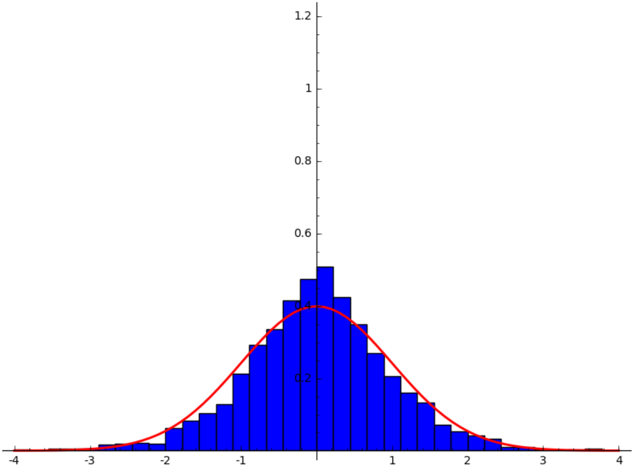


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod{d}$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L : \mathbb{Q}] = d$,

$d = 31$

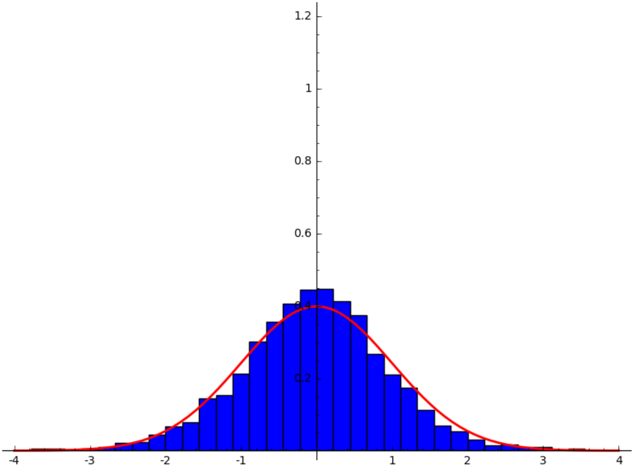


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod d$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L:\mathbb{Q}] = d$,

$d = 41$

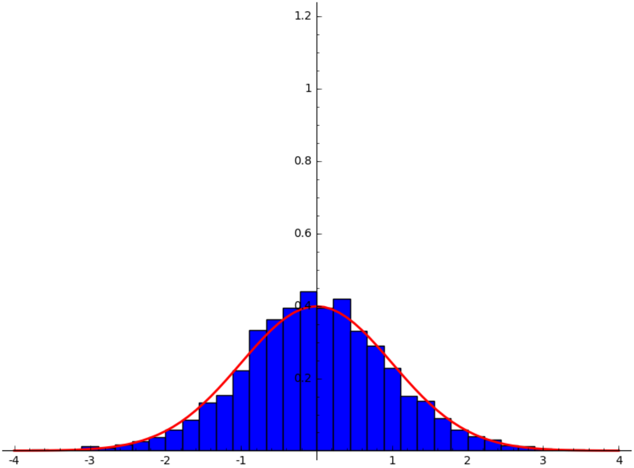


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod{d}$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L:\mathbb{Q}] = d$,

$d = 53$

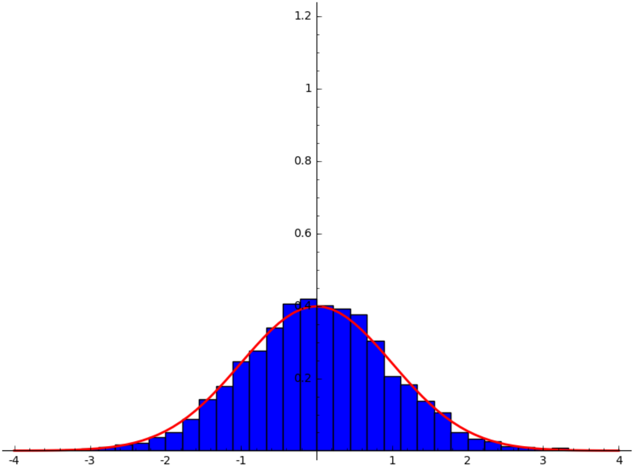


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod d$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L : \mathbb{Q}] = d$,

$d = 97$

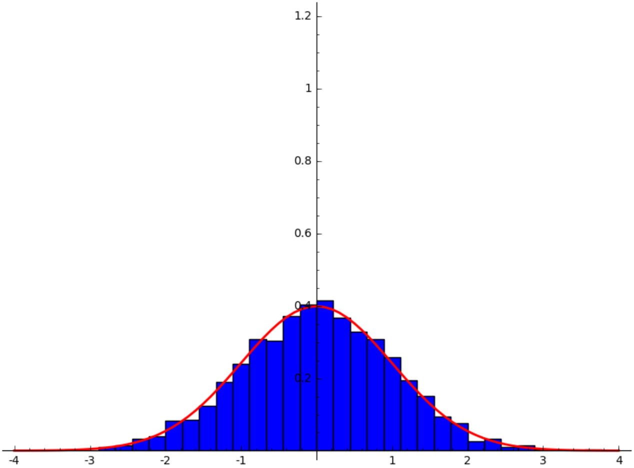


The distribution of θ-coefficients for fixed d

The spikiness seems to disappear as d grows:

$E = 11A1$, $m \equiv 1 \pmod{d}$, $L \subset \mathbb{Q}(\boldsymbol{\mu}_m)$, $[L : \mathbb{Q}] = d$,

$d = 293$



The distribution of θ-coefficients for fixed d

Questions

1. *Does the distribution $\Lambda_{E,d}(t)$ make sense (converge) for fixed d?*

The distribution of θ-coefficients for fixed d

Questions

1. *Does the distribution $\Lambda_{E,d}(t)$ make sense (converge) for fixed d?*
2. *If so, is $\Lambda_{E,d}(t)$ bounded?*

The distribution of θ-coefficients for fixed d

Questions

1. *Does the distribution $\Lambda_{E,d}(t)$ make sense (converge) for fixed d?*
2. *If so, is $\Lambda_{E,d}(t)$ bounded?*
3. *If it's not bounded, what is the behavior near $t = 0$?*

The distribution of θ-coefficients for fixed d

Questions

1. *Does the distribution $\Lambda_{E,d}(t)$ make sense (converge) for fixed d?*
2. *If so, is $\Lambda_{E,d}(t)$ bounded?*
3. *If it's not bounded, what is the behavior near $t = 0$?*
4. *Do the $\Lambda_{E,d}(t)$ converge to a limiting distribution as d grows?*

The distribution of θ-coefficients for fixed d

Questions

1. *Does the distribution $\Lambda_{E,d}(t)$ make sense (converge) for fixed d?*
2. *If so, is $\Lambda_{E,d}(t)$ bounded?*
3. *If it's not bounded, what is the behavior near $t = 0$?*
4. *Do the $\Lambda_{E,d}(t)$ converge to a limiting distribution as d grows?*
5. *If so, is $\lim_{d\to\infty} \Lambda_{E,d}(t)$ the normal distribution with variance 1?*

The distribution of θ-coefficients for fixed d

Questions

1. *Does the distribution $\Lambda_{E,d}(t)$ make sense (converge) for fixed d?*
2. *If so, is $\Lambda_{E,d}(t)$ bounded?*
3. *If it's not bounded, what is the behavior near $t = 0$?*
4. *Do the $\Lambda_{E,d}(t)$ converge to a limiting distribution as d grows?*
5. *If so, is $\lim_{d\to\infty} \Lambda_{E,d}(t)$ the normal distribution with variance 1?*
6. *How does $\Lambda_{E,d}(t)$ depend on E?*

“Expectation" of L-function vanishing

Fix E. We originally expected that $\Lambda_{E,d}(t)$ would be the normal distribution with variance 1, but the data contradicts this.

“Expectation" of L-function vanishing

Fix E. We originally expected that $\Lambda_{E,d}(t)$ would be the normal distribution with variance 1, but the data contradicts this.

We next expected that the $\Lambda_{E,d}(t)$ would be bounded independently of d. We suspect this may not be true either.

“Expectation" of L-function vanishing

Fix E. We originally expected that $\Lambda_{E,d}(t)$ would be the normal distribution with variance 1, but the data contradicts this.

We next expected that the $\Lambda_{E,d}(t)$ would be bounded independently of d. We suspect this may not be true either.

When d is prime, $\chi(\theta_L) = 0$ if and only if $(d-1)/2$ of the θ coefficients take a specified value. For general $d > 2$, we need the θ-coefficients to lie in a sub-lattice of codimension $\varphi(d)/2$.

This all leads to the following heuristic:

“Expectation" of L-function vanishing

Heuristic

Suppose $\Lambda_{E,d}(t) \ll_E t^{-a}$ *for some* $a \geq 0$. *Then there is a constant* γ_E *depending only on* E *such that*

$$\text{“Exp”}[L(E,\chi,1)=0] \leq \left(\frac{d}{\varphi(m)} \cdot \frac{\gamma_E}{\log(m)}\right)^{\frac{\varphi(d)}{4}-a}$$

where d *is the order of* χ *and* m *its conductor.*

This should hold for all χ of order greater than 2.

Consequences of the heuristic, small d

Heuristic

If $\Lambda_{E,d}(t) \ll_E t^{-a}$ *then* $\text{“Exp”}[L(E,\chi,1)=0] \leq \left(\frac{d}{\varphi(m)} \cdot \frac{\gamma_E}{\log(m)}\right)^{\frac{\varphi(d)}{4}-a}$

Example ($d = 3$)

$$\sum_{\chi \text{ order } 3,\ \text{conductor} < X} \text{“Exp”}[L(E,\chi,1)=0] \ll \sum_{m=2}^{X} (\log(m)\varphi(m))^{a-1/2}$$

$$\ll X^{1/2+a}.$$

If $a > 0$ this is consistent with the prediction of David-Fearnley-Kisilevsky.

Consequences of the heuristic, small d

Heuristic

If $\Lambda_{E,d}(t) \ll_E t^{-a}$ *then* $\text{“Exp”}[L(E,\chi,1)=0] \leq \left(\frac{d}{\varphi(m)} \cdot \frac{\gamma_E}{\log(m)}\right)^{\frac{\varphi(d)}{4}-a}$

Example ($d = 5$)

$$\sum_{\chi \text{ order } 5,\ \text{conductor} < X} \text{“Exp”}[L(E,\chi,1)=0] \ll \sum_{m=2}^{X} (\log(m)\varphi(m))^{a-1}$$

$$\ll X^a \log\log X.$$

If $a > 0$ this is consistent with the prediction of David-Fearnley-Kisilevsky.

Consequences of the heuristic, small d

Heuristic

If $\Lambda_{E,d}(t) \ll_E t^{-a}$ *then* "Exp"$[L(E,\chi,1)=0] \leq \left(\frac{d}{\varphi(m)} \cdot \frac{\gamma_E}{\log(m)}\right)^{\frac{\varphi(d)}{4}-a}$

Example ($d = 7$)

$$\sum_{\chi \text{ order } 7, \text{ conductor} < X} \text{"Exp"}[L(E,\chi,1)=0] \ll \sum_{m=2}^{X} (\log(m)\varphi(m))^{a-3/2}$$

$$\ll X^{a-1/2}$$

If $0 < a < 1/2$ this is consistent with the prediction of David-Fearnley-Kisilevsky.

Consequences of the heuristic: all large d

Proposition

Suppose $t : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ *is a function, and* $t(d) \gg \log(d)$. *Then*

$$\sum_{d\,:\,t(d)>1} \sum_{\chi \text{ of order } d \text{ and conductor } m} \left(\frac{d}{\varphi(m)} \cdot \frac{\gamma_E}{\log(m)} \right)^{t(d)}$$

converges.

Consequences of the heuristic: all large d

Proposition

Suppose $t : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$ *is a function, and* $t(d) \gg \log(d)$. *Then*

$$\sum_{d\,:\,t(d)>1} \sum_{\chi \text{ of order } d \text{ and conductor } m} \left(\frac{d}{\varphi(m)} \cdot \frac{\gamma_E}{\log(m)} \right)^{t(d)} \quad \textit{converges.}$$

Applying this with $t(d) = \varphi(d)/4 - a$ shows

Heuristic

If $\Lambda_{E,d}(t) \ll_E t^{-a}$ *then*

$$\sum_{d\,:\,\varphi(d)>4+4a} \sum_{\chi \text{ order } d} \text{“Exp”}[L(E,\chi,1)=0\text{”}] \quad \textit{converges.}$$

For $a < 1/2$ this leads to the conjectures stated at the beginning.