Derived Algebraic Geometry XI: Descent Theorems

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Introduction

Let $f : U \to X$ be a map of schemes, and let $p, q : U \times_X U \to U$ be the two projection maps. A *descent datum* is a quasi-coherent sheaf $\mathcal{F}$ on U, together with an isomorphism $p^* \mathcal{F} \to q^* \mathcal{F}$ which satisfies a suitable cocycle condition. There is a functor from the category of quasi-coherent sheaves on X to the category of descent data, given by $\mathcal{F}_0 \mapsto (f^* \mathcal{F}_0, \alpha)$, where α denotes the evident isomorphism

$$p^*(f^* \mathcal{F}_0) \simeq (f \circ p)^* \mathcal{F}_0 = (f \circ q)^* \mathcal{F}_0 \simeq q^*(f^* \mathcal{F}_0).$$

The classical theory of *faithfully flat descent* guarantees that this functor is an equivalence of categories whenever the map f is faithfully flat and quasi-compact. This is the basis for an important technique in algebraic geometry: one can often reduce questions about X (or about quasi-coherent sheaves on X) to questions about U, which may be easier to answer.

In this paper, we will study some descent principles in spectral algebraic geometry. For simplicity, we will mainly restrict our attention to statements about *affine* spectral schemes (the passage from local to global is fairly routine). Our starting point is the ∞-categorical analogue of the equivalence of categories described above: if $A^\bullet$ is a flat hypercovering of an $\mathbb{E}_\infty$-ring A, then the canonical map

$$\mathrm{Mod}_A(\mathrm{Sp}) \to \varprojlim \mathrm{Mod}_{A^\bullet}(\mathrm{Sp})$$

is an equivalence of ∞-categories, where $\mathrm{Mod}_R(\mathrm{Sp})$ denotes the ∞-category of R-module spectra (which we will henceforth denote simply by Mod_R). This was proven in [40] as a consequence of the following more general result: if A is a connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ is an A-linear ∞-category which admits an excellent t-structure, then the construction $B \mapsto \mathrm{LMod}_B(\mathcal{C})$ satisfies descent for the flat topology (Theorem VII.6.12). Our proof of this result made essential use of the t-structure to reduce to a statement at the level of abelian categories.

One of our first main goals in this paper is to prove an analogous descent theorem (Theorem 5.4), which asserts that *any* A-linear ∞-category $\mathcal{C}$ satisfies descent with respect to the étale topology (that is, we need not assume that $\mathcal{C}$ admits an excellent t-structure). In §5, we will use this result to establish a categorified analogue of étale descent. The collection of all A-linear ∞-categories can itself be regarded as an ∞-category LinCat_A, depending functorially on A. We will show that the construction $A \mapsto \mathrm{LinCat}_A$ is a sheaf with respect to the étale topology (Theorem 5.13). Moreover, our methods can be used to show that many natural conditions on A-linear ∞-categories can be tested locally for the étale topology. We will illustrate this in §6 by reproving a result of Toën, which asserts that the condition of being compactly generated can tested locally for the étale topology (we refer the reader to [58] for another proof, and for a discussion of some applications to the theory of cohomological Brauer groups).

The first few sections of this paper are devoted to developing some general tools for proving these types of descent theorems. The basic observation (which we explain in §3) is that if $\mathcal{F}$ is a functor defined on the category of commutative rings (or some variant, such as an ∞-category of ring spectra) which is a sheaf for both the Nisnevich topology and the finite étale topology, then $\mathcal{F}$ is also a sheaf for the étale topology (Theorem 3.7). This criterion is quite useful, because Nisnevich and finite étale descent can sometimes be established by very different methods:

- The work of Morel and Voevodsky gives a simple test for Nisnevich descent in terms of Mayer-Vietoris squares." We will prove a version of their result in §2, after giving a general review of the Nisnevich topology in §1.
- Questions about finite étale coverings can often be reduced to questions about Galois coverings. We will sketch this reduction in §4; not needed for any of the later results of this paper, it seems of interest in its own right.

In §8, we explain how to globalize some of the above ideas. For example, since the theory of linear ∞-categories satisfies étale descent, we can associate to any spectral Deligne-Mumford stack $\mathfrak{X}$ an ∞-category

$\mathrm{QStk}(\mathfrak{X})$ of *quasi-coherent stacks* on $\mathfrak{X}$. When $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} A$ is affine, a quasi-coherent stack is simply given by an A-linear ∞-category. In the general case, a quasi-coherent stack on $\mathfrak{X}$ is a rule which assigns a quasi-coherent stack to any affine chart of $\mathfrak{X}$. Our main result is that, in many cases, a quasi-coherent stack on $\mathfrak{X}$ can be recovered from its ∞-category of global sections (Theorem 8.6). We will also prove an analogous statement where the spectral Deligne-Mumford stack $\mathfrak{X}$ is replaced by a more general functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (Theorem 8.11). The precise formulation of this second result requires a discussion of descent properties of linear ∞-categories with respect to the flat topology, which we carry out in §7.

In the final section of this paper (§9), we return to a discussion of linear ∞-categories equipped with a t-structure. We will show that the data of a t-structure on linear ∞-category is of a local nature (with respect to the flat topology), and use this to introduce the notion of a t-structure on a quasi-coherent stack. Our main result (Theorem 9.12) asserts that if X is a geometric stack (in the sense of Definition VIII.3.4.1), then giving a quasi-coherent stack $\mathcal{C}$ on X together with an excellent t-structure is equivalent to giving an ∞-category $\Gamma(X; \mathcal{C})$ tensored over $\mathrm{QCoh}(X)$, which equipped with a t-structure for which the action of flat objects of $\mathrm{QCoh}(X)$ is t-exact. This can be regarded as a categorification of the fact that for an affine scheme X, the category of quasi-coherent sheaves on X is equivalent to the category of modules over the commutative ring $\Gamma(X; \mathcal{O}_X)$. However, our result requires a much weaker hypothesis: we require affineness only for the diagonal $X \to X \times X$, rather than for X itself.

Notation and Terminology

We will use the language of ∞-categories freely throughout this paper. We refer the reader to [37] for a general introduction to the theory, and to [38] for a development of the theory of structured ring spectra from the ∞-categorical point of view. For convenience, we will adopt the following reference conventions:

(T) We will indicate references to [37] using the letter T.

(A) We will indicate references to [38] using the letter A.

(V) We will indicate references to [39] using the Roman numeral V.

(VII) We will indicate references to [40] using the Roman numeral VII.

($VIII$) We will indicate references to [41] using the Roman numeral VIII.

(IX) We will indicate references to [42] using the Roman numeral IX.

For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [37].

If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^{\simeq}$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all non-invertible morphisms. We will say that a map of simplicial sets $f : S \to T$ is *left cofinal* if, for every right fibration $X \to T$, the induced map of simplicial sets $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ is a homotopy equivalence of Kan complexes (in [37], we referred to a map with this property as *cofinal*). We will say that f is *right cofinal* if the induced map $S^{op} \to T^{op}$ is left cofinal: that is, if f induces a homotopy equivalence $\mathrm{Fun}_T(T, X) \to \mathrm{Fun}_T(S, X)$ for every *left* fibration $X \to T$. If S and T are ∞-categories, then f is left cofinal if and only if for every object $t \in T$, the fiber product $S \times_T T_{t/}$ is weakly contractible (Theorem T.4.1.3.1).

1 Nisnevich Coverings

In [46], Nisnevich introduced the *completely decomposed topology* (now called the *Nisnevich topology*) associated to a Noetherian scheme X of finite Krull dimension. The Nisnevich topology on X is intermediate between the Zariski and étale topologies, sharing some of the pleasant features of each. In this paper, we will need an analogue of the Nisnevich topology in the setting of possibly non-Noetherian schemes. In the Noetherian setting, one can define a Nisnevich covering to be a collection of étale maps $\{p_\alpha : U_\alpha \to X\}$

having the property that, for every point $x \in X$, there exists an index α and a point $\overline{x} \in U_\alpha$ such that $p_\alpha(\overline{x}) = x$ and p_α induces an isomorphism of residue fields $\kappa_x \to \kappa_{\overline{x}}$. To handle the general case, we need a slightly more complicated definition.

Definition 1.1. Let R be a commutative ring. We will say that a collection of étale ring homomorphisms $\{\phi_\alpha : R \to R_\alpha\}_{\alpha \in A}$ is an *Nisnevich covering* of R if there exists a finite sequence of elements $a_1, \ldots, a_n \in R$ with the following properties:

(1) The elements $a_1, \ldots, a_n$ generate the unit ideal in R.

(2) For $1 \leq i \leq n$, there exists an index α and a ring homomorphism ψ which fits into a commutative diagram

$$\begin{array}{ccc} & R_\alpha & \\ {\scriptstyle \phi_\alpha}\nearrow & & \searrow{\scriptstyle \psi} \\ R & \longrightarrow & R[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1}). \end{array}$$

Remark 1.2. If the commutative ring R is Noetherian, then Definition 1.1 recovers the classical definition of Nisnevich covering (see Proposition 1.14).

Example 1.3. For any commutative ring R, the one-element family of maps $\{\mathrm{id}_R : R \to R\}$ is a Nisnevich covering of R.

Example 1.4. Let R be the zero ring. Then the empty collection $\emptyset$ is a Nisnevich covering of R (take $n = 0$ in Definition 1.1).

Remark 1.5. Let $\{\phi_\alpha : R \to R_\alpha\}_{\alpha \in A}$ be a Nisnevich covering of a commutative ring R. Suppose we are given a family of étale maps $\{\psi_\beta : R \to R_\beta\}_{\beta \in B}$ with the following property: for each $\alpha \in A$, there exists $\beta \in B$ and a commutative diagram

$$\begin{array}{ccc} & R_\beta & \\ {\scriptstyle \psi_\beta}\nearrow & & \searrow \\ R & \xrightarrow{\phi_\alpha} & R_\alpha. \end{array}$$

Then $\{\psi_\beta : R \to R_\beta\}_{\beta \in B}$ is also a Nisnevich covering of R.

Remark 1.6. Let $\{R \to R_\alpha\}_{\alpha \in A}$ be a Nisnevich covering of a commutative ring R. Then there exists a finite subset $A_0 \subseteq A$ such that $\{R \to R_\alpha\}_{\alpha \in A_0}$ is also a Nisnevich covering of R.

Remark 1.7. Let $\{R \to R_\alpha\}_{\alpha \in A}$ be a Nisnevich covering of a commutative ring R, and suppose we are given a ring homomorphism $R \to R'$. Then the collection of induced maps $\{R' \to R' \otimes_R R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R'.

Lemma 1.8. *Let R be a commutative ring and let $\{R \to R_\alpha\}_{\alpha \in A}$ be a collection of étale maps. Suppose that there exists a nilpotent ideal $I \subseteq R$ such that the family of induced maps $\{R/I \to R_\alpha/IR_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R/I. Then $\{R \to R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R.*

Proof. Choose a sequence of elements $a_1, \ldots, a_n \in R$ which generate the unit ideal in R/I and a collection of commutative diagrams

$$\begin{array}{ccc} & R_{\alpha_i} & \\ \nearrow & & \searrow{\scriptstyle \phi_i} \\ R & \longrightarrow & (R/I)[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1}). \end{array}$$

Since each R_{α_i} is étale over R and I is a nilpotent ideal, we can lift ϕ_i to an R-algebra map $\psi_i : R_{\alpha_i} \rightarrow R[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1})$. Let J denote the ideal $(a_1, \ldots, a_n)$; since the a_i generate the unit ideal in R/I, we have $I + J = R$. It follows that $R \subseteq (I+J)^m \subseteq I^m + J$ for every integer n. Choosing n sufficiently large, we have $I^m = 0$ so that $R = J$; this proves that $\{R \rightarrow R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R. □

Lemma 1.9. *Let R be a commutative ring containing an element x. A collection of étale maps $\{R \rightarrow R_\alpha\}_{\alpha \in A}$ is a Nisnevich cover of R if and only if the following conditions are satisfied:*

(1) *The maps $\{R/xR \rightarrow R_\alpha/xR_\alpha\}_{\alpha \in A}$ are a Nisnevich cover of the quotient ring R/xR.*

(2) *The maps $\{R[\frac{1}{x}] \rightarrow R_\alpha[\frac{1}{x}]\}_{\alpha \in A}$ are a Nisnevich cover of the commutative ring $R[\frac{1}{x}]$.*

Proof. The necessity of conditions (1) and (2) follows from Lemma 1.7. For the converse, suppose that conditions (1) and (2) are satisfied. Using (2), we deduce that there exists a sequence of elements $\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n \in R[\frac{1}{x}]$ which generates the unit ideal, a sequence of indices $\alpha_1, \ldots, \alpha_n \in A$, and a sequence of commutative diagrams

$$\begin{array}{ccc} & R_{\alpha_i}[\frac{1}{x}] & \\ \nearrow & & \searrow^{\phi_i} \\ R[\frac{1}{x}] & \longrightarrow & R[\frac{1}{xa_i}]/(\overline{a}_1, \ldots, \overline{a}_{i-1}). \end{array}$$

Multiplying each $\overline{a}_i$ by a sufficiently large power of x, we may assume that $\overline{a}_i$ is the image of an element $a_i \in R$. Let J denote the ideal $(xa_1, \ldots, xa_n) \subseteq R$. Since the $\overline{a}_i$ generate the unit ideal in $R[\frac{1}{x}]$, the ideal J contains x^k for $k \gg 0$. It follows that x generates a nilpotent ideal in R/J. Using (1) and Lemma 1.8, we deduce that the maps $\{R/J \rightarrow R_\alpha/JR_\alpha\}_{\alpha \in A}$ determine a Nisnevich covering of R/J. We may therefore choose elements $b_1, \ldots, b_m \in R$ which generate the unit ideal in R/J together with commutative diagrams

$$\begin{array}{ccc} & R_{\beta_j}/JR_{\beta_j} & \\ \nearrow & & \searrow^{\psi_j} \\ R/J & \longrightarrow & R[\frac{1}{b_j}]/(J + (b_1, \ldots, b_{j-1})). \end{array}$$

Then the sequence $xa_1, xa_2, \ldots, xa_n, b_1, \ldots, b_m$ generates the unit ideal in R; using the maps $\{\phi_i\}_{1 \le i \le n}$ and $\{\psi_j\}_{1 \le j \le m}$, we see that the family of maps $\{R \rightarrow R_\alpha\}$ determines a Nisnevich covering of R. □

Proposition 1.10. *Let R be a commutative ring, and suppose we are given a Nisnevich covering $\{R \rightarrow R_\alpha\}_{\alpha \in A}$ of R. Assume furthermore that for each $\alpha \in A$, we are given a Nisnevich covering $\{R_\alpha \rightarrow R_{\alpha,\beta}\}$. Then the family of composite maps $\{R \rightarrow R_{\alpha,\beta}\}$ is a Nisnevich covering of R.*

Proof. Choose a sequence $a_1, \ldots, a_n \in R$ which generate the unit ideal and commutative diagrams

$$\begin{array}{ccc} & R_{\alpha_i} & \\ \nearrow & & \searrow \\ R & \longrightarrow & R[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1}). \end{array}$$

We prove by induction on n that the maps $\{R \rightarrow R_{\alpha,\beta}\}$ form a Nisnevich covering of R. If $n = 0$, then the unit ideal and zero ideal of R coincide, so that $R \simeq 0$ and the result is obvious (Example 1.4 and Remark 1.5). Otherwise, we may assume by the inductive hypothesis that the family of maps $\{R/a_1R \rightarrow R_{\alpha,\beta}/a_1R_{\alpha,\beta}\}$ is a Nisnevich covering of R/a_1R. According to Lemma 1.9, it will suffice to show that the maps $\{R[\frac{1}{a_1}] \rightarrow R_{\alpha,\beta}[\frac{1}{a_1}]\}$ are a Nisnevich covering of $R[\frac{1}{a_1}]$. Using Remark 1.7, we see that $\{R[\frac{1}{a_1}] \rightarrow R[\frac{1}{a_1}] \otimes_{R_{\alpha_1}} R_{\alpha_1,\beta}\}$ is a Nisnevich covering; the desired result now follows from Remark 1.5. □

Lemma 1.11. *Let R be a commutative ring containing an element a, and suppose we are given a flat map $\phi : R \to R'$. If the induced maps $R[\frac{1}{a}] \to R'[\frac{1}{a}]$ and $R/aR \to R'/aR'$ are faithfully flat, then ϕ is faithfully flat.*

Proof. Let M be a (discrete) R-module such that $M \otimes_R R' \simeq 0$; we wish to show that $M \simeq 0$. We have $M \otimes_R R'[\frac{1}{a}] \simeq 0$ so that, by the faithful flatness of $R'[\frac{1}{a}]$ over $R[\frac{1}{a}]$, we obtain $M[\frac{1}{a}] \simeq 0$. In other words, every element of M is annihilated by a power of a. If $M \neq 0$, then there exists a nonzero submodule $M' \subseteq M$ which is annihilated by a. Since R' is flat over R, we conclude that $M' \otimes_R R'$ is a submodule of $M \otimes_R R' \simeq 0$, so that $M' \otimes_R (R'/aR') \simeq 0$. The faithful flatness of $R/aR \to R_1/aR_1$ guarantees that $M'/aM' \simeq 0$. Since a annihilated M', it follows that $M' \simeq 0$, contrary to our assumption. ☐

Remark 1.12. Every Nisnevich covering $\{R \to R_\alpha\}_{\alpha \in A}$ is a covering with respect to the étale topology. To see this, choose a sequence of elements $a_1, \ldots, a_n \in R$ which generate the unit ideal and a collection of commutative diagrams

$$\begin{array}{ccc} & R_{\alpha_i} & \\ \nearrow & & \searrow \\ R & \longrightarrow & R[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1}). \end{array}$$

We prove by descending induction on i that the flat maps

$$R/(a_1, \ldots, a_i) \to \prod_{j > i} R_{\alpha_j}/(a_1, \ldots, a_i) R_{\alpha_j}$$

are faithfully flat. This is obvious when $i = n$, and the inductive step follows from Lemma 1.11. Taking $i = 0$, we obtain the desired result.

Remark 1.13. Let R be a commutative ring and let $a_1, \ldots, a_n \in R$ be elements which generate the unit ideal. Then the collection of maps $\{R \to R[\frac{1}{a_i}]\}_{1 \leq i \leq n}$ is a Nisnevich covering of R. In other words, every Zariski covering of R is also a Nisnevich covering.

For the next result, we need to introduce a bit of notation. For every commutative ring R, we let $\mathrm{Spec}^{\mathrm{Z}}(R)$ denote the Zariski spectrum of R: this is a topological space whose points are prime ideals $\mathfrak{p} \subset R$. For each point $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}(R)$, we let $\kappa(\mathfrak{p})$ denote the fraction field of the quotient $R/\mathfrak{p}$.

Proposition 1.14. *Let R be a Noetherian ring, and suppose we are given a collection of étale maps $\{\phi_\alpha : R \to R_\alpha\}_{\alpha \in A}$. The following conditions are equivalent:*

1. *The maps $\{R \to R_\alpha\}_{\alpha \in A}$ determine a Nisnevich covering of R.*
2. *For every $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}(R)$, there exists an index $\alpha \in A$ and a prime ideal $\mathfrak{q} \in \mathrm{Spec}^{\mathrm{Z}}(R_\alpha)$ such that $\mathfrak{p} = \phi_\alpha^{-1}\mathfrak{q}$ and the induced map $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ is an isomorphism.*

Proof. We first show that (1) $\Rightarrow$ (2) (which does not require our assumption that R is Noetherian). Choose a sequence of elements $a_1, \ldots, a_n \in R$ which generate the unit ideal and a collection of commutative diagrams

$$\begin{array}{ccc} & R_{\alpha_i} & \\ \nearrow & & \searrow \\ R & \longrightarrow & R[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1}). \end{array}$$

Let $\mathfrak{p}$ be a prime ideal of R. Since $\mathfrak{p} \neq R$, there exists an integer $i \leq n$ such that $a_1, \ldots, a_{i-1} \in \mathfrak{p}$ but $a_i \notin \mathfrak{p}$. Then $\mathfrak{p}$ is the inverse image of a prime ideal $\mathfrak{p}' \in R[\frac{1}{a_i}]/(a_1, \ldots, a_{i-1})$, which also has an inverse image $\mathfrak{q} \in \operatorname{Spec}^{\mathrm{Z}}(R_{\alpha_i})$. We have a commutative diagram of fields

$$\begin{array}{ccc} & \kappa(\mathfrak{q}) & \\ \nearrow & & \searrow \\ \kappa(\mathfrak{p}) & \xrightarrow{\qquad\qquad} & \kappa(\mathfrak{p}'). \end{array}$$

Since the lower horizontal map is an isomorphism, we conclude that $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{q})$.

Now suppose that (2) is satisfied; we will prove (1). Let X be the collection of all ideals $I \subseteq R$ for which the maps $\{R/I \to R_\alpha/IR_\alpha\}_{\alpha \in A}$ do not form a Nisnevich covering of R/I; we will prove that X is empty. Otherwise, X contains a maximal element I (since R is Noetherian). Replacing R by R/I, we may assume that X does not contain any nonzero ideal of R.

Let J be the nilradical of R. Since R is Noetherian, J is nilpotent. If $J \neq 0$, then $J \notin X$ and assertion (1) follows from Lemma 1.8. We may therefore assume that $J = 0$; that is, the ring R is reduced.

If $R = 0$ there is nothing to prove. Otherwise, since R is Noetherian, it contains finitely many associated primes $\eta_0, \eta_1, \ldots, \eta_k$. Reordering if necessary, we may assume that η_0 is not contained in η_i for $i > 0$. Choose an element $x \in R$ which belongs to $\eta_1 \cap \cdots \cap \eta_k$ but not to η_0. Then $x \neq 0$, so the principal ideal (x) does not belong to X. Using Lemma 1.9, we can replace R by $R[\frac{1}{x}]$ and thereby reduce to the case where R is a reduced ring with a unique associated prime, and therefore an integral domain.

Let κ denote the fraction field of R. Using assumption (2), we conclude that there exists an index $\alpha \in A$ and a prime ideal $\mathfrak{q} \subseteq R_\alpha$ such that $\phi_\alpha^{-1}\mathfrak{q} = (0) \subseteq R$ and $\kappa(\mathfrak{q}) \simeq \kappa$. In particular, we have an R-algebra map $R_\alpha \to \kappa(\mathfrak{q}) \to \kappa$. Since R_α is finitely presented as an R-algebra, this map factors through $R[\frac{1}{y}] \subseteq \kappa$ for some nonzero element $y \in R$. It follows that the single map $\{R[\frac{1}{y}] \to R_\alpha[\frac{1}{y}]\}$ is a Nisnevich covering of $R[\frac{1}{y}]$. Since the principal ideal (y) does not belong to X, Lemma 1.9 implies that the maps $\{R \to R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R, as desired. $\square$

Remark 1.15. Let $R = \varinjlim_\gamma R(\gamma)$ be a filtered colimit of commutative rings $R(\gamma)$, and suppose we are given a Nisnevich covering $\{R \to R_\alpha\}_{\alpha \in A}$ where the set A is finite (in view of Remark 1.6, this assumption is harmless). Then there exists an index γ and a Nisnevich covering $\{R(\gamma) \to R(\gamma)_\alpha\}_{\alpha \in A}$ such that $R_\alpha \simeq R \otimes_{R(\gamma)} R(\gamma)_\alpha$ for each $\alpha \in A$.

Remark 1.16. The theory of Nisnevich coverings (as set forth in Definition 1.1) is uniquely determined by Remark 1.15, Remark 1.6, and Proposition 1.14. That is, suppose we are given a commutative ring R and a collection of étale maps $\{R \to R_\alpha\}_{\alpha \in A}$. We can realize R as a filtered colimit of subrings which are finitely generated over $\mathbf{Z}$, and therefore Noetherian. It follows that $\{R \to R_\alpha\}_{\alpha \in A}$ is a Nisnevich covering of R if and only if there exists a finite subset $A_0 \subseteq A$, a subring $R' \subseteq R$ which is finitely generated over $\mathbf{Z}$, and a collection of étale maps $\{R' \to R'_\alpha\}_{\alpha \in A_0}$ satisfying condition (2) of Proposition 1.14, such that $R_\alpha \simeq R \otimes_{R'} R'_\alpha$ for $\alpha \in A_0$.

2 Nisnevich Excision

Let A be an $\mathbb{E}_\infty$-ring. We let $\mathrm{CAlg}_A^{\mathrm{Nis}}$ denote the full subcategory of CAlg_A spanned by the étale A-algebras. According to Theorem A.7.5.0.6, the construction $B \mapsto \pi_0 B$ determines an equivalence from $\mathrm{CAlg}_A^{\mathrm{Nis}}$ to (the nerve of) the ordinary category of étale $\pi_0 A$-algebras. Let $B \in \mathrm{CAlg}_A^{\mathrm{Nis}}$ and let

$$\mathcal{C} \subseteq (\mathrm{CAlg}_B^{\mathrm{Nis}})^{op} \simeq ((\mathrm{CAlg}_A^{\mathrm{Nis}})^{op})_{/B}$$

be a sieve. We will say that $\mathcal{C}$ is a *Nisnevich covering sieve* if it contains a collection of morphisms $\phi_\alpha : B \to B_\alpha$ for which the underlying maps of commutative rings $\pi_0 B \to \pi_0 B_\alpha$ determine a Nisnevich covering, in the sense of Definition 1.1. It follows from Proposition 1.10 and Remark 1.7 that the collection of Nisnevich covering sieves determines a Grothendieck topology on $(\mathrm{CAlg}_A^{\mathrm{Nis}})^{op}$, which we will refer to as the *Nisnevich* *topology.* We let $\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}$ denote the full subcategory of $\mathcal{P}((\mathcal{C}_A^{\mathrm{Nis}})^{op}) = \mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{Nis}}, \mathcal{S})$ spanned by those functions which are sheaves with respect to the Nisnevich topology.

Our goal in this section is to prove a result of Morel and Voevodsky, which characterizes $\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}$ as the ∞-category of presheaves which satisfy a certain excision property (Theorem 2.9). Note that $(\mathrm{CAlg}_A^{\mathrm{Nis}})^{op}$ can be identified with (the nerve of) the category of affine schemes which are étale over $\pi_0 A$. In order to formulate our result, we will need a slight enlargement of this category, which includes some non-affine schemes as well.

Notation 2.1. Let R be a commutative ring. We let Test_R denote the category of R-schemes X which admit a quasi-compact open immersion $X \to \operatorname{Spec} R'$, for some étale R-algebra R'.

Remark 2.2. In what follows, we can afford some flexibility in the exact definition of the category Test_R. For example, we could replace Test_R by the larger category of all quasi-compact, separated étale R-schemes without substantially changing the exposition below.

Remark 2.3. For every commutative ring R, we let $\operatorname{Spec}^c R$ denote the Zariski spectrum of R, regarded as a scheme. Let A be an $\mathbb{E}_\infty$-ring and let $R = \pi_0 A$. Then the construction $B \mapsto \operatorname{Spec}^c \pi_0 B$ determines a fully faithful embedding $(\mathrm{CAlg}_A^{\mathrm{Nis}})^{op} \to \mathrm{N}(\mathrm{Test}_R)$.

We now extend the definition of the Nisnevich topology to the category Test_R.

Definition 2.4. Let R be a commutative ring and let $X \in \mathrm{Test}_R$. We will say that a sieve $\mathcal{C} \subseteq (\mathrm{Test}_R)_{/X}$ is *Nisnevich covering* if, for every affine open subset $\operatorname{Spec}^c R' \subseteq X$, there exists a Nisnevich covering $\{R' \to R'_\alpha\}_{\alpha \in A}$ such that each of the induced maps $\operatorname{Spec}^c R'_\alpha \to X$ belongs to $\mathcal{C}$.

Definition 2.5. Let R be a commutative ring. We will say that a functor $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ satisfies *Nisnevich excision* if the following conditions are satisfied:

(1) The space $\mathcal{F}(\emptyset)$ is contractible.

(2) Let $f : X' \to X$ be a morphism in Test_R and let $U \subseteq X$ be a quasi-compact open subscheme. Let $X - U$ denote the reduced closed subscheme of X complementary to U, and assume that the induced map $X' \times_X (X - U) \to X - U$ is an isomorphism of schemes. Then the diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(X') \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U \times_X X') \end{array}$$

is a pullback square in $\mathcal{S}$.

We will say that $\mathcal{F}$ satisfies *affine Nisnevich excision* if condition (1) is satisfied and condition (2) is satisfied whenever the morphism $f : X' \to X$ is affine.

Remark 2.6. Let R be a commutative ring. The collection of all functors $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ satisfying Nisnevich excision (affine Nisnevich excision) is closed under limits.

Our main results can now be stated as follows:

Proposition 2.7. *Let R be a commutative ring. The collection of covering sieves described in Definition 2.4 determines a Grothendieck topology on the category* Test_R.

We will refer to the Grothendieck topology of Definition 2.4 as the *Nisnevich topology* on Test_R.

Proposition 2.8. *Let A be an $\mathbb{E}_\infty$-ring and let $R = \pi_0 A$. The fully faithful embedding* $\mathrm{CAlg}_A^{\mathrm{Nis}} \to \mathrm{N}(\mathrm{Test}_R)^{op}$ *of Remark 2.3 induces an equivalence of ∞-categories* $\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}} \simeq \mathcal{S}\mathrm{hv}(\mathrm{Test}_R)$.

Theorem 2.9 (Morel-Voevodsky). *Let R be a commutative ring and let $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology of Definition 2.4.*

(2) *The functor $\mathcal{F}$ satisfies Nisnevich excision.*

(3) *The functor $\mathcal{F}$ satisfies affine Nisnevich excision.*

We begin with the proofs of Propositions 2.7 and 2.8; the proof of Theorem 2.9 will be given near the end of this section.

Proof of Proposition 2.7. Example 1.3 implies that for $X \in \mathrm{Test}_R\, R$, the sieve $(\mathrm{Test}_R)_{/X} \subseteq (\mathrm{Test}_R)_{/X}$ is covering. Remark 1.7 shows that the collection of covering sieves is stable under pullback. To complete the proof, we must show that if $X \in \mathrm{Test}_R$ and if $\mathcal{C}, \mathcal{C}' \subseteq (\mathrm{Test}_R)_{/X}$ are sieves such that $\mathcal{C}$ is covering and $f^* \mathcal{C}' \subseteq (\mathrm{Test}_R)_{/X'}$ is covering for every map $f : X' \to X$ in $\mathcal{C}$, then $\mathcal{C}'$ is also covering. Let $\mathrm{Spec}^c R'$ be an affine open subset of X. Since $\mathcal{C}$ is covering, there exists a Nisnevich covering $\{R' \to R'_\alpha\}$ such that each of the induced maps $f_\alpha : \mathrm{Spec}^c R'_\alpha \to X$ belongs to $\mathcal{C}$. Since $f_\alpha^* \mathcal{C}'$ is covering, we can choose further Nisnevich coverings $\{R'_\alpha \to R'_{\alpha,\beta}\}$ such that each $\mathrm{Spec}^c R'_{\alpha,\beta} \to X$ belongs to $\mathcal{C}'$. It follows from Proposition 1.10 that the maps $\{R' \to R'_{\alpha,\beta}\}$ determine a Nisnevich covering of R'. We conclude that $\mathcal{C}'$ is also a covering sieve, as desired. □

To prove Proposition 2.8, we must show that a Nisnevich sheaf on Test_R is determined by its values on affine R-schemes. In what follows, we will identify $\mathrm{CAlg}_R^{\mathrm{Nis}}$ with a full subcategory of $\mathrm{N}(\mathrm{Test}_R)^{op}$ (see Remark 2.3).

Proposition 2.10. *Let R be a commutative ring and let $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ be a functor. Then $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology (of Definition 2.4) if and only if the following conditions are satisfied:*

(1) *The restriction $\mathcal{F}_0 = \mathcal{F} \,|\, \mathrm{CAlg}_R^{\mathrm{Nis}}$ is a sheaf with respect to the Nisnevich topology.*

(2) *The functor $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_0$.*

Proof of Proposition 2.8. Using Theorem A.7.5.0.6, we may assume without loss of generality that A is discrete. The desired result then follows from Propositions 2.10 and T.4.3.2.15. □

Proof of Proposition 2.10. Suppose first that conditions (1) and (2) are satisfied. Let $X \in \mathrm{Test}_R$ and let $\mathcal{C} \subseteq (\mathrm{Test}_R)_{/X}$ be a covering sieve; we must show that the canonical map $\mathcal{F}(X) \to \varprojlim_{U \in \mathcal{C}} \mathcal{F}(U)$ is an equivalence. Let $\mathcal{D}$ be the the full subcategory of $(\mathrm{Test}_R)_{/X}$ spanned by those maps $U \to X$ where U is affine. Using condition (2), we deduce that $\mathcal{F} \,|\, \mathcal{C}^{op}$ is a right Kan extension of $\mathcal{F} \,|(\mathcal{C} \cap \mathcal{D})^{op}$. It will therefore suffice to show that $\mathcal{F}(X) \simeq \varprojlim_{U \in \mathcal{C} \cap \mathcal{D}} \mathcal{F}(U)$ (Lemma T.4.3.2.7). Condition (1) implies that $\mathcal{F} \,|(\mathcal{C} \cap \mathcal{D})^{op}$ is a right Kan extension of $\mathcal{F} \,|\, \mathcal{D}^{op}$. Using Lemma T.4.3.2.7 again, we are reduced to proving that the map $\mathcal{F}(X) \to \varprojlim_{U \in \mathcal{D}} \mathcal{F}(U)$ is an equivalence, which follows immediately from (2).

Now suppose that $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology. We first prove that $\mathcal{F}$ satisfies (2). Let $X \in \mathrm{Test}_R$ and let $\mathcal{D} \subseteq (\mathrm{Test}_R)_{/X}$ be defined as above; we wish to show that the map $\mathcal{F}(X) \to \varprojlim_{U \in \mathcal{D}} \mathcal{F}(U)$ is an equivalence. Choose a covering of X by open affine subsets $U_1, \ldots, U_n \subseteq X$. Let $\mathcal{I}$ denote the partially ordered collection of nonempty subsets of $\{1, \ldots, n\}$, and for $I \in \mathcal{I}$ let U_I denote the intersection $\bigcap_{i \in I} U_i$. Using Remark 1.13, we see that the open subsets U_i generate a covering sieve $\mathcal{C} \subseteq (\mathrm{Test}_R)_{/X}$. The map $\mathrm{N}(\mathcal{I}) \to \mathrm{N}(\mathcal{C})^{op}$ is right cofinal. Since $\mathcal{F}$ is a sheaf, we obtain homotopy equivalences $\mathcal{F}(X) \to \varprojlim_{V \in \mathcal{C}} \mathcal{F}(V) \to \varprojlim_{I \in \mathcal{I}} \mathcal{F}(U_I)$. We have maps

$$\mathcal{F}(X) \xrightarrow{u} \varprojlim_{U \in \mathcal{D}} \mathcal{F}(U) \xrightarrow{v} \varprojlim_{U \in (\mathcal{C} \cap \mathcal{D})} \mathcal{F}(U) \xrightarrow{w} \varprojlim_{I \in \mathcal{I}} \mathcal{F}(U_I).$$

The above argument shows that wvu is a homotopy equivalence. The map $\mathrm{N}(\mathcal{I}) \to \mathrm{N}(\mathcal{C} \cap \mathcal{D})^{op}$ is right cofinal so that w is a homotopy equivalence. To complete the proof of (2), it will suffice to show that v is a homotopy equivalence. Using Lemma T.4.3.2.7, we are reduced to proving that $\mathcal{F} \mid \mathrm{N}(\mathcal{D})^{op}$ is a right Kan extension of $\mathcal{F} \mid \mathrm{N}(\mathcal{C} \cap \mathcal{D})^{op}$. Unwinding the definitions, this amounts to the following assertion: if $X' \to X$ is a morphism in Test_R such that X' is affine, $\mathcal{C}' \subseteq (\mathrm{Test}_R)_{/X'}$ is the sieve generated by the open subschemes $U'_i = U_i \times_X X'$, and $\mathcal{D}' \subseteq (\mathrm{Test}_R)_{/X'}$ is the full subcategory spanned by those maps $V \to X'$ where V is affine, then the map $\mathcal{F}(X') \to \varprojlim_{V \in \mathcal{C}' \cap \mathcal{D}'} \mathcal{F}(V)$ is a homotopy equivalence. The construction $I \mapsto U'_I = \bigcap_{i \in I} U'_i$ determines a right cofinal map $\mathrm{N}(\mathcal{I}) \to \mathrm{N}(\mathcal{C}' \cap \mathcal{D}')^{op}$; it will therefore suffice to show that the map $\mathcal{F}(X') \to \varprojlim_{I \in \mathcal{I}} \mathcal{F}(U'_I)$ is a homotopy equivalence. This map factors as a composition

$$\mathcal{F}(X') \to \varprojlim_{V \in \mathcal{C}'} \mathcal{F}(V) \to \varprojlim_{I \in \mathcal{I}} \mathcal{F}(U'_i).$$

The first map is a homotopy equivalence since $\mathcal{F}$ is a sheaf (note that $\mathcal{C}'$ is a covering sieve on X', by Remark 1.13), and the second is a homotopy equivalence since $\mathrm{N}(\mathcal{I}) \to \mathrm{N}(\mathcal{C}')^{op}$ is right cofinal. This completes the proof that $\mathcal{F}$ satisfies (2).

We now claim that $\mathcal{F}$ satisfies (1). Let $X = \mathrm{Spec}^{\mathrm{c}} R' \in \mathrm{Test}_R$, and suppose we are given a covering sieve $\mathcal{C}_0 \subseteq (\mathrm{CAlg}_{R'}^{\mathrm{Nis}})^{op}$; we wish to show that the map $\theta : \mathcal{F}(X) \to \varprojlim_{R'' \in \mathcal{C}_0} \mathcal{F}(\mathrm{Spec}^{\mathrm{c}} R'')$ is a homotopy equivalence. Let $\mathcal{C} \subseteq (\mathrm{Test}_R)_{/X}$ be the sieve generated by the maps $\mathrm{Spec}^{\mathrm{c}} R'' \to X$, where $R'' \in \mathcal{C}_0$. The map θ factors as a composition

$$\mathcal{F}(X) \xrightarrow{\theta'} \varprojlim_{U \in \mathcal{C}} \mathcal{F}(U) \xrightarrow{\theta''} \varprojlim_{R'' \in \mathcal{C}_0} \mathcal{F}(\mathrm{Spec}^{\mathrm{c}} R'').$$

Since $\mathcal{C}$ is a covering sieve, the map θ' is a homotopy equivalence. The map θ'' is a homotopy equivalence by Lemma T.4.3.2.7, since $\mathcal{F} \mid \mathrm{N}(\mathcal{C})$ is a right Kan extension of $\mathcal{F} \mid \mathcal{C}_0$ by virtue of assumption (2). □

We now turn to the proof of Theorem 2.9. We first spell out the sheaf condition on a functor $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ in more explicit terms.

Lemma 2.11. *Let R be a commutative ring and let $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ be a functor. Then $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology if and only if the following conditions are satisfied:*

(1) *For every finite collection of objects $\{X_i\}_{1 \le i \le n}$ in Test_R, the canonical map*

$$\mathcal{F}(\coprod_{1 \le i \le n} X_i) \to \prod_{1 \le i \le n} \mathcal{F}(X_i)$$

is a homotopy equivalence.

(2) *Let $f : X_0 \to X$ be a morphism in Test_R which generates a covering sieve on X, and let $X_\bullet$ be the Čech nerve of f. Then the canonical map $\mathcal{F}(X) \to \varprojlim \mathcal{F}(X_\bullet)$ is a homotopy equivalence.*

Proof. Let $X \in \mathrm{ExSpec}(R)$. Using the quasi-compactness of X and Remark 1.6, we see that every covering sieve $\mathcal{C} \subseteq \mathrm{ExSpec}(R)_{/X}$ contains another covering sieve $\mathcal{C}' \subseteq \mathcal{C}$ which is finitely generated. Let S denote the collection of all morphisms $X' \to X$ in Test_R which generate a covering sieve on X (with respect to the Nisnevich topology). The set S satisfies the hypotheses of Proposition VII.5.1; moreover, the topology on $\mathrm{N}(\mathrm{Test}_R)$ described in Proposition VII.5.1 coincides with the Nisnevich topology of Definition 2.4. The desired result now follows from Proposition VII.5.7. □

We will prove Theorem 2.9 in general by reducing to the case where R is a Noetherian ring of finite Krull dimension. For this, we need to understand how the ∞-category $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$ of Nisnevich sheaves behaves as a functor of the commutative ring R.

Notation 2.12. Let $f : R' \to R$ be a map of commutative rings. Then f induces a functor $F : \mathrm{Test}_R\, R') \to \mathrm{Test}_R$, given by the formula $X \mapsto X \times_{\mathrm{Spec^c}\, R'} \mathrm{Spec^c}\, R$. Composition with F induces a pushforward functor $f_* : \mathcal{P}(\mathrm{N}(\mathrm{Test}_R)) \to \mathcal{P}(\mathrm{N}(\mathrm{Test}_R\, R')))$, which carries $\mathcal{S}\mathrm{hv}(\mathrm{Test}_R)$ into $\mathcal{S}\mathrm{hv}(\mathrm{Test}_R\, R'))$.

Let R be a commutative ring, and let $X \in \mathrm{ExSpec}(R)$. Then there exists a finitely generated subring $R' \subseteq R$ and an object $X' \in \mathrm{ExSpec}(R')$ such that $X \simeq X' \times_{\mathrm{Spec^c}\, R'} \mathrm{Spec^c}\, R$. Using Remark 1.15, we see that if we are given a Nisnevich covering $\phi : X_0 \to X$ in $\mathrm{ExSpec}(R)$, then we can choose a finitely generated subring $R' \subseteq R$ and a Nisnevich covering $\phi' : X'_0 \to X'$ such that $\phi = F(\phi')$, where F is as in Notation 2.12. Combining this observation with Lemma 2.11, we obtain the following:

Lemma 2.13. *Let R be a commutative ring and let $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology.*

(2) *For every finitely generated commutative ring R' and every ring homomorphism $f : R' \to R$, the direct image $f_* \mathcal{F} \in \mathcal{P}(\mathrm{N}(\mathrm{Test}_R))$ is a sheaf with respect to the Nisnevich topology.*

Notation 2.14. Let X be a Noetherian scheme. For every point $x \in X$, we let $\mathrm{ht}\, x$ denote the dimension of the local ring $\mathcal{O}_{X,x}$. Note that if $X = \mathrm{Spec^c}\, R$ is the spectrum of a commutative ring R, then $\mathrm{ht}\, x$ is the height of the prime ideal $\mathfrak{p} \subseteq R$ corresponding to x.

We will need the following two basic facts about heights of points on Noetherian schemes:

(i) If y is another point of X which is contained in the closure of the point x, then $\mathrm{ht}\, y \geq \mathrm{ht}\, x$, with equality if and only if $x = y$.

(ii) Let $f : X \to X'$ be an étale map of Noetherian schemes. For each $x \in X$, we have $\mathrm{ht}\, x = \mathrm{ht}\, f(x)$.

Notation 2.15. Let R be a commutative ring and let $X \in \mathrm{Test}_R$. We let $\chi_U : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ denote the functor given by the formula

$$\chi_U(V) = \mathrm{Hom}_{\mathrm{Test}_R}(V, U).$$

Note that χ_U is a sheaf with respect to the Nisnevich topology (in fact, it is a sheaf with respect to the étale topology), and that χ_U satisfies Nisnevich excision.

Definition 2.16. Let R be a Noetherian ring, let $X \in \mathrm{Test}_R$, and let $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ be a functor equipped with a map $\theta : \mathcal{F} \to \chi_X$. For every morphism $f : U \to X$ in Test_R, we let $\mathcal{F}_f(U)$ denote the inverse image $\mathcal{F}(U) \times_{\mathrm{Hom}_{\mathrm{Test}_R}(U,X)} \{f\}$. Let $n \geq -1$ be an integer. We will say that θ is *weakly n-connective* if the following condition is satisfied:

($*$) Let $f : U \to X$ be a map in Test_R. Let $x \in U$ be a point suppose we are given a map of spaces $S^k \to \mathcal{F}_f(U)$, where $-1 \leq k < n - \mathrm{ht}\, x$. Then there exists a map $g : U' \to U$, a point $x' \in U$ with $g(x') = x$ such that g induces an isomorphism of residue fields $\kappa(x) \to \kappa(x')$, and the composite map $S^k \to \mathcal{F}_f(U) \to \mathcal{F}_{fg}(U')$ is nullhomotopic (when $k = -1$, this means that $\mathcal{F}_{fg}(U')$ is nonempty).

Remark 2.17. In the situation of Definition 2.16, suppose that $\theta : \mathcal{F} \to \chi_X$ is weakly n-connective. For every map $U \to X$ in Test_R, the pullback map $\mathcal{F} \times_{\chi_X} \chi_U \to \chi_U$ is also weakly n-connective.

Lemma 2.18. *Let R be a Noetherian ring, let $X \in \mathrm{Test}_R$, and let $\theta : \mathcal{F} \to \chi_X$ be a morphism in $\mathrm{Fun}(\mathrm{N}(\mathrm{Test}_R)^{op}, \mathcal{S})$. Let $n \geq 0$. Then θ is weakly n-connective if and only if the following conditions are satsified:*

(1) *For every map $f : U \to X$ and every point $x \in U$ of height $\leq n$, there exists a map $g : U' \to U$ in Test_R and a point $x' \in U'$ such that $x = g(x')$, the map of residue fields $\kappa(x) \to \kappa(x')$ is an isomorphism, and $\mathcal{F}_{fg}(U')$ is nonempty.*

(2) *If $n > 0$, then for every pair of maps $\chi_U \to \mathcal{F}$, $\chi_V \to \mathcal{F}$, the induced map $\chi_U \times_{\mathcal{F}} \chi_V \to \chi_U \times_{\chi_X} \chi_V \simeq \chi_{U \times_X V}$ is $(n-1)$-connective.*

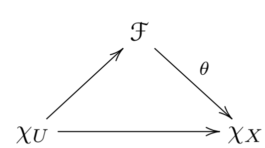
Proof. Suppose first that θ is weakly n-connective. Condition (1) is obvious (take $k = -1$ in Definition 2.16). To prove (2), assume that $n > 0$ and that we are given maps $\alpha : \chi_U \to \mathcal{F}$, $\beta : \chi_V \to \mathcal{F}$. We wish to show that the map $\theta' : \chi_U \times_{\mathcal{F}} \chi_V \to \chi_{U \times_X V}$ satisfies condition $(*)$ of Definition 2.16. To this end, suppose we are given a map $f : W \to U \times_X V$ in Test_R, a point $x \in W$, and a map of spaces $\eta : S^k \to \mathcal{F}'_f(W)$, where $\mathcal{F}' = \chi_U \times_{\mathcal{F}} \chi_V$ such that $-1 \leq k < n - 1 - \mathrm{ht}\, x$. Let f' be the induced map $W \to X$. Note that $\mathcal{F}'_f(W)$ can be identified with the space of paths joining the two points of $\mathcal{F}_{f'}(W)$ determines by α and β. Then η determines a map $\eta_0 : S^{k+1} \to \mathcal{F}_{f'}(W)$. Using our assumption that θ is weakly n-connective, we can choose a map $g : W' \to W$ and a point $x' \in W'$ such that $g(x') = x$, $\kappa(x) \simeq \kappa(x')$, and the composite map $S^{k+1} \stackrel{\eta_0}{\to} \mathcal{F}_{f'}(W) \to \mathcal{F}_{f'g}(W')$ is nullhomotopic. Since $\mathcal{F}'_{fg}(W')$ is homotopy equivalent to the space of paths in $\mathcal{F}_{f'g}(W')$ joining the points determined by α and β, we deduce that the composite map $S^k \stackrel{\eta}{\to} \mathcal{F}'_f(W) \to \mathcal{F}'\, fg(W')$ is also nullhomotopic, as desired.

Now suppose that conditions (1) and (2) are satisfied. We must show that θ satisfies condition $(*)$ of Definition 2.16. Choose a map $f : U \to X$ in Test_R, a point $x \in U$, and a map of spaces $\eta : S^k \to \mathcal{F}_f(U)$, where $-1 \leq k < n - \mathrm{ht}\, x$. We wish to prove that there exists a map $g : U' \to U$ and a point $x' \in U'$ such that $g(x') = x$, $\kappa(x) \simeq \kappa(x')$, and the composite map $S^k \to \mathcal{F}_f(U) \to \mathcal{F}_{fg}(U')$ is nullhomotopic. If $k = -1$, this follows from condition (1). Otherwise, we can write S^k as a homotopy pushout $* \coprod_{S^{k-1}} *$. Then η determines a pair of maps $\alpha, \beta : \chi_U \to \mathcal{F}$ such that $\theta \circ \alpha$ and $\theta \circ \beta$ are induced by f. Let $\mathcal{F}' = \chi_U \times_{\mathcal{F}} \chi_U$. The restriction of η to the equator of S^k gives a map $S^{k-1} \to \mathcal{F}'_\delta(U)$, where $\delta : U \to U \times_X U$ is the diagonal map. Using condition (2), we deduce the existence of a map $g : U' \to U$ and a point $x' \in U'$ such that $g(x') = x$, $\kappa(x) \simeq \kappa(x')$, and the induced map $S^{k-1} \to \mathcal{F}'_\delta(U) \to \mathcal{F}'_{\delta g}(U')$ is nullhomotopic. Unwinding the definitions, we deduce that $S^k \to \mathcal{F}_f(U) \to \mathcal{F}\, fg(U')$ is nullhomotopic, as desired. □

Lemma 2.19. *Let R be a Noetherian ring, let $X \in \mathrm{Test}_R$, and let $\theta : \mathcal{F} \to \chi_X$ be a natural transformation of functors $\mathcal{F}, \chi_X : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$. Suppose that θ exhibits χ_X as the sheafification of $\mathcal{F}$ with respect to the Nisnevich topology. Then θ is weakly n-connective for each $n \geq 0$.*

Proof. The proof proceeds by induction on n. We will show that θ satisfies the criteria of Lemma 2.18. Condition (1) follows immediately from our assumption that θ is an effective epimorphism after Nisnevich sheafificiation. To verify (2), we may assume that $n > 0$. Choose maps $\chi_U \to \mathcal{F}$ and $\chi_V \to \mathcal{F}$. Since sheafification is left exact, the induced map $\theta' : \chi_U \times_{\mathcal{F}} \chi_V \to \chi_{U \times_X V}$ exhibits $\chi_{U \times_X V}$ as the sheafification of $\chi_U \times_{\mathcal{F}} \chi_V$ with respect to the Nisnevich topology. It follows from the inductive hypothesis that θ' is weakly $(n-1)$-connective, as desired. □

Lemma 2.20. *Let R be a Noetherian ring, let $X \in \mathrm{Test}_R$, and let $\theta : \mathcal{F} \to \chi_X$ be a weakly n-connective morphism in $\mathcal{C} = \mathrm{Fun}(\mathrm{N}(\mathrm{Test}_R)^{op}, \mathcal{S})$. Assume that $\mathcal{F}$ satisfies affine Nisnevich excision. Then there exists a finite set of points $x_1, \ldots, x_m \in X$ of height $> n$ and a commutative diagram*



where $U = X - \bigcup_{1 \leq i \leq m} \overline{\{x_i\}}$.

Proof. The proof proceeds by induction on n. When $n = -1$, we take $\eta_1, \ldots, \eta_m$ to be the set of generic points of X, so that $U = \emptyset$ and the existence of the desired map $\chi_U \to \mathcal{F}$ follows from our assumption that $\mathcal{F}(\emptyset)$ is contractible. Assume now that $n \geq 0$ and that the result is known for the integer $n - 1$, so that we can choose a commutative diagram

$$\begin{array}{ccc} & \mathcal{F} & \\ {}^{\phi}\nearrow & & \searrow^{\theta} \\ \chi_U & \longrightarrow & \chi_X \end{array}$$

where $U = X - \bigcup_{1 \le i \le m} \overline{\{x_i\}}$ where the points x_i have height $\geq n$. Reordering the points η_i if necessary, we may assume that $x_1, x_2, \ldots, x_k$ have height n while $x_{k+1}, \ldots, x_m$ have height $> n$. We assume that this data has been chosen so that k is as small as possible. We will complete the induction by showing that $k = 0$. Otherwise, the point x_1 has height n. Since θ is weakly n-connective, there exists a map $f : X' \to X$ and a point $x' \in X'$ such that $f(x') = x_1$, $\kappa(x) \simeq \kappa(x')$, and $\mathcal{F}_f(X')$ is nonempty. We may therefore choose a commutative diagram

$$\begin{array}{ccc} & \mathcal{F} & \\ {}^{\psi}\nearrow & & \searrow^{\theta} \\ \chi_{X'} & \xrightarrow{f} & \chi_X \end{array}$$

Replacing X' by an open subset if necessary, we may suppose that f induces an isomorphism from $f^{-1}\overline{\{x_1\}}$ to an open subset $V \subseteq \overline{\{x_1\}}$ (here we endow these closed subsets with the reduced scheme structure). Our maps $\phi : \chi_U \to \mathcal{F}$ and $\psi : \chi_{X'} \to \mathcal{F}$ determine a map $\theta' : \chi_U \times_{\mathcal{F}} \chi_{X'} \to \chi_{U'}$. Using Lemma 2.18, we see that θ' is weakly $(n-1)$-connective. Applying the inductive hypothesis, we deduce that there exists a finite collection of points $y_1, \ldots, y_{m'} \in U'$ of height $\geq n$ and a commutative diagram

$$\begin{array}{ccc} & \chi_U \times_{\mathcal{F}} \chi_{X'} & \\ \nearrow & & \searrow \\ \chi_W & \longrightarrow & \chi_{U'}, \end{array}$$

where W is the open subscheme $U' - \bigcup \overline{\{y_j\}}$ of U'. Replacing X' by the open subscheme $X' - \bigcup \overline{\{y_j\}}$ (which contains x', since x' is a point of height n and therefore cannot lie in the closure of any other point of height n), we may assume that $W = U'$, so that the maps $\chi_U \xrightarrow{\phi} \mathcal{F} \xleftarrow{\psi} \chi_{X'}$ induce homotopic maps from $\chi_{U'}$ into $\mathcal{F}$. Shrinking X' further if necessary, we may assume that X' is affine (so that the map $X' \to X$ is affine). Regard $U \cup V$ as an open subscheme of X; since $\mathcal{F}$ satisfies affine Nisnevich excision, the diagram

$$\begin{array}{ccc} \mathcal{F}(U \cup V) & \longrightarrow & \mathcal{F}(X') \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U') \end{array}$$

is a pullback square in $\mathcal{S}$. It follows that ϕ extends to a map ϕ' fitting into a commutative diagram

$$\begin{array}{ccc} & \mathcal{F} & \\ {}^{\phi'}\nearrow & & \searrow^{\theta} \\ \chi_{U \cup V} & \longrightarrow & \chi_X, \end{array}$$

contradicting the minimality of k. □

Lemma 2.21. *Let R be a Noetherian ring of finite Krull dimension, let $X \in \mathrm{Test}_R$, and let $\theta : \mathcal{F} \to \chi_X$ be a natural transformation of functors $\mathcal{F}, \chi_X : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$. Assume that $\mathcal{F}$ satisfies Nisnevich excision and that θ exhibits χ_X as the sheafification of $\mathcal{F}$ with respect to the Nisnevich topology. Then θ admits a section.*

Proof. Combine Lemmas 2.19 and 2.20. □

Lemma 2.22. *Let R be a Noetherian ring of finite Krull dimension, let $X \in \mathrm{Test}_R$, and let $\theta : \mathcal{F} \to \chi_X$ be a morphism in $\mathcal{C} = \mathrm{Fun}(\mathrm{N}(\mathrm{Test}_R)^{op}, \mathcal{S})$. Assume that $\mathcal{F}$ satisfies affine Nisnevich excision and that θ exhibits χ_X as the Nisnevich sheafification of $\mathcal{F}$. Then the mapping space $\mathrm{Map}_{\mathcal{C}_{/\chi_X}}(\chi_X, \mathcal{F})$ is contractible.*

Proof. We will prove by induction on k that the mapping space $\mathrm{Map}_{\mathcal{C}_{/\chi_X}}(\chi_X, \mathcal{F})$ is k-connective. If $k > 0$, then it suffices to show that for every pair of maps $f, g : \chi_X \to \mathcal{F}$ in $\mathcal{C}_{/\chi_X}$, the mapping space $\mathrm{Map}_{\mathcal{C}_{/\chi_X}}(\chi_X, \chi_X \times_{\mathcal{F}} \chi_X)$ is $(k-1)$-connective. It will therefore suffice to treat the case $k = 0$, which follows from Lemma 2.21. □

Proof of Theorem 2.9. We first show that (1) $\Rightarrow$ (2). Assume that $\mathcal{F} : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ is a sheaf with respect to the Nisnevich topology. We claim that $\mathcal{F}$ satisfies Nisnevich excision. Using Example 1.4, we see that the empty sieve is a covering of $\emptyset \in \mathrm{Test}_R$, so that $\mathcal{F}(\emptyset)$ is contractible. We need to verify the second condition of Definition 2.5. We first establish the following special case:

- $(*)$ Assume that $\mathcal{F}$ is a Nisnevich sheaf. If $X \in \mathrm{Test}_R$ is the union of a pair of quasi-compact open subschemes $U, V \subseteq X$, then the diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cap V) \end{array}$$

  is a pullback square in $\mathcal{S}$.

To prove this, we let $\mathcal{C} \subseteq (\mathrm{Test}_R)_{/X}$ be the sieve generated by U and V. Note that the diagram

$$U \leftarrow U \cap V \to V$$

determines a right cofinal map $\Lambda^2_2 \to \mathrm{N}(\mathcal{C})^{op}$. Since $\mathcal{C}$ is a covering sieve, we obtain homotopy equivalences

$$\mathcal{F}(X) \to \varprojlim_{W \in \mathcal{C}} \mathcal{F}(W) \simeq \mathcal{F}(U) \times_{\mathcal{F}(U \cap V)} \mathcal{F}(V)$$

as desired.

Let us now treat the general case: suppose that $f : X' \to X$ is a morphism in Test_R and $U \subseteq X$ a quasi-compact open subset such that $X' \times X(X - U) \simeq X - U$. Let $\mathcal{C} \subseteq (\mathrm{Test}_R)_{/X}$ be the sieve generated by U and X'. We first claim that $\mathcal{C}$ is a covering sieve. To prove this, we may assume without loss of generality that $X = \mathrm{Spec}\, R'$ is affine. Since U is quasi-compact, the closed subset $X - U \subseteq X$ is defined by a finitely generated ideal $I = (a_1, \ldots, a_n) \subseteq R'$ (not necessarily a radical ideal). Since the inclusion $\mathrm{Spec}^{\mathrm{c}}\, R'/I$ factors through X', there exists a finite sequence of elements $\overline{b}_1, \ldots, \overline{b}_m \in R'/I$ which generate the unit ideal, such that each of the induced maps $\mathrm{Spec}^{\mathrm{c}}(R'/I)[\frac{1}{\overline{b}_j}] \to X$ factors through an affine open subscheme $\mathrm{Spec}^{\mathrm{c}}\, R'_j \subseteq X'$. Choose elements $b_j \in R'$ lifting $\overline{b}_j$ for $1 \leq j \leq m$. Then the sequence $a_1, \ldots, a_n, b_1, \ldots, b_j$ generates the unit ideal in R', and exhibit $\{R' \to R'[\frac{1}{a_i}], R' \to R'_j\}$ as a Nisnevich covering of R'. Since each $\mathrm{Spec}^{\mathrm{c}}\, R'[\frac{1}{a_i}] \to X$ factors through U and each $\mathrm{Spec}^{\mathrm{c}}\, R'_j \to X$ factors through X', we conclude that $\mathcal{C}$ is covering as desired.

Let $U' = U \times_X X'$. Let $X_\bullet$ denote the Čech nerve of f and $U_\bullet$ the Čech nerve of the induced map $U' \to U$, so that $X_0 = X'$ and $U_0 = U'$. There is a canonical inclusion of (augmented) simplicial objects $U_\bullet \hookrightarrow X_\bullet$ of $(\mathrm{Test}_R)_{/X}$, which determines a map $\phi : \mathrm{N}(\mathbf{\Delta}_+)^{op} \times \Delta^1 \to \mathrm{N}(\mathrm{Test}_R)$. Let $\mathcal{J} \subseteq \mathrm{N}(\mathbf{\Delta}_+)^{op} \times \Delta^1$ be the full subcategory obtained by omitting the final object. The functor ϕ carries the final object of $\mathrm{N}(\mathbf{\Delta}_+)^{op} \times \Delta^1$ to X, and can therefore be identified with a map

$$\phi' : \mathcal{J} \to \mathrm{N}(\mathrm{Test}_R)_{/X}.$$

We note that ϕ' factors through $\mathcal{C}$. We claim that the map $\phi' : \mathcal{J} \to \mathcal{C}$ is left cofinal. We will prove this using the criterion of Theorem T.4.1.3.1: it suffices to show that if $V \to X$ is a morphism belonging to the sieve $\mathcal{C}$, then the ∞-category $\mathcal{J} \times_{\mathcal{C}} \mathcal{C}_{V/}$ is weakly contractible. There are two cases to consider:

- (i) The map $V \to X$ does not factor through U. Let $S = \mathrm{Hom}_{(\mathrm{Test}_R)_{/X}}(V, X')$. Since the map $V \to X$ belongs to $\mathcal{C}$ and does not factor through U, it must factor through X'; thus the set S is nonempty. Consider the functor $g : \mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ which carries a finite linearly ordered set $[n]$ to the set $S^{[n]}$ of all maps from $[n]$ into S. The functor g is classified by a left fibration $\mathcal{D} \to \mathrm{N}(\mathbf{\Delta})^{op}$. Unwinding the definitions, we obtain an equivalence $\mathcal{J} \times_{\mathcal{C}} \mathcal{C}_{V/} \simeq \mathcal{D}$. To show that $\mathcal{D}$ is weakly contractible, it suffices to show that the colimit of the induced diagram $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{N}(\mathcal{S}\mathrm{et}) \subseteq \mathcal{S}$ is contractible (Proposition T.3.3.4.5). Note that g is the Čech nerve of the map $S \to *$; it is therefore contractible provided since S is nonempty.

- (ii) The map $V \to X$ factors through U. Let $\mathcal{D}$ be as in (i). Unwinding the definitions, we see that $\mathcal{J} \times_{\mathcal{C}} \mathcal{C}_{V/}$ is isomorphic to $(\mathcal{D} \times \Delta^1) \coprod_{\mathcal{D} \times \{0\}} \mathcal{D}^{\triangleright}$, which is obviously weakly contractible.

Consider the maps

$$\mathcal{F}(X) \to \varprojlim_{J \in \mathcal{J}} \mathcal{F}(\phi(J)) \to \varprojlim_{V \in \mathcal{C}} \mathcal{F}(V).$$

Since ϕ' is left cofinal, the second map is a homotopy equivalence. Since $\mathcal{F}$ is a Nisnevich sheaf, the composite map is a homotopy equivalence. It follows that $\mathcal{F} \circ \phi^{op}$ is a limit diagram in $\mathcal{S}$. We have a canonical isomorphism $\mathcal{J} \simeq (\mathrm{N}(\mathbf{\Delta})^{op} \times \Delta^1) \coprod_{\mathrm{N}(\mathbf{\Delta})^{op} \times \{0\}} (\mathrm{N}(\mathbf{\Delta}_+)^{op} \times \{0\})$. Applying the results of §T.4.2.3 to this decomposition, we get an equivalence

$$\varprojlim_{J \in \mathcal{J}} \mathcal{F}(\phi(J)) \simeq \mathcal{F}(U) \times_{\varprojlim \mathcal{F}(U_\bullet)} \varprojlim \mathcal{F}(X_\bullet).$$

Consider the diagram

$$\begin{array}{ccccc} \mathcal{F}(X) & \longrightarrow & \varprojlim \mathcal{F}(X_\bullet) & \longrightarrow & \mathcal{F}(X') \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \varprojlim \mathcal{F}(U_\bullet) & \longrightarrow & \mathcal{F}(U'). \end{array}$$

The above argument shows that the square on the left is a pullback. We wish to show that the outer square is a pullback. It will therefore suffice to show that the square on the right is a pullback. We can identify this square with a limit of diagrams σ_n :

$$\begin{array}{ccc} \mathcal{F}(X_n) & \longrightarrow & \mathcal{F}(X') \\ \downarrow & & \downarrow \\ \mathcal{F}(U_n) & \longrightarrow & \mathcal{F}(U') \end{array}$$

induced by the diagonal map $X' \to X_n$ and the inclusion $U_n \subseteq X_n$. Since X' is étale over X (because both X' and X are étale over $\mathrm{Spec}^{\mathrm{c}}\, R$), the map $X' \to X_n$ is an open immersion. Moreover, the condition that $X' \times_X (X - U) \simeq X - U$ guarantees that X_n is the union of U_n and the image of X'. It follows from $(*)$ that the diagram σ_n is a pullback square. This completes the proof that $\mathcal{F}$ satisfies Nisnevich excision.

It is clear that if $\mathcal{F}$ satisfies Nisnevich excision, then it satisfies affine Nisnevich excision. To complete the proof, let us assume that $\mathcal{F}$ satisfies affine Nisnevich excision; we will show that $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology. Using Lemma 2.13, we can reduce to the case where the commutative ring R is finitely generated; in particular, we may assume that R is a Noetherian ring of finite Krull dimension. Let $\mathcal{F}'$ be the sheafification of $\mathcal{F}$ with respect to the Nisnevich topology. The first part of the proof shows that $\mathcal{F}'$ satisfies Nisnevich excision. We wish to prove that for each $X \in \mathrm{Test}_R$, the map $\mathcal{F}(X) \to \mathcal{F}'(X)$ is a homotopy equivalence. Fix a point $\eta \in \mathcal{F}'(X)$, corresponding to a map $\chi_X \to \mathcal{F}'$. Then the homotopy fiber of $\mathcal{F}(X) \to \mathcal{F}'(X)$ over the point η can be identified with the space of sections of the induced map $\theta : \mathcal{F} \times_{\mathcal{F}'} \chi_X \to \chi_X$. Note that $\mathcal{F} \times_{\mathcal{F}'} \chi_X$ satisfies affine Nisnevich excision (Remark 2.6) and that θ exhibits χ_X as the Nisnevich sheafification of $\mathcal{F} \times_{\mathcal{F}'} \chi_X$ (since sheafification is left exact). The desired result now follows from Lemma 2.22. □

We conclude this section with another application of Lemma 2.20.

Lemma 2.23. *Let R be a Noetherian ring, let $X \in \mathrm{Test}_R$, and let $\theta : \mathcal{F} \to \chi_X$ be an n-connective morphism in $\mathcal{S}\mathrm{hv}(\mathrm{Test}_R)$. Then θ is weakly n-connective.*

Proof. This follows immediately by induction on n, using the criteria of Lemma 2.18. □

Theorem 2.24. *Let R be a Noetherian ring of Krull dimension $\leq n$. Then for each object $X \in \mathrm{Test}_R$, the ∞-topos $\mathcal{S}\mathrm{hv}(\mathrm{Test}_R)_{/\chi_X}$ is of homotopy dimension $\leq n$ and locally of homotopy dimension $\leq n$.*

Proof. Since $\mathcal{S}\mathrm{hv}(\mathrm{Test}_R)_{/\chi_X}$ is generated under colimits by objects of the form χ_U for $U \in (\mathrm{Test}_R)_{/X}$, it will suffice to prove that $\mathcal{S}\mathrm{hv}(\mathrm{Test}_R)_{/\chi_X}$ has homotopy dimension $\leq n$. Let $\mathcal{F}$ be an n-connective object of $\mathcal{S}\mathrm{hv}(\mathrm{Test}_R)_{/\chi_X}$. Then the underlying map $\theta : \mathcal{F} \to \chi_X$ is weakly n-connective (Lemma 2.23). Since R has Krull dimension $\leq n$, the scheme X does not contain any points of height $> n$. Note that $\mathcal{F}$ satisfies Nisnevich excision (Theorem 2.9). It follows from Lemma 2.20 that θ admits a section. □

Corollary 2.25. *Let A be an E_∞-ring such that $\pi_0 A$ is a Noetherian ring of finite Krull dimension. Then Postnikov towers in the ∞-topos $\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}$ are convergent. In particular, the ∞-topos $\mathcal{S}\mathrm{hv}_A^{\mathrm{Nis}}$ is hypercomplete.*

Proof. Combine Theorem 2.24, Proposition T.7.2.1.10, and Proposition 2.8. □

3 A Criterion for Étale Descent

Let R be an $\mathbb{E}_\infty$-ring and let $\mathcal{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$. Our goal in this section is to show that $\mathcal{F}$ satisfies for arbitrary étale coverings if and only if it satisfies descent for Nisnevich coverings and for finite étale coverings (Theorem 3.7). The idea of the proof is simple: we can use Nisnevich descent to reduce to the Henselian case, in which case every étale covering can be refined to a finite étale covering (Corollary 3.14).

Our first step is to introduce the finite étale topology.

Proposition 3.1. *Let $f : A \to A'$ be a map of $\mathbb{E}_1$-rings which exhibits A' as a flat left A-module. The following conditions are equivalent:*

(1) *The map f exhibits $\pi_0 A'$ as a finitely presented left module over $\pi_0 A$.*

(2) *The map f exhibits $\pi_0 A'$ as a finitely generated projective left module over $\pi_0 A$.*

(3) *The map f exhibits $\tau_{\geq 0} A'$ as a finitely generated projective left module over $\tau_{\geq 0} A$.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows from Proposition A.7.2.2.18. The implication (1) $\Rightarrow$ (2) is obvious. Conversely, suppose that $\pi_0 A'$ is finitely presented as a left $\pi_0 A$-module. Since $\pi_0 A'$ is flat over $\pi_0 A$, Lazard's theorem (Theorem A.7.2.2.15) guarantees that $\pi_0 A'$ can be realized as a filtered colimit $\varinjlim M_\alpha$ of finitely generated free left modules over $\pi_0 A$. It follows that the isomorphism $\pi_0 A' \simeq \varinjlim M_\alpha$ factors through some M_α, so that $\pi_0 A'$ is a retract of M_α (as a left $\pi_0 A'$-module) and is therefore a finitely generated projective module. □

Definition 3.2. We say that a map $f : A \to A'$ of E_∞-rings is *finite flat* if it satisfies the equivalent conditions of Proposition 3.1. We say that f is *finite étale* if it is both finite flat and étale .

Warning 3.3. A flat map of commutative rings $A \to A'$ which exhibits A' as a finitely generated A-module need not be finite flat. For example, suppose that X is a totally disconnected compact Hausdorff space, $x \in X$ is a point, A is the ring of locally constant $\mathbf{C}$-valued functions on X, and $e : A \to \mathbf{C}$ is given by evaluation at x. Then e is surjective and flat, but is finite flat only if x is an isolated point of X.

Remark 3.4. Let $f : A \to A'$ be an étale map of $\mathbb{E}_\infty$-rings. If f exhibits $\pi_0 A'$ as a finitely generated module over $\pi_0 A$, then it is finite étale . To prove this, it suffices to show that $\pi_0 A'$ is finitely presented as a module over $\pi_0 A$ (Proposition 3.1). It follows from the structure of étale algebras (Proposition VII.7.14) that there exists a finitely generated subring $R \subseteq \pi_0 A$ and an étale R-algebra R' such that $\pi_0 A' \simeq \pi_0 A \otimes_R R'$. Enlarging R if necessary, we can ensure that R' is a finitely generated R-module. Since R is Noetherian, the algebra R' is finitely presented as an R-module. It follows that $\pi_0 A'$ is finitely presented as a $\pi_0 A$-module.

We now summarize some properties of the class of finite flat morphisms:

Lemma 3.5. (1) *Every equivalence $f : A \to A'$ of $\mathbb{E}_\infty$-rings is finite flat.*

(2) *The collection of finite flat morphisms in* CAlg *is closed under composition.*

(3) *Suppose we are given a pushout diagram of $\mathbb{E}_\infty$-rings*

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{g} & B'. \end{array}$$

If f is finite flat, then g is finite flat.

(4) *Given a finite collection $f_i : A_i \to A'_i$ of finite flat morphisms, the induced map $\prod_i A_i \to \prod_i A'_i$ is finite flat.*

Proposition 3.6. *Let R be an $\mathbb{E}_\infty$-ring. Then there exists a Grothendieck topology on the ∞-category $(\mathrm{CAlg}_R^{ét})^{op}$ with the following property: for any étale R-algebra R', a sieve on $(\mathrm{CAlg}_{R'}^{ét})^{op}$ is covering if and only if it contains a finite collection of finite étale maps $R' \to R'_\alpha$ in CAlg_R such that the induced map $R' \to \prod_\alpha R'_\alpha$ is faithfully flat.*

Proof. The desired result follows from Proposition VII.5.1, applied to the collection S of all morphisms in $\mathrm{CAlg}_R^{\text{ét}}$ which are finite flat, faithfully flat, and étale (the hypotheses of Proposition VII.5.1 are satisfied by virtue of Lemma 3.5 and Proposition VII.5.4). □

We will refer to the Grothendieck topology of Proposition 3.6 as the *finite étale topology.* We can now state our main result:

Theorem 3.7. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathcal{F} : \mathrm{CAlg}_R^{ét} \to \mathcal{S}$ be a functor. Then $\mathcal{F}$ is a sheaf with respect to the étale topology if and only if the following conditions are satisfied:*

(1) *The functor $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology.*

(2) *The functor $\mathcal{F}$ is a sheaf with respect to the finite étale topology.*

The proof of Theorem 3.7 will require some general facts about Henselian rings. Recall that a local commutative ring R with maximal ideal $\mathfrak{m}$ is said to be *Henselian* if, for every étale R-algebra R', every map of R-algebras $R' \to R/\mathfrak{m}$ can be lifted to a map $R' \to R$ (see Definition VII.7.1). We will briefly review some basic facts about Henselian rings (see [49] for a more detailed discussion).

Notation 3.8. If R is a commutative ring and we are given a pair of commutative R-algebras A and B, we let $\mathrm{Hom}_R(A, B)$ denote the set of R-algebra homomorphisms from A to B.

Proposition 3.9. *Let R be a Henselian local ring with maximal ideal $\mathfrak{m}$ and let R' be an étale R-algebra. Then the reduction map $\theta_{R'} : \mathrm{Hom}_R(R', R) \to \mathrm{Hom}_R(R', R/\mathfrak{m})$ is bijective.*

Proof. The definition of a Henselian local ring guarantees that $\theta_{R'}$ is surjective. For injectivity, suppose we are given two R-algebra maps $f, g : R' \to R$ with $\theta(f) = \theta(g)$. Since R' is étale over R, the multiplication map $m : R' \otimes_R R' \to R'$ induces an isomorphism $(R' \otimes_R R')[\frac{1}{e}] \simeq R'$ for some idempotent element $e \in R' \otimes_R R'$. The maps f and g determine a map $u : R' \otimes_R R' \to R$. Since $\theta(f) = \theta(g)$, the composite map $u' : R' \otimes_R R' \to R \to R/\mathfrak{m}$ factors through m, so that $u'(e)$ is invertible in $R/\mathfrak{m}$. Since R is local, we conclude that $u(e) \in R$ is invertible, so that u also factors through m; this proves that $f = g$. □

We will need a few stability properties for the class of Henselian rings:

Proposition 3.10. *Let $R \to A$ be a finite étale map between local commutative rings. If R is Henselian, then A is also Henselian.*

Remark 3.11. Proposition 3.10 can be generalized: if R is a Henselian local ring and A is a local R-algebra which is finitely generated as an R-module, then A is also Henselian. We refer the reader to [49] for a proof.

Lemma 3.12. *Let $f : R \to R'$ be an étale map of commutative rings which exhibits R' as a projective R-module of rank n. Then there exists a faithfully flat finite étale morphism $R \to A$ such that $R' \otimes_R A \simeq A^n$.*

Proof. We proceed by induction on n. If $n = 0$, we can take $A = R$. Assume $n > 0$. Then f is faithfully flat. Replacing R by R', we can assume that f admits a left inverse $g : R' \to R$. Since f is étale , the map g determines a decomposition $R' \simeq R \times R''$. Then R'' is finite étale of rank $(n-1)$ over R. By the inductive hypothesis, we can choose a faithfully flat finite étale map $R \to A$ such that $R'' \otimes_R A \simeq A^{n-1}$. It follows that $R' \otimes_R A \simeq A^n$ as desired. □

Lemma 3.13. *Let $R \to A$ be a finite étale map of commutative rings, and let $\phi : A \to A'$ be a ring homomorphism. Then there exists a ring homomorphism $R \to R'$ and a map $\psi : A' \to R' \otimes_R A$ with the following universal property: for every commutative R-algebra B, composition with ψ induces a bijection*

$$\mathrm{Hom}_R(R', B) \to \mathrm{Hom}_A(A', B \otimes_R A).$$

Moreover, if A' is étale over A, then R' is étale over R.

Proof. The assertion is local on R (with respect to the étale topology, say). We may therefore reduce to the case where the finite étale map $R \to A$ splits, so that $A \simeq R^n$ (Lemma 3.12). Then A' is isomorphic to a product $A'_1 \times \cdots \times A'_n$ of R-algebras. Let $R' = A'_1 \otimes \cdots \otimes A'_n$, and let $\psi : A' \to R' \otimes_R A \simeq R'^n$ be the product of the evident maps $A'_i \to R'$. It is easy to see that ψ has the desired property. If A' is étale over A, then each A'_i is étale over R, so that R' is also étale over R. □

Proof of Proposition 3.10. Let $\mathfrak{m}$ denote the maximal ideal of R; since A is local, $\mathfrak{m}A$ is the unique maximal ideal of A. Choose an étale map $A \to A'$ and an A-algebra map $\phi_0 : A' \to A/\mathfrak{m}A$. We wish to prove that ϕ_0 lifts to a map $A' \to A$. Choose an étale map $R \to R'$ as in Lemma 3.13. Then ϕ_0 is classified by a map $\psi_0 : R' \to R/\mathfrak{m}$. Since R is Henselian, we can lift ψ_0 to a map $\psi : R' \to R$, which classifies a lifting $\phi : A' \to A$ of ϕ_0. □

Corollary 3.14. *Let R be a Henselian local ring. Suppose we are given a faithfully flat étale map $R \to R'$. Then there exists an idempotent element $e \in R'$ such that $R'[\frac{1}{e}]$ is local, faithfully flat over R, and finitely generated as an R-module.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R and set $k = R/\mathfrak{m}$. Since R' is faithfully flat over R, the quotient $R'/\mathfrak{m}R'$ is a nontrivial étale k-algebra. We can therefore choose a finite separable extension k' of k and a surjective k-algebra map $\phi_0 : R'/\mathfrak{m}R' \to k'$. Choose a filtration

$$k = k_0 \hookrightarrow k_1 \hookrightarrow \cdots \hookrightarrow k_n = k'$$

where each k_{i+1} has the form $k_i[x_i]/(f_i(x_i))$ for some monic polynomial f_i (in fact, we may assume that $n = 1$, by the primitive element theorem, but we will not need to know this). We lift this to a sequence of algebra extensions

$$R = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n$$

where $A_{i+1} = A_i[x_i]/(\overline{f}_i(x_i))$ for some monic polynomial $\overline{f}_i$ lifting f_i. Since k' is separable over k, each derivative $\frac{\partial f_i(x_i)}{\partial x_i}$ is invertible in k_{i+1}. It follows that $\frac{\partial \overline{f}_i(x_i)}{\partial x_i}$ is invertible in A_{i+1}, so that each A_{i+1} is a finite étale extension of A_i. Set $A = A_n$, so that A is a finite étale extension of R. Note that A is a local ring with maximal ideal $\mathfrak{m}A$ and residue field $A/\mathfrak{m}A = k'$. The map ϕ_0 together with the quotient map $A \to A/\mathfrak{m}A$ amalgamate to give an A-algebra map $\psi_0 : A \otimes_R R' \to A/\mathfrak{m}A$. Since A is Henselian (Proposition 3.10), the map ψ_0 lifts to an A-algebra map $\psi : A \otimes_R R' \to A$, which we can identify with a map $\phi : R' \to A$ lifting ϕ_0. Since ϕ_0 is surjective, the map ϕ is surjective modulo $\mathfrak{m}$ and therefore surjective by Nakayama's lemma (since A is a finitely generated R-module). Since R' and A are both étale over R, the map ϕ is an étale surjection. It follows that $A \simeq R'[\frac{1}{e}]$ for some idempotent element $e \in R'$. □

Definition 3.15. Let A be an $\mathbb{E}_\infty$-ring. We will say that A is *Henselian* if the commutative ring $\pi_0 A$ is a Henselian local ring.

Remark 3.16. Let R be an $\mathbb{E}_\infty$-ring, and let $\mathrm{CAlg}_R^{\text{ét}}$ denote the full subcategory of CAlg_R spanned by the étale R-algebras. Every object $R' \in \mathrm{CAlg}_R^{\text{ét}}$ is a compact object of CAlg_R (Corollary A.7.5.4.4). Applying Proposition T.5.3.5.11, we obtain a fully faithful embedding $\mathrm{Ind}(\mathrm{CAlg}_R^{\text{ét}}) \to \mathrm{CAlg}_R$. We will say that an R-algebra A is Ind-*étale* if it belongs to the essential image of this functor.

Remark 3.17. Let A be an $\mathbb{E}_\infty$-ring. We will say that a collection of étale maps $\{A \to A_\alpha\}$ is a *Nisnevich covering* if the collection of induced maps $\{\pi_0 A \to \pi_0 A_\alpha\}$ is a Nisnevich covering of the commutative ring $\pi_0 A$, in the sense of Definition 1.1.

Proposition 3.18. *Let R be an $\mathbb{E}_\infty$-ring and let A be an* Ind-*étale R-algebra. The following conditions are equivalent:*

(1) *The $\mathbb{E}_\infty$-ring A is Henselian.*

(2) *Let A' be an étale A-algebra, and suppose we are given a Nisnevich covering $\{A' \to A'_\alpha\}$. Then every morphism $A' \to A$ in CAlg_A factors through A'_α, for some index α.*

(3) *Let R' be an étale R-algebra and suppose we are given a Nisnevich covering $\{R' \to R'_\alpha\}$. Then every morphism $R' \to A$ in CAlg_R factors through R'_α, for some index α.*

(4) *Let $h_A : (\mathrm{CAlg}_R^{\mathrm{Nis}})^{op} \to \mathcal{S}$ be the functor represented by A (so that $h_A(R') = \mathrm{Map}_{\mathrm{CAlg}_R}(R', A)$). Then h_A factors as a composition*

$$(\mathrm{CAlg}_R^{\acute{e}t})^{op} \xrightarrow{j} \mathcal{P}((\mathrm{CAlg}_R^{\acute{e}t})^{op}) \xrightarrow{L} \mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}} \xrightarrow{f^*} \mathcal{S},$$

where j denotes the Yonead embedding, L a left adjoint to the inclusion (given by sheafification with respect to the Nisnevich topology), and f^* is a geometric morphism of ∞-topoi.

Remark 3.19. It follows from Proposition 3.18 that conditions (3) and (4) depend only on the structure of the $\mathbb{E}_\infty$-ring A, and not on the map $\phi : R \to A$.

Lemma 3.20. *Let A be a Henselian $\mathbb{E}_\infty$-ring, let $\mathfrak{m} \subseteq \pi_0 A$ be the maximal ideal, and let $k = (\pi_0 A)/\mathfrak{m}$ denote the residue field of A. For every étale A-algebra A', the canonical map*

$$\mathrm{Map}_{\mathrm{CAlg}_A}(A', A) \to \mathrm{Hom}_k(A' \otimes_R k, k)$$

is a homotopy equivalence. In particular, $\mathrm{Map}_{\mathrm{CAlg}_A}(A', A)$ is homotopy equivalent to a finite set.

Proof. Using Theorem A.7.5.0.6 we can replace A by $\pi_0 A$, thereby reducing to the statement of Proposition 3.9. □

Proof of Proposition 3.18. We first show that (1) ⇒ (2). Assume that $\pi_0 A$ is a local Henselian ring with maximal ideal $\mathfrak{m}$, and let $k = (\pi_0 A)/\mathfrak{m}$ denote the residue field of A. Suppose we are given an étale R-algebra R', a morphism $R' \rightarrow A$ in CAlg_R, and a Nisnevich covering $\{R' \rightarrow R'_\alpha\}$. We wish to prove that one of the spaces $\mathrm{Map}_{\mathrm{CAlg}_{R'}}(R'_\alpha, A)$ is nonempty. Using Lemma 3.20, we are reduced to proving that one of the mapping spaces $\mathrm{Hom}_k(R'_\alpha \otimes_{R'} k, k)$ is nonempty. This follows immediately from Proposition 1.14, since the collection of maps $\{k \rightarrow R'_\alpha \otimes_{R'} k\}$ form a Nisnevich covering of the field k.

Now suppose that (2) is satisfied; we will prove (1). If $A = 0$, then the empty set is a Nisnevich covering of $\pi_0 A$, contradicting assumption (2). If $a \in A$, then the pair of maps $\{A \rightarrow A[\frac{1}{a}], A \rightarrow A[\frac{1}{1-a}]\}$ determines a Nisnevich covering of $\pi_0 A$. It then follows from (2) that either a or $(1-a)$ is invertible in $\pi_0 A$. It follows that the commutative ring $\pi_0 A$ is local; let $\mathfrak{m}$ denote its maximal ideal. Let B be an étale $\pi_0 A$-algebra and suppose we are given a $\pi_0 A$-algebra map $\phi_0 : B \rightarrow (\pi_0 A)/\mathfrak{m}$; we wish to show that ϕ_0 can be lifted to a $\pi_0 A$-algebra map $B \rightarrow \pi_0 A$. Replacing B by a localization if necessary, we may assume that ϕ_0 is the only $\pi_0 A$-algebra map from B to $(\pi_0 A)/\mathfrak{m}$. In this case, it suffices to show that the map $\pi_0 A \rightarrow B$ admits a left inverse. Since B is finitely presented as an $\pi_0 A$-algebra, we can lift ϕ_0 to a map $\phi_1 : B \rightarrow \pi_0 A/I$, where I is a proper ideal generated by finitely many elements $a_1, \ldots, a_n \in \pi_0 A$. Using Theorem A.7.5.0.6, we may assume that $B = \pi_0 A'$ for some étale A-algebra A'. By construction, the collection of maps $\{A \rightarrow A[\frac{1}{a_i}], A \rightarrow A'\}$ is a Nisnevich covering. Note that since $a_i \in I \subseteq \mathfrak{m}$, there cannot exist an A-algebra map $A[\frac{1}{a_i}] \rightarrow A$. Using (2), we deduce the existence of an A-algebra map $A' \rightarrow A$, so that the induced map $B \rightarrow \pi_0 A$ admits a left inverse as desired.

The implication (2) ⇒ (3) is obvious, and the converse follows from Remark 1.15 (since A can be obtained as a filtered colimit of étale R-algebras). We will complete the proof by showing that (3) ⇔ (4). According to Theorem T.5.1.5.6, the map $h_A : (\mathrm{CAlg}_R^{\text{ét}})^{op} \rightarrow \mathcal{S}$ is homotopic to a composition $F^* \circ j$, where $F^* : \mathcal{P}((\mathrm{CAlg}_R^{\text{ét}})^{op}) \rightarrow \mathcal{S}$ preserves small colimits; moreover, F^* is unique up to equivalence. Since h_A is left exact, the functor F^* is also left exact (Proposition T.6.1.5.2). We wish to show that F^* factors through the localization functor L if and only if condition (3) is satisfied. This follows immediately from Proposition T.6.2.3.20. □

In the situation of Proposition 3.18, the geometric morphism $f^* : \mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}} \rightarrow \mathcal{S}$ is uniquely determined by A, provided that it exists. In fact, we can be more precise.

Definition 3.21. Let R be an $\mathbb{E}_\infty$-ring. We will say that a object $A \in \mathrm{CAlg}_R$ is a *Henselization* of R if A is Henselian and Ind-étale over A. We let $\mathrm{CAlg}_R^{\mathrm{Hens}}$ denote the full subcategory of CAlg_R spanned by the Henselizations of R.

Corollary 3.22. *Let R be an $\mathbb{E}_\infty$-ring, and let $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}, \mathcal{S})$ denote the ∞-category of geometric morphisms $f^* : \mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}} \rightarrow \mathcal{S}$. Then the construction of Proposition 3.18 determines an equivalence $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}, \mathcal{S})$ with $\mathrm{CAlg}_R^{\mathrm{Hens}}$.*

Proof. Theorem T.5.1.5.6 and Proposition T.6.1.5.2 determine an equivalence of $\mathrm{Fun}^*(\mathcal{P}((\mathrm{CAlg}_R^{\text{ét}})^{op}), \mathcal{S})$ with the ∞-category $\mathrm{Ind}(\mathrm{CAlg}_R^{\text{ét}})$, which we can identify with the full subcategory of CAlg_R spanned by those R-algebras which are Ind-étale over R. Using Proposition 3.18, we see that this restricts to an equivalence of $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}, \mathcal{S})$ with $\mathrm{CAlg}_R^{\mathrm{Hens}}$. □

Using Corollary 3.22, we can obtain a very explicit description of the Kan complex $\mathrm{Fun}^*(\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}, \mathcal{S})^{\simeq}$ of points of the ∞-topos $\mathcal{S}\mathrm{hv}_R^{\mathrm{Nis}}$.

Notation 3.23. Let R be a commutative ring. We define a subcategory $\mathrm{Field}_R^{\mathrm{sep}} \subseteq \mathrm{Ring}_R$ as follows:

- The objects of $\mathrm{Field}_R^{\mathrm{sep}}$ are given by ring homomorphisms $\phi : R \rightarrow k$, where k is a field which is a separable algebraic extension of the residue field $\kappa(\mathfrak{p})$ of R at some prime ideal $\mathfrak{p} \subseteq R$.

- The moprhisms in $\mathrm{Field}^{\mathrm{sep}}_R$ are given by isomorphisms of R-algebras.

Let R be an $\mathbb{E}_\infty$-ring and let A be a Henselization of R. Then $\pi_0 A$ is a local ring with maximal ideal $\mathfrak{m}$. Since A is Ind-etale over R, the residue field $\pi_0 A/\mathfrak{m}$ is unramified over the commutative ring $\pi_0 R$, and therefore a separable algebraic extension of the residue field $\kappa(\mathfrak{p})$ for some $\mathfrak{p} \subseteq \pi_0 R$. The construction $A \mapsto (\pi_0 R)/\mathfrak{m}$ determines a functor

$$\Phi : (\mathrm{CAlg}^{\mathrm{Hens}}_R)^{\simeq} \to \mathrm{N}(\mathrm{Field}^{\mathrm{sep}}_{\pi_0 R}).$$

Proposition 3.24. *Let R be an $\mathbb{E}_\infty$-ring. Then the functor $\Phi : (\mathrm{CAlg}^{\mathrm{Hens}}_R)^{\simeq} \to \mathrm{N}(\mathrm{Field}^{\mathrm{sep}}_{\pi_0 R})$ defined above is a homotopy equivalence of Kan complexes.*

Remark 3.25. Let R be an $\mathbb{E}_\infty$-ring. Proposition 3.24 asserts that every separable extension k of every residue field of $\pi_0 R$ has the form $\pi_0 A/\mathfrak{m}$, for some Henselian $\mathbb{E}_\infty$-ring A which is Ind-étale over R. We will refer to A as the *Henselization of* R at k.

Proof. Using Theorem A.7.5.0.6, we may reduce to the case where R is discrete. We first prove that Φ is fully faithful. Let A and A' be Henselizations of R, let $\mathfrak{m} \subseteq A$ and $\mathfrak{m}' \subseteq A'$ denote maximal ideals, and set $k = A/\mathfrak{m}$ and $k' = A'/\mathfrak{m}'$. Unwinding the definitions, we must show that every R-algebra isomorphism of k with k' can be lifted uniquely to an R-algebra isomorphism of A with A'. Since A is Henselian, we have

$$\mathrm{Hom}_R(R', A) \simeq \mathrm{Hom}_A(R' \otimes_R A, A) \simeq \mathrm{Hom}_A(R' \otimes_R A, k) \simeq \mathrm{Hom}_k(R' \otimes_R k, k) \simeq \mathrm{Hom}_R(R', k)$$

for every étale R-algebra R'. Passing to filtered colimits, we deduce that the restriction map $\mathrm{Hom}_R(R', A) \to \mathrm{Hom}_R(R', k)$ is bijective whenever R' is Ind-etale over R. In particular, every R-algebra homomorphism $\phi_0 : k' \to k$ induces a map $A' \to k' \stackrel{\phi_0}{\to} k$, which lifts uniquely to a map $\phi : A' \to A$. If ϕ_0 is invertible, the same argument shows that ϕ_0^{-1} lifts to an R-algebra map $\psi : A \to A'$. The compositions $\phi \circ \psi$ and $\psi \circ \phi$ lift the identity maps from k and k' to themselves; by uniqueness we deduce that $\phi \circ \psi = \mathrm{id}_A$ and $\psi \circ \phi = \mathrm{id}_{A'}$. It follows that ϕ is an isomorphism of R-algebras.

We now prove that Φ is essentially surjective. Let $k \in \mathrm{Field}^{\mathrm{sep}}_R$, and let $\mathcal{C}$ denote the category whose objects are étale R-algebras R' equipped with an R-algebra homomorphism $\epsilon_{R'} : R' \to k$. The category $\mathcal{C}$ admits finite colimits, and is therefore filtered. We let $A = \varinjlim_{(R', \epsilon_{R'}) \in \mathcal{C}} R'$, so that A is an Ind-étale R-algebra. By construction, we have a canonical map $\epsilon : A \to k$. This implies in particular that $A \neq 0$. We next claim that A is a local ring with maximal ideal $\mathfrak{m} = \ker(\epsilon)$. To prove this, choose an arbitrary element $x \notin \mathfrak{m}$; we will show that x is invertible in A. To prove this, choose a representation of x as the image of an element $x_0 \in R'$, for some étale R-algebra R' equipped with a map $\epsilon_{R'} : R' \to k$. Then $\epsilon_{R'}(x_0) = \epsilon(x) \neq 0$, so that $\epsilon_{R'}$ factors through $R'[\frac{1}{x_0}]$. It follows that the map $R' \to A$ also factors through $R'[\frac{1}{x_0}]$, so that the image of x_0 in A is invertible.

We now claim that ϵ induces an isomorphism $A/\mathfrak{m} \to k$. This map is injective by construction. To prove the surjectivity, choose an arbitrary element $y \in k$; we will prove that y belongs to the image of ϵ. Since k is a separable algebraic extension of some residue field of R, the element y satisfies a polynomial equation $f(y) = 0$ where the coefficients of f lie in R, and the discriminant Δ of f invertible in k. Then $R' = R[Y, \frac{1}{\Delta}]/(f)$ is an étale R-algebra equipped with a map $R' \to k$ given by $Y \mapsto y$. It follows that y belongs to the image of ϵ as desired.

To complete the proof, it will suffice to show that the local ring A is Henselian. Let B be an étale A-algebra equipped with an A-algebra homomorphism $f_0 : B \to k$; we wish to prove that f_0 can be lifted to a map $f : B \to A$. Using the structure theory of étale morphisms (Proposition VII.7.14), we can write $B = A \otimes_{R'} B_0$, where R' is an étale R-algebra equipped with a map $\epsilon_{R'} : R' \to k$, and B_0 is étale over R'. Then f_0 determines a map $B_0 \to k$ extending $\epsilon_{R'}$. It follows that the canonical map $R' \to A$ factors through B_0; this factorization determines a map $B \to A$ having the desired properties. □

Remark 3.26. Let R be an $\mathbb{E}_\infty$-ring, let $\mathcal{C} = (\mathrm{CAlg}^{\text{ét}}_R)^{op}$, and let $\mathcal{X}$ be the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by those functors $\mathcal{F} : \mathrm{CAlg}^{\text{ét}}_R \to \mathcal{S}$ which are sheaves with respect to the Nisnevich and finite étale topologies.

It follows from Lemma T.7.3.2.3 that $\mathcal{X}$ is an accessible left exact localization of $\mathcal{P}(\mathcal{C})$. Since $\mathcal{X}$ can be obtained as $S^{-1}\mathcal{P}(\mathcal{C})$ where S consists of monomorphisms, we deduce from Proposition T.6.2.2.17 that $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})$, where we regard $\mathcal{C}$ as endowed with the coarsest Grothendieck topology which is finer than both the Nisnevich and finite étale topologies.

We now turn to the proof of Theorem 3.7.

Proof of Theorem 3.7. The "only if" direction is obvious. To prove the converse, we wish to show that $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})$ is contained in the ∞-category of étale sheaves on $\mathcal{C} = (\mathrm{CAlg}_R^{\text{ét}})^{op}$ (here $\mathcal{X}$ is defined as in Remark 3.26). Let $L : \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S}) \to \mathcal{X}$ be a left adjoint to the inclusion, and let $j : (\mathrm{CAlg}_R^{\text{ét}})^{op} \to \mathrm{Fun}(\mathrm{CAlg}_R^{\text{ét}}, \mathcal{S})$ be the Yoneda embedding. Note that j takes values in $\mathcal{X}$ (Theorem VII.5.14). Using Proposition T.6.2.3.20, we are reduced to proving the following: for every collection of morphisms $\{R' \to R_\alpha\}$ which generate a covering sieve with respect to the étale topology, the induced map $\theta : \coprod_\alpha j(R_\alpha) \to j(R')$ is an effective epimorphism in $\mathcal{X}$. Without loss of generality we may replace R by R', and thereby reduce to the case where $j(R')$ is a final object of $\mathcal{X}$.

Note that the Grothendieck topology on $\mathcal{C}$ is finitary; consequently, to prove that θ is an effective epimorphism in $\mathcal{X}$, it will suffice to prove that $\eta^*(\theta)$ is an effective epimorphism in $\mathcal{S}$, for every geometric morphism $\eta^* : \mathcal{X} \to \mathcal{S}$ (Theorem VII.4.1). Note that η^* determines a geometric morphism $\eta'^* : \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_R^{\mathrm{Nis}})^{op}) \to \mathcal{S}$. According to Corollary 3.22, the point η'^* is determined by a Henselian R-algebra A which is a filtered colimit $\varinjlim A_\beta$ of étale R-algebras A_β. In particular, A is local. It follows that there exists an index α such that the induced map $A \to A \otimes_R R_\alpha$ is faithfully flat. We will complete the proof by showing that $\eta^* j(R_\alpha)$ is nonempty.

According to Corollary 3.14, there exists an idempotent element $e \in \pi_0(A \otimes_R R_\alpha)$ such that $A' = (A \otimes_R R_\alpha)[\frac{1}{e}]$ is a faithfully flat finite étale A-algebra. Consequently, there exists an index β and an idempotent $e_\beta \in \pi_0(A_\beta \otimes_R R_\alpha)$ such that $A'_\beta = (A_\beta \otimes_R R_\alpha)[e_\beta^{-1}]$ is a faithfully flat finite étale A_β-algebra. It follows that the map $j(A'_\beta) \to j(A_\beta)$ is an effective epimorphism in $\mathcal{X}$. Since $\eta^* j(A_\beta)$ can be identified with $\mathrm{Map}_{\mathrm{CAlg}_R}(A_\beta, A) \neq \emptyset$, we conclude that $\eta^* j(A'_\beta)$ is nonempty. The map $R_\alpha \to A'_\beta$ induces a map of spaces $\eta^* j(A'_\beta) \to \eta^* j(R_\alpha)$, so that $\eta^* j(R_\alpha)$ is nonempty as desired. □

4 Galois Descent

Let R be an $\mathbb{E}_\infty$-ring. In §3, we saw that a presheaf $\mathcal{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ satisfies descent for étale coverings if and only if it satisfies descent for both Nisnevich and finite étale coverings. The former condition is very concrete: it is equivalent to Nisnevich excision, by virtue of Theorem 2.9. Our goal in this section is to obtain a concrete interpretation of the second condition as well (Theorem 4.23). The main observation is that every finite étale covering can be refined to a Galois covering (Lemma 4.21).

We begin by reviewing the Galois theory of commutative rings. For a much more general discussion, we refer the reader to [7].

Proposition 4.1. *Let G be a finite group acting on a commutative ring R. The following conditions are equivalent:*

(1) *For every nonzero commutative ring A, the group G acts freely on the set* $\mathrm{Hom}_{\mathrm{Ring}}(R, A)$.

(2) *Let $\mathfrak{p}$ be a prime ideal of R and $G_\mathfrak{p} \subseteq G$ the stabilizer group of $\mathfrak{p}$. Then $G_\mathfrak{p}$ acts faithfully on the residue field $\kappa(\mathfrak{p})$.*

Proof. Assume first that (1) is satisfied. Choose a point $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}(R)$ and let $\phi : R \to \kappa(\mathfrak{p})$ be the canonical map. For any $g \in G_\mathfrak{p}$, the action of g on $\mathrm{Hom}_{\mathrm{Ring}}(R, \kappa(\mathfrak{p}))$ is given by the inverse of the action of g on $\kappa(\mathfrak{p})$. Since the action of G is free, we conclude that any nontrivial element $g \in G_\mathfrak{p}$ must act nontrivially on $\kappa(\mathfrak{p})$.

Conversely, suppose that condition (2) is satisfied, and let A be a nonzero commutative ring. We wish to prove that G acts freely on $\mathrm{Hom}_{\mathrm{Ring}}(R, A)$. Choose a homomorphism $\psi : A \to k$ where k is a field; composition with ψ induces a G-equivariant map $\mathrm{Hom}_{\mathrm{Ring}}(R, A) \to \mathrm{Hom}_{\mathrm{Ring}}(R, k)$. It will therefore suffice to show that G acts freely on $\mathrm{Hom}_{\mathrm{Ring}}(R, k)$. Let $\theta : R \to k$ be a ring homomorphism which is invariant under an element $g \in G$. Then $g \in G_{\mathfrak{p}}$ for $\mathfrak{p} = \ker(\theta)$, and the map θ factors as a composition

$$R \xrightarrow{\theta'} \kappa(\mathfrak{p}) \xrightarrow{\theta''} k.$$

Since θ'' injective and θ is g-invariant, we conclude that g acts trivially on the field $\kappa(\mathfrak{p})$; condition (2) then guarantees that g is the identity element of G. □

Definition 4.2. We will say that an action of a finite group G on a commutative ring R is *free* if it satisfies the equivalent conditions of Proposition 4.1.

Notation 4.3. Let R be a commutative ring acted on by a finite group G, and let $R' = \prod_{g \in G} R$. We define ring homomorphisms $\phi_0, \phi_1 : R \to R'$ by the formulas

$$\phi_0(r)_g = r \qquad \phi_1(r)_g = g(r).$$

Together these maps determine a ring homomorphism

$$\phi : \mathrm{Tor}_0^{R^G}(R, R) \to R',$$

where $R^G \subseteq R$ is the ring of G-invariant elements of R.

Lemma 4.4. *Let G be a group of order n which acts faithfully on a field k, and let $\phi_0, \phi_1 : k \to k'$ be as in Notation 4.3. Then there exists a finite sequence $x_1, \ldots, x_n \in k$ such that the images $\phi_1(x_i)$ form a basis for k', regarded as a k-vector space via the homomorphism ϕ_0.*

Proof. It is clear that k' has dimension n over k. Consequently, it will suffice to show that k' is generated as a k-vector space by elements of the form $\phi_1(x)$. In other words, we must show that the map $\phi : \mathrm{Tor}_0^{k^G}(k, k) \to k'$ is surjective. In the ring k', we have a unique decomposition $1 = \sum_{g \in G} e_g$, where each e_g is a nonzero idempotent element corresponding to projection of $k' = \prod_{g \in G} k$ onto the gth factor. The elements e_g form a basis for k' as a k-vector space; it will therefore suffice to show that each e_g belongs to the image of ϕ. If $h \neq g$, then since $h^{-1}g$ acts nontrivially on k we can choose an element $x \in k$ such that $h(x) \neq g(x)$. Then

$$y_h = (\phi_1(x) - \phi_0(h(x)))\phi_0(\frac{1}{g(x) - h(x)})$$

belongs to the image of ϕ; note that the gth coordinate of y_h is equal to 1, and the hth coordinate vanishes. It follows that $e_g = \prod_{h \neq g} y_h$ also belongs to the image of ϕ, as desired. □

Proposition 4.5. *Let G be a finite group acting on freely on a commutative ring R, and let $R^G \subseteq R$ denote the subring consisting of G-invariant elements. Then:*

(1) *The inclusion $R^G \hookrightarrow R$ is finite and étale.*

(2) *Let $R' = \prod_{g \in G} R$ be as in Notation 4.3. Then the map $\phi : \mathrm{Tor}_0^{R^G}(R, R) \to R'$ is an isomorphism.*

Proof. Consider the following weaker version of assertion (1):

(1′) The inclusion $R^G \hookrightarrow R$ is faithfully flat.

Since the diagonal $\phi_0 : R \to R'$ is finite étale, assertions (1′) and (2) imply (1) by faithfully flat descent. Assertions (1′) and (2) are local on $\mathrm{Spec}^{\mathrm{Z}}\, R^G$; we may therefore replace R^G by its localization at some prime and thereby reduce to the case where R^G is a local ring with a unique maximal ideal $\mathfrak{m}$.

Note that R integral over R^G: every element $x \in R$ is a solution to the polynomial equation

$$\prod_{g \in G} (X - g(x)) = 0,$$

whose coefficients are G-invariant. We may therefore write R as a union of subalgebras R_α which are finitely generated as R^G-modules. It follows from Nakayama's lemma that each quotient $R_\alpha/\mathfrak{m}R_\alpha$ is nonzero, so that the direct limit $R/\mathfrak{m}R \simeq \varinjlim R_\alpha/\mathfrak{m}R_\alpha$ is nonzero. We conclude that $\mathfrak{m}$ is contained in a maximal ideal of R.

Choose a maximal ideal $\mathfrak{n} \subset R$ containing $\mathfrak{m}$. For each $g \in G$, let $\mathfrak{n}^g$ denote the inverse image of $\mathfrak{n}$ under the action of g on R. We next claim that R is a semi-local ring, whose maximal ideals are precisely those of the form $\mathfrak{n}^g$ for $g \in G$. To see this, it suffices to show that if $x \in R$ is an element which does not belong to any of the ideals $\mathfrak{n}^g$, then x is invertible in R. Note that for each $g \in G$, our assumption $x \notin \mathfrak{n}^g$ is equivalent to $g(x) \notin \mathfrak{n}$. Since $\mathfrak{n}$ is a prime ideal, we conclude that $y = \prod_{g \in G} g(x) \notin \mathfrak{n}$. Then y is a G-invariant element of R which does not belong to $\mathfrak{m}$, so that y is invertible in R^G and therefore x is invertible in R.

Let $H \subseteq G$ be the stabilizer of the ideal $\mathfrak{n} \subset R$ and let n be the order of H. Our assumption that G acts freely on R guarantees that H acts freely on the residue field $R/\mathfrak{n}$. Using Lemma 4.4, we deduce the existence of elements $\overline{x}_1, \ldots, \overline{x}_n \in R/\mathfrak{n}$ whose images under the map ϕ_1 of Notation 4.3 form a basis for $\prod_{h \in H}(R/\mathfrak{n})$ as a vector space over $R/\mathfrak{n}$. Choose elements $x_i \in \bigcap_{g \in G - H} \mathfrak{n}^g \subseteq R$ which reduce to the elements $\overline{x}_i$ modulo $\mathfrak{n}$, and let $g_1, \ldots, g_m \in G$ be a set of representatives for the set of right cosets G/H. For each maximal ideal $\mathfrak{n}'$ of R, the images $\phi_1(g_i(x_j))$ form a basis for the vector space $\prod_{g \in G} R/\mathfrak{n}'$. Using Nakayama's lemma, we conclude that the elements $\phi_1(g_i(x_j))$ form a basis of R' (viewed as an R-module via ϕ_0).

Let G act on R' via the formula

$$g(\{r_{g'}\}_{g' \in G}) = \{g(r_{g'g^{-1}})\}_{g' \in G}.$$

The map $\{r_{i,j}\} \mapsto \sum_{i,j} \phi_0(r_{i,j})\phi_1(g_i(x_j))$ determines a G-equivariant isomorphism

$$\bigoplus_{1 \leq i \leq m, 1 \leq j \leq n} R \to R'.$$

Passing to fixed points, we see that R is freely generated by the elements $g_i(x_j)$ as an R^G-module, which immediately implies both (1′) and (2). □

Let R be a commutative ring equipped with a free action of a finite group G. Proposition 4.5 guarantees that R is flat over R^G. If $R^\bullet$ is the cosimplicial commutative ring obtained as the Čech nerve of the inclusion $R^G \hookrightarrow R$, then we have a canonical isomorphism

$$R^k \simeq \prod_{g_1, \ldots, g_k \in G} R.$$

Using faithfully flat descent, we obtain the following result:

Proposition 4.6. *Let R be a commutative ring equipped with a free action of a finite group G. Then the construction $M \mapsto M \otimes_{R^G} R$ determines an equivalence of categories from the category of (discrete) R^G-modules to the category of (discrete) R-modules equipped with a compatible action of G. Moreover, if M is any R^G-module, the augmented cochain complex*

$$M \to R^0 \otimes_{R^G} M \to R^1 \otimes_{R^G} M \to \cdots$$

associated to the cosimplicial abelian group $R^\bullet \otimes_{R^G} M$ is acyclic.

Remark 4.7. Let R be a commutative ring equipped with a free action of a finite group G, and let N be an R-module equipped with a compatible action of the group G. Unwinding the definitions, we see that the cochain complex appearing in Proposition 4.6 is the standard complex for computing the group cohomology $\mathrm{H}^*(G; N)$. Proposition 4.6 gives

$$\mathrm{H}^n(G; N) \simeq \begin{cases} 0 & \text{if } n > 0 \\ M & \text{if } n = 0. \end{cases}$$

where $M \simeq N^G$ is an R^G-module (unique up to canonical isomorphism) such that $M \otimes_{R^G} R \simeq N$.

We now adapt some of the above ideas to the ∞-categorical setting.

Notation 4.8. Let G be a discrete group. Then G is a monoid object in the category of sets, and therefore determines a simplicial set BG (see §A.4.1.2). The simplicial set BG is a Kan complex with a unique vertex, and can therefore be viewed as an object of $\mathcal{S}_*$. We refer to BG as the *classifying space* of G.

Remark 4.9. According to Proposition T.7.2.2.12, the classifying space BG is determined up to equivalence by the requirements that BG is 1-connective, 1-truncated, and the fundamental group of BG (taken with respect to its base point) is isomorphic to G.

Definition 4.10. Let $\mathcal{C}$ be an ∞-category. A *G-equivariant object* of $\mathcal{C}$ is a map of simplicial sets $BG \to \mathcal{C}$.

Remark 4.11. Evaluation at the base point of BG determines a forgetful functor $\theta : \mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$. We will generally abuse notation by identifying a G-equivariant object C of $\mathcal{C}$ with its image $\theta(C) \in \mathcal{C}$. In this situation, we will also say that the group G *acts on* the object $\theta(C) \in \mathcal{C}$ (the action itself is given by the object $C \in \mathrm{Fun}(BG, \mathcal{C})$).

Definition 4.12. Let $\mathcal{C}$ be an ∞-category and let $F : BG \to \mathcal{C}$ be a functor which determines an action of G on the object $F(*) = X \in \mathcal{C}$. We let X^G denote a limit of the diagram F (if such a limit exists).

Remark 4.13. The notation of Definition 4.12 is abusive: the object X^G depends on the G-equivariant object $F : BG \to \mathcal{C}$, and not only on the underlying object $X \in \mathcal{C}$.

Let $\mathcal{C}$ be an ∞-category which admits small limits, let G be a discrete group, and suppose we are given a G-equivariant object $BG \to \mathcal{C}$, corresponding to an action of G on an object $X \in \mathcal{C}$. We can regard BG as a simplicial object in the category of sets: in particular, BG determines a simplicial space $S_\bullet$ with $S_n \simeq G^n$. According to Example T.A.2.9.31, we can identify BG with the colimit of the diagram $S_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{S}$. It follows that X^G can be identified with the limit of a cosimplicial object of $X^\bullet \in \mathcal{C}$, where each X^n is a limit of the induced diagram

$$S_n \to BG \to \mathcal{C}.$$

Since S_n is discrete, we obtain $X^n = \prod_{g_1,\ldots,g_n \in G} X$.

Now suppose that $\mathcal{C}$ is the ∞-category of spectra. Applying Variant A.1.2.4.7, we deduce the existence of a spectral sequence of abelian groups $\{E_r^{p,q}, d_r\}_{r \geq 1}$ with

$$E_1^{p,q} \simeq \prod_{g_1,\ldots,g_p \in G} \pi_{-q} X.$$

Note that each $\pi_{-q}X$ is an abelian group acted on by G; unwinding the definitions, we see that the differential d_1 is the standard differential in the cochain complex

$$\pi_{-q}X \to \prod_{g \in G} \pi_{-q}X \to \prod_{g,g' \in G} \pi_{-q}X \to \cdots$$

which computes the cohomology of the group G with coefficients in $\pi_{-q}X$. We therefore obtain a canonical isomorphism $E_2^{p,q} \simeq \mathrm{H}^p(G; \pi_{-q}X)$.

In good cases, the spectral sequence described above will converge to the homotopy groups $\pi_{-p-q}X^G$. For example, Corollary A.1.2.4.10 yields the following result:

Proposition 4.14. *Let G be a discrete group, and let $BG \to \mathrm{Sp}$ be a G-equivariant object of the ∞-category of spectra whose underlying spectrum is X. Assume that for every integer n and for each $k > 0$, the cohomology group $\mathrm{H}^k(G; \pi_n X)$ vanishes. Then for every integer n, the map $\pi_n(X^G) \to \pi_n X$ is injective and its image is the group of G-invariant elements of $\pi_n X$.*

Corollary 4.15. *Let G be a finite group, and suppose we are given a G-equivariant object $BG \to \mathrm{CAlg}$, which we view as an $\mathbb{E}_\infty$-ring R equipped with an action of G. Suppose that G acts freely on $\pi_0 R$ (in the sense of Definition 4.2). Then:*

(1) *For every integer n, the map $\pi_n R^G \to \pi_n R$ is injective, and its image is the subgroup of G-invariant elements of $\pi_n R$.*

(2) *The map $R^G \to R$ is finite étale.*

(3) *The canonical map $R \otimes_{R^G} R \to \prod_{g \in G} R$ is an equivalence.*

Proof. Combine Propositions 4.5, 4.6, and 4.14. □

Definition 4.16. Let G be a finite group acting on an $\mathbb{E}_\infty$-ring R. We will say that the action is *free* if the induced action of G on the commutative ring $\pi_0 R$ is free, in the sense of Definition 4.2.

Let $f : R \to R'$ be a map of $\mathbb{E}_\infty$-rings and G a finite group. We will say that f is a *Galois extension* (with Galois group G) if there exists a free action of G on R' such that f factors as a composition

$$R \simeq R'^G \to R'.$$

Remark 4.17. Let $f : R \to R'$ be a Galois extension $\mathbb{E}_\infty$-rings. The Galois group G is not uniquely determined by f. For example, if $R' \simeq R^n$, then any group G of order n can serve as a Galois group for f.

Remark 4.18. The notion of Galois extension introduced in Definition 4.16 is very restrictive. For a much more general analogue of Galois theory in this context, we refer the reader to [50].

Remark 4.19. Using Theorem A.7.5.0.6, we see that a map $f : R \to R'$ of E_∞-rings is a Galois extension if and only if the induced map of commutative rings $\pi_0 R \to \pi_0 R'$ is a Galois extension.

We now establish the existence of a good supply of Galois extensions.

Lemma 4.20. *Let $f : R \to R'$ be a finite étale map of $\mathbb{E}_\infty$-rings, and suppose that $\pi_0 R'$ is a projective $\pi_0 R$-module of rank n. Then there exists a map $R \to A$ which is finite étale and faithfully flat such that $R' \otimes_R A \simeq A^n$.*

Proof. Using Theorem A.7.5.0.6, we can reduce to the case where R and R' are discrete, in which case the desired result was established as Lemma 3.12. □

Lemma 4.21. *Let R be an $\mathbb{E}_\infty$-ring and let $f : R \to R'$ be a faithfully flat, finite étale morphism. Then there exists a map $g : R' \to R''$ such that the composite map $R \to R''$ is a Galois extension.*

Proof. For each $i \geq 0$, there exists a largest open subset U_i of $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$ over which the localization of the module $\pi_0 R'$ has rank i. Then $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$ is the disjoint union of the open sets U_i, so each U_i is also closed and therefore has the form $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)[\frac{1}{e_n}]$ where e_i is some idempotent element of $\pi_0 R$. Since R' is faithfully flat over R, we have $e_0 = 0$. Let n be the least common multiple of the set $\{i : e_i \neq 0\}$. Replacing R' by $\prod_i R'[\frac{1}{e_i}]^{\frac{n}{i}}$, we can assume that $\pi_0 R'$ has constant rank n over $\pi_0 R$.

Let $R'^{\otimes n}$ be the n-fold tensor power of R' over R. For every pair of integers $1 \leq i < j \leq n$, the multiplication on the ith and jth tensor factors induces a map of étale R-algebras $f_{i,j} : R'^{\otimes n} \to R'^{\otimes (n-1)}$. Since $f_{i,j}$ admits a section, it induces an equivalence $R'^{\otimes(n-1)} \simeq R'^{\otimes n}[\frac{1}{\epsilon_{i,j}}]$ for some idempotent elements $\epsilon_{i,j} \in \pi_0 R'^{\otimes n}$. Let $R'' = R'^{\otimes n}[\frac{1}{\prod_{i,j} \epsilon_{i,j}}]$. Then R'' carries an action of the symmetric group Σ_n (as one can see, for example, by reducing to the discrete case using Theorem A.7.5.0.6) in the ∞-category CAlg_R. To complete the proof, it suffices to show that Σ_n acts freely on $\pi_0 R''$ and that the induced map $R \to R''^{\Sigma_n}$ is an equivalence. This assertion is local on R; we may therefore invoke Lemma 4.20 to reduce to the case where $R' = R^n$. In this case, $R'' \simeq \prod_{\sigma \in \Sigma_n} R$ and the desired result is obvious. □

Definition 4.22. Let R be an E_∞-ring and let $\mathcal{F} : \mathrm{CAlg}^{\text{ét}}_{R/} \to \mathcal{S}$ be a functor. We will say that $\mathcal{F}$ *satisfies Galois descent* if, for every object $A \in \mathrm{CAlg}^{\text{ét}}_R$ equipped with a free action of a finite group G, the canonical map $\mathcal{F}(A^G) \to \mathcal{F}(A)^G$ is a homotopy equivalence.

We can now state the main result of this section.

Theorem 4.23. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathcal{F} : \mathrm{CAlg}_R^{\acute{e}t} \to \mathcal{S}$ be a functor. Then $\mathcal{F}$ is a sheaf with respect to the finite étale topology if and only if the following conditions are satisfied:*

(1) *The functor $\mathcal{F}$ satisfies Galois descent.*

(2) *The functor $\mathcal{F}$ commutes with finite products.*

Corollary 4.24. *Let R be an $\mathbb{E}_\infty$-ring and let $\mathcal{F} : \mathrm{CAlg}_R^{\acute{e}t} \to \mathcal{S}$ be a functor. Then $\mathcal{F}$ is a sheaf with respect to the étale topology if and only if the following conditions are satisfied:*

(1) *The functor $\mathcal{F}$ satisfies Galois descent.*

(2) *The functor $\mathcal{F}$ is a sheaf with respect to the Nisnevich topology.*

Proof. Combine Theorem 4.23 and Theorem 3.7 (note that if $\mathcal{F} : \mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ is a sheaf for the Nisnevich topology, then $\mathcal{F}$ commutes with finite products). □

Proof of Theorem 4.23. Let G be a finite group acting on an object $A \in \mathrm{CAlg}_R^{\text{ét}}$. Write BG as the colimit of simplicial space $S_\bullet$ (with $S_n \simeq G^n$) and let $A^\bullet$ be the cosimplicial $\mathbb{E}_\infty$-ring given by

$$A^n \simeq \varinjlim_{S_n} A \simeq \prod_{g_1,\ldots,g_n \in G} A,$$

so that $A^G \simeq \varprojlim A^\bullet$. Using Corollary 4.15, we see that $A^\bullet$ can be identified with the Čech nerve of the map $A^G \to A$. Similarly, we see that $\mathcal{F}(A)^G$ can be identified with the limit of the cosimplicial space $[n] \mapsto \prod_{g_1,\ldots,g_n \in G} \mathcal{F}(A)$. If $\mathcal{F}$ commutes with finite products, then this cosimplicial object is given by $[n] \mapsto \mathcal{F}(A^n)$. We have proven:

($*$) If $\mathcal{F}$ commutes with finite products, then $\mathcal{F}$ satisfies Galois descent if and only if, for every Galois extension $A^G \to A$ with Čech nerve $A^\bullet$ in $\mathrm{CAlg}_R^{\text{ét}}$, the induced map $\mathcal{F}(A^G) \to \varprojlim \mathcal{F}(A^\bullet)$ is a homotopy equivalence.

If $\mathcal{F}$ is a sheaf with respect to the finite étale topology, then Proposition VII.5.7 implies that $\mathcal{F}$ commutes with finite products and satisfies the criterion of ($*$), and therefore satisfies Galois descent.

Conversely, suppose that $\mathcal{F}$ commutes with finite products and satisfies Galois descent. We wish to prove that $\mathcal{F}$ is a sheaf with respect to the finite étale topology. We proceed as in the proof of Proposition VII.5.7. Let $A \in \mathrm{CAlg}_R^{\text{ét}}$ and let $\mathcal{C}^{(0)} \subseteq \mathrm{CAlg}_A^{\text{ét}}$ be a sieve on A; we wish to prove that the map $\mathcal{F}(A) \to \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(0)}$ is an equivalence. Suppose first that $\mathcal{C}^{(0)}$ is generated by a Galois extension $f : A \to A^0$. Let $A^\bullet$ be the Čech nerve of f. It follows from ($*$) that the canonical map $\mathcal{F}(A) \to \varprojlim \mathcal{F}(A^\bullet)$. The desired result now follows follows from the observation that $A^\bullet$ is given by a right cofinal map $\mathrm{N}(\mathbf{\Delta}) \to \mathcal{C}^{(0)}$.

Suppose now that $\mathcal{C}^{(0)}$ is generated by a single map $f : A \to A'$ which is finite étale and faithfully flat. Using Lemma 4.21, we see that there exists a map $A' \to A''$ such that A'' is a Galois extension of A. Let $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(0)}$ be the sieve generated by A'', so that $\mathcal{F}(A) \simeq \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(1)}$ by the above argument. In view of Lemma T.4.3.2.7, it will suffice to show that $\mathcal{F} \,|\, \mathcal{C}^{(0)}$ is a right Kan extension of $\mathcal{F} \,|\, \mathcal{C}^{(1)}$. Choose a map $A' \to B$ and let $\mathcal{C}^{(2)}$ be the pullback of the sieve $\mathcal{C}^{(1)}$ to $\mathrm{CAlg}_B^{\text{ét}}$; we wish to prove that $\mathcal{F}(B) \simeq \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(2)}$. This follows from the above argument, since $\mathcal{C}^{(2)}$ is generated by the Galois extension $B \to B \otimes_A A''$.

Now suppose that $\mathcal{C}^{(0)}$ is generated by a finite collection of finite étale morphisms $\{A \to A_i\}_{1 \leq i \leq n}$ such that the induced map $A \to \prod_i A_i$ is faithfully flat. Let $A' = \prod_i A_i$ and let $\mathcal{C}^{(1)}$ denote the sieve generated by the the map $A \to A'$. The above argument shows that $\mathcal{F}(A) \simeq \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(1)}$. To prove that $\mathcal{F}(A) \simeq \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(0)}$, it will suffice to show that $\mathcal{F} \,|\, \mathcal{C}^{(1)}$ is a right Kan extension of $\mathcal{F} \,|\, \mathcal{C}^{(0)}$. Fix an object $g : A \to B$ in $\mathcal{C}^{(1)}$, and let $\mathcal{C}^{(2)}$ be the pullback of $\mathcal{C}^{(0)}$ to $\mathrm{CAlg}_B^{\text{ét}}$; we wish to prove that $\mathcal{F}(B) \simeq \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(2)}$. By construction,

g factors through some map $g_0 : \prod_i A_i \to B$, so that g_0 determines a decomposition $B \simeq \prod_i B_i$. Let $T \subseteq \{1, \ldots, n\}$ denote the collection of indices i such that $B_i \neq 0$. Let $\mathcal{C}' \subseteq \mathcal{C}^{(2)}$ be the full subcategory spanned by those morphisms $B \to B'$ which factor through some B_i, where $B' \neq 0$. Note that in this case B_i is uniquely determined and the index i belongs to T; it follows that $\mathcal{C}'$ decomposes as a disjoint union of full subcategories $\coprod_{i \in T} \mathcal{C}'_i$. Each of the categories $\mathcal{C}'_i$ contains the projection $B \to B_i$ as an initial object, so the inclusion $\{B_i\}_{i \in T} \hookrightarrow \mathcal{C}'$ is right cofinal. It follows that

$$\mathcal{F}(B) \simeq \mathcal{F}(\prod_{i \in T} B_i) \simeq \prod_{i \in T} \mathcal{F}(B_i) \simeq \varprojlim \mathcal{F} \,|\, \mathcal{C}' .$$

In view of Lemma T.4.3.2.7, we are reduced to proving that $\mathcal{F} \,|\, \mathcal{C}^{(2)}$ is a right Kan extension of $\mathcal{F} \,|\, \mathcal{C}'$. To see this, choose an object $B \to B'$ in $\mathcal{C}^{(2)}$; we wish to show that $\mathcal{F}(B') \simeq \varprojlim \mathcal{F} \,|(\mathcal{C}'_{B'/})$. Let $B'_i = B_i \otimes_B B'$ for $1 \leq i \leq n$, and let T' be the collection of indices i for which $B'_i \neq 0$. Then $\mathcal{C}'_{B'/}$ decomposes as a disjoint union $\coprod_{i \in T'} (\mathcal{C}'_{B'/})_i$, each of which has a final object (given by the map $B' \to B'_i$). It follows that

$$\varprojlim \mathcal{F} \,|(\mathcal{C}'_{/B'}) \simeq \prod_{i \in T'} \mathcal{F}(B'_i) \simeq \mathcal{F}(\prod_{i \in T'} B'_i) \simeq \mathcal{F}(B'),$$

as desired.

We now treat the case of a general covering sieve $\mathcal{C}^{(0)} \subseteq \mathrm{CAlg}^{\text{ét}}_A$. By definition, there exists a finite collection of finite étale maps maps $f_i : A \to A_i$ which generate a covering sieve $\mathcal{C}^{(1)} \subseteq \mathcal{C}^{(0)}$. The above argument shows that $\mathcal{F}(A) \simeq \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(1)}$. To complete the proof, it will suffice (by Lemma T.4.3.2.7) to show that $\mathcal{F} \,|\, \mathcal{C}^{(0)}$ is a right Kan extension of $\mathcal{F} \,|\, \mathcal{C}^{(1)}$. Unwinding the definitions, we must show that for every map $g : A \to B$ in $\mathcal{C}^{(0)}$, we have $\mathcal{F}(A) \simeq \varprojlim \mathcal{F} \,|(g^* \,\mathcal{C}^{(1)})$. This is clear, since $g^* \,\mathcal{C}^{(1)}$ is a covering sieve on B generated by finitely many morphisms $B \to A_i \otimes_A B$. □

5 Linear ∞-Categories and Étale Descent

According to Theorem VII.6.1, the construction $A \mapsto \mathrm{Mod}_A$ satisfies descent with respect to the flat topology on the ∞-category CAlg of $\mathbb{E}_\infty$-rings. In this section, we will consider an analogous situation. Suppose that we fix an $\mathbb{E}_\infty$-ring k and an k-linear ∞-category $\mathcal{C}$. Our goal in this section is to prove Theorem 5.4, which asserts that the construction $A \mapsto \mathrm{LMod}_A(\mathcal{C})$ satisfies descent for the étale topology on the the ∞-category CAlg_k of $\mathbb{E}_\infty$-algebras over k (in the case where k is connective and $\mathcal{C}$ admits an excellent t-structure, this follows from Theorem VII.6.12). In fact, we will prove a more general result, which does not require so much commutativity from the ring spectrum k.

Notation 5.1. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are colimit preserving functors. We let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by the stable ∞-categories. We can identify the ∞-category Sp of spectra with a commutative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ with the ∞-category $\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ (see Proposition A.6.3.2.15). According to Theorem A.6.3.5.14, the construction $A \mapsto \mathrm{LMod}_A(\mathrm{Sp})$ determines a symmetric monoidal functor $\mathrm{Alg}(\mathrm{Sp}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$. Passing to algebra objects (and using Theorem A.5.1.2.2), we obtain a functor

$$\mathrm{Alg}^{(2)} \simeq \mathrm{Alg}(\mathrm{Alg}(\mathrm{Sp})) \to \mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}) \to \mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}).$$

We let LinCat denote the fiber product $\mathrm{Alg}^{(2)} \times_{\mathrm{Alg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{LMod}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. We will refer to LinCat as the *∞-category of linear ∞-categories*.

There is an evident categorical fibration $\theta : \mathrm{LinCat} \to \mathrm{Alg}^{(2)}$. By construction, the fiber of θ over an $\mathbb{E}_2$-ring k can be identified with the ∞-category LinCat_k of k-linear ∞-categories described in Definition VII.6.2. We may therefore think of LinCat as an ∞-category whose objects are pairs $(k, \mathcal{C})$, where k is an $\mathbb{E}_2$-ring and $\mathcal{C}$ is a k-linear ∞-category (that is, a presentable ∞-category which is left-tensored over the monoidal ∞-category LMod_k).

Remark 5.2. The categorical fibration $\theta : \mathrm{LinCat} \to \mathrm{Alg}^{(2)}$ is both a Cartesian fibration and a coCartesian fibration. If $f : k \to k'$ is a map of $\mathbb{E}_2$-rings, then f determines a pair of adjoint functors

$$\mathrm{LinCat}_k \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{LinCat}_{k'} .$$

Here the functor f_* is the forgetful functor which allows us to regard every k'-linear ∞-category as an k-linear ∞-category, and f^* is its left adjoint. If $\mathcal{C}$ is an k-linear ∞-category, then its image under the functor f^* is given by $\mathrm{LMod}_{k'} \otimes_{\mathrm{LMod}_k} \mathcal{C} \simeq \mathrm{LMod}_{k'}(\mathcal{C})$ (see Theorem A.6.3.4.6).

Definition 5.3. Let k be an $\mathbb{E}_2$-ring. We let Alg_k denote the ∞-category of $\mathbb{E}_1$-algebras over k. Recall that an $\mathbb{E}_1$-algebra A over k is said to be *quasi-commutative* if $\pi_0 A$ is central in the ring $\pi_* A$. Given a quasi-commutative k-algebra A, we let $(\mathrm{Alg}_k)^{\text{ét}}_{A/}$ denote the full subcategory of $(\mathrm{Alg}_k)_{A/}$ spanned by those maps of $f : A \to A'$ which exhibit A' as étale over A (see Definition A.7.5.1.4). According to Theorem A.7.5.0.6, the construction $A' \mapsto \pi_0 A'$ determines an equivalence

$$(\mathrm{Alg}_k)^{\text{ét}}_{A/} \to \mathrm{N}(\mathrm{Ring}^{\text{ét}}_{\pi_0 A/});$$

here $\mathrm{Ring}^{\text{ét}}_{\pi_0 A/}$ denotes the ordinary category of étale algebras over the commutative ring commutative $\pi_0 A$.

Let $\mathcal{X}$ be an arbitrary ∞-category. We will say that a functor $\mathcal{F} : (\mathrm{Alg}_k)^{\text{ét}}_{A/} \to \mathcal{X}$ is a *sheaf with respect to the étale topology* (*finite étale topology, Nisnevich topology*) if the associated functor $\mathrm{N}(\mathrm{Ring}^{\text{ét}}_{\pi_0 A/}) \to \mathcal{X}$ is a sheaf with respect to the étale topology (finite étale topology, Nisnevich topology).

More generally, we say that a functor $\mathcal{F} : \mathrm{Alg}_k \to \mathcal{X}$ is a *sheaf with respect to the étale topology* (*finite étale topology, Nisnevich topology*) if, for every quasi-commutative k-algebra A, the composite functor

$$(\mathrm{Alg}_k)^{\text{ét}}_{A/} \to \mathrm{Alg}_k \xrightarrow{\mathcal{F}} \mathcal{X}$$

is a sheaf with respect to the étale topology (finite étale topology, Nisnevich topology).

We can now state our main result.

Theorem 5.4. *Let k be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a k-linear ∞-category, and let $\chi : \mathrm{Alg}_k \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classify the coCartesian fibration $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}_k$ (so that $\chi(A) \simeq \mathrm{LMod}_A(\mathcal{C})$). Then χ is a sheaf with respect to the étale topology.*

Most of the remainder of this section is devoted to the proof of Theorem 5.4. In view of Theorem 3.7, it will suffice to prove the following pair of results:

Proposition 5.5. *Let k be an $\mathbb{E}_2$-ring and $\mathcal{C}$ a k-linear ∞-category. Then the functor $\chi : \mathrm{Alg}_k \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifying the coCartesian fibration $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}_k$ is a sheaf with respect to the finite étale topology.*

Proposition 5.6. *Let k be an $\mathbb{E}_2$-ring and $\mathcal{C}$ a k-linear ∞-category. Then the functor $\chi : \mathrm{Alg}_k \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifying the coCartesian fibration $\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}_k$ is a sheaf with respect to the Nisnevich topology.*

We begin with the proof of Proposition 5.5.

Lemma 5.7. *Let k be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a k-linear ∞-category, and let $\{A_i\}$ be a finite collection of $\mathbb{E}_1$-algebras over k having product $A \in \mathrm{Alg}_k$. Then the canonical functor*

$$F : \mathrm{LMod}_A(\mathcal{C}) \to \prod_i \mathrm{LMod}_{A_i}(\mathcal{C})$$

is an equivalence of ∞-categories.

Proof. The functor F has a right adjoint G, which carries a collection of module objects $M_i \in \mathrm{LMod}_{A_i}(\mathcal{C})$ to the product $\prod_i M_i$ (regarded as a left A-module object of $\mathcal{C}$). For any object $M \in \mathcal{C}$, the unit map

$$M \to (GF)M \simeq \prod_i (A_i \otimes_A M) \simeq (\prod_i A_i) \otimes_A M \simeq A \otimes_A M \simeq M$$

is an equivalence; this proves that F is fully faithful. To show that F is an equivalence, it suffices to show that G is conservative. Suppose we are given a morphism α in $\prod_i \mathrm{LMod}_{A_i}(\mathcal{C})$, given by a collection of maps $\alpha_i : M_i \to N_i$ in $\mathrm{LMod}_{A_i}(\mathcal{C})$. Then each α_i is a retract of $G(\alpha)$ (when regarded as a morphism in the ∞-category $\mathcal{C}$). Consequently, if $G(\alpha)$ is an equivalence, then each α_i is an equivalence; this proves that G is conservative as desired. □

Proof of Proposition 5.5. Fix a quasi-commutative $\mathbb{E}_1$-algebra A over k. We wish to prove that $\chi_A = \chi | (\mathrm{Alg}_k)^{\text{ét}}_{A/}$ is a sheaf with respect to the finite étale topology. In view of Proposition VII.5.7, it will suffice to show the following:

- (i) The functor χ_A commutes with products.
- (ii) Let $f : B \to B^0$ be a faithfully flat finite étale morphism in $(\mathrm{Alg}_k)^{\text{ét}}_{A/}$, and let $B^\bullet$ be the Čech nerve of f (in the ∞-category $((\mathrm{Alg}_k)^{\text{ét}}_{A/})^{op}$). Then the canonical map $\chi(B) \to \varprojlim \chi(B^\bullet)$ is an equivalence of ∞-categories.

Assertion (i) follows from Lemma 5.7. To establish (ii), we let $\mathrm{LinCat}_k = \mathrm{LMod}_{\mathrm{LMod}_k}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ denote the ∞-category of k-linear ∞-categories, and note that LinCat_k is right-tensored over the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories (see §A.4.3.2). For any $\mathbb{E}_1$-algebra R over k, we can identify $\mathrm{LMod}_R(\mathcal{C}) \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$ with a morphism object $\mathrm{Mor}_{\mathrm{LinCat}_k}(\mathrm{RMod}_R, \mathcal{C})$ (see Theorem A.6.3.4.1). Assertion (ii) is therefore a consequence of the following:

- (ii') Let $f : B \to B^0$ be as in (ii). Then the canonical map

  $$\varinjlim \mathrm{LMod}_{B^\bullet} \to \mathrm{LMod}_B$$

  is an equivalence in LinCat_k.

Corollary A.4.2.3.5 implies that the forgetful functor $\mathrm{LinCat}_k \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves small colimits. Consequently, to prove (ii'), it will suffice to show that LMod_B is a colimit of the diagram $\mathrm{LMod}_{B^\bullet}$ in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. The construction $[n] \mapsto \mathrm{LMod}_{B^\bullet}$ determines a functor $\mathrm{N}(\mathbf{\Delta}_+)^{op} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$, which is classified by a coCartesian fibration $q : \mathcal{X} \to \mathrm{N}(\mathbf{\Delta}_+)^{op}$ whose fibers are given by $\mathcal{X}_{[n]} \simeq \mathrm{LMod}_{B^n}$ (here we adopt the convention that $B^{-1} = B$). The functor q is also a Cartesian fibration, classified by a functor $u : \mathrm{N}(\mathbf{\Delta}_+) \to \mathcal{P}\mathrm{r}^{\mathrm{R}}$. To prove ($ii'$), it will suffice to show that u is a limit diagram. Equivalently, we must show that the composite functor $\mathrm{N}(\mathbf{\Delta}_+) \to \mathcal{P}\mathrm{r}^{\mathrm{R}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ is a limit diagram (Theorem T.5.5.3.18). For this, it will suffice to show that u satisfies the criteria of Corollary A.6.2.4.3:

1. Let $G : \mathrm{LMod}_B \to \mathrm{LMod}_{B^0}$ be the right adjoint to the forgetful functor $\mathrm{LMod}_{B^0} \to \mathrm{LMod}_B$. Then G preserves geometric realizations of G-split simplicial objects. In fact, we claim that G preserves all colimits. To prove this, it will suffice to show that the composite functor

   $$\mathrm{LMod}_B \xrightarrow{G} \mathrm{LMod}_{B^0} \to \mathrm{Sp}$$

   preserves all colimits (Corollary A.4.2.3.5). This composite functor is right adjoint to the functor $\mathrm{Sp} \to \mathrm{LMod}_B$ given by the action of Sp on the object $B^0 \in \mathrm{LMod}_B$. It will therefore suffice to show that B^0 is perfect as a left B-module. Since B^0 is flat over B, we have $B^0 \simeq B \otimes_{\tau_{\geq 0} B} \tau_{\geq 0} B^0$; it therefore suffices to show that $\tau_{\geq 0} B^0$ is a perfect left module over $\tau_{\geq 0} B$, which follows from Proposition 3.1.

(2) For every morphism $[m] \to [n]$ in $\mathbf{\Delta}_+$, the diagram

$$\begin{array}{ccc} \mathrm{LMod}_{B^m} & \longrightarrow & \mathrm{LMod}_{B^{m+1}} \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{B^n} & \longrightarrow & \mathrm{LMod}_{B^{n+1}} \end{array}$$

determined by u is left adjointable. Passing to left adjoints everywhere, we are reduced to proving that the diagram of forgetful functors

$$\begin{array}{ccc} \mathrm{LMod}_{B^{n+1}} & \longrightarrow & \mathrm{LMod}_{B^n} \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{B^{m+1}} & \longrightarrow & \mathrm{LMod}_{B^m} \end{array}$$

is right adjointable. Unwinding the definitions, we must show that if M is a left B^n-module, then the canonical maps

$$\mathrm{Ext}^k_{B^n}(B^{n+1}, M) \to \mathrm{Ext}^k_{B^m}(B^{m+1}, M)$$

are isomorphisms. Equivalently, we must show that the canonical map $B^n \otimes_{B^m} B^{m+1} \to B^{n+1}$ is an equivalence of left B^n-modules. This follows from Theorem A.7.5.1.11, since we have a pushout diagram

$$\begin{array}{ccc} B^m & \longrightarrow & B^{m+1} \\ \downarrow & & \downarrow \\ B^n & \longrightarrow & B^{n+1} \end{array}$$

in $(\mathrm{Alg}_k)^{\text{ét}}_{A/}$.

(3) The functor G is conservative. As in the proof of (1), it suffices to show that the composite functor $\mathrm{LMod}_B \xrightarrow{G} \mathrm{LMod}_{B^0} \to \mathrm{Sp}$ is conservative. This composite functor is given by the formula $M \mapsto (B^0)^\vee \otimes_B M$, where $(B^0)^\vee$ is the perfect right B-module dual to B^0. Since B^0 is a retract of a finitely generated free left B-module, the dual $(B^0)^\vee$ is a retract of a finitely generated free right B-module, and therefore flat over B. It follows that

$$\pi_n((B^0)^\vee \otimes_B M) \simeq \pi_0(B^0)^\vee \otimes_{\pi_0 B} \pi_n M.$$

If $M \neq 0$, then $\pi_n M \neq 0$ for some integer n, so that $\pi_n((B^0)^\vee \otimes_B M) \neq 0$ (note that $\pi_0(B^0)^\vee$ is the $\pi_0 B$-linear dual of $\pi_0 B^0$, and therefore faithfully flat over $\pi_0 B$).

□

The rest of this section is devoted to the proof of Proposition 5.6. In what follows, we fix an $\mathbb{E}_2$-algebra k, a k-linear ∞-category $\mathcal{C}$, and a quasi-commutative $\mathbb{E}_1$-algebra A over k. Let $R = \pi_0 A$ and let Test_R be the category defined in §2. Using Theorem A.7.5.0.6, we see that the construction $A' \mapsto \mathrm{Spec}^{\mathrm{Z}}(\pi_0 A')$ determines a fully faithful embedding $i : (\mathrm{Alg}_k)^{\text{ét}}_{A/} \to \mathrm{N}(\mathrm{Test}_R)^{op}$. Let $\chi : (\mathrm{Alg}_k)^{\text{ét}}_{A/} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Proposition 5.6 and let $\chi' : \mathrm{N}(\mathrm{Test}_R)^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be a right Kan extension of χ along i. The functor χ' assigns to each object $X \in \mathrm{Test}_R$ an ∞-category which we will denote by $\mathrm{QCoh}(X; \mathcal{C})$, and to each map of R-schemes $f : X \to Y$ in Test_R a functor $f^* : \mathrm{QCoh}(Y; \mathcal{C}) \to \mathrm{QCoh}(X; \mathcal{C})$.

Remark 5.8. Suppose we are given a pushout diagram σ :

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ B' & \longrightarrow & C' \end{array}$$

in $(\mathrm{Alg}_k)^{\text{ét}}_{A/}$. Then the associated diagram

$$\begin{array}{ccc} \mathrm{LMod}_B(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_C(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{B'}(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_{C'}(\mathcal{C}) \end{array}$$

is right adjointable. In other words, for any object $M \in \mathrm{LMod}_C(\mathcal{C})$, the canonical map $B' \otimes_B M \to C' \otimes_C M$ is an equivalence in $\mathrm{LMod}_{B'}(\mathcal{C})$. This follows immediately from the observation that σ induces an equivalence $B' \otimes_B C \to C'$.

Lemma 5.9. *The forgetful functor $\theta : (\mathrm{Alg}_k)^{\acute{e}t}_{A/} \to \mathrm{Sp}$ is a* Sp*-valued sheaf with respect to the étale topology on $(\mathrm{Alg}_k)^{\acute{e}t}_{A/}$.*

Proof. We will show that θ satisfies the criterion of Proposition VII.5.7. It is obvious that θ commutes with finite products. Let $f : B \to B^0$ be a morphism in $(\mathrm{Alg}_k)^{\text{ét}}_{A/}$, and let $B^\bullet$ be the Čech nerve of f; we wish to show that the canonical map $B \to \varprojlim B^\bullet$ is an equivalence of spectra. In view of Corollary A.1.2.4.10, it will suffice to show that for each $n \geq 0$, the unnormalized chain complex associated to the cosimplicial abelian group $\pi_n B^\bullet$ is an acyclic resolution of $\pi_n B$. That is, we must show the exactness of the sequence

$$0 \to \pi_n B \to \pi_n B^0 \to \pi_n B^1 \to \cdots$$

This follows from the classical theory of faithfully flat descent: the sequence of $\pi_0 B$-modules becomes exact after tensoring with the faithfully flat $\pi_0 B$-module $\pi_0 B^0$. □

As a first approximation to Proposition 5.6, we prove:

Lemma 5.10. *Let $X \in \mathrm{Test}_R$, and suppose that X is covered by a pair of open subsets $U, V \subseteq X$. Then the diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{QCoh}(X; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(U; \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(V; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(U \cap V; \mathcal{C}) \end{array}$$

is a pullback square.

Remark 5.11. Note that if X is empty, then

$$\mathrm{QCoh}(X; \mathcal{C}) \simeq \mathrm{QCoh}(\mathrm{Spec}\, A'; \mathcal{C}) \simeq \mathrm{LMod}_{A'}(\mathcal{C})$$

is a contractible Kan complex, where $A' = 0 \in (\mathrm{Alg}_k)^{\text{ét}}_{A/}$. According to Theorem T.7.3.5.2, Lemma 5.10 is equivalent to the statement that the construction $X \mapsto \mathrm{QCoh}(X; \mathcal{C})$ is a sheaf with respect to the Zariski topology on Test_R.

Proof. Let $\mathcal{Z}_X$ denote the ∞-category

$$(\mathrm{Alg}_k)^{\text{ét}}_{A/} \times_{\mathrm{N}(\mathrm{Test}_R)^{op}} \mathrm{N}((\mathrm{Test}_R)_{/X})^{op},$$

and define $\mathcal{Z}_U$, $\mathcal{Z}_V$, and $\mathcal{Z}_{U\cap V}$ similarly. Let $\chi : \mathrm{Alg}(k)^{\text{ét}}_{A/} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Proposition 5.6. We wish to prove that

$$\begin{array}{ccc} \varprojlim \chi | \mathcal{Z}_X & \longrightarrow & \varprojlim \chi | \mathcal{Z}_U \\ \downarrow & & \downarrow \\ \varprojlim \chi | \mathcal{Z}_V & \longrightarrow & \varprojlim \chi | \mathcal{Z}_{U\cap V} \end{array}$$

is an equivalence. Let $\mathcal{Z}'_X$ be the full subcategory of $\mathcal{Z}_X$ spanned by those objects belonging to $\mathcal{Z}_U$ or $\mathcal{Z}_V$. We have a pushout diagram of simplicial sets

$$\begin{array}{ccc} \mathcal{Z}_{U\cap V} & \longrightarrow & \mathcal{Z}_V \\ \downarrow & & \downarrow \\ \mathcal{Z}_U & \longrightarrow & \mathcal{Z}'_X . \end{array}$$

Using the results of §T.4.2.3, we are reduced to proving that the restriction map $\varprojlim \chi | \mathcal{Z}_X \to \varprojlim \chi | \mathcal{Z}'_X$ is an equivalence. In view of Lemma T.4.3.2.7, it will suffice to show that $\chi | \mathcal{Z}_X$ is a right Kan extension of $\chi | \mathcal{Z}'_X$.

Since $U \cup V = X$, there exists a finite collection of affine open subsets $W_1, W_2, \ldots, W_n \subseteq X$ such that $X = \bigcup_i W_i$ and each W_i is contained either in U or in V. Let $\mathcal{Z}''_X$ denote the union of the subcategories $\mathcal{Z}_{W_i} \subseteq \mathcal{Z}_X$. We will complete the proof by showing that $\chi | \mathcal{Z}_X$ is a right Kan extension of $\chi | \mathcal{Z}''_X$. To this end, choose a map $f : X' \to X$ in Test_R where X' is affine, and let $W'_i = W_i \times_X X'$ for $1 \leq i \leq n$. For each $I \subseteq \{1, \ldots, n\}$, let $W'_I = \bigcap_{i \in I} W'_i \subseteq X'$. Let S be the collection of all nonempty subsets of $\{1, \ldots, n\}$. We note that the construction $I \mapsto W'_I$ determines a right cofinal map

$$\mathrm{N}(S) \to \mathcal{Z}''_X \times_{\mathrm{N}((\mathrm{Test}_R)_{/X})} \mathrm{N}((\mathrm{Test}_R)_{/X'}).$$

It will therefore suffice to show that the canonical map

$$F : \mathrm{QCoh}(X'; \mathcal{C}) \to \varprojlim_{I \in S} \mathrm{QCoh}(W'_I; \mathcal{C})$$

is an equivalence of ∞-categories.

Note that each W'_I is affine, and therefore has the form $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 A'_I)$ for an essentially unique $A'_I \in \mathrm{Alg}(k)^{\text{ét}}_{A/}$. Let $A' = A'_\emptyset$, so we can identify F with the base-change map $\mathrm{LMod}_{A'}(\mathcal{C}) \to \varprojlim_{I \in S} \mathrm{LMod}_{A'_I}(\mathcal{C})$. Note that F has a right adjoint G, which carries a compatible family of objects $\{M_I \in \mathrm{LMod}_{A'_I}(\mathcal{C})\}$ to the limit $\varprojlim_{I \in S} M_I \in \mathrm{LMod}_{A'}(\mathcal{C})$. We first show that the counit transformation $F \circ G \to \mathrm{id}$ is an equivalence. In other words, we claim that for every object $\{M_I \in \mathrm{LMod}_{A'_i}(\mathcal{C})\}_{I \in S} \in \varprojlim \mathrm{LMod}_{A'_I}(\mathcal{C})$ and each $J \in S$, the canonical map

$$A'_J \otimes_{A'} \varprojlim_{I \in S} M_I \to M_J$$

is an equivalence. Since the relative tensor product preserves finite limits, we can rewrite the left hand side as $\varprojlim_{I \in S} A'_J \otimes_{A'} M_I$. Note that for each $I \in S$, the diagram

$$\begin{array}{ccc} A' & \longrightarrow & A'_I \\ \downarrow & & \downarrow \\ A'_J & \longrightarrow & A'_{I\cup J} \end{array}$$

is a pushout square in $(\mathrm{Alg}_k)^{\text{ét}}_{A/}$, so Remark 5.8 implies that the canonical map $A'_J \otimes_{A'} M_I \to A'_{I\cup J} \otimes_{A'_I} M_I \simeq M_{I\cup J}$ is an equivalence. It follows that the functor $I \mapsto A'_J \otimes_{A'} M_I$ is a right Kan extension of its restriction to $\mathrm{N}(S_J)$, where $S_J = \{I \in S : J \subseteq I\}$. Since S_J contains J as a final object, we obtain

$$\varprojlim_{I\in S} A'_J \otimes_{A'} M_I \simeq \varprojlim_{I\in S_J} A'_J \otimes_{A'} M_I \simeq A'_J \otimes_{A'} M_J \simeq M_J$$

as desired.

We now claim that the unit transformation $u : \mathrm{id} \to G \circ F$ is an equivalence. In other words, for every object $M \in \mathrm{LMod}_{A'}(\mathcal{C})$, the canonical map $M \to \varprojlim_{I\in S} A'_I \otimes_{A'} M$ is an equivalence in $\mathrm{LMod}_{A'}(\mathcal{C})$. Since the relative tensor product preserves finite limits, it suffices to show that the map $A' \to \varprojlim_{I\in S} A'_I$ is an equivalence of right A'-modules. Let $\mathcal{Y} \subseteq (\mathrm{Alg}(k)^{\text{ét}}_{A'/})^{op}$ be the sieve generated by the objects $\{A'_{\{i\}}\}_{1\leq i\leq n}$. The construction $I \mapsto A'_I$ determines a left cofinal functor $S^{op} \to \mathcal{Y}$; we are therefore reduced to proving that the canonical map $A' \to \varprojlim_{B\in\mathcal{Y}^{op}} B$ is an equivalence, which follows from Lemma 5.9. □

Lemma 5.12. *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

in Test_R*. Then the associated diagram of ∞-categories σ :*

$$\begin{array}{ccc} \mathrm{QCoh}(Y;\mathcal{C}) & \longrightarrow & \mathrm{QCoh}(X;\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(Y';\mathcal{C}) & \xrightarrow{g} & \mathrm{QCoh}(X';\mathcal{C}) \end{array}$$

is right adjointable.

Proof. We proceed in several steps.

(1) Assume that X, Y, and Y' are affine (so that X' is also affine). In this case, the desired result is a reformulation of Remark 5.8.

(2) Assume that Y and Y' are affine. Since X is quasi-compact, we can write X as a union of affine open subsets $\{U_i \subseteq X\}_{1\leq i\leq n}$. We proceed by induction on n. If $n \leq 1$, then X is affine and the desired result follows from (1). Assume therefore that $n > 1$. Let $U = U_1$ and $V = \bigcup_{1<i\leq n} U_i$, and set $U' = U \times_X X'$ and $V' = V \times_X X'$. Lemma 5.10 implies that, as an object of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \widehat{\mathcal{C}\mathrm{at}}_\infty)$, the diagram σ can be written as a fiber product $\sigma_U \times_{\sigma_{U\cap V}} \sigma_V$, where σ_U denotes the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Y;\mathcal{C}) & \longrightarrow & \mathrm{QCoh}(U;\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(Y';\mathcal{C}) & \longrightarrow & \mathrm{QCoh}(U';\mathcal{C}) \end{array}$$

and the diagrams σ_V and $\sigma_{U\cap V}$ are defined similarly. The inductive hypothesis guarantees that σ_U, σ_V, and $\sigma_{U\cap V}$ are right adjointable. It follows from Corollary A.6.2.3.18 that σ is also right adjointable.

(3) Assume that the map g is affine. Since Y is quasi-compact, we can choose an open cover of Y by affine subsets $\{U_i \subseteq Y\}_{1\leq i\leq n}$. We proceed by induction on n. If $n \leq 1$, then Y is affine. Since g is affine, Y' is also affine and the desired result follows from (2). If $n > 1$, we let $U = U_1$ and $V = \bigcup_{1<i\leq n} U_i$.

Using Lemma 5.10, we deduce that, as an object of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \widehat{\mathcal{C}\mathrm{at}}_\infty)$, the diagram σ can be written as a fiber product $\sigma_U \times_{\sigma_{U \cap V}} \sigma_V$, where σ_U denotes the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(U; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(X \times_Y U; \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(Y' \times_Y U; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(X' \times_Y U; \mathcal{C}) \end{array}$$

and the diagrams σ_V and $\sigma_{U \cap V}$ are defined similarly. The inductive hypothesis guarantees that σ_U, σ_V, and $\sigma_{U \cap V}$ are right adjointable. It follows from Corollary A.6.2.3.18 that σ is also right adjointable.

(4) Let the diagram σ be arbitrary. Since Y' is quasi-compact, we can choose an open cover of Y' by affine subsets $\{U_i \subseteq Y'\}_{1 \leq i \leq n}$. We proceed by induction on n. If $n \leq 1$, then Y' is affine. It follows in particular that g is affine (since Y is separated) so that the desired result follows from (3). If $n > 1$, we let $U = U_1$ and $V = \bigcup_{1 < i \leq n} U_i$. Using Lemma 5.10, we deduce that, as an object of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \widehat{\mathcal{C}\mathrm{at}}_\infty)$, the diagram σ can be written as a fiber product $\sigma_U \times_{\sigma_{U \cap V}} \sigma_V$, where σ_U denotes the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(Y; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(X; \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(U; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(X' \times_{Y'} U; \mathcal{C}) \end{array}$$

and the diagrams σ_V and $\sigma_{U \cap V}$ are defined similarly. The inductive hypothesis guarantees that σ_U, σ_V, and $\sigma_{U \cap V}$ are right adjointable. It follows from Corollary A.6.2.3.18 that σ is also right adjointable.

□

Proof of Proposition 5.6. According to Proposition 2.10, it will suffice to show that the construction $X \mapsto \mathrm{QCoh}(X; \mathcal{C})$ is a Nisnevich sheaf on $\mathrm{N}(\mathrm{Test}_R)$. By virtue of Theorem 2.9, it will suffice to show that this construction satisfies affine Nisnevich excision. Since $\mathrm{QCoh}(X; \mathcal{C}) \simeq \Delta^0$ when X is empty (Remark 5.11), we need only show that if $f : X' \to X$ is an affine morphism in Test_R and $U \subseteq X$ is a quasi-compact open subset such that $X' - U' \simeq X - U$ where $U' = U \times_X X'$, then the diagram σ :

$$\begin{array}{ccc} \mathrm{QCoh}(X; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(X'; \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(U; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(U'; \mathcal{C}) \end{array}$$

is a pullback diagram of ∞-categories.

Since X is quasi-compact, we can write X as a union of affine open subsets $\{V_i \subseteq X\}_{1 \leq i \leq m}$. We proceed by induction on m. If $m \geq 2$, we set $V = V_1$ and $W = \bigcup_{1 < i \leq m} V_i$. Lemma 5.10 implies that σ is equivalent to the fiber product $\sigma_V \times_{\sigma_{V \cap W}} \sigma_W$, where σ_V is the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(V; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(X' \times_X V; \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(U \cap V; \mathcal{C}) & \longrightarrow & \mathrm{QCoh}(U' \times_X V; \mathcal{C}) \end{array}$$

and the diagrams $\sigma_{V \cap W}$ and σ_W are defined similarly. The inductive hypothesis guarantees that σ_V, σ_W, and $\sigma_{V \cap W}$ are pullback diagrams in $\widehat{\mathcal{C}\mathrm{at}}_\infty$, so that σ is also a pullback diagram. We may therefore assume that $m \leq 1$, so that X and X' are affine.

Let $F : \mathrm{QCoh}(X; \mathcal{C}) \to \mathrm{QCoh}(X'; \mathcal{C}) \times_{\mathrm{QCoh}(U;\mathcal{C})} \mathrm{QCoh}(U'; \mathcal{C})$ be the functor determined by σ; we wish to show that F is an equivalence. Since U is quasi-compact, there exists a finite covering of U by affine open subsets $\{U_i \subseteq U\}_{1 \leq i \leq n}$. For $I \subseteq \{0, 1, \ldots, n\}$, let

$$U_I = \begin{cases} \bigcap_{i \in I} U_i & \text{if } 0 \notin I \\ X' \times_X \bigcap_{i \in I - \{0\}} U_i & \text{if } 0 \in I. \end{cases}$$

Let S denote the collection of all nonempty subsets of $\{0, \ldots, n\}$, partially ordered by inclusion. Using Lemma 5.10, we deduce that the canonical maps

$$\mathrm{QCoh}(U; \mathcal{C}) \to \varprojlim_{I \neq \emptyset, 0 \notin I} \mathrm{QCoh}(U_I; \mathcal{C})$$

$$\mathrm{QCoh}(U'; \mathcal{C}) \to \varprojlim_{I \neq \emptyset, 0 \in I} \mathrm{QCoh}(U'_I; \mathcal{C})$$

are equivalences. Consequently, the results of §T.4.2.3 imply that that the fiber product

$$\mathrm{QCoh}(X'; \mathcal{C}) \times_{\mathrm{QCoh}(U;\mathcal{C})} \mathrm{QCoh}(U'; \mathcal{C})$$

can be identified with the limit $\varprojlim_{I \in S} \mathrm{QCoh}(U_I; \mathcal{C})$.

Each of the schemes U_I is affine, and therefore of the form $\mathrm{Spec}\, \pi_0 B_I$ for some $B_I \in (\mathrm{Alg}_k)^{\text{ét}}_{A/}$; let $B = B_\emptyset$. We therefore have $\mathrm{QCoh}(U_I; \mathcal{C}) \simeq \mathrm{LMod}_{B_I}(\mathcal{C})$. As in the proof of Lemma 5.10, the functor F has a right adjoint, which carries a compatible system of objects $\{M_I \in \mathrm{LMod}_{B_I}(\mathcal{C})\}$ to the object $\varprojlim M_I \in \mathrm{LMod}_B(\mathcal{C})$. We first claim that the unit map $u : \mathrm{id} \to G \circ F$ is an equivalence. In other words, we claim that for each object $M \in \mathrm{LMod}_B(\mathcal{C})$, the canonical map $M \to \varprojlim_{I \in S} B_I \otimes_B M$ is an equivalence in $\mathrm{LMod}_B(\mathcal{C})$. Since tensor product over B commutes with finite limits, it will suffice to show that the canonical map $\theta : B \to \varprojlim_{I \in S} B_I$ is an equivalence of right B-modules.

Let $\phi : (\mathrm{Alg}_k)^{\text{ét}}_{A/} \to \mathrm{Sp}$ be the forgetful functor, and let $\Gamma : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathrm{Sp}$ be a right Kan extension of ϕ along the fully faithful embedding $\mathrm{Alg}(k)^{\text{ét}}_{A/} \to \mathrm{N}(\mathrm{Test}_R)^{op}$ given by $B \mapsto \mathrm{Spec}^{\mathrm{Z}}(\pi_0 B)$. Lemma 5.9 guarantees that ϕ is a Nisnevich sheaf, so that Γ is a sheaf with respect to the Nisnevich topology on Test_R (Proposition 2.10). In particular, Γ satisfies Nisnevich excision, so that the canonical maps

$$\Gamma(U) \to \varprojlim_{I \neq \emptyset, 0 \notin I} B_I$$

$$\Gamma(U') \to \varprojlim_{I \neq \emptyset, 0 \in I} B_I$$

are equivalences. Using the results of §T.4.2.3, we can identify θ with the canonical map $\Gamma(X) \to \Gamma(X') \times_{\Gamma(U')} \Gamma(U)$, which is an equivalence by virtue of our assumption that Γ satisfies Nisnevich excision.

To complete the proof, it will suffice to show that the functor $G : \mathrm{QCoh}(X'; \mathcal{C}) \times_{\mathrm{QCoh}(U';\mathcal{C})} \mathrm{QCoh}(U; \mathcal{C}) \to \mathrm{QCoh}(X; \mathcal{C})$ is conservative. Consider the diagram

$$\begin{array}{ccc} U' & \xrightarrow{j'} & X' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X. \end{array}$$

Let $f^* : \mathrm{QCoh}(X; \mathcal{C}) \to \mathrm{QCoh}(X'; \mathcal{C})$ denote the functor associated to f and let f_* denote its left adjoint, and define adjoint pairs (f'^*, f'_*), (j^*, j_*), and (j'^*, j'_*) similarly. We can think of an object of the fiber product $\mathrm{QCoh}(X'; \mathcal{C}) \times_{\mathrm{QCoh}(U';\mathcal{C})} \mathrm{QCoh}(U; \mathcal{C})$ as a triple

$$(M_{X'} \in \mathrm{QCoh}(X'; \mathcal{C}), M_{U'} \in \mathrm{QCoh}(U'; \mathcal{C}), M_U \in \mathrm{QCoh}(U; \mathcal{C}))$$

equipped with equivalences $j'^{*}M_{X'} \simeq M_{U'} \simeq f'^{*}M_U$. In terms of this identification, G is given informally by the formula

$$G(M_{X'}, M_{U'}, M_U) = f_* M_{X'} \times_{f_* j'_* M_{U'}} j_* M_U \in \mathrm{QCoh}(X; \mathcal{C}).$$

Since G is an exact functor between stable ∞-categories, it will suffice to show that if $G(M_{X'}, M_{U'}, M_U) \simeq 0$, then $(M_{X'}, M_{U'}, M_U) \simeq 0$. Using Lemma 5.12, we deduce that the counit map $v : j'^{*} j'_* M_{U'} \to M_{U'}$ is an equivalence. Let $h : f_* M_{X'} \to f_* j'^{*} M_{U'}$ be the canonical map, and consider the diagram

$$j^* f_* M_{X'} \xrightarrow{j^*(h)} j^* f_* j'_* M_{U'} \xrightarrow{h'} f'_* j'^{*} j'{*} M_{U'} \xrightarrow{f'_*(v)} f'_* M_{U'} \simeq f'_* j'^{*} M_{X'}.$$

Lemma 5.12 guarantees that the composition and h' are equivalences. Since $f'_*(v)$ is an equivalence, we concude that $j^*(h)$ is an equivalence. It follows that the projection map $j^* G(M_{X'}, M_{U'}, M_U) \to j^* j_* M_U$ is an equivalence. If $G(M_{X'}, M_{U'}, M_U) \simeq 0$, then we conclude that $M_U \simeq j^* j_* M_U \simeq j^* G(M_{X'}, M_{U'}, M_U) \simeq 0$. Then $M_{U'} \simeq f'^{*} M_U \simeq 0$, so that $f_* M_{X'} \simeq G(M_{X'}, M_{U'}, M_U) \simeq 0$. Since X' and X are affine, the pushforward functor $\mathrm{QCoh}(X'; \mathcal{C}) \to \mathrm{QCoh}(X; \mathcal{C})$ can be identified with the forgetful functor $\mathrm{LMod}_{B'}(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{C})$ associated to some map $B \to B'$ in $(\mathrm{Alg}_k)^{\text{ét}}_{A/}$, and is therefore conservative. It follows that $M_{X'} \simeq 0$, so that the object $(M_{X'}, M_{U'}, M_U) \in \mathrm{QCoh}(X'; \mathcal{C}) \times_{\mathrm{QCoh}(U'; \mathcal{C})} \mathrm{QCoh}(U; \mathcal{C})$ is zero as desired. □

Using Theorem 5.4, we can prove a descent property for the functor $k \mapsto \mathrm{LinCat}_k$ itself. First, let us introduce a bit of terminology. For every $\mathbb{E}_2$-ring k, we let $(\mathrm{Alg}^{(2)})^{\text{ét}}_{k/}$ denote the full subcategory of $\mathrm{Alg}^{(2)}_{k/}$ spanned by the étale maps $k \to k'$. According to Theorem A.7.5.0.6, the functor $k' \mapsto \pi_0 k'$ determines an equivalence from $(\mathrm{Alg}^{(2)})^{\text{ét}}_{k/}$ to the nerve of the ordinary category of commutative rings which are étale over $\pi_0 k$. We will say that a functor $\mathcal{F} : (\mathrm{Alg}^{(2)})^{\text{ét}}_{k/} \to \mathcal{C}$ is a *$\mathcal{C}$-valued sheaf for the étale topology* if the corresponding functor $\mathrm{N}(\mathrm{Ring}^{\text{ét}}_{\pi_0 k/}) \to \mathcal{C}$ is a sheaf for the étale topology. We say that a functor $\mathrm{Alg}^{(2)} \to \mathcal{C}$ is a *$\mathcal{C}$-valued sheaf for the étale topology* if, for every $\mathbb{E}_2$-ring k, the induced map $(\mathrm{Alg}^{(2)})^{\text{ét}}_{k/} \to \mathcal{C}$ is a $\mathcal{C}$-valued sheaf for the étale topology.

Theorem 5.13 (Effective Descent for Linear ∞-Categories)**.** *Let $\chi : \mathrm{Alg}^{(2)} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor classifying the coCartesian fibration $\theta : \mathrm{LinCat} \to \mathrm{Alg}^{(2)}$. Then χ is a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf with respect to the étale topology.*

We first need an analogue of Lemma 5.7.

Lemma 5.14. *Let $\{A_i\}_{1 \leq i \leq n}$ be a finite collection of $\mathbb{E}_2$-rings having product A. Then the canonical map*

$$\phi : \mathrm{LinCat}_A \to \prod_{1 \leq i \leq n} \mathrm{LinCat}_{A_i}$$

is an equivalence of ∞-categories.

Proof. We will prove that ϕ satisfies the hypotheses of Lemma VII.5.17:

(*a*) Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism in LinCat_A whose image in each LinCat_{A_i} is an equivalence. We wish to show that F is an equivalence. Using Lemma 5.7, we deduce that $\mathcal{C}$ and $\mathcal{D}$ can be identified with the products $\prod_i \mathrm{LMod}_{A_i}(\mathcal{C})$ and $\prod_i \mathrm{LMod}_{A_i}(\mathcal{D})$, respectively. Since the induced map $\mathrm{LMod}_{A_i}(\mathcal{C}) \to \mathrm{LMod}_{A_i}(\mathcal{D})$ is an equivalence for each index i, we conclude that F induces an equivalence of ∞-categories $f : \mathcal{C} \to \mathcal{D}$.

(*b*) Suppose we are given a finite collection of objects $(A_i, \mathcal{C}_i)$ in LinCat. For each index i, let $\mathcal{D}_i$ denote the ∞-category $\mathcal{C}_i$, regarded as an A-linear category, and set $\mathcal{D} = \prod_i \mathcal{D}_i$. We wish to prove that for each index i, the canonical map $\mathrm{LMod}_{A_i} \otimes_{\mathrm{LMod}_A} \mathcal{D} \to \mathcal{C}_i$ is an equivalence. We have

$$\mathrm{LMod}_{A_i}(\mathcal{D}) \simeq \mathrm{LMod}_{A_i}(\prod_j \mathcal{D}_j) \simeq \prod_j \mathrm{LMod}_{A_i}(\mathcal{D}_j) \simeq \prod_j \mathrm{LMod}_{A_i \otimes_A A_j}(\mathcal{C}_j).$$

For $i \neq j$, the tensor product $A_i \otimes_A A_j$ is trivial, so that $\mathrm{LMod}_{A_i \otimes_A A_j}(\mathcal{C}_j)$ is a contractible Kan complex. For $i = j$, the tensor product $A_i \otimes_A A_j$ is equivalent to A_i, so that the forgetful functor $\mathrm{LMod}_{A_i \otimes_A A_j}(\mathcal{C}_j) \to \mathcal{C}_j$ is an equivalence. Passing to the product over i, we obtain the desired result.

□

Proof of Theorem 5.13. Let k be an $\mathbb{E}_2$-ring; we wish to prove that $\chi_k = \chi | (\mathrm{Alg}^{(2)})^{\text{ét}}_{k/}$ is a sheaf with respect to the étale topology. It will suffice to show that χ_k satisfies the hypotheses of Proposition VII.5.7. It follows from Lemma 5.14 that the functor χ_k commutes with finite products. To complete the proof, let us suppose that $f : A \to A^0$ is a faithfully flat morphism in $(\mathrm{Alg}^{(2)})^{\text{ét}}_{k/}$, and let $A^\bullet$ denote its Čech nerve. We wish to show that the canonical map $\chi(A) \to \varprojlim \chi(A^\bullet)$ is an equivalence of ∞-categories. We proceed by showing that this functor satisfies the conditions of Lemma VII.5.17:

(*a*) Fix an morphism $F : \mathcal{C} \to \mathcal{D}$ in $\chi(A) = \mathrm{LinCat}_A$ whose image in LinCat_{A^0} is an equivalence. It follows that F induces an equivalence of cosimplicial ∞-categories $\mathrm{LMod}_{A^\bullet}(\mathcal{C}) \to \mathrm{LMod}_{A^\bullet}(\mathcal{D})$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ \varprojlim \mathrm{LMod}_{A^\bullet}(\mathcal{C}) & \longrightarrow & \varprojlim \mathrm{LMod}_{A^\bullet}(\mathcal{D}), \end{array}$$

where the vertical maps are equivalences of ∞-categories by Theorem 5.4. It follows that F is an equivalence of ∞-categories.

(*b*) Suppose we are given a diagram $X^\bullet : \mathrm{N}(\mathbf{\Delta}) \to \mathrm{LinCat}$ lying over the cosimplicial $\mathbb{E}_2$-ring $A^\bullet$, and write $X^\bullet = (A^\bullet, \mathcal{C}^\bullet)$. Then $X^\bullet$ can be extended to a θ-limit diagram $\overline{X}^\bullet$ with $\overline{X}^{-1} = (A, \mathcal{C})$. We must show that if $X^\bullet$ carries every morphism in $\mathrm{N}(\mathbf{\Delta})$ to a θ-coCartesian morphism in $\mathcal{C}$, then $\overline{X}$ has the same property. This follows from the calculation

$$\begin{aligned} \mathrm{LMod}_{A^0}(\mathcal{C}) &\simeq \mathrm{LMod}_{A^0}(\varprojlim \mathcal{C}^\bullet) \\ &\simeq \varprojlim \mathrm{LMod}_{A^0}(\mathcal{C}^\bullet) \\ &\simeq \varprojlim \mathcal{C}^{\bullet+1} \\ &\simeq \mathcal{C}^0. \end{aligned}$$

□

For later use, we record one other useful consequences of the ideas developed in this section:

Proposition 5.15. *Let k be an $\mathbb{E}_2$-ring, $\mathcal{C}$ a k-linear ∞-category, A a quasi-commutative $\mathbb{E}_1$-algebra over k, and R the commutative ring $\pi_0 A$. For every map $f : X' \to X$ in Test_R, the induced map $f_* : \mathrm{QCoh}(X'; \mathcal{C}) \to \mathrm{QCoh}(X; \mathcal{C})$ preserves small colimits.*

Proof. For each quasi-compact open subset $U \subseteq X'$, let f^U_* be the composition of the restriction functor $\mathrm{QCoh}(X'; \mathcal{C}) \to \mathrm{QCoh}(U; \mathcal{C})$ with the pushforward functor $\mathrm{QCoh}(U; \mathcal{C}) \to \mathrm{QCoh}(X; \mathcal{C})$. We prove by induction on n that if $U \subseteq X'$ can be written as a union of n affine open subsets, then f^U_* preserves small colimits. If $n > 1$, then the desired result follows from inductive hypothesis (since Lemma 5.10 implies that for any pair of quasi-compact open subsets $U, V \subseteq X'$, we have an equivalence $f^{U \cup V}_* \to f^U_* \times_{f^{U \cap V}_*} f^V_*$). We may therefore assume that $n \leq 1$. Replacing X' by U, we may assume that $X' = U$ and that the map f is affine.

To prove that f_* preserves small colimits, it will suffice to show that for every map $g : Y \to X$ where Y is affine, the composite map $g^* f_* : \mathrm{QCoh}(X'; \mathcal{C}) \to \mathrm{QCoh}(X; \mathcal{C})$ preserves small colimits. Using Lemma 5.12 we can replace X by Y and X' by $X' \times_X Y$, and thereby reduce to the case where $X = \mathrm{Spec}(\pi_0 B)$ is affine. Since f is affine, we also know that $X' = \mathrm{Spec}(\pi_0 B')$ is affine. Then f_* can be identified with the forgetful functor $\mathrm{LMod}_{B'}(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{C})$, which preserves small colimits by Corollary A.4.2.3.5. □

6 Compactly Generated ∞-Categories

Recall that a presentable ∞-category $\mathcal{C}$ is said to be *compactly generated* if it is generated (under small colimits) by the full subcategory $\mathcal{C}_c \subseteq \mathcal{C}$ consisting of compact objects. Our goal in this section is to prove that, in the setting of linear ∞-categories, this condition can be tested locally for the étale topology. More precisely, we have the following result:

Theorem 6.1. *Let A be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a A-linear ∞-category. Suppose that there exists an étale covering $\{A \to A_\alpha\}$ of A such that each of the ∞-categories $\mathrm{LMod}_{A_\alpha}(\mathcal{C})$ is compactly generated. Then $\mathcal{C}$ is compactly generated.*

We will give the proof of Theorem 6.1 at the end of this section. To put this result in context, we need a few simple observations about the permanence of compact generation.

Proposition 6.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories. Assume that F has a right adjoint $G : \mathcal{D} \to \mathcal{C}$ which is conservative and preserves small filtered colimits. If $\mathcal{C}$ is compactly generated, then $\mathcal{D}$ is compactly generated.*

Proof. Since G preserves filtered colimits, the functor F preserves compact objects (Proposition T.5.5.7.2). Let $\mathcal{E} \subseteq \mathcal{D}$ be the smallest full subcategory which contains all compact objects of $\mathcal{D}$ and is closed under colimits. Since $\mathcal{C}$ is compactly generated, we conclude that $F(\mathcal{C}) \subseteq \mathcal{E}$. The ∞-category $\mathcal{E}$ is presentable, so Corollary T.5.5.2.9 guarantees that the inclusion $\mathcal{E} \subseteq \mathcal{D}$ has a right adjoint ϕ. To show that $\mathcal{E} = \mathcal{D}$, it will suffice to show that ϕ is conservative. Let $\alpha : M \to M'$ be a morphism in $\mathcal{D}$ such that $\phi(\alpha)$ is an equivalence. Then α induces a homotopy equivalence $\theta_E : \mathrm{Map}_{\mathcal{D}}(E, M) \simeq \mathrm{Map}_{\mathcal{D}}(E, M')$ whenever $E \in \mathcal{E}$. In particular, the map θ_E is a homotopy equivalence whenever E belongs to the essential image of F, so that $G(\alpha)$ is an equivalence. Since G is conservative, we conclude that α is an equivalence as desired. ☐

Corollary 6.3. *Let k be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a k-linear ∞-category, and let $f : A \to B$ be a map of $\mathbb{E}_1$-algebras over k. If $\mathrm{LMod}_A(\mathcal{C})$ is compactly generated, then $\mathrm{LMod}_B(\mathcal{C})$ is compactly generated.*

Proof. We have a commutative diagram of ∞-categories and forgetful functors

$$\begin{array}{ccc} \mathrm{LMod}_B(\mathcal{C}) & \xrightarrow{G} & \mathrm{LMod}_A(\mathcal{C}) \\ & \searrow \quad \swarrow & \\ & \mathcal{C}. & \end{array}$$

The vertical maps are conservative and preserve small colimits (see Corollary A.4.2.3.3). It follows that G is also conservative and preserves small colimits. The desired result now follows from the criterion of Proposition 6.2. ☐

Notation 6.4. Let LinCat be the ∞-category of pairs $(A, \mathcal{C})$, where A is an $\mathbb{E}_2$-ring and $\mathcal{C}$ is an A-linear ∞-category (see Notation 5.1). We let $\mathrm{LinCat}^{\mathrm{cg}}$ denote the full subcategory spanned by those pairs $(A, \mathcal{C})$ where the underlying ∞-category of $\mathcal{C}$ is compactly generated. We will show below that if $\mathcal{C}$ is a compactly generated A-linear ∞-category $\phi : A \to B$ is a map of $\mathbb{E}_2$-rings, then $\mathrm{LMod}_B(\mathcal{C})$ is a compactly generated B-linear ∞-category. It follows that the coCartesian fibration $\theta : \mathrm{LinCat} \to \mathrm{Alg}^{(2)}$ restricts to a coCartesian fibration $\theta^{\mathrm{cg}} : \mathrm{LinCat}^{\mathrm{cg}} \to \mathrm{Alg}^{(2)}$; moreover, a morphism in $\mathrm{LinCat}^{\mathrm{cg}}$ is θ_0-coCartesian if and only if it is θ-coCartesian.

The following result is essentially a restatement of Theorem 6.1.

Proposition 6.5. *Let $\chi^{\mathrm{cg}} : \mathrm{Alg}^{(2)} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classify the coCartesian fibration $\theta^{\mathrm{cg}} : \mathrm{LinCat}^{\mathrm{cg}} \to \mathrm{Alg}^{(2)}$ of Notation 6.4. Then χ^{cg} is a sheaf with respect to the étale topology.*

More informally: the construction $k \mapsto \mathrm{LinCat}_k^{\mathrm{cg}}$ satisfies étale descent.

Proof. Combine Theorems 6.1 and 5.13. □

The remainder of this section is devoted to the proof of Theorem 6.1. We begin by reviewing some generalities concerning the tensor product of presentable ∞-categories. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the subcategory whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits. In §A.6.3.1, we described a symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$: if $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ is universal among presentable ∞-categories which are equipped with a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which preserves colimits separately in each variable. This tensor product can be described more explicitly: as an ∞-category, $\mathcal{C} \otimes \mathcal{D}$ is equivalent to the full subcategory $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{op}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{D}^{op}, \mathcal{C})$ spanned by those functors which preserve small limits (see Proposition A.6.3.1.16). Suppose now that $\mathcal{D}$ is a compactly generated ∞-category, and let $\mathcal{D}_c \subseteq \mathcal{D}$ be the full subcategory spanned by the compact objects. Using Proposition T.5.3.5.10, we deduce that the restriction functor $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}_c^{op}, \mathcal{C})$ is fully faithful; moreover, its essential image is the full subcategory $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}_c^{op}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{D}_c^{op}, \mathcal{C})$ spanned by the left exact functors (Proposition T.5.5.1.9).

Remark 6.6. Let $\mathcal{D}$ be a compactly generated presentable ∞-category and let $F : \mathcal{C} \to \mathcal{C}'$ be a colimit-preserving functor between presentable ∞-categories. Then F induces a functor

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}_c^{op}, \mathcal{C}) \simeq \mathcal{C} \otimes \mathcal{D} \to \mathcal{C}' \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}_c^{op}, \mathcal{C}').$$

If F is left exact, then this functor is simply given by composition with F (see §V.1.1 for a discussion in the special case where F is a geometric morphism of ∞-topoi).

Remark 6.7. Let $\mathcal{C}$ be a monoidal ∞-category and $\mathcal{M}$ an ∞-category which is right-tensored over $\mathcal{C}$. Assume that $\mathcal{C}$ and $\mathcal{M}$ are presentable, and that the tensor product operations

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{M} \times \mathcal{C} \to \mathcal{M}$$

preserve colimits separately in each variable. Then the second multiplication induces a colimit-preserving functor $F : \mathcal{M} \otimes \mathcal{C} \to \mathcal{M}$, which admits a right adjoint $G : \mathcal{M} \to \mathcal{M} \otimes \mathcal{C} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}, \mathcal{M})$ (by Corollary T.5.5.2.9). Unwinding the definitions, we see that G is given by the formula $G(M)(C) = M^C$, where $M^C \in \mathcal{M}$ is characterized by the universal property $\mathrm{Map}_{\mathcal{M}}(N, M^C) \simeq \mathrm{Map}_{\mathcal{M}}(N \otimes C, M)$.

Proposition 6.8. *Let $\mathcal{C}$ be a monoidal ∞-category, and let $U : \mathcal{M} \to \mathcal{N}$ be a functor between ∞-categories right tensored over $\mathcal{C}$. Assume that:*

(1) *The ∞-categories $\mathcal{C}$, $\mathcal{M}$, and $\mathcal{N}$ are presentable.*

(2) *The tensor product functors*

$$\mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \mathcal{M} \times \mathcal{C} \to \mathcal{M} \qquad \mathcal{N} \times \mathcal{C} \to \mathcal{N}$$

preserve small colimits separately in each variable.

(3) *The functor U preserves small colimits and finite limits.*

(4) *The ∞-category $\mathcal{C}$ is compactly generated, and every compact object $C \in \mathcal{C}$ admits a right dual (see §A.4.2.5).*

Then the diagram σ :

$$\begin{array}{ccc} \mathcal{M} \otimes \mathcal{C} & \xrightarrow{F} & \mathcal{M} \\ \downarrow{\scriptstyle \mathrm{id} \otimes U} & & \downarrow{\scriptstyle U} \\ \mathcal{N} \otimes \mathcal{C} & \xrightarrow{F'} & \mathcal{N} \end{array}$$

is right adjointable.

Proof. Let $\mathcal{C}_c$ denote the full subcategory of $\mathcal{C}$ spanned by the compact objects. Using Remark 6.6, we can identify $\mathcal{M} \otimes \mathcal{C}$ with $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_c^{op}, \mathcal{M})$ and $\mathcal{N} \otimes \mathcal{C}$ with $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_c^{op}, \mathcal{N})$. Condition (3) guarantees that, under this identification, the functor $U \otimes \mathrm{id}_{\mathcal{C}}$ is given by composition with U. Using Remark 6.7, we see that F admits a right adjoint G, given by the formula $G(M)(C) = M^C$ for $M \in \mathcal{M}$ and $C \in \mathcal{C}_c$. Similarly, F' admits a right adjoint G', given by the formula $G'(N)(C) = N^C$. Unwinding the definitions, we see that the right adjointability of σ is equivalent to the assertion that for each $M \in \mathcal{M}$ and each $C \in \mathcal{C}_c$, the canonical map $U(M^C) \to U(M)^C$ is an equivalence. Since C is right dualizable, both sides can be identified with $U(M \otimes C^\vee) \simeq U(M) \otimes C^\vee$. □

Proposition 6.9. *Let $U : \mathcal{M} \to \mathcal{N}$ and $V : \mathcal{C} \to \mathcal{D}$ be colimit-preserving functors between presentable ∞-categories. Assume that $\mathcal{C}$ and $\mathcal{D}$ are compactly generated, that V carries compact objects of $\mathcal{C}$ to compact objects of $\mathcal{D}$, and that U is left exact. Then the diagram σ :*

$$\begin{array}{ccc} \mathcal{M} \otimes \mathcal{C} & \xrightarrow{F} & \mathcal{M} \otimes \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{N} \otimes \mathcal{C} & \xrightarrow{F'} & \mathcal{N} \otimes \mathcal{D} \end{array}$$

is right adjointable.

Proof. Let $\mathcal{C}_c \subseteq \mathcal{C}$ and $\mathcal{D}_c \subseteq \mathcal{D}$ denote the full subcategories spanned by the compact objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. We can identify σ with a diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_c^{op}, \mathcal{M}) & \xrightarrow{F} & \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}_c^{op}, \mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_c^{op}, \mathcal{N}) & \xrightarrow{F'} & \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}_c^{op}, \mathcal{N}). \end{array}$$

Under this identification, the right adjoints to F and F' are given by composition with the functor V, and the vertical maps are given by composition with U. It is clear that the resulting diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_c^{op}, \mathcal{M}) & \xleftarrow{\circ V} & \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}_c^{op}, \mathcal{M}) \\ \downarrow{\scriptstyle U\circ} & & \downarrow{\scriptstyle U\circ} \\ \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_c^{op}, \mathcal{N}) & \xleftarrow{\circ V} & \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}_c^{op}, \mathcal{N}). \end{array}$$

commutes up to equivalence. □

Theorem 6.10. *Let $\mathcal{C}$ be a monoidal ∞-category and $\mathcal{N}$ an ∞-category which is left-tensored over $\mathcal{C}$. Assume that:*

(*a*) *The ∞-categories $\mathcal{N}$ and $\mathcal{C}$ are compactly generated and presentable.*

(*b*) *The tensor product functors*

$$\mathcal{C} \times \mathcal{N} \to \mathcal{N} \qquad \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

preserve small colimits separately in each variable and carry compact objects to compact objects.

(*c*) *The ∞-category $\mathcal{C}$ is stable.*

(*d*) *Every compact object $C \in \mathcal{C}$ admits a right dual.*

Then the construction $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{C}} \mathcal{N}$ determines a functor $G : \mathrm{RMod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which preserves small limits.

Proof. Let K be a small simplicial set and $q : K \to \mathrm{RMod}_{\mathcal{C}}(\mathcal{P}r^{\mathrm{L}})$ a diagram. We wish to prove that the canonical map $u : G(\varprojlim q) \to \varprojlim(G \circ q)$ is an equivalence. Consider the functor $p : \mathrm{N}(\mathbf{\Delta})^{op} \times K \to \mathcal{P}r^{\mathrm{L}}$ given by the formula $p([n], v) = \mathrm{Bar}_{\mathcal{C}}(q(v), \mathcal{N})_n \simeq q(v) \otimes \mathcal{C}^{\otimes n} \otimes \mathcal{N}$, so that u can be identified with the natural map

$$\varinjlim_{[n] \in \mathbf{\Delta}^{op}} \varprojlim_{v \in K} p([n], v) \to \varprojlim_{v \in K} \varinjlim_{[n] \in \mathbf{\Delta}^{op}} p([n], v)$$

in $\mathcal{P}r^{\mathrm{L}}$. In view of Proposition A.6.2.3.19, it will suffice to verify the following:

($*$) Let $e : v \to v'$ be an edge in K, so that $q(e)$ can be identified with a colimit-preserving functor $\mathcal{M} \to \mathcal{M}'$ between presentable ∞-categories right-tensored over $\mathcal{C}$. Then for every morphism $\alpha : \langle m \rangle \to \langle n \rangle$, the diagram σ :

$$\begin{array}{ccc} \mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_n & \longrightarrow & \mathrm{Bar}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})_m \\ \downarrow & & \downarrow \\ \mathrm{Bar}_{\mathcal{C}}(\mathcal{M}', \mathcal{N})_n & \longrightarrow & \mathrm{Bar}_{\mathcal{C}}(\mathcal{M}', \mathcal{N})_m \end{array}$$

is right adjointable.

The collection of all morphisms $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ which satisfy ($*$) is clearly stable under composition. It will therefore suffice to prove ($*$) in two special cases:

(1) Assume that $\alpha(0) = 0$. Let $\mathcal{X} = \mathcal{C}^{\otimes n} \otimes \mathcal{N}$ and $\mathcal{Y} = \mathcal{C}^{\otimes m} \otimes \mathcal{N}$, so that α induces a colimit-preserving functor $\mathcal{X} \to \mathcal{Y}$; we can identify the diagram σ of ($*$) with the square

$$\begin{array}{ccc} \mathcal{M} \otimes \mathcal{X} & \longrightarrow & \mathcal{M} \otimes \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{M}' \otimes \mathcal{X} & \longrightarrow & \mathcal{N}' \otimes \mathcal{Y}. \end{array}$$

Using (a) and (b), we deduce that that $\mathcal{X}$ and $\mathcal{Y}$ are compactly generated and that the functor $\mathcal{X} \to \mathcal{Y}$ preserves compact objects. According to Proposition 6.9, to show that σ is right adjointable it suffices to prove that the functor $f : \mathcal{M} \to \mathcal{M}'$ is left exact. This is clear, since f is right exact and the ∞-categories $\mathcal{M}$ and $\mathcal{M}'$ are stable (since they are modules over the stable ∞-category $\mathcal{C}$).

(2) Assume that α is the canonical inclusion $[m] \hookrightarrow [0] \star [m] \simeq [m+1]$. Let $\mathcal{X} = \mathcal{M} \otimes \mathcal{C}^{\otimes m} \mathcal{N}$ and $\mathcal{X}' = \mathcal{M}' \otimes \mathcal{C}^{\otimes m} \mathcal{N}$; we regard $\mathcal{X}$ and $\mathcal{X}'$ as right-tensored over $\mathcal{C}$ via the right action of $\mathcal{C}$ on $\mathcal{M}$ and $\mathcal{M}'$. Then we can identify σ with the diagram

$$\begin{array}{ccc} \mathcal{X} \otimes \mathcal{C} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{X}' \otimes \mathcal{C} & \longrightarrow & \mathcal{X}'. \end{array}$$

Using assumption (d) and Proposition 6.8, we see that the diagram σ is right adjointable provided that the evident functor $f : \mathcal{X} \to \mathcal{X}'$ is left exact. This again follows from the right-exactness of f, since the ∞-categories $\mathcal{X}$ and $\mathcal{X}'$ are stable (since they are right modules for the stable ∞-category $\mathcal{C}$).

□

Corollary 6.11. *Let A be an E_∞-ring and let $\mathcal{C}$ be a compactly generated A-linear ∞-category. Then $\mathcal{C}$ satisfies flat hyperdescent (see Definition VII.6.8).*

Proof. Using Lemma 5.7 and Proposition VII.5.12, we are reduced to proving the following:

$(*)$ Let $B^\bullet : \mathrm{N}(\mathbf{\Delta}_s) \to \mathrm{CAlg}_{A/}$ be a flat hypercovering of an A-algebra B. Then the canonical map

$$\mathrm{Mod}_B \otimes_{\mathrm{Mod}_A} \mathcal{C} \to \varprojlim (\mathrm{Mod}_{B^\bullet} \otimes_{\mathrm{Mod}_A} \mathcal{C})$$

is an equivalence of ∞-categories.

Using Theorem 6.10, we are reduced to proving that $\mathrm{Mod}_B \simeq \varprojlim \mathrm{Mod}_{B^\bullet}$, which follows from Corollary VII.6.13. □

Theorem 6.1 can be reformulated as follows:

Proposition 6.12. *Let k be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be a k-linear ∞-category. Define a functor $\chi : \mathrm{Alg}^{(2)}_{k/} \to \mathcal{S}$ by the formula*

$$\chi(A) = \begin{cases} \{*\} & \text{if } \mathrm{LMod}_A(\mathcal{C}) \text{ is compactly generated} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then χ is a sheaf with respect to the étale topology (see Definition 5.3).

Remark 6.13. The well-definedness of the functor χ appearing in the statement of Proposition 6.12 is well-defined.

To prove Proposition 6.12, it will suffice to show that the functor $\chi : \mathrm{Alg}^{(2)}_{k/} \to \mathcal{S}$ is a sheaf with respect to both the finite étale topology and the Nisnevich topology (see Theorem 3.7). In the former case, this amounts to the following observation:

Lemma 6.14. *Let A be an $\mathbb{E}_2$-ring, $\mathcal{C}$ an A-linear ∞-category, and suppose we are given a faithfully flat, finite étale map $A \to \prod_{i \in I} A_i$ for some finite set I. If each of the ∞-categories $\mathrm{LMod}_{A_i}(\mathcal{C})$ is compactly generated, then $\mathrm{LMod}_A(\mathcal{C})$ is compactly generated.*

Proof. Let $B = \prod_i A_i$. According to Lemma 5.7, $\mathrm{LMod}_B(\mathcal{C})$ is equivalent to $\prod_i \mathrm{LMod}_{A_i}(\mathcal{C})$ and therefore compactly generated. Let $F : \mathrm{LMod}_B(\mathcal{C}) \to \mathrm{LMod}_A(\mathcal{C})$ be the forgetful functor. The functor F fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{LMod}_B(\mathcal{C}) & \xrightarrow{F} & \mathrm{LMod}_A(\mathcal{C}) \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

where the vertical maps are conservative and preserve small colimits (see Corollary A.4.2.3.3). It follows that F preserves small colimits, and therefore admits a right adjoint G (Corollary T.5.5.2.9). According to Proposition 6.2, it will suffice to show that G is conservative and preserves small colimits. Equivalently, we must show that the composite functor $\mathrm{LMod}_A(\mathcal{C}) \xrightarrow{G} \mathrm{LMod}_B(\mathcal{C}) \to \mathcal{C}$ is conservative and preserves small colimits. Since B is perfect as a left A-module, this functor is given by $M \mapsto B^\vee \otimes_A M$. It is clear that this functor preserves small colimits, and it is conservative since B (and therefore $B^\vee$) is faithfully flat as a A-module. □

To complete the proof of Proposition 6.12, we must show that for every map of $\mathbb{E}_2$-rings $k \to A$, the restriction $\chi|(\mathrm{Alg}^{(2)})^{\text{ét}}_{A/}$ is a sheaf with respect to the Nisnevich topology. Replacing $\mathcal{C}$ by $\mathrm{LMod}_A(\mathcal{C})$, we can reduce to the case $k = A$. For the remainder of this section, we will will fix the $\mathbb{E}_2$-ring A and the A-linear ∞-category $\mathcal{C}$. Set $R = \pi_0 A$, and let Test_R be as in Notation 2.1. For $X \in \mathrm{Test}_R$, we define an ∞-category $\mathrm{QCoh}(X; \mathcal{C})$ as in the proof of Proposition 5.6. Let $\chi' : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ be a right Kan extension of $\chi|(\mathrm{Alg}^{(2)})^{\text{ét}}_{A/}$. More concretely, if $X \in \mathrm{Test}_R$, then $\chi'(X)$ is contractible if $\mathrm{QCoh}(Y; \mathcal{C})$ is compactly generated for every map $Y \to X$ in Test_R such that Y is affine, and empty otherwise. According to Theorem 2.9, it will suffice to show that χ' satisfies affine Nisnevich excision.

Lemma 6.15. *Let $X \in \mathrm{Test}_R$ and let $M \in \mathrm{QCoh}(X; \mathcal{C})$. The following conditions are equivalent:*

(1) *The object* $M \in \mathrm{QCoh}(X; \mathcal{C})$ *is compact.*

(2) *For every map* $f : Y \subseteq X$ *in* Test_R *where* Y *is affine, the pullback* $f^* M \in \mathrm{QCoh}(Y; \mathcal{C})$ *is compact.*

(3) *There exists an open covering* $\{U_i\}_{1 \leq i \leq n}$ *of* X *such that the image of* M *in* $\mathrm{QCoh}(U_i; \mathcal{C})$ *is compact for* $1 \leq i \leq n$.

Proof. We first show that (1) ⇒ (2). Assume that M is compact. Let $f : Y \subseteq X$ be a map in Test_R where Y is affine. We wish to prove that $f^* M$ is compact. For this, it suffices to observe that the functor f_* preserves filtered colimits (Proposition 5.15)

The implication (2) ⇒ (3) is obvious. We prove that (3) ⇒ (1). Let $\{U_i\}_{1 \leq i \leq n}$ be as in (3). For every nonempty subset $I \subseteq \{1, \ldots, n\}$ let $U_I = \bigcap_{i \in I} U_i$ and $j_I : U_I \to X$ be the inclusion, so that each $j_I^* M \in \mathrm{QCoh}(U_I; \mathcal{C})$ is compact. Let $N \in \mathrm{QCoh}(X; \mathcal{C})$ be a filtered colimit of objects N_α, and consider the diagram

$$\begin{array}{ccc} \varinjlim \mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(M, N_\alpha) & \longrightarrow & \mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(M, N) \\ \downarrow & & \downarrow \\ \varinjlim \varprojlim_{I \neq \emptyset} \mathrm{Map}_{\mathrm{QCoh}(U_I;\mathcal{C})}(j_I^* M, j_I^* N_\alpha) & \longrightarrow & \varprojlim_{I \neq \emptyset} \mathrm{Map}_{\mathrm{QCoh}(U_I;\mathcal{C})}(j_I^* M; j_I^* N). \end{array}$$

The vertical maps are equivalences by Lemma 5.10, and the bottom horizontal map is an equivalence since each $j_I^* M$ is compact (note that filtered colimits commute with finite limits in $\mathcal{S}$). It follows that the upper horizontal map is an equivalence, which shows that M is a compact object of $\mathrm{QCoh}(X; \mathcal{C})$. □

Notation 6.16. If $X \in \mathrm{Test}_R$ and $j : U \hookrightarrow X$ is a quasi-compact open immersion, we let $\mathrm{QCoh}(X, X - U; \mathcal{C})$ denote the full subcategory of $\mathrm{QCoh}(X; \mathcal{C})$ spanned by those objects M such that $j^* M \simeq 0$.

Lemma 6.17. *Let* $X \in \mathrm{Test}_R$ *and let* $j : U \hookrightarrow X$ *be a quasi-compact open immersion. Then* $\mathrm{QCoh}(X; \mathcal{C})$ *is compactly generated if the following conditions are satisfied:*

(1) *The* ∞*-category* $\mathrm{QCoh}(U; \mathcal{C})$ *is generated (under small colimits) by objects of the form* $j^* M$*, where* M *is a compact object of* $\mathrm{QCoh}(X; \mathcal{C})$.

(2) *The* ∞*-category* $\mathrm{QCoh}(X, X - U; \mathcal{C})$ *is compactly generated.*

If X *is affine, then the converse holds.*

Proof. Suppose first that (1) and (2) are satisfied. Let $\mathcal{E} \subseteq \mathrm{QCoh}(X; \mathcal{C})$ be the smallest full subcategory which contains all compact objects of $\mathrm{QCoh}(X; \mathcal{C})$ and is stable under small colimits; we wish to prove that $\mathcal{E} = \mathrm{QCoh}(X; \mathcal{C})$. By the adjoint functor theorem, the inclusion $\mathcal{E} \subseteq \mathrm{QCoh}(X; \mathcal{C})$ admits a right adjoint G; it will suffice to show that G is conservative. Since G is an exact functor between stable ∞-categories, this is equivalent to the assertion that if $M \in \mathrm{QCoh}(X; \mathcal{C})$ is such that $G(M) \simeq 0$, then $M \simeq 0$.

Let T be a right adjoint to the inclusion $\mathrm{QCoh}(X, X - U; \mathcal{C}) \hookrightarrow \mathrm{QCoh}(X; \mathcal{C})$. For every object $M \in \mathrm{QCoh}(X; \mathcal{C})$, we have a canonical fiber sequence

$$T(M) \to M \to j_* j^* M.$$

Since j_* preserves small colimits (Proposition 5.15), we conclude that T preserves small colimits, so that every compact object of $\mathrm{QCoh}(X, X - U; \mathcal{C})$ is also compact as an object of $\mathrm{QCoh}(X; \mathcal{C})$. Combining this observation with (2), we deduce that $\mathrm{QCoh}(X, X - U; \mathcal{C}) \subseteq \mathcal{E}$. If $G(M) \simeq 0$, then $T(M) \simeq 0$ so the unit map $M \to j_* j^* M$ is an equivalence. If $M \neq 0$, then $j^* M \neq 0$, so that assumption (1) guarantees a nonzero map $u : j^* M' \to j^* M$ where $M' \in \mathrm{QCoh}(X; \mathcal{C})$ is compact. Then u is adjoint to a nonzero map $M' \to j_* j^* M \simeq M$, which contradicts our assumption that $G(M) \simeq 0$.

Now suppose that X is affine and $\mathrm{QCoh}(X; \mathcal{C})$ is compactly generated. We first prove (1). Let $\mathcal{E}' \subseteq \mathrm{QCoh}(U; \mathcal{C})$ be the smallest full subcategory of $\mathrm{QCoh}(U; \mathcal{C})$ which is closed under small colimits and contains

j^*M for all compact objects $M \in \mathrm{QCoh}(X;\mathcal{C})$. We wish to prove that $\mathcal{E}' = \mathrm{QCoh}(U;\mathcal{C})$. As above, we note that the inclusion $\mathcal{E}' \hookrightarrow \mathrm{QCoh}(U;\mathcal{C})$ admits a right adjoint G'; it will suffice to show that G' is conservative. Assume otherwise; then there exists a nonzero object $N \in \mathrm{QCoh}(U;\mathcal{C})$ such that $\mathrm{Map}_{\mathrm{QCoh}(U;\mathcal{C})}(j^*M, N) \simeq \mathrm{Map}_{\mathrm{QCoh}(X;\mathcal{C})}(M, j_*N) \simeq 0$ for all compact objects $M \in \mathrm{QCoh}(X;\mathcal{C})$. Since $\mathrm{QCoh}(X;\mathcal{C})$ is compactly generated, we conclude that $j_*N \simeq 0$. Since Lemma 5.12 guarantees that j_* is fully faithful, we conclude that $N \simeq 0$ and obtain a contradiction.

We now prove (2). Let $\mathcal{E}''$ be the smallest full subcategory of $\mathrm{QCoh}(X, X-U;\mathcal{C})$ which is closed under filtered colimits and contains every compact object of $\mathrm{QCoh}(X, X-U;\mathcal{C})$. We wish to prove that $\mathcal{E}'' = \mathrm{QCoh}(X, X-U;\mathcal{C})$. As above, we note that the inclusion $\mathcal{E}'' \hookrightarrow \mathrm{QCoh}(X, X-U;\mathcal{C})$ admits a right adjoint G''. We are reduced to proving that if $N \in \mathrm{QCoh}(X, X-U;\mathcal{C})$ is such that $G''(N) \simeq 0$, then $N \simeq 0$. Assume otherwise. Then, since $\mathrm{QCoh}(X;\mathcal{C})$ is compactly generated, there exists a compact object $M \in \mathrm{QCoh}(X;\mathcal{C})$ and a nonzero map $\alpha : M \to N$. We wish to show that M can be chosen to lie in $\mathrm{QCoh}(X, X-U;\mathcal{C})$.

Since X is affine, it has the form $\mathrm{Spec}(\pi_0 B)$ for some étale map $A \to B$ of $\mathbb{E}_2$-rings. Because U is quasi-compact, the closed subset $X - U \subseteq X$ is defined by a finitely generated ideal $(x_1, \ldots, x_n) \subseteq \pi_0 B$. Let U_1 be the complement of the closed subset determined by x_1, and $j_1 : U_1 \to X$ the corresponding inclusion. The composite map

$$M \xrightarrow{\alpha} N \to (j_1)_* j_1^* N$$

is zero, since $j_1^* N \simeq 0$. We note that $(j_1)_* j_1^* N \simeq B[\frac{1}{x_1}] \otimes_B N$ and that $B[\frac{1}{x_1}]$ can be identified (in the monoidal ∞-category of left B-modules) with the colimit of the sequence

$$B \xrightarrow{x_1} B \xrightarrow{x_1} \cdots ,$$

where each map is given by left multiplication by x_1. Since M is compact, we conclude that the composite map

$$M \xrightarrow{\alpha} N \xrightarrow{x_1^{k_1}} N$$

is nullhomotopic for k sufficiently large. In other words, α factors as a composition

$$M \to B'/B'x_1^{k_1} \otimes_{B'} M \xrightarrow{\alpha_1} N,$$

where $B'/B'x_1^{k_1}$ denotes the left B'-module obtained as the cofiber of the map from B' to itself given by right multiplication by $x_1^{k_1}$.

Let $M_1 = B'/B'x_1^{k_1} \otimes_{B'} M$. Then M_1 is the cofiber of a certain map from M to itself, and therefore compact. Repeating the above argument, we deduce that α_1 factors as a composition

$$M_1 \to B/Bx_2^{k_2} \otimes_{B'} M_1 \xrightarrow{\alpha_2} N.$$

Continuing in this way, we conclude that α factors as a composition

$$M \to Q \otimes_{B'} M \xrightarrow{\alpha_n} N,$$

where $Q = \bigotimes_{0 \leq i < n} B/Bx_{n-i}^{k_{n-i}}$ (here the tensor product is formed in the monoidal ∞-category LMod_B). Note that $j^*(Q \otimes_B M) \simeq 0$, since $j_i^*(B/Bx_i^{k_i})$ vanishes for $1 \leq i \leq n$. It follows that $Q \otimes_B M$ is a compact object of $\mathrm{QCoh}(X, X-U;\mathcal{C})$ equipped with a nonzero map $Q \otimes_B M \to N$, contradicting our assumption that $G''(N) \simeq 0$. □

Lemma 6.18. *Let $X \in \mathrm{Test}_R$ be affine, let $j : U \to X$ be a quasi-compact open immersion, and let $\mathcal{X} \subseteq \mathrm{QCoh}(U;\mathcal{C})$ be the full subcategory spanned by objects of the form j^*M, where $M \in \mathrm{QCoh}(X;\mathcal{C})$ is compact. If $\mathrm{QCoh}(X;\mathcal{C})$ is compactly generated, then $\mathcal{X}$ is a stable subcategory of $\mathrm{QCoh}(U;\mathcal{C})$.*

Proof. It is obvious that $\mathcal{X}$ is stable under suspension and desuspension. To complete the proof, it will suffice to show that if f is a morphism in $\mathcal{X}$, then the cofiber of f (formed in the stable ∞-category $\mathrm{QCoh}(U;\mathcal{C})$) also belongs to $\mathcal{X}$. Without loss of generality, we can assume that f is a morphism from j^*M to j^*M', where M and M' are compact objects of $\mathrm{QCoh}(X;\mathcal{C})$. Then f is adjoint to a morphism $\lambda : M \to j_*j^*M'$. Let K be the fiber of the unit map $M' \to j_*j^*M'$, so that $K \in \mathrm{QCoh}(X, X-U;\mathcal{C})$. Lemma 6.17 implies that we can write K as a filtered colimit $\varinjlim K_\alpha$, where each K_α is a compact object of $\mathrm{QCoh}(X, X-U;\mathcal{C})$ (and hence also a compact object of $\mathrm{QCoh}(X;\mathcal{C})$). For each index α, let M'_α denote the cofiber of the induced map $K_\alpha \to M'$, so that $\varinjlim M'_\alpha \simeq j_*j^*M'$. Since M is compact, the map λ factors as a composition

$$M \xrightarrow{\lambda'} M'_\alpha \to \varinjlim M'_\alpha \simeq j_*j^*M'.$$

Then $j^*(\lambda') \simeq f$, so that $\mathrm{cofib}(f) \simeq j^* \mathrm{cofib}(\lambda') \in \mathcal{X}$. □

Our next lemma (and its proof) are essentially taken from [54].

Lemma 6.19. *Let $X \in \mathrm{Test}_R$ be affine, let $j : U \to X$ be a quasi-compact open immersion, and assume that $\mathrm{QCoh}(X;\mathcal{C})$ is compactly generated. For every compact object $M \in \mathrm{QCoh}(U;\mathcal{C})$, the direct sum $M \oplus M[1]$ belongs to the full subcategory $\mathcal{X}$ described in Lemma 6.18.*

Proof. Lemma 6.18 implies that $\mathcal{X}$ admits finite limits, so that the inclusion $\mathcal{X} \hookrightarrow \mathrm{QCoh}(U;\mathcal{C})$ induces a fully faithful embedding $\alpha : \mathrm{Ind}(\mathcal{X}) \to \mathrm{QCoh}(U;\mathcal{C})$ which commutes with small colimits. Lemma 6.17 implies that α is an equivalence of ∞-categories. Consequently, every compact object of $\mathrm{QCoh}(U;\mathcal{C})$ is a retract of an object belonging to $\mathcal{X}$. In particular, we have a direct sum decomposition $M \oplus M' \simeq j^*N$, for some compact object $N \in \mathrm{QCoh}(X;\mathcal{C})$. Let $f : M \oplus M' \to M \oplus M'$ be the direct sum of the zero map $M \to 0 \to M$ with the identity map $\mathrm{id}_{M'} : M' \to M'$. The proof of Lemma 6.18 shows that we can write $f = j^*(\overline{f})$, for some map $\overline{f} : N \to N'$ between compact objects of $\mathrm{QCoh}(X;\mathcal{C})$. Then $M \oplus M[1] \simeq \mathrm{cofib}(f) \simeq j^* \mathrm{cofib}(\overline{f}) \in \mathcal{X}$. □

Lemma 6.20. *Let $f : X' \to X$ be a morphism in Test_R where X' is affine. Let $U \subseteq X$ be a quasi-compact open subset such that f induces an isomorphism $X' - U' \to X - U$, where $U' = U \times_X X'$. If $\mathrm{QCoh}(X';\mathcal{C})$ and $\mathrm{QCoh}(U;\mathcal{C})$ are compactly generated, then $\mathrm{QCoh}(X;\mathcal{C})$ is compactly generated.*

Proof. We will show that the open immersion $j : U \to X$ satisfies the hypotheses of Lemma 6.17:

(1) We claim that the ∞-category $\mathrm{QCoh}(U;\mathcal{C})$ is generated (under small colimits) by objects of the form j^*M, where M is a compact object of $\mathrm{QCoh}(X;\mathcal{C})$. Since $\mathrm{QCoh}(U;\mathcal{C})$ is compactly generated by assumption, it will suffice to show that for each compact object $N \in \mathrm{QCoh}(U;\mathcal{C})$, the direct sum $N \oplus N[1]$ can be lifted to a compact object of $\mathrm{QCoh}(X;\mathcal{C})$. Let $f' : U' \to U$ be the restriction of f; Lemma 6.15 implies that f'^* carries compact objects of $\mathrm{QCoh}(U;\mathcal{C})$ to compact objects of $\mathrm{QCoh}(U';\mathcal{C})$. Proposition 5.6 guarantees that the forgetful functor

$$\mathrm{QCoh}(X;\mathcal{C}) \to \mathrm{QCoh}(U;\mathcal{C}) \times_{\mathrm{QCoh}(U',\mathcal{C})} \mathrm{QCoh}(X';\mathcal{C}).$$

In particular, an object $M \in \mathrm{QCoh}(X;\mathcal{C})$ is compact if and only if j^*M and f^*M are compact (from which it follows that the image of M in $\mathrm{QCoh}(U';\mathcal{C})$ is compact, by Lemma 6.15). Consequently, to lift $N \oplus N[1]$ to a compact object of $\mathrm{QCoh}(X)$, it will suffice to lift $f'^*N \oplus f'^*N[1]$ to a compact object of $\mathrm{QCoh}(X')$. Since $\mathrm{QCoh}(X';\mathcal{C})$ is compactly generated, the existence of such a lifting is guaranteed by Lemma 6.19.

(2) We claim that $\mathrm{QCoh}(X, X-U;\mathcal{C})$ is compactly generated. Using Proposition 5.6, we conclude that the pullback functor f^* induces an equivalence of ∞-categories $\mathrm{QCoh}(X, X-U;\mathcal{C}) \to \mathrm{QCoh}(X', X'-U';\mathcal{C})$. It will therefore suffice to show that $\mathrm{QCoh}(X', X'-U';\mathcal{C})$ is compactly generated. Since $\mathrm{QCoh}(X';\mathcal{C})$ is compactly generated by assumption, this follows from Lemma 6.17).

□

Proof of Proposition 6.12. In view of Theorem 3.7, Lemma 6.14, and Theorem 2.9, it will suffice to show that the functor $\chi' : \mathrm{Test}_R \to \mathcal{S}$ satisfies affine Nisnevich excision. In other words, we must show that if $f : X' \to X$ is a morphism in Test_R such that X' is affine, $U \subseteq X$ is a quasi-compact open subset with $X' - U' \simeq X - U$ where $U' = U \times_X X'$, and both $\chi'(X')$ and $\chi'(U)$ are nonempty, then $\chi'(X)$ is nonempty. Unwinding the definitions, we must show that if $Y \to X$ is a map in Test_R where Y is affine, then $\mathrm{QCoh}(Y; \mathcal{C})$ is compactly generated. Replacing X by Y, X' by $X' \times_X Y$, and U by $U \times_X Y$, we may assume that X is affine and we wish to show that $\mathrm{QCoh}(X; \mathcal{C})$ is compactly generated. Using Lemma 6.20, we may reduce to showing that $\mathrm{QCoh}(U; \mathcal{C})$ is compactly generated. Since U is quasi-compact, we can write U as the union of finitely many affine open subsets $\{U_i \subseteq U\}_{1 \leq i \leq n}$. For $0 \leq j \leq n$, let $U(j) = \bigcup_{i \leq j} U_i$. We prove by induction on j that $\mathrm{QCoh}(U(j); \mathcal{C})$ is compactly generated. When $j = 0$, $U(j)$ is empty and $\mathrm{QCoh}(U(j); \mathcal{C})$ is a contractible Kan complex, so there is nothing to prove. If $j > 0$, we apply Lemma 6.20 to the morphism $U_j \to U(j)$ and the open subset $U(j-1) \subseteq U(j)$. □

We conclude this section by proving an analogue of Proposition 6.12, which asserts that the property of compactness itself enjoys étale descent.

Proposition 6.21. *Let A be an $\mathbb{E}_2$-ring, let $\mathcal{C}$ be a A-linear ∞-category, and let $M \in \mathcal{C}$ be an object. Define $\chi : \mathrm{Alg}^{(2)}_{A/} \to \mathcal{S}$ by the formula*

$$\chi(B) = \begin{cases} \{*\} & \text{if } B \otimes_A M \in \mathrm{LMod}_B(\mathcal{C}) \text{ is compact} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then χ is a sheaf with respect to the étale topology.

Proof. According to Theorem 3.7, it will suffice to show that χ is a sheaf with respect to the finite étale topology and with respect to the Nisnevich topology. We first consider the finite étale topology. Suppose we are given a faithfully flat étale map $u : B \to B'$ in $\mathrm{Alg}^{(2)}_{A/}$ and that $B' \otimes_A M$ is a compact object of $\mathrm{LMod}_{B'}(\mathcal{C})$. The proof of Lemma 6.14 shows that the forgetful functor $\mathrm{LMod}_{B'}(\mathcal{C}) \to \mathrm{LMod}_B(\mathcal{C})$ preserves compact objects, so that $B' \otimes_A M$ is a compact object of $\mathrm{LMod}_B(\mathcal{C})$. Consider the cofiber sequence of B-modules

$$B \to B' \to K.$$

Since u is finite étale , K is a projective B-module of finite rank, hence the above sequence splits. It follows that $B \otimes_A M$ is a direct summand of $B' \otimes_A M$ in the ∞-category $\mathrm{LMod}_B(\mathcal{C})$ and therefore compact.

We now show that the functor χ satisfies Nisnevich descent. We must show that for every map of $\mathbb{E}_2$-rings $f : A \to B$, the restriction $\chi | (\mathrm{Alg}^{(2)})^{\text{ét}}_{B/}$ is a sheaf with respect to the Nisnevich topology. Replacing $\mathcal{C}$ by $\mathrm{LMod}_B(\mathcal{C})$, we can reduce to the case $A = B$. Let $R = \pi_0 A$. For $X \in \mathrm{Test}_R$, we let $\mathrm{QCoh}(X; \mathcal{C})$ be defined as in the proof of Proposition 5.6, M_X the image of M in $\mathrm{QCoh}(X; \mathcal{C})$, and $\theta_X : \mathrm{QCoh}(X; \mathcal{C}) \to \mathcal{S}$ the functor corepresented by M_X. Let $\chi' : \mathrm{N}(\mathrm{Test}_R)^{op} \to \mathcal{S}$ be given by the formula

$$\chi'(X) = \begin{cases} \{*\} & \text{if } M_X \in \mathrm{QCoh}(X; \mathcal{C}) \text{ is compact} \\ \emptyset & \text{otherwise.} \end{cases}$$

For each morphism $f : Y \to X$ in Test_R, the pushforward functor $f_* : \mathrm{QCoh}(Y; calC) \to \mathrm{QCoh}(X; \mathcal{C})$ preserves small colimits (Proposition 5.15), so that the pullback functor $f^* : \mathrm{QCoh}(X; \mathcal{C}) \to \mathrm{QCoh}(Y; \mathcal{C})$ carries compact objects to compact objects (Proposition T.5.5.7.2); it follows that χ' is well-defined. According to Theorem 2.9, it will suffice to show that χ' satisfies Nisnevich excision.

Let $f : X' \to X$ be an étale map in Test_R and $U \subseteq X$ a quasi-compact open subset such that f induces an isomorphism $X' \times_X (X - U) \to X - U$. Using Theorems 2.9 and 5.4, we deduce that the diagram of

∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(X;\mathcal{C}) & \xrightarrow{\phi} & \mathrm{QCoh}(X';\mathcal{C}) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\ \mathrm{QCoh}(U;\mathcal{C}) & \xrightarrow{\phi'} & \mathrm{QCoh}(U \times_X X';\mathcal{C}) \end{array}$$

is a pullback square. It follows that the functor θ_X can be described as a pullback

$$(\theta_U \circ \psi) \times_{\theta_{U \times_X X'} \circ \phi' \circ \psi} (\theta_{X'} \circ \phi).$$

If θ_U, $\theta_{U \times_X X'}$ and $\theta_{X'}$ preserve filtered colimits, it follows that θ_X preserves filtered colimits as well. $\square$

7 Flat Descent

In §5 we proved two main results:

(a) If k is an $\mathbb{E}_2$-ring and $\mathcal{C}$ is a k-linear ∞-category, then the functor $A \mapsto \mathrm{LMod}_A(\mathcal{C})$ is a sheaf for the étale topology (Theorem 5.4).

(b) The functor $k \mapsto \mathrm{LinCat}_k$ is a sheaf for the étale topology (Theorem 5.13).

Our proof of (a) made essential use of special properties of the étale topology, but the deduction of (b) from (a) was essentially formal. In this section, we will carry out an analogous deduction in the setting of the flat topology. This will require two changes in our approach:

- We will restrict our attention to the case of linear ∞-categories over $\mathbb{E}_\infty$-rings, rather than arbitrary $\mathbb{E}_2$-rings.
- Since we do not expect assertion (a) to hold for the flat topology in general, we must restrict our attention to a special class of k-linear ∞-categories: namely, those for which the analogue of assertion (a) holds. This is a mild restriction: see Example 7.2.

Our first step is to formalize these restrictions.

Notation 7.1. Let LinCat denote the ∞-category of linear ∞-categories (see Notation 5.1): the objects of LinCat are pairs $(k, \mathcal{C})$, where k is an $\mathbb{E}_2$-ring and $\mathcal{C}$ is a k-linear ∞-category. We let $\mathrm{LinCat}^\flat$ denote the full subcategory of

$$\mathrm{CAlg} \times_{\mathrm{Alg}^{(2)}} \mathrm{LinCat}$$

spanned by those pairs $(k, \mathcal{C})$, where k is an $\mathbb{E}_\infty$-ring and $\mathcal{C}$ is a k-linear ∞-categories which satisfies flat hyperdescent (see Definition VII.6.8). For every E_∞-ring k, we let $\mathrm{LinCat}^\flat_k$ denote the fiber product $\mathrm{LinCat}^\flat \times_{\mathrm{CAlg}} \{k\}$ (that is, the full subcategory of LinCat_k spanned by those k-linear ∞-categories which satisfy flat hyperdescent).

Example 7.2. Let k be an $\mathbb{E}_\infty$-ring and $\mathcal{C}$ a k-linear ∞-category. If $\mathcal{C}$ is compactly generated, then $\mathcal{C} \in \mathrm{LinCat}^\flat_k$ (Corollary 6.11). If k is connective and $\mathcal{C}$ admits an excellent t-structure, then $\mathcal{C} \in \mathrm{LinCat}^\flat_k$ (Theorem VII.6.12).

Remark 7.3. Let A be an $\mathbb{E}_\infty$-ring. Then the full subcategory $\mathrm{LinCat}^{\mathrm{desc}}_A \subseteq \mathrm{LinCat}_A$ is stable under small limits.

Remark 7.4. It follows immediately from the definitions that if $f : A \to B$ is a map of $\mathbb{E}_\infty$-rings and $\mathcal{C}$ is an A-linear ∞-category which satisfies flat hyperdescent, then the ∞-category $\mathrm{LMod}_B(\mathcal{C})$ also satisfies flat hyperdescent. It follows that the coCartesian fibration $\theta_0 : \mathrm{LinCat} \to \mathrm{Alg}^{(2)}$ induces to a coCartesian fibration $\theta : \mathrm{LinCat}^\flat \to \mathrm{CAlg}$. Moreover, a morphism in $\mathrm{LinCat}^\flat$ is θ-coCartesian if and only if its image in LinCat is θ_0-coCartesian.

We can now state the main result of this section.

Theorem 7.5. *Let* $\theta : \mathrm{LinCat}^{\flat} \to \mathrm{CAlg}$ *be the coCartesian fibration of Theorem 5.13, and let* $\chi : \mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ *classify* θ*. Then* χ *is a hypercomplete sheaf with respect to the flat topology on* CAlg^{op}*.*

The rest of this section is devoted to the proof of Theorem 7.5. We begin by proving the following weaker result.

Proposition 7.6. *Let* $\chi : \mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ *be defined as in Theorem 7.5. Then* χ *is a sheaf for the flat topology.*

The proof of Proposition 7.6 relies on the following observation.

Lemma 7.7. *Let* $f : A \to A'$ *be a map of* $\mathbb{E}_{\infty}$*-rings. Let* $\mathcal{C}'$ *be an* A'*-linear* ∞*-category, and let* $\mathcal{C} = f_* \mathcal{C}'$ *denote the underlying* A*-linear* ∞*-category. If* $\mathcal{C}'$ *has flat descent (hyperdescent), then* $\mathcal{C}$ *has flat descent (hyperdescent).*

Proof. We will give the proof for hyperdescent; the case of descent is handled in the same way. Using Proposition VII.5.12 and Lemma 5.7, we are reduced to proving the following:

- $(*)$ Let $B^{\bullet} : \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{CAlg}_{A/}$ be a flat hypercovering. Then the functor $\bullet \mapsto \mathrm{LMod}_{B^{\bullet}}(\mathcal{C})$ defines a limit diagram $\chi : \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$.

Let $B'^{\bullet} = A' \otimes_A B^{\bullet}$ be the image of $B^{\bullet}$ in $\mathrm{CAlg}_{A'/}$, so that we can identify χ with the functor $\bullet \mapsto \mathrm{LMod}_{B'^{\bullet}}(\mathcal{C}')$. Since $\mathcal{C}'$ has flat hyperdescent, we conclude that χ is a limit diagram. □

Remark 7.8. Lemma 7.7 implies that that the forgetful functor $\theta : \mathrm{LinCat}^{\flat} \to \mathrm{CAlg}$ is a Cartesian fibration, and that a morphism in $\mathrm{LinCat}^{\flat}$ is θ-Cartesian if and only if its image in LinCat is θ_0-Cartesian (where $\theta_0 : \mathrm{LinCat} \to \mathrm{Alg}^{(2)}$ denotes the forgetful functor).

Proof of Proposition 7.6. We will show that χ satisfies the hypotheses of Proposition VII.5.7. Note that if we are given a finite collection of objects $(k_i, \mathcal{C}_i) \in \mathrm{LinCat}$ where each k_i is an $\mathbb{E}_{\infty}$-ring, then the $\prod_i k_i$-linear ∞-category $\mathcal{C} = \prod_i \mathcal{C}_i$ satisfies flat hyperdescent if and only if each $\mathcal{C}_i$ satisfies flat hyperdescent. Using Lemma 5.14, we deduce htat χ commutes with finite products. To complete the proof, suppose we are given a faithfully flat map of $\mathbb{E}_{\infty}$-rings $f : A \to A^0$ having Čech nerve $A^{\bullet}$. We wish to show that the induced map $\chi(A) \to \varprojlim \chi(A^{\bullet})$ is an equivalence of ∞-categories. We proceed by showing that this functor satisfies the conditions of Lemma VII.5.17:

- (a) Fix an morphism $F : \mathcal{C} \to \mathcal{D}$ in $\chi(A) = \mathrm{LinCat}^{\flat}_A$ whose image in $\mathrm{LinCat}^{\flat}_{A^0}$ is an equivalence. It follows that F induces an equivalence of cosimplicial ∞-categories $\mathrm{LMod}_{A^{\bullet}}(\mathcal{C}) \to \mathrm{LMod}_{A^{\bullet}}(\mathcal{D})$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ \varprojlim \mathrm{LMod}_{A^{\bullet}}(\mathcal{C}) & \longrightarrow & \varprojlim \mathrm{LMod}_{A^{\bullet}}(\mathcal{D}) \end{array}$$

  where the vertical maps are equivalences of ∞-categories (since $\mathcal{C}$ and $\mathcal{D}$ satisfy flat hyperdescent). It follows that F is an equivalence of ∞-categories.

- (b) Suppose we are given a diagram $X^{\bullet} : \mathrm{N}(\mathbf{\Delta}) \to \mathrm{LinCat}^{\flat}$ lying over the cosimplicial $\mathbb{E}_{\infty}$-ring $A^{\bullet}$, and write $X^{\bullet} = (A^{\bullet}, \mathcal{C}^{\bullet})$. Then $X^{\bullet}$ can be extended to a θ'-limit diagram $\overline{X}^{\bullet}$ with $\overline{X}^{-1} = (A, \mathcal{C})$, where θ' is the projection $\mathrm{CAlg} \times_{\mathrm{Alg}^{(2)}} \mathrm{LinCat} \to \mathrm{CAlg}$. Note that $\mathcal{C} \simeq \varprojlim \mathcal{C}^{\bullet}$; it follows from Lemma 7.7 and Remark 7.3 that $\mathcal{C}$ satisfies flat hyperdescent, so that $\overline{X}^{\bullet}$ can be identified with a diagram in

LinCat$^\flat$. To complete the proof, it will suffice to show that if $X^\bullet$ carries every morphism in $\mathrm{N}(\mathbf{\Delta})$ to a θ-coCartesian morphism in $\mathcal{C}$, then $\overline{X}$ has the same property. This follows from the calculation

$$\begin{aligned}\mathrm{LMod}_{A^0}(\mathcal{C}) &\simeq \mathrm{LMod}_{A^0}(\varprojlim \mathcal{C}^\bullet) \\ &\simeq \varprojlim \mathrm{LMod}_{A^0}(\mathcal{C}^\bullet) \\ &\simeq \varprojlim \mathcal{C}^{\bullet+1} \\ &\simeq \mathcal{C}^0.\end{aligned}$$

□

To deduce Theorem 7.5 from Proposition 7.6, we will need some generalities concerning hypercompleteness for sheaves of ∞-categories.

Lemma 7.9. *Let $\mathcal{X}$ be an ∞-topos containing an object X. The following conditions are equivalent:*

(1) *The object $X \in \mathcal{X}$ is hypercomplete.*

(2) *For every ∞-connective morphism $E \to E'$ in $\mathcal{X}$, the map $\mathrm{Map}_{\mathcal{X}}(E', X) \to \mathrm{Map}_{\mathcal{X}}(E, X)$ is surjective on connected components.*

Proof. The implication (1) ⇒ (2) is obvious. Suppose that (2) is satisfied. We wish to prove that for every ∞-connective morphism $\alpha : E \to E'$, the map $\theta_\alpha : \mathrm{Map}_{\mathcal{X}}(E', X) \to \mathrm{Map}_{\mathcal{X}}(E, X)$ is a homotopy equivalence. We will prove that θ_α is n-connective using induction on n, the case $n = -1$ being vacuous. Since θ_α is surjective on connected components (by (2)), it will suffice to show that the diagonal map

$$\mathrm{Map}_{\mathcal{X}}(E', X) \to \mathrm{Map}_{\mathcal{X}}(E', X) \times_{\mathrm{Map}_{\mathcal{X}}(E,X)} \mathrm{Map}_{\mathcal{X}}(E', X) \simeq \mathrm{Map}_{\mathcal{X}}(E' \coprod_E E', X)$$

is $(n-1)$-connected. This follows from the inductive hypothesis, since the codiagonal $E' \coprod_E E' \to E'$ is also ∞-connective. □

Lemma 7.10. *Let $\mathcal{X}$ be an ∞-topos. Then the collection of hypercomplete objects of $\mathcal{X}$ is closed under small coproducts.*

Proof. Suppose we are given a collection of hypercomplete objects $\{X_\alpha\}_{\alpha \in A}$ having coproduct $X \in \mathcal{X}$. We wish to prove that X is hypercomplete. According to Lemma 7.9, it will suffice to show that if $\phi : E \to E'$ is an ∞-connective morphism in $\mathcal{X}$, then the induced map $\mathrm{Map}_{\mathcal{X}}(E', X) \to \mathrm{Map}_{\mathcal{X}}(E, X)$ is surjective on connected components. Fix a morphism $f : E \to X$. For each index α, let E_α denote the fiber product $X_\alpha \times_X E$, so that the induced map $E_\alpha \to E'$ admits a factorization

$$E_\alpha \stackrel{g_\alpha}{\to} E'_\alpha \stackrel{h_\alpha}{\to} E'$$

where g_α is an effective epimorphism and h_α is a monomorphism. Let $E'' = \coprod_\alpha E'_\alpha$, so that the maps h_α induce a map $\psi : E'' \to E'$. We claim that ψ is an equivalence. Since ϕ factors through ψ, we deduce that ψ is an effective epimorphism. It will therefore suffice to show that the diagonal map

$$\coprod_\alpha E'_\alpha E'' \to E'' \times_{E'} E'' \simeq \coprod_{\alpha,\beta} E'_\alpha \times_{E'} E'_\beta$$

is an equivalence. Because each h_α is a monomorphism, each of the diagonal maps $E'_\alpha \to E'_\alpha \times_{E'} E'_\alpha$ is an equivalence; we are therefore reduced to proving that $E'_\alpha \times_{E'} E'_\beta \simeq \emptyset$ for $\alpha \neq \beta$. This follows from the existence of an effective epimorphism

$$\emptyset \simeq E_\alpha \times_E E_\beta \to E_\alpha \times_{E'} E_\beta \to E'_\alpha \times_{E'} E'_\beta.$$

This completes the proof that ψ is an equivalence, so that we can identify ϕ with the coproduct of morphisms $\phi_\alpha : E_\alpha \to E'_\alpha$. To prove that f factors through ϕ, it suffices to show that each restriction $f|E_\alpha$ factors through ϕ_α. This follows from our assumption that X_α is hypercomplete, since ϕ_α is a pullback of ϕ and therefore ∞-connective. $\square$

Lemma 7.11. *Let $\mathcal{X}$ be an ∞-topos, and let $f : U \to X$ be an effective epimorphism in $\mathcal{X}$. Assume that U is hypercomplete. Then the following conditions are equivalent:*

(1) *The object X is hypercomplete.*

(2) *The fiber product $U \times_X U$ is hypercomplete.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, since the full subcategory $\mathcal{X}^\wedge \subseteq \mathcal{X}$ spanned by the hypercomplete objects is closed under small limits. We will prove that (2) $\Rightarrow$ (1). Let $L : \mathcal{X} \to \mathcal{X}^\wedge$ be a left adjoint to the inclusion, so that L is left exact (see §T.6.5.2). Let $U_\bullet$ be a Čech nerve of the map $U \to X$, so that $LU_\bullet$ is a Čech nerve of the induced map $LU \to LX$. Using assumption (2) and our assumption that U is hypercomplete, we deduce that U_0 and U_1 are both hypercomplete, so that $U_n \simeq U_1 \times_{U_0} \cdots \times_{U_0} U_1$ is hypercomplete for all $n \geq 0$. It follows that the canonical map $U_\bullet \to LU_\bullet$ is an equivalence. In particular, we obtain an equivalence

$$X \simeq |U_\bullet| \simeq |LU_\bullet| \simeq LX,$$

so that X is also hypercomplete (the last equivalence here results from the observation that $LU \to LX$ is an effective epimorphism, since it can be identified with the composition of f with the ∞-connective map $X \to LX$). $\square$

Proposition 7.12. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and is equipped with a Grothendieck topology. Assume that every object $C \in \mathcal{C}$ represents a functor $e_C : \mathcal{C}^{op} \to \mathcal{S}$ which is a hypercomplete sheaf on $\mathcal{C}$. Let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}$ be a sheaf on $\mathcal{C}$. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ is hypercomplete.*

(2) *For every pair of objects $C, C' \in \mathcal{C}$ and maps $\eta : e_C \to \mathcal{F}$, $\eta' : e_{C'} \to \mathcal{F}$, the fiber product $e_C \times_{\mathcal{F}} e_{C'}$ is hypercomplete.*

(3) *For every object $C \in \mathcal{C}$ and every pair of maps $\eta, \eta' : e_C \to \mathcal{F}$, the equalizer of the diagram*

$$e_C \underset{\eta'}{\overset{\eta}{\rightrightarrows}} \mathcal{F}$$

is hypercomplete.

Proof. The implication (1) $\Rightarrow$ (3) is clear, since the collection of hypercomplete objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is stable under small limits. The implication (3) $\Rightarrow$ (2) follows from the observation that $e_C \times_{\mathcal{F}} e_{C'}$ can be identified with the equalizer of the pair of maps

$$\mathcal{F} \leftarrow e_C \leftarrow e_{C \times C'} \to e_{C'} \to \mathcal{F}.$$

We will prove that (2) $\Rightarrow$ (1). Let $\mathcal{F}' = \coprod_{\eta \in \mathcal{F}(C)} e_C$, so we have an effective epimorphism $\mathcal{F}' \to \mathcal{F}$. Lemma 7.10 implies that $\mathcal{F}'$ is hypercomplete. In view of Lemma 7.11, it will suffice to prove that the fiber product $\mathcal{F}' \times_{\mathcal{F}} \mathcal{F}'$ is hypercomplete. This fiber product can be identified with the coproduct

$$\coprod_{\eta \in \mathcal{F}(C), \eta' \in \mathcal{F}(C')} e_C \times_{\mathcal{F}} e_{C'}$$

which is hypercomplete by virtue of assumption (2) and Lemma 7.10. $\square$

Lemma 7.13. *The ∞-category $\mathcal{C}\mathrm{at}_\infty$ is generated (under small colimits) by the objects $\Delta^0, \Delta^1 \in \mathcal{C}\mathrm{at}_\infty$.*

Proof. Let $\mathcal{C}$ be the smallest full subcategory of $\mathcal{C}\mathrm{at}_\infty$ which contains Δ^0 and Δ^1 and is closed under small colimits. Let $\overline{\mathcal{C}} \subseteq \mathcal{S}\mathrm{et}_\Delta$ be the full subcategory spanned by those simplicial sets K which are categorically equivalent to ∞-categories belonging to $\mathcal{C}$. We wish to prove that $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$, or equivalently that $\overline{\mathcal{C}} = \mathcal{S}\mathrm{et}_\Delta$. Since $\overline{\mathcal{C}}$ is stable under filtered colimits in $\mathcal{S}\mathrm{et}_\Delta$, it will suffice to show that $\overline{\mathcal{C}}$ contains every finite simplicial set K. We proceed by induction on dimension n of K and the number of nondegenerate n-simplices of K. If $K = \emptyset$, then there is nothing to prove. Otherwise, we have a homotopy pushout diagram

$$\begin{array}{ccc} \partial \Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ K_0 & \longrightarrow & K \end{array}$$

where $\partial \Delta^n$ and K_0 belong to $\overline{\mathcal{C}}$ by the inductive hypothesis. Since $\mathcal{C}$ is stable under pushouts, $\overline{\mathcal{C}}$ is stable under homotopy pushouts; consequently, to show that $K \in \overline{\mathcal{C}}$, it will suffice to show that $\Delta^n \in \overline{\mathcal{C}}$. If $n > 2$, then the inclusion $\Lambda^n_1 \subseteq \Delta^n$ is a categorical equivalence. It therefore suffices to show that $\Lambda^n_1 \in \overline{\mathcal{C}}$, which follows from the inductive hypothesis. We are therefore reduced to the case $n \leq 1$: that is, we must show that $\Delta^0, \Delta^1 \in \overline{\mathcal{C}}$, which is obvious. $\square$

To state our next result, we need to introduce a bit of notation. Suppose that $\mathcal{C}$ is a small ∞-category, and let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{C}\mathrm{at}_\infty$ be a presheaf of ∞-categories on $\mathcal{C}$, classified by a Cartesian fibration $p : \widetilde{\mathcal{C}} \to \mathcal{C}$. According to Proposition T.3.3.3.1, we can identify $\varprojlim \mathcal{F}$ with the full subcategory of $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \widetilde{\mathcal{C}})$ spanned by the Cartesian sections of p. Let $X, Y \in \varprojlim \mathcal{F} \subseteq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C}, \widetilde{\mathcal{C}})$ so that the pair (X, Y) determines a functor $\mathcal{C} \to \mathcal{C}' = \mathrm{Fun}(\partial \Delta^1, \widetilde{\mathcal{C}}) \times_{\mathrm{Fun}(\partial \Delta^1, \mathcal{C})} \mathcal{C}$. We let $\mathcal{D}$ denote the fiber product $\mathrm{Fun}(\Delta^1, \widetilde{\mathcal{C}}) \times_{\mathcal{C}'} \mathcal{C}$. The projection $\mathcal{D} \to \mathcal{C}$ is a right fibration, whose fiber over an object $C \in \mathcal{C}$ can be identified with the Kan complex $\mathrm{Hom}_{\widetilde{\mathcal{C}}_C}(X(C), Y(C))$ (see §T.1.2.2). This right fibration is classified by a functor $\underline{\mathrm{Map}}_{\mathcal{F}}(X, Y) : \mathcal{C}^{op} \to \mathcal{S}$, given informally by the formula $\underline{\mathrm{Map}}_{\mathcal{F}}(X, Y)(C) = \mathrm{Map}_{\mathcal{F}(C)}(X(C), Y(C))$. Let $\mathcal{D}_0$ be the full subcategory of $\mathcal{D}$ whose fiber over an object $C \in \mathcal{C}$ is given by the full subcategory of $\mathrm{Hom}_{\widetilde{\mathcal{C}}_C}(X(C), Y(C))$ spanned by the equivalences in $\widetilde{\mathcal{C}}_C$. The projection $\mathcal{D}_0 \to \mathcal{C}$ is also a right fibration, classified by a functor $\underline{\mathrm{Map}}^{\sim}_{\mathcal{F}}(X, Y) : \mathcal{C}^{op} \to \mathcal{S}$.

Proposition 7.14. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and is equipped with a Grothendieck topology. Assume that for every object $C \in \mathcal{C}$, the functor $e_C : \mathcal{C}^{op} \to \mathcal{S}$ represented by C is a hypercomplete sheaf on $\mathcal{C}$. Let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{C}\mathrm{at}_\infty$ be a $\mathcal{C}\mathrm{at}_\infty$-valued sheaf on $\mathcal{C}$. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ is hypercomplete.*

(2) *For every object $C \in \mathcal{C}$ and every pair of objects $X, Y \in \mathcal{F}(C) \simeq \varprojlim \mathcal{F} | (\mathcal{C}_{/C})^{op}$, the functor*

$$\underline{\mathrm{Map}}_{\mathcal{F}}(X, Y) : (\mathcal{C}_{/C})^{op} \to \mathcal{S}$$

is a hypercomplete sheaf on $\mathcal{C}_{/C}$.

Proof. Suppose first that (1) is satisfied; we will prove (2). Replacing $\mathcal{C}$ by $\mathcal{C}_{/C}$, we may suppose that $X, Y \in \varprojlim \mathcal{F}$. For every simplicial set K, let $\mathcal{F}^K$ denote the composition

$$\mathcal{C}^{op} \stackrel{\mathcal{F}}{\to} \mathcal{C}\mathrm{at}_\infty \stackrel{\mathrm{Fun}(K, \bullet)}{\longrightarrow} \mathcal{C}\mathrm{at}_\infty .$$

and let $*$ denote the constant functor $\mathcal{F}^{\emptyset} : \mathcal{C}^{op} \to \mathcal{C}\mathrm{at}_\infty$ taking the value Δ^0. Then the pair (X, Y) determines a natural transformation $* \to \mathcal{F}^{\partial \Delta^0}$, and $\underline{\mathrm{Map}}_{\mathcal{F}}(X, Y)$ can be identified with the fiber product $* \times_{\mathcal{F}^{\partial \Delta^1}} \mathcal{F}^{\Delta^1}$. Since $\mathcal{F}$ is hypercomplete, we deduce that $\mathcal{F}^{\Delta^1}$, $\mathcal{F}^{\partial \Delta^1}$, and $* \simeq \mathcal{F}^{\emptyset}$ are hypercomplete. It follows that

$\underline{\mathrm{Map}}_{\mathcal{F}}(X,Y)$ is a hypercomplete $\mathcal{C}\mathrm{at}_\infty$-valued sheaf on $\mathcal{C}$, and therefore a hypercomplete $\mathcal{S}$-valued sheaf on $\mathcal{C}$ (since the inclusion $\mathcal{S} \subseteq \mathcal{C}\mathrm{at}_\infty$ preserves small limits).

Now assume (2). Fix an object $C \in \mathcal{C}$, and let $f : x \to y$ be a morphism in $\mathcal{F}(C)$. Since $\mathcal{F}$ is a sheaf, we deduce that f is an equivalence if and only if there exists a covering sieve $\{C_\alpha \to C\}$ on C such that the image of f under each of the induced functors $\mathcal{F}(C) \to \mathcal{F}(C_\alpha)$ is an equivalence. Combining this observation with (2) and Lemma VIII.3.1.20, we deduce:

$(*)$ For every object $C \in \mathcal{C}$ and every pair of objects $X, Y \in \mathcal{F}(C) \simeq \varprojlim \mathcal{F}\,|(\mathcal{C}_{/C})^{op}$, the functor

$$\underline{\mathrm{Map}}^{\sim}_{\mathcal{F}}(X,Y) : (\mathcal{C}_{/C})^{op} \to \mathcal{S}$$

is a hypercomplete sheaf on $\mathcal{C}_{/C}$.

For every ∞-category $\mathcal{D}$, let $\chi_{\mathcal{D}} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ be the functor corepresented by $\mathcal{D}$. Let $\mathcal{C}\mathrm{at}'_\infty$ denote the full subcategory of $\mathcal{C}\mathrm{at}_\infty$ spanned by those ∞-categories $\mathcal{D}$ for which the composite functor

$$\mathcal{F}_{\mathcal{D}} : \mathcal{C}^{op} \xrightarrow{\mathcal{F}} \mathcal{C}\mathrm{at}_\infty \xrightarrow{\chi_{\mathcal{D}}} \mathcal{S}$$

is a hypercomplete sheaf on $\mathcal{C}$. We wish to prove that $\mathcal{C}\mathrm{at}'_\infty = \mathcal{C}\mathrm{at}_\infty$. Since the collection of hypercomplete sheaves is stable under small limits in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ and the construction $\mathcal{D} \mapsto \mathcal{F}_{\mathcal{D}}$ carries colimits to limits, we conclude that $\mathcal{C}\mathrm{at}'_\infty \subseteq \mathcal{C}\mathrm{at}_\infty$ is stable under small colimits. By virtue of Lemma 7.13, it will suffice to show that $\Delta^0, \Delta^1 \in \mathcal{C}\mathrm{at}'_\infty$. The inclusion $\Delta^0 \in \mathcal{C}\mathrm{at}'_\infty$ follows from $(*)$ together with Proposition 7.12. It follows that $\partial\,\Delta^1 \in \mathcal{C}\mathrm{at}'_\infty$, so that $\mathcal{F}_{\partial\,\Delta^1}$ is a hypercomplete sheaf on $\mathcal{C}$. Applying Lemma VIII.3.1.20 to the restriction map $\mathcal{F}_{\Delta^1} \to \mathcal{F}_{\partial\,\Delta^1}$, we deduce that $\mathcal{F}_{\Delta^1}$ is hypercomplete so that $\Delta^1 \in \mathcal{C}\mathrm{at}'_\infty$ as desired (the hypotheses of Lemma VIII.3.1.20 are satisfied by virtue of assumption (2)). □

Proof of Theorem 7.5. Proposition 7.6 shows that χ is a sheaf with respect to the flat topology. Note that every object of CAlg corepresents a hypercomplete sheaf on CAlg^{op} (Theorem VII.5.14). We will complete the proof by showing that χ satisfies the criterion of Proposition 7.14. Namely, we must show that for every $\mathbb{E}_\infty$-ring A and every pair of objects $\mathcal{C}, \mathcal{D} \in \chi(A) \simeq \mathrm{LinCat}^\flat_A$, the functor $\underline{\mathrm{Map}}_\chi(\mathcal{C}, \mathcal{D}) : \mathrm{CAlg}_A \to \widehat{\mathcal{S}}$ is a hypercomplete sheaf with respect to the flat topology on CAlg_A. Note that $\underline{\mathrm{Map}}_\chi(\mathcal{C}, \mathcal{D})$ can be identified with the composition

$$\mathrm{CAlg}_A \xrightarrow{F} \mathrm{LinCat}_A \xrightarrow{F'} \widehat{\mathcal{S}},$$

where F is the functor given informally by $F(B) = \mathrm{LMod}_B(\mathcal{D})$ and F' is the functor corepresented by $\mathcal{C} \in \mathrm{LinCat}_A$. It therefore suffices to show that F is a hypercomplete LinCat_A-valued sheaf on CAlg_A. Let $F'' : \mathrm{LinCat}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the forgetful functor. Since F'' is conservative and preserves limits (Corollary A.3.4.3.2), it will suffice to show that $F'' \circ F : \mathrm{CAlg}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ is a hypercomplete $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf. This is equivalent to our assumption that $\mathcal{D}$ satisfies flat hyperdescent. □

8 Quasi-Coherent Stacks

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. According to Proposition VIII.2.7.18, there are two different ways of understanding the theory of quasi-coherent sheaves on $\mathfrak{X}$:

(*a*) A quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ can be viewed as a sheaf of spectra on $\mathcal{X}$ equipped with an action of the structure sheaf $\mathcal{O}_{\mathcal{X}}$.

(*b*) A quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ can be viewed as a rule which assigns to each R-point $\eta : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$ an R-module $\mathcal{F}(\eta) \in \mathrm{Mod}_R$, depending functorially on η.

The equivalence of these two definitions relies on the fact that the construction $R \mapsto \mathrm{Mod}_R$ is a sheaf with respect to the étale topology. In §5, we proved a categorified version of this result: according to Theorem 5.13, the construction $R \mapsto \mathrm{LinCat}_R$ is a sheaf with respect to the étale topology. In this section, we will use Theorem 5.13 to globalize the theory of linear ∞-categories. For each spectral Deligne-Mumford stack $\mathfrak{X}$, we will introduce an ∞-category $\mathrm{QStk}(\mathfrak{X})$, which we call the ∞-category of *quasi-coherent stacks* on $\mathfrak{X}$. The definition is suggested by (b): essentially, a quasi-coherent stack on $\mathfrak{X}$ is a rule which assigns to each R-point $\eta : \mathrm{Spec}^{\text{ét}} R \to \mathfrak{X}$ an R-linear ∞-category $\mathcal{C}_\eta \in \mathrm{LinCat}_R$, which depends functorially on the point η.

In the special case where $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ is an affine spectral Deligne-Mumford stack, the global sections functor $\mathcal{F} \mapsto \Gamma(\mathfrak{X}; \mathcal{F})$ induces an equivalence of ∞-categories $\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}_R$. Our main goal in this section is show that the analogous statement for quasi-coherent stacks is valid under much weaker hypotheses: if $\mathfrak{X}$ is quasi-compact and the diagonal of $\mathfrak{X}$ is quasi-affine, then every quasi-coherent stack on $\mathfrak{X}$ can be recovered from its ∞-category of global sections, regarded as an ∞-category tensored over $\mathrm{QCoh}(\mathfrak{X})$ (Theorem 8.6).

We begin by giving a more precise definition of the notion of quasi-coherent stack.

Construction 8.1. Let LinCat denote the ∞-category of linear ∞-categories (see Notation 5.1). We let θ denote the projection map $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{Alg}^{(2)}} \mathrm{LinCat} \to \mathrm{CAlg}^{\mathrm{cn}}$, and let

$$\mathrm{QStk} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$$

be the functor obtained by applying the construction of Remark VIII.2.7.7 to the coCartesian fibration θ. The functor QStk is characterized up to equivalence by the fact that it preserves limits and the fact that the composition $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \xrightarrow{\mathrm{QStk}} \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifies the coCartesian fibration θ.

Given a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we will refer to the $\mathrm{QStk}(X)$ as the *∞-category of quasi-coherent stacks on X*. If $\mathfrak{X}$ is a spectral Deligne-Mumford stack and $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is the functor represented by $\mathfrak{X}$ (given by $X(R) = \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}^{\text{ét}} R, \mathfrak{X})$), then we set $\mathrm{QStk}(\mathfrak{X}) = \mathrm{QStk}(X)$ and refer to $\mathrm{QStk}(\mathfrak{X})$ as the *∞-category of quasi-coherent stacks on $\mathfrak{X}$*.

Notation 8.2. If $f : X \to Y$ is a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, we let f^* denote the induced functor $\mathrm{QStk}(Y) \to \mathrm{QStk}(X)$.

Remark 8.3. If $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a functor classifying by a left fibration $\mathcal{X} \to \mathrm{CAlg}^{\mathrm{cn}}$, then $\mathrm{QStk}(X)$ can be identified with the full subcategory of $\mathrm{Fun}_{\mathrm{Alg}^{(2)}}(\mathcal{X}, \mathrm{LinCat})$ spanned by those functors which carry each morphism in $\mathcal{C}$ to a q-coCartesian morphism in LinCat, where $q : \mathrm{LinCat} \to \mathrm{Alg}^{(2)}$ denotes the projection.

We can think of the objects of $\mathcal{X}$ as given by pairs (A, η), where $A \in \mathrm{CAlg}^{\mathrm{cn}}$ and $\eta \in X(A)$. Consequently, we may think of an object $\mathcal{C} \in \mathrm{QStk}(X)$ as a rule which assigns to each point $\eta \in X(A)$ an R-linear ∞-category $\mathcal{C}_\eta$, which is functorial in the following sense: if $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings and η has image $f_!(\eta) \in X(B)$, then we are given a canonical equivalence of $\mathcal{C}_{f_!(\eta)}$ with

$$\mathrm{Mod}_B \otimes_{\mathrm{Mod}_A} \mathcal{C}_\eta \simeq \mathrm{LMod}_B(\mathcal{C}_\eta).$$

Remark 8.4. Let $f : X \to Y$ be a map in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which induces an equivalence after sheafification with respect to the étale topology. Then the induced functor $f^* : \mathrm{QStk}(Y) \to \mathrm{QStk}(X)$ is an equivalence of ∞-categories. This follows immediately from Theorem 5.13 (and the characterization of the functor QStk supplied by Proposition VIII.2.7.6).

Construction 8.5. Let $\mathcal{U}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors X for which the ∞-category $\mathrm{QCoh}(X)$ is presentable. Note that $\mathcal{U}$ contains all of those functors which are representable by spectral Deligne-Mumford stacks (Proposition VIII.2.3.13) or geometric stacks (Proposition VIII.3.4.17). Define a functor $\mathrm{QStk}' : \mathcal{U}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ by the formula

$$\mathrm{QStk}'(X) = \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}).$$

Let $j : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{U}^{op}$ denote the Yoneda embedding, and note that the functors $\mathrm{QStk} \circ j$ and $\mathrm{QStk}' \circ j$ are canonically equivalent. Since QStk is a right Kan extension of $\mathrm{QStk} \circ j$ along j (Proposition VIII.2.7.6), we deduce the existence of a natural transformation of functors

$$\mathrm{QStk}' \to \mathrm{QStk} \,|\, \mathcal{U}^{op} .$$

In particular, for each $X \in \mathcal{U}$ we obtain a functor

$$\Phi_X : \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{QStk}(X).$$

This functor assigns to an ∞-category $\mathcal{C}$ tensored over $\mathrm{QCoh}(X)$ the quasi-coherent stack given by

$$(\eta \in X(R)) \mapsto \mathrm{Mod}_R \otimes_{\mathrm{QCoh}(X)} \mathcal{C} .$$

If $\mathfrak{X}$ is a spectral Deligne-Mumford stack representing a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, then we will denote the functor Φ_X by $\Phi_{\mathfrak{X}} : \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{QStk}(\mathfrak{X})$.

We are now ready to state our first main result.

Theorem 8.6. *Let $\mathfrak{X}$ be a quasi-geometric spectral Deligne-Mumford stack (see §VIII.3.3.8). Then the functor $\Phi_{\mathfrak{X}} : \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{QStk}(\mathfrak{X})$ of Construction 8.5 admits a fully faithful right adjoint. In particular, $\Phi_{\mathfrak{X}}$ is essentially surjective.*

Lemma 8.7. *Suppose we are given a diagram of spectral Deligne Mumford stacks σ :*

$$\begin{array}{ccc} \mathfrak{U}' & \longrightarrow & \mathfrak{X}' \\ \downarrow & & \downarrow f \\ \mathfrak{U} & \xrightarrow{j} & \mathfrak{X} \end{array}$$

where j is a quasi-compact open immersion, f is étale and quasi-affine, where σ is simultaneously a pullback square and a pushout square. Then the canonical functor

$$F : \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{U})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \times_{\mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{U}')}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X}')}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$$

is fully faithful.

Proof. We wish to show that the canonical map

$$F : \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{U})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \times_{\mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{U}')}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X}')}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$$

is an equivalence of ∞-categories. Let us identify objects of the target of F with triples $(\mathcal{C}_U, \mathcal{C}_{X'}, \alpha)$, where $\mathcal{C}_U$ is a presentable ∞-category tensored over $\mathrm{QCoh}(\mathfrak{U})$, $\mathcal{C}_{X'}$ is a presentable ∞-category tensored over $\mathrm{QCoh}(\mathfrak{X}')$, and α is an equivalence

$$\mathrm{QCoh}(\mathfrak{U}') \otimes_{\mathrm{QCoh}(\mathfrak{U})} \mathcal{C}_U \simeq \mathrm{QCoh}(\mathfrak{U}') \otimes_{\mathrm{QCoh}(\mathfrak{X}'} \mathcal{C}_{X'} .$$

Let us denote either of these equivalent ∞-categories by $\mathcal{C}_{U'}$. The functor F admits a right adjoint G, given by $(\mathcal{C}_U, \mathcal{C}_{X'}, \alpha) \mapsto \mathcal{C}_U \times_{\mathcal{C}_{X'}} \mathcal{C}_X$. We will show that the unit map $u : \mathrm{id} \to G \circ F$ is an equivalence of functors from $\mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ to itself (so that the functor F is fully faithful).

Let $A \in \mathrm{QCoh}(\mathfrak{X})$ denote the direct image of the structure sheaf of $\mathfrak{X}'$, and let $B \in \mathrm{QCoh}(\mathfrak{X})$ denote the direct image of the structure sheaf of $\mathfrak{U}$. Then A and B are commutative algebra objects of $\mathrm{QCoh}(\mathfrak{X})$. Since j and f are quasi-affine, we have canonical equivalences

$$\mathrm{QCoh}(\mathfrak{X}') \simeq \mathrm{Mod}_A(\mathrm{QCoh}(\mathfrak{X})) \qquad \mathrm{QCoh}(\mathfrak{U}') \simeq \mathrm{Mod}_{A \otimes B}(\mathrm{QCoh}(\mathfrak{X})) \qquad \mathrm{QCoh}(\mathfrak{U}) \simeq \mathrm{Mod}_B(\mathrm{QCoh}(\mathfrak{X}))$$

(Corollary VIII.2.5.16). It follows that the base change functors

$$\mathcal{C} \mapsto \mathrm{QCoh}(\mathfrak{X}') \otimes_{\mathrm{QCoh}(\mathfrak{X})} \mathcal{C} \qquad \mathcal{C} \mapsto \mathrm{QCoh}(\mathfrak{U}) \otimes_{\mathrm{QCoh}(\mathfrak{X})} (\mathcal{C}$$

are given by

$$\mathcal{C} \mapsto \mathrm{LMod}_A(\mathcal{C}) \qquad \mathcal{C} \mapsto \mathrm{LMod}_B(\mathcal{C})$$

(see Theorem A.6.3.4.6). To prove that u is an equivalence, we must show that for every presentable ∞-category $\mathcal{C}$ tensored over $\mathrm{QCoh}(\mathfrak{X})$, the canonical map

$$f : \mathcal{C} \to \mathrm{LMod}_A(\mathcal{C}) \times_{\mathrm{LMod}_{A \otimes B}(\mathcal{C})} \mathrm{LMod}_B(\mathcal{C})$$

is an equivalence of ∞-categories. We will identify objects of the fiber product $\mathrm{LMod}_A(\mathcal{C}) \times_{\mathrm{LMod}_{A\otimes B}(\mathcal{C})} \mathrm{LMod}_B(\mathcal{C})$ with triples (M, N, β) where M is a left A-module objects of $\mathcal{C}$, N is a left B-module object of $\mathcal{C}$, and $\beta : B \otimes M \simeq A \otimes N$ is an equivalence of left $A \otimes B$-module objects of $\mathcal{C}$. The functor f has a right adjoint g, given by

$$g(M, N, \beta) = M \times_{B \otimes M} N.$$

We first claim that the unit map $\mathrm{id}_{\mathcal{C}} \to g \circ f$ is an equivalence. For this, it suffices to show that for each object $C \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} C & \longrightarrow & A \otimes C \\ \downarrow & & \downarrow \\ B \otimes C & \longrightarrow & A \otimes B \otimes C \end{array}$$

is a pullback diagram in $\mathcal{C}$. Since $\mathcal{C}$ is stable, it suffices to show that this diagram is a pushout square. Since the action $\mathrm{QCoh}(\mathfrak{X})$ on $\mathcal{C}$ preserves colimits in each variable, we are reduced to proving that the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X}} & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \otimes B \end{array}$$

is a pushout square in $\mathrm{QCoh}(\mathfrak{X})$. Using stability again, we are reduced to proving that this diagram is a pullback square. This follows from our assumption that σ is a pushout diagram.

To complete the proof that the functor f is an equivalence, it will suffice to show that the functor g is conservative. Suppose we are given a triple $(M, N, \beta) \in \mathrm{LMod}_A(\mathcal{C}) \times_{\mathrm{LMod}_{A\otimes B}(\mathcal{C})} \mathrm{LMod}_B(\mathcal{C})$ with $g(M, N, \beta) \simeq 0$. Since j is an open immersion, the object $B \in \mathrm{QCoh}(\mathfrak{X})$ is idempotent. It follows that the canonical map $M \to B \otimes M$ becomes an equivalence after tensoring with B, so that the projection map $g(M, N, \beta) \to N$ becomes an equivalence after tensoring with B. Since N is a B-module, we deduce that

$$N \simeq B \otimes N \simeq B \otimes g(M, N, \beta) \simeq B \otimes 0 \simeq 0.$$

It follows that $B \otimes M \simeq A \otimes N \simeq 0$, so that the projection map $g(M, N, \beta) \to M$ is an equivalence. Since $g(M, N, \beta) \simeq 0$, we deduce that $M \simeq 0$, so that (M, N, β) is a zero object of the fiber product $\mathrm{LMod}_A(\mathcal{C}) \times_{\mathrm{LMod}_{A\otimes B}(\mathcal{C})} \mathrm{LMod}_B(\mathcal{C})$ as desired. □

Lemma 8.8. *Let $\mathfrak{X}$ be a quasi-geometric spectral Deligne-Mumford stack. If $\mathfrak{X}$ admits a scallop decomposition, then the functor $\Phi_{\mathfrak{X}} : \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}r^{\mathrm{L}}) \to \mathrm{QStk}(\mathfrak{X})$ of Construction 8.5 is fully faithful.*

Proof. We will prove by induction on n that the map $\Phi_{\mathfrak{X}}$ is fully faithful whenever $\mathfrak{X}$ admits a scallop decomposition

$$\emptyset = \mathfrak{X}_0 \hookrightarrow \mathfrak{X}_1 \hookrightarrow \cdots \hookrightarrow \mathfrak{X}_n = \mathfrak{X}$$

of length n. When $n=0$, the result is trivial. To carry out the inductive step, we set $\mathfrak{U} = \mathfrak{X}_{n-1}$ and choose a pullback daigram

$$\begin{array}{ccc} \mathfrak{U} & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow \\ \mathfrak{U} & \longrightarrow & \mathfrak{X} \end{array}$$

satisfying the hypotheses of Lemma 8.7. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) & \xrightarrow{\Phi_{\mathfrak{X}}} & \mathrm{QStk}(\mathfrak{X}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{U})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \times_{\mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{U}')}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X}')}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) & \longrightarrow & \mathrm{QStk}(\mathfrak{U}) \times_{\mathrm{QCoh}(\mathfrak{U}')} \mathrm{QCoh}(\mathfrak{X}'). \end{array}$$

Here the right vertical map is an equivalence (see Remark 8.4) and the left vertical map is fully faithful by virtue of Lemma 8.7. We are therefore reduced to proving that the bottom horizontal map is fully faithful. For this, it suffices to show that the functors $\Phi_{\mathfrak{U}}$, $\Phi_{\mathfrak{U}'}$, and $\Phi_{\mathfrak{X}'}$ are fully faithful. In the first two cases, this follows from the inductive hypothesis. The functor $\Phi_{\mathfrak{X}'}$ is an equivalence, since $\mathfrak{X}'$ is affine. □

Proof of Theorem 8.6. We first prove that assertion (a) holds under the assumption that $\mathfrak{X}$ is a geometric spectral Deligne-Mumford stack (that is, we assume that $\mathfrak{X}$ is quasi-compact and that the diagonal of $\mathfrak{X}$ is affine). Choose an étale surjection $f : \mathfrak{X}_0 \to \mathfrak{X}$, where $\mathfrak{X}_0 = \mathrm{Spec}^{\text{ét}} A$ is affine. Let $\mathfrak{X}_\bullet$ be the Čech nerve of f. Then $\Phi_{\mathfrak{X}}$ can be identified with the map

$$\mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{QStk}(\mathfrak{X}) \simeq \varprojlim \mathrm{QStk}(X_\bullet) \simeq \varprojlim \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X}_\bullet)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}).$$

We wish to show that this functor has a fully faithful right adjoint. Let $\mathfrak{X}_{-1} = \mathfrak{X}$, so that the construction $[n] \mapsto \mathfrak{X}_n$ determines an augmented simplicial object of Stk. We will complete the proof by showing that the cosimplicial ∞-category $Q(\mathfrak{X}_\bullet) = \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X}_\bullet)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ satisfies the hypotheses of Corollary A.6.2.4.3:

(1) Let $f^* : Q(\mathfrak{X}) = Q(\mathfrak{X}_{-1}) \to Q(\mathfrak{X}_0) \simeq \mathrm{LinCat}_A$ be the pullback functor associated to f. We claim that the ∞-category $Q(\mathfrak{X})$ admits limits of f^*-split cosimplicial objects, which are preserved by the functor f^*. In fact, $Q(\mathfrak{X}) \simeq \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ admits all small limits (since $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ admits small limits). Since the map $f : \mathfrak{X}_0 \to \mathfrak{X}$ is quasi-affine, we have a canonical equivalence $\mathrm{QCoh}(\mathfrak{X}_0) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X))$ for $\mathcal{A} = f_* A \in \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{X}))$ (Proposition VIII.3.2.5). It follows that

$$f^* \mathcal{C} \simeq \mathrm{QCoh}(\mathfrak{X}_0) \otimes_{\mathrm{QCoh}(\mathfrak{X})} \mathcal{C} \simeq \mathrm{LMod}_{\mathcal{A}}(\mathcal{C})$$

(Theorem A.6.3.4.6). Using Theorem A.6.3.4.1, we deduce that the functor f^* commutes with small limits.

(2) For every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_+$, we claim that the diagram

$$\begin{array}{ccc} Q(\mathfrak{X}_m) & \longrightarrow & Q(\mathfrak{X}_{m+1}) \\ \downarrow & & \downarrow \\ Q(\mathfrak{X}_n) & \longrightarrow & Q(\mathfrak{X}_{n+1}) \end{array}$$

is right adjointable. By virtue of Lemma VII.6.15, it will suffice to show that the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(\mathfrak{X}_m) & \longleftarrow & \mathrm{QCoh}(\mathfrak{X}_{m+1}) \\ \uparrow & & \uparrow \\ \mathrm{QCoh}(\mathfrak{X}_n) & \longleftarrow & \mathrm{QCoh}(\mathfrak{X}_{n+1}) \end{array}$$

is a pushout square in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. This follows from Corollary VIII.3.2.6, since the projection maps $\mathfrak{X}_{m+1} \to \mathfrak{X}_m$ are quasi-affine.

We now treat the general case. Once again, we choose an étale surjection $f : \mathfrak{X}_0 \to \mathfrak{X}$ where $\mathfrak{X}_0$ is affine, and let $\mathfrak{X}_\bullet$ denote the Čech nerve of f. Since $\mathfrak{X}$ is quasi-geometric, each of the spectral Deligne-Mumford stacks $\mathfrak{X}_n$ is quasi-affine. It follows from the first part of the proof that the functors $\Phi_{\mathfrak{X}_n}$ are essentially surjective. Combining this with Lemma 8.8, we deduce that each $\Phi_{\mathfrak{X}_n}$ is an equivalence of ∞-categories. It follows as before that $\Phi_{\mathfrak{X}}$ is equivalent to the composite map

$$\mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X})}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{QStk}(\mathfrak{X}) \simeq \varprojlim \mathrm{QStk}(X_\bullet) \simeq \varprojlim \mathrm{Mod}_{\mathrm{QCoh}(\mathfrak{X}_\bullet)}(\mathcal{P}\mathrm{r}^{\mathrm{L}});$$

the rest of the proof is identical to the case treated above. □

Corollary 8.9. *Let $\mathfrak{X}$ be a quasi-geometric spectral Deligne-Mumford stack. Assume that $\mathfrak{X}$ admits a scallop decomposition. Then the functor $\Phi_{\mathfrak{X}}$ is an equivalence of ∞-categories.*

Proof. Combine Theorem 8.6 with Lemma 8.8. □

We now generalize Theorem 8.6 to other types of stacks. For this, we will need to restrict our attention to linear ∞-categories satisfying descent.

Notation 8.10. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We let $\mathrm{QStk}^\flat(X)$ denote the full subcategory of $\mathrm{QStk}(X)$ spanned by those quasi-coherent stacks $\mathcal{C}$ on X such that for each $\eta \in X(A)$, the pullback $\eta^* \mathcal{C} \in \mathrm{LinCat}_A$ belongs to the full subcategory $\mathrm{LinCat}^\flat_A \subseteq \mathrm{LinCat}_A$ of Notation 7.1 (that is $\eta^* \mathcal{C}$ satisfies flat hyperdescent). If $\mathrm{QCoh}(X)$ is presentable, we let $\mathrm{Mod}^\flat_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ denote the inverse image of the subcategory $\mathrm{QStk}^\flat(X) \subseteq \mathrm{QStk}(X)$ under the functor Φ_X of Construction 8.5. Note that Φ_X restricts to a functor $\Phi^\flat_X : \mathrm{Mod}^\flat_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{QStk}^\flat(X)$. If X is representable by a quasi-affine spectral Deligne-Mumford stack, then the functor $\Phi^\flat_X$ is an equivalence (Corollary 8.9).

Theorem 8.11. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor satisfying the following pair of conditions: Assume that:*

- *(a) There exists a connective $\mathbb{E}_\infty$-ring A and a map $f : \mathrm{Spec}^{\mathrm{f}} A \to X$ which is an effective epimorphism with respect to the flat topology.*
- *(b) The diagonal $X \to X \times X$ is representable and quasi-affine (that is, X is quasi-geometric in the sense of Definition VIII.3.3.1).*

(These conditions are satisfied if X is a geometric stack, in the sense of Definition VIII.3.4.1.) Then $\mathrm{QCoh}(X)$ is a presentable ∞-category, and the functor $\Phi^\flat_X : \mathrm{Mod}^\flat_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{QStk}^\flat(X)$ admits a fully faithful right adjoint.

Lemma 8.12. *Let $u : X \to Y$ be a representable, quasi-affine natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and assume that $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$ are presentable ∞-categories. Then the forgetful functor $\theta : \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{Mod}_{\mathrm{QCoh}(Y)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ carries $\mathrm{Mod}^\flat_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ to $\mathrm{Mod}^\flat_{\mathrm{QCoh}(Y)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.*

Proof. Let $\mathcal{C} \in \mathrm{Mod}^\flat_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. Choose a connective $\mathbb{E}_\infty$-ring A and a point $\eta \in Y(A)$; we wish to show that the A-linear ∞-category $\eta^*\theta(\mathcal{C})$ satisfies flat descent. For every map of $\mathbb{E}_\infty$-rings $A \to B$, let X_B denote the fiber product $\mathrm{Spec}^{\mathrm{f}} B \times_Y X$ and η_B the image of η in $Y(B)$. Using Corollary VIII.3.2.6, we obtain a canonical equivalence

$$\eta^*_B \theta(\mathcal{C}) \simeq \mathrm{QCoh}(X_B) \otimes_{\mathrm{QCoh}(X)} \mathcal{C}.$$

Let $\Gamma(X_B; \bullet)$ denote the right adjoint to the functor Φ_{X_B}, and let $\mathcal{C}_B = \Phi_{X_B}(\mathrm{QCoh}(X_B) \otimes_{\mathrm{QCoh}(X)} \mathcal{C})$ denote the pullback of $\Phi_X(\mathcal{C})$ to X_B. Since u is quasi-affine, the functor Φ_{X_B} is an equivalence of ∞-categories

(Corollary 8.9). It follows that we can recover $\mathrm{QCoh}(X_B) \otimes_{\mathrm{QCoh}(X)} \mathcal{C}$ as the global sections $\Gamma(X_B; \mathcal{C}_B)$. Now suppose we are given a flat hypercovering $B^{\bullet}$ of B; we wish to show that the induced map

$$\mathrm{QCoh}(X_B) \otimes_{\mathrm{QCoh}(X)} \mathcal{C} \to \varprojlim \mathrm{QCoh}(X_{B^{\bullet}}) \otimes_{\mathrm{QCoh}(X)} \mathcal{C}.$$

Equivalently, we must show that the canonical map

$$\theta : \Gamma(X_B; \mathcal{C}_B) \to \varprojlim \Gamma(X_{B^{\bullet}}, \mathcal{C}_{B^{\bullet}})$$

is an equivalence of ∞-categories. Since u is quasi-affine, the functor X_B is representable by a quasi-affine spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For each object $U \in \mathcal{X}$, let $X_{B,U}$ denote the functor represented by $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U)$, and set $X_{B^{\bullet},U} = X_{B,U} \times_{X_B} X_{B^{\bullet}}$. Let $\mathcal{C}_{B,U}$ denote the pullback of $\mathcal{C}_B$ to $X_{B,U}$, and define $\mathcal{C}_{B^{\bullet},U}$ similarly. We will prove that the map

$$\theta_U : \Gamma(X_{B,U}; \mathcal{C}_B) \to \varprojlim \Gamma(X_{B^{\bullet},U}, \mathcal{C}_{B^{\bullet},U})$$

is an equivalence for each $U \in \mathcal{X}$. The collection of those $U \in \mathcal{X}$ for which θ_U is an equivalence is stable under colimits in $\mathcal{X}$. We may therefore reduce to the case where $U \in \mathcal{X}$ is affine, in which case the desired result follows from our assumption that $\mathcal{C}$ belongs to $\mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. □

Proof of Proposition 8.11. The proof is essentially identical to that of Theorem 8.6. Choose a map $f : \mathrm{Spec}^{\mathrm{f}} A \to X$ as in (a), and let $X_{\bullet}$ be the Čech nerve of f. Let $\mathcal{Z}^{\bullet}$ denote the cosimplicial ∞-category given by $\mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X_{\bullet})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X)} & \longrightarrow & \varprojlim \mathcal{Z}^{\bullet} \\ \downarrow{\scriptstyle \Phi^{\flat}_X} & & \downarrow \\ \mathrm{QStk}^{\flat}(X) & \longrightarrow & \varprojlim \mathrm{QStk}^{\flat}(X_{\bullet}). \end{array}$$

Since f is an effective epimorphism with respect to the flat topology, the canonical map $|X_{\bullet}| \to X$ induces an equivalence after sheafification for the flat topology; it follows from Proposition 5.13 that the lower horizontal map is an equivalence of ∞-categories. Using (b), we see that each X_n is quasi-affine; it follows from Corollary 8.9 that the right vertical map is an equivalence. Combining these facts, we can identify $\Phi^{\flat}_X$ with the upper horizontal map

$$\mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \varprojlim \mathcal{Z}^{\bullet}.$$

Let us extend $\mathcal{Z}^{\bullet}$ to an augmented cosimplicial object $\overline{\mathcal{Z}}^{\bullet}$ with $\overline{\mathcal{Z}}^{-1} = \mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. We will complete the proof by showing that $\overline{\mathcal{Z}}^{\bullet}$ satisfies the hypotheses of Corollary A.6.2.4.3:

(1) Let $f^* : \mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{LinCat}^{\flat}_A$ be the pullback functor associated to f. We claim that the ∞-category $\mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ admits limits of f^*-split cosimplicial objects, which are preserved by the functor f^*. Note that f^* is the restriction of a forgetful functor $U : \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{LinCat}_A$. As in the proof of Theorem 8.6, we see that $\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ admits small limits and that the forgetful functor $\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{LinCat}_B$ preserves small limits, for every point $\eta \in X(B)$. Since $\mathrm{LinCat}^{\flat}_B$ is closed under small limits in LinCat_B (Remark 7.3), we conclude that $\mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ is stable under small limits in $\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ and also that f^* preserves small limits.

(2) For every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_+$, we claim that the diagram σ :

$$\begin{array}{ccc} \mathcal{X}^m & \longrightarrow & \mathcal{X}^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{X}^n & \longrightarrow & \mathcal{X}^{n+1} \end{array}$$

is right adjointable. For every functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, let $Q(Y) = \mathrm{Mod}_{\mathrm{QCoh}(Y)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. The proof of Theorem 8.6 shows that the diagram τ :

$$\begin{array}{ccc} Q(X_m) & \longrightarrow & Q(X_{m+1}) \\ \downarrow & & \downarrow \\ Q(X_n) & \longrightarrow & Q(X_{n+1}) \end{array}$$

is right adjointable. To complete the proof, it will suffice to show that for each $n \geq 0$, the canonical map

$$F : \mathrm{Mod}_{\mathrm{QCoh}(X_n)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{Mod}_{\mathrm{QCoh}(X_{n+1})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$$

admits a right adjoint G which carries $\mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X_{n+1})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ to $\mathrm{Mod}^{\flat}_{\mathrm{QCoh}(X_n)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. Since the projection $X_{n+1} \to X_n$ is quasi-affine, this follows from Lemma 8.12.

□

Definition 8.13. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We will say that an object $M \in \mathrm{QCoh}(X)$ is *perfect* if, for every point $\eta \in X(A)$, the image $\eta^* M \in \mathrm{Mod}_A$ is a perfect A-module.

Definition 8.14. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We will say that X is a *perfect stack* if the following conditions are satisfied:

(a) The functor X is a sheaf for the flat topology.

(b) There exists a connective $\mathbb{E}_\infty$-ring A and a map $\mathrm{Spec}^{\mathrm{f}} A \to X$ which is an effective epimorphism for the flat topology.

(c) The diagonal map $X \to X \times X$ is quasi-affine.

(d) The ∞-category $\mathrm{QCoh}(X)$ is compactly generated.

(e) The unit object $\mathcal{O}_X \in \mathrm{QCoh}(X)$ is compact (equivalently, the global sections functor $\Gamma(X, \bullet)$: $\mathrm{QCoh}(X) \to \mathrm{Sp}$ commutes with filtered colimits).

Remark 8.15. Definition 8.14 is a variant of the notion of perfect stack introduced by Ben-Zvi, Francis, and Nadler in [2]. The definition given here is slightly more general: we require that the diagonal of X is quasi-affine rather than affine.

Lemma 8.16. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $M \in \mathrm{QCoh}(X)$. The following conditions are equivalent:*

(1) *The object M is perfect.*

(2) *The object M is dualizable with respect to the symmetric monoidal structure on $\mathrm{QCoh}(X)$.*

If X is perfect, then these conditions are also equivalent to:

(3) *The object M is a compact in $\mathrm{QCoh}(X)$.*

Proof. The equivalence of (1) and (2) is Proposition VIII.2.7.28. Assume that X is perfect. We first prove that (2) ⇒ (3). Assume that M admits a dual $M^\vee$. Then for any filtered diagram $\{N_\alpha\}$ in $\mathrm{QCoh}(X)$, we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{QCoh}(X)}(M, \varinjlim N_\alpha) &\simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, M^\vee \otimes \varinjlim N_\alpha) \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \varinjlim (M^\vee \otimes N_\alpha)) \\ &\simeq \varinjlim \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, M^\vee \otimes N_\alpha) \\ &\simeq \varinjlim \mathrm{Map}_{\mathrm{QCoh}(X)}(M, N_\alpha) \end{aligned}$$

so that (3) is satisfied. We complete the proof by showing that (3) $\Rightarrow$ (1). Let $\eta \in X(A)$, we wish to show that $\eta^*(M)$ is a perfect A-module. The point η determines a map $f : Y \to X$, where Y is the functor corepresented by A. Since X is quasi-geometric, the map f is quasi-affine, so that $f_* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ commutes with small colimits (Proposition VIII.2.5.12). It follows that $\eta^* \simeq f^*$ preserves compact objects (Proposition T.5.5.7.2), so that $\eta^*(M)$ is a perfect A-module as desired. $\square$

Remark 8.17. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a perfect stack and let $\mathcal{C} \in \mathrm{LMod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. If $M \in \mathrm{QCoh}(X)$ is perfect and $N \in \mathcal{C}$ is compact, then $M \otimes N$ is a compact object of $\mathcal{C}$. Indeed, for any filtered diagram $\{N'_\alpha\}$ in $\mathcal{C}$, we have homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(M \otimes N, \varinjlim N'_\alpha) &\simeq \mathrm{Map}_{\mathcal{C}}(N, M^\vee \otimes \varinjlim N'_\alpha) \\ &\simeq \mathrm{Map}_{\mathcal{C}}(N, \varinjlim (M^\vee \otimes N'_\alpha)) \\ &\simeq \varinjlim \mathrm{Map}_{\mathcal{C}}(N, M^\vee \otimes N'_\alpha) \\ &\simeq \varinjlim \mathrm{Map}_{\mathcal{C}}(M \otimes N, N'_\alpha). \end{aligned}$$

Proposition 8.18. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a perfect stack, and let $\mathcal{C} \in \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. If $\mathcal{C}$ is compactly generated, then $\mathcal{C}$ lies in the essential image of the fully faithful embedding $\mathrm{QStk}^\flat(X) \to \mathrm{LMod}^\flat_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of Theorem 8.11.*

Proof. We first show that $\mathcal{C} \in \mathrm{LMod}^\flat_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. Let $\eta \in X(A)$; we wish to show that $\eta^* \mathcal{C} \in \mathrm{LinCat}_A$ satisfies flat descent. Note that $\eta^* \mathcal{C} \simeq \mathrm{Mod}_A \otimes_{\mathrm{QCoh}(X)} \mathcal{C}$. Since X is perfect, $\mathrm{QCoh}(X)$ is compactly generated and the collection of compact objects in $\mathrm{QCoh}(X)$ is stable under tensor products (Lemma 8.16). Moreover, the ∞-categories $\mathcal{C}$ and Mod_A are also compactly generated, and the tensor product functors

$$\mathrm{Mod}_A \times \mathrm{QCoh}(X) \to \mathrm{QCoh}(X) \qquad \mathrm{QCoh}(X) \times \mathcal{C} \to \mathcal{C}$$

preserve compact objects. It follows that $\eta^* \mathcal{C}$ is compactly generated (see Remark A.6.3.1.8), and therefore satisfies flat hyperdescent (Corollary 6.11).

Let $\Phi^\flat_X$ be the functor of Theorem 8.11 and let Γ denote its right adjoint. To show that $\mathcal{C}$ lies in the essential image of Γ, it will suffice to show that the unit map $\mathcal{C} \to (\Gamma \circ \Phi^\flat_X)(\mathcal{C})$ is an equivalence. Choose $f : X_0 \to X$ as in the proof of Theorem 8.11, and let $X_\bullet$ be the Čech nerve of f. Unwinding the definitions, we must show that the map

$$\mathcal{C} \to \varprojlim \mathrm{QCoh}(X_\bullet) \otimes_{\mathrm{QCoh}(X)} \mathcal{C}$$

is an equivalence. This follows immediately from Theorem 6.10, since $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$. $\square$

For every perfect stack X, we let $\Gamma(X, \bullet) : \mathrm{QStk}^\flat(X) \to \mathrm{LMod}^{\mathrm{desc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ be the right adjoint to $\Phi^\flat_X$ whose existence is guaranteed by Theorem 8.11.

Proposition 8.19. *Let $f : Y \to X$ be a map between perfect stacks, and let $\mathcal{C} \in \mathrm{QStk}(X)$ have the following property: for every point $\eta \in X(A)$, the pullback $\eta^* \mathcal{C} \in \mathrm{LinCat}_A$ is compactly generated (in particular, $\mathcal{C} \in \mathrm{QStk}^\flat(X)$ by Corollary 6.11). Then the canonical map*

$$\theta : \mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(X)} \Gamma(X; \mathcal{C}) \to \Gamma(Y; f^* \mathcal{C})$$

is an equivalence in $\mathrm{Mod}_{\mathrm{QCoh}(Y)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Proof. Choose a map $X_0 \to X$ as in the proof of Theorem 8.11, and let $X_\bullet$ be its Čech nerve. For each $n \geq 0$, let $Y_n = X_n \times_X Y$, let $f_n : Y_n \to X_n$ be the projection map, and let $\mathcal{C}_n \in \mathrm{QStk}(X_n)$ be the pullback of $\mathcal{C}$. Then $\Gamma(X; \mathcal{C}) \simeq \varprojlim \Gamma(X_\bullet; \mathcal{C}_\bullet)$ and $\Gamma(Y; \mathcal{C}) \simeq \varprojlim \Gamma(Y_\bullet; f^*_\bullet \mathcal{C}_\bullet)$. Note that each X_n is quasi-affine. Using Corollary 8.9 we obtain equivalences $\Gamma(X_n; \mathcal{C}_n) \simeq \mathrm{QCoh}(X_n) \otimes_{\mathrm{QCoh}(X_0)} \Gamma(X_0; \mathcal{C}_0)$, from which it follows that $\Gamma(X_n; \mathcal{C}_n)$ is compactly generated. Using Theorem 6.10, we deduce that θ is a limit of maps

$$\theta_n : \mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(X)} \Gamma(X_n; \mathcal{C}_n) \to \Gamma(Y_n; f^*_n \mathcal{C}).$$

Since the maps $X_n \to X$ are quasi-affine, we can use Corollary VIII.3.2.6 to identify the domain of θ_n with $\mathrm{QCoh}(Y_n) \otimes_{\mathrm{QCoh}(X_n)} \Gamma(X_n; \mathcal{C}_n)$. We may therefore replace X by X_n and Y by Y_n and thereby reduce to the case where the ∞-category $\Gamma(X; \mathcal{C})$ is compactly generated.

It is easy to see that $\Phi_Y^{\flat}(\theta)$ is an equivalence. To show that θ is an equivalence, it will suffice to show that both $\mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(X)} \Gamma(X; \mathcal{C})$ and $\Gamma(Y; f^* \mathcal{C})$ lie in the essential image of the fully faithful embedding $\Gamma(Y; \bullet)$. This is obvious in the second case. In the first case, it follows from Proposition 8.18, since $\mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(X)} \Gamma(X; \mathcal{C})$ is compactly generated (Remark A.6.3.1.8). □

Remark 8.20. In the situation of Proposition 8.19, suppose that $\Gamma(X; \mathcal{C})$ is compactly generated. Then Remark A.6.3.1.8 guarantees that $\Gamma(Y; f^* \mathcal{C}) \simeq \mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(X)} \Gamma(X; \mathcal{C})$ is also compactly generated.

Variant 8.21. Let $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ denote the full subcategory spanned by those functors which are sheaves with respect to the flat topology. Let $f : Y \to X$ be a morphism in $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$, where X is a perfect stack and Y can be obtained as a small colimit of corepresentable functors (where the colimit is formed in the ∞-category $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$); it follows in particular that $\mathrm{QCoh}(Y)$ is presentable. Let $\mathcal{C} \in \mathrm{QStk}^{\flat}(X)$ be such that $\Gamma(X; \mathcal{C})$ is compactly generated. Then we have an equivalence

$$\mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(X)} \Gamma(X; \mathcal{C}) \simeq \Gamma(Y; f^* \mathcal{C})$$

in the following sense: for any $\mathcal{M} \in \mathrm{Mod}_{\mathrm{QCoh}(Y)}(\mathcal{P}r^{\mathrm{L}})$, the canonical map

$$\mathrm{Map}_{\mathrm{Mod}_{\mathrm{QCoh}(Y)}(\mathcal{Z})}(\mathcal{M}, \mathrm{QCoh}(Y) \otimes_{\mathrm{QCoh}(X)} \Gamma(X; \mathcal{C})) \to \mathrm{Map}_{\mathrm{QStk}(Y)}(\Phi_Y \mathcal{M}, f^* \mathcal{C})$$

is a homotopy equivalence. Indeed, both sides commute with passage to small colimits in Y (for the left side, this follows from Theorem 6.10). We may therefore reduce to the case where $Y = \mathrm{Spec}^{\mathrm{f}} A$, for some connective $\mathbb{E}_\infty$-ring A. In this case case the desired result follows from Proposition 8.19.

Corollary 8.22 (Ben-Zvi-Francis-Nadler)**.** *Suppose we are given a pullback diagram*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}$$

in $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$. Assume that X and X' are perfect, and that Y can be written as a small colimit of corepresentable functors. Then the induced functor

$$\theta : \mathrm{QCoh}(X') \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y')$$

is an equivalence. If, in addition, Y is perfect, then Y' is also perfect.

Proof. For each point $\eta \in X(A)$, we let X'_η denote the fiber product $X' \times_X \mathrm{Spec}^{\mathrm{f}} A$. Set $\mathcal{C} = \Phi_X(\mathrm{QCoh}(X')) \in \mathrm{QStk}(X)$. Using Theorem A.6.3.4.6, we deduce that $\mathcal{C}$ can be described by the formula $\eta^* \mathcal{C} = \mathrm{QCoh}(X'_\eta)$. Passing to the limit, we deduce that for any $\mathcal{M} \in \mathrm{Mod}_{\mathrm{QCoh}(Y)}(\mathcal{P}r^{\mathrm{L}})$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{\mathrm{QCoh}(Y)}(\mathcal{P}r^{\mathrm{L}})}(\mathcal{M}, \mathrm{QCoh}(Y')) \to \mathrm{Map}_{\mathrm{QStk}(Y)}(\Phi_Y \mathcal{M}, g^* \mathcal{C}).$$

Combining this observation with Variant 8.21, we deduce that θ is an equivalence.

Assume now that $\mathrm{QCoh}(Y)$ is perfect. Remark A.6.3.1.8 implies that the ∞-category $\mathrm{QCoh}(Y') \simeq \mathrm{QCoh}(X) \times_{\mathrm{QCoh}(Y)} \mathrm{QCoh}(X')$ is compactly generated. Moreover, for any pair of compact objects $M \in \mathrm{QCoh}(X')$, $N \in \mathrm{QCoh}(Y)$, the image of $M \otimes N$ in $\mathrm{QCoh}(Y')$ is compact. Taking M and N to be the unit objects of $\mathrm{QCoh}(X')$ and $\mathrm{QCoh}(Y)$, respectively, we conclude that the unit object of $\mathrm{QCoh}(Y')$ is compact. Since Y' is obviously satisfies conditions (a) through (c) of Definition 8.14, it follows that Y' is perfect as desired. □

9 Descent for t-Structures

In §5 and §7 we have studied the descent properties of the theory R-linear ∞-categories as the ring spectrum R is allowed to vary. In this section, we will consider analogous questions for R-linear ∞-categories equipped with t-structures. If we require all of our t-structures to be excellent, then the resulting theory satisfies descent with respect to the flat topology (Theorem 9.5). Consequently, we can globalize to obtain reasonable theory of t-structures on quasi-coherent stacks. Our main result (Theorem 9.12) is a local-to-global principle, which asserts that for every geometric stack X the following data are equivalent:

- Quasi-coherent stacks on X equipped with excellent t-structures.
- Presentable stable ∞-categories $\mathcal{C}$ equipped excellent t-structures and an action of $\mathrm{QCoh}(X)$, for which every flat object of $\mathrm{QCoh}(X)$ determines an exact functor from $\mathcal{C}$ to itself.

Notation 9.1. Let LinCat be the ∞-category of linear ∞-categories (see Notation 5.1): the objects of LinCat are pairs $(A, \mathcal{C})$, where $A \in \mathrm{Alg}^{(2)}$ is an $\mathbb{E}_2$-ring and $\mathcal{C}$ is a A-linear ∞-category. We let $\mathrm{LinCat}^{\mathrm{cn}}$ denote the full subcategory of LinCat spanned by those pairs $(A, \mathcal{C})$, where A is connective. Let PSet denote the category of partially ordered sets (which we do not require to be small). We define a functor T from the homotopy category of $\mathrm{LinCat}^{\mathrm{cn}}$ to PSet as follows: as follows:

(a) The functor T associates to each pair $(A, \mathcal{C}) \in \mathrm{LinCat}^{\mathrm{cn}}$ the partially ordered set of all presentable subcategories $\mathcal{C}_{\geq 0}$ which are closed under colimits and extensions (in other words, the collection of all accessible t-structures on $\mathcal{C}$; see Definition A.1.4.5.11).

(b) Given a morphism $\alpha : (A, \mathcal{C}) \to (B, \mathcal{D})$ in $\mathrm{LinCat}^{\mathrm{cn}}$, corresponding to an A-linear functor $f : \mathcal{C} \to \mathcal{D}$ the induced map $T(\alpha) : T(A, \mathcal{C}) \to T(B, \mathcal{D})$ carries a subcategory $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ to the smallest subcategory $\mathcal{D}_{\geq 0} \subseteq \mathcal{D}$ which contains $f(\mathcal{C}_{\geq 0})$ and is stable under colimits and extensions (it follows from Proposition A.1.4.5.11 that $\mathcal{D}_{\geq 0}$ is also presentable).

We can identify N(PSet) with a full subcategory of $\widehat{\mathcal{C}\mathrm{at}_\infty}$. Under this identification, T corresponds to a functor $\mathrm{LinCat}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which classifies a coCartesian fibration $p : \mathrm{LinCat}^t \to \mathrm{LinCat}^{\mathrm{cn}}$. We can think of LinCat^t with an ∞-category whose objects are triples $(A, \mathcal{C}, \mathcal{C}_{\geq 0})$ where A is a connective $\mathbb{E}_2$-ring, $\mathcal{C}$ is an A-linear ∞-category, and $\mathcal{C}_{\geq 0}$ determines an accessible t-structure on $\mathcal{C}$.

Recall that a t-structure on a presentable stable ∞-category $\mathcal{C}$ is said to be *excellent* if it is both left and right complete, and the truncation functors on $\mathcal{C}$ commute with filtered colimits (see Definition VII.6.9). We let $\mathrm{LinCat}^{\mathrm{exc}}$ denote the full subcategory of LinCat^t spanned by those triples $(A, \mathcal{C}, \mathcal{C}_{\geq 0})$ such that $\mathcal{C}_{\geq 0}$ determines an excellent t-structure on $\mathcal{C}$.

Note that for every morphism $\alpha : (A, \mathcal{C}) \to (B', \mathcal{D})$ in $\mathrm{LinCat}^{\mathrm{cn}}$, the map of partially ordered sets $T(\alpha) : T(k, \mathcal{C}) \to T(k', \mathcal{D})$ admits a right adjoint, which carries a full subcategory $\mathcal{D}_{\geq 0} \subseteq \mathcal{D}$ to its inverse image in $\mathcal{C}$ (this inverse image is clearly stable under colimits and extensions, and is presentable by virtue of Proposition T.5.5.3.12). This proves the following:

Lemma 9.2. *The map* $p : \mathrm{LinCat}^t \to \mathrm{LinCat}^{\mathrm{cn}}$ *is a Cartesian fibration. Moreover, a morphism*

$$(A, \mathcal{C}, \mathcal{C}_{\geq 0}) \to (A', \mathcal{C}', \mathcal{C}'_{\geq 0})$$

in LinCat^t *is p-Cartesian if and only if it induces an equivalence* $\mathcal{C}_{\geq 0} \simeq \mathcal{C} \times_{\mathcal{C}'} \mathcal{C}'_{\geq 0}$.

Lemma 9.3. *The composite functor*

$$\mathrm{LinCat}^t \xrightarrow{p} \mathrm{LinCat}^{\mathrm{cn}} \xrightarrow{q} \mathrm{Alg}^{(2),\mathrm{cn}}$$

is both a Cartesian fibration and a coCartesian fibration. Moreover, a morphism α in LinCat^t *is $(q \circ p)$-Cartesian ($(q \circ p)$-coCartesian) if and only if α is p-Cartesian (p-coCartesian) and $p(\alpha)$ is q-Cartesian (q-coCartesian).*

Proof. Combine Proposition T.2.4.1.3, Lemma 9.2, and Remark 5.2. □

Lemma 9.4. *Let $p : \mathrm{LinCat}^t \to \mathrm{Alg}^{(2),\mathrm{cn}}$ be the forgetful functor, and let $q : \mathrm{LinCat}^{\mathrm{exc}} \to \mathrm{Alg}^{(2),\mathrm{cn}}$ be the restriction of p. Then q is a Cartesian fibration and a coCartesian fibration. Moreover, a morphism α in $\mathrm{LinCat}^{\mathrm{exc}}$ is q-Cartesian (q-coCartesian) if and only if it is p-coCartesian.*

Proof. In view of Lemma 9.3, it will suffice to prove the following:

(a) Let $\alpha : (A, \mathcal{C}, \mathcal{C}_{\geq 0}) \to (B, \mathcal{D}, \mathcal{D}_{\geq 0})$ be an p-Cartesian morphism in LinCat^t. If $\mathcal{D}_{\geq 0}$ determines an excellent t-structure on $\mathcal{D}$, then $\mathcal{C}_{\geq 0}$ determines an excellent t-structure on $\mathcal{C}$.

(b) Let $\alpha : (A, \mathcal{C}, \mathcal{C}_{\geq 0}) \to (B, \mathcal{D}, \mathcal{D}_{\geq 0})$ be an p-coCartesian morphism in LinCat^t. If $\mathcal{C}_{\geq 0}$ determines an excellent t-structure on $\mathcal{C}$, then $\mathcal{D}_{\geq 0}$ determines an excellent t-structure on $\mathcal{D}$.

Assertion (a) is obvious, since the condition that α be p-Cartesian implies that α induces a t-exact equivalence $\mathcal{C} \to \mathcal{D}$. Assertion ($b$) is a special case of Proposition VII.6.20. □

Theorem 9.5. *Let $\chi : \mathrm{Alg}^{(2),\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}_\infty}$ classify the coCartesian fibration $q : \mathrm{LinCat}^{\mathrm{exc}} \to \mathrm{Alg}^{(2),\mathrm{cn}}$, and let χ_0 denote the restriction of χ to $\mathrm{CAlg}^{\mathrm{cn}}$. Then χ_0 is a hypercomplete sheaf with respect to the flat topology.*

Proof. Let $\chi_1 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classify the coCartesian fibration $\mathrm{LinCat}^\flat \to \mathrm{CAlg}^{\mathrm{cn}}$. Theorem VII.6.12 implies that the forgetful functor $\mathrm{LinCat}^t \to \mathrm{LinCat}^{\mathrm{cn}}$ carries $\mathrm{LinCat}^{\mathrm{exc}} \times_{\mathrm{Alg}^{(2),\mathrm{cn}}} \mathrm{CAlg}^{\mathrm{cn}}$ into $\mathrm{LinCat}^\flat$. This functor preserves coCartesian morphisms and therefore induces a natural transformation $\chi_0 \to \chi_1$. For every ∞-category $\mathcal{X}$, let $\chi_0^{\mathcal{X}} : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be the composition of χ_0 with the functor $\mathrm{Map}_{\widehat{\mathcal{C}\mathrm{at}}_\infty}(\mathcal{X}, \bullet)$, and define $\chi_1^{\mathcal{X}}$ similarly. We wish to prove that each $\chi_0^{\mathcal{X}}$ is a hypercomplete sheaf on $\mathrm{CAlg}_1^{\mathrm{cn}}$. According to Lemma 7.13, it will suffice to prove this result in the cases $\mathcal{X} = \Delta^0$ and $\mathcal{X} = \Delta^1$.

We begin by treating the case $\mathcal{X} = \Delta^0$. We have a natural transformation $\alpha : \chi_0^{\Delta^0} \to \chi_1^{\Delta^0}$, where $\chi_1^{\Delta^0}$ is a hypercomplete sheaf by Theorem 7.5. We will prove that $\chi_0^{\Delta^0}$ is a hypercomplete sheaf using Lemma VIII.3.1.20. Unwinding the definitions, we must show the following:

($*$) Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be an A-linear ∞-category with flat hyperdescent. Let $F : \mathrm{CAlg}_A^{\mathrm{cn}} \to \mathrm{N}(\widehat{\mathcal{S}\mathrm{et}})$ be the functor which associates to each connective A-algebra B the collection of excellent t-structures on $\mathrm{LMod}_B(\mathcal{C})$. Then F is a hypercomplete sheaf with respect to the flat topology on $\mathrm{CAlg}_A^{\mathrm{cn}}$.

Since F takes discrete values, it will suffice to show that F is a sheaf with respect to the flat topology. We will prove that F satisfies the hypotheses of Proposition VII.5.7. Remark VII.6.11 shows that F preserves finite products. To complete the proof, it will suffice to show that for every faithfully flat morphism $B \to B^0$ of connective A-algebras having Čech nerve $B^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}_A$, the canonical map $F(B) \to \varprojlim F(B^\bullet)$ is a homotopy equivalence. Replacing $\mathcal{C}$ by $\mathrm{LMod}_B(\mathcal{C})$, we can assume that $A = B$. Fix an object in the limit $\varprojlim_{[n] \in \mathbf{\Delta}} F(B^n)$, corresponding to a compatible family of excellent t-structures on the ∞-categories $\mathrm{LMod}_{B^n}(\mathcal{C})$. We wish to prove that there exists a unique t-structure on $\mathcal{C}$ such that

$$\mathrm{LMod}_{B^n}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{\geq 0} = \mathrm{LMod}_{B^n}(\mathcal{C})_{\geq 0}$$

for each $n \geq 0$. The uniqueness is clear: since B^0 is faithfully flat over A, Lemma VII.6.28 asserts that if such a t-structure on $\mathcal{C}$ exists, then we can recover $\mathcal{C}_{\geq 0}$ as the inverse image of $\mathrm{LMod}_{B^0}(\mathcal{C})_{\geq 0}$ under the free module functor $\mathrm{Free} : \mathcal{C} \to \mathrm{LMod}_{B^0}(\mathcal{C})$. This inverse image is presentable by Proposition T.5.5.3.12 and is obviously stable under colimits and extensions, and therefore determines an accessible t-structure on $\mathcal{C}$ (Proposition A.1.4.5.11). To complete the verification of ($*$), it will suffice to verify the following points:

(a) The subcategory $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ determines an excellent t-structure on $\mathcal{C}$.

(b) For each $n \geq 0$, the subcategory $\mathrm{LMod}_{B^n}(\mathcal{C})_{\geq 0}$ is generated (under colimits and extensions) by the essential image of $\mathcal{C}_{\geq 0}$.

Using Lemma VII.6.15, we conclude that for every morphism $[m] \to [n]$ in $\mathbf{\Delta}_+$, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{LMod}_{B^m}(\mathcal{C})^{op} & \longrightarrow & \mathrm{LMod}_{B^{m+1}}(\mathcal{C})^{op} \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{B^n}(\mathcal{C})^{op} & \longrightarrow & \mathrm{LMod}_{B^{n+1}}(\mathcal{C})^{op} \end{array}$$

is left adjointable. Since $\mathcal{C}$ satisfies flat descent, the augmented cosimplicial ∞-category $\mathrm{LMod}_{B^\bullet}(\mathcal{C})$ is a limit diagram in $\widehat{\mathcal{C}\mathrm{at}}_\infty$. It follows from Theorem A.6.2.4.2 that we can identify $\mathrm{LMod}_B(\mathcal{C})$ with the ∞-category of modules over a comonad U on $\mathrm{LMod}_{B^0}(\mathcal{C})$, where U is obtained by composing the face map $d^1 : \mathrm{LMod}_{B^0}(\mathcal{C}) \to \mathrm{LMod}_{B^1}(\mathcal{C})$ with the right adjoint to the face map $d^0 : \mathrm{LMod}_{B^0}(\mathcal{C}) \to \mathrm{LMod}_{B^1}(\mathcal{C})$. The right adjoint to d^0 is t-exact (Proposition VII.6.20), and the flatness of B^1 over B^0 implies that d^1 is also t-exact; it follows that U is a t-exact functor from $\mathrm{LMod}_{B^0}(\mathcal{C})$ to itself. Applying Proposition VII.6.20, we deduce that $\mathcal{C} \simeq \mathrm{LMod}_U(\mathrm{LMod}_{B^0}(\mathcal{C})^{op})^{op}$ admits a t-structure $(\mathrm{Free}^{-1}\,\mathrm{LMod}_{B^0}(\mathcal{C}), \mathrm{Free}^{-1}\,\mathrm{LMod}_{B^0}(\mathcal{C})_{\leq 0}) = (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Moreover, since $\mathrm{LMod}_{B^0}(\mathcal{C})$ is left and right complete, we conclude that the t-structure on $\mathcal{C}$ is left and right complete. Since Free preserves filtered colimits and $\mathrm{LMod}_{B^0}(\mathcal{C})_{\leq 0}$ is closed under filtered colimits, we conclude that $\mathcal{C}_{\leq 0} = \mathrm{Free}^{-1}\,\mathrm{LMod}_{B^0}(\mathcal{C})_{\leq 0}$ is closed under filtered colimits; this completes the proof of (a).

To prove (b), it suffices to treat the case where $n = 0$. Fix an object $M \in \mathrm{LMod}_{B^0}(\mathcal{C})_{\geq 0}$; we wish to show that M can be written as a colimit of objects belonging to $\mathrm{Free}(\mathcal{C}_{\geq 0})$. Let M_0 be the image of M in $\mathcal{C}$. We have $M \simeq B^0 \otimes_{B^0} M \simeq |\,\mathrm{Sqr}_{B^0}(B^0, M)_\bullet|$, where each $\mathrm{Sqr}_{B^0}(B^0, M)_n \simeq \mathrm{Free}((B^0)^{\otimes n} \otimes M_0)$. It will therefore suffice to show that each $(B^0)^{\otimes n} \otimes M_0$ belongs to $\mathcal{C}_{\geq 0}$. Since B^0 is connective, it suffices to show that $M_0 \in \mathcal{C}_{\geq 0}$: that is, that $\mathrm{Free}(M_0) \simeq U(M) \in \mathrm{LMod}_{B^0}(\mathcal{C})_{\geq 0}$. This follows from the t-exactness of the functor U.

We now complete the proof of Theorem 9.5 by showing that $\chi_0^{\Delta^1}$ is a hypercomplete sheaf with respect to the flat topology. There is an evident natural transformation $\beta : \chi_0^{\Delta^1} \to \chi_0^{\partial\,\Delta^1} \times_{\chi_1^{\partial\,\Delta^1}} \chi_1^{\Delta^1}$. Using Theorem 7.5 together with the first part of the proof, we deduce that the fiber product $\chi^{\partial\,\Delta^1} \times_{\chi_0^{\partial\,\Delta^1}} \chi_0^{\Delta^1}$ is a hypercomplete sheaf for the flat topology. According to Lemma VIII.3.1.20, we are reduced to proving the following:

$(*')$ Let A be a connective E_∞-ring, let $f : \mathcal{C} \to \mathcal{D}$ be a map of A-linear ∞-categories, and let $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ and $\mathcal{D}_{\geq 0} \subseteq \mathcal{D}$ be excellent t-structures. Let $F' : \mathrm{CAlg}_A \to \mathcal{S}$ be defined by the formula

$$F'(B) = \begin{cases} \Delta^0 & \text{if } f(\mathrm{LMod}_B(\mathcal{C})_{\geq 0}) \subseteq \mathrm{LMod}_B(\mathcal{D})_{\geq 0} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then F' is a hypercomplete sheaf with respect to the flat topology.

This is an immediate consequence of Proposition VII.6.30. □

Construction 9.6. Let $\chi_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be the functor appearing in Theorem 9.5 (given by $\chi_0(A) = \mathrm{LinCat}_A^{\mathrm{exc}}$). We let

$$\mathrm{QStk}^{\mathrm{exc}} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$$

be a left Kan extension of χ_0 along the Yoneda embedding $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op}$. Equivalently, we can describe $\mathrm{QStk}^{\mathrm{exc}}$ as the functor obtained by applying the construction of Remark VIII.2.7.7 to the coCartesian fibration

$$\mathrm{LinCat}^{\mathrm{exc}} \times_{\mathrm{Alg}^{(2),\mathrm{cn}}} \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}.$$

Given a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, we will refer to the $\mathrm{QStk}^{\mathrm{exc}}(X)$ as the *∞-category of quasi-coherent stacks on X with excellent t-structure.*

Remark 9.7. The forgetful functor $\mathrm{LinCat}^{\mathrm{exc}} \to \mathrm{LinCat}$ determines a natural transformation $\mathrm{QStk}^{\mathrm{exc}} \to \mathrm{QStk}$ of functors $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. This functor factors through the subfunctor $\mathrm{QStk}^\flat \subseteq \mathrm{QStk}$, by Theorem VII.6.12.

Notation 9.8. If $f : X \to Y$ is a morphism in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, we let f^* denote the induced functor $\mathrm{QStk}^{\mathrm{exc}}(Y) \to \mathrm{QStk}^{\mathrm{exc}}(X)$.

Remark 9.9. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We can informally think of an object $\mathcal{C} \in \mathrm{QStk}^{\mathrm{exc}}(X)$ as a rule which assigns to every point $\eta \in X(A)$ a pair $(\mathcal{C}_\eta, \mathcal{C}_{\eta,\geq 0})$, where $\mathcal{C}_\eta$ is an A-linear ∞-category and $\mathcal{C}_{\eta,\geq 0}$ determines an excellent t-structure on $\mathcal{C}_\eta$.

Remark 9.10. Let $f : X \to Y$ be a map in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which induces an ∞-connective map after sheafification with respect to the flat topology. It follows from Theorem 9.5 that the induced functor $f^* : \mathrm{QStk}^{\mathrm{exc}}(Y) \to \mathrm{QStk}^{\mathrm{exc}}(X)$ is an equivalence of ∞-categories.

Notation 9.11. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ denote the symmetric monoidal ∞-category introduced in Notation VIII.4.6.1: the objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ are pairs $(\mathcal{C}, \mathcal{C}_{\geq 0})$, where $\mathcal{C}$ is a presentable stable ∞-category and $\mathcal{C}_{\geq 0}$ determines an accessible t-structure on $\mathcal{C}$.

If X is a geometric stack, then we can regard $\mathrm{QCoh}(X)$ (with its natural t-structure) as a commutative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$. Let $\mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$ denote the full subcategory of $\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$ spanned by those pairs $(\mathcal{C}, \mathcal{C}_{\geq 0})$, where $\mathcal{C}_{\geq 0}$ determines an excellent t-structure on $\mathcal{C}$.

Let A be a connective $\mathbb{E}_\infty$-ring and suppose we are given a point $\eta \in X(A)$. Then η determines an affine map $\mathrm{Spec}^{\mathrm{f}} A \to X$. Since X is geometric, this map is affine and determines a commutative algebra object $\mathcal{A} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$. If $(\mathcal{C}, \mathcal{C}_{\geq 0}) \in \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$, Theorem A.6.3.4.6 gives a canonical equivalence

$$\begin{aligned} \mathrm{Mod}_A \otimes_{\mathrm{QCoh}(X)} \mathcal{C} &\simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X)) \otimes_{\mathrm{QCoh}(X)} \mathcal{C} \\ &\simeq \mathrm{LMod}_{\mathcal{A}}(\mathcal{C}). \end{aligned}$$

Since $\mathcal{A}$ is connective, Proposition VII.6.20 implies that $\mathrm{LMod}_{\mathcal{A}}(\mathcal{C})$ inherits a t-structure from that of $\mathcal{C}$, which is excellent if the t-structure on $\mathcal{C}$ is excellent. This construction is functorial in A, and gives us a lifting of the functor $\Phi_X : \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}) \to \mathrm{QStk}(\mathcal{C})$ of Construction 8.5 to a functor

$$\Psi^{\mathrm{exc}}_X : \mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}) \to \mathrm{QStk}^{\mathrm{exc}}(\mathcal{C}).$$

We can now formulate our main result.

Theorem 9.12. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack. Then the functor $\Phi^{\mathrm{exc}}_X : \mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}) \to \mathrm{QStk}^{\mathrm{exc}}(X)$ admits a fully faithful right adjoint $\Gamma(X, \bullet) : \mathrm{QStk}^{\mathrm{exc}}(X) \to \mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$. Moreover, an object $(\mathcal{C}, \mathcal{C}_{\geq 0}) \in \mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$ belongs to the essential image of $\Gamma(X; \bullet)$ if and only if the following condition is satisfied:*

- $(*)$ *For every flat quasi-coherent sheaf $M \in \mathrm{QCoh}(X)$, the construction $N \mapsto M \otimes N$ determines a t-exact functor from $\mathcal{C}$ to itself.*

Proof. Since X is a geometric stack, we can choose a map $f : X_0 \to X$ which is an effective epimorphism of sheaves with respect to the flat topology, where $X_0 = \mathrm{Spec}^{\mathrm{f}} A$ is affine. Let $X_\bullet$ be the Čech nerve of f, so that $X_\bullet \simeq \mathrm{Spec}\, A^\bullet$ for some cosimplicial $\mathbb{E}_\infty$-ring $A^\bullet$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}) & \xrightarrow{\Phi^{\mathrm{exc}}_X} & \mathrm{QStk}^{\mathrm{exc}}(X) \\ \downarrow & & \downarrow \\ \varprojlim \mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X_\bullet)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}) & \longrightarrow & \varprojlim \mathrm{QStk}^{\mathrm{exc}}(X_\bullet). \end{array}$$

The bottom horizontal map is an equivalence since each X_n is affine, and the right vertical map is an equivalence since $\mathrm{QStk}^{\mathrm{exc}}$ is a sheaf with respect to the flat topology (Theorem 9.5). It follows that Φ^{exc}_X is equivalent to the functor

$$\mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}) \to \varprojlim \mathrm{LMod}^{\mathrm{exc}}_{\mathrm{QCoh}(X_\bullet)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}).$$

We first construct the right adjoint $\Gamma(X;\bullet)$ to Φ_X^{exc}. Fix an object of $\varprojlim \mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X_\bullet)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$, consisting of a cosimplicial $A^\bullet$-linear ∞-category $\mathcal{C}^\bullet$ and a compatible collection of full subcategories $\mathcal{C}^n_{\geq 0} \subseteq \mathcal{C}^n$ which determine an excellent t-structure on each $\mathcal{C}^n$. Let $\mathcal{C} = \varprojlim \mathcal{C}^\bullet$, regarded as an ∞-category left-tensored over $\mathrm{QCoh}(X)$, and let $\mathcal{C}_{\geq 0} = \varprojlim \mathcal{C}^\bullet_{\geq 0}$, regarded as a full subcategory of $\mathcal{C}$. We will abuse notation by identifying $(\mathcal{C}^\bullet, \mathcal{C}^\bullet_{\geq 0})$ with an object of $\mathrm{QStk}^{\mathrm{exc}}(X)$. It is easy to see that for $(\mathcal{D}, \mathcal{D}_{\geq 0}) \in \mathrm{Mod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$, the mapping space $\mathrm{Map}_{\mathrm{QStk}^{\mathrm{exc}}(X)}(\Phi_X^{\mathrm{exc}}(\mathcal{D}, \mathcal{D}_{\geq 0}), (\mathcal{C}^\bullet, \mathcal{C}^\bullet_{\geq 0}))$ is equivalent to the full subcategory of

$$\mathrm{Map}_{\mathrm{QStk}(X)}(\Psi_X(\mathcal{D}), \mathcal{C}^\bullet) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})}(\mathcal{D}, \mathcal{C})$$

spanned by those functors which carry $\mathcal{D}_{\geq 0}$ to $\mathcal{C}_{\geq 0}$. To complete the proof of the existence of the functor $\Gamma(X;\bullet)$ (and to verify that $\Gamma(X; (\mathcal{C}^\bullet, \mathcal{C}^\bullet_{\geq 0}) \simeq (\mathcal{C}, \mathcal{C}_{\geq 0}))$, it will suffice to show that $\mathcal{C}_{\geq 0}$ determines an excellent t-structure on $\mathcal{C}$.

It follows from Proposition T.5.5.3.13 that $\mathcal{C}_{\geq 0}$ is a presentable ∞-category, which is evidently stable under colimits and extensions in $\mathcal{C}$, and therefore determines an accessible t-structure on $\mathcal{C}$ (Proposition A.1.4.5.11). Let $F : \mathcal{C} \to \mathcal{C}^0$ be the evident functor and $G : \mathcal{C}^0 \to \mathcal{C}$ a right adjoint to F. Applying Theorem A.6.2.4.2 to the cosimplicial ∞-category $\mathcal{C}^\bullet$, we deduce that $\mathcal{C}$ can be identified with the ∞-category of comodules over the comonad $U = F \circ G$ on $\mathcal{C}^0$, which is given by the composite functor

$$\mathcal{C}^0 \xrightarrow{\phi} \mathcal{C}^1 \xrightarrow{\psi} \mathcal{C}^0 .$$

We claim that each of these functors is t-exact. For the functor ψ, this is obvious (since we can identify ψ with the forgetful functor $\mathrm{LMod}_{A^1}(\mathcal{C}^0) \to \mathcal{C}^0$). For the functor ϕ, we must instead contemplate the base change functor $\mathcal{C}^0 \to \mathrm{LMod}_{A^1}(\mathcal{C}^0)$ given by $M \mapsto A^1 \otimes_{A^0} M$. Since A^1 is flat over A^0, the underlying functor from $\mathcal{C}^0$ to itself can be realized as a filtered colimit of functors of the form $M \mapsto M^n$ (Theorem A.7.2.2.15), and is therefore t-exact (since the t-structure on $\mathcal{C}^0$ is assumed to be excellent). Applying Proposition VII.6.20, we deduce that the t-structure determined by $\mathcal{C}_{\geq 0}$ is both right and left complete. Moreover, $\mathcal{C}_{\leq 0}$ can be identified with the fiber product $\mathcal{C} \times_{\mathcal{C}^0} \mathcal{C}^0_{\leq 0}$, and is therefore closed under filtered colimits (since the t-structure on $\mathcal{C}^0$ is assumed to be excellent). This completes the verification that Φ_X^{exc} admits a right adjoint.

We now prove that the functor $\Gamma(X;\bullet)$ is fully faithful. In other words, we claim that for every object $(\mathcal{C}^\bullet, \mathcal{C}^\bullet_{\geq 0}) \in \mathrm{QStk}(X)$, the counit map

$$\Phi_X^{\mathrm{exc}} \Gamma(X; \mathcal{C}^\bullet) \to (\mathcal{C}^\bullet, \mathcal{C}^\bullet_{\geq 0})$$

is an equivalence in $\mathrm{QStk}(X)$. Let $(\mathcal{C}, \mathcal{C}_{\geq 0})$ be as above and let $(\mathcal{D}, \mathcal{D}_{\geq 0})$ be its image in $\mathrm{QStk}^{\mathrm{exc}}(X_0)$; we wish to show that the canonical map $\theta : (\mathcal{D}, \mathcal{D}_{\geq 0}) \to (\mathcal{C}^0, \mathcal{C}^0_{\geq 0})$ is an equivalence in $\mathrm{QStk}^{\mathrm{exc}}(X_0)$. Using Notation 9.11 and Theorem 8.11, we see that the underlying map of ∞-categories $\mathcal{D} \to \mathcal{C}^0$ is an equivalence. To complete the proof, it will suffice to show that if $M \in \mathcal{D}$ is such that $\theta(M) \in \mathcal{C}^0_{\geq 0}$, then $M \in \mathcal{D}_{\geq 0}$. Write $\mathcal{A} = f_* \mathcal{O}_X \in \mathrm{CAlg}(\mathrm{QCoh}(X_0))$, so that $\mathcal{D} \simeq \mathrm{LMod}_{\mathcal{A}}(\mathcal{C})$ (see Notation 9.11) is endowed with the t-structure described in Proposition VII.6.20. It will therefore suffice to show that the image of M in $\mathcal{C}$ belongs to $\mathcal{C}_{\geq 0}$: in other words, we wish to prove that the functor $G : \mathcal{C}^0 \to \mathcal{C}$ is right t-exact. This is equivalent to the assertion (verified above) that $U = F \circ G : \mathcal{C}^0 \to \mathcal{C}^0$ is right t-exact.

We next prove that the essential image of the functor $\Gamma(X;\bullet)$ consists of objects satisfying condition $(*)$. Suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0}) \simeq \Gamma(X; \mathcal{C}^\bullet)$ is as above, that $M \in \mathrm{QCoh}(X)$ is flat and that $N \in \mathcal{C}_{\leq 0}$; we wish to prove that $M \otimes N \in \mathcal{C}_{\leq 0}$. According to Proposition VII.6.20, this is equivalent to the requirement that $F(M \otimes N) \simeq f^* M \otimes F(N)$ belongs to $\mathcal{C}^0_{\leq 0}$. Since M is flat, $f^* M \in \mathrm{Mod}_A$ is flat and can therefore be realized as a filtered colimit of free A-modules of finite rank (Theorem A.7.2.2.15). Since $\mathcal{C}^0_{\leq 0}$ is stable under filtered colimits, we are reduced to proving that $A^n \otimes F(N) \simeq F(N)^n \in \mathcal{C}^0_{\leq 0}$ for $n \geq 0$, which follows from our assumption that $N \in \mathcal{C}_{\leq 0}$.

We now complete the proof by showing that if an object $(\mathcal{C}, \mathcal{C}_{\geq 0}) \in \mathrm{LMod}^{\mathrm{exc}}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$ satisfies condition $(*)$, then $(\mathcal{C}, \mathcal{C}_{\geq 0})$ lies in the essential image of the functor $\Gamma(X;\bullet)$. Let $\mathcal{A} = f_* A \in \mathrm{CAlg}(\mathrm{QCoh}(X))$, and let $\mathcal{A}^\bullet$ be the Čech nerve of the unit map $\mathcal{O}_X \to \mathcal{A}$ in $\mathrm{CAlg}(\mathrm{QCoh}(X))^{\mathrm{op}}$, so that $\mathrm{QCoh}(X_n) \simeq \mathrm{Mod}_{\mathcal{A}^n}(\mathrm{QCoh}(X))$. Let us identify $\Phi_X^{\mathrm{exc}}(\mathcal{C}, \mathcal{C}_{\geq 0})$ with the cosimplicial object $(\mathcal{C}^\bullet, \mathcal{C}^\bullet_{\geq 0})$, where $\mathcal{C}^\bullet$ denotes the cosimplicial ∞-category given by $\mathrm{LMod}_{\mathcal{A}^\bullet}(\mathcal{C})$ and $\mathcal{C}^\bullet_{\geq 0}$ determines the t-structure on $\mathcal{C}^\bullet$ described in Proposition VII.6.20. In view of the description of $\Gamma(X; \mathcal{C}^\bullet)$ given above, it will suffice to verify the following:

(a) The canonical map $\mathcal{C} \to \varprojlim \mathcal{C}^\bullet \simeq \varprojlim \mathrm{LMod}_{\mathcal{A}^\bullet}(\mathcal{C})$ is an equivalence of ∞-categories.

(b) The full subcategory $\mathcal{C}_{\geq 0}$ is the inverse image of $\mathcal{C}^0_{\geq 0}$ under the forgetful functor $F : \mathcal{C} \to \mathcal{C}^0$.

We first prove (b). Unwinding the definitions, we must show that an object $M \in \mathcal{C}$ belongs to $\mathcal{C}_{\geq 0}$ if and only if $\mathcal{A} \otimes M \in \mathcal{C}$ belongs to $\mathcal{C}_{\geq 0}$. The "only if" direction is obvious, since $\mathcal{A} \in \mathrm{QCoh}(X)_{\geq 0}$. To prove the "if" direction, suppose that $\mathcal{A} \otimes M \in \mathcal{C}_{\geq 0}$; we wish to show that $M \in \mathcal{C}_{\geq 0}$. Consider the cofiber sequence

$$\mathcal{A} \otimes \tau_{\geq 0} M \to \mathcal{A} \otimes \mathcal{M} \to \mathcal{A} \otimes \tau_{\leq -1} M.$$

Since $\mathcal{A}$ is connective, the first two terms belong to $\mathcal{C}_{\geq 0}$, so that $\mathcal{A} \otimes \tau_{\leq -1} M$ belongs to $\mathcal{C}_{\geq 0}$. Form a cofiber sequence

$$\mathcal{O}_X \to \mathcal{A} \to \mathcal{A}'$$

in $\mathrm{QCoh}(X)$. Since the map $f : X_0 \to X$ is affine and faithfully flat, the object $\mathcal{A}' \in \mathrm{QCoh}(X)$ is flat (see Lemma VII.6.22). Tensoring this cofiber sequence with $\tau_{\leq -1} M$, we obtain a cofiber sequence

$$\tau_{\leq -1} M \to \mathcal{A} \otimes \tau_{\leq -1} M \to \mathcal{A}' \otimes \tau_{\leq -1} M.$$

Since $\mathcal{A} \otimes \tau_{\leq -1} M \in \mathcal{C}_{\geq 0}$, we obtain for $n \geq 0$ an epimorphism

$$\pi_{-n}(\mathcal{A}' \otimes \tau_{\leq -1} M) \to \pi_{-n-1}(\tau_{\leq -1} M) \simeq \pi_{-n-1} M$$

in the abelian category $\mathcal{C}^\heartsuit$. We now prove by induction on n that $\tau_{\leq -1} M \in \mathcal{C}_{\leq -n-1}$. In the case $n = 0$, this is clear. For the inductive step, assume that that $\tau_{\leq -1} M \in \mathcal{C}_{\leq -n-1}$, so that $\mathcal{A} \otimes \tau_{\leq -1} M \in \mathcal{C}_{\leq -n-1}$ by virtue of (2). It follows that $\pi_{-n}(\mathcal{A}' \otimes \tau_{\leq -1} M) \simeq 0$, so the epimorphism above proves that $\pi_{-n-1}(\tau_{\leq -1} M) \simeq 0$ and therefore $\tau_{\leq -1} M \in \mathcal{C}_{\leq -n-2}$. We therefore conclude that $\tau_{\leq -1} M \in \bigcap_n \mathcal{C}_{\leq -n}$. Since the t-structure on $\mathcal{C}$ is right complete, $\tau_{\leq -1} M \simeq 0$ so that $M \in \mathcal{C}_{\geq 0}$ as desired.

It remains to prove (a). Using (the dual of) Corollary A.6.2.4.3, we are reduced to verifying the following:

(a') The forgetful functor $F : \mathcal{C} \to \mathcal{C}^0$ preserves limits of F-split cosimplicial objects.

(a'') The forgetful functor F is conservative.

To prove (a''), it suffices to show that if $M \in \mathcal{C}$ and $F(M) \simeq 0$, then $M \simeq 0$. Using assertion (b), we deduce that $F(M) \in \bigcap_n \mathcal{C}^0_{\geq n}$ implies that $M \in \bigcap_n \mathcal{C}_{\geq n}$, so that $M \simeq 0$ since $\mathcal{C}$ is left complete.

It remains to prove (a'). Assumption ($*$) guarantees that tensor product with $\mathcal{A}$ determines a t-exact functor from $\mathcal{C}$ to itself, which restricts to an exact functor T from the abelian category $\mathcal{C}^\heartsuit$ to itself. Similarly, tensor product with $\mathcal{A}'$ determines an exact functor T' from $\mathcal{C}^\heartsuit$ to itself, and we have an exact sequence of functors

$$0 \to \mathrm{id} \to T \to T' \to 0$$

It follows that the functor T is conservative.

Let $M^\bullet$ be an F-split cosimplicial object of $\mathcal{C}$. Then $\mathcal{A} \otimes M^\bullet$ is a split cosimplicial object of $\mathcal{C}$. For every integer n, we obtain a cosimplicial object $\pi_n M^\bullet$ of $\mathcal{C}^\heartsuit$, which has an associated cochain complex

$$\pi_n M^0 \xrightarrow{d_n} \pi_n M^1 \to \pi_n M^2 \to \cdots$$

After applying the functor T, this cochain complex becomes a split exact resolution of $\pi_n \varprojlim(\mathcal{A} \otimes M^\bullet)$. Since T is exact and conservative, we conclude that the cochain complex above is acyclic in positive degrees. Let

$M = \varprojlim M^{\bullet}$; it follows from Corollary A.1.2.4.10 that for each $n \in \mathbf{Z}$, the canonical map $M \rightarrow M^0$ induces an isomorphism $\pi_n M \simeq \ker(d_n)$ in the abelian category $\mathcal{C}^{\heartsuit}$. We claim that the induced map $F(M) \rightarrow \varprojlim F(M^{\bullet})$ is an equivalence: in other words, that $\mathcal{A} \otimes M \simeq \varprojlim(\mathcal{A} \otimes M^{\bullet})$. Since the t-structure on $\mathcal{C}$ is right and left complete, it will suffice to show that the induced map $T(\pi_n M) \simeq \pi_n(\mathcal{A} \otimes M) \rightarrow \pi_n \varprojlim(\mathcal{A} \otimes M^{\bullet})$ is an equivalence for each integer n. This follows immediately Corollary A.1.2.4.10 and the exactness of the functor T. □

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