TANNAKA DUALITY FOR GEOMETRIC STACKS

1. Introduction

Let X and S denote algebraic stacks of finite type over the field $\mathbf{C}$ of complex numbers, and let X^{an} and S^{an} denote their analytifications (which are stacks in the complex analytic setting). Analytification gives a functor

$$\phi : \mathrm{Hom}_{\mathbf{C}}(S, X) \to \mathrm{Hom}(S^{\mathrm{an}}, X^{\mathrm{an}}).$$

It is natural to ask when ϕ is an equivalence.

In the case where X and S are projective schemes, a satisfactory answer was obtained long ago. In this case, both algebraic and analytic maps may be classified by their graphs, which are closed in the product $X \times S$. One may then deduce that any analytic map is algebraic by applying Serre's GAGA theorem (see [6]) to $X \times S$.

If S is a projective scheme and X is the classifying stack of the algebraic group GL_n, then $\mathrm{Hom}(S, X)$ classifies vector bundles on S. If S is a proper scheme, then any analytic vector bundle on S is algebraic (again by Serre's GAGA theorem), and one may again deduce that ϕ is an equivalence.

By combining the above methods, one can deduce that ϕ is an equivalence whenever X is given globally as a quotient of a separated algebraic space by the action of a linear algebraic group (and S is proper). The main motivation for this paper was to find a more natural hypothesis on X which forces ϕ to be an equivalence. We will show that this is the case whenever X is *geometric*: that is, when X is quasi-compact and the diagonal morphism $X \to X \times X$ is affine. More precisely, we have the following:

Theorem 1.1. *Let S be a Deligne-Mumford stack which is proper over $\mathbf{C}$, and let X be a geometric stack of finite type over $\mathbf{C}$. Then the analytification functor ϕ is an equivalence of categories.*

Our method of proving Theorem 1.1 is perhaps more interesting than the theorem itself. The basic idea is to show that if X is a geometric stack, then there exists a Tannakian characterization for morphisms $f : S \to X$, in both the algebraic and analytic categories. More precisely, we will show that giving a morphism f is equivalent to specifying a "pullback functor" f^* from coherent sheaves on X to coherent sheaves on S. We will then be able to deduce Theorem 1.1 by applying Serre's GAGA theorem to S.

This paper was originally intended to be included in the more ambitious paper [4], which studies the analogous formulation of Tannaka duality in derived algebraic geometry. However, since the derived setting offers a host of additional technical difficulties, it seemed worthwhile to write a separate account in the simpler case considered here.

I would like to thank Brian Conrad for offering several suggestions and corrections after reading an earlier version of this paper.

2. Notation

Throughout this paper, the word *topos* shall mean Grothendieck topos. If S is a topos, then we shall usually refer to the objects of S as *sheaves on S*. Similarly we shall speak of *sheaves of groups*, *sheaves of rings*, and so forth, rather than *group objects* or *ring objects* of S. This terminology is justified by the fact that any category S is equivalent to the category of representable presheaves on S, and that when S is a topos then the representable presheaves are precisely those presheaves which are sheaves with respect to the canonical topology (see [1]).

Throughout this note, *ring* shall mean *commutative ring with identity.* If A is a ring, then we will write $\mathcal{M}_A$ for the category of A-modules. More generally, if $\mathcal{A}$ is a sheaf of rings on a topos S, then $\mathcal{M}_{\mathcal{A}}$ shall denote the category of sheaves of $\mathcal{A}$-modules.

If $\mathcal{F}$ is an abelian sheaf (on some topos S) which admits an action of a ring of endomorphisms A, and $M \in \mathcal{M}_A$, then we shall write $\mathcal{F} \otimes_A M$ for the *external tensor product* of $\mathcal{F}$ by M over A. In other words, $\mathcal{F} \otimes_A M$ is the sheafification (with respect to the canonical topology) of the presheaf $V \mapsto \mathcal{F}(V) \otimes_A M$ on S.

Throughout this paper, the term *algebraic stack* shall mean Artin stack (not necessarily of finite type) over $\operatorname{Spec} \mathbf{Z}$. If X is an algebraic stack, we will write QC_X for the category of quasi-coherent sheaves on X. If X is locally Noetherian, then we will write Coh_X for the category of coherent sheaves on X.

3. Geometric Stacks

Definition 3.1. An algebraic stack X is *geometric* if it is quasi-compact and the diagonal morphism

$$X \to X \times X = X \times_{\operatorname{Spec} \mathbf{Z}} X$$

is representable and affine.

Remark 3.2. The terminology we have just introduced is borrowed from [7], with one modification: we include a hypothesis of quasi-compactness in our definition of a geometric stack.

Remark 3.3. Let X be a geometric stack. Since X is quasi-compact, there exists a smooth surjection $\operatorname{Spec} A \to X$. Since the diagonal $X \to X \times X$ is affine, the fiber product $\operatorname{Spec} A \times_X \operatorname{Spec} A = \operatorname{Spec} B$ is affine. Moreover, the pair of objects $(\operatorname{Spec} A, \operatorname{Spec} B)$ are part of a groupoid object in the category of affine schemes. Algebraically, this means that the pair of rings (A, B) are endowed with the structure of a *Hopf algebroid*. This Hopf algebroid is commutative (in the sense that the rings A and B are commutative) and smooth (in the sense that either of the natural maps $A \to B$ is smooth).

Conversely, any commutative, smooth Hopf algebroid gives rise to a geometric stack in a natural way. It follows that this entire paper could be written using the language of Hopf algebroids, rather than algebraic stacks. We will avoid following this course, since the language of algebraic stacks seems more intuitive and notationally simpler. We refer the reader to [3] for a discussion which touches upon some of the ideas of this paper, written in the language of Hopf algebroids.

Remark 3.4. In Definition 3.1, we are free to replace the absolute product $X \times X = X \times_{\operatorname{Spec} \mathbf{Z}} X$ with the fiber product $X \times_Y X$ for any separated scheme Y which admits a map from X.

Example 3.5. Any quasi-compact, separated scheme (or algebraic space) is a geometric stack.

Example 3.6. The classifying stack of any smooth, affine group scheme is a geometric stack.

Example 3.7. Call a morphism $X \to S$ of algebraic stacks *relatively geometric* if $X \times_S \operatorname{Spec} A$ is a geometric stack, for any morphism $\operatorname{Spec} A \to S$. Then a composition of relatively geometric morphisms is relatively geometric. Applying this to the particular case where $S = BG$ is the classifying stack of an smooth affine group scheme, we deduce that the quotient of any separated algebraic space by the action of a smooth affine group scheme is geometric.

The main theme of this paper is that if X is a geometric stack, then X has "enough" quasi-coherent sheaves. As an illustration of this principle, we prove the following:

Theorem 3.8. *Let X be a geometric stack. Then the left-bounded derived category of quasi-coherent sheaves on X is naturally equivalent to the full subcategory of the left-bounded derived category of (smooth-étale) $\mathcal{O}_X$-modules which have quasi-coherent cohomologies.*

Proof. Let QC_X denote the abelian category of quasi-coherent sheaves on X. It will suffice to show that QC_X has enough injective objects, and that if $I \in \mathrm{QC}_X$ is injective and $M \in \mathrm{QC}_X$ is arbitrary, then $\mathrm{Ext}^i(M, I) = 0$ for all $i > 0$. Here the Ext-group is computed in the larger category of all (smooth-étale) $\mathcal{O}_X$-modules.

Since X is quasi-compact, we may choose a smooth surjection $p : U \to X$, where $U = \operatorname{Spec} A$ is an affine scheme. If $N \in \mathrm{QC}_X$, then we may choose an injection $p^* N \to I$, where I is a quasi-coherent sheaf on U corresponding to an injective A-module. Since p^* is exact, $p_* I$ is injective. We claim that the adjoint morphism $N \to p_* I$ is a monomorphism. For this, it suffices to show that each of the maps $N \to p_* p^* N$ and

$p_* p^* N \to p_* I$ are monomorphisms. For the first map, this follows from the fact that U is a flat covering of X. For the second, we note that since X is geometric, p is an affine morphism so that p_* is an exact functor when restricted to quasi-coherent sheaves.

The above argument shows that QC_X has enough injectives, and that in fact every injective is a direct summand of a quasi-coherent sheaf having the form $p_* I$. Next, we note that $\mathrm{Ext}^i(M, p_* I) = \mathrm{Ext}^i(p^* M, I)$ (since I has vanishing higher direct images under p). It now suffices to show that for any quasi-coherent sheaf N on U, we have $\mathrm{Ext}^i(N, I) = 0$ for $i > 0$, where the Ext-group is computed in the category of smooth-étale $\mathcal{O}_U$-modules. Since U is affine, there exists a resolution $P_\bullet$ of N such that each P_i is a direct sum of copies of $\mathcal{O}_U$. Since $\mathrm{Ext}^j(\mathcal{O}_U, I) = \mathrm{H}^j(U, I) = 0$ for $j > 0$, and since $\mathrm{Ext}^j(\bullet, I)$ carries arbitrary direct sums into direct products, we deduce that $\mathrm{Ext}^i(N, I)$ is the ith cohomology group of the complex $\mathrm{Hom}(P_\bullet, I)$. This cohomology group vanishes for $i > 0$ since $P_\bullet$ is acyclic in positive degrees and I is obtained from an injective A-module. □

If X is a Noetherian geometric stack, then we can say even more: X has "enough" coherent sheaves. This follows from a well-known argument, but we include the proof for lack of a reference:

Lemma 3.9. *Let X be an algebraic stack which is Noetherian and geometric. Then QC_X is equivalent to the category of* Ind-*objects of the full subcategory $\mathrm{Coh}_X \subseteq \mathrm{QC}_X$.*

Proof. We first prove that if M is a coherent sheaf on X, then M is a compact object of QC_X. Indeed, suppose that $\{N_\alpha\}$ is some filtered system of quasi-coherent sheaves on X with colimit N.

Choose a smooth surjection $p : U \to X$, where $U = \operatorname{Spec} A$ is affine. Since p_* and p^* commute with filtered colimits, we have a filtered system of short exact sequences

$$\{0 \to N_\alpha \to p_* p^* N_\alpha \to p_* p^* p_* p^* N_\alpha\}$$

having filtered colimit

$$0 \to N \to p_* p^* N \to p_* p^* p_* p^* N.$$

Using these short exact sequences, we see that in order to prove that $\mathrm{Hom}(M, N) \simeq \mathrm{colim}\{\mathrm{Hom}(M, N_\alpha)\}$, it suffices to prove the analogous result for the filtered systems $\{p_* p^* N_\alpha\}$ and $\{p_* p^* p_* p^* N_\alpha\}$. In other words, we may reduce to the case where the filtered system $\{N_\alpha\}$ is the direct image of a filtered system $\{P_\alpha\}$ of quasi-coherent sheaves on U; let P be the colimit of this system. In this case, we have

$$\mathrm{Hom}(M, N) = \mathrm{Hom}(p^* M, P) = \mathrm{colim}\{\mathrm{Hom}(p^* M, P_\alpha)\} = \mathrm{colim}\{\mathrm{Hom}(M, p_* P_\alpha)\}.$$

Here the second equality follows from the fact that $p^* M$ is coherent, and therefore corresponds to a finitely presented A-module. We remark that this last argument also shows that any compact object of QC_X is coherent.

By formal nonsense, we obtain a fully faithful embedding $\mathrm{Ind}(\mathrm{Coh}_X) \to \mathrm{QC}_X$. To complete the proof, it suffices to show that every quasi-coherent sheaf M on X is a filtered colimit of coherent subsheaves. For this, we write $p^* M$ as a filtered colimit of coherent subsheaves $\{P_\alpha \subseteq p^* M\}$ on U. Then $M \subseteq p_* p^* M = \bigcup\{p_* P_\alpha\}$. Set $M_\alpha = M \cap p_* P_\alpha$. Then the natural map $p^* M_\alpha \to p^* M$ factors through P_α, so it follows that M_α is a coherent subsheaf of M. Clearly M is the union of the filtered family of subobjects $\{M_\alpha\}$. □

Remark 3.10. The proof does not really require that X is geometric; really all that is needed is that the diagonal morphism $X \to X \times X$ is a quasi-compact, quasi-separated relative algebraic space.

4. Maps into Algebraic Stacks

The main goal of this paper is to prove Theorem 1.1, which furnishes a comparison between the categories $\mathrm{Hom}(S, X)$ and $\mathrm{Hom}(S^{\mathrm{an}}, X^{\mathrm{an}})$. As an intermediate step, we will define a category $\mathrm{Hom}(S^{\mathrm{an}}, X)$. This category will be equivalent to $\mathrm{Hom}(S^{\mathrm{an}}, X^{\mathrm{an}})$ by construction, and we will be reduced to comparing $\mathrm{Hom}(S, X)$ with $\mathrm{Hom}(S^{\mathrm{an}}, X)$.

Let $(S, \mathcal{O}_S)$ be *any* ringed topos, and $\mathcal{F}$ any covariant functor from commutative rings to groupoids. The assignment $U \mapsto \mathcal{F}(\mathcal{O}_S(U))$ determines a presheaf of groupoids $\mathscr{H}\mathrm{om}_0(S, \mathcal{F})$ on the topos S. This presheaf of groupoids may or may not be a stack on S; in either case, there always exists a stack $\mathscr{H}\text{om}(S, \mathcal{F})$ on S which is initial among stacks equipped with a morphism

$$\mathscr{H}\text{om}_0(S, \mathcal{F}) \to \mathscr{H}\text{om}(S, \mathcal{F}).$$

We will refer to $\mathscr{H}\text{om}(S, \mathcal{F})$ as the *stackification* of $\mathscr{H}\text{om}_0(S, \mathcal{F})$. The groupoid of global sections of $\mathscr{H}\text{om}(S, \mathcal{F})$ will be denoted by $\text{Hom}(S, \mathcal{F})$. In the case where $\mathcal{F}$ is represented by an algebraic stack X, we will also write $\mathscr{H}\text{om}(S, X)$ and $\text{Hom}(S, X)$. We note that this is an abuse of notation, because these morphism spaces depend on the sheaf of rings $\mathcal{O}_S$ and not only on the underlying topos S.

Example 4.1. Let $(S, \mathcal{O}_S)$ be the étale topos of a Deligne-Mumford stack, and let X be an arbitrary algebraic stack. Then our definition of $\text{Hom}(S, X)$ is equivalent to the usual definition.

The definition given above is not of much use unless we have some means of calculating $\text{Hom}(S, X)$ in terms of a presentation of X. This requires an additional hypothesis on the ringed topos $(S, \mathcal{O}_S)$ which we now introduce:

Definition 4.2. A ringed topos $(S, \mathcal{O}_S)$ is *local for the étale topology* if it has the following property: for any $E \in S$ and any finite set of étale ring homomorphism $\{\mathcal{O}_S(E) \to R_i\}$, having the property that the induced map

$$\mathcal{O}_S(E) \to \prod_i R_i$$

is faithfully flat, there exist morphisms $E_i \to E$ in S and factorizations $\mathcal{O}_S(E) \to R_i \to \mathcal{O}_S(E_i)$ having the property that the induced map

$$\coprod E_i \to E$$

is an epimorphism.

Remark 4.3. In fact, there exists a canonical choice for E_i. For any ring homomorphism $\mathcal{O}_S(E) \to R$, the functor $U \mapsto \text{Hom}_{\mathcal{O}_S(E)}(R, \mathcal{O}_S(U))$ is representable by an object $U_0 \to E$. If we imagine that R is presented over $\mathcal{O}_S(E)$ by generators and relations, then U_0 may be thought of as the "sheaf of solutions" to the corresponding equations.

Remark 4.4. If S has enough points, then $(S, \mathcal{O}_S)$ is local for the étale topology if and only if the stalk $\mathcal{O}_{S,s}$ at any point s of S is a strictly Henselian local ring. In particular, this implies that each stalk $\mathcal{O}_{S,s}$ is local so that $(S, \mathcal{O}_S)$ is a locally ringed topos in the usual sense.

It follows that the étale topos of a Deligne-Mumford stack is local for the étale topology. Similarly, if $(S, \mathcal{O}_S)$ is the underlying topos of a complex analytic space, then $(S, \mathcal{O}_S)$ is local for the étale topology.

Remark 4.5. Suppose that $(S, \mathcal{O}_S)$ is local for the étale topology, and that X is representable by a scheme. Then $\text{Hom}(S, X)$ may be identified with the category of morphisms from S to X in the 2-category of locally ringed topoi.

Let $\mathcal{F}$ be any groupoid-valued functor defined on commutative rings, and let $\mathcal{F}'$ denote the stackification of $\mathcal{F}$ with respect to the étale topology. If $(S, \mathcal{O}_S)$ is local for the étale topology, then the natural map $\mathscr{H}\text{om}(S, \mathcal{F}) \to \mathscr{H}\text{om}(S, \mathcal{F}')$ is an equivalence of stacks on S.

In particular, let us suppose that X is an algebraic stack equipped with an smooth atlas $p : U \to X$, so that $(U, U \times_X U)$ extends naturally to a groupoid object in the category of algebraic spaces. This groupoid object represents a functor $\mathcal{F}$ from rings to groupoids, and X represents the stackification of the functor $\mathcal{F}$ with respect to the étale topology. If $(S, \mathcal{O}_S)$ is local for the étale topology, then we get $\text{Hom}(S, X) \simeq \text{Hom}(S, \mathcal{F})$. It follows that $\text{Hom}(S, X)$ can be computed in terms of any atlas for X. More concretely, this means that:

- Locally on S, any morphism $f : S \to X$ factors through U.
- Given any two morphisms $f, g : S \to U$, any isomorphism $\alpha : p \circ f \simeq p \circ g$ is induced locally by a factorization $S \xrightarrow{h} U \times_X U \to U \times U$ of $f \times g$.

Remark 4.6. If $(S, \mathcal{O}_S)$ is the underlying topos of a complex analytic space (or complex-analytic orbifold) and X is any algebraic stack of finite type over $\mathbf{C}$, then $\mathrm{Hom}_{\mathbf{C}}(S, X) \simeq \mathrm{Hom}(S, X^{\mathrm{an}})$. To prove this, we note that equality holds when X is a scheme or algebraic space, essentially by the definition of the analytification functor. In the general case, both sides are computed in the same way from a presentation of X.

We conclude this section with a brief discussion of the pullback functor f^* determined by a map $f : S \to X$. Suppose that $(S, \mathcal{O}_S)$ is a ringed topos, X an algebraic stack, and $f : S \to X$ is any morphism. Locally on S, the morphism f admits a factorization $S \to \operatorname{Spec}\Gamma(S, \mathcal{O}_S) \to X$, and we may define f^* as the composite of the usual pullback functor $\mathrm{QC}_X \to \mathrm{QC}_{\operatorname{Spec}\Gamma(S,\mathcal{O}_S)} = \mathcal{M}_{\Gamma(S,\mathcal{O}_S)}$, followed by the functor

$$M \mapsto \mathcal{O}_S \otimes_{\Gamma(S,\mathcal{O}_S)} M$$

from $\mathcal{M}_{\Gamma(S,\mathcal{O}_S)}$ to $\mathcal{M}_{\mathcal{O}_S}$. This local construction is natural and therefore makes sense even when f does not factor through $\operatorname{Spec}\Gamma(S, \mathcal{O}_S)$. Moreover, the functor f^* is compatible with tensor products in the sense that there exist natural isomorphisms

$$\gamma_{M,N} : f^*(M \otimes N) \simeq f^*M \otimes f^*N$$

$$\epsilon : f^* \mathcal{O}_X \simeq \mathcal{O}_S\,.$$

The functor f^* and the coherence data $\{\gamma_{M,N}, \epsilon\}$ enjoy a number of additional properties which the next section will place in a more formal context.

5. Abelian Tensor Categories

The main step in the proof of Theorem 1.1 is Theorem 5.11, which asserts roughly that a geometric stack X is determined by the category QC_X. In order to make a more precise statement, we must first decide what sort of object QC_X is. The relevant definitions and the statement of our main result, Theorem 5.11, will be given in this section.

Recall that a *symmetric monoidal category* is a category $\mathcal{C}$ equipped with a tensor product bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is coherently unital, associative, and commutative. This means that there exists an object $1 \in \mathcal{C}$ and natural isomorphisms

$$1 \otimes M \simeq M \simeq M \otimes 1$$

$$(M \otimes N) \otimes P \simeq M \otimes (N \otimes P)$$

$$M \otimes N \simeq N \otimes M.$$

These isomorphisms are required to specify a number of coherence conditions: for a discussion, we refer the reader to [5]. These conditions are evidently satisfied in the cases of relevance to us, and will not play an important role in this paper.

Remark 5.1. The commutativity and associativity isomorphisms are part of the data of a symmetric monoidal category. However, we will abuse notation and simply refer to $(\mathcal{C}, \otimes)$ or $\mathcal{C}$ as a symmetric monoidal category.

We also recall that a *Grothendieck abelian category* is an abelian category with a generator which satisfies the axiom (AB5) of [2]: the existence and exactness of (small) filtered colimits.

Definition 5.2. An *abelian tensor category* is a symmetric monoidal category $(\mathcal{C}, \otimes)$ with the following properties:

1. The underlying category $\mathcal{C}$ is an abelian category.
2. For any fixed object $M \in \mathcal{C}$, the functor $N \mapsto M \otimes N$ commutes with finite colimits. Equivalently, the tensor product operation $\otimes$ is additive and right-exact.

We shall say that $(\mathcal{C}, \otimes)$ is *complete* if $\mathcal{C}$ is a Grothendieck abelian category and the functor $N \mapsto M \otimes N$ commutes with *all* (small) colimits, for each fixed object $M \in \mathcal{C}$.

Remark 5.3. If $(\mathcal{C}, \otimes)$ is an abelian tensor category such that the underlying category $\mathcal{C}$ is Grothendieck, then $(\mathcal{C}, \otimes)$ is complete if and only if for each $M \in \mathcal{C}$, the functor

$$N \mapsto M \otimes N$$

has a right adjoint

$$N \mapsto \mathrm{Hom}(M, N).$$

The "if" direction is easy and the reverse implication follows from the adjoint functor theorem.

If $(\mathcal{C}, \otimes)$ is an abelian tensor category and $M \in \mathcal{C}$, then we shall say that M is *flat* if the functor $N \mapsto M \otimes N$ is an exact functor. We shall say that $(\mathcal{C}, \otimes)$ is *tame* if it has the following property: for any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathcal{C}$ such that M'' is flat, and any $N \in \mathcal{C}$, the induced sequence

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is also exact. Any abelian tensor category which has enough flat objects to set up a theory of flat resolutions is tame: this follows from vanishing of the group $\mathrm{Tor}_1(M'', N)$. We will need to work with abelian tensor categories which do not satisfy the latter condition; however, all of the abelian tensor categories which we will encounter will be tame.

Lemma 5.4. *Let $(\mathcal{C}, \otimes)$ be a tame abelian tensor category, and let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence in $\mathcal{C}$. Suppose that M'' is flat. Then M is flat if and only if M' is flat.

Proof. Let

$$0 \to N' \to N \to N'' \to 0$$

be any short exact sequence in $\mathcal{C}$. A simple diagram chase shows that $M \otimes N' \to M \otimes N$ is a monomorphism if and only if $M' \otimes N' \to M' \otimes N$ is a monomorphism. □

An *algebra* in $\mathcal{C}$ is a commutative monoid in $\mathcal{C}$: that is, it is an object $A \in \mathcal{C}$ equipped with a commutative and associative multiplication $A \otimes A \to A$ and a unit $1 \to A$ (here $1 \in \mathcal{C}$ denotes the unit for the tensor product) satisfying the usual identities.

Lemma 5.5. *Let $(\mathcal{C}, \otimes)$ be a tame abelian tensor category containing an algebra A. The following conditions are equivalent:*

1. *The algebra A is flat, and $A \otimes M = 0$ implies $M = 0$.*
2. *The unit morphism $u : 1 \to A$ is a monomorphism, and the cokernel of u is flat.*

Proof. Let us first suppose that (1) is satisfied and prove (2). This part of the argument will not require the assumption that $\mathcal{C}$ is tame. In order to prove that u is a monomorphism, it suffices to prove that u is a monomorphism after tensoring with A (since tensor product with A cannot annihilate the kernel of u unless the kernel of u is zero). But $u \otimes A : A \to A \otimes A$ is split by the multiplication $A \otimes A \to A$.

A similar argument proves that the cokernel A' of u is flat. Since $u \otimes A$ is split injective, $A' \otimes A$ is a direct summand of $A \otimes A$. It follows that $A' \otimes A$ is flat. Let

$$0 \to N' \to N \to N'' \to 0$$

be any exact sequence in $\mathcal{C}$. Tensoring with A', we obtain an exact sequence

$$0 \to K \to N' \otimes A' \to N \otimes A' \to N'' \otimes A' \to 0.$$

Since $A' \otimes A$ is flat, we deduce (from the flatness of A) that $K \otimes A = 0$, so that our hypothesis implies that $K = 0$.

Now suppose that (2) is satisfied. Since the cokernel of u is flat, A is an extension of flat objects of $\mathcal{C}$ and therefore flat by Lemma 5.4. Suppose that $A \otimes M = 0$. Since the cokernel of u is flat, the assumption that $\mathcal{C}$ is tame implies that $1 \otimes M \to A \otimes M$ is a monomorphism, so that $M \simeq 1 \otimes M \simeq 0$. □

Definition 5.6. Let $(\mathcal{C}, \otimes)$ be a tame abelian tensor category. An algebra $A \in \mathcal{C}$ is *faithfully flat* if the equivalent conditions of Lemma 5.5 are satisfied.

We now give some examples of abelian tensor categories.

Example 5.7. Let $(S, \mathcal{O}_S)$ be a ringed topos. Then the usual tensor product operation endows the category $\mathcal{M}_{\mathcal{O}_S}$ with the structure of a complete abelian tensor category. Moreover, $\mathcal{M}_{\mathcal{O}_S}$ is tame. This follows from the fact that $\mathcal{O}_S$ has enough flat sheaves to set up a good theory of Tor-functors.

If S has enough points, then a sheaf of $\mathcal{O}_S$-modules ($\mathcal{O}_S$-algebras) is flat (faithfully flat) if and only its stalk at every point $s \in S$ is flat (faithfully flat) as an $\mathcal{O}_{S,s}$-module (algebra).

Example 5.8. Let X be an algebraic stack. Then the category QC_X, equipped with its usual tensor structure, is a complete abelian tensor category. However, we must distinguish between two potentially different notions of flatness. We will call an object $M \in \mathrm{QC}_X$ *globally flat* if it is flat in the sense defined above: that is, $N \mapsto M \otimes N$ is an exact functor from QC_X to itself. We shall call an object $M \in \mathrm{QC}_X$ *locally flat* if it is flat in the usual algebro-geometric sense: that is, for morphism $f : \operatorname{Spec} A \to X$, the A-module $\Gamma(\operatorname{Spec} A, f^*M)$ is flat. It is easy to see that any locally flat module is globally flat, but the converse is unclear.

If X is geometric, then any globally flat object $M \in \mathrm{QC}_X$ is locally flat. To prove this, let us choose a smooth surjection $p : U \to X$, where $U = \operatorname{Spec} A$ is affine. To show that M is locally flat, we need to show that p^*M is flat as a quasi-coherent sheaf on U. In other words, we need to show that the functor $N \mapsto p^*M \otimes N$ is an exact functor from QC_U to itself. It suffices to prove the exactness after composing with the pullback functor $\mathrm{QC}_U \to \mathrm{QC}_{U \times_X U}$. Using the appropriate base-change formula, this is equivalent to the assertion that the functor

$$N \mapsto p^* p_* (p^* M \otimes N)$$

is exact. Making use of the natural push-pull isomorphism $p_*(p^*M \otimes N) \simeq M \otimes p_* N$, we are reduced to proving the exactness of the functor

$$N \mapsto p^*(M \otimes p_* N).$$

This is clear, since the functor is a composite of the exact functors p_*, p^*, and $M \otimes \bullet$.

From the equivalence of local and global flatness, we may deduce that QC_X is tame whenever X is a geometric stack.

Our next goal is to describe the appropriate notion of functor between abelian tensor categories.

Definition 5.9. Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be abelian tensor categories. An *additive tensor functor* F^* from $\mathcal{C}$ to $\mathcal{C}'$ is a symmetric monoidal functor (that is, a functor which is compatible with the symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{C}'$ up to natural isomorphism; see [5] for a discussion) which commutes with finite colimits (this latter condition is equivalent to the condition that F^* be additive and right-exact).

If $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ are complete, then we shall say that F^* is *continuous* if it commutes with all colimits.

We shall say that F^* is *tame* if it possesses the following additional properties:

- If $M \in \mathcal{C}$ is flat, then $F^*M \in \mathcal{C}'$ is flat.
- If
$$0 \to M' \to M \to M'' \to 0$$
is a short exact sequence in $\mathcal{C}$ and M'' is flat, then the induced sequence
$$0 \to F^*M' \to F^*M \to F^*M'' \to 0$$
is exact in $\mathcal{C}'$.

Remark 5.10. Let F^* be an additive tensor functor between abelian tensor categories $\mathcal{C}$ and $\mathcal{C}'$. Since F^* is a symmetric monoidal functor, it carries algebra objects in $\mathcal{C}$ to algebra objects in $\mathcal{C}'$. If, in addition, F^* is tame, then it carries faithfully flat algebras in $\mathcal{C}$ to faithfully flat algebras in $\mathcal{C}'$: this is clear from the second characterization given in Lemma 5.5.

If $\mathcal{C}$ and $\mathcal{C}'$ are complete, tame, abelian tensor categories, then we shall let $\mathrm{Hom}_{\otimes}(\mathcal{C}, \mathcal{C}')$ denote the groupoid of continuous, tame, additive tensor functors from $\mathcal{C}$ to $\mathcal{C}'$ (where the morphisms are given by isomorphisms of symmetric monoidal functors). It is a full subcategory of the groupoid of all monoidal functors from $\mathcal{C}$ to $\mathcal{C}'$. We remark that the notation is slightly abusive: the category $\mathrm{Hom}_{\otimes}(\mathcal{C}, \mathcal{C}')$ depends on the symmetric monoidal structures on $\mathcal{C}$ and $\mathcal{C}'$, and not only on the underlying categories.

If $(S, \mathcal{O}_S)$ is a ringed topos, X an algebraic stack, and $f : S \to X$ is any morphism, then in the last section we constructed an associated pullback functor $f^* : \mathrm{QC}_X \to \mathcal{M}_{\mathcal{O}_S}$. From the local description of f^*, it is easy to see that f^* is a continuous, tame, additive tensor functor from QC_X to $\mathcal{M}_{\mathcal{O}_S}$. We are now prepared to state the main result of this paper:

Theorem 5.11. *Suppose that $(S, \mathcal{O}_S)$ is a ringed topos which is local for the étale topology and that X is a geometric stack. Then the functor*

$$f \mapsto f^*$$

induces an equivalence of categories

$$T : \mathrm{Hom}(S, X) \to \mathrm{Hom}_{\otimes}(\mathrm{QC}_X, \mathcal{M}_{\mathcal{O}_S}).$$

The proof of Theorem 5.11 will occupy the next four sections of this paper.

Remark 5.12. Let X and S be arbitrary algebraic stacks, and define $\mathrm{Hom}'_{\otimes}(\mathrm{QC}_X, \mathrm{QC}_S) \subseteq \mathrm{Hom}_{\otimes}(\mathrm{QC}_X, \mathrm{QC}_S)$ to be the full subcategory consisting of tensor functors which carry flat objects of QC_X to *locally flat* objects of QC_S (that is, objects of QC_S which are flat according the usual definition). In particular, if every flat object of QC_S is locally flat (for example, if S is geometric), then $\mathrm{Hom}'_{\otimes}(\mathrm{QC}_X, \mathrm{QC}_S) = \mathrm{Hom}_{\otimes}(\mathrm{QC}_X, \mathrm{QC}_S)$.

We note that $\mathrm{Hom}(S, X)$ and $\mathrm{Hom}'_{\otimes}(\mathrm{QC}_X, \mathrm{QC}_S)$ are both stacks with respect to the smooth topology on S. Consequently, to prove that $\mathrm{Hom}(S, X) \simeq \mathrm{Hom}'_{\otimes}(\mathrm{QC}_X, \mathcal{M}_{\mathcal{O}_S})$ we may work locally on S and thereby reduce to the case where S is affine scheme. In this case, the result follows from Theorem 5.11, at least when X is geometric. Consequently, Theorem 5.11 and Example 5.8 imply that the functor

$$X \mapsto \mathrm{QC}_X$$

is a fully faithful embedding of the 2-category of geometric stacks into the 2-category of tame, complete abelian tensor categories.

Unfortunately, it seems very difficult to say anything about the essential image of this functor: that is, to address the question of when an abelian tensor category arises as the category of quasi-coherent sheaves on a geometric stack.

6. The Proof that T is Faithful

In this section we give the argument for the first and easiest step in the proof of Theorem 5.11: showing that T is faithful. Since $\mathrm{Hom}(S, X)$ is a groupoid, this reduces to the following assertion: if $F : S \to X$ is any morphism, and α any automorphism of F such that the natural transformation $T\alpha : F^* \to F^*$ is the identity, then α is the identity.

Let $(S, \mathcal{O}_S)$ be any ringed topos and $p : U \to X$ a morphism of algebraic stacks, and let $F : S \to X$ be any morphism. We note that the $\mathcal{O}_X$-algebra morphism $\mathcal{O}_X \to p_* \mathcal{O}_U$ acquires a canonical section after pullback to U. We deduce the existence of a natural map θ from the set of factorizations $\{f : S \to U | p \circ f = F\}$ to the set of sections of the algebra homomorphism $\mathcal{O}_S \to F^* p_* \mathcal{O}_U$. The crucial observation is the following:

Lemma 6.1. *If $(S, \mathcal{O}_S)$ is local for the étale topology and p is affine, then θ is bijective.*

Proof. The assertion is local on S. We may therefore suppose that F factors through some smooth morphism $V \to X$, where V is an affine scheme. Replacing X by V and U by $U \times_X V$ (and noting that the formation of p_* is compatible with the flat base change $V \to X$), we may reduce to the case in which X and U are affine schemes. In this situation, the result is obvious. □

Let us now return to the setting of Theorem 5.11. Since X is quasi-compact, there exists a smooth surjection $p : U \to X$, where U is an affine scheme. Since the diagonal of X is affine, p is an affine morphism. Let $\mathcal{A} = F^* p_* \mathcal{O}_U$. The condition that α be the identity is local on S; since p is surjective, we may suppose the existence of a factorization $S \xrightarrow{f} U \xrightarrow{p} X$ for F. Let $\overline{f} = \theta f : \mathcal{A} \to \mathcal{O}_S$ be the morphism of sheaves of algebras classifying f.

The morphism α induces a factorization $S \to U \times_X U \to X$, which is classified by the $\mathcal{O}_S$-algebra map

$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \stackrel{1 \otimes T\alpha(\mathcal{A})}{\to} \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \stackrel{\overline{f} \otimes \overline{f}}{\to} \mathcal{O}_S .$$

If $T\alpha$ is the identity, then this $\mathcal{O}_S$-algebra map coincides with the map $\theta(\Delta \circ f)$ classifying the composition

$$S \xrightarrow{f} U \xrightarrow{\Delta} U \times_X U.$$

Since θ is injective, we deduce that α is the identity.

7. The Proof that T is Full

Our next goal is to prove that the functor T is full. Concretely, this means that given any pair of morphisms $F, G : S \to X$ and any isomorphism $\beta : F^* \to G^*$, there exists an isomorphism $\alpha : F \to G$ with $T\alpha = \beta$.

Since we have already shown that α is uniquely determined, it suffices to construct α locally on S. We may therefore suppose that F factors as $S \xrightarrow{f} U \xrightarrow{p} X$, where $U = \operatorname{Spec} A$ is affine and p is a smooth surjection. Similarly, we may suppose that G factors as $S \xrightarrow{g} V \xrightarrow{q} X$, where $V = \operatorname{Spec} B$ is affine and q is a smooth surjection. Of course, we could take $U = V$ and $p = q$, but this would lead to unnecessary confusion.

Let $\mathcal{A} = F^* p_* \mathcal{O}_U$ and $\mathcal{B} = G^* q_* \mathcal{O}_V$. Then $\mathcal{A}$ and $\mathcal{B}$ are sheaves of $\mathcal{O}_S$-algebras, and

$$\mathcal{A} \otimes \mathcal{B} \stackrel{\beta}{\simeq} G^* p_* \mathcal{O}_U \otimes G^* q_* \mathcal{O}_V \simeq G^* r_* \mathcal{O}_W,$$

where $r : W = U \times_X V \to X$ is the natural projection. The sections f and g induce morphisms $\mathcal{A} \to \mathcal{O}_S$, $\mathcal{B} \to \mathcal{O}_S$ of sheaves of algebras. Tensoring them together, we obtain a morphism $\eta : G^* r_* \mathcal{O}_W \to \mathcal{O}_S$ which classifies a morphism $h : S \to W$. It is clear from the construction that h induces an isomorphism $\alpha : F \to G$.

To complete the proof, it will suffice to show that $T\alpha = \beta$. In other words, we must show that for any $M \in \mathrm{QC}_X$, the induced maps $T\alpha(M), \beta(M) : F^* M \to G^* M$ coincide. Since F factors through $r : W \to X$, the sheaf $F^* M$ is a direct factor of $F^* r_* r^* M$. Therefore we may suppose that $M = r_* N$ for some $N \in \mathrm{QC}_W$. Choosing a surjection $N' \to N$ with N' free, we may reduce to the case where N is free (since F^* is right exact). Using the fact that F^* commutes with direct sums, we may reduce to the case where $N = \mathcal{O}_W$. In this case, $F^* M \simeq \mathcal{A} \otimes_A C$, $G^* M \simeq \mathcal{B} \otimes_B C$. We now observe that both $T\alpha$ and β are implemented by the isomorphism $\mathcal{A} \otimes_A C \simeq \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B} \simeq \mathcal{B} \otimes_B C$.

8. Interlude

Before we can complete the proof of Theorem 5.11, we need to introduce some additional terminology and establish some lemmas.

Definition 8.1. Let S be a topos, $\mathcal{A}$ a sheaf of rings on S. A sheaf of $\mathcal{A}$-modules $\mathcal{F}$ is *locally finitely presented* if S admits a covering by objects $\{U_\alpha\}$ such that each $\mathcal{F} | U_\alpha$ is isomorphic to $\mathcal{A} \otimes_{\mathcal{A}(U_\alpha)} F_\alpha$ for some finitely presented $\mathcal{A}(U_\alpha)$-module F_α. We will say that $\mathcal{F}$ is *locally projective* if each F_α may be chosen to be a (finitely generated) projective module over $\mathcal{A}(U_\alpha)$.

Similarly, if $\mathcal{B}$ is a sheaf of $\mathcal{A}$-algebras on S, then we shall say that $\mathcal{B}$ is *smooth* over $\mathcal{A}$ if S admits a covering by objects $\{U_\alpha\}$ such that $\mathcal{B} | U_\alpha \simeq \mathcal{A} | U_\alpha \otimes_{\mathcal{A}(U_\alpha)} R_\alpha$, where each R_α is a smooth $\mathcal{A}(U_\alpha)$-algebra.

Lemma 8.2. *Suppose that $(S, \mathcal{O}_S)$ is a ringed topos which is local for the étale topology. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras which is smooth and faithfully flat over $\mathcal{O}_S$. Then, locally on S, there exists a section $\mathcal{A} \to \mathcal{O}_S$.*

Proof. Since the assertion is local on S, we may suppose that $\mathcal{A} \simeq \mathcal{O}_S \otimes_{\Gamma(S, \mathcal{O}_S)} A$, where A is a smooth $\Gamma(S, \mathcal{O}_S)$-algebra. Consequently, the map $\operatorname{Spec} A \to \operatorname{Spec} \Gamma(S, \mathcal{O}_S)$ is open. Thus, there exist finitely many global sections $\{s_1, \ldots, s_n\}$ of $\mathcal{O}_S$ such that $A[\frac{1}{s_i}]$ is faithfully flat over $\Gamma(S, \mathcal{O}_S)[\frac{1}{s_i}]$, and

$$A \to A[\frac{1}{s_1}] \times \ldots \times A[\frac{1}{s_n}]$$

is faithfully flat. Let $\mathcal{I}$ denote the ideal sheaf of $\mathcal{O}_S$ generated by $\{s_1, \ldots, s_n\}$. Then $\mathcal{O}_S/\mathcal{I}$ is annihilated by tensor product with $A[\frac{1}{s_i}]$ for each i, and therefore annihilated by tensor product with A. Since $\mathcal{A}$ is faithfully flat, we deduce that $\mathcal{O}_S/\mathcal{I} = 0$. Thus, the global sections $\{s_1, \ldots, s_n\}$ generate the unit ideal sheaf. Shrinking S further, we may suppose that the s_i generate the unit ideal in $\Gamma(S, \mathcal{O}_S)$. This implies that $\operatorname{Spec} A \to \operatorname{Spec} \Gamma(S, \mathcal{O}_S)$ is surjective. Since A is smooth over $\Gamma(S, \mathcal{O}_S)$, we deduce the existence of a section $A \to R$, where R is étale and faithfully flat over $\Gamma(S, \mathcal{O}_S)$. Since $\mathcal{O}_S$ is local for the étale topology, we may (after shrinking S) assume the existence of a section $R \to \Gamma(S, \mathcal{O}_S)$. The composite homomorphism $A \to \Gamma(S, \mathcal{O}_S)$ induces the desired section $\mathcal{A} \to \mathcal{O}_S$. □

Lemma 8.3. *Let $\mathcal{A}$ be a sheaf of rings on a topos S, and let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules. Then $\mathcal{F}$ is locally projective if and only if it is locally finitely presented and, for each $U \in S$, the functor $\mathscr{H}\mathrm{om}(\mathcal{F}|U, \bullet)$ is an exact functor from $\mathcal{M}_{\mathcal{A}|U}$ to itself.*

Proof. The "only if" direction is obvious. For the converse, we may locally choose a surjection $\mathcal{A}^n \to \mathcal{F}$. The hypothesis implies that the identity map $\mathcal{F} \to \mathcal{F}$ admits a lifting $\mathcal{F} \to \mathcal{A}^n$, at least locally on S, so we may write $\mathcal{A}^n \simeq \mathcal{F} \oplus \mathcal{F}'$. Consequently, $P = \Gamma(S, \mathcal{F})$ is a direct summand of $\Gamma(S, \mathcal{A}^n)$ and is therefore a finitely generated, projective $\Gamma(S, \mathcal{A})$-module; let $P' = \Gamma(S, \mathcal{F}')$ denote the complementary factor. Let $\mathcal{P} = \mathcal{A} \otimes_{\Gamma(S,\mathcal{A})} P$ and $\mathcal{P}' = \mathcal{A} \otimes_{\Gamma(S,\mathcal{A})} P$.

The isomorphisms $P \to \Gamma(S, \mathcal{F})$ and $P' \to \Gamma(S, \mathcal{F}')$ induce maps $\alpha : \mathcal{P} \to \mathcal{F}$ and $\alpha' : \mathcal{P}' \to \mathcal{F}'$. The direct sum $\alpha \oplus \alpha'$ is the isomorphism

$$\mathcal{P} \oplus \mathcal{P}' \simeq \mathcal{A} \otimes_{\Gamma(S,\mathcal{A})} (P \oplus P') \simeq \mathcal{A}^n .$$

It follows that α and α' are both isomorphisms, so that $\mathcal{F} \simeq \mathcal{P}$ is locally projective. □

Lemma 8.4. *Let S be a topos, $\mathcal{A}$ a sheaf of rings on S, $\mathcal{B}$ a faithfully flat sheaf of $\mathcal{A}$-algebras, and $\mathcal{F}$ a locally finitely presented sheaf of $\mathcal{A}$-modules. Then $\mathcal{F}$ is locally projective (as a sheaf of $\mathcal{A}$-modules) if and only if $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{B}$ is locally projective (as a sheaf of $\mathcal{B}$-modules).*

Proof. The "only if" direction is obvious. For the converse, suppose that $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{B}$ is locally projective. Since $\mathcal{F}$ is locally finitely presented, the assumption that $\mathcal{B}$ is flat over $\mathcal{A}$ implies that

$$\mathscr{H}\mathrm{om}_{\mathcal{M}_{\mathcal{A}}}(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathscr{H}\mathrm{om}_{\mathcal{M}_{\mathcal{B}}}(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{G} \otimes_{\mathcal{A}} \mathcal{B}).$$

The exactness of the latter functor implies the exactness of the former, since $\mathcal{B}$ is faithfully flat over $\mathcal{A}$. Now we simply apply Lemma 8.3. □

Lemma 8.5. *Let $(S, \mathcal{O}_S)$ be a ringed topos, let $f : \mathcal{B}' \to \mathcal{B}$ be a map of $\mathcal{O}_S$-algebras, and let $\mathcal{A}$ be a faithfully flat $\mathcal{O}_S$-algebra. Suppose that the map $f \otimes_{\mathcal{O}_S} \mathcal{A} : \mathcal{B}' \otimes_{\mathcal{O}_S} \mathcal{A} \simeq \mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{A}$ extends to an isomorphism*

$$\mathcal{B}' \otimes_{\mathcal{O}_S} \mathcal{A} \simeq (\mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{A}) \times \mathcal{C}$$

for some $\mathcal{A}$-algebra $\mathcal{C}$. Then f extends to an isomorphism

$$\mathcal{B}' \simeq \mathcal{B} \times \mathcal{C}_0$$

of $\mathcal{O}_S$-algebras.

Proof. Since $f \otimes_{\mathcal{O}_S} \mathcal{A}$ is surjective, f is surjective. Let $\mathcal{I} \subseteq \mathcal{B}'$ denote the kernel of f. The hypothesis implies that $\mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A}$ is generated by an idempotent. It follows that for any $\mathcal{A}$-algebra $\mathcal{A}'$, the ideal sheaf $\mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A}'$ is generated by a idempotent section

$$e_{\mathcal{A}'} \in \Gamma(S, \mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A}').$$

We note that the idempotent $e_{\mathcal{A}'}$ is uniquely determined and independent of the $\mathcal{A}$-algebra structure on $\mathcal{A}'$. In particular, we deduce that the image of $e_{\mathcal{A}}$ under the two natural maps

$$\Gamma(S, \mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A}) \to \Gamma(S, \mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A})$$

coincide. Since $\mathcal{A}$ is faithfully flat, we deduce that $e_{\mathcal{A}}$ belongs to the image of the injection $\Gamma(S, \mathcal{I}) \to \Gamma(S, \mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{A})$. Let us denote its preimage by $e \in \Gamma(S, \mathcal{I})$. One deduces readily that e is an idempotent which generates $\mathcal{I}$, which gives rise to the desired product decomposition for $\mathcal{B}'$. □

Lemma 8.6. *Let $(S, \mathcal{O}_S)$ be a ringed topos, and let $\mathcal{A}$ and $\mathcal{B}$ be sheaves of $\mathcal{O}_S$-algebras. Suppose that $\mathcal{A}$ is faithfully flat over $\mathcal{O}_S$ and that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B}$ is smooth over $\mathcal{A}$. Then, locally on S, we may find finitely many global sections $\{b_1, \ldots, b_n\}$ of $\mathcal{B}$ which generate the unit ideal of $\Gamma(S, \mathcal{B})$, with the property that each $\mathcal{B}[\frac{1}{b_i}]$ is smooth over $\mathcal{O}_S$.*

Proof. Let Ω denote the sheaf of relative differentials of $\mathcal{B}$ over $\mathcal{O}_S$. In other words, Ω is the sheafification of the presheaf $U \mapsto \Omega_{\mathcal{B}(U)/\mathcal{O}_S(U)}$, where $\Omega_{R/R'}$ denotes the module of relative differentials of R over R'. In particular, if $\mathcal{B} = \mathcal{O}_S \otimes_{\Gamma(S,\mathcal{O}_S)} R$, then $\Omega = \mathcal{O}_S \otimes_{\Gamma(S,\mathcal{O}_S)} \Omega_{R/\Gamma(S,\mathcal{O}_S)}$.

The sheaf $\Omega \otimes_{\mathcal{B}} (\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B})$ is isomorphic to the sheaf of relative differentials of $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B}$ over $\mathcal{A}$, and therefore locally projective. Consequently, Ω is locally projective as a $\mathcal{B}$-module by Lemma 8.4.

We may suppose that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{B}$ is generated over $\mathcal{A}$ by finitely many global section $\{x_1, \ldots, x_n\}$ of $\mathcal{B}$. The corresponding map $\mathcal{O}_S[x_1, \ldots, x_n] \to \mathcal{B}$ becomes surjective after tensoring with $\mathcal{A}$. Since $\mathcal{A}$ is faithfully flat, we deduce that $\mathcal{O}_S[x_1, \ldots, x_n] \to \mathcal{B}$ is surjective. Thus, the induced map on differentials $\mathcal{B}^n \to \Omega$ is surjective. Let P be the kernel of this map, so that P is a locally projective $\mathcal{B}$-module.

Locally on S, we may find sections $\{b_1, \ldots, b_m\}$ of $\mathcal{B}$ such that $P[\frac{1}{b_i}]$ is a finitely generated, free $\mathcal{B}[\frac{1}{b_i}]$-module. Replacing $\mathcal{B}$ by $\mathcal{B}[\frac{1}{b_i}]$, we may reduce to the case where P is free of rank $(n-k)$.

Let $\mathcal{I}$ denote the kernel of the induced map $\mathcal{O}_S[x_1, \ldots, x_n] \to \mathcal{B}$. Then $\mathcal{I}/\mathcal{I}^2 \simeq P$. Locally on S, we may find sections $\{f_1, \ldots, f_k\}$ of $\mathcal{I}$ which freely generate P over $\mathcal{B}$. Localizing S further, we may suppose that $f_1, \ldots, f_k \in \Gamma(S, \mathcal{O}_S)[x_1, \ldots, x_n]$. Let $a_1, \ldots, a_m \in \Gamma(S, \mathcal{O}_S)[x_1, \ldots, x_n]$ denote the determinants of $k \times k$ minors of the Jacobian matrix for $(f_1, \ldots, f_k)$. The assumption that the differentials $\{df_i\}$ generate a summand of $\mathcal{B}^n$ implies that $a_1, \ldots, a_m$ generate the unit ideal sheaf in $\mathcal{B}$. Consequently, working locally on S and replacing $\mathcal{B}$ by $\mathcal{B}[\frac{1}{a_i}]$, we may suppose that some a_i is invertible in $\mathcal{B}$.

Let $\mathcal{B}' = \mathcal{O}_S[x_1, \ldots, x_n][\frac{1}{a_i}]/(f_1, \ldots, f_k)$. Then, by construction, $\mathcal{B}'$ is smooth and we have a surjection $p : \mathcal{B}' \to \mathcal{B}$. To prove that $\mathcal{B}$ is smooth, it suffices to show that p can be extended to an isomorphism $\mathcal{B}' \simeq \mathcal{B} \times \mathcal{C}_0$. By Lemma 8.5, it will suffice to show that $f \otimes_{\mathcal{O}_S} \mathcal{A}$ can be extended to an isomorphism $\mathcal{B}' \otimes_{\mathcal{O}_S} \mathcal{A} \simeq (\mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{A}) \times \mathcal{C}$. The latter assumption is local on S, so we may suppose that $\mathcal{B}' \otimes_{\mathcal{O}_S} \mathcal{A} \simeq \mathcal{A} \otimes_{\Gamma(S,\mathcal{A})} R'$ and $\mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{A} \simeq \mathcal{A} \otimes_{\Gamma(S,\mathcal{A})} R$, where R and R' are smooth $\Gamma(S, \mathcal{A})$-algebras of relative dimension $(n-k)$ over $\Gamma(S, \mathcal{A})$, and that $f \otimes_{\mathcal{O}_S} \mathcal{A}$ is induced by a surjection $g : R' \to R$.

We note that g induces a closed immersion $\operatorname{Spec} R \to \operatorname{Spec} R'$ of smooth $\operatorname{Spec} \Gamma(S, \mathcal{A})$-schemes having the same relative dimension over $\operatorname{Spec} \Gamma(S, \mathcal{A})$. It follows that this closed immersion is also an open immersion, so that g extends to an isomorphism $R' \simeq R \times R''$, which evidently gives rise to the desired factorization of $\mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{A}$. □

Remark 8.7. It seems likely that it is possible to take $\{b_1, \ldots, b_n\} = \{1\}$ in the statement of Lemma 8.6. However, we were unable to prove this without the assumption that S has enough points. Lemma 8.6 will be sufficient for our application.

9. The Proof that T is Essentially Surjective

In this section, we will complete the proof of Theorem 5.11 by showing that the functor T is essentially surjective. In other words, we must show that if $F^* : \mathrm{QC}_X \to \mathcal{M}_{\mathcal{O}_S}$ is a tame, continuous, additive tensor functor, then F^* is the pullback functor associated to some morphism $F : S \to X$. Since we have already shown that T is fully faithful, the morphism F is uniquely determined; it therefore suffices to construct F locally on S.

Let $p : U \to X$ be a smooth surjection, where $U = \operatorname{Spec} A$ is affine. Let $\mathcal{A} = F^* p_* \mathcal{O}_U$. Since F^* is an additive tensor functor, $\mathcal{A}$ is a sheaf of $\mathcal{O}_S$-algebras. Moreover, the isomorphism $A \simeq \Gamma(U, \mathcal{O}_U)$ induces a morphism $A \to \Gamma(S, \mathcal{A})$. Let $U \times_X U \simeq \operatorname{Spec} B$ so that $p_* \mathcal{O}_U \otimes_{\mathcal{O}_X} p_* \mathcal{O}_U \simeq p_* \mathcal{O}_U \otimes_A B$. Since F^* commutes with all colimits, it commutes with external tensor products, so that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \simeq \mathcal{A} \otimes_A B$. Since B is smooth over A, we deduce that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A}$ is smooth over $\mathcal{A}$. Since F^* is a tame functor, $\mathcal{A}$ is faithfully flat over $\mathcal{O}_S$ (see Remark 5.10). Applying Lemma 8.6, we deduce (possibly after shrinking S) the existence of finitely many global sections $\{a_1, \ldots, a_n\}$ of $\mathcal{A}$, which generate the unit ideal of $\Gamma(S, \mathcal{A})$, such that $\mathcal{A}[\frac{1}{a_i}]$ is smooth. Let $\mathcal{A}'$ denote the product of the $\mathcal{O}_S$-algebras $\mathcal{A}[\frac{1}{a_i}]$. Then $\mathcal{A}'$ is smooth and faithfully flat over $\mathcal{O}_S$, so there exists a section $\mathcal{A}' \to \mathcal{O}_S$. Composing with the natural map $\mathcal{A} \to \mathcal{A}'$, we deduce the existence of a section $s : \mathcal{A} \to \mathcal{O}_S$.

The composite map $A \to \Gamma(S, \mathcal{A}) \xrightarrow{s} \Gamma(S, \mathcal{O}_S)$ induces a morphism $g : S \to \operatorname{Spec} A = U$. We claim that the composition $G = p \circ g : S \to X$ has the desired properties. To prove this, we must exhibit an isomorphism $G^* \simeq F^*$ of additive tensor functors.

Let M be a quasi-coherent sheaf on X. Then $p_* p^* M \simeq (p_* \mathcal{O}_U) \otimes_A \Gamma(U, M)$, so that $F^* p_* p^* M \simeq \mathcal{A} \otimes_A \Gamma(U, M)$. Composing with the adjunction morphism $M \to p_* p^* M$ and using the section $s : \mathcal{A} \to \mathcal{O}_S$, we deduce the existence of a natural transformation $\beta_M : F^* M \to \mathcal{O}_S \otimes_A M = G^* M$. It is easy to see that this is a map of additive tensor functors. To complete the proof, it suffices to show that β_M is an isomorphism for every M.

Since $\mathcal{A}$ is faithfully flat, to show that β_M is an isomorphism it suffices to show that $\beta_M \otimes_{\mathcal{O}_S} \mathcal{A} = \beta_{M \otimes p_* \mathcal{O}_U}$ is an isomorphism. In particular, we may suppose that $M = p_* N$ for some $\mathcal{O}_U$-module N. Since both F^* and G^* are right exact and commute with all direct sums, we may reduce to the case where $N = \mathcal{O}_U$. In this case one can easily compute that $F^* M \simeq \mathcal{A} \simeq G^* M$ and that β_M corresponds to the identity map.

10. The Proof of Theorem 1.1

The goal of this section is to show that Theorem 5.11 implies Theorem 1.1. The main difficulty that needs to be overcome is that Theorem 5.11 is concerned with categories of quasi-coherent sheaves, which are rather unwieldy. We therefore specialize to the case where X is a *Noetherian* geometric stack. In this case, we have a well-behaved subcategory of coherent sheaves $\mathrm{Coh}_X \subseteq \mathrm{QC}_X$. We let $\mathrm{Coh}_S \subseteq \mathcal{M}_{\mathcal{O}_S}$ denote the category of locally finitely presented $\mathcal{O}_S$-modules. If $\mathcal{O}_S$ is a coherent sheaf of rings in the usual sense, then this agrees with the usual notion of a coherent $\mathcal{O}_S$-module. If $\mathcal{O}_S$ is not coherent, then Coh_S need not be an abelian subcategory of $\mathcal{M}_{\mathcal{O}_S}$, but this does not impact any of the statements which follow.

By Lemma 3.9, the inclusion $\mathrm{Coh}_X \subseteq \mathrm{QC}_X$ induces an equivalence of categories $\mathrm{Ind}(\mathrm{Coh}_X) \simeq \mathrm{QC}_X$. It follows that the category of continuous, additive tensor functors from QC_X to $\mathcal{M}_{\mathcal{O}_S}$ is equivalent to the category $\mathcal{C}$ of additive tensor functors from Coh_X to $\mathcal{M}_{\mathcal{O}_S}$. By Theorem 5.11, $\mathrm{Hom}(S, X)$ is equivalent to the full subcategory of $\mathcal{C}$, consisting of those functors $\mathrm{Coh}_X \to \mathcal{M}_{\mathcal{O}_S}$ which admit continuous, tame extensions to QC_X. Since Theorem 5.11 implies that any such functor has the form f^* for some map $S \to X$, it must carry coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_S$-modules (this is immediate from the construction of f^*). Consequently, we may deduce the following "coherent" version of Theorem 5.11:

Corollary 10.1. *Suppose that $(S, \mathcal{O}_S)$ is ringed topos which is local for the étale topology, such that $\mathcal{O}_S$ is coherent. Let X be a Noetherian geometric stack, and let $\mathrm{Hom}(\mathrm{Coh}_X, \mathrm{Coh}_S)$ denote the groupoid of additive tensor functors from Coh_X to Coh_S. Then the natural functor*

$$\mathrm{Hom}(S, X) \to \mathrm{Hom}(\mathrm{Coh}_X, \mathrm{Coh}_S)$$

is fully faithful, and its essential image consists of precisely those functors F which extend to continuous, tame, additive tensor functors $\hat{F} : \mathrm{QC}_X \to \mathcal{M}_{\mathcal{O}_S}$.

Remark 10.2. It is unfortunate that there does not seem to be any simple criterion on the functor F which may be used to test whether or not $\hat{F}$ is tame.

We are now ready to give the proof of Theorem 1.1.

Proof. We have already noted that $\mathrm{Hom}(S^{\mathrm{an}}, X^{\mathrm{an}}) = \mathrm{Hom}_{\mathbf{C}}(S^{\mathrm{an}}, X)$. Consequently, it will suffice to prove that the natural functors $\mathrm{Hom}(S, X) \to \mathrm{Hom}(S^{\mathrm{an}}, X)$ and $\mathrm{Hom}(S, \operatorname{Spec} \mathbf{C}) \to \mathrm{Hom}(S^{\mathrm{an}}, \operatorname{Spec} \mathbf{C})$ are equivalences. We will focus on the former (the latter is just the special case where $X = \operatorname{Spec} \mathbf{C}$).

We are now free to drop the assumption that X is of finite type over $\mathbf{C}$ (or even that X is an algebraic stack over $\mathbf{C}$); all we will need to know is that X is Noetherian and geometric. We can therefore apply Corollary 10.1 to deduce that $\mathrm{Hom}(S^{\mathrm{an}}, X)$ and $\mathrm{Hom}(S, X)$ are equivalent to full subcategories $\mathcal{C}_0 \subseteq \mathcal{C} = \mathrm{Hom}(\mathrm{Coh}_X, \mathrm{Coh}_{S^{\mathrm{an}}})$ and $\mathcal{C}'_0 \subseteq \mathcal{C}' = \mathrm{Hom}(\mathrm{Coh}_X, \mathrm{Coh}_S)$.

Using an appropriate generalization of Serre's GAGA theorem, we may deduce that Coh_S is equivalent to $\mathrm{Coh}_{S^{\mathrm{an}}}$ as an abelian tensor category. It follows that we may identify $\mathcal{C}$ with $\mathcal{C}'$. To complete the proof, it suffices to show that the subcategories $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{C}'_0 \subseteq \mathcal{C}'$ coincide (under the identification of $\mathcal{C}$ with $\mathcal{C}'$). Let $F : \mathrm{Coh}_X \to \mathrm{Coh}_S$ be any additive tensor functor. Then F admits continuous extensions

$$\hat{F} : \mathrm{QC}_X \to \mathcal{M}_{\mathcal{O}_S}$$
$$\hat{F}' : \mathrm{QC}_X \to \mathcal{M}_{\mathcal{O}_{S^{\mathrm{an}}}}.$$

We must show that $\hat{F}$ is tame if and only if $\hat{F}'$ is tame.

The essential point is to observe that $\hat{F}'M = p^*\hat{F}M$, where $p : (S^{\mathrm{an}}, \mathcal{O}_{S^{\mathrm{an}}}) \to (S, \mathcal{O}_S)$ is the natural map. It follows immediately that if $\hat{F}$ is tame, then $\hat{F}'$ is tame. For the converse, let us suppose that $\hat{F}'$ is tame. Let $M \in \mathrm{QC}_X$ be flat. Then $\hat{F}'M \in \mathcal{M}_{\mathcal{O}_{S^{\mathrm{an}}}}$ is flat, so that the stalk $(\hat{F}'M)_s = (\hat{F}M)_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S^{\mathrm{an}},s}$ is flat over $\mathcal{O}_{S^{\mathrm{an}},s}$ at every closed point s of S. Since $\mathcal{O}_{S^{\mathrm{an}},s}$ is faithfully flat over $\mathcal{O}_{S,s}$, we deduce that $(\hat{F}M)_s$ is flat over $\mathcal{O}_{S,s}$ at every closed point s of S. Since $\hat{F}M$ is a quasi-coherent sheaf on S, this implies that $\hat{F}M$ is flat.

Let us now suppose that

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of quasi-coherent sheaves on X, with M'' flat. We wish to show that this sequence remains exact after applying the functor $\hat{F}$. It suffices to show that the map $p : \hat{F}M' \to \hat{F}M$ is injective. The kernel of p is quasi-coherent. Consequently, if the kernel of p is nonzero, then it has a nonzero stalk at some closed point $s \in S$. Since $\mathcal{O}_{S^{\mathrm{an}},s}$ is faithfully flat over $\mathcal{O}_{S,s}$, we deduce that the map $\hat{F}'M' \to \hat{F}'M$ is not injective at the point $s \in S^{\mathrm{an}}$, a contradiction. □

Remark 10.3. The same method may be used to prove comparison theorems in the formal and rigid-analytic contexts. In the latter case, one must take care to employ the proper definition of $\mathrm{Hom}(S, X)$ when S is a rigid analytic space, making use of the étale topology on S rather than the usual Tate topology.

Remark 10.4. Theorem 1.1 is not necessarily true if the stack X is not assumed to be geometric. For example, it can fail if X is the classifying stack of an abelian variety.

References

[1] Artin, M., Grothendieck, A., and J.L. Verdier. *Théorie des Topos et Cohomologie Etale des Schémas (SGA 4).* Lecture Notes in Mathematics 269, Springer-Verlag, 1972.

[2] Grothendieck, A. *Sur Quelques Points d'Algébre Homologique.* Tohoku Math. J. (2) 9, 1957, p. 119-221.

[3] Hai, P.H. *Tannaka-Krein Duality for Hopf Algebroids.* Available for download: math.QA/0206113.

[4] Lurie, Jacob. *Geometric Derived Stacks.* In preparation.

[5] MacLane, S. *Categories for the Working Mathematician, Second Edition.* Graduate Texts in Mathematics 5, Springer-Verlag, 1998.

[6] Serre, J.P. *Géométrie Algébrique et Géométrie Analytique.* Ann. Inst. Fourier, Grenoble 6, 1955-1956, p. 1-42.

[7] Toën, B. and G. Vezzosi. *"Brave New" Algebraic Geometry and Global Derived Moduli Spaces of Ring Spectra.* Available for download: math.AT/0309145.