

1. INTRODUCTION

In these notes we shall prove some of the basic facts concerning Hadamard spaces (metric spaces of non-positive curvature). In particular we shall give a detailed proof of the Cartan-Hadamard theorem, which applies even to exotic cases such as non-positively curved orbifolds.

A few remarks about material not covered here. We shall say nothing about Riemannian manifolds of non-positive curvature. We shall not touch on any results requiring assumptions about *negative* (as opposed to non-positive) curvature. And we shall say very little about groups which act on Hadamard spaces.

2. BASIC DEFINITIONS

Definition 2.1. A *Hadamard space* is a nonempty complete metric space (X, d) with the property that for any pair of points $x, y \in X$, there exists a point $m \in X$ such that

$$d(z, m)^2 + \frac{d(x, y)^2}{4} \leq \frac{d(z, x)^2 + d(z, y)^2}{2}$$

for any $z \in X$.

Applying the definition with $z = x$, we deduce that $d(x, m)^2 \leq \frac{d(x, y)^2}{4}$, so that $d(x, m) \leq \frac{d(x, y)}{2}$. Similarly $d(y, m) \leq \frac{d(x, y)}{2}$. Thus $d(x, m) + d(y, m) \leq d(x, y)$. By the triangle inequality, equality must hold, so that $d(x, m) = d(y, m) = \frac{d(x, y)}{2}$. We say that m is the *midpoint* of x and y . Furthermore, m is uniquely determined by this property. Indeed, suppose that $d(x, m') = d(y, m') = \frac{d(x, y)}{2}$. Applying the definition with $z = m'$, we deduce that $d(m, m') \leq 0$, so that $m = m'$.

Let X be any metric space. A *geodesic of speed D* in X is a map $p : [t_-, t_+] \rightarrow X$ with the property that for any $t \in [t_-, t_+]$, there exists a positive constant ϵ such that $d(p(s), p(s')) = D|s - s'|$ for any $s, s' \in [t_-, t_+]$ such that $t - \epsilon \leq s, s' \leq t + \epsilon$. We call a map p a *geodesic* if it is a geodesic of speed D for some D , which is obviously uniquely determined. If $d(p(t_-), p(t_+)) = D(t_+ - t_-)$, then we shall say that p is a *minimizing geodesic*.

Proposition 2.2. *Let X be a Hadamard space, and let $x, y \in X$ with $d(x, y) = D$. Then there is a unique minimizing geodesic $p : [0, 1] \rightarrow X$ of speed D with $p(0) = x$ and $p(1) = y$. Furthermore, for any point z and any $0 \leq t \leq 1$, we have the inequality $d(z, p(t))^2 \leq (1 - t)d(z, x)^2 + td(z, y)^2 - t(1 - t)D^2$.*

Proof. Without loss of generality we may assume $D = 1$, to simplify our notations. We first define p on all rational numbers of the form $\frac{a}{2^n}$ by induction on n : if a is odd and $n > 0$, then we let $p(\frac{a}{2^n})$ be the midpoint of $p(\frac{a+1}{2^n})$ and $p(\frac{a-1}{2^n})$. Of course we begin by setting $f(0) = x$ and $f(1) = y$.

Now we claim that p satisfies the stated inequality whenever t is a dyadic rational. For $t = 0$ or $t = 1$ this is obvious; in general we proceed by induction. In establishing the claim for $t = \frac{a}{2^n}$, we may assume that it holds already for $\frac{a+1}{2^n}$ and $\frac{a-1}{2^n}$. Note that $d(f(\frac{a+1}{2^n}), f(\frac{a-1}{2^n})) = \frac{1}{2^{n-1}}$, by an easy induction.

Then we have the inequalities

$$d(z, p(\frac{a+1}{2^n}))^2 \leq \frac{2^n - a - 1}{2^n} d(z, x)^2 + \frac{a+1}{2^n} d(z, y)^2 - \frac{2^n - a - 1}{2^n} \frac{a+1}{2^n}$$

$$d(z, p(\frac{a-1}{2^n}))^2 \leq \frac{2^n - a + 1}{2^n} d(z, x)^2 + \frac{a-1}{2^n} d(z, y)^2 - \frac{2^n - a + 1}{2^n} \frac{a-1}{2^n}$$

By the definition of a Hadamard space we have

$$d(z, f(\frac{a}{2^n}))^2 \leq \frac{d(z, f(\frac{a+1}{2^n}))^2 + d(z, f(\frac{a-1}{2^n}))^2}{2^n} - \frac{1}{2^{2n}}$$

Combining these inequalities, we obtain the desired result. An easy induction shows that $d(x, p(t)) = t$, $d(p(t), y) = 1 - t$ for any dyadic rational t . It follows that $d(p(t), p(t'))^2 \leq (1-t)t'^2 + t(1-t')^2 - t(1-t) = (t-t')^2$, so that $d(f(t), f(t')) \leq |t-t'|$. Without loss of generality $t \leq t'$, so we get $1 = d(x, y) \leq d(x, p(t)) + d(p(t), p(t')) + d(p(t'), y) \leq t + (t' - t) + (1 - t') = 1$, so we must have equality throughout and $d(p(t), p(t')) = |t - t'|$ for all dyadic rationals $t, t' \in [0, 1]$.

In particular, p is uniformly continuous on the dyadic rationals. Since X is complete, it follows that f admits a unique continuous extension to $[0, 1]$. By continuity, it follows that $d(p(t), p(t')) = |t - t'|$ for all $t, t' \in [0, 1]$, and that the inequality in the statement of the proposition holds for all $t \in [0, 1]$.

The uniqueness is clearly forced by our construction. \square

Another formulation of the inequality of the lemma is that if $p : [t_-, t_+] \rightarrow X$ is a minimizing geodesic of speed D and $x \in X$, then the function $d(x, p(t))^2 - (Dt)^2$ is a convex function of t ; the existence of geodesics with this property characterizes Hadamard spaces among all complete metric spaces. Note also that the convexity of this function is independent of the chosen interval of definition $[t_-, t_+]$, since replacing this interval with $[t_- + c, t_+ + c]$ modifies the function $d(x, p(t))^2 - (Dt)^2$ by an affine function of t .

If x and y are points in a Hadamard space X , we shall denote the minimizing geodesic joining x to y by $p_{\overline{xy}}$.

Remark 2.3. If X is a Hadamard space under a metric d , then it is a Hadamard space under any metric obtained by multiplying d by a constant (positive) factor.

Remark 2.4. Given any finite collection of Hadamard spaces $\{X_i\}_{i \in I}$, the Cartesian product $X = \prod_{i \in I} X_i$ is Hadamard, with metric given by

$$d(\{x_i\}, \{x'_i\})^2 = \sum_{i \in I} d_{X_i}(x_i, x'_i)$$

In particular, a one-point space is Hadamard.

3. COMPARISON TRIANGLES

Let X be a metric space in which every pair of points may be joined by a unique minimizing geodesic. For any triple of points $x, y, z \in X$, we let $\Delta(x, y, z)$ denote the Euclidean triangle (homeomorphic to a circle) with side lengths $d(x, y)$, $d(y, z)$, $d(z, x)$. By hypothesis, there is a natural map $\phi : \Delta(x, y, z) \rightarrow X$ which sends the vertices to x , y , and z , and sends the sides isometrically to the minimizing geodesics joining x , y , and z . By abuse of notation, we shall generally not distinguish between points of $\Delta(x, y, z)$ and their images in X .

We may regard $\Delta(x, y, z)$ as a metric space (via its embedding in the Euclidean plane); let us denote the metric by d_Δ . We shall say that $x, y, z \in X$ form a *hyperbolic triangle* if ϕ is distance-decreasing.

Proposition 3.1. *A nonempty, complete metric space is Hadamard if and only if every three points $x, y, z \in X$ form a hyperbolic triangle.*

Proof. Suppose every three points of X form a hyperbolic triangle. Take any pair of points $x, y \in X$ and join them by a minimizing geodesic; let m be the midpoint of that geodesic. Then for any point $z \in X$, the square of the distance from z to m in $\Delta(x, y, z)$ is precisely $\frac{d(z,x)^2 + d(z,y)^2}{2} - \frac{d(x,y)^2}{4}$. If ϕ is distance-decreasing then we deduce that X is Hadamard.

For the converse, we suppose that X is Hadamard. Pick $x, y, z \in X$, and let $a, b \in \Delta(x, y, z)$. We must show that $d(a, b) \leq d_\Delta(a, b)$. This is clear if a and b lie on the same side of $\Delta(x, y, z)$; without loss of generality we may assume that a lies on the side containing x and y , while b lies on the side containing y and z . Let $d_{\Delta'}$ denote the metric on the Euclidean triangle $\Delta(a, y, z)$. Then it suffices to show that $d(a, b) \leq d_{\Delta'}(a, b) \leq d_\Delta(a, b)$.

The first inequality follows directly from Proposition 2.2, which also shows that $d(a, z) \leq d_\Delta(a, z)$. This shows that the angle of $\Delta(a, y, z)$ at y is smaller than the angle of $\Delta(x, y, z)$ at y . This in turn implies the second inequality, since the distance between two points in Euclidean space that are at fixed distances from y varies monotonically with the angle between the segments joining them to y . \square

Corollary 3.2. *Let X be a Hadamard space, and let $x, y, z \in X$. Then the distance from the midpoint of x and y to the midpoint of x and z is less than or equal to $\frac{d(y,z)}{2}$.*

Corollary 3.3. *Let X be a Hadamard space, and let $x, y, x', y' \in X$. Let m be the midpoint of x and y , m' the midpoint of x' and y' . Then $d(m, m') \leq \frac{d(x,x') + d(y,y')}{2}$.*

Proof. Let m'' denote the midpoint of x and y' . Then $d(m, m') \leq d(m, m'') + d(m'', m')$. Now apply the previous corollary. \square

4. ANGLES

Let X be a Hadamard space. Let $p : [0, s] \rightarrow X$ and $p' : [0, s'] \rightarrow X$ be geodesics of speeds D and D' with $p(0) = p'(0) = x$. Then for any $t \in (0, s]$, $t' \in (0, s']$, we let $\alpha(t, t') \in [0, \pi]$ denote the angle at the vertex x in the triangle $\Delta(x, p(t), p'(t'))$. This is determined by the law of cosines:

$$\cos(\alpha(t, t')) = \frac{(tD)^2 + (t'D')^2 - d(p(t), p'(t'))^2}{2(tD)(t'D')}$$

If $t_0 \leq t$ and $t'_0 \leq t'$, then $d(p(t_0), p'(t'_0)) \leq d_\Delta(p(t_0), p'(t'_0)) = (t_0D)^2 + (t'_0D')^2 - 2(t_0D)(t'_0D') \cos(\alpha(t_0, t'_0))$, where the latter equality again follows from the law of cosines. This in turn implies $\cos(\alpha(t_0, t'_0)) \geq \cos(\alpha(t, t'))$ so that $\alpha(t_0, t'_0) \leq \alpha(t, t')$. Thus the angles $\alpha(t, t')$ must approach some infimum as $t, t' \rightarrow 0$; we refer to this infimum as the *angle* between p and p' at x and denote it by $\angle_{p,p'}$.

Remark 4.1. This definition makes sense under much weaker hypotheses. However, we shall only make use of angles between geodesics in Hadamard spaces, which enjoy many pleasant properties.

Our first goal is to establish the following ‘‘triangle-inequality’’ for angles:

Theorem 4.2. *Let $p : [0, s] \rightarrow X$, $p' : [0, s'] \rightarrow X$, $p'' : [0, s''] \rightarrow X$ be three geodesics with $p(0) = p'(0) = p''(0) = x$. Then $\angle_{p,p''} \leq \angle_{p,p'} + \angle_{p',p''}$.*

Proof. Without loss of generality we may assume that the geodesics are unit-speed. We must show that for every $\epsilon > 0$, there exist $t \in (0, s]$ and $t'' \in (0, s'']$ such that the vertex angle of $\Delta(x, p(t), p''(t''))$ at x is less than or equal to $\angle_{p,p'} + \angle_{p',p''} + \epsilon$. Take t and t'' so small that for any $t' \leq t + t''$, the vertex angles of $\Delta(x, f(t), f'(t'))$ and $\Delta(x, f'(t'), f''(t''))$ at x are less than $\angle_{p,p'} + \frac{\epsilon}{2}$ and $\angle_{p',p''} + \frac{\epsilon}{2}$.

If $\angle_{p,p'} + \angle_{p',p''} + \epsilon > \pi$ there is nothing to prove. Otherwise it suffices to show the inequality

$$t^2 + t''^2 - 2tt'' \cos(\angle_{p,p'} + \angle_{p',p''} + \epsilon) \geq d(p(t), p''(t''))^2$$

Let Δ denote the Euclidean triangle having a vertex x with angle $\angle_{p,p'} + \angle_{p',p''} + \epsilon$ and adjacent side lengths t, t'' ; call the endpoints of these adjacent sides y and z . Then we wish to show that $d(p(t), p''(t'')) \leq d_{\Delta}(y, z)$. Let w be the length of the segment from x to the a point w on the side of Δ opposite to x which divides the vertex angle at x into pieces of size $\angle_{p,p'} + \frac{\epsilon}{2}$ and $\angle_{p',p''} + \frac{\epsilon}{2}$. By the triangle inequality, it suffices to show that $d(p(t), p'(t')) + d(p'(t'), p''(t'')) \leq d_{\Delta}(y, z) = d_{\Delta}(y, w) + d_{\Delta}(w, z)$. This follows immediately from the inequalities $d(p(t), p'(t')) \leq d_{\Delta}(y, w)$ and $d(p'(t'), p''(t'')) \leq d_{\Delta}(w, z)$. \square

Proposition 4.3. *Let $p_- : [0, s_-] \rightarrow X$ and $p_+ : [0, s_+] \rightarrow X$ be geodesics of the same speed D with $p_-(0) = p_+(0)$. Define $p : [-s_-, s_+] \rightarrow X$ by letting $p(t) = p_-(-t)$ for $t \leq 0$, $p(t) = p_+(t)$ for $t \geq 0$. Then p is a geodesic if and only if $\angle_{p_-, p_+} = \pi$.*

Proof. For simplicity of notation, we may take $D = 1$. Then p is a geodesic (necessarily of speed D) if and only if $d(p(t), p(t')) = |t - t'|$ for all t sufficiently close to 0. This inequality clearly holds if $t, t' \geq 0$ or $t, t' \leq 0$. Therefore we may take $t \leq 0 \leq t'$ without loss of generality. If t and t' are sufficiently close to zero, then we do know that $d(p(t), p(t')) \leq d(p(t), p(0)) + d(p(t'), p(0)) = (-t) + t'$. If the inequality is not strict, then the vertex angle of $\Delta(p(0), p(t), p(t'))$ at $p(0)$ is not equal to π so that $\angle_{p_+, p_-} < \pi$. The converse follows by the same argument. \square

Remark 4.4. Let X be a Hadamard space, and let $x, y, z \in X$. Then the vertex angle α of the geodesic triangle spanned by x, y , and z in X is \leq the corresponding angle in the Euclidean triangle $\Delta(x, y, z)$. Consequently we have the “law of cosines”, which now takes the form of an *inequality*

$$d(y, z)^2 \geq d(x, y)^2 + d(x, z)^2 - 2d(x, y)d(x, z) \cos(\alpha)$$

5. FLAT TRIANGLES

Proposition 5.1. *Let $x, y, z \in X$, and join x, y , and z by geodesic segments to form a geodesic triangle. The sum of the vertex angles of this triangle is at most π .*

Proof. Each of the vertex angles is less than or equal to the corresponding angle in $\Delta(x, y, z)$. \square

If $x, y, z \in X$ are such that the sum of the vertex angles of the corresponding geodesic triangle is exactly π , then we shall say that (x, y, z) is in *Euclidean position*. This is a very special circumstance, as we shall now demonstrate.

Lemma 5.2. *Let X be a Hadamard space, let $x, y, z \in X$, and let w lie on the minimizing geodesic from y to z . If (x, y, z) is in Euclidean position, then (x, y, w) and (x, w, z) are in Euclidean position.*

Proof. By the triangle inequality, the sum of the vertex angles at w in the geodesic triangles spanned by (x, y, w) and (x, w, z) is at least π . Similarly, the sum of the vertex angles at x in the triangles spanned by (x, y, w) and (x, w, z) is at least as great as the vertex angle at x in the triangle spanned by (x, y, z) . Since the sum of the vertex angles in the triangle spanned by (x, y, z) is equal to π , the sum of the vertex angles in the triangles spanned by (x, y, w) and (x, w, z) is at least 2π . Since the total angle is at most π for each triangle, equality must hold in both cases. \square

Lemma 5.3. *Let X be a Hadamard space, let $x, y, z \in X$, and let w lie on the minimizing geodesic from y to z . Let Δ' the Euclidean triangle containing y which is obtained by subdividing $\Delta(x, y, z)$ by adding a segment joining x to w . If (x, y, z) is in Euclidean position, then the natural bijection $\Delta' \simeq \Delta(x, y, w)$ is an isometry.*

Proof. The side lengths of the sides containing y in both triangles are the same. Thus it suffices to show that the vertex angles of these two triangles at y are the same. But since (x, y, z) and (x, y, w) are both in Euclidean position, the vertex angles in both of these Euclidean triangles coincides with the angle between the corresponding geodesics in X . \square

Lemma 5.4. *Let X be a Hadamard space, and let $x, y, z \in X$ be such that (x, y, z) is in Euclidean position. Then the distance-decreasing map $\Delta(x, y, z) \rightarrow X$ is an isometry.*

Proof. Suppose $w, w' \in \Delta(x, y, z)$; we wish to show $d(w, w') = d_\Delta(w, w')$. Assume first that w' is a vertex. If w lies on an edge containing w' then the claim is obvious. If not, then the claim follows immediately from the last lemma.

Now suppose that w' is not a vertex. Without loss of generality, w' lies on the side of $\Delta(x, y, z)$ which is opposite x and w lies in the triangle $\Delta(x, y, w')$. By the preceding lemma, we may replace (x, y, z) by (x, y, w') , thereby reducing to the case where w' is a vertex. \square

For $x, y, z \in X$, we let $\overline{\Delta}(x, y, z)$ denote the triangle $\Delta(x, y, z)$ together with its interior (homeomorphic to a disk), metrized as a subset of the Euclidean plane.

Theorem 5.5. *Let X be a Hadamard space, $x, y, z \in X$. Then (x, y, z) is in Euclidean position if and only if the natural distance-decreasing map $\phi : \Delta(x, y, z) \rightarrow X$ prolongs to an isometry $\overline{\phi} : \overline{\Delta}(x, y, z) \rightarrow X$. Such an isometry is automatically unique.*

Proof. The “if” part of the theorem is obvious. The uniqueness is also clear: for any $w \in \overline{\Delta}(x, y, z)$, let $w' \in \Delta(x, y, z)$ denote the intersection of the line joining x and w with the segment joining y and z . Then we must have $\overline{\phi}(w) = p_{xw'}(\frac{d_\Delta(x, w)}{d_\Delta(x, w')})$. We will use this as our definition of $\overline{\phi}$; to complete the proof, it suffices to show that $\overline{\phi}$ is an isometry.

If (x, y, z) is in Euclidean position, then ϕ is an isometry. It follows easily from this that $\overline{\phi}|_{\Delta(x, y, z)} = \phi$. To complete the proof, it suffices to show that $\overline{\phi}$ is an isometry. Choose $u, v \in \overline{\Delta}(x, y, z)$: we must show $d_\Delta(u, v) = d(\overline{\phi}(u), \overline{\phi}(v))$. We have seen that there is no harm in subdividing the triangle spanned by (x, y, z) .

Therefore we may replace y and z by u' and v' . In this case $u, v \in \Delta(x, y, z)$ and the result follows from the fact that ϕ is an isometry. \square

6. CONVEXITY

Let X be a Hadamard space. A subset $K \subseteq X$ is *convex* if K contains the unique minimizing geodesic joining any two points $x, y \in K$. Note that a subset $K \subseteq X$ is a Hadamard space in its own right if and only if it is nonempty, closed and convex.

A real-valued function f on a Hadamard space is *convex* if, for any pair of points $x, y \in X$ and any $t \in [0, 1]$, we have $f(p_{\overline{xy}}(t)) \leq (1-t)f(x) + tf(y)$. If f is continuous, this is equivalent to requiring that $f(m) \leq \frac{f(x)+f(y)}{2}$, where m is the midpoint of x and y . The convex functions are closed under addition, multiplication by nonnegative scalars, and formation of pointwise suprema. The following fact will be needed again and again in the arguments to come:

Lemma 6.1. *Let X be a Hadamard space, and let $p, p' : [t_-, t_+]$ be minimizing geodesics. Then $d(p(t), p'(t))$ is a convex function of t .*

Proof. Without loss of generality we may assume $t_- = 0, t_+ = 1$, and it suffices to show $d(p(t), p'(t)) \leq (1-t)d(p(0), p'(0)) + td(p(1), p'(1))$. Let p'' be the minimizing geodesic joining $p(0)$ to $p'(1)$. Then Theorem 3.1 gives the inequalities

$$\begin{aligned} d(p(t), p''(t)) &\leq p_{\Delta(p(0), p'(0), p'(1))}(p(t), p''(t)) = (1-t)d(p(0), p(1)) \\ d(p''(t), p'(t)) &\leq p_{\Delta(p(0), p(1), p'(1))}(p''(t), p'(t)) = td(p'(0), p'(1)) \end{aligned}$$

Adding these inequalities and applying the triangle inequality, we obtain the desired result. \square

Lemma 6.2. *Let X be a Hadamard space, and f a real-valued function on X . Then f is convex if and only if every point of X has a (convex) neighborhood on which the restriction of f is convex.*

In other words, convexity is a *local* property.

Proof. The “only if” is clear. To prove the “if” direction, it suffices to show that f is convex when restricted to geodesics in X , so we may assume that X is the interval $[0, 1]$. By assumption, $[0, 1]$ has an open covering by sub-intervals on which f is convex. By compactness, we may assume finitely many subintervals are used. By induction, we can reduce to the case where only two such sub-intervals are used: say f is convex on $[0, s]$ and $(s', 1]$ with $s' < s$. Subtracting a linear function if necessary, we may assume $f(0) = f(1) = 0$; our goal is to show that $f(t) \leq 0$ for all $t \in [0, 1]$. Without loss of generality $t \in [0, s]$; suppose $f(t) > 0$. By convexity, we get $f(t') > 0$ for all $t' \in [t, s]$. Replacing t by some t' if necessary, we may assume that $t > s'$. Now choose $t' \in (t, s)$. The convexity of $f|_{[0, s]}$ shows that $f(t') \geq \frac{t'}{t}f(t) > f(s')$. Similarly, the convexity of $f|_{(s', 1]}$ shows that $f(t) \geq \frac{1-t}{1-t'}f(t') > f(t')$, a contradiction. \square

Theorem 6.3. *Let X be a Hadamard space. Then every geodesic in X is minimizing.*

Proof. Let $p : [0, 1] \rightarrow X$ be a geodesic of speed D . Let $p' : [0, 1] \rightarrow X$ be the unique minimizing geodesic joining $p(0)$ to $p(1)$. Consider the function $f(t) = d(p(t), p'(t))$. Locally on the interval $[0, 1]$, p is a minimizing geodesic. From Lemma 6.1 it follows that f is a convex function locally. Thus f is a convex function. Since

$f(0) = f(1) = 0$, the nonnegativity of f implies that f is identically zero, so that $p = p'$. \square

If K is a subset of a metric space X and $x \in X$, we let $d(x, K)$ denote the infimum of $d(x, y)$ over all $y \in K$.

Proposition 6.4. *Let X be a Hadamard space and let K be a convex subset. Then the function $x \mapsto d(x, K)$ is continuous and convex.*

Proof. Continuity is obvious. For the convexity, note that if $d(x, K) = D$ and $d(y, K) = D'$, then there are points $k, k' \in K$ such that $d(x, k) \leq D + \epsilon$, $d(y, k') \leq D' + \epsilon$ for any $\epsilon > 0$. The function $f(t) = d(p_{\overline{xy}}(t)p_{\overline{k k'}}(t))$ is convex by Lemma 6.1. Since the image of $p_{\overline{k k'}}$ is contained in K , we see immediately that $d(p_{\overline{xy}}(t), K) \leq (1-t)D + tD' + \epsilon$. Since this holds for any $\epsilon > 0$, we obtain the desired result. \square

Proposition 6.5. *Let X be a Hadamard space, and let K be a closed convex subset of X . For any $x \in X$, there is a unique point $\pi_K(x) \in K$ such that $d(x, \pi_K(x)) = d(x, K)$.*

Proof. Let $D = d(x, K)$. If $y, z \in K$ with midpoint m and $d(x, y), d(x, z) \leq D + \epsilon$, then $D^2 \leq d(x, m)^2 \leq (D + \epsilon)^2 - \frac{d(y, z)^2}{4}$. Thus we obtain $d(y, z)^2 \leq 8D\epsilon + 4\epsilon^2$. This immediately implies the uniqueness of $\pi_K(x)$. It also shows that any sequence of points $y_i \in K$ such that $d(x, y_i)$ approaches D is necessarily a Cauchy sequence. Since X is complete and K is closed, this sequence has a limit in K , which is a point y such that $d(x, y) = D$. \square

Before we reach the main result about the retraction π_K , we need a lemma.

Lemma 6.6. *Let X be a Hadamard space and K a convex subset; let $x \in X$ and let $p : [0, 1] \rightarrow K$ be a geodesic with $p(0) = \pi_K(x)$. Let $p' : [0, 1] \rightarrow X$ denote the geodesic joining $\pi_K(x)$ to x . Then $\angle_{p, p'} \geq \frac{\pi}{2}$.*

Proof. Suppose $\angle_{p, p'} < \alpha < \frac{\pi}{2}$. Then there exist $t, t' > 0$ such that the vertex angle of the triangle $\Delta(\pi_K(x), p(\epsilon), p'(t'))$ at $\pi_K(x)$ is $\leq \alpha$ whenever $\epsilon < t$. Let p have speed D and p' speed D' . Then the law of cosines implies that

$d(p(\epsilon), p'(t'))^2 \leq (D\epsilon)^2 + (D't')^2 - 2(D\epsilon)(D't') \cos(\alpha)$. Taking ϵ sufficiently small, we obtain $d(p(\epsilon), p'(t')) < D't'$. Then by the triangle inequality we get $d(p(\epsilon), x) \leq d(p(\epsilon), p'(t')) + d(p'(t'), x) < D't' + D'(1-t') = D'$. This contradicts the definition of $\pi_K(x)$. \square

Proposition 6.7. *Let X be a Hadamard space and K a closed convex subset of X . Then π_K is a distance-decreasing retraction $X \rightarrow K$.*

Proof. Let $x, x' \in X$; we wish to show that $d(\pi_K(x), \pi_K(x')) \leq d(x, x')$. Let $p : [0, D] \rightarrow X$ and $p' : [0, D'] \rightarrow X$ be unit speed geodesics joining $\pi_K(x)$ to x and $\pi_K(x')$ to x' . Let $A = d(\pi_K(x), \pi_K(x'))$, and let p'' denote a geodesic joining $\pi_K(x)$ to $\pi_K(x')$.

Lemma 6.6 implies that $\angle_{p, p''}, \angle_{p', p''} \leq \frac{\pi}{2}$.

Choose a small number $\epsilon > 0$, let q denote the unit speed geodesic joining $\pi_K(x)$ to $p'(\epsilon D')$, and let $\alpha = \angle_{p, q}$. Let $d(\pi_K(x), p'(\epsilon D')) = A + \epsilon D'$. Note that the triangle inequality shows $|\epsilon'| \leq \epsilon$.

The law of cosines gives

$$\cos(\alpha) \geq \frac{A^2 + (A + \epsilon' D')^2 - \epsilon^2 D'^2}{2A(A + \epsilon' D')}$$

Thus α approaches zero as ϵ approaches zero. Assume that ϵ is chosen sufficiently small so that $\cos(\alpha) \leq 1 - \frac{\alpha^2}{2}$ and $\frac{1}{A + \epsilon' D'} \geq \frac{1}{A}(1 - \epsilon' \frac{D'}{A})$. Then we deduce

$$\begin{aligned} 1 - \frac{\alpha^2}{2} &\geq \frac{2A^2 + 2A\epsilon' D' + (\epsilon'^2 - \epsilon^2)D'^2}{2A^2} (1 - \epsilon' \frac{D'}{A}) \\ &= 1 - \frac{\epsilon'^2 D'^2}{2A^2} - \frac{\epsilon^2 D'^2}{2A^2} + (\epsilon')(\epsilon^2 - \epsilon'^2) \frac{D'^3}{2A^3} \end{aligned}$$

so that

$$\frac{D'^2 \epsilon^2}{A^2} \geq \frac{\alpha^2}{2}$$

and $\alpha \leq \frac{2D'\epsilon}{A}$.

Now we apply the law of cosines to the triangle spanned by x , $p(D\epsilon)$, and $p'(D'\epsilon)$. The vertex angle of this triangle is at least $\frac{\pi}{2} - \alpha$, so that

$$A'^2 \geq (A + \epsilon' D')^2 + (\epsilon D)^2 - 2(A + \epsilon' D')(\epsilon D) \cos(\frac{\pi}{2} - \alpha)$$

where $A' = d(p(D\epsilon), p'(D'\epsilon))$. Thus $A'^2 \geq A^2 + 2A\epsilon' D' + \delta$, where δ is second-order in ϵ . Now $\epsilon' \geq 0$ since the angle of $\angle_{p', p''} \geq \frac{\pi}{2}$. Thus we deduce that $A' \geq A + \delta$ for some (other) δ which is second order in ϵ .

Let $f(t) = d(p(Dt), p'(Dt'))$. Then f is a convex function. Since $f(0) = A$, we deduce $f(t) \geq A + \frac{\delta}{\epsilon}$ for all $t \geq \epsilon$. For fixed t , we may take the limit as ϵ approaches zero to deduce $f(t) \geq A$. In particular, taking $t = 1$, we deduce that $d(x, x') \geq A$, as desired. \square

7. FLAT SPACES

Let X be a Hilbert space with norm $\|\cdot\|$, and let $d(x, y) = \|x - y\|$. Then X is a Hadamard space, and in fact we have

$$d(z, m)^2 = \frac{d(z, x)^2 + d(z, y)^2}{2} - \frac{d(x, y)^2}{4}$$

whenever $m = \frac{x+y}{2}$ is the midpoint between x and y . In general, we shall call a Hadamard space *flat* if the above equality holds. Clearly, any nonempty closed convex subset of a flat Hadamard space is flat; hence any nonempty closed convex subset of a Hilbert space is flat. We shall show that the converse holds. First, we need a lemma:

Lemma 7.1. *Let X be a flat Hadamard space, and let $p : [0, 1] \rightarrow X$ be a geodesic of speed D . Then for any point z and any $t \in [0, 1]$, we have*

$$d(z, p(t))^2 = (1-t)d(z, p(0))^2 + td(z, p(1))^2 - t(1-t)D^2$$

Proof. By hypothesis, this holds when $t = \frac{1}{2}$. By induction we can establish equality when t is any dyadic rational. The result for arbitrary $t \in [0, 1]$ follows by continuity. \square

Theorem 7.2. *Let X be a flat Hadamard space. Then X is isometric to a (nonempty) closed convex subset of a Hilbert space.*

Proof. Choose an arbitrary base point $e \in X$. Let $V(X)$ denote the free (real) vector space generated by the elements of X . We define a symmetric bilinear map $b : V(X) \times V(X) \rightarrow \mathbb{R}$ by setting $B(x, y) = d(e, x)^2 + d(e, y)^2 - d(x, y)^2$ on generators and extending by linearity.

The first thing to note is that B is “affine” as a function on X . Namely, Lemma 7.1 implies that if $p : [0, 1] \rightarrow X$ is a geodesic, then $B(y, p(t)) = (1 - t)B(y, p(0)) + tB(y, p(1))$ for any $y \in X$, and hence for any $y \in V(X)$. In other words, p is an affine function modulo the kernel of the bilinear form B .

We now claim that B is positive semidefinite. We must show that $B(v, v) \geq 0$ for any $v \in V(X)$. Write

$$v = \sum_i \alpha_i y_i + \sum_j \beta_j z_j$$

where the y_i and z_j are elements of X , each $\alpha_i \geq 0$, and each $\beta_j \leq 0$. If there are at least two coefficients α_i , then we may combine them by replacing $\alpha_i y_i + \alpha_{i'} y_{i'}$ by $(\alpha_i + \alpha_{i'}) y_{i''}$, where

$$y_{i''} = p_{\overline{y_i y_{i'}}} \left(\frac{\alpha_{i'}}{\alpha_i + \alpha_{i'}} \right)$$

Since B is affine, this replacement does not change the value of B . In this manner we may reduce to the case where $v = \alpha y + \beta z$, with $\alpha \geq 0$, $\beta \leq 0$.

By the triangle inequality, $|d(e, x) - d(e, y)| \leq d(x, y)$. Thus $(d(e, x) - d(e, y))^2 \leq d(x, y)^2$. Multiplying by $2\alpha\beta$, we obtain

$$2\alpha\beta d(e, x)^2 - 4\alpha\beta d(e, x)d(e, y) + 2\alpha\beta d(e, y)^2 - 2\alpha\beta d(y, z)^2 \geq 0$$

Adding $2(\alpha d(e, x) + \beta d(e, y))^2 \geq 0$, we obtain

$$\begin{aligned} B(v, v) &= \alpha^2 B(x, x) + 2\alpha\beta B(x, y) + \beta^2 B(y, y) \\ &= 2\alpha^2 d(e, y)^2 + 2\alpha\beta d(e, x)^2 + 2\alpha\beta d(e, y)^2 - 2\alpha\beta d(x, y)^2 + 2\beta^2 d(e, y)^2 \\ &\geq 0 \end{aligned}$$

as desired.

Thus $V(X)$ is a pre-Hilbert space with respect to B . Let $\overline{V}(X)$ denote its completion. Our proof will be completed if we show that the natural map $\phi : X \rightarrow \overline{V}(X)$ is an isometry (we employ the convention that $2|v|^2 = B(v, v)$). For this, note that $2|x - y|^2 = B(x - y, x - y) = B(x, x) + B(y, y) - 2B(x, y) = 2d(e, x)^2 + 2d(e, y)^2 - 2d(e, x)^2 - 2d(e, y)^2 + 2d(x, y)^2 = 2d(x, y)^2$, so that $d(y, z) = |y - z|$ as desired. \square

Corollary 7.3. *Let X be a normed vector space, and set $d(x, y) = |x - y|$. Then X is a Hadamard space if and only if it is a Hilbert space.*

Proof. It is clear that the only candidate for the midpoint of x and y is $\frac{x+y}{2}$. Thus X is a Hadamard space if and only if it is complete and

$$\left| z - \frac{x+y}{2} \right|^2 + \left| \frac{x-y}{2} \right|^2 \leq \frac{|z-x|^2 + |z-y|^2}{2}$$

Changing variables by setting $z = 2a$, $x = a + b$, $y = a - b$, our inequality may be rewritten as

$$|a|^2 + |b|^2 \leq \frac{|a+b|^2 + |a-b|^2}{2}$$

If this inequality holds for all $a, b \in V$, then we deduce also

$$|a + b|^2 + |a - b|^2 \leq \frac{|2a|^2 + |2b|^2}{2} = 2(|a|^2 + |b|^2)$$

Thus equality must hold, so that X is flat. Thus X is isometric to a closed convex subset of a Hilbert space which contains the origin without loss of generality. The inclusion of X into this Hilbert space is linear, so that X is a closed subspace of a Hilbert space and therefore a Hilbert space. \square

Let X be an arbitrary Hadamard space. Note that any intersection of closed convex subsets of X is a closed convex subset of X . Thus every subset $K \subseteq X$ is contained in a least closed convex subset of X , called the *convex hull* of K ; this set is nonempty if K is nonempty.

Conjecture 7.4. Let X be a Hadamard space and let K be a nonempty subset of X . Then the convex hull of K is flat if and only if every triple of points (x, y, z) belonging to K is in Euclidean position.

???

8. FIXED POINTS

In this section we prove the Cartan-Tits fixed point theorem. This result was applied to symmetric spaces of noncompact type to deduce the conjugacy of maximal compact subgroups of a real reductive Lie group.

Let X be a Hadamard space, and let K be a bounded subset of X . For $x \in X$, there is a least real number r such that $K \subseteq \{y \in X : d(x, y) \leq r\}$. We shall denote this real number r by $r(x)$; it is the supremum of the distances of x to all the elements of K . As a supremum of convex functions, $r(x)$ is a convex function of x . In fact we can say more: if m is the midpoint of x and y , then $r(m)^2 \leq \frac{r(x)^2 + r(y)^2}{2} - \frac{d(x, y)^2}{4}$.

Let R denote the infimum of the values of $r(x)$ over all $x \in X$. Let $\{x_i\}$ be a sequence of elements of X such that $r(x_i)$ approaches R . If we choose i sufficiently large, so that $r(x_j)^2 \leq R + \epsilon$ for all $j \geq i$, then the inequality above shows that $d(x_j, x_k)^2 \leq 4\epsilon$ for any $j, k \geq i$. It follows that the sequence $\{x_i\}$ is Cauchy and approaches a limit x such that $r(x) = R$. The inequality above then shows that x is unique. We say that x is the *circumcenter* of K and $r(x) = R$ is the *circumradius* of K .

The circumcenter of a bounded set K is canonically determined by K . It follows that any isometry of X which leaves K invariant must also leave its circumcenter invariant. This immediately implies the following theorem:

Theorem 8.1. *Let X be a Hadamard space, and let a group G act on X by isometries. Suppose that for some $x \in X$, the orbit Gx is bounded. Then the set $X^G = \{y \in X : (\forall g \in G)[gy = y]\}$ is a Hadamard space.*

Proof. It is obvious that X^G is closed and convex. The real content of the theorem is the assertion that X^G is nonempty. So consider a point $x \in X$ such that the orbit Gx is bounded. Clearly this orbit is also G -invariant. Thus its circumcenter is a fixed point for G . \square

A typical application occurs if G is a compact group, so that Gx is compact and therefore bounded for any $x \in X$.

9. LOCAL COMPACTNESS

The goal of this section is to prove the following result:

Theorem 9.1. *Let X be a locally compact Hadamard space, and let K be a subset of X . Then K is compact if and only if it is closed and bounded.*

Proof. The “only if” direction is obvious. For the converse, it suffices to show that for any $x \in X$, the subsets $X_r = \{x' \in X : d(x, x') \leq r\}$ are compact.

We first claim that if X_r is compact for some $r \geq 0$, then $X_{r'}$ is compact for some $r' > r$. Since X is locally compact, X_r has a compact neighborhood. If this neighborhood does not contain any $X_{r'}$, then we may select a sequence of points $\{x_i\}$ such that the distances $d(x, x_i)$ converge to r , but the x_i do not converge to a point of X_r . For each i , let $r_i = d(x_i, x)$, and let $p_i : [0, r_i] \rightarrow X$ be a minimizing geodesic joining x to x_i . Without loss of generality we may assume that all $x_i \notin X_r$, so that $r_i > r$. Let $x'_i = p_i(r)$. Then $x'_i \in X_r$. Since X_r is compact by assumption, we may (by passing to a subsequence if necessary) assume that the x'_i converge to a point $x' \in X_r$. Now $d(x', x_i) \leq d(x', x'_i) + d(x'_i, x_i) \leq d(x', x'_i) + (r_i - r)$ which converges to zero as i increases. It follows that the x_i converge to $x' \in X_r$, contrary to our assumption.

Now consider the set of all r such that X_r is compact. Clearly this set is closed-downwards and contains 0. We have just shown that this set is open in $[0, \infty)$. We now invoke the least-upper-bound property of the real numbers. To complete the proof, it suffices to show that for any $r > 0$, the compactness of all X_s for $s < r$ implies the compactness of X_r .

To show that X_r is compact we proceed as follows: consider any sequence $\{x_i\}$ contained in X_r . Let $r_i = d(x_i, x)$, and let $p_i : [0, r_i] \rightarrow X$ denote the unit speed geodesic joining x and x_i . We may assume without loss of generality that the r_i converge to r . For any rational number $0 < \alpha < 1$, the points $p_i(\alpha r_i)$ are contained in the compact set X_{q_r} . Repeatedly passing to subsequences, we may assume that for every such rational number α , the points $p_i(\alpha r_i)$ converge to a point $p(\alpha)$. Now $d(p(\alpha), p(\alpha')) \leq d(p(\alpha), p_i(\alpha r_i)) + d(p_i(\alpha r_i), p_i(\alpha' r_i)) + d(p_i(\alpha' r_i), p(\alpha')) \leq d(p(\alpha), p_i(\alpha r_i)) + |\alpha' - \alpha|r + d(p_i(\alpha' r_i), p(\alpha'))$. Taking the limit as i increases, we deduce that $d(p(\alpha), p(\alpha')) \leq |\alpha' - \alpha|r$, so that p is uniformly continuous. Since X is complete, p prolongs uniquely to a continuous map $p : [0, 1] \rightarrow X$. Now one easily checks that the sequence $\{x_i\}$ converges to $p(1)$. \square

A slight variant on this argument can be used to prove that closed, bounded subsets of complete Riemannian manifolds are compact.

10. BOUNDARY AT INFINITY

In this section, we show how to construct a natural “compactification” of a Hadamard space X by adjoining a “sphere at infinity”. However, let us begin very generally. Let X be an arbitrary metric space (not necessarily complete, or even separated). One usually constructs the completion of X via equivalence classes of Cauchy sequences in X . However, there is an alternative construction: let $C(X)$ denote the collection of all continuous real-valued functions on X , considered as a Banach space via the supremum norm. Then $x \mapsto d(x, \cdot)$ gives a continuous map $\psi : X \rightarrow C(X)$, and we have the following:

Proposition 10.1. *The map ψ exhibits the closure of $f(X)$ in $C(X)$ as a completion of X .*

Proof. By the triangle inequality, ψ is distance decreasing. On the other hand, since $|d(x, y) - d(x, x)| = d(x, y)$, we see that ψ is an isometry onto its image. Since $C(X)$ is complete, the closure of $\psi(X)$ is complete. \square

Now suppose that we wish to compactify a metric space (X, d) . Since every compact metric space is complete, we may regard the above construction as a “first step” toward the compactification of X . In order to proceed, it is natural to try to form the closure of X in other function spaces. However, in order to obtain a space larger than the completion of X , we will have to consider function spaces in which X does *not* embed isometrically.

Let $C(X \times X)$ denote the space of continuous real-valued functions on $X \times X$, given the topology of convergence on bounded subsets. Note that if X is locally compact, this coincides with the topology of convergence on compact subsets by Theorem 9.1. As before we have a natural map $\phi : X \rightarrow C(X \times X)$, given by $x \mapsto d(x, y) - d(x, z)$. Let \bar{X} denote the closure of the image of X , and denote the map from X to \bar{X} also by ϕ . Note that since $X \times X$ is a union of countably many bounded subsets, the topology on $C(X \times X)$ may be defined by countably many norms and so sequences shall suffice to define the topology on $C(X \times X)$.

Theorem 10.2. *If X is locally compact, then \bar{X} is compact.*

Proof. Since $|d(x, y) - d(x, z)| \leq d(y, z)$ and $|(d(x, y) - d(x, z)) - (d(x, y') - d(x, z'))| \leq d(y, y') + d(z, z')$, we see that $\phi(X)$ is bounded and equicontinuous in $C(X \times X)$. Since X is locally compact, the topology on $C(X \times X)$ is the compact open topology. The compactness of \bar{X} now follows from Ascoli’s theorem. \square

For the next assertion we will need a lemma.

Lemma 10.3. *Let X be a metric space in which any pair of points may be joined by a minimizing geodesic. Choose a point $x \in X$, and let $B_r(x)$ denote the closed ball of radius r about x . Let U denote the open subset of \bar{X} consisting of those functions h satisfying $|h(y, z) - d(x, y) + d(x, z)| < 2r$ for $(y, z) \in B_r(x) \times B_r(x)$. Then $\phi^{-1}(U)$ is the open ball of radius r about x .*

Proof. It is clear that $d(x', x) < r$ implies $\phi(x') \in U$. For the converse, let us suppose that $d(x', y) - d(x', z)$ lies in U . If $d(x', x) > r$, then we may choose a point x'' such that $d(x'', x) = r$, $d(x'', x') = d(x, x') - r$. Applying the definition of U in the case $y = x''$, $z = x$, we deduce that $2r = |d(x', x'') - d(x', x) - d(x', x'')| < 2r$, a contradiction. \square

Theorem 10.4. *Let X be a metric space in which any two points can be joined by a minimizing geodesic. Then $\phi : X \rightarrow \bar{X}$ is a homeomorphism onto its image.*

Proof. Choose any $x \in X$ and any $r > 0$; we will produce an open subset $U \subseteq C(X \times X)$ whose preimage under ϕ is contained in a closed ball $B_r(x)$ of radius r about x . It suffices to take for U those functions h with the property that $|h(y, z) - (d(x, y) - d(x, z))| < 2r$ for $(y, z) \in B_r(x) \times B_r(x)$. \square

Lemma 10.5. *Let K be a compact Hausdorff space, and let $U \subseteq K$ be a dense, locally compact subset. Then U is open.*

Proof. We will show that U contains a neighborhood of any point $x \in U$. Let V_0 be a neighborhood of x in U whose closure $\overline{V_0}$ in U is compact. It will suffice to show that V_0 is open in K . We know that $V_0 = V \cap U$ for some open $V \subseteq K$.

Since $\overline{V_0}$ is compact and K is Hausdorff, $\overline{V_0}$ is closed in K . Thus $\overline{V_0}$ is the closure of V_0 in K . Since $V \cap \overline{V_0} \subseteq V \cap U = V_0$, we deduce that V_0 is closed in V . Thus $V - V_0$ is an open set. Since U is dense in K and $(V - V_0) \cap U = \emptyset$, we deduce that $V - V_0 = \emptyset$ so that V_0 is open in K . This completes the proof. \square

Theorem 10.6. *Let X be a locally compact metric space in which any two points can be joined by a unique minimizing geodesic. Then $\phi : X \rightarrow \overline{X}$ is an open immersion.*

Proof. Since ϕ is a homeomorphism onto its image, this follows immediately from Lemma 10.5. \square

Suppose a sequence of points $\{x_i\}$ of X is such that $\{\phi(x_i)\}$ converges in \overline{X} . If the $\{x_i\}$ belong to a bounded set U , then for any ϵ we may choose i so large that $|d(x_j, y) - d(x_j, z) - d(x_k, y) + d(x_k, z)| \leq \epsilon$ for all $j, k \geq i, y, z \in U$. In particular, we may take $y = x_j, z = x_k$, and we deduce that the sequence $\{x_j\}$ is Cauchy. Thus, if X is complete, we conclude that any point in the ‘‘boundary’’ $\overline{X} - \phi(X)$ is the limit of an unbounded sequence of elements of X .

For the remainder of this section we shall assume that X is a Hadamard space. In this case we may identify X with a subset of \overline{X} (an open subset if X is locally compact). Note that for any function $h \in \overline{X}$, we have $h(x, y) + h(y, z) = h(x, z)$. It follows that we may write $h(x, y) = h'(x) - h'(y)$ for some continuous function $h' : X \rightarrow \mathbb{R}$, and the function h' is well-defined up to an additive constant. We shall call a function $h' : X \rightarrow \mathbb{R}$ which arises in this way a *horofunction* on X .

Lemma 10.7. *Let X be a Hadamard space, $y \in X$. For every unit speed geodesic $p : [0, \infty) \rightarrow X$, the limit $d(p(t), y) - t$ exists as t approaches infinity.*

Proof. The $t \mapsto d(p(t), y) - t$ is bounded in absolute value by $d(x, y)$. Moreover, it is decreasing by the triangle inequality. \square

We will also need to know that the function $d(p(t), y) - t$ approaches its limit sufficiently quickly.

Lemma 10.8. *Let X be a Hadamard space, $K \subseteq X$ a bounded subset, and $x \in X$ a point. For every constant ϵ , there exists a constant $t_0 > 1$ such that for every $T > t_0$ and every unit speed geodesic $p : [0, T] \rightarrow X$ with $p(0) = x$, we have the inequality $d(p(t_0), y) - t_0 \leq d(p(T), y) - T + \epsilon$.*

Proof. Suppose given $t_0 > 1, T > t_0^2$, and a geodesic $p : [0, T] \rightarrow X$ with $p(0) = x$. Let $h(t) = d(p(t), y) - t$. For fixed y , let us apply the fact that the function $d(y, p(t))^2 - t^2$ is convex in t . We get

$$d(y, p(t_0))^2 - t_0^2 \leq \frac{T - t_0}{T} d(x, y)^2 + \frac{t_0}{T} (d(y, p(T))^2 - T^2)$$

so that

$$2t_0 h(t_0) \leq (h(t_0) + t_0)^2 - t_0^2 \leq d(x, y)^2 + h(T) t_0 \left(\frac{h(T)}{T} + 2 \right)$$

. Using the fact that $h(t) \leq d(x, y)$, we obtain $h(t_0) - h(T) \leq \frac{d(x, y)^2}{t_0}$. Now if y lies in a bounded set K , then $d(x, y)$ is bounded and we get the desired result. \square

Theorem 10.9. *Let X be a Hadamard space, and let $p : [0, \infty) \rightarrow X$ be a unit speed geodesic. Then $p(t)$ approaches a limit in \overline{X} as t approaches ∞ .*

Proof. We must show that the function $h_t(y, z) = d(p(t), y) - d(p(t), z)$ converges uniformly on bounded subsets of $X \times X$ as t approaches infinity. It suffices to show that the function $h'_t(y) = d(p(t), y) - t$ converges uniformly on bounded subsets of X as t approaches infinity. This follows immediately from Lemma 10.8. \square

Theorem 10.10. *Let X be a Hadamard space, $x \in X$, and let $\{x_i\}$ be an unbounded sequence in X . Let $p_i : [0, t_i] \rightarrow X$ denote unit speed geodesic joining x to x_i . Then the following are equivalent:*

- *The p_i converge uniformly on each interval $[0, t]$. (Since the sequence $\{x_i\}$ is unbounded, p_i is defined on $[0, t]$ for all but finitely many values of i .)*
- *The sequence $\{x_i\}$ approaches a limit in \overline{X} .*

Proof. Suppose that the p_i converge uniformly on each interval $[0, t]$. Let $p : [0, \infty) \rightarrow X$ denote their limit. We see immediately that p is a unit speed geodesic, hence $p(t)$ approaches some limit $h \in \overline{X}$. We shall show that h is a limit of the sequence $\{x_i\}$ in \overline{X} . For this, it suffices to show that the functions $y \mapsto d(x_i, y) - t_i$ converge uniformly on bounded subsets to the limiting value C_y of $y \mapsto d(p(t), y) - t$.

Fix a bounded set $K \subseteq X$ and $\epsilon > 0$. Then there exists a constant t' such that $d(x_i, y) - t_i \leq d(p_i(t'), y) - t' \leq d(x_i, y) - t_i + \epsilon$ and $C_y \leq d(p(t'), y) - t' \leq C_y + \epsilon$ for almost every i and all $y \in K$. Choosing i sufficiently large, we may ensure that $d(p_i(t'), p(t')) \leq \epsilon$. Then we get $|d(p_i(t'), y) - t' - C_y| \leq 3\epsilon$, as desired.

For the converse, let us assume that the sequence $\{x_i\}$ converges in \overline{X} . We will fix $T > 0$ and show that the geodesics $p_i|_{[0, T]}$ converge uniformly. Since the p_i are unit speed geodesics, the images of all $p_i|_{[0, T]}$ are contained in a closed ball $B_T(x)$ of radius T about x . The convergence of the x_i in \overline{X} shows that for every $\epsilon > 0$, there exists an integer i such that $|d(x_j, y) - t_j - (d(x_k, y) - t_k)| \leq \epsilon$ for all $j, k > i$ and all $y \in B_T(x)$. In particular, taking $y = p_j(t)$ for $t < T$, we get $d(x_k, p_j(t)) \leq t_k - t + \epsilon$. On the other hand, we have $d(x, p_j(t)) = t$. Thus we obtain

$$\begin{aligned} d(p_k(t), p_j(t))^2 &\leq \frac{t_k - t}{t_k} t^2 + \frac{t}{t_k} (t_k - t + \epsilon)^2 - t(t_k - t) \\ &= 2t\epsilon + \frac{t\epsilon^2 - 2t^2\epsilon}{t_k} \leq 2T\epsilon + \epsilon^2 \end{aligned}$$

The desired result follows. \square

Corollary 10.11. *Let X be a Hadamard space, $x \in X$, and $x' \in \overline{X} - X$. Then there exists a unique unit-speed geodesic $p : [0, \infty) \rightarrow X$ such that $p(0) = x$ and $p(t)$ approaches x' as t approaches infinity.*

Proof. The theorem establishes the existence. For the uniqueness, suppose that two geodesics $p, p' : [0, \infty)$ satisfy $p(0) = p'(0)$ and both approach the same limit in \overline{X} . Then the sequence of points $p(1), p'(2), p(3), \dots$ approaches the same limit in \overline{X} . It follows that the sequence of geodesics $p|_{[0, 1]}, p'|_{[0, 2]}, p|_{[0, 3]}, \dots$ converge uniformly on any interval $[0, t]$. This implies that $p|_{[0, t]} = p'|_{[0, t]}$. Since this is true for all t , we get $p = p'$. \square

Thus \overline{X} may be thought of as constructed from X by adjoining endpoints to all geodesics originating at a given point of X . Of course, our definition of \overline{X} does

not presuppose the choice of such a point. This raises the question of when two geodesics p and q with *different* points of origin have the same limit in \overline{X} . We now show that this is true if and only if p and p' are *asymptotic*, meaning that the $d(p(t), p'(t))$ is bounded as t approaches infinity.

Theorem 10.12. *Let X be a Hadamard space, and let $p, p' : [0, \infty) \rightarrow X$ be two unit speed geodesics. Then p and p' have the same limit in \overline{X} if and only if p and p' are asymptotic.*

Proof. First suppose that p and p' have the same limit in \overline{X} . Let p_i denote the unit speed geodesic joining $p(0)$ to $p'(i)$. Then since the $p'(i)$ converge in \overline{X} , the geodesic segments p_i converge uniformly on compact subsets to a geodesic with the same limit in \overline{X} . By uniqueness, we deduce that the p_i converge uniformly to p . The function $d(p_i(t), p'(t))$ is convex and vanishes for $t = i$, so it is bounded by $d(p(0), p'(0))$ for all t . Taking i sufficiently large, we deduce that $d(p(t), p'(t)) \leq d(p(0), p'(0))$, and so p is asymptotic to p' .

Now suppose that p is asymptotic to p' . Let $p'' : [0, \infty) \rightarrow X$ be a unit speed geodesic with $p''(0) = p(0)$, such that p'' and p' approach the same limit in \overline{X} . Then p'' is asymptotic to p' , so p'' is asymptotic to p . The function $t \mapsto d(p(t), p''(t))$ is convex, bounded, and vanishes for $t = 0$. It follows that this function is identically zero, so that $p = p''$. Thus p and p' approach the same limit in \overline{X} . \square

To conclude this section, we shall give a nice characterization of the horofunctions on a Hadamard space X .

Theorem 10.13. *Let X be a Hadamard space, and let h be a real-valued function on X . Then h is a horofunction if and only if the following conditions are satisfied:*

- *The function h is convex.*
- *The inequality $h(x) \leq d(x, y) + h(y)$ is satisfied for all $x, y \in X$.*
- *For any $x \in X$ and any $r > 0$, the function h attains its infimum on $B_r(x)$ at a unique point y , and $h(y) = h(x) - r$.*

Proof. First suppose that h is a horofunction. Choose any point $x_0 \in X$. Adding a constant to h if necessary, we may assume that $h(z) = h'(z, x_0)$ for some $h' \in \overline{X} - X$. Thus there is an unbounded sequence of points $\{a_i\}$ in X such that the functions $z \mapsto d(a_i, z) - d(a_i, x_0)$ converge uniformly to h on bounded subsets of X . From this, the first two properties follow immediately.

To verify the third property, we may assume that the horofunction h is associated to a unit speed geodesic $p : [0, \infty) \rightarrow X$ with $f(0) = x$. If we normalize h so that $h(x) = 0$, it is evident that $h(p(t)) = -t$ for every t . Moreover, the inequality $|h(x) - h(y)| \leq d(x, y)$ shows that $h(y) \geq -r$ for all $y \in B_r(X)$. So it suffices to show that $h(y) = -r$ implies that $y = p(r)$. Indeed, choose $\epsilon > 0$. Then for large t , we have $d(y, p(t)) \leq t - r + \epsilon$. Thus

$$d(y, p(r))^2 \leq \frac{t-r}{t} d(y, x)^2 + \frac{r}{t} (t-r+\epsilon)^2 - r(t-r) \leq 2\epsilon + \frac{r\epsilon^2 - 2r\epsilon}{t}$$

which becomes arbitrarily small as ϵ approaches zero.

Now let us prove the converse: any function h satisfying the above conditions is a horofunction. Pick any point $x \in X$. For every $t \geq -h(x)$, let $p_x(t)$ denote the point of $B_{t+h(x)}(x)$ where h achieves its infimum $-t$. Then $p_x(-h(x)) = x$. We first claim that $p_x : [-h(x), \infty) \rightarrow X$ is a geodesic. For this, it suffices to

show that $p_x|[-h(x), T]$ is the minimizing geodesic joining x to $p_x(T)$. For we have $d(p_x(t), p_x(T)) = T - t$, so $h(p(t)) \leq h(p_x(T)) + T - t = -t$. Since $p(t) \in B_t(x)$, we deduce $p(t) = p_x(t)$. Note that $p_x(t)$ is also equal to the projection of x into the ‘‘horoball’’ $X_r = \{y : h(y) \leq -t\}$.

Fix $x \in X$; we may assume that h normalized so that $h(x) = 0$. The proof will be completed if we show that h is the horofunction associated to the geodesic f . In other words, we must show that for each $y \in X$, $h(y)$ is the limit of $h_t(y) = d(f_x(t), y) - t$ as t approaches infinity. Since $h_t(y) = d(f_x(t), y) + h(f_x(t))$, we get $h(y) \leq h_t(y)$ for every value of t . We will obtain the reverse inequality in the limit by a more subtle argument.

Choose $t \geq 0, -h(y)$. Then $p_y|[-h(y), t]$ is a geodesic joining y to its projection $\pi_{X_t}(y)$. Since projections decrease distances, we get $d(p_y(t), p_x(t)) \leq d(x, y)$, which is therefore bounded as t approaches infinity. On the other hand, the distances $d(x, p_x(t)) = t$, $d(y, p_y(t)) = t + h(y)$, and $d(y, p_x(t)) \geq t - d(x, y)$ grow without bound as t approaches infinity. It follows that the vertex angle at $p_x(t)$ of the geodesic triangle spanned by $(x, y, p_x(t))$ approaches zero as t approaches infinity. On the other hand, the vertex angle at $f_x(t)$ of the geodesic triangle $(x, f_x(t), f_y(t))$ is at least $\frac{\pi}{2}$. It follows from the triangle inequality that for any $\epsilon > 0$, the vertex angle α_t at $p_x(t)$ of $(y, p_x(t), p_y(t))$ satisfies $\alpha_t \geq \frac{\pi}{2} - \epsilon$ for t sufficiently large. On the other hand, the vertex angle β_t of $(y, p_t(x), p_t(y))$ at $p_t(y)$ is at least $\frac{\pi}{2}$. Since the sum of the angles of a geodesic triangle in X can be at most π , we deduce that

$$|\beta_t - \frac{\pi}{2}|, |\alpha_t - \frac{\pi}{2}| \leq \epsilon$$

for t sufficiently large.

Now, applying the law of cosines, we obtain

$$\begin{aligned} d(y, p_x(t))^2 &\geq d(p_y(t), p_x(t))^2 + d(y, p_y(t))^2 - 2d(p_y(t), p_x(t))d(y, p_y(t)) \cos(\beta_t) \\ (h_t(y) + t)^2 &\geq (t + h(y))^2 - 2d(p_y(t), p_x(t))(t + h(y)) \cos(\beta_t) \\ 2h_t(y) + \frac{h_t(y)^2}{t} &\geq 2h(y) + \frac{h(y)^2}{t} - 2d(f_y(t), f_x(t))(1 + \frac{h(y)}{t}) \cos(\beta_t) \end{aligned}$$

Taking the limit as t approaches infinity, we obtain the desired inequality. \square

Example 10.14. Let X be a Hilbert space. Then \overline{X} is the ‘‘oriented projective space’’ obtained by adjoining to X a point at infinity for every equivalence class of geodesic rays in X . Two geodesics are asymptotic if and only if they are parallel. The horofunctions are precisely the linear functionals on X .

11. A CHARACTERIZATION OF HADAMARD SPACES

Our goal for the remainder of these notes is the proof of the Cartan-Hadamard theorem. Roughly speaking, this asserts that if a space satisfies non-positive curvature conditions *locally*, then its universal cover has non-positive curvature *globally*.

In this section, we shall begin our work by proving a very weak version of the Cartan-Hadamard theorem. Later, we will do some hard work to show that the hypotheses of this weak version are satisfied, and thereby deduce that certain metric spaces are Hadamard.

Theorem 11.1. *Let (X, d) be a complete, nonempty metric space with the following properties:*

- For each $x \in X$, there exists a continuous function $g_x : X \times [0, 1] \rightarrow X$ such that for each $y \in X$, $g_x|_{\{y\} \times [0, 1]}$ is a geodesic of speed $d(x, y)$ joining x to y .
- Every point $x \in X$ contains a closed neighborhood which is a Hadamard space.

Then X is a Hadamard space.

The proof will occupy the rest of this section. First, note that since X is Hadamard in a neighborhood of each point, the angle between any two geodesics originating from a point is well-defined and has the usual properties.

By a *geodesic triangle* in X we shall mean a collection of three vertices and three geodesic segments joining the vertices pairwise. We have not yet shown that such a triangle is determined by its vertices, but for simplicity of notation we shall simply denote such a triangle by the triple of vertices $(x, y, z) \in X^3$.

Definition 11.2. A geodesic triangle Δ in X is *good* if each of its vertex angles is \leq the corresponding angle of a Euclidean triangle with the same side lengths.

Theorem 11.3. Let $\Delta = (x, y, z)$ be a geodesic triangle in X , and let w be a point on the segment joining y to z . Choose a geodesic joining x to w . If the triangles (x, y, w) and (x, w, z) are both good, then (x, y, z) is good.

Proof. Let Δ_0 denote a Euclidean triangle with the same side lengths as the triangle (x, y, w) , Δ_1 a Euclidean triangle with the same side lengths as (x, w, z) . The vertex angles of Δ_0 and Δ_1 are both larger than that of their counterparts in X , whose sum is $\geq \pi$ by the triangle inequality. Let Q denote the plane quadrilateral obtained by joining Δ_0 and Δ_1 on opposite sides of the segment \overline{xw} . Then by assumption (and the triangle inequality in the case of the vertex x), the angles of the geodesic triangle (x, y, z) are majorized by the corresponding angles of the quadrilateral Q . The Euclidean triangle Δ with the same side lengths as (x, y, z) may also be obtained from the quadrilateral Q by “straightening” the angle at w . Since the quadrilateral is concave at w , elementary plane geometry shows that the other angles are only increased by this “straightening”. \square

Theorem 11.4. Let $x, y, z \in X$, and assume that y and z are joined by a geodesic segment $p : [0, 1] \rightarrow X$. Join x to y and z using the geodesic segments $t \mapsto g_x(y, t)$ and $t \mapsto g_x(z, t)$. Then the triangle (x, y, z) is good.

Proof. Consider the function $h : [0, 1] \times [0, 1] \rightarrow X$ given by $h(t, s) = g_x(p(t), s)$. Then the image of h is compact, so there exists $\epsilon > 0$ such that the closed ball of radius ϵ around any point in the image of h is a Hadamard space. Choose δ so that $d(h(t, s), h(t', s')) \leq \epsilon$ whenever $|t - t'|, |s - s'| \leq \delta$, and choose a large integer n such that $\frac{1}{n} \leq \delta$.

Let $\Delta_{i,j}$ denote the triangle spanned by $(h(\frac{i}{n}, \frac{j}{n}), h(\frac{i}{n}, \frac{j+1}{n}), h(\frac{i+1}{n}, \frac{j+1}{n}))$ and let $\Delta'_{i,j}$ denote the triangle spanned by $(h(\frac{i}{n}, \frac{j}{n}), h(\frac{i+1}{n}, \frac{j}{n}), h(\frac{i+1}{n}, \frac{j+1}{n}))$. The vertices of each of these triangles are contained in Hadamard ϵ -balls of X , hence there exist unique minimizing geodesics joining them. Since each of these triangles is contained in a Hadamard space, each is good. Now the triangle (x, y, z) is obtained by gluing these triangles together. Thus (x, y, z) is good by repeated application of Theorem 11.3. \square

We can now give a proof of Theorem 11.1.

Proof. Since X is nonempty and complete, it suffices to show that every pair of points $y, z \in X$ possess a midpoint with the appropriate properties. Join y and z by a geodesic $p : [0, 1] \rightarrow X$, and let $m = p(\frac{1}{2})$. To complete the proof, we must show that $d(x, m)^2 \leq \frac{d(x, y)^2}{2} + \frac{d(x, z)^2}{2} - \frac{d(y, z)^2}{4}$.

Using the function g_x , we obtain geodesic triangles (x, y, m) and (x, m, z) which share a common side joining x to m . By Theorem 11.4, each of these triangles is hyperbolic. Let α and β denote the vertex angles of (x, y, m) and (x, m, z) at the vertex m . Then by the law of cosines we obtain the inequality

$$\begin{aligned} d(x, y)^2 &\geq d(x, m)^2 + d(y, m)^2 - 2d(x, m)d(y, m) \cos(\alpha) \\ d(x, z)^2 &\geq d(x, m)^2 + d(z, m)^2 - 2d(x, m)d(z, m) \cos(\beta) \end{aligned}$$

By the triangle inequality, we have $\alpha + \beta \geq \pi$, so that

$$2d(x, m)d(y, m) \cos(\alpha) + 2d(x, m)d(z, m) \cos(\beta) = d(x, m)(b-a)(\cos(\alpha) + \cos(\beta)) \leq 0$$

Adding this to the above inequalities, we obtain the desired result. \square

12. METRIZED TOPOI

We now introduce the rather strange notion of a metric topos. The reader who does not like the theory of topoi is invited to replace “topos” by “space” throughout our discussion. In this case, many of our arguments become quite a bit simpler, both technically and psychologically. However, the extra generality afforded by the notion of a metric topos will allow our version of the Cartan-Hadamard theorem to be applied to orbifolds and other exotic beasts.

Let us emphasize that the definition of a metric topos is *not* intended to be a useful general notion. It is simply a nice way of encoding the relevant structure of a topos which is “locally Hadamard” (and it is these topoi that we shall be interested in).

Let X be a (Grothendieck) topos. By a *metric* on X we shall mean a process which associates to each geometric morphism $p : [t_-, t_+] \rightarrow X$ of an interval into X , a length $L(p) \in [0, \infty]$, subject to the following conditions:

- Let $p : [t_-, t_+] \rightarrow X$ and $p' : [t'_-, t'_+] \rightarrow X$ be two geometric morphisms. If there is a homeomorphism $q : [t_-, t_+] \simeq [t'_-, t'_+]$ such that $p \simeq p' \circ q$, then $L(p) = L(p')$.
- If $p : [t_0, t_n] \rightarrow X$ is a geometric morphism and $t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n$, then $L(p) = L(p|[t_0, t_1]) + \dots + L(p|[t_{n-1}, t_n])$.

In the statement above we understand that a sum of terms is infinite if any individual term is infinite.

Note that given any geometric morphism $X \rightarrow Y$, a metric on Y induces by composition a metric on X . In particular, given any morphism $E \rightarrow E'$ of sheaves on X and any metric on the “slice topos” $X_{E'}$, we obtain by composition a metric on X_E . Thus we obtain a presheaf on X which associates to each object E in X the collection of all metrics on X_E . (Strictly speaking, we should worry that this collection may not be a set. We shall ignore this issue.)

Theorem 12.1. *For any topos X , the presheaf of metrics on X is a sheaf.*

Proof. It is clear that to specify a metric on a disjoint union of topoi is equivalent to specifying a metric on each topos individually. Thus it suffices to check that for any epimorphism $j : E \rightarrow E'$ in X , we obtain a bijection from the collection of

metrics on $X_{E'}$ to the collection of metrics X_E for which the pair of “restrictions” to $X_{E \times_{E'} E}$ agree. Replacing X by $X_{E'}$, we may assume that E' is the terminal object of X (to simplify notations).

To show that the natural map is injective, we must show that any metric L on X is determined by its restriction to X_E . Indeed, given any geometric morphism $p : [t_-, t_+] \rightarrow X$, we obtain a sheaf $p^*(E)$ on $[t_-, t_+]$ with nonempty stalks. Then we may subdivide $[t_-, t_+]$ into intervals $[t_0, t_1], \dots, [t_{n-1}, t_n]$ over which the sheaf p^*E admits sections. Then $L(f)$ is determined by $L(p|[t_i, t_{i+1}])$, which is in turn determined by the metric on X_E since $p|[t_i, t_{i+1}]$ factors through X_E .

To show surjectivity, we must show that any metric L_E on X_E whose “restrictions” to $X_{E \times E}$ agree is the “restriction” of a metric on X . We define a metric L on X as follows: given a geometric morphism $f : [t_-, t_+] \rightarrow X$, we may subdivide $[t_-, t_+]$ into subintervals $[t_0, t_1], \dots, [t_{n-1}, t_n]$ such that $p|[t_i, t_{i+1}]$ factors through a geometric morphism $p_i : [t_i, t_{i+1}] \rightarrow X_E$. We then set $L(f) = L_E(p_0) + \dots + L_E(p_{n-1})$. The compatibility condition on L_E guarantees that $L(f)$ does not depend on the factorizations chosen. One checks easily that $L(f)$ does not depend on the subdivision chosen, and is a metric on X . \square

Let X be a metric topos, and let $p : [t_-, t_+] \rightarrow X$ be a geometric morphism. Then we define $L(p)$ to be the supremum of $L(p|[t_-, t])$ as t ranges over (t_-, t_+) .

Remark 12.2. For any space Y , the collection of geometric morphisms from Y into a topos X form a category. In general, morphisms in this category need not be isomorphisms. However, we shall have no need for this feature, and shall consider only *invertible* transformations between geometric morphisms.

Definition 12.3. Let X be a topos with a metric L . Let $\text{Hom}_{\leq D}([0, 1], X)$ denote the groupoid of geometric morphisms from $[0, 1]$ to X having length $\leq D$. Similarly, let $\text{Hom}_{\leq D}([0, 1), X)$ denote the groupoid of geometric morphisms f from $[0, 1)$ to X such that $L(f) \leq D$. There is a natural “restriction” functor $T : \text{Hom}_{\leq D}([0, 1], X) \rightsquigarrow \text{Hom}_{\leq D}([0, 1), X)$.

We shall say that X is

- *weakly separated* if T is faithful for every $D \geq 0$.
- *separated* if T is fully faithful for every $D \geq 0$.
- *complete* if T is an equivalence of categories for every $D \geq 0$.

Lemma 12.4. *Let X be a weakly separated (separated, complete) metrized topos, and let E be a locally constant sheaf on X . Then X_E is weakly separated (separated, complete). If E surjects onto the terminal sheaf, then the converse holds.*

Proof. For any morphism $p : [0, 1] \rightarrow X$ or $p : [0, 1) \rightarrow X$, the pullback p^*E is locally constant, hence constant. Let $M(p)$ denote its constant value (for example, we may take $M(p)$ to be the stalk of this sheaf at zero); then M induces functors from $\text{Hom}_b([0, 1], X)$ and $\text{Hom}_b([0, 1), X)$ to the category of sets. The category $\text{Hom}_b([0, 1], X_E)$ has as objects pairs $p \in \text{Hom}_b([0, 1], X)$ and $m \in M(p)$, and we have a similar description for $\text{Hom}_b([0, 1), X_E)$. Moreover it is clear that for $f : [0, 1] \rightarrow X$, $M(p|[0, 1)) \simeq M(p)$. From this description it is clear that the induced functor $T_E : \text{Hom}_b([0, 1], X) \rightarrow \text{Hom}_b([0, 1), X)$ is faithful (fully faithful, an equivalence) if T is. The converse follows also, noting that if E surjects onto the terminal sheaf then $M(p)$ is nonempty for any p . \square

The fundamental examples of metric topoi are those which are associated to metric spaces (X, d) . For any metric space (X, d) , we may consider X as a topological space, and the category of sheaves on X is a topos which we shall denote also by X . The topos X has a natural metric, which may be described as follows:

Given a continuous map $p : [t_-, t_+] \rightarrow X$, we define the *length* $L(f)$ of f to be the supremum over all subdivisions $t_- = t_0 < t_1 < \dots < t_n = t_+$ of $[t_-, t_+$ of the quantity $d(p(t_0), p(t_1)) + \dots + d(p(t_{n-1}), p(t_n))$.

Remark 12.5. In view of the triangle inequality, this sum will only increase as we pass to finer subdivisions of $[t_-, t_+]$. Thus the supremum exists in a strong sense.

Let us now investigate the possibility of obtaining metric spaces from metric topoi. Let X be a metric topos. We will let $|X|$ denote the collection of isomorphism classes of points of X . We shall assume that this collection forms a set (this will be automatic in all cases of interest). We define a function $d : |X| \times |X| \rightarrow [0, \infty]$ as follows: $d(x, y)$ is the infimum of $L(p)$, where f ranges over the collection of all paths $p : [0, 1] \rightarrow X$ with $p\{0\} \simeq x$ and $p\{1\} \simeq y$. Then it is clear that d has the following properties:

- For all $x \in |X|$, $d(x, x) = 0$.
- For all $x, y, z \in |X|$, $d(x, z) \leq d(x, y) + d(y, z)$.

Thus d defines a (pseudo)-metric on $|X|$; we will consider $|X|$ as a topological space with the induced topology.

Lemma 12.6. *Let X be a metric topos. Then:*

- *The pseudo-metric d on $|X|$ is interior. That is, $d(x, y)$ is the infimum of the d -length of all continuous paths from x to y , for any $x, y \in |X|$.*
- *If X is complete, then $(|X|, d)$ is complete (but not necessarily separated).*

Proof. Let $x, y \in |X|$, and let D be the infimum of the length of all paths from x to y . Then we obviously have $d(x, y) \leq D$. The reverse inequality follows since $d(x, y)$ is the infimum of the length of all paths joining x and y in X , and any such path yields a path in $|X|$ having the same length.

Now suppose that X is complete. Let $\{x_i\}$ be a Cauchy sequence in $|X|$. Passing to a subsequence if necessary, we may assume that $d(x_i, x_{i+1}) \leq \frac{1}{2^{i+1}}$. It follows that we can find a path $p_i : [1 - \frac{1}{2^{i+1}}, 1 - \frac{1}{2^i}] \rightarrow X$ of length $\leq \frac{1}{2^i}$. with $p_i\{1 - \frac{1}{2^i}\} \simeq x_i$, $p_i\{1 - \frac{1}{2^{i+1}}\} \simeq x_{i+1}$ for each i . Concatenating, we get a path $p : [0, 1] \rightarrow X$ of bounded length. Since X is complete, we may extend this (up to isomorphism) to a map from $[0, 1]$ to X having the same length; then $p\{1\}$ is a limit of the Cauchy sequence $\{x_i\}$ in $|X|$. \square

Example 12.7. Suppose (X, d) is a metric space. Then we may consider X as a metric topos. The space $|X|$ is equal to X as a set. However, the metric is different; the distance between two points in $|X|$ is the infimum of the length of all continuous paths joining them in X . This is at least as great as the distance between the two points in X , and is in general larger. However, if X has the property that any two points can be joined by a minimizing geodesic, then the two metrics agree.

We define a map $p : [t_-, t_+] \rightarrow X$ into a metric topos to be a *geodesic of speed* D if $L(p|[t, s]) = D(s - t)$ for all $t \leq s$, $t, s \in [t_-, t_+]$. In case X is the metric topos associated to a metric space, this is a slightly weaker condition than the usual

definition of a geodesic, which is equivalent in all cases of interest (for example, if X is Hadamard).

A *closed geodesic* is a geodesic $p : [t_-, t_+] \rightarrow X$ together with an isomorphism $\phi : p|_{\{t_-\}} \simeq p|_{\{t_+\}}$. A closed geodesic is *trivial* if $t_+ = t_-$ and ϕ is the identity.

13. LOCALLY METRIC TOPOI

We can say very little about metric topoi in general. However, we can say much more about metric topoi which are associated to metric spaces. In this section, we will investigate metric topoi which are of this nature *locally*.

Definition 13.1. A metric space X is *convex* if any two points of X can be joined by a minimizing geodesic, and for any pair of geodesics $p, p' : [0, 1] \rightarrow X$, the function $d(p(t), p'(t))$ is convex. A metric space X is *locally complete* if for every point $x \in X$, there exists $r > 0$ such that the closed ball $B_r(x)$ is complete.

Note that in a convex metric space, an open ball around any point is also convex. The only tricky point is to show that a minimizing geodesic p whose endpoints lie in the ball is entirely contained in the ball, and this follows from the convexity statement in the degenerate case where p' is the “constant” geodesic at the center of the ball. It follows that a convex metric space has a basis for its topology consisting of open subsets which themselves convex.

Definition 13.2. A metric topos X is *locally convex* (*locally complete*) if the terminal object of X admits a covering by sheaves $\{E_\alpha\}$ such that each metric topos X_{E_α} is associated to a convex (locally complete) metric space.

Remark 13.3. Since convex (locally complete) metric spaces admit bases for their topology consisting of convex (locally complete) open sets, it follows that a locally convex (locally complete) metric topos possesses a set of generators $\{E_\alpha\}$ such that each X_{E_α} is isomorphic to a metric topos associated to a convex (locally complete) metric space.

Remark 13.4. Since the property of local completeness for metric spaces is a local property, a metric topos is locally complete if and only if it satisfies the following two conditions:

- There is cover of the terminal object by sheaves $\{E_\alpha\}$ such that each X_{E_α} is a metric topos associated to a metric space.
- For *any* metric space V and any etale isometry $\pi : V \rightarrow X$, V is locally complete.

In particular, if a metric topos is both locally convex *and* locally complete, then it is covered by etale isometries $\pi : V_\alpha \rightarrow X$ where V_α are convex, locally complete metric spaces.

Lemma 13.5. *Let X be a separated metric topos, let V be a metric space, let $\pi : V \rightarrow X$ be an etale isometry. Suppose $x \in V$ is such that the closed ball $B_r(x)$ of radius r about x is complete. Then for any path $p : [0, 1] \rightarrow X$ with $L(p) \leq r$ and any isomorphism $\alpha_0 : p|_{\{0\}} \simeq \pi|_{\{x\}}$, there exists a path $p' : [0, 1] \rightarrow V$ with $p'(0) = x$ and an isomorphism $\alpha : p \simeq \pi \circ p'$ which prolongs α_0 .*

Proof. Let E be the sheaf on X which classifies the etale map π . Constructing the pair (p', α) is equivalent to finding a global section of the sheaf $p^*(E)$. The

isomorphism α_0 provides us with an element s_0 of the stalk of this sheaf at 0. We need to show that s_0 can be extended to a global section of $p^*(E)$.

Consider the collection of all pairs (U, s_U) , where $U \subseteq [0, 1]$ is closed downwards and s_U is a section of $p^*(E)$ over U whose germ at 0 is s_0 . This collection is partially ordered: let $(U, s_U) \leq (V, s_V)$ mean that $U \subseteq V$ and $s_V|_U = s_U$. It is clear that this collection satisfies the hypotheses of Zorn's lemma, so it has a maximal element (U, s_U) . If $U = [0, 1]$, then we are done (with the existence half of the proof). Otherwise, U is either of the form $[0, t)$ or $[0, t]$ for some $t \leq 1$. The latter case is impossible, since a section of a sheaf over $[0, t]$ can always be extended to a section over an open neighborhood of $[0, t]$. Therefore $U = [0, t)$ and s_U classifies a map $p'_{<t} : [0, t) \rightarrow V$. Since p has length $\leq r$, so does $p'_{<t}$. Then, since the closed ball $B_r(x)$ is complete, it follows that $p'_{<t}$ can be extended to a map $p'_{\leq t} : [0, t] \rightarrow X$ having the same length. Since X is separated, it follows that the isomorphism $p|_{[0, t)} \simeq \pi \circ p'_{<t}$ can be extended uniquely to an isomorphism $p|_{[0, t]} \simeq \pi \circ p'_{\leq t}$. This contradicts the maximality of U and completes the proof of the existence.

For the uniqueness, suppose we are given two global sections s, s' of $p^*(E)$ which have the same germ at 0. If $s \neq s'$, then there is a maximal open interval $[0, t)$ over which s and s' agree. Since V is Hausdorff, the agreement of the maps $[0, t) \rightarrow V$ corresponding to s and s' forces the agreement of these maps on $[0, t]$. Then the fact that X is weakly separated implies that $s|[0, t] = s'|[0, t]$, a contradiction. \square

Theorem 13.6. *Let X be a metric topos which admits an etale covering by metric spaces. Then there exists a unique (up to unique isomorphism) geometric morphism $\phi : X \rightarrow |X|$ with the property that for any $x \in |X|$, the composite map $\{x\} \rightarrow X \rightarrow |X|$ is the identity.*

Proof. Let $\{E_\alpha\}$ be a collection of generators for X such that each X_{E_α} is isomorphic to a metric topos associated to a convex metric space V_α . Then we obtain a map $V_\alpha \rightarrow |X|$ which sends a point $x \in V_\alpha$ to the isomorphism class of the point x . This map decreases distances, so it is continuous. Consequently we get a canonical geometric morphism $\pi_\alpha : X_{E_\alpha} \rightarrow |X|$. This construction is completely natural in E_α ; hence by descent we obtain a map $\pi : X \rightarrow |X|$ with the desired property. \square

Theorem 13.7. *Let X be a separated, locally convex, locally complete metric topos. Assume that X has no nontrivial closed geodesics. Then $\phi : X \rightarrow |X|$ is an isomorphism of metric topoi.*

Proof. We will first show that ϕ is an etale isometry. Since X admits an etale covering by maps $\pi : V \rightarrow X$ with V a convex, locally complete metric space, it suffices to show that for any such π , the map $\phi \circ \pi$ is an etale isometry. Choose any point $x \in V$, and choose $r > 0$ such that the closed ball $B_{3r}(x)$ of radius $3r$ about x is complete. We get an induced map of open balls $\pi' : B_{<r}(x) \rightarrow B_{<r}(\pi(x))$; it suffices to show that π' is an isometry.

First we show that π' is surjective. Choose $y \in |X|$ with $d(\pi(x), y) < r$. Then there exists a path $p : [0, 1] \rightarrow X$ with $p|_{\{0\}} \simeq \pi(x)$, $p|_{\{1\}} \simeq y$, and $L(p) < r$. By Lemma 13.5, we deduce that p factors through $B_r(U)$, which shows that y lies in the image of π' .

We now show that π' preserves distances. It is obvious that π' is a contraction. Choose $y, z \in V$. To show that $d(\pi'(y), \pi'(z)) \geq d(y, z)$, it suffices to show that for every $\epsilon < r$, we have $d(\pi'(y), \pi'(z)) + \epsilon \geq d(y, z)$. Now there exists a path

$p : [0, 1] \rightarrow X$ with $p|_{\{0\}} \simeq y$, $p|_{\{1\}} \simeq z$, and $L(p) \leq d(\pi'(x), \pi'(y)) + \epsilon$. Since $d(\pi'(x), \pi'(y)) \leq d(x, y) \leq 2r$, we have $L(f) \leq 3r$. Applying Lemma 13.5, we deduce that p factors as $\pi \circ p'$ for some map $p' : [0, 1] \rightarrow V$ with $p'(0) = y$. If $p'(1) = z$, then $L(f) \geq d(y, z)$ and we are done. Otherwise, we may join $p'(1)$ to z by a geodesic $p'' : [0, 1] \rightarrow X$. Since $\pi'(p'(1)) = \pi'(1)$, there is an isomorphism $\pi \circ p''|_{\{0\}} \simeq \pi \circ p''|_{\{1\}}$, so that $\pi \circ p''$ is a nontrivial closed geodesic in X , a contradiction.

Now we claim that ϕ is an isomorphism. We have established that ϕ is an étale isometry, so that ϕ exhibits X as the total space of some stack on $|X|$. Any nontrivial automorphism of a point of X yields a nontrivial closed geodesic. Thus the automorphism group of any point of X is trivial, so that ϕ exhibits X as the total space of some sheaf on $|X|$. Since ϕ is bijective on points by construction, it is an isomorphism. \square

From this, we see that if X is separated, locally convex, locally complete, and has no nontrivial closed geodesics, then $(|X|, d)$ is a separated pseudo-metric space. Indeed, suppose $x, y \in |X|$ satisfy $d(x, y) = 0$. Then any open set in $|X|$ which contains x also contains y . Since $|X|$ and X are isometric, we may ensure that such an open ball is actually a metric space, hence separated.

14. THE MAIN STEP

In this section, we shall establish a technical result which is the key step in the proof of the Cartan-Hadamard theorem.

Let $0 = t_0 < t_1 < \dots < t_n = 1$ be a partition of the interval $[0, 1]$. Suppose that for $0 \leq i < n$, we are given metric spaces V_i which are convex and locally complete, together with convex open subsets $V_i^+ \subseteq V_i$ for $0 \leq i < n - 1$ and $V_i^- \subseteq V_i$ for $0 < i < n$, and isometries $\gamma_i : V_i^+ \rightarrow V_{i+1}^-$ for $0 \leq i \leq n - 1$. We shall suppose this data fixed throughout this section.

Suppose given an interval $[s_-, s_+] \subseteq [0, 1]$ and a collection of paths $p_i : [s_-, s_+] \cap [t_i, t_{i+1}] \rightarrow V_i$ for $0 \leq i < n$. We shall say that $p = \{p_i\}$ is a *broken geodesic of speed D* if the following conditions hold:

- Each p_i is a geodesic of speed D .
- For $0 < i < n$ such that $t_i \in [s_-, s_+]$, there exists a real number $\delta > 0$ and a geodesic $q_i : [t_i - \delta, t_i + \delta] \rightarrow V_{i-1}^+$ of speed D such that $q_i|_{[t_i - \delta, t_i]} = p_i|_{[t_i - \delta, t_i]}$ and $\gamma_i \circ q_i|_{[t_i, t_i + \delta]} = p_{i+1}|_{[t_i, t_i + \delta]}$.

Note that the functions q_i are uniquely determined; their existence means simply that p_{i+1} picks up where p_i left off, and continues as a geodesic.

Lemma 14.1. *Let p and p' be broken geodesics defined on $[s_-, s_+]$. Let $f : [s_-, s_+]$ be given by $f(t) = d(p_i(t), p'_i(t))$, where $t \in [t_i, t_{i+1}]$. Then f is a well-defined, convex function of t .*

Proof. Since convexity is a local property, it suffices to work locally near each $t \in [s_-, s_+]$. If $t = t_i$, then we check the well-definedness and convexity using the functions q_i, q'_i asserted to exist in the definition of a broken geodesic. For other values of t , the result is immediate. \square

Theorem 14.2. *Let p be a broken geodesic of speed D defined on $[0, 1]$. Then there exists a constant ϵ such that for any pair of points $x \in V_0, y \in V_{n-1}$ such that*

$d(x, p_0(0)), d(y, p_{n-1}(1)) \leq \epsilon$, there is a broken geodesic defined on $[0, 1]$ joining x to y .

Proof. Choose ϵ so that the closed ball of radius 2ϵ about $p_i(t)$ is complete for $t \in [t_i, t_{i+1}]$, and this ball is contained in V_i^+ for $t = t_{i+1}$ if $i < n$, and contained in V_i^- for $t = t_i$ if $i > 0$.

Now consider the following assertion $Q(k)$ concerning a real number k :

- For any interval $[s_-, s_+] \subseteq [0, 1]$ with $s_+ - s_- \leq k$, $s_- \in [t_i, t_{i+1}]$, $s_+ \in [t_j, t_{j+1}]$, and any pair of points $x \in V_i$, $y \in V_j$ with

$$d(x, p_i(s_-)), d(y, p_j(s_+)) < \epsilon$$

there exists a broken geodesic joining x and y defined on $[s_-, s_+]$.

Choose δ so that $D\delta < \epsilon$ and $\delta < |t_{i+1} - t_i|$ for each i , then we claim $Q(\delta)$ holds. For suppose $c_+ - c_- \leq \delta$, with $c_- \in [t_i, t_{i+1}]$ and $c_+ \in [t_j, t_{j+1}]$. If $i = j$, then we can just choose a geodesic joining x to y in V_i . Otherwise, we must have $j = i + 1$. Then $d(y, p_j(t_j)) \leq d(y, p_j(c_+)) + d(p_j(c_+), p_j(t_j)) \leq 2\epsilon$, so that $p_j(t_j) \in V_j^-$. Then we may join x to $\gamma_i^{-1}(y)$ by a geodesic, which uniquely determines a broken geodesic from x to y .

We have thus shown that there is a positive constant k such that $Q(k)$ holds. If we show that $Q(1)$ holds, then the theorem will be proved. To complete the proof, we shall show that $Q(k)$ implies $Q(\frac{3k}{2})$ for all k . So assume that $Q(k)$ holds, and let $[s_-, s_+] \subseteq [0, 1]$ with $s_+ - s_- \leq \frac{3k}{2}$, and let x, y be as above. Let $s = \frac{2s_-}{3} + \frac{s_+}{3}$ and $s' = \frac{s_-}{3} + \frac{2s_+}{3}$. Let us suppose that $s_- \in [t_i, t_{i+1}]$, $s \in [t_j, t_{j+1}]$, $s' \in [t_k, t_{k+1}]$, $s_+ \in [t_l, t_{l+1}]$. We construct a sequence of auxiliary points $\{a_m\}_{m \geq 0}$ in V_j and $\{b_m\}_{m \geq 0}$ in V_k , such that $d(a_m, p_j(s)) \leq \epsilon$, $d(b_m, p_k(s')) \leq \epsilon$. Let $a_0 = p_j(s)$, $b_0 = p_k(s')$. Assuming that a_n and b_n have been constructed, the assumption $Q(k)$ enables us to construct broken geodesics q^n defined on $[s_-, s']$ joining x to b_n , and r^n defined on $[s, s_+]$ joining a_n to y . We let $a_{n+1} = (q^n)_j(s)$, $b_{n+1} = (r^n)_k(s')$. Since the distance between q^n and p is bounded by ϵ at its endpoints, it is bounded by ϵ everywhere by Lemma 14.1 and hence $d(a_{n+1}, p_j(s)) \leq \epsilon$, and similarly $d(b_{n+1}, p_k(s')) \leq \epsilon$.

Now we claim that $d(a_n, a_{n+1}) \leq \frac{\epsilon}{2^{n+1}}$ and $d(b_n, b_{n+1}) \leq \frac{\epsilon}{2^{n+1}}$. This is proven simultaneously by induction on n . For $n = 0$, we apply Lemma 14.1 to the geodesics q^0 and $p|_{[s_-, s']}$ to deduce $d(a_0, a_1) \leq \frac{d(x, p_i(s_-)) + d(b_0, b_0)}{2} \leq \frac{\epsilon}{2}$, and similarly $d(b_0, b_1) \leq \frac{\epsilon}{2}$. In the general case, we apply the same argument to the pair of geodesics q^n and q^{n-1} to deduce the inequality $d(a_n, a_{n+1}) \leq \frac{\epsilon}{2^{n+1}}$, and to r^n and r^{n-1} to deduce the inequality $d(b_n, b_{n+1}) \leq \frac{\epsilon}{2^{n+1}}$.

It follows that the sequences $\{a_n\}$ and $\{b_n\}$ converge to points $a \in V_j$, $b \in V_k$. A convexity argument using Lemma 14.1 now shows that the broken geodesics $\{q^n\}$ and $\{r^n\}$ converge uniformly to broken geodesics q joining x to b , and r joining a to y . Moreover, continuity forces $q_j(s) = a$, $r_k(s') = b$. It follows that $q|_{[s, s']}$ and $r|_{[s, s']}$ are both broken geodesics joining a to b . Using Lemma 14.1 once again, we deduce that $q|_{[s, s']} = r|_{[s, s']}$. Thus we can glue q and r along $[s, s']$, and thereby obtain a broken geodesic joining x to y . This completes the proof. \square

Lemma 14.3. *Let p and p' be broken geodesics of speed D and D' with $p_0(0) = p'_0(0)$. Then $D' \leq D + d(p_{n-1}(1), p'_{n-1}(1))$.*

Proof. By convexity, we have $d(p_0(\epsilon), p'_0(\epsilon)) \leq \epsilon d(p_{n-1}(1), p'_{n-1}(1))$ for $\epsilon \leq t_1$. Thus it suffices to show that $D'\epsilon \leq D\epsilon + d(p_0(\epsilon), p'_0(\epsilon))$. This follows immediately from the triangle inequality since $D'\epsilon = d(p'_0(\epsilon), p'_0(0))$ and $D\epsilon = d(p_0(\epsilon), p_0(0))$. \square

Lemma 14.4. *Let p be a broken geodesic, and let ϵ denote the constant of Theorem 14.2. Let p', p'' be broken geodesics of speeds D' and D'' with*

$$d(p_0(0), p'_0(0)), d(p_0(0), p''_0(0)), d(p_{n-1}(1), p'_{n-1}(1)), d(p_{n-1}(1), p''_{n-1}(1)) \leq \epsilon$$

Then $|D' - D''| \leq d(p'_0(0), p''_0(0)) + d(p'_{n-1}(1), p''_{n-1}(1))$.

Proof. Theorem 14.2 implies that there exists a broken geodesic q of speed D joining $p'_0(0)$ to $p''_{n-1}(0)$. Then Lemma 14.3 yields the inequalities

$$D' \leq D + d(p'_{n-1}(1), p''_{n-1}(1))$$

$$D \leq D'' + d(p'_0(0), p''_0(0))$$

Adding these inequalities, we obtain $D' \leq D'' + d(p'_0(0), p''_0(0)) + d(p'_{n-1}(1), p''_{n-1}(1))$. The reverse inequality is proved in the same way. \square

We now want to establish a version of Theorem 14.2 which implies that the geodesics depend continuously on their endpoints. First we must formulate the meaning of this statement. Let Y be a topological space. A *broken Y -geodesic* shall mean a collection of continuous maps $P_i : Y \times [t_i, t_{i+1}] \rightarrow V_i$ such that for each $y \in Y$, the restriction $\{P_i|_{\{y\} \times [t_i, t_{i+1}]}\}$ is a broken geodesic.

Lemma 14.1 shows that a broken geodesic defined on $[0, 1]$ is uniquely determined by its endpoints. Hence any Y -geodesic P is uniquely determined by the restrictions $P_0|_{Y \times \{0\}}$ and $P_{n-1}|_{Y \times \{1\}}$. We are interested in the following question: given two maps $\psi_0 : Y \rightarrow V_0$ and $\psi_1 : Y \rightarrow V_{n-1}$, when can they be joined by a broken Y -geodesic?

Theorem 14.5. *Let Y be a topological space, let $\psi_0 : Y \rightarrow V_0$ and $\psi_1 : Y \rightarrow V_{n-1}$ be two continuous maps. Suppose that there is a broken geodesic p joining $\psi_0(y)$ and $\psi_1(y)$ for some point $y \in Y$. Then there is an open neighborhood U of y and a broken U -geodesic joining $\psi_0|_U$ and $\psi_1|_U$.*

Proof. Let ϵ be the constant of Theorem 14.2. We now take U to be any neighborhood of y such that $\psi_j(U)$ is contained in an ϵ ball around $\psi_j(y)$ for $j = 0, 1$. Then Theorem 14.2 allows us to define, for each $u \in U$, a broken geodesic p^u joining $\psi_0(u)$ to $\psi_1(u)$. We now define $P_i : U \times [t_i, t_{i+1}] \rightarrow V_i$ by the formula $P_i(u, t) = p_i^u(t)$. The only thing that requires proof is that P_i is continuous. By Lemma 14.4, each p^u has speed at most $D + 2\epsilon$, where D is the speed of p . Moreover, by convexity, we see that the distance between $P_i(u, t)$ and $P_i(u', t)$ is bounded by the larger of $d(\psi_0(u), \psi_0(u'))$ and $d(\psi_1(u), \psi_1(u'))$. Thus $d(P_i(u, t), P_i(u', t')) \leq d(\psi_0(u), \psi_0(u')) + d(\psi_1(u), \psi_1(u')) + |t - t'|(D + 2\epsilon)$. This shows that P_i is continuous and completes the proof. \square

15. SHEAVES OF GEODESICS

Our goal now is to find conditions which will guarantee that a metric topos is free of nontrivial closed geodesics. Closed geodesics are a global phenomenon; in order to investigate them using local hypotheses we shall need to consider other geodesics as well. That is the goal of this section.

Let X be a metric topos, Y a topological space, and let $\phi_0, \phi_1 : Y \rightarrow X$ be two geometric morphisms. We let \mathcal{F}_Y denote the stack on Y which assigns to each open set $U \subseteq Y$ the groupoid whose objects consist of geometric morphisms $\Phi : U \times [0, 1] \rightarrow X$ together with isomorphisms $\alpha_0 : \Phi|_{U \times \{0\}} \simeq \phi_0|_U$ and $\alpha_1 : \Phi|_{U \times \{1\}} \simeq \phi_1|_U$, such that $\Phi|_{\{y\} \times [0, 1]}$ is a geodesic for all $y \in U$. Note that \mathcal{F}_Y depends on the morphisms ϕ_0 and ϕ_1 as well as on the space Y , but for simplicity we shall neglect this fact in our notation.

Lemma 15.1. *Let X be a metric topos which admits an etale covering by metric spaces, let $f : Y \rightarrow X$ be a geometric morphism from a topological space Y , and let $\alpha : f \simeq f$ be an automorphism of f . If $\alpha|_{\{y\}}$ is the identity for some point $y \in Y$, then $\alpha|_U$ is the identity for some neighborhood U of y .*

Proof. Choose an etale isometry $\pi : V \rightarrow X$ where V metric space ball in some Hadamard space and $f|_{\{y\}}$ lifts to V . Shrinking Y if necessary, we may assume that f factors through a map $f' : Y \rightarrow V$. Then to give an automorphism of f is precisely to give a lifting of $f \times f$ along the map $V \times_X V \rightarrow V \times V$. Such a lifting is determined by the corresponding map $Y \rightarrow V \times_X V$, where the latter space is etale over V . It follows that if such two such automorphisms agree at a point of Y , then they agree in a neighborhood of that point. \square

Lemma 15.2. *Let X be a separated metric topos which admits an etale covering by metric spaces, and let $f : [0, 1] \rightarrow X$ be a geometric morphism of finite length. Suppose given an automorphism $\alpha : f \simeq f$. If α induces the identity at any point of $[0, 1]$, then α is the identity.*

Proof. The automorphisms of f form a sheaf on $[0, 1]$. Thus there exists a maximal open set U over which α is the identity. Since X is assumed separated, we see that U is closed. Since $[0, 1]$ is connected, it suffices to prove that U is nonempty. This follows immediately from the preceding lemma, since α is the identity when restricted to some point. \square

Theorem 15.3. *For every pair of geometric morphisms $\phi_0, \phi_1 : Y \rightarrow X$, the stack \mathcal{F}_Y is a sheaf.*

Proof. We must show that objects of $\mathcal{F}_Y(U)$ have no nontrivial automorphisms. If Φ is such an object, then an automorphism α of Φ consists of a natural transformation from Φ^* to itself satisfying certain conditions. Thus α is determined by its restrictions to the intervals $\{y\} \times [0, 1]$ for $y \in U$. Since we must have $\alpha|_{\{y\} \times \{0\}}$ the identity, it follows from Lemma 15.2 that α is the identity. \square

Note that given a map $f : Y' \rightarrow Y$, composition with f induces a natural map of sheaves $f^* \mathcal{F}_Y \rightarrow \mathcal{F}_{Y'}$.

Lemma 15.4. *Let X be a locally complete, locally convex metric topos, and let $\phi_0, \phi_1 : Y \rightarrow X$ be two geometric morphisms, where Y is topological space. For any $y \in Y$, the natural map $k : (\mathcal{F}_Y)_y \rightarrow \mathcal{F}_{\{y\}}$ is bijective.*

Proof. Let us examine an element $s \in \mathcal{F}_{\{y\}}$. This consists of a geodesic $p : [0, 1] \rightarrow X$ of some speed D which joins $\phi_0(y)$ to $\phi_1(y)$. For some subdivision $0 = t_0 \leq t_1 \leq \dots \leq t_n = 1$ of the interval $[0, 1]$, we may find etale isometries $\pi_i : V \rightarrow X$ such that there are isomorphisms $\beta_i : p|_{[t_i, t_{i+1}]} \simeq \pi_i \circ p_i$ for some map $p_i : [t_i, t_{i+1}] \rightarrow V_i$, where V_i is a convex, locally complete metric space. The isomorphisms β_i and

β_{i+1} compose to give an isomorphism $\pi_i \circ p_i|_{\{t_{i+1}\}} \simeq \pi_{i+1} \circ p_{i+1}|_{\{t_{i+1}\}}$, which corresponds to a point z_i of $V_i \times_X V_{i+1}$ lying over $(p_i(t_{i+1}), p_{i+1}(t_{i+1}))$. Since $V_i \times_X V_{i+1}$ is etale over V_i and V_{i+1} , we can find an open neighborhood of z_i which maps isomorphically to convex open subsets $V_i^+ \subseteq V_i$ and $V_{i+1}^- \subseteq V_{i+1}$. Let $\gamma_i : V_i^+ \rightarrow V_{i+1}^-$ denote the induced isometry.

In the terminology of the last section, p determines a broken geodesic. Moreover, replacing Y by a neighborhood of Y , we may assume that there are isomorphisms $\phi_0 \circ \pi_0 \circ \psi_0$ and $\phi_1 = \pi_{n-1} \circ \psi_1$ for some continuous maps $\psi_0 : Y \rightarrow V_0$, $\psi_1 : Y \rightarrow V_{n-1}$. Then by Theorem 14.5, we can after shrinking Y further assume that there is a broken Y -geodesic joining ψ_0 to ψ_1 . This obviously determines a section of \mathcal{F}_Y which restricts to the element of $\mathcal{F}_{\{y\}}$ that we started with. This proves that k is surjective.

Let us now prove that k is injective. Shrinking Y if necessary, it suffices to show that any two sections of $S, S' \in \mathcal{F}_Y$ which restrict to $s \in \mathcal{F}_{\{y\}}$ agree in some neighborhood of Y .

Suppose we are given any global section of \mathcal{F}_Y represented by a geometric morphism $\Phi : Y \times [0, 1] \rightarrow X$ and isomorphisms $\alpha_i : \Phi|_{Y \times \{i\}} \simeq \phi_i$ ($i = 0, 1$). Let us define a *rigidification* of $(\Phi, \alpha_0, \alpha_1)$ to be a collection of continuous $P_i : Y \times [t_i, t_{i+1}] \rightarrow V_i$ with $\gamma_i \circ P_i|_{Y \times \{t_{i+1}\}} = P_{i+1}|_{Y \times \{t_{i+1}\}}$ such that $P_0|_{Y \times \{0\}} = \psi_0$, $P_{n-1}|_{Y \times \{1\}} = \psi_1$, together with isomorphisms $\pi_i \circ P_i \simeq \Phi|_{Y \times [t_i, t_{i+1}]}$ which are compatible with the isomorphisms γ_i . One checks easily that the rigidifications of S and S' on $\{y\}$ extend to rigidifications in a neighborhood of $\{y\}$. Replacing Y by this neighborhood, we may represent S and S' by broken Y -geodesics joining ψ_0 to ψ_1 . A convexity argument shows that these broken Y -geodesics must coincide, so that $S = S'$. \square

Theorem 15.5. *Let X be a separated, locally complete, locally convex metric topos, and let $\phi : Y \rightarrow X$ be a geometric morphism. For any map $f : Y' \rightarrow Y$, the natural map $f^* \mathcal{F}_Y \rightarrow \mathcal{F}_{Y'}$ is an isomorphism.*

Proof. It suffices to check on stalks, and then we can use Lemma 15.4. \square

16. \mathcal{F}_Y IS HAUSDORFF

In this section, we shall prove that \mathcal{F}_Y is a Hausdorff sheaf. Recall that this means that for any $U \subseteq Y$ and any pair of sections in $\mathcal{F}_Y(U)$, the maximal open subset of U on which the sections agree is also closed in U . First, we shall need a number of lemmas.

Lemma 16.1. *Let V be a convex metric space, and let $\pi : V' \rightarrow V$ be an etale map. Suppose that given any path $p : [0, 1] \rightarrow V'$ of finite length, there is at most one extension of p to $[0, 1]$. Then V' is Hausdorff.*

Proof. We must show that any two distinct points of $x, y \in V'$ have disjoint open neighborhoods. If $\pi(x) \neq \pi(y)$ we are done. Otherwise, since π is etale, there exists a neighborhood U of $\pi(x) = \pi(y)$ and neighborhoods U_x, U_y of x and y such that $\pi|_{U_x} \rightarrow U$ and $\pi|_{U_y} \rightarrow U$ are isomorphisms. Shrinking U , we may assume that U is a convex metric space. It now suffices to show that U_x and U_y are disjoint. For suppose $z \in U_x \cap U_y$. Let $p : [0, 1] \rightarrow V$ be a path of finite length joining $\pi(z)$ to $\pi(x) = \pi(y)$, and let p_x and p_y denote its lifts to U_x and U_y . Since $p_x(1) = x \neq y = p_y(1)$, p_x and p_y are not identically equal. On the other hand, we

do have $p_x(0) = z = p_y(z)$. Since π is etale and $\pi \circ p_x = p = \pi \circ p_y$, there is some maximal open interval $[0, t)$ on which $p_x = p_y$. But then the hypothesis shows that $p_x(t) = p_y(t)$, a contradiction. \square

Lemma 16.2. *Let X be a weakly separated locally convex metric topos, and let $\pi : V \rightarrow X$ and $\pi' : V' \rightarrow X$ be etale maps where V and V' are convex metric spaces. Then $V \times_X V'$ is a Hausdorff topological space.*

Proof. We first note that $V \times_X V'$ is a topological space. Indeed, it is etale over V , so it is the total space of some stack on V . But points of $V \times_X V'$ have no automorphisms, so that stack is a sheaf.

It now suffices to show that a path $p : [0, 1) \rightarrow V \times_X V'$ admits at most one extension to $[0, 1]$. If such an extension is to exist, the composite map $[0, 1) \rightarrow V \times_X V'$ must admit an extension to $[0, 1]$. Then we have two maps $p_0, p_1 : [0, 1] \rightarrow V$, and an isomorphism of $\pi \circ f_0 \simeq \pi \circ f_1$ over $[0, 1)$. We need to show that this isomorphism admits a unique extension over $[0, 1]$, which follows immediately from the assumption that X is weakly separated. \square

Lemma 16.3. *Let X be a weakly separated, locally convex metric topos. Let $f, g : Y \rightarrow X$ be two geometric morphisms, let $\alpha, \beta : f \simeq g$ be two isomorphisms, and suppose that $\alpha|_U = \beta|_U$ for some dense subset $U \subseteq Y$. Then $\alpha = \beta$.*

Proof. Using the isomorphism $\beta : f \simeq g$, we can reduce to the case where $f = g$ and β is the identity. The assertion is local on Y , so we may assume that f factors as a composite of a map $f' : Y \rightarrow V$ and some etale map $\psi : V \rightarrow X$, where V is a convex metric space. Then to give an automorphism of f is to give a lifting of the map $f' \times f' : Y \rightarrow V \times V$ to the space $V \times_X V$, which is etale over V . To prove the lemma, it suffices to know that $V \times_X V$ is Hausdorff, so that two morphisms to $V \times_X V$ which agree on a dense subset agree everywhere. \square

The analogous statement for separatedness is much more subtle. We shall prove only a weak version of it.

Lemma 16.4. *Let V_0 and V_1 be convex metric spaces, and let V be a Hausdorff space admitting etale projections $\pi_0 : V \rightarrow V_0$ and $\pi_1 : V \rightarrow V_1$ so that the induced metrics agree. Suppose that for each map $p : [0, 1) \rightarrow V$ having bounded length, if $\pi_0 \circ p$ and $\pi_1 \circ p$ both extend continuously to $[0, 1]$, then p extends continuously to $[0, 1]$.*

Let Y be any space, and let $\phi_0 : Y \times [0, 1] \rightarrow V_0$, $\phi_1 : Y \times [0, 1] \rightarrow V_1$ and $\psi : Y \rightarrow V$ be continuous maps such that $\pi_i \psi = \phi_i|_{Y \times \{0\}}$, and such that $\phi_i|_{\{y\} \times [0, 1]}$ have finite length. Let U be a dense subset of Y , and suppose $\phi : U \times [0, 1] \rightarrow V$ is a continuous map such that $\pi_i \circ \phi = \phi_i|_{U \times [0, 1]}$, $\phi|_{U \times \{0\}} = \psi|_U$. Then ϕ admits a continuous extension to $Y \times [0, 1]$ having the same properties.

Proof. Since V is Hausdorff, it suffices to show the existence of an extension of ϕ to $Y \times [0, 1]$. The compatibility conditions then follow immediately since they are known over the dense subset $U \times [0, 1]$. Let \mathcal{F}_i denote the sheaf on $Y \times [0, 1]$ which classifies factorizations of ϕ_i through π_i . Then each \mathcal{F}_i is a Hausdorff sheaf and we are given sections of these sheaves over the dense subset $U \times [0, 1]$. Thus there is some largest open set U' of $Y \times [0, 1]$ over which these sections extend; U' is independent of i because it may also be characterized as the largest open subset to which ϕ continuously extends. We shall denote this maximal extension of ϕ by ϕ' .

Let $U'' = \{(y, t) : \{y\} \times [0, t] \subseteq U'\}$. Note that U'' is open and contains $U \times [0, 1]$. If $U'' = Y \times [0, 1]$, we are done. Otherwise, choose some $y \in Y$ such that $\{y\} \times [0, 1]$ is not contained in U' , and let t be the largest value such that $\{y\} \times [0, t] \subseteq U'$. The path $\phi'|_{\{y\} \times [0, t]}$ satisfy the conditions stated in the hypotheses, hence $\phi'|_{\{y\} \times [0, t]}$ extends to $\{y\} \times [0, t]$. Thus we obtain a section of the sheaf \mathcal{F}_i over the set $\{y\} \times [0, t]$. Let us denote the germ at (y, t) by $s_{y,t}$; this germ extends to a section s over some open neighborhood $W \times [t - \epsilon, t + \epsilon]$ of (y, t) . Shrinking W if necessary, we may assume that $W \times \{t - \epsilon\} \subseteq U''$. Now we claim that, for W sufficiently small, the section s and the map $\phi'|_{U''}$ agree on $U'' \cap (W \times [t - \epsilon, t + \epsilon])$. Every point (w, t') of this intersection can be joined to $(w, t - \epsilon)$ by a path contained in the intersection. Since \mathcal{F}_i is a Hausdorff sheaf, it suffices to show that the two sections coincide over $W \times \{t - \epsilon\}$. By construction, they coincide at the point $(y, t - \epsilon)$; thus they coincide on a neighborhood of this point. Replacing W by a smaller neighborhood of y , we obtain the desired result.

Now we can glue the section s and the map $\phi'|_{U''}$, so as to obtain a section of \mathcal{F}_i over the open set $U'' \cup (W \times (t - \epsilon, t + \epsilon))$. By the maximality of U' , this implies that $W \times (t - \epsilon, t + \epsilon)$ is contained in U' . But then $\{y\} \times [0, t] \subseteq U'$, a contradiction. \square

Lemma 16.5. *Let X be a separated, locally convex metric topos. Let Y be a topological space, U a dense subspace, $\Phi, \Phi' : Y \times [0, 1] \times X$ two geometric morphisms whose restrictions to any $y \times [0, 1]$ have bounded length, and $\alpha_0 : \Phi|_{Y \times \{0\}} \simeq \Phi'|_{Y \times \{0\}}$ an isomorphism. Suppose that α_U is an isomorphism between $\Phi|_{U \times [0, 1]}$ and $\Phi'|_{U \times [0, 1]}$ such that $\alpha_U|_{U \times \{0\}} = \alpha_0|_{U \times \{0\}}$. Then α_U can be extended uniquely to an isomorphism $\alpha : \Phi \simeq \Phi'$ such that $\alpha|_{Y \times \{0\}} = \alpha_0$.*

Proof. The uniqueness is automatic by Lemma 16.3. In view of the uniqueness, it suffices to prove the result locally on Y . Thus we may find a natural number n and etale maps $\psi_i : V_i \rightarrow X$, $\psi'_i : V'_i \rightarrow X$ for $0 \leq i < n$ such that V_i and V'_i are convex metric spaces and there are isomorphisms $\beta_i : \Phi|_{Y \times [\frac{i}{n}, \frac{i+1}{n}]} \simeq \psi_i \circ \Phi_i$, $\beta'_i : \Phi'|_{Y \times [\frac{i}{n}, \frac{i+1}{n}]} \simeq \psi'_i \circ \Phi'_i$ where $\Phi_i : Y \times [\frac{i}{n}, \frac{i+1}{n}] \rightarrow V_i$ and $\Phi'_i : Y \times [\frac{i}{n}, \frac{i+1}{n}] \rightarrow V'_i$ are continuous maps.

We will construct the isomorphism $\alpha_i = \alpha|_{[0, \frac{i}{n}]}$ by induction on i . The isomorphism α_0 is given to us from the start. Assuming that α_{i-1} has been constructed for $i \leq n$, we see that α_{i-1} induces an isomorphism between $\psi_i \circ \Phi_i|_{Y \times \{\frac{i}{n}\}}$ and $\psi'_i \circ \Phi'_i|_{Y \times \{\frac{i}{n}\}}$. Moreover, the existence of α_U gives an extension of this isomorphism to $U \times [\frac{i}{n}, \frac{i+1}{n}]$. The result now follows from Lemma 16.4, applied to the sets V_i , V'_i , and $V_i \times_X V'_i$. \square

Theorem 16.6. *Let X be a separated locally convex metric topos, and let $\phi_0, \phi_1 : Y \rightarrow X$ be a geometric morphisms. Then \mathcal{F}_Y is a Hausdorff sheaf.*

Proof. Replacing Y by an open subset if necessary, it suffices to show that the maximal open subset U of Y on which two global sections $s, s' \in \mathcal{F}_Y(U)$ agree is closed. In view of Lemma 15.5, we may replace Y by the closure of U , thereby reducing to the case where U is dense. Represent s and s' by morphisms $\Phi, \Phi' : Y \times [0, 1] \rightarrow X$. Then $\Phi|_{Y \times \{0\}} \simeq \phi_0 \simeq \Phi'|_{Y \times \{0\}}$, and this isomorphism extends over all of $U \times [0, 1]$. By Lemma 16.5, this isomorphism extends uniquely over all of $Y \times [0, 1]$. Moreover, Lemma 16.3 shows that this isomorphism is compatible with the isomorphisms $\Phi|_{Y \times \{1\}} \simeq \phi_1 \simeq \Phi'|_{Y \times \{1\}}$, since this compatibility is known over the dense subset U . It follows that $s = s'$ and we are done. \square

17. THE SHEAF \mathcal{F}_Y IS EXTENDABLE

We wish to show that the sheaves \mathcal{F}_Y are locally constant. In order to prove this, we first establish the following special case:

Theorem 17.1. *Let X be a complete, locally convex, locally complete metric topos. Let $Y = [0, 1]$, and let $\phi_0, \phi_1 : Y \rightarrow X$ be geometric morphisms of finite length. Then any section of \mathcal{F}_Y over $(0, 1)$ can be extended to a global section.*

This in turn will rest on the following lemma:

Lemma 17.2. *Let \mathcal{F} be a Hausdorff sheaf on $[0, 1] \times [0, 1]$. Suppose that any germ of \mathcal{F} at a point $(0, t)$ can be extended to a section of \mathcal{F} over $[0, 1] \times \{t\}$. Then any section of \mathcal{F} over $0 \times [0, 1]$ can be extended to a global section of \mathcal{F} .*

Proof. Let s_0 be a section of \mathcal{F} over $0 \times [0, 1]$. If s_0 cannot be extended to a global section of \mathcal{F} , then there is some maximal interval $[0, t)$ such that s_0 can be extended to a section s_t of \mathcal{F} over $[0, t) \times [0, 1]$. For each $y \in [0, 1]$, let s'_y denote the (unique) section of \mathcal{F} over $[0, 1] \times \{y\}$ which agrees with s_0 at $(0, y)$, and let $s'_{t,y}$ denote the stalk of s'_y at the point (t, y) . Consider for each y an open ball V_y containing (t, y) and a section s''_y of \mathcal{F} over V_y which agrees with $s'_{t,y}$ at (t, y) . We claim that the sections s_t and $\{s''_y\}_{y \in Y}$ glue to give a section of \mathcal{F} over an open set containing $[0, t] \times [0, 1]$, which will be a contradiction.

First, we show that s_t and s''_y have the same restriction to $([0, t) \times [0, 1]) \cap V_y$. Since the intersection is convex and \mathcal{F} is Hausdorff, suffices to prove that the stalks agree at a single point. We may take for that point $(t - \epsilon, y)$ for any sufficiently small ϵ .

Now we show that s''_y and $s''_{y'}$ have the same restriction to $V_y \cap V_{y'}$. If the intersection is empty there is nothing to prove. If not, then the intersection is convex, and again it suffices to demonstrate agreement at a single point. But this is clear from what we proved above since the intersection $V_y \cap V_{y'} \cap ([0, t) \times [0, 1])$ is nonempty. \square

Now we give the proof of Theorem 17.2.

Proof. Represent a section of \mathcal{F}_Y over $[0, 1]$ by a geometric morphism $\Phi : [0, 1] \times [0, 1] \rightarrow X$, together with the required isomorphisms $\alpha_0 : \Phi|_{[0, 1] \times \{0\}} \simeq \phi_0$ and $\alpha_1 : \Phi|_{[0, 1] \times \{1\}} \simeq \phi_1$. It will suffice to prove that Φ can be extended, up to isomorphism, to a geometric morphism defined on $[0, 1] \times [0, 1]$. For then α_0 and α_1 extend uniquely due to the assumption that X is separated, and $\Phi|_{\{1\} \times [0, 1]}$ is a geodesic by continuity.

An easy convexity argument shows that for each $t \in [0, 1]$, the restriction of Φ to $[0, 1] \times \{t\}$ has length bounded by the maximum of $L(\phi_0)$ and $L(\phi_1)$. Since X is complete, this restriction extends uniquely (up to unique isomorphism) to a geodesic $\phi_t : [0, 1] \rightarrow X$. Then there is an etale morphism $\pi_t : V_t \rightarrow X$ such that $\phi_t|_{[1-\epsilon_t, 1]}$ factors as $\pi_t \circ \psi_t$ for some continuous map $\psi_t : [1-\epsilon_t, 1] \rightarrow V_t$, where V_t is a locally complete convex metric space and $\epsilon_t > 0$. Choose r so that the closed ball of radius $2r$ about $\psi_t(1)$ in V_t is complete. Shrinking ϵ_t if necessary, we may assume $\epsilon_t L(\phi) < r$. Let U_t be an open subset of $[0, 1]$ so that $\Phi|_{\{1-\epsilon_t\} \times U_t}$ has length $< r$. Let \mathcal{F} denote the sheaf on $[1-\epsilon, 1] \times U_t$ corresponding to liftings of Φ along $\pi_t : V_t \rightarrow X$. Then \mathcal{F} is a Hausdorff sheaf, and ϕ_t supplies an element of the stalk $\mathcal{F}_{(1-\epsilon_t, t)}$. Using Lemma 13.5, we can lift this to a section of \mathcal{F} along $\{1-\epsilon_t\} \times U_t$.

The same lemma shows that the hypotheses of Lemma 17.2 are satisfied, so that (applying that lemma countably many times) this section extends to a global section of \mathcal{F} . In other words, the restriction of Φ to $[1 - \epsilon_t, 1] \times U_t$ lifts to V_t ; moreover the image of the corresponding map $\Phi_t : [1 - \epsilon_t, 1] \times U_t \rightarrow V_t$ is contained in the closed ball of radius $2r$ about $\phi_t(1 - \epsilon)$. If we endow $[1 - \epsilon, 1] \times U_t$ with the metric given by $d((y_0, t_0), (y_1, t_1)) = L(\phi|[y_0, y_1]) + |t_0 - t_1|L(\Phi\{[1 - \epsilon_t] \times [0, 1]\})$ (for $y_0 \leq y_1$), then one may easily check that the map Φ_t is a contraction, hence uniformly continuous. Therefore Φ_t extends continuously to $[1 - \epsilon_t, 1] \times U_t$.

To complete the proof, it suffices to show that for different choices of $t \in [0, 1]$, the extensions $\Phi_t : [1 - \epsilon_t] \times U_t \rightarrow X$ are canonically isomorphic on their overlaps. For this, we apply Lemma 17.2 to the sheaf of isomorphisms between them, where the hypotheses are verified using the fact that X is separated. \square

18. THE SHEAF \mathcal{F}_Y IS LOCALLY CONSTANT

Lemma 18.1. *Let Y be a connected topological space, \mathcal{F} a Hausdorff sheaf on Y . Suppose that for any $y \in Y$, any germ of \mathcal{F} at y can be extended to the entire space Y . Then \mathcal{F} is a constant sheaf.*

Proof. Let M be the collection of global sections of \mathcal{F} . Clearly every continuous function from an open set $U \subseteq Y$ to M (considered as a discrete set) determines a section of \mathcal{F} over U . We will show that the corresponding map of sheaves is an isomorphism.

Given two continuous functions $f, g : U \rightarrow M$, and consider any $u \in U$. If the sections of \mathcal{F} associated to f and g agree, then $f(u)$ and $g(u)$ must have the same stalk at u . Since \mathcal{F} is Hausdorff, $f(u)$ and $g(u)$ agree on a set which is both closed and open. Since this set is nonempty and Y is connected, we conclude that $f(u) = g(u)$. As this holds for every $u \in U$, we get $f = g$.

For the reverse surjectivity, consider any section $s \in \mathcal{F}(U)$. For each $u \in U$, let $f(u)$ be an element of M whose stalk at u coincides with the stalk of s at u . It will suffice to show that f is a continuous function. For this, it suffices to show that for each $u \in U$, the set $V_u = \{v \in U : f(v) = f(u)\}$ is closed and open in U . For this, it suffices to show that V_u is open subset of U on which s and $f(u)$ agree. For if s and $f(u)$ agree at a point v , then $f(v)$ and $f(u)$ agree at v , hence everywhere on Y ; the converse is obvious. \square

Let us now consider the following situation: let V be a convex metric space, let \mathcal{F} be a Hausdorff sheaf on V . Assume that for any continuous path $p : [0, 1] \rightarrow V$ having finite length, any section of $p^*\mathcal{F}$ over $[0, 1)$ can be extended to a section over $[0, 1]$. It follows by an easy argument that for such a sheaf, $p^*\mathcal{F}$ is constant on $[0, 1]$. Thus, we get an induced bijection $p_* : \mathcal{F}_{p(0)} \rightarrow \mathcal{F}_{p(1)}$.

Lemma 18.2. *If $p : [0, 1] \rightarrow V$ and $p' : [0, 1] \rightarrow V$ are two paths having finite length with $p(0) = p'(0) = x$, $p(1) = p'(1) = y$, then $p_* = p'_* : \mathcal{F}_x \rightarrow \mathcal{F}_y$.*

Proof. Define $h : [0, 1] \times [0, 1] \rightarrow V$ so that $h(s, t)$ is a geodesic (of some speed) joining $p(t)$ to $p'(t)$. The convexity of the metric of V shows that h is continuous.

Consider any germ $s_x \in \mathcal{F}_x$. Then we may consider s_x as an element of the stalk $(h^*\mathcal{F})_{(0,0)}$. The local constancy of $f^*\mathcal{F}$ allows us to extend s_x to $\{0\} \times [0, 1]$. Applying Lemma 17.2, we deduce the existence of an extension s of s_x to $[0, 1] \times [0, 1]$. Since \mathcal{F} is Hausdorff, this extension is unique. Moreover, we see that $f_*(s_x)$

is the stalk of s at the point $(0, 1)$, and $g_*(s_x)$ is the stalk of s at $(1, 1)$, which implies the desired result. \square

Theorem 18.3. *The sheaf \mathcal{F} is constant on V .*

Proof. We must show that any element of a stalk can be extended to a global section. Pick $y \in Y$ and an element s_y of the stalk \mathcal{F}_y . We will show that s_y can be extended to a global section of \mathcal{F} . Consider the partially ordered set whose elements are pairs (U, s_U) , where U is a connected open subset of Y containing y and s_U a section of \mathcal{F} over U whose germ at y is s_y . This partial order is obviously nonempty and inductive, so by Zorn's lemma it has a maximal element (U, s_U) . Suppose $U \neq Y$. Pick a point $x \in Y$, $x \notin U$, and let $p : [0, 1] \rightarrow Y$ be a path joining y to x having finite length. Let t be the smallest element of $[0, 1]$ such that $p(t) \notin U$. Replacing x by $p(t)$ and reparametrizing p , we may assume $p([0, 1]) \subseteq U$. Let $s_x = p_*s_y$, and choose an open ball U' around x such that s_x lifts to a section $s_{U'}$ on U' . We claim that s_U and $s_{U'}$ agree on $U \cap U'$. This will contradict the maximality of U and proves the theorem.

To show that $s_U|_{U \cap U'} = s_{U'}|_{U \cap U'}$, it suffices to show that they agree at every point $z \in U \cap U'$. Then we may choose paths $q : [0, 1] \rightarrow X$, $q' : [0, 1] \rightarrow X$ such that q is a path of finite length joining y to z in U , and q' is a path of finite length joining x to z in V . Then we see that $(s_U)_z = q_*s_y$, $(s_{U'})_z = q'_*s_x = q'_*p_*s_y = q''_*s_y$, where q'' is the path obtained by concatenating p and q' . Then Lemma 18.2 implies that $(s_V)_z = (s_U)_z$, as desired. \square

Theorem 18.4. *Let X be a complete, locally complete, locally convex metric topos, and let $\phi_0, \phi_1 : Y \rightarrow X$ be a geometric morphism from a topological space Y into X . Then the sheaf \mathcal{F}_Y is locally constant.*

Proof. The assertion is local on Y , so without loss of generality we may assume that $\phi : Y \rightarrow X$ factors through an etale map $\psi : V \rightarrow X$, where V is a convex metric space. Since the formation of \mathcal{F}_Y is compatible with base change, it suffices to prove this result when $Y = V$. By Theorem 16.6, the sheaf \mathcal{F}_Y is Hausdorff, and by Theorem 17.2 it verifies the conditions required for the above discussion. Thus Theorem 18.3 proves the desired result. \square

19. CONSEQUENCES

Lemma 19.1. *Let X be a complete, locally convex, locally complete metric topos. Suppose that X is simply connected. Then every closed geodesic in X is trivial.*

Proof. Fix a point $x : \{*\} \rightarrow X$. For any topological geometric morphism $\phi_0 : Y \rightarrow X$, let \mathcal{F}_Y denote the sheaf corresponding to the pair of geometric morphisms (ϕ_0, ϕ_1) , where ϕ_1 is the composite of x with the constant map $Y \rightarrow \{*\}$.

Since X admits a covering family of topological spaces, we obtain by descent a sheaf \mathcal{F}_X on X . By Theorem 18.4, \mathcal{F}_X is a locally constant sheaf on X . Since X is simply connected, \mathcal{F} is a constant sheaf. The stalk of this sheaf at the point x is, by Lemma 15.5, the set of isomorphism classes of closed geodesics at x . In particular, taking the trivial closed geodesic at x , we obtain a canonical global section s of \mathcal{F}_X .

Now let $Y = [0, 1]$, and let $p : Y \rightarrow X$ denote a closed geodesic with $p|_{\{0\}} \simeq x \simeq p|_{\{1\}}$. There is a natural section $s' \in F_Y(Y)$ over Y , which associates to each point $t \in [0, 1]$ the geodesic obtained from $p|_{[0, t]}$ by reparametrization. The germ of this section at the point $0 \in Y$ is determined by the restriction $p|_{\{0\}}$ which coincides

with the inclusion of the point x . It follows that $s' = p^*s$ in a neighborhood of a . Since \mathcal{F}_Y is locally constant, we get $s' = p^*s$ everywhere. Now restrict to the point $1 \in Y$; we deduce that the closed geodesic p is isomorphic to the trivial closed geodesic at x . \square

Theorem 19.2. *Let X be a complete, locally convex, locally complete metric topos. Then X is the metric topos associated to a complete metric space.*

Proof. This follows immediately by combining Theorem 13.7 and Lemma 19.1. \square

Remark 19.3. If X is a locally convex metric topos, then X is locally isomorphic to a simply connected (even contractible) topological space. It follows that X is a disjoint union of its connected components, each of which admits a universal cover. If, in addition, X is complete and locally complete, then its universal cover shares those properties so that Theorem 19.2 can be applied.

Before proceeding to the Hadamard case, there is one more loose end to tie up. The constancy of the sheaves \mathcal{F}_Y proves the existence of many geodesics. However, it does not immediately guarantee that these geodesics are length-minimizing. But this is indeed the case.

Lemma 19.4. *Let X be a complete, locally complete, locally convex metric topos. Let $p : [0, 1] \rightarrow X$ be path in X . Then there exists a geodesic $p' : [0, 1] \rightarrow X$ with $p'(0) \simeq p(0)$, $p'(1) \simeq p(1)$, and $L(p') \leq L(p)$.*

Proof. Let $x = p|_{\{0\}}$, $Y = [0, 1]$, $\phi_0 = p$, and ϕ_1 be the constant map with value x . Let \mathcal{F}_Y denote the associated sheaf on Y . We have shown that \mathcal{F}_Y is locally constant and that its stalk at 0 consists of the closed geodesics joining x to itself. The trivial closed geodesic lifts uniquely to a global of \mathcal{F}_Y , which we can think of as a continuous family of geodesics $p'_s : [0, 1] \rightarrow X$ joining $p(s)$ to $p(0)$. Let $A = \{s \in [0, 1] : L(p'_s) \leq L(p|[0, s])\}$. Clearly $0 \in A$, and to prove the lemma it suffices to show that $1 \in A$. Since $L(p'_s)$ and $L(p|[0, s])$ are both continuous functions of s , the set A is closed. Since the interval $[0, 1]$ is connected, it suffices to show that A is open.

Suppose $s \in A$. Then there exists a decomposition $0 = t_0 < t_1 < \dots < t_n = s$, a collection of etale isometries $\pi_i : V_i \rightarrow X$, and convex open subsets $V_i^+ \subseteq V_i$ and $V_{i+1}^- \subseteq V_{i+1}$, together with isomorphisms $\pi_i|_{V_i^+} \simeq \pi_{i+1}|_{V_{i+1}^-}$ such that p'_s comes from a broken geodesic. Then, as in the proof of Lemma 15.4, we see that $p'_{s'}$ comes from a broken geodesic for all s' sufficiently close to s . Now Lemma 14.4 implies that $L(p'_{s'}) \leq L(p'_s) + \delta_{s,s'}$, where $\delta_{s,s'}$ is the the local distance between the endpoints of p'_s and $p'_{s'}$. This in turn is bounded above by the length of $p|[s, s']$. Combining this with the inequality $L(p'_s) \leq L(p|[0, s])$, we obtain the desired result (even with room to spare if $s' < s$). \square

Definition 19.5. A metric topos X is *locally Hadamard* if it admits an etale cover by metric spaces which are isomorphic to open balls in Hadamard spaces.

Since Hadamard spaces are convex and locally complete, it follows that a locally Hadamard metric topos is locally convex and locally complete.

Theorem 19.6. *Let X be a complete, simply connected, locally Hadamard metric topos. Then X is the metric topos associated to a Hadamard space.*

Proof. We have seen that $\pi : X \rightarrow |X|$ is an isomorphism. Thus it suffices to show that the metric space $|X|$ is a Hadamard space. Since X is connected, $|X|$ is nonempty. Since X is complete, $|X|$ is complete.

Fix a point x of X , and consider the sheaf \mathcal{F}_X corresponding to the pair of maps $\phi_0, \phi_1 : X \rightarrow X$ where ϕ_0 is the identity and ϕ_1 has the constant value x . The stalk of the sheaf \mathcal{F}_X at x consists of all closed geodesics joining x to itself. Since we have seen that these are all trivial, we deduce that \mathcal{F}_X is the terminal sheaf. Thus it has a unique global section s . This supplies the function g_x required for Theorem 11.1. The only tricky point is that we need to show that the geodesics supplied by g_x are minimizing. By construction, the distance between two points $x, y \in |X|$ is the infimum of the lengths of all paths joining x to y . Hence it suffices to show that this infimum is achieved by the geodesic which joins x and y . This follows easily from 19.4. \square

It follows that any complete, connected, locally Hadamard metric topos has a Hadamard space as its universal cover. In particular, any complete, connected metric space which is locally isometric to a Hadamard space has a Hadamard space as its universal cover. This applies, for example, to complete, connected Riemannian manifolds of non-positive sectional curvatures: this is the original formulation of the Hadamard-Cartan theorem.