

INTRODUCTION TO AXIOMATIC REASONING

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CONTENTS

Part 1. About our seminar-course “Axiomatic Reasoning.”	1
1. How to comprehend all this? And why?	3
2. Comments on the expected format of our seminar	3
3. Here are the specific slants to the subject that my chaired sessions will take.	4
Part 2. Skeletal notes for today’s session: <i>The evolution of definitions and axioms, from ancient Greek philosophy and mathematics to Hilbert.</i>	6
4. Venerable formats for reasoned argument and demonstration	6
5. <i>Euclid’s Elements, Book I</i>	8
6. <i>Hilbert’s Euclidean Geometry</i>	11
7. <i>George Birkhoff’s Axioms for Euclidean Geometry</i>	14

Part 1. About our seminar-course “Axiomatic Reasoning.”

The etymological root of the word *axiom* is the Greek $\alpha\xi\iota\omicron\mu\alpha$ meaning ‘what is fitting.’ The concept *axiom* is often taken to mean ‘self-evident assertion,’ but we will take a much broader view, allowing it to encompass frame-creating assertions ranging from ‘common notions’ (Euclid’s favorite) to ‘rules,’ ‘postulates,’ ‘hypotheses,’ and even ‘definitions’ if they play a suitable role in the ensuing discussion.

Axioms as a tool, a way of formulating a reasoned argument, a way of making explicit one’s ‘priors’ or prior assumptions, a way of stipulating assertions very clearly so as to investigate their consequences, of organizing beliefs; a way of. . . in short, reasoning, has been with us for a long time, and in various guises.

Axiomatic frameworks offer striking transparency and help open to view the lurking assumptions and presumptions that might otherwise be unacknowledged. This mode of thought has been with us at least since Aristotle.

Axioms in formal (and even sometimes in somewhat informal) structures constitute an 'MO' of mathematics at least since Euclid, but surely earlier as well (despite, curiously, the lack of any earlier record of it). We will see how the very core of meaning and use of *axiom* in mathematics has undergone quite an evolution, through Euclid, his later commentators, Hilbert's revision of the notion of axiom, and the more contemporary set theorists.

Axioms are standard structures as they appear in models in the sciences, sometimes occurring as proclaimed 'laws': borrowing that word from its *legal* roots. Newton's Laws act as axioms for Classical Mechanics, the fundamental laws of thermodynamics for Thermodynamics.

Similarly for Economics: Axiomatic Utility Theory is very well named where the 'axioms' play more the role of *desiderata* which may or may not be realizable, especially in the face of the variously named 'paradoxes' and 'Impossibility Theorems.' Nevertheless this axiomatic format provides us with an enormously useful and powerful tool to understand forces at play in Economics.

Noam Chomsky's *Structural linguistics* set the stage for a very axiomatic approach to language acquisition and use (with interesting later critique) as does the vast tradition of rule-based grammars (as Professor Sen will be discussing later in this seminar).

Rules in games, and in the formal set-ups in mathematical game theory have their distinct qualities. Even more of 'distinct quality' is the subtle manner in which it is sometimes understood that rules are *not* expected to be strictly obeyed; e.g., as in the composition of a sonnet.

In the Bayesian mode of inductive reasoning, the 'priors' (as the Bayesians call them)—which are, in effect, input axioms—are constantly re-assessed in connection with the flow of further incoming data ("the data educates the priors" as they sometimes say). This is also quite a distinctive way of dealing with one's axioms!

We live these days at a time when computer programs, governed by *algorithms*—hence a specific form of axiomatic reasoning—make selections for us, recommendations, choices, and sometimes critical decisions. The question of when, and how, more flexible modes of human judgment should combine with, and possibly mitigate, axiom-driven

decision processes is a daily concern, and something that we might address, at least a bit, in our seminar¹.

Most curiously, axiomatic structure has come up in various reflections regarding moral issues. (This, by the way, happens more than merely the *golden rule*: we will read a bit of Spinoza's *Ethics* which is set up in the formal mode of mathematical discourse, complete with Postulates, Definitions, and Theorems.)

1. HOW TO COMPREHEND ALL THIS? AND WHY?

My motivation for participating in this seminar-course (as was my motivation for participating in our previous two seminar courses, *Reasoning via Models* and *Utilty*, with my two co-teachers, Amartya Sen and Eric Maskin) is that it gives me the opportunity to focus for a semester of close reading and discussion on vital 'organizing frameworks of thought' with Amartya, Eric, and the students of our course—the discussion enriched by their (and your!) broad experience, both practical and theoretical.

I have often found it fruitful to *live with* a concept for a significant length of time—without specifying a particular goal other than to become at home with, intimate with, the concept in broad terms in its various facts and its various moods. Such an experience can provide resonances which enrich thoughts that one may have, or can connect with ideas that one encounters, years later. Everyone in this room has interacted, in one way or another, with our subject—axiomatic reasoning—in one form or another, and this seminar is a way for us to learn about different experiences and viewpoints towards it.

2. COMMENTS ON THE EXPECTED FORMAT OF OUR SEMINAR

Except for this first (“introductory”) session and the final (“wrapping up”) session, each of our other sessions will be ‘chaired’ by one of the professors. To say that we each ‘chair’ a session means that we expect full involvement of students in discussions and also, at times, presentations. That is, besides a final paper for the course, we may request a (usually very short) presentation on the part of some volunteers.

For example, in anticipation of each of my chaired sessions, a student (or a team of students) may be asked (or may volunteer) to give a

¹Relevant to this discussion is Stephen Wolfram's *A New Kind of Science* (<https://www.wolframscience.com/>) in which it is proposed that one simply replace equations in scientific laws with algorithms.

twenty-minute presentation of specific related material, and to offer handouts related to that material for the class to read—preferably beforehand.

Here are some ‘axioms’ regarding the ‘final paper’ of this course:

- The substance of the paper should be ‘topical,’ in the sense that it should be directly related to the discussions and reading that we have done.
- It should investigate some issue that you actually want to know about, or care deeply about,
- and (it would be great if it can) make use of your own expertise and experience.

As soon as you have an idea for what topic, or direction, your final paper might take on, please feel free to consult with Amartya, Eric or me, (or all of us) for comments and suggestions.

3. HERE ARE THE SPECIFIC SLANTS TO THE SUBJECT THAT MY CHAIRED SESSIONS WILL TAKE.

- (i) The ‘evolution’ of definitions and axioms, from ancient Greek philosophy and mathematics to Hilbert.

Readings:

- (a) Aristotle: Prior Analytics (Chapter 1 of Book I) https://en.wikisource.org/wiki/Prior_Analytics
- (b) Euclid’s *Definitions, Postulates, Common Notions*: Pages 6,7 of Euclid’s Elements <http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf>
- (c) Hilbert: Hilbert’s Axioms: <http://web.mnstate.edu/peil/geometry/C2EuclidNonEuclid/hilberts.htm>
- (d) George Birkhoff’s axioms for plane geometry: <http://web.mnstate.edu/peil/geometry/C2EuclidNonEuclid/birkhoffs.htm>

- (ii) The introduction of ‘axiomatic formats’ in philosophy, and—curiously—in theology: “TheoLogic” in the sense of Ontological Arguments; queries about Gödel’s Ontological Argument, and about the axiomatic structure of the *Ethics* of Spinoza². Formal logic, and issues regarding foundation(s) of mathematics.

Readings:

- (a) Some parts of *Ontological Arguments*, Stanford Encyclopedia of Philosophy: <https://plato.stanford.edu/entries/ontological-arguments/#StAnsOntArg>
- (b) Take a look at: *Logicomix: An Epic Search for Truth*. Apostolos Doxiadis and Christos H. Papadimitriou with art by Alecos Papadatos and Annie di Donna.
- (c) Martin Davis, *Gödel’s incompleteness Theorem*, Notices of the AMS **53** (2006) 414-418 <http://www.ams.org/notices/200604/fea-davis.pdf>

- (iii) What is a set? What is a number? Peano’s axioms, more of Gödel’s incompleteness Theorem. A discussion of Models and the model-theoretic approach to axioms.

Readings:

- (a) [F-G] *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931* Edited by Jean van Heijenoort:
- **AST**: *Axiomatized Set Theory* footnote 9 (p.209) and pp.290-301 of [F-G],
 - **LCI**: *Logico-Combinatorial Investigations* pp.252-263 of [F-G].

² (which is impenetrable to me)

Part 2. Skeletal notes for today's session: *The evolution of definitions and axioms, from ancient Greek philosophy and mathematics to Hilbert.*

Here is Socrates lecturing to Adeimantus in Plato's *Republic* VI.510c,d:

... the men who work in geometry, calculation, and the like treat as known the odd and the even, the figures, three forms of angles, and other things akin to these in each kind of inquiry.

*These things they make **hypotheses** and don't think it worthwhile to give any further account of them to themselves or others as though they were clear to all. Beginning from them, they ... make the arguments for the sake of the square itself and the diagonal itself, not for the sake of the diagonal they draw, and likewise with the rest. These things themselves that they mold and draw—shadows and images in water—they now use as images, seeking to see those things themselves, that one can see in no other way than with thought.*

4. VENERABLE FORMATS FOR REASONED ARGUMENT AND DEMONSTRATION

Ancient organizational schemes of logic, such as the *Organon* of Aristotle, have been vastly influential and have been—even if largely implicit—the armature of the way in which we formulate assertions, ask questions, and reach conclusions in mathematics as in everything else. Aristotle begins his discussion in the *Prior Analytics* by setting for himself quite a task: to pin down *demonstration* “and for the sake of demonstrative science,” to:

...define, what is a proposition, what a term, and what a syllogism, also what kind of syllogism is perfect, and what imperfect; lastly, what it is for a thing to be, or not to be...

Aristotle gets to this job right away, and offers this succinct definition of *proposition* neatly distinguishing between those propositions involving universal, no, or existential quantification:

A proposition then is a sentence which affirms or denies something of something, and this is universal, or particular, or indefinite.

and, turning to *syllogism*, the main object of exploration of the *Organon*, he characterizes *syllogism* as:

discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so.

Since *definition*, defined by Aristotle as: *an account which signifies what it is to be for something*³ plays such a vital role in mathematics, the notion deserves close attention. Mathematics seems to require as strict lack-of-ambiguity in its assertions as possible, and therefore maximal clarity in its definitions. But perhaps—since ambiguity is sometimes unavoidable—it is better to say that any ambiguity should be unambiguously labeled as such.

The nature, and role, of *definition* in mathematical usage has evolved in remarkable ways. We will be discussing this in more detail later this session, but consider the first two definitions in Book I of Euclid's *Elements*.⁴

- (i) A point is that which has no part.
- (ii) A line is breadthless length.

and their counterparts in Hilbert's rewriting of Euclid's *Elements*, which begins with:

Let us consider three distinct systems of things. The things composing the first system, we will call **points** and designate them by the letters A, B, C, . . . ; those of the second, we will call **straight lines** and designate

³ a puzzling definition: *logos ho to ti ên einai sêmainei*

⁴ These 'Elements' have quite an impressive spread, starting with the proclamation that a point is characterized by the property of 'having no part,' and ending with its last three books, deep into the geometry of solids, their volumes, and the five Platonic solids. It is tempting to interpret this choice of ending for the *Elements* as something of a response to the curious interchange between Socrates and Glaucon in Plato's *Republic* (528a-d) where the issue was whether Solid Geometry should precede Astronomy, and whether the mathematicians had messed things up.

It also would be great to know exactly how—in contrast—the *Elements* of Hippocrates of Chios ended. (It was written over a century before Euclid's *Elements* but, unfortunately, has been lost.)

them by the letters a, b, c, . . . The points are called the **elements of linear geometry**; the points and straight lines, **the elements of plane geometry**. . .

One might call Euclid's and Hilbert's formulations **primordial definitions** since they spring ab ovo—i.e., from nothing. Or at least from 'things' not in the formalized arena of mathematics, such as Hilbert's "*system of things*". Euclid's definitions of *point* and *line* seem to be whittling these concepts into their pure form from some more materially graspable context (e.g., where lines have breadth)⁵ while for Hilbert the essence of *point and line* is their relationship one to the other.

Once one allows the bedrock of—say—Set Theory, definitions are often 'delineations of structure,' cut out by means of quantifiers and predicates but making use of set theoretic, or at least priorly defined objects. E.g. *A circle is a set of points equi-distant from a single point in the Euclidian plane.* We will discuss this in a moment.

The essential roles that 'definition' play for us are: to delineate the objects of interest to be studied; to encapsulate; to abbreviate; and to focus.

But now let us compare the vastly different axiomatic formats of Euclidean geometry, as conceived by Euclid, David Hilbert, George Birkhoff, and—if there's time—we may also consider the viewpoint of the 'Erlangen Program' (regarding the formulation of Euclidean Geometry).

5. *Euclid's Elements, Book I*

(i) **Definitions**

- (a) A point is that which has no part.
- (b) A line is breadthless length.
- (c) The extremities of a line are points.
- (d) A straight line is a line which lies evenly with the points on itself.
- (e) A surface is that which has length and breadth only.
- (f) The extremities of a surface are lines.
- (g) A plane surface is a surface which lies evenly with the straight lines on itself.

⁵ I want to thank Eva Brann for pointing this out.

- (h) (Def'n. 13) A boundary is that which is an extremity of anything.
- (i) (Def'n. 14) A figure is that which is contained by any boundary or boundaries.
- (j) (Def'n. 15) A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;
- (k) (Def'n. 16) And the point is called the centre of the circle.

(ii) **Postulates**

- (a) To draw a straight line from any point to any point.
- (b) To produce a finite straight line continuously in a straight line.
- (c) To describe a circle with any centre and distance.
- (d) That all right angles are equal to one another.
- (e) (*'Fifth Postulate':*) That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

(iii) **Common Notions**

- (a) Things which are equal to the same thing are also equal to one another.
- (b) If equals be added to equals, the wholes are equal.
- (c) If equals be subtracted from equals, the remainders are equal.
- (d) Things which coincide with one another are equal to one another.
- (e) The whole is greater than the part.

A. Topics that we might discuss regarding Euclid:

- Euclid's text does indeed have actual postulates, definitions, demonstrations; and the format of his argumentation is explicit, the parts articulated. (There are some conjectured scenarios of how this discursive reasoned mode of mathematical

practice emerged, and some controversy about it, as well.) Such a format of demonstration is in contrast to what we know, for example, about Babylonian or Egyptian Mathematics, It would be interesting to consider what explicit formats for argumentation are in evidence in Indian and Chinese Mathematics, and how it compares with the Greek practices⁶

- Euclid’s definitions are—one might say—*descriptive* rather than *generative*. Euclid seems to be happy assuming that there is a certain object—let’s call it the Euclidean plane—and he’s just naming and describing its components. All this structure (presumably) precedes his discussion (e.g., “a line is breadthless length”) and his discussion merely alludes to it, rather than generates it. A modern axiomatic attitude would view such a definition as utterly unusable!
- Issues of uniqueness are perhaps implied, but not specifically mentioned. E.g.:
 - The following ‘familiar’ definitions, *not at all in the spirit of the definitions in Book I* appear in later interpretations of Euclid:

A straight line is [implied: uniquely] determined by two of its points.

Or:

A straight line segment is the [implied: unique] curve of shortest distance between its endpoints.
 - Or uniqueness, for example, of that *central point* in Def’ns 15 and 16?
- For me, the most striking fact about these definitions is that they don’t rely on set theoretic vocabulary. We moderns immediately think ‘sets,’ ‘subsets,’ ‘membership in sets,’ etc. and tend to build our structures starting with sets as substrate.
- Euclid has no vocabulary *at all* for ‘continuous motion,’ ‘transformation,’ ‘function’ except as these issues are introduced in the *Postulates* and/or when one triangle is “applied” to another. Discuss the ‘Erlangen Program.’

⁶ At least by report (via Simplicius) it seems to be the case with even in the earliest Greek mathematics that we know (Hippocrates of Chios) there is some explicit form of argumentation and not pure assertion.

On the Postulates:

- The emphasis is on Construction rather than Existence.
- Regarding The Fifth Postulate: the minute one questions its independence, one is on the way to model-formation, but, of course, this is not at all in the spirit of Euclid.

On the Common Notions:

- These are closest to modern axiomatics, formulating rules regarding the terms *equal* and *greater than*. The intuitive notion—that, say, two angles are *equal* if there is a Euclidean transformation bringing one exactly onto the other—is utterly absent from Euclid’s vocabulary.

6. *Hilbert’s Euclidean Geometry*

Hilbert’s axiom system is constructed with six **primitive notions**:

(i) three primitive terms:

- point;
- line;
- plane;

and

(ii) three primitive relations:

- *Betweenness*, a ternary relation linking points;
- *Lies on (Containment)*, three binary relations, one linking points and straight lines, one linking points and planes, and one linking straight lines and planes;
- *Congruence*, two binary relations, one linking line segments and one linking angles. Note that line segments, angles, and triangles may each be defined in terms of points and straight lines, using the relations of betweenness and containment. All points, straight lines, and planes in the following axioms are distinct unless otherwise stated.

And there are these structures and axioms:

- (iii) **Incidence** For every two points A and B there exists a line a that contains them both. . .
- (iv) **Order** If a point B lies between points A and C , B is also between C and A , and there exists a line containing the distinct points A, B, C . . . If A and C are two points of a line, then there exists at least one point B lying between A and C . Of any three points situated on a line, there is no more than one which lies between the other two.

Pasch's Axiom: Let A, B, C be three points not lying in the same line and let L be a line lying in the plane ABC and not passing through any of the points A, B, C . Then, if the line L passes through a point of the segment AB , it will also pass through either a point of the segment BC or a point of the segment AC .

- (v) **Congruence** If A, B are two points on a line L , and if A' is a point upon the same or another line L' , then, upon a given side of A' on the straight line L' , we can always find a point B' so that the segment AB is congruent to the segment $A'B$. . .

(vi) **Continuity**

- *Axiom of Archimedes:* If AB and CD are any segments then there exists a number n such that n segments CD constructed contiguously from A , along the ray from A through B , will pass beyond the point B .
- *Axiom of line completeness:* . . .

A. Topics that we might discuss regarding Hilbert:

- Hilbert's Axioms offer an articulation very different from Euclid's: the triple *definitions/postulates/common notions* being replaced by *primitive terms/primitive relations/structures and axioms*.

- The common notions (i.e., logical pre-structures like ‘equality’) are implicitly assumed rather than formulated.
- Modern quantification is explicit. E.g., the ‘incidence Axiom’ calls up universal and existential quantification: \forall points A, B, \exists a line through A and B .
- Geometry as a structure—following the Erlangen Program.
- Most importantly, Hilbert expresses his axioms in Set theoretic vocabulary.

But if one uses Set Theory as a ‘substrate’ on which to build the structures of mathematics, as in the classical *Grundlagen der Mathematik* of Bernays and Hilbert, one must tangle with all the definitional questions that are faced by Set Theory (starting with: *what is a set?* and continuing with the discussion generated by the work of Frege, Russell, etc.)⁷.

And then compare all this with the discussion about the existence of infinite sets in Bernays-Hilbert’s *Grundlagen der Mathematik, Vol. I*:

... reference to non-mathematical objects can not settle the question whether an infinite manifold exists; the question must be solved within mathematics itself. But how should one make a start with such a solution? At first glance it seems that something impossible is being demanded here: to present infinitely many individuals is impossible in principle; therefore an infinite domain of individuals as such can only be indicated through its structure, i. e., through relations holding among its elements. In other words: a proof must be given that for this domain certain formal relations can be satisfied. The existence of an infinite domain of individuals can not be represented in any other way than through the satisfiability of certain logical formulas...

⁷ For example, go back to Dedekind’s marvelous idea of capturing the notion of *infinite* by discussing self-maps (this notion popularized by people checking into Hilbert’s hotel). You might formulate Dedekind’s idea this way: a set S is **infinite** if it admits an injective but non-surjective self-map... and then confuse yourself by trying to figure out how this compares with the property that S admits a surjective but non-injective self-map.

From Hilbert's *On The Infinite*:

... We encounter a completely different and quite unique conception of the notion of infinity in the important and fruitful method of ideal elements. The method of ideal elements is used even in elementary plane geometry. The points and straight lines of the plane originally are real, actually existent objects. One of the axioms that hold for them is the axiom of connection: one and only one straight line passes through two points. It follows from this axiom that two straight lines intersect at most at one point. There is no theorem that two straight lines always intersect at some point, however, for the two straight lines might well be parallel. Still we know that by introducing ideal elements, viz., infinitely long lines and points at infinity, we can make the theorem that two straight lines always intersect at one and only one point come out universally true. These ideal "infinite" elements have the advantage of making the system of connection laws as simple and perspicuous as possible. Moreover, because of the symmetry between a point and a straight line, there results the very fruitful principle of duality for geometry...

$\angle x$

7. George Birkhoff's Axioms for Euclidean Geometry

Undefined Elements and Relations:

- points A, B, \dots
- sets of points called lines $\ell, m \dots$
- distance between any two points A and B : a nonnegative real number $d(A, B)$ such that $d(A, B) = d(B, A)$.
- angle formed by three ordered points A, O, B ($A \neq O, B \neq O$): $\angle AOB$ a real number (mod 2π). The point O is called the **vertex** of the angle.
- **Postulate I.** (Postulate of Line Measure) The points A, B, \dots of any line ℓ can be placed into one-to-one correspondence with

the real numbers x so that $|xA - xB| = xd(A, B)$ for all points A, B .

- **Postulate II.** (Point-Line Postulate) One and only one line ℓ contains two given points P, Q ($P \neq Q$).
- **Postulate III.** (Postulate of Angle Measure) The half-lines $\ell, m \dots$ through any point O can be put into one-to-one correspondence with the real numbers $a \pmod{2\pi}$, so that, if $A \neq O$ and $B \neq O$ are points of ℓ and m , respectively, the difference $am - al \pmod{2\pi}$ is $\angle AOB$. Furthermore if the point B on m varies continuously in a line ℓ not containing the vertex O , the number am varies continuously also.
- **Postulate IV.** (Similarity Postulate) If in two triangles

$$\triangle ABC, \triangle A'B'C'$$

and for some constant $k > 0$,

$$d(A', B') = kd(A, B), \quad d(A', C') = kd(A, C),$$

and

$$\angle B'A'C' = \pm \angle BAC,$$

then also

$$d(B', C') = kd(B, C), \quad \angle A'B'C' = \pm \angle ABC,$$

and

$$\angle A'C'B' = \pm \angle ACB.$$

Defined Terms:

A point B is between A and C ($A \neq C$), if $d(A, B) + d(B, C) = d(A, C)$. The half-line ℓ' with endpoint O is defined by two points O, A in line ℓ ($A \neq O$) as the set of all points A' of ℓ such that O is not between A and A' . The points A and C , together with all point B between A and C , for segment AC . If A, B, C are three distinct points, the segments AB, BC, CA are said to form a triangle $\triangle ABC$ with sides AB, BC, CA and vertices A, B, C .

A. Topics that we might discuss regarding Birkhoff:

- The big distinction between the three ‘Euclidean axiom-formulations’ (Euclid’s, Hilbert’s, and Birkhoff’s) is—I believe—in the implicitly assumed substrates that ground each of the axiom systems:
 - Euclid *assumes* that we are—at least vaguely—familiar with the basic nature of ‘Euclidean Space’ and his mission is to describe it more precisely and give terminology so that we may offer reasoned arguments about its features and make constructions in it.
 - Hilbert—in effect—generates his Euclidean space by relational axioms, depending on the substrate (undiscussed explicitly) of set theory.
 - Birkhoff brings in (in a way fundamental to his approach) metric considerations; hence his ‘substrate’ includes quite explicitly the system of real numbers.

All three axiom systems fall under the general rubric of “synthetic geometries,” i.e., set-ups that formulate conditions regarding essentially geometric features. This is in contrast to ‘analytic geometry’ which would set things up by working in the substrate of \mathbb{R}^2 or \mathbb{R}^3 and providing its geometric definitions in purely algebraic language. Birkhoff’s axioms move closer to that, but are still (interestingly) synthetic. As already mentioned, there is also the ‘Erlangen Program,” which takes a somewhat different slant—if there’s time—we can discuss.