1. Match the following vector fields to the pictures, below. Explain your reasoning.

(Notice that in some of the pictures all of the vectors have been uniformly scaled so that the picture is more clear. Also notice that there are eight vector fields but only six pictures. There’s probably a reason behind this.)

Here are the possible vector fields:

(a) $\mathbf{F}(x, y) = \langle 1, x \rangle$
(b) $\mathbf{F}(x, y) = \langle -y, x \rangle$
(c) $\mathbf{F}(x, y) = \langle y, x \rangle$
(d) $\mathbf{F}(x, y) = \langle 2x, -2y \rangle$
(e) $\nabla f$, where $f(x, y) = x^2 + y^2$
(f) $\nabla f$, where $f(x, y) = \sqrt{x^2 + y^2}$
(g) $\nabla f$, where $f(x, y) = xy$
(h) $\nabla f$, where $f(x, y) = x^2 - y^2$

2. Recall that the gradient of a function is a vector normal to the level curve of this function. Explain how this confirms your identification of the pictures for vector fields (e) through (h), above.
3. Here are some examples in three-dimensional space. The vector fields are

(a) \( \mathbf{F} = \langle x, y, z \rangle \) 
(b) \( \mathbf{F} = \langle 0, 1, 0 \rangle \)

(c) \( \mathbf{F} = \langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}}, 0 \rangle \) 
(d) \( \mathbf{F} = \langle -\frac{x}{(x^2+y^2+z^2)^{3/2}}, -\frac{y}{(x^2+y^2+z^2)^{3/2}}, -\frac{z}{(x^2+y^2+z^2)^{3/2}} \rangle \)
Let’s start with Problem #35 in Section 13.1 of Stewart. The flow lines or streamlines of a vector field are the path of a particle whose velocity vector field is the given vector field. Here’s the example:

\[
\mathbf{F}(x, y) = \langle x, -y \rangle = x \mathbf{i} - y \mathbf{j}
\]

Can you tell which direction the flow goes in each case?

4. We’d like to find \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) for which \( \mathbf{r}'(t) = \mathbf{F} \). This amounts to solving the equations

\[
\begin{align*}
  x'(t) &= x \\
  y'(t) &= -y
\end{align*}
\]

Do this to find \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \).
5. Above are two sketches of the vector field \( \mathbf{F}(x, y) = \langle -y, 1 \rangle = -y \mathbf{i} + \mathbf{j} \). Sketch the flow lines and repeat the previous problem to find \( \mathbf{r}(t) \) for which \( \mathbf{r}'(t) = \mathbf{F}(x, y) \).
6. Consider two vector fields $\mathbf{F}_1(x, y) = (2xy + 2, x^2 + 1)$, $\mathbf{F}_2(x, y) = (-y, x)$. Use Clairaut’s theorem to tell which one is the gradient vector field for a smooth function $f$? Find the corresponding function $f$.

7. Consider vector field $\mathbf{F}_2(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. Note that the vector field is not defined at origin. Does this vector field satisfy the Clairaut’s test? Is this a gradient vector field for smooth function?
Vector Fields – Answers and Solutions

1. (I) This is vector fields (d) and (h). (First notice that these two vector fields are the same!) We can see this by noticing that the vectors should point down when \( y > 0 \) and up when \( y < 0 \), and field (I) is the only one that does this.

   (II) This is vector field (a). Notice that this vector field always has a positive rightward component, which is true only of Field (II).

   (III) This is vector field (f). Both (e) and (f) are vector fields that point radially outward, so they are Fields (III) and (VI). But which is which? Notice that the vector field in (e) is \( \nabla f = \langle 2x, 2y \rangle \), which has length \( 2r = 2\sqrt{x^2 + y^2} \). On the other hand, the vector field in (f) is \( \nabla f = \langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \rangle = \langle \frac{x}{r}, \frac{y}{r} \rangle \), which has length 1. Thus (e) is Field (VI), the field with the vectors that increase in length as the distance from the origin increases, while (f) is Field (III), the field with vectors all the same magnitude.

   (IV) This is vector field (b). Look, for example, at the vectors on the axes. On the \( x \)-axis, the vector field is \( F(x, 0) = \langle 0, x \rangle \), a vector that points vertically up (if \( x > 0 \)) or down (if \( x < 0 \)). This narrows our choices to Fields (IV) or (V). On the \( y \)-axis, the vector field is \( F(0, y) = \langle -y, 0 \rangle \), a vector that points to the left (if \( y > 0 \)) or to the right (if \( y < 0 \)). This eliminates Field (V) and confirms Field (IV).

   (V) This is vector field (c) and (g), by an analysis that is very similar to the one in Field (IV). (Notice that (c) and (g) are the same!)

   (VI) This is vector field (e). See (III) for the explanation.

2. Four of the vector fields are (explicitly) gradient fields (we’ll be able to tell later that vector fields (a) and (b) are not gradient fields). Since \( \nabla f \) is perpendicular to the level curves of \( f \), we should be able to see this in the vector field. Here I’ve re-drawn the four vector fields in question with some level curves drawn in as well – note the perpendicularity!

Field (I) and \( f(x, y) = x^2 - y^2 = k \) for \( k = 0, \pm 2, \pm 4, \pm 6, \pm 8 \)
Field (III) and $f(x, y) = \sqrt{x^2 + y^2} = k$ with $k = 1, 2, 3, 4$

Field (V) with $f(x, y) = xy = k$ for $k = 0, \pm 1, \pm 2, \pm 3$
Field (VI) and \( f(x, y) = x^2 + y^2 = k \) with \( k = 1, 2, 3, 4, 9, \) and 16

3. Here are the answers, with a few tips:

(a) This is the top left vector field. The vector at the point \((x, y, z)\) is the radial vector – the same as the vector from the origin to this point (namely \(\langle x, y, z \rangle\)).

(b) This is the bottom right vector field. Each vector is the same.

(c) This is the top right vector field. Note that there is no “z” component, so the arrows are all parallel to the \(xy\)-plane. Notice that in cylindrical coordinates, this vector field is \(\mathbf{F} = \langle \sin \theta, -\cos \theta, 0 \rangle\). This means two things: first, the vectors are all unit vectors (length 1), and second, the vectors are tangent to circles (and perpendicular to the radial vector \(\langle x, y, 0 \rangle = \langle r \cos \theta, r \sin \theta, 0 \rangle\)).

(d) This is the bottom left vector field. Like vector field (a), this vector field is a radial vector field (parallel to \(\langle x, y, z \rangle\)). It is a negative multiple of \(\langle x, y, z \rangle\), so the vector field points toward the origin. The scale is by a factor of \(\rho^{-3}\), so vector fields far from the origin are much shorter than those closer.

4. We’re solving two simple ordinary differential equations:

\[
\frac{dx}{dt} = x \quad \text{and} \quad \frac{dy}{dt} = -y.
\]

The solutions are \(x(t) = Ae^t\) and \(y(t) = Be^{-t}\). (Often we call these constants something like \(x_0 = A\) and \(y_0 = B\) to indicate their meaning: they are the values of \(x\) and \(y\) at \(t = 0\).) We get \(\mathbf{r}(t) = \langle x_0 e^t, y_0 e^{-t} \rangle\) for some constants \(x_0\) and \(y_0\).

Notice that we can solve for \(e^t = x/x_0 = y_0/y\), so we can determine that these flowlines lie on the curve \(xy = x_0y_0\).
5. Here the two differential equations are

\[
\frac{dx}{dt} = -y \quad \text{and} \quad \frac{dy}{dt} = 1.
\]

The solution to the second is \( y(t) = t + y_0 \), from which the first differential equation becomes

\[
\frac{dx}{dt} = -y = -t - y_0.
\]

The solution to this is \( x(t) = -\frac{1}{2}t^2 - y_0t + x_0 \), so we get \( \mathbf{r}(t) = (-\frac{1}{2}t^2 - y_0t + x_0, t + y_0) \), where the constants \( x_0 \) and \( y_0 \) have the natural meaning (they are the position at \( t = 0 \): \( \mathbf{r}(0) = (x_0, y_0) \)). Below we sketch some of these trajectories together with the original vector field:

\[
\mathbf{F}(x, y) = (-y, 1) = -yi + j
\]

with flow lines with \((x_0, y_0) = (n, 0)\).

Notice that we can solve for \( t = y - y_0 \), so we can determine that these flowlines lie on the curve \( x = \frac{1}{2}(y - y_0)^2 - y_0(y - y_0) + x_0 \), which is a parabola opening to the left (as we’ve seen in the sketches).
6. Any gradient vector field \( \mathbf{F} = \langle P(x, y), Q(x, y) \rangle \) must satisfy \( P_y = Q_x \) from Clairaut’s theorem. This is called Clairaut’s test. Direct computation shows that \( \mathbf{F}_2 \) does not satisfy the Clairaut’s test.

For \( \mathbf{F}_1 \), assume \( \nabla f = \mathbf{F}_1 \). Then

\[
\begin{align*}
    f_x &= 2xy + 2, \\
    f_y &= x^2 + 1.
\end{align*}
\]

From \( f_x = 2xy + 2 \), we know \( f(x, y) = x^2y + 2x + C(y) \) by taking antiderivative with respect to \( x \). So \( f_y = x^2 + C'(y) \), \( C'(y) = 1 \). Then we have \( C(y) = y + C_0 \), where \( C_0 \) is a constant. So \( f(x, y) = x^2y + 2x + y + C_0 \).

7. Direct computation shows that \( \mathbf{F} \) satisfies the Clairaut’s test. But it’s not a gradient vector field. Because it’s flow lines are circles, which are closed curves. This can not happen for gradient vector field because the value of the function always increases along the flow lines generated by its gradient vector fields.