Derived Algebraic Geometry VII: Spectral Schemes

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Contents

Introduction

Our goal in this paper is to introduce a variant of algebraic geometry, which we will refer to as *spectral algebraic geometry*. We will take as our starting point Grothendieck's theory of schemes. Recall that a scheme is a pair $(X, \mathcal{O}_X)$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of commutative rings on X, and the pair $(X, \mathcal{O}_X)$ is locally (with respect to the topology of X) isomorphic to the Zariski spectrum of a commutative ring. We can regard a commutative ring R as a set equipped with addition and multiplication maps

$$a : R \times R \to R \qquad m : R \times R \to R$$

which are required to satisfy certain identities. For certain applications (particularly in algebraic topology), it is useful to consider a variation, where R is equipped with a topology. Roughly speaking, a (connective) $\mathbb{E}_\infty$*-ring* is a space X equipped with continuous addition and multiplication maps

$$a : X \times X \to X \qquad m : X \times X \to X$$

which are required to satisfy the same identities up to (coherent) homotopy. The theory of $\mathbb{E}_\infty$-rings is a robust generalization of commutative algebra: in particular, the basic formal constructions needed to set up the theory of schemes (such as localization) make sense in the setting of $\mathbb{E}_\infty$-rings. We will use this observation to introduce the notion of a *spectral scheme*: a mathematical object which is obtained by "gluing together" a collection of (connective) $\mathbb{E}_\infty$-rings, just as a scheme is obtained by "gluing together" a collection of commutative rings.

The collection of commutative rings can be organized into a category Ring. That is, to every pair of commutative rings R and R', we can associate a set $\mathrm{Hom}_{\mathrm{Ring}}(R, R')$ of ring homomorphisms from R to R'. The analogous statement for $\mathbb{E}_\infty$-rings is more complicated: to every pair of $\mathbb{E}_\infty$-rings R and R', we can associate a *space* $\mathrm{Map}(R, R')$ of morphisms from R to R'. Moreover, these mapping spaces are equipped with a composition products $\mathrm{Map}(R, R') \times \mathrm{Map}(R', R'') \to \mathrm{Map}(R, R'')$, which are associative (and unital) up to coherent homotopy. To adequately describe this type of structure, it is convenient to use the language of ∞*-categories* developed in [40]. The collection of all $\mathbb{E}_\infty$-rings is naturally organized into an ∞-category which we will denote by CAlg, which contains (the nerve of) the category Ring as a full subcategory.

Let us now outline the contents of this paper. Recall first that the category of schemes can be realized as a subcategory of the category of *ringed spaces*, whose objects are pairs $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X. Our first goal will be to introduce a suitable ∞-categorical version of this category. In §1, we will introduce the ∞-category RingTop of *spectrally ringed* ∞*-topoi*. The objects of RingTop are given by pairs $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}_\mathcal{X}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

In §2, we will introduce the notion of a *spectral scheme*. The collection of spectral schemes is organized into an ∞-category, which we regard as a subcategory of the ∞-category RingTop of spectrally ringed ∞-topoi. This subcategory admits a number of characterizations (see Definitions 2.2, 2.7, and 2.27) which we will show to be equivalent. We will also explain the relationship between our theory of spectral schemes and the classical theory of schemes (Proposition 2.37).

Let $(X, \mathcal{O}_X)$ be a scheme. Recall that X is said to be *quasi-compact* if every open covering of X has a finite subcovering, and *quasi-separated* if the collection of quasi-compact open subsets of X is closed under pairwise intersections. In §3, we will generalize these conditions to our ∞-categorical setting by introducing the notion of a *coherent* ∞*-topos*. If $\mathcal{X}$ is the ∞-topos of sheaves on a topological space X, then $\mathcal{X}$ is coherent if and only if X is quasi-compact and quasi-separated. For every spectral scheme $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, the ∞-topos $\mathcal{X}$ is *locally coherent* (because coherence is automatic in the affine case), and the coherence of $\mathcal{X}$ is an important hypothesis for almost any nontrivial application.

Our theory of coherent ∞-topoi is an adaptation of the classical theory of coherent topoi (see, for example, [28]). A theorem of Deligne asserts that every coherent topos $\mathcal{X}$ has enough points: that is, that there exists a collection of geometric morphisms $\{f_\alpha : \mathcal{S}\mathrm{et} \to \mathcal{X}\}$ such that a morphism ϕ in $\mathcal{X}$ is invertible if and only if each pullback $f_\alpha^*(\phi)$ is a bijection of sets. In §4, we will prove an ∞-categorical analogue of this statement (Theorem 4.1). As an application, we prove a connectivity result for the geometric realization of a hypercovering (Theorem 4.20) which is useful in §5.

To every connective $\mathbb{E}_\infty$-ring R, one can associate a spectral scheme $\mathrm{Spec}_{\mathrm{Z}}(R) \in \mathrm{SpSch}$. Consequently, every spectral scheme $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ represents a functor X on the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings, given by the formula $X(R) = \mathrm{Map}_{\mathrm{SpSch}}(\mathrm{Spec}_{\mathrm{Z}}(R), \mathfrak{X})$. In §5, we will show that if $\mathfrak{X}$ is 0-*localic* (meaning that the underlying ∞-topos of $\mathcal{X}$ can be realized as the category of sheaves on a topological space), then X is a sheaf with respect to the flat topology (Theorem 5.15). The proof makes use of the fact that the flat topology is *subcanonical* on the ∞-category of $\mathbb{E}_\infty$-rings: that is, that every corepresentable functor is a sheaf with respect to the flat topology. We will deduce this subcanonicality from a more general result concerning descent for modules over $\mathbb{E}_\infty$-rings, which is proven in §6.

In classical algebraic geometry, the category of schemes can be regarded as a full subcategory of a larger 2-category of *Deligne-Mumford stacks*. In §8, we will introduce the notion of a *spectral Deligne-Mumford stack*. Our definition involves the notion of a *strictly Henselian* sheaf of $\mathbb{E}_\infty$-rings, which is generalization of the classical theory of strictly Henselian rings; we include a brief review of the classical theory in §7. As with spectral schemes, we can think of spectral Deligne-Mumford stacks as mathematical objects obtained by "gluing together" connective $\mathbb{E}_\infty$-rings. The difference lies in the nature of the gluing: in the setting of spectral Deligne-Mumford stacks, we replace the Zariski topology by the (far more flexible) étale topology on $\mathbb{E}_\infty$-rings. The collection of all spectral Deligne-Mumford stacks is organized into an ∞-category Stk, and there is an evident functor SpSch $\to$ Stk. In §9, we will show that this functor is fully faithful when restricted to the ∞-category of 0-localic spectral schemes.

Remark 0.1. Our theory of spectral algebraic geometry is closely related to the theory of homotopical algebraic geometry introduced by Toën and Vezzosi, and there is substantial overlap between their work (see [68], [69], [70], and [71]) and the ideas treated in this paper. Perhaps the primary difference in our presentation is that we stick closely to the classical view of scheme as a kind of ringed space, while Toën and Vezzosi make use of the "functor of points" philosophy which identifies an algebro-geometric object X with the underlying functor $R \mapsto \mathrm{Hom}(\mathrm{Spec}\, R, X)$.

Notation and Terminology

This paper will make extensive use of the theory of ∞-categories, as developed in [40]. We will also need the theory of structured ring spectra, which is presented from an ∞-categorical point of view in [41]. Finally, we will make use of the theory of geometries developed in [42], and earlier paper in this series. For convenience, we will adopt the following reference conventions:

(T) We will indicate references to [40] using the letter T.

(A) We will indicate references to [41] using the letter A.

(V) We will indicate references to [42] using the Roman numeral V.

For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [40].

Let R be a commutative ring. We let $\mathrm{Spec}^{\mathrm{Z}}\, R$ denote the collection of all prime ideals in R. We will refer to $\mathrm{Spec}^{\mathrm{Z}}\, R$ as the *Zariski spectrum* of R. We regard $\mathrm{Spec}^{\mathrm{Z}}\, R$ as endowed with the *Zariski topology*: a set $U \subseteq \mathrm{Spec}^{\mathrm{Z}}\, R$ is open if and only if there exists an ideal $I \subseteq R$ such that $U = \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}\, R : I \not\subseteq \mathfrak{p}\}$. This topology has a basis of open sets given by $U_x = \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}}\, R : x \notin \mathfrak{p}\}$, where x ranges over the collection of elements of R.

If R is an $\mathbb{E}_\infty$-ring, we let $\mathrm{Spec}^{\mathrm{Z}}\, R$ denote the Zariski spectrum $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$ of the commutative ring $\pi_0 R$.

We will occasionally need the following result from commutative algebra:

Proposition 0.2. *Let $f : R \to R'$ be an étale map of commutative rings. Then f induces an open map of topological spaces* $\mathrm{Spec}^{\mathrm{Z}}(R') \to \mathrm{Spec}^{\mathrm{Z}}(R)$.

Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O}_{\mathcal{X}}$ be a sheaf on $\mathcal{X}$ with values in an ∞-category $\mathcal{C}$ (that is, a functor $\mathcal{X}^{op} \to \mathcal{C}$ which preserves small limits). For each object $U \in \mathcal{X}$, we let $\mathcal{O}_{\mathcal{X}} |U$ denote the composite functor

$$(\mathcal{X}_{/U})^{op} \to \mathcal{X}^{op} \stackrel{\mathcal{O}_{\mathcal{X}}}{\to} \mathcal{C},$$

which we regard as a sheaf on $\mathcal{X}_{/U}$ with values in $\mathcal{C}$.

We will say that a functor $f : \mathcal{C} \to \mathcal{D}$ between ∞-categories is *left cofinal* if, for every object $D \in \mathcal{D}$, the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible (in [40], we referred to a functor with this property as *cofinal*; see Theorem T.4.1.3.1). We will say that f is *right cofinal* if the induced map $\mathcal{C}^{op} \to \mathcal{D}^{op}$ is left cofinal, so that f is right cofinal if and only if the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible for each $D \in \mathcal{D}$.

Notation 0.3. If k is an $\mathbb{E}_\infty$-ring, we let $\mathrm{CAlg}_k = \mathrm{CAlg}(\mathrm{Mod}_k(\mathrm{Sp}))$ denote the ∞-category of commutative algebra objects in the ∞-category $\mathrm{Mod}_k(\mathrm{Sp})$; we will refer to the objects of CAlg_k as *k-algebras*. If k is connective, we let $\mathrm{CAlg}_k^{\mathrm{cn}}$ denote the full subcategory of CAlg_k spanned by the *connective* k-algebras.

Notation 0.4. We let ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are ∞-topoi and whose morphisms are functors $f^* : \mathcal{X} \to \mathcal{Y}$ which preserve small colimits and finite limits. We let ${}^{\mathrm{R}}\mathcal{T}\mathrm{op} \simeq {}^{\mathrm{L}}\mathcal{T}\mathrm{op}^{op}$ denote the full subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ whose objects are ∞-topoi and whose morphisms are functors $f_* : \mathcal{X} \to \mathcal{Y}$ which admit left exact left adjoints.

Notation 0.5. If $\mathcal{X}$ is an ∞-topos, we let $\mathcal{X}^\wedge$ denote its hypercompletion: that is, the full subcategory of $\mathcal{X}$ spanned by the hypercomplete objects. See §T.6.5.2 for more details.

Acknowledgements

The material of §3 evolved out of a conversation with Dustin Clausen. I thank him for correcting some of my misconceptions about the theory of coherent topoi. This paper also owes a lot to conversations with Bertrand Toën and Gabrielle Vezzosi, and from studying their own writings on the subject ([68], [69], [70], and [71]).

1 Sheaves of Spectra

Recall that a *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X. The category of schemes can be regarded as a (non-full) subcategory of the category of ringed spaces. As a starting point for our theory of spectral algebraic geometry, we will introduce an ∞-categorical analogue of the notion of a ringed space. In this section, we will study pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}_{\mathcal{X}}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We begin by considering sheaves of spectra in general.

Definition 1.1. Let $\mathcal{X}$ be an ∞-topos. A *sheaf of spectra* on $\mathcal{X}$ is a sheaf on $\mathcal{X}$ with values in the ∞-category Sp of spectra: that is, a functor $\mathcal{O} : \mathcal{X}^{op} \to \mathrm{Sp}$ which preserves small limits. We let $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$ spanned by the sheaves of spectra on $\mathcal{X}$.

Remark 1.2. For any ∞-topos $\mathcal{X}$, the functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ induces a forgetful functor $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$ (the last equivalence being induced by the Yoneda embedding $\mathcal{X} \to \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{S})$, whose essential image is $\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})$ by Proposition T.5.5.2.2). We claim that this functor exhibits $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ as a stabilization of the ∞-category $\mathcal{X}$. Writing Sp as the homotopy limit of the tower

$$\cdots \to \mathcal{S}_* \stackrel{\Omega}{\to} \mathcal{S}_* \stackrel{\Omega}{\to} \mathcal{S}_*,$$

we deduce that $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is the homotopy inverse limit of the tower

$$\cdots \to \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X}) \stackrel{\Omega}{\to} \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X}) \stackrel{\Omega}{\to} \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X}).$$

It now suffices to observe that we have a canonical equivalence $\mathcal{X}_* \simeq \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X})_* \simeq \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X})$.

Remark 1.3. For any ∞-topos $\mathcal{X}$, the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is a full subcategory of $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$. Since the ∞-category Sp is stable, we deduce that $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$ is stable (Proposition A.1.1.3.1). Since $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is closed under limits and translation in $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$, we conclude that $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is stable (Lemma A.1.1.3.3).

Remark 1.4. For every ∞-topos $\mathcal{X}$, the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is presentable (Remark V.1.1.5).

Remark 1.5. Let $\mathcal{X}$ be an ∞-topos. Composing the forgetful functor functor $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$ with the truncation functor $\tau_{\leq 0} : \mathcal{X} \to \mathcal{X}$, we obtain a functor $\pi_0 : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \tau_{\leq 0}\,\mathcal{X}$. More generally, for any integer n, we let $\pi_n : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \tau_{\leq 0}\,\mathcal{X}$ denote the composition of the functor π_0 with the shift functor $\Omega^n : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Note that π_n can also be described as the composition

$$\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X}) \xrightarrow{\Omega^{n-2}} \mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}_*}(\mathcal{X}) \simeq \mathcal{X}_* \xrightarrow{\pi_2} \tau_{\leq 0}\,\mathcal{X}\,.$$

It follows that π_n can be regarded as a functor from the homotopy category $\mathrm{h}\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ to the category of abelian group objects in the topos of discrete objects of $\mathcal{X}$.

Definition 1.6. For every integer n, the functor $\Omega^{\infty-n} : \mathrm{Sp} \to \mathcal{S}$ induces a functor $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$, which we will also denote by $\Omega^{\infty-n}$. We will say that an object $M \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is *coconnective* if $\Omega^{\infty} M$ is a discrete object of $\mathcal{X}$. We will say that a sheaf of spectra $M \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is *connective* if the homotopy groups $\pi_n M$ vanish for $n < 0$ (equivalently, M is connective if the object $\Omega^{\infty-m} M \in \mathcal{X}$ is m-connective for every integer m). We let $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$ denote the the full subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ spanned by the connective objects, and $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$ the full subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ spanned by the coconnective objects.

Proposition 1.7. *Let $\mathcal{X}$ be an ∞-topos.*

(1) *The full subcategories $(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}, \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0})$ determine an accessible t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$.*

(2) *The t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is compatible with filtered colimits (that is, the full subcategory*

$$\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0} \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$$

is closed under filtered colimits).

(3) *The t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is right complete.*

(4) *The functor π_0 of Remark 1.5 determines an equivalence of categories from the heart of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ to the category of abelian group objects in the underlying topos of $\mathcal{X}$.*

Warning 1.8. The t-structure on $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is not left complete in general. For example, there may exist nonzero objects $M \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ whose homotopy groups $\pi_n M$ vanish for all integers n. However, such objects do not exist if $\mathcal{X}$ is hypercomplete. The ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is left complete if Postnikov towers in $\mathcal{X}$ are convergent; for example, if $\mathcal{X}$ is locally of finite homotopy dimension (see §T.7.2.1).

Proof of Proposition 1.7. It follows from Proposition A.1.4.3.3 that $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ admits an accessible t-structure given by the pair $(\mathcal{C}, \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0})$, where $\mathcal{C}$ is the collection of objects $M \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ for which the mapping space $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(M, \Omega(N))$ is contractible for every coconnective object $N \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$. Fix $M \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$; using Remark 1.2 we can identify M with a sequence of pointed objects $M(n) \in \mathcal{X}_*$ and equivalences $\gamma_n : M(n) \simeq \Omega M(n+1)$. Set $M'(n) = \tau_{\leq n-1} M(n)$; the equivalences γ_n induce equivalences $\gamma'_n : M'(n) \simeq \Omega M'(n+1)$, so we can regard $\{M'(n)\}$ as an object $M' \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. We have a canonical map $M \to M'$. If N is a coconnective object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$, then we have

$$\begin{aligned}
\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(M, \Omega N) &\simeq \varprojlim \mathrm{Map}_{\mathcal{X}_*}(M(n), \Omega^{\infty+1-n} N) \\
&\simeq \varprojlim \mathrm{Map}_{\mathcal{X}_*}(M'(n), \Omega^{\infty+1-n} N) \\
&\simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})}(M', N).
\end{aligned}$$

On the other hand, $\Omega^{-1}M'$ is a coconnective object of $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$. It follows that $M \in \mathcal{C}$ if and only if $M' \simeq 0$. This is equivalent to the requirement that each $M'(n) \simeq \tau_{\leq n-1}M(n)$ is a final object of $\mathcal{X}_*$: that is, the requirement that each $M'(n)$ is n-connective. This proves that $\mathcal{C} = \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$ so that assertion (1) holds.

We observe that the loop functor $\Omega : \mathcal{X}_* \to \mathcal{X}_*$ preserves filtered colimits (Example T.7.3.4.7), so that $\Omega^{\infty+1} : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{X}_*$ preserves filtered colimits for each n. It follows that the homotopy fiber of $\Omega^{\infty+1}$ (over the zero object $* \in \mathcal{X}_*$) is closed under filtered colimits, so that (2) is satisfied. It follows easily that $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$ is stable under countable coproducts. Any object $M \in \bigcap_n \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leq -n}$ has the property that $\Omega^{\infty-n}M \in \mathcal{X}_*$ is final for each n, so that M is a zero object of $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$. Assertion (3) now follows from Proposition A.1.2.1.19.

The heart of the ∞-category $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ can be identified, as a full subcategory of the homotopy inverse of the tower

$$\cdots \to \mathcal{X}_* \xrightarrow{\Omega_*} \mathcal{X}_* \xrightarrow{\Omega_*} \mathcal{X}_*,$$

with the homotopy inverse limit of the tower

$$\cdots \to \mathcal{EM}_2(\mathcal{X}) \xrightarrow{\Omega} \mathcal{EM}_1(\mathcal{X}) \xrightarrow{\Omega} \mathcal{X},$$

where $\mathcal{EM}_n(\mathcal{X}) \subseteq \mathcal{X}_*$ denotes the full subcategory spanned by the Eilenberg-MacLane objects (that is, objects which are both n-truncated and n-connective; see Definition T.7.2.2.1). Assertion (4) follows from the observation that $\mathcal{EM}_n(\mathcal{X})$ is equivalent to the nerve of the category of abelian group objects of the underlying topos of $\mathcal{X}$ for $n \geq 2$ (Proposition T.7.2.2.12). □

Remark 1.9. Let $g^* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Then g^* is left exact, and therefore induces a functor $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \simeq \mathrm{Stab}(\mathcal{X}) \to \mathrm{Stab}(\mathcal{Y}) \simeq \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})$. We will abuse notation by denoting this functor also by g^*. It is a left adjoint to the pushforward functor $g_* : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})$, given by pointwise composition with $g^* : \mathcal{X} \to \mathcal{Y}$.

Since $g^* : \mathcal{X} \to \mathcal{Y}$ preserves n-truncated objects and n-connective objects for every integer n, we conclude that the functor $g^* : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})$ is t-exact: that is, it carries $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$ into $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})_{\geq 0}$ and $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$ into $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})_{\leq 0}$. It follows that g_* is left t-exact: that is, $g_* \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y})_{\leq 0} \subseteq \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$. The functor g_* usually fails to be right t-exact.

Our next objective is to describe a symmetric monoidal structure on the ∞-category $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$. Roughly speaking, this symmetric monoidal structure is given by pointwise tensor product. However, this operation does not preserve the property of being a sheaf. Consequently, it will be convenient to first discuss the process of sheafification.

Remark 1.10. Let $\mathcal{D}$ and $\mathcal{C}$ be small ∞-categories, and assume that $\mathcal{D}$ admits finite colimits. Composition with the Yoneda embeddings $\mathcal{D}^{op} \to \mathcal{P}(\mathcal{D}^{op})$ and $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ yields functors

$$\mathrm{Fun}^*(\mathcal{P}(\mathcal{D}^{op}), \mathcal{P}(\mathcal{C}^{op})) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}^{op}, \mathcal{P}(\mathcal{C}^{op})) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Ind}(\mathcal{D})) \leftarrow \mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D})).$$

Here $\mathrm{Fun}^*(\mathcal{P}(\mathcal{D}^{op}), \mathcal{P}(\mathcal{C}^{op}))$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{P}(\mathcal{D}^{op}), \mathcal{P}(\mathcal{C}^{op}))$ spanned by those functors which preserve small colimits and finite limits, $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{D}^{op}, \mathcal{P}(\mathcal{C}^{op}))$ the full subcategory of $\mathrm{Fun}(\mathcal{D}^{op}, \mathcal{P}(\mathcal{C}^{op}))$ spanned by those functors which preserve finite limits,and $\mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))$ the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))$ spanned by those functors which preserve filtered colimits. Each of these functors is an equivalence of ∞-categories (see Propositions T.6.1.5.2 and T.5.3.5.10; the middle equivalence is an isomorphism of simplicial sets obtained by identifying both sides with a full subcategory of $\mathrm{Fun}(\mathcal{D}^{op} \times \mathcal{C}, \mathcal{S})$).

Assume that both $\mathcal{C}$ and $\mathcal{D}$ admit finite colimits, so that $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Ind}(\mathcal{D})$ are compactly generated presentable ∞-categories. The presheaf ∞-categories $\mathcal{P}(\mathcal{C}^{op})$ and $\mathcal{P}(\mathcal{D}^{op})$ are classifying ∞-topoi for $\mathrm{Ind}(\mathcal{C})$-valued and $\mathrm{Ind}(\mathcal{D})$-valued sheaves, respectively. The above argument shows that every geometric morphism between classifying ∞-topoi arises from a functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ which preserves filtered colimits. Put more informally, every natural operation which takes $\mathrm{Ind}(\mathcal{C})$-valued sheaves and produces $\mathrm{Ind}(\mathcal{D})$-valued sheaves is determined by a functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ which preserves filtered colimits.

Suppose now that we are given ∞-categories $\mathcal{C}$ and $\mathcal{D}$ which admit finite colimits, and let $f : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ be a functor which preserves filtered colimits. Remark 1.10 guarantees the existence of an induced functor $\theta : \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{D})}(\mathcal{X})$ for an arbitrary ∞-topos $\mathcal{X}$, which depends functorially on $\mathcal{X}$. In the special case where $\mathcal{X} = \mathcal{P}(\mathcal{U})$ is an ∞-category of presheaves on some small ∞-category $\mathcal{U}$, we can write down the functor θ very explicitly: it fits into a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X}) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{U}^{op}, \mathrm{Ind}(\mathcal{C})) & \xrightarrow{\circ f} & \mathrm{Fun}(\mathcal{U}^{op}, \mathrm{Ind}(\mathcal{D})), \end{array}$$

where the vertical maps are equivalences of ∞-categories given by composition with the Yoneda embedding $\mathcal{U} \to \mathcal{P}(\mathcal{U})$. More generally, if we assume only that we are given a geometric morphism $\mathcal{P}(\mathcal{U}) \to \mathcal{X}$, then we obtain a larger (homotopy commutative) diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X}) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{D})}(\mathcal{X}) \\ \uparrow & & \uparrow \\ \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{P}(\mathcal{U})) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{D})}(\mathcal{P}(\mathcal{U})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{U}^{op}, \mathrm{Ind}(\mathcal{C})) & \xrightarrow{\circ f} & \mathrm{Fun}(\mathcal{U}^{op}, \mathrm{Ind}(\mathcal{D})). \end{array}$$

The existence of this diagram immediately implies the following result:

Lemma 1.11. *Let $\mathcal{U}$ be a small ∞-category and suppose we are given a geometric morphism of ∞-topoi $g^* : \mathcal{P}(\mathcal{U}) \to \mathcal{X}$. Let $\mathcal{C}$ be a small ∞-category which admits finite colimits, and let $T_{\mathcal{C}}$ denote the functor $\mathrm{Fun}(\mathcal{U}^{op}, \mathrm{Ind}(\mathcal{C})) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{P}(\mathcal{U})) \to \mathcal{S}\mathrm{hv}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{X})$ induced by g^*. Let $\mathcal{D}$ be another small ∞-category which admits finite colimits, and define $T_{\mathcal{D}}$ similarly. Suppose that $f : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is a functor which preserves small filtered colimits. Then if $\alpha : M \to N$ is a morphsim in $\mathrm{Fun}(\mathcal{U}^{op}, \mathrm{Ind}(\mathcal{C}))$ such that $T_{\mathcal{C}}(\alpha)$ is an equivalence, then the induced map $\alpha' : (f \circ M) \to (f \circ N)$ has the property that $T_{\mathcal{D}}(\alpha')$ is an equivalence.*

Lemma 1.12. *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{C}$ a presentable ∞-category. Then the inclusion $i : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$ admits a left adjoint L.*

Proof. The proof does not really require the fact that $\mathcal{X}$ is an ∞-topos, only that $\mathcal{X}$ is a presentable ∞-category. That is, we may assume without loss of generality that $\mathcal{X} = \mathrm{Ind}_{\kappa}(\mathcal{X}_0)$, where κ is a regular cardinal and $\mathcal{X}_0$ is a small ∞-category which admits κ-small colimits. Then i is equivalent to the composition

$$\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \stackrel{G_{\mathcal{C}}}{\to} \mathrm{Fun}'(\mathcal{X}_0^{op}, \mathcal{C}) \stackrel{i'}{\subseteq} \mathrm{Fun}(\mathcal{X}_0^{op}, \mathcal{C}) \stackrel{G'_{\mathcal{C}}}{\to} \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}),$$

where $\mathrm{Fun}'(\mathcal{X}_0^{op}, \mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathcal{X}_0^{op}, \mathcal{C})$ spanned by those functors which preserve κ-small limits, $G_{\mathcal{C}}$ is the functor given by restriction along the Yoneda embedding $j : \mathcal{X}_0 \to \mathcal{X}$, and $G'_{\mathcal{C}}$ is given by right Kan extension along j. The functor $G_{\mathcal{C}}$ is an equivalence of ∞-categories (Proposition T.5.5.1.9), and the functor $G'_{\mathcal{C}}$ admits a left adjoint (given by composition with j). Consequently, it suffices to show that the inclusion i' admits a left adjoint. This follows immediately from Lemmas T.5.5.4.17, T.5.5.4.18, and T.5.5.4.19. $\square$

Lemma 1.13. *Let $\mathcal{X}$ be an ∞-topos, and let $f : \mathcal{C} \to \mathcal{D}$ be a functor between compactly generated presentable ∞-categories. Assume that f preserves small filtered colimits. Let $L_{\mathcal{C}} : \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ and $L_{\mathcal{D}} : \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{D}) \to \mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathcal{X})$ be left adjoints to the inclusion functors. Then composition with f determines a functor $F : \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{D})$ which carries $L_{\mathcal{C}}$-equivalences to $L_{\mathcal{D}}$-equivalences.*

Remark 1.14. In the situation of Lemma 1.13, the functor F descends to a functor $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) \to \mathrm{Shv}_{\mathcal{D}}(\mathcal{X})$, given by the composition $L_{\mathcal{D}} \circ F$. This is simply another avatar of the construction arising from Remark 1.10.

Proof. We use notation as in the proof of Lemma 1.12. For κ sufficiently large, the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ is stable under limits, so that (by Proposition T.6.1.5.2) we have a geometric morphism $g^* : \mathcal{P}(\mathcal{X}_0) \to \mathcal{X}$. Then the functor $L_{\mathcal{C}}$ can be realized as the composition of the restriction functor $r_{\mathcal{C}} : \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{X}_0^{op}, \mathcal{C})$ with the functor $T_{\mathcal{C}} : \mathrm{Fun}(\mathcal{X}_0^{op}, \mathcal{C}) \simeq \mathrm{Shv}_{\mathcal{C}}(\mathcal{P}(\mathcal{X}_0)) \to \mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ induced by g^*, and we can similarly write $L_{\mathcal{D}} = T_{\mathcal{D}} \circ r_{\mathcal{D}}$. If α is a morphism in the ∞-category $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$ such that $L_{\mathcal{C}}(\alpha) = T_{\mathcal{C}}(r_{\mathcal{C}}(\alpha))$ is an equivalence, then Lemma 1.11 shows that $L_{\mathcal{D}}(F(\alpha)) = T_{\mathcal{D}}(r_{\mathcal{D}}(F\alpha))$ is an equivalence, as required. □

We will regard the ∞-category Sp of spectra as endowed with the smash product monoidal structure defined in §A.6.3.2. This symmetric monoidal structure induces a symmetric monoidal structure on the ∞-category $\mathrm{Fun}(K, \mathrm{Sp})$, for any simplicial set K (Remark A.2.1.3.4); we will refer to this symmetric monoidal structure as the *pointwise smash product monoidal structure.*

Proposition 1.15. *Let $\mathcal{X}$ be an ∞-topos, and let $L : \mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ be a left adjoint to the inclusion. Then L is compatible with the pointwise smash product monoidal structure, in the sense of Definition A.2.2.1.6: that is, if $f : M \to M'$ is an L-equivalence in $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$ and $N \in \mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$, then the induced map $M \otimes N \to M' \otimes N$ is also an L-equivalence in $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$. Consequently, the ∞-category $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ inherits the structure of a symmetric monoidal ∞-category, with respect to which L is a symmetric monoidal functor (Proposition A.2.2.1.9).*

Proof. Apply Lemma 1.13 to the tensor product functor $\otimes : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$. □

We will henceforth regard the ∞-category $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ as endowed with the symmetric monoidal structure of Proposition 1.15, for any ∞-topos $\mathcal{X}$. We will abuse terminology by referring to this symmetric monoidal structure as the *smash product* symmetric monoidal structure.

Proposition 1.16. *Let $\mathcal{X}$ be an ∞-topos, and let $L : \mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp}) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ be a left adjoint to the inclusion. Regard $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$ as endowed with the t-structure induced by the natural t-structure on Sp. Then:*

(1) *The functor L is t-exact: that is, L carries $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp}_{\geq 0})$ into $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$ and $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp}_{\leq 0})$ into $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$.*

(2) *The smash product symmetric monoidal structure on $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ is compatible with the t-structure on $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$.*

Proof. The construction of Lemma 1.12 shows that (for sufficiently large κ) we can factor L as the composition of a restriction functor $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp}) \to \mathrm{Fun}(\mathcal{X}_0^{op}, \mathrm{Sp})$ with the functor $\mathrm{Fun}(\mathcal{X}_0^{op}, \mathrm{Sp}) \simeq \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{P}(\mathcal{X}_0)) \to \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ induced by a geometric morphism $g^* : \mathcal{P}(\mathcal{X}_0) \to \mathcal{X}$. Assertion (1) now follows from Remark 1.9. To prove (2), we show that if we are given a finite collection of connective objects $\{X_i\}_{1 \leq i \leq n}$ of $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$, then the tensor product $X_1 \otimes \cdots \otimes X_n$ is connective. Choose fiber sequences

$$X'_i \to X_i \to X''_i \to X'_i[1]$$

in $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$, where $X'_i \in \mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp}_{\geq 0})$ and $X''_i \in \mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp}_{\leq -1})$. It follows from (1) that $LX'_i \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$ and $LX''_i \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leq -1}$. We have fiber sequences

$$LX'_i \to LX_i \to LX''_i \to LX'_i[1]$$

in $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$. Since $LX_i \simeq X_i$ is connective, we deduce that the map $LX'_i \to LX_i \simeq X_i$ is an equivalence for every index i. Using Proposition 1.15, we deduce that the tensor product $X_1 \otimes \cdots \otimes X_n$ in the ∞-category $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ can be written as $L(X'_1 \otimes \cdots \otimes X'_n)$. By virtue of (1), it will suffice to show that $X'_1 \otimes \cdots \otimes X'_n$ is a connective object of $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp})$, which follows from the fact that the smash product monoidal structure on Sp is compatible with its t-structure (Lemma A.7.1.1.7). □

Definition 1.17. Let $\mathcal{X}$ be an ∞-topos. A *sheaf of* $\mathbb{E}_\infty$*-rings* on $\mathcal{X}$ is a functor $\mathcal{X}^{op} \to \mathrm{CAlg}$ which preserves small limits; we will denote by $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ the full subcategory of $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{CAlg})$ spanned by the sheaves of $\mathbb{E}_\infty$-rings.

Remark 1.18. Since the forgetful functor $\mathrm{CAlg} = \mathrm{CAlg}(\mathrm{Sp}) \to \mathrm{Sp}$ is conservative and preserves small limits (see Lemma A.3.2.2.6 and Corollary A.3.2.2.5), we have a canonical equivalence of ∞-categories (in fact, an isomorphism of simplicial sets) $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \simeq \mathrm{CAlg}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$.

Remark 1.19. Let $\mathcal{X}$ be an ∞-topos and $\mathcal{O} : \mathcal{X}^{op} \to \mathrm{CAlg}$ a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Composing with the forgetful functor $\mathrm{CAlg} \to \mathrm{Sp}$, we obtain a sheaf of spectra on $\mathcal{X}$; we will generally abuse notation by denoting this sheaf of spectra also by $\mathcal{O}$. In particular, we can define homotopy groups $\pi_n \mathcal{O}$ as in Remark 1.5. These homotopy groups have a bit more structure in this case: $\pi_0 \mathcal{O}$ is a commutative ring object in the underlying topos of $\mathcal{X}$, while each $\pi_n \mathcal{O}$ has the structure of a $\pi_0 \mathcal{O}$-module.

Definition 1.20. Let $\mathcal{X}$ be an ∞-topos. We will say that a sheaf $\mathcal{O}$ of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ is *connective* if it is connective when regarded as a sheaf of spectra on $\mathcal{X}$: that is, if the homotopy groups $\pi_n \mathcal{O}$ vanish for $n < 0$. We let $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\geq 0}$ denote the full subcategory of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ spanned by the connective sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

The following result is useful for working with connective sheaves of spectra on an ∞-topos $\mathcal{X}$.

Proposition 1.21. *Let $\mathcal{C}$ be a compactly generated presentable ∞-category. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under the formation of colimits, and which is generated under small colimits by compact objects of $\mathcal{C}$. Let $\mathcal{X}$ be an ∞-topos. Then:*

(1) *The ∞-category $\mathcal{C}_0$ is presentable and compactly generated.*

(2) *The inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ admits a right adjoint g which commutes with filtered colimits.*

(3) *Composition with g determines a functor $G : \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X})$.*

(4) *The functor G admits a fully faithful left adjoint F.*

Proof of Proposition 1.21. Since $\mathcal{C}_0$ is stable under small colimits in $\mathcal{C}$, the inclusion $i : \mathcal{C}_0 \subseteq \mathcal{C}$ preserves small colimits so that i admits a right adjoint $g : \mathcal{C} \to \mathcal{C}_0$ by Corollary T.5.5.2.9. Let $\mathcal{D} \subseteq \mathcal{C}_0$ be the full subcategory spanned by those objects of $\mathcal{C}_0$ which are compact in $\mathcal{C}$. Any such object is automatically compact in $\mathcal{C}_0$, so we have a fully faithful embedding $q : \mathrm{Ind}(\mathcal{D}) \to \mathcal{C}_0$ (Proposition T.5.3.5.11). Since $\mathcal{C}_0$ is generated under small colimits by objects of $\mathcal{D}$, we deduce that q is an equivalence of ∞-categories; this proves (1). Moreover, it shows that the collection of compact objects in $\mathcal{C}_0$ is an idempotent completion of $\mathcal{D}$; since $\mathcal{D}$ is already idempotent complete, we deduce that every compact object of $\mathcal{C}_0$ is also compact in $\mathcal{C}$. Assertion (2) now follows from Proposition T.5.5.7.2. Assertion (3) is obvious (since g preserves small limits; see Proposition T.5.2.3.5).

Let $L : \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \to \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ be a left adjoint to the inclusion, and define L_0 similarly. We observe that G is equivalent to the composition

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \stackrel{G'}{\to} \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}_0) \stackrel{L_0}{\to} \mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X}),$$

where G' is given by composition with g. It follows that G admits a left adjoint F, which can be described as the composition

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \stackrel{L}{\leftarrow} \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}) \supseteq \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}_0) \supseteq \mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X}).$$

To complete the proof, it suffices to show that F is fully faithful. In other words, we wish to show that for every object $\mathcal{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X})$, the unit map $\mathcal{F} \to (G \circ F)(\mathcal{F})$ is an equivalence. In other words, we wish to show that the map $\alpha : \mathcal{F} \to L\mathcal{F}$ becomes an equivalence after applying the functor G'. Since $G'(\mathcal{F}) \simeq \mathcal{F}$ and $G'(L\mathcal{F})$ belong to $\mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X})$, this is equivalent to the requirement that $G'(\alpha)$ is an L_0-equivalence in the ∞-category $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C}_0)$. This follows from (3) and Lemma 1.13, since α is an L-equivalence in $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$. □

Remark 1.22. In the situation of Proposition 1.21, an object $\mathcal{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ belongs to the essential image of the full faithful embedding $\mathcal{S}\mathrm{hv}_{\mathcal{C}_0}(\mathcal{X}) \rightarrow \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X})$ if and only if the canonical map $G(\mathcal{F}) \rightarrow \mathcal{F}$ is an L-equivalence in $\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$, where L denotes a left adjoint to the inclusion $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})$.

Example 1.23. The full subcategory $\mathrm{Spt}_{\geq 0} \subseteq \mathrm{Spt}$ of connective spectra is stable under small colimits in Spt, and is generated under small colimits by the sphere spectrum $S \in \mathrm{Spt}_{\geq 0}$ (which is a compact object of the ∞-category Spt). Consequently, Proposition 1.21 gives a fully faithful embedding $F : \mathcal{S}\mathrm{hv}_{\mathrm{Spt}_{\geq 0}}(\mathcal{X}) \rightarrow \mathcal{S}\mathrm{hv}_{\mathrm{Spt}}(\mathcal{X})$ for every ∞-topos $\mathcal{X}$. Let $\mathcal{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Spt}}(\mathcal{X})$, so that we have an exact triangle

$$\tau_{\geq 0} \mathcal{F} \xrightarrow{\phi} \mathcal{F} \rightarrow \tau_{\leq -1} \mathcal{F} \rightarrow (\tau_{\geq 0} \mathcal{F})[1]$$

in the ∞-category $\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Spt})$. Let $L : \mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Spt}) \rightarrow \mathcal{S}\mathrm{hv}_{\mathrm{Spt}}(\mathcal{X})$ be a left adjoint to the inclusion. According to Remark 1.22, the object $\mathcal{F}$ belongs to the essential image of F if and only if $L(\phi)$ is an equivalence. Since the functor L is t-exact, this is equivalent to the requirement that $\mathcal{F} \in \mathcal{S}\mathrm{hv}_{\mathrm{Spt}}(\mathcal{X})_{\geq 0}$: that is, the functor F induces an equivalence $\mathcal{S}\mathrm{hv}_{\mathrm{Spt}_{\geq 0}}(\mathcal{X}) \rightarrow \mathcal{S}\mathrm{hv}_{\mathrm{Spt}}(\mathcal{X})_{\geq 0}$.

Example 1.24. Let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings, and let $\mathrm{CAlg}^{\mathrm{cn}} \subseteq \mathrm{CAlg}$ denote the full subcategory of CAlg spanned by the connective $\mathbb{E}_\infty$-rings. Then $\mathrm{CAlg}_{\geq 0}$ is stable under small colimits in CAlg (it is the essential image of the colocalization functor given by passage to the connective cover; see Proposition A.7.1.3.13). It is generated under small colimits by the compact object $\mathrm{Sym}^*(S)$, where S denotes the sphere spectrum and $\mathrm{Sym}^* : \mathrm{Sp} \rightarrow \mathrm{CAlg}$ denotes a left adjoint to the forgetful functor. Proposition 1.21 gives a fully faithful embedding $F : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X}) \rightarrow \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ for every ∞-topos $\mathcal{X}$. Let $\mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and let $\tau_{\geq 0} \mathcal{O} \in \mathrm{Fun}(\mathcal{X}^{op}, \mathrm{CAlg})$ be the presheaf of $\mathbb{E}_\infty$-rings obtained by pointwise passage to the connective cover. Let $\mathcal{O}' \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ be a sheafification of the presheaf $\tau_{\geq 0} \mathcal{O}$, so that the evident map $\tau_{\geq 0} \mathcal{O} \rightarrow \mathcal{O}$ induces a map of sheaves $\alpha : \mathcal{O}' \rightarrow \mathcal{O}$. According to Remark 1.22, the sheaf $\mathcal{O}$ belongs to the essential image of F if and only if α is an equivalence. Let $u : \mathrm{CAlg} \rightarrow \mathrm{Sp}$ denote the forgetful functor. Since u preserves small limits, composition with u induces a forgetful functor $U : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \rightarrow \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Since u is conservative, the functor U is also conservative, so that α is an equivalence if and only if $U(\alpha)$ is an equivalence. Since u preserves filtered colimits, Lemma 1.13 implies that $U(\mathcal{O}')$ can be identified with a sheafification of $u \circ \tau_{\geq 0} \mathcal{O} \simeq \tau_{\geq 0}(u \circ \mathcal{O})$. Example 1.23 guarantees that $U(\alpha)$ is an equivalence if and only if $U(\mathcal{O})$ is connective as a sheaf of spectra. Combining these observations, we deduce that $\mathcal{O}$ is connective if and only if it belongs to the essential image of $\mathcal{F}$. In other words, the functor F induces an equivalence $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}}(\mathcal{X}) \rightarrow \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\geq 0}$.

We conclude this section by describing the notion of a sheaf of $\mathbb{E}_\infty$-rings using the formalism of geometries developed in [42].

Definition 1.25. Let $k \in \mathrm{CAlg}$ be an $\mathbb{E}_\infty$-ring. We let $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)$ denote the full subcategory of CAlg_k spanned by the compact objects. If A is a compact k-algebra, we let $\mathrm{Spec}\, A$ denote the corresponding object of $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$. If k is connective, we let $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ denote the full subcategory of $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)$ spanned by the connective compact k-algebras.

We will view $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)$ and $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ as discrete geometries: that is, we will say that a morphism in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)$ is *admissible* if it is an equivalence, and we will say that a collection of admissible morphisms $\{\phi_\alpha : \mathrm{Spec}\, B_\alpha \rightarrow \mathrm{Spec}\, A\}$ generates a *covering sieve* if one of the morphsims ϕ_α admits a section.

In the special case where k is the sphere spectrum (regarded as an initial object of CAlg), we will denote the geometries $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)$ and $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ by $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}$ and $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}$, respectively.

Remark 1.26. Let k be an $\mathbb{E}_\infty$-ring. For any ∞-topos $\mathcal{X}$, Remark V.1.1.7 furnishes equivalences of ∞-categories

$$\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}_k}(\mathcal{X}) \simeq \mathrm{Str}_{\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)}(\mathcal{X}).$$

That is, a $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)$-structure on $\mathcal{X}$ can be identified with a sheaf on $\mathcal{X}$ with values in the ∞-category CAlg_k of $\mathbb{E}_\infty$-algebras over k. If k is connective, the same argument gives an equivalence

$$\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}_k}(\mathcal{X}) \simeq \mathrm{Str}_{\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k)}(\mathcal{X}) :$$

that is, a $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ can be identified with a sheaf of *connective* $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Definition 1.27. Let k be an $\mathbb{E}_\infty$-ring. We let $\mathrm{RingTop}(k)$ denote opposite of the ∞-category ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k))$ of Definition V.1.4.8. Concretely, the objects of $\mathrm{RingTop}(k)$ are given by pairs $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, where $\mathcal{X}$ is an ∞-topos and (by Remark 1.26) $\mathcal{O}_\mathcal{X}$ is a CAlg_k-valued sheaf on $\mathcal{X}$. A morphism $f : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ in $\mathrm{RingTop}(k)$ can be identified with a pair (f^*, α), where $f^* : \mathcal{Y} \to \mathcal{X}$ is a geometric morphism of ∞-topoi and $\alpha : f^* \mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ is a morphism of CAlg_k-valued sheaves on $\mathcal{X}$.

If k is a connective $\mathbb{E}_\infty$-ring, we let $\mathrm{RingTop}(k)^{\mathrm{cn}}$ denote the opposite of the ∞-category ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k))$. We can identify $\mathrm{RingTop}(k)^{\mathrm{cn}}$ with the full subcategory of $\mathrm{RingTop}(k)$ spanned by those pairs $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ where $\mathcal{O}_\mathcal{X}$ is connective. The inclusion $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k) \to \mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)$ induces a functor $\mathrm{RingTop}(k) \to \mathrm{RingTop}(k)^{\mathrm{cn}}$ which is right adjoint to the above identification. Concretely, this right adjoint is given by the formula $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \mapsto (\mathcal{X}, \tau_{\geq 0} \mathcal{O}_\mathcal{X})$.

In the special case where k is the sphere spectrum (regarded as an initial object of CAlg), we will denote $\mathrm{RingTop}(k)$ and $\mathrm{RingTop}(k)^{\mathrm{cn}}$ by RingTop and $\mathrm{RingTop}^{\mathrm{cn}}$, respectively. We will refer to RingTop as the *∞-category of spectrally-ringed ∞-topoi.*

Remark 1.28. Let k be a connective $\mathbb{E}_\infty$-ring. Let $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ denote the full subcategory of CAlg_k^{op} spanned by those k-algebras of the form $\mathrm{Sym}^* M$, where M is a free k-module of finite rank. Then $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ can be regarded as a discrete pregeometry. Using Proposition A.7.2.5.27, we see that the inclusion $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}(k) \hookrightarrow \mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ exhibits $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ as a geometric envelope of $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$. When k is the sphere spectrum, we will denote $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}(k)$ by $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{disc}}$.

2 Spectral Schemes

Our goal in this section is to introduce an ∞-categorical generalization of the classical notion of scheme, which we will refer to as a *spectral scheme*. Recall that the category of schemes can be regarded as a subcategory (which is not full) of the category of ringed spaces. In §1, we introduced the ∞-category RingTop of *spectrally ringed ∞-topoi*. The notion of spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ generalizes the classical notion of ringed space in two ways:

(*a*) Rather than considering sheaves on topological spaces, we consider sheaves on ∞-topos. Every topological space X determines an ∞-topos $\mathcal{S}\mathrm{hv}(X)$. Moreover, the construction $X \mapsto \mathcal{S}\mathrm{hv}(X)$ is determines a fully faithful embedding from the (nerve of the) category of sober topological spaces to the ∞-category of ∞-topoi (recall that a topological space X is said to be *sober* if every irreducible closed subset of X contains a unique generic point: this condition is always satisfied if X is Hausdorff, or if X is the underlying topological space of a scheme).

(*b*) Rather than considering sheaves with values in the ordinary category of commutative rings, we consider sheaves with values in the ∞-category CAlg of $\mathbb{E}_\infty$-rings.

Remark 2.1. Every ringed space $(X, \mathcal{O}_X)$ determines a spectrally ringed ∞-topos $(\mathcal{S}\mathrm{hv}(X), \mathcal{O})$ via the following procedure:

(*i*) Let Ring denote the category of commutative rings. Using Proposition V.1.1.12, we see that giving a sheaf on X with values in the category Ring is equivalent to giving a sheaf $\mathcal{O}$ on $\mathcal{S}\mathrm{hv}(X)$ with values in the ∞-category N(Ring).

(*ii*) According to Proposition A.7.1.3.18, we can identify N(Ring) with the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}$ spanned by the *discrete* $\mathbb{E}_\infty$-rings. Note that N(Ring) is closed under limits in $\mathrm{CAlg}^{\mathrm{cn}}$, so that $\mathcal{O}$ can be identified with a sheaf of connective $\mathbb{E}_\infty$-rings on $\mathcal{S}\mathrm{hv}(X)$.

(*iii*) Using Example 1.24, we can identify $\mathcal{O}$ with a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{S}\mathrm{hv}(X)$ (beware that this identification is not compatible with passage to global sections: when viewed as a CAlg-valued sheaf on $\mathcal{S}\mathrm{hv}(X)$ the values of $\mathcal{O}$ are generally not connective).

If X is sober, we can recover X as the topological space of points of the ∞-topos $\mathcal{S}\mathrm{hv}(X)$, and its structure sheaf $\mathcal{O}_X$ is given by $\pi_0\,\mathcal{O}$.

In this section, we will define a subcategory SpSch $\subseteq$ RingTop, which we will refer to as the *∞-category of spectral schemes*. To help cement the reader's intuition, we begin by describing some of the objects of this ∞-category:

Definition 2.2 (Spectral Schemes: Preliminary Definition)**.** A 0-*localic spectral scheme* is a spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ which satisfies the following conditions:

(1) There exists a topological space X and an equivalence of ∞-topos $\mathcal{S}\mathrm{hv}(X) \simeq \mathcal{X}$. We will use this equivalence to identify $\pi_0\,\mathcal{O}_{\mathcal{X}}$ with a sheaf of commutative rings $\mathcal{O}_X$ on X, and each higher homotopy group $\pi_n\,\mathcal{O}_{\mathcal{X}}$ as a sheaf of $\mathcal{O}_X$-modules on X.

(2) The ringed space $(X, \mathcal{O}_X)$ is a scheme, in the sense of classical algebraic geometry.

(3) Each $\pi_n\,\mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf of modules on the scheme $(X, \mathcal{O}_X)$, in the sense of classical algebraic geometry.

(4) The zeroth space $\Omega^\infty\,\mathcal{O}_{\mathcal{X}}$ is hypercomplete (when viewed as a sheaf of spaces on X; see §T.6.5.2).

Remark 2.3. Condition (4) of Definition 2.2 plays a purely technical role, and can safely be ignored by the reader. In most cases of interest, it is automatically satisfied. For example, if $(X, \mathcal{O}_X)$ is a Noetherian scheme of finite Krull dimension, then every object of $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(X)$ is hypercomplete (see §T.7.2.4). In general, if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a spectrally ringed ∞-topos which satisfies conditions (1), (2), and (3) of Definition 2.2, then we can replace $\mathcal{O}_{\mathcal{X}}$ by its hypercompletion to obtain a spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}})$ which satisfies conditions (1), (2), (3) and (4) (without changing the underlying scheme).

In order to make a systematic study of the theory of spectral schemes, it will be convenient to formulate Definition 2.2 in a different way. We would like to mimic the classical definition of scheme as closely as possible. Recall that a ringed space $(X, \mathcal{O}_X)$ is a scheme if and only if X can be covered by open sets $\{U_\alpha\}$ such that each of the ringed spaces $(U_\alpha, \mathcal{O}_X\,|U_\alpha)$ is isomorphic (in the category of ringed spaces) to the Zariski spectrum of a commutative ring. Our definition of spectral scheme will be essentially the same, except that we will replace the category of commutative rings by the larger category of connective $\mathbb{E}_\infty$-rings. To make this precise, we will need to understand how to generalize the definition of the Zariski spectrum to the setting of $\mathbb{E}_\infty$-rings. This will lead us to a different notion of spectral scheme (Definition 2.7), which we will prove to be equivalent to Definition 2.2 at the end of this section (Theorem 2.40).

We begin with a review of the Zariski topology in classical algebraic geometry. Let R be a commutative ring R. For every element $r \in R$, we let (r) denote the ideal generated by r. If r is not a unit, then $(r) \neq R$, so (by Zorn's lemma) (r) is contained in a maximal ideal $\mathfrak{m} \subset R$. We say that R is *local* if R contains a unique maximal ideal $\mathfrak{m}_R$. The above reasoning shows that $\mathfrak{m}_R$ can be described as the collection of non-invertible elements of R. The ring R is local if and only if the collection of non-units $R - R^\times$ forms an ideal in R. Since $R - R^\times$ is clearly closed under multiplication by elements of R, this is equivalent to the requirement that $R - R^\times$ is a submonoid of R (with respect to addition). That is, R is local if and only if the following pair of conditions is satisfied:

(*a*) The element 0 belongs to $R - R^\times$. In other words, 0 is not a unit in R: this is equivalent to the requirement that R is nontrivial; that is, that $0 \neq 1$ in R.

(*b*) If $r, r' \in R - R^\times$, then $r + r' \in R - R^\times$. Equivalently, if $r + r'$ is a unit, then either r or r' is a unit. This is equivalent to the following apparently weaker condition: if $s \in R$, then either s or $1 - s$ is a unit in R (to see this, take $s = \frac{r}{r+r'}$, so that s is invertible if and only if r is invertible and $1 - s \simeq \frac{r'}{r+r'}$ is invertible if and only if r' is invertible).

If R and R' are local commutative rings, then we say that a ring homomorphism $f : R \to R'$ is *local* if it carries $\mathfrak{m}_R$ into $\mathfrak{m}_{R'}$: that is, if an element $x \in R$ is invertible if and only if its image $f(x) \in R'$ is invertible.

All of these notions admit generalizations to the setting of sheaves of commutative rings on a topological space X. We say that a sheaf of commutative rings $\mathcal{O}$ on X is *local* if, for every point $x \in X$, the stalk $\mathcal{O}_x$ is a local commutative ring. Similarly, a map of local sheaves of commutative rings $\mathcal{O} \to \mathcal{O}'$ is *local* if, for every point $x \in X$, the induced map of stalks $\mathcal{O}_x \to \mathcal{O}'_x$ is a local homomorphism of local commutative rings.

We can think of a sheaf of commutative rings $\mathcal{O}$ on a topological space X as a commutative ring *object* in the category $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$ of sheaves of sets on X. The locality of $\mathcal{O}$ can then be formulated in terms which are entirely internal to the topos $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(X)$. Moreover, this formulation makes sense in an arbitrary topos:

Definition 2.4. Let $\mathcal{X}$ be a Grothendieck topos with final object $\mathbf{1}$, and let $\mathcal{O}$ be a commutative ring object of $\mathcal{X}$. We will say that $\mathcal{O}$ is *local* if the following conditions are satisfied:

(*a*) The sheaf $\mathcal{O}$ is locally nontrivial. That is, if $0, 1 : \mathbf{1} \to \mathcal{O}$ denote the multiplicative and additive identity in $\mathcal{O}$, then the fiber product $\mathbf{1} \times_{\mathcal{O}} \mathbf{1}$ is an initial object of $\mathcal{X}$.

(*b*) Let $\mathcal{O}^{\times}$ denote the group object of $\mathcal{X}$ given by the units of $\mathcal{O}$, so that we have a pullback diagram

$$\begin{array}{ccc} \mathcal{O}^{\times} & \longrightarrow & \mathcal{O} \times \mathcal{O} \\ \downarrow & & \downarrow{\scriptstyle m} \\ \mathbf{1} & \xrightarrow{1} & \mathcal{O} \end{array}$$

where m denotes the multiplication on $\mathcal{O}$, and let $e : \mathcal{O}^{\times} \to \mathcal{O}$ denote the canonical inclusion. Then the maps e and $1 - e$ determine an effective epimorphism $\mathcal{O}^{\times} \coprod \mathcal{O}^{\times} \to \mathcal{O}$.

If $\alpha : \mathcal{O} \to \mathcal{O}'$ is a map between local commutative ring objects of $\mathcal{X}$, then we say that α is *local* if the diagram

$$\begin{array}{ccc} \mathcal{O}^{\times} & \longrightarrow & \mathcal{O}'^{\times} \\ \downarrow & & \downarrow \\ \mathcal{O} & \longrightarrow & \mathcal{O}' \end{array}$$

is a pullback square.

Definition 2.4 can be adapted to the ∞-categorical setting in a straightforward way:

Definition 2.5. Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We will say that $\mathcal{O}$ is *local* if $\pi_0\,\mathcal{O}$ is local, when viewed as a commutative ring object of the underlying topos of $\mathcal{X}$ (see Definition 2.4). If $\alpha : \mathcal{O} \to \mathcal{O}'$ is a morphism between local objects of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$, then we will say that α is *local* if it induces a local morphism of commutative ring objects $\pi_0\,\mathcal{O} \to \pi_0\,\mathcal{O}'$.

Let RingTop denote the ∞-category of spectrally ringed ∞-topoi. We define a subcategory $\mathrm{RingTop}_{\mathrm{Zar}}$ as follows:

(*i*) A spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ belongs to $\mathrm{RingTop}_{\mathrm{Zar}}$ if and only if $\mathcal{O}_{\mathcal{X}}$ is a local sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

(*ii*) A morphism of spectrally ringed ∞-topoi $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ belongs to $\mathrm{RingTop}_{\mathrm{Zar}}$ if and only if the induced map $\alpha : f^*\,\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is a local map between sheaves of $\mathbb{E}_\infty$-rings.

We will refer to $\mathrm{RingTop}_{\mathrm{Zar}}$ as the *∞-category of locally spectrally ringed ∞-topoi.* We let $\mathrm{RingTop}^{\mathrm{cn}}_{\mathrm{Zar}} = \mathrm{RingTop}_{\mathrm{Zar}} \cap \mathrm{RingTop}^{\mathrm{cn}}$. If k is an $\mathbb{E}_\infty$-ring, we define $\mathrm{RingTop}(k)_{\mathrm{Zar}}$ to be the fiber product

$$\mathrm{RingTop}(k) \times_{\mathrm{RingTop}} \mathrm{RingTop}_{\mathrm{Zar}} \subseteq \mathrm{RingTop}(k),$$

and if k is connective we let $\mathrm{RingTop}(k)^{\mathrm{cn}}_{\mathrm{Zar}}$ denote the fiber product

$$\mathrm{RingTop}(k) \times_{\mathrm{RingTop}} \mathrm{RingTop}^{\mathrm{cn}}_{\mathrm{Zar}} \subseteq \mathrm{RingTop}(k)^{\mathrm{cn}}.$$

Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectrally ringed ∞-topos. Evaluating $\mathcal{O}_\mathcal{X}$ on the final object of $\mathcal{X}$, we obtain an $\mathbb{E}_\infty$-ring, which we will denote by $\Gamma(\mathcal{X}; \mathcal{O}_\mathcal{X})$. The construction $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \mapsto \Gamma(\mathcal{X}; \mathcal{O}_\mathcal{X})$ determines a functor $\Gamma : \mathrm{RingTop} \to \mathrm{CAlg}^{op}$, which is left adjoint to the canonical inclusion

$$\mathrm{CAlg}^{op} \simeq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{S})^{op} \simeq \mathrm{RingTop} \times_{\mathcal{R}\mathcal{T}\mathrm{op}} \{\mathcal{S}\} \hookrightarrow \mathrm{RingTop}.$$

The starting point for our theory of spectral algebraic geometry is the following observation:

Proposition 2.6. *The functor* $\Gamma | \mathrm{RingTop}_{\mathrm{Zar}} : \mathrm{RingTop}_{\mathrm{Zar}} \to \mathrm{CAlg}^{op}$ *admit a right adjoint.*

Proposition 2.6 asserts that for every $\mathbb{E}_\infty$-ring R, there exists a locally spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ and a map $\theta : R \to \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$ with the following universal property: for every locally spectrally ringed ∞-topos $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$, composition with θ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{RingTop}_{\mathrm{Zar}}}((\mathcal{Y}, \mathcal{O}_\mathcal{Y}), (\mathcal{X}, \mathcal{O}_\mathcal{X})) \to \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{Y}; \mathcal{O}_\mathcal{Y})).$$

The spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is uniquely determined up to equivalence; and we will denote it by $\mathrm{Spec}^{\mathrm{Z}}(R)$. We will refer to $\mathrm{Spec}^{\mathrm{Z}}(R)$ as the *spectrum of R with respect to the Zariski topology.*

We now give a different version of Definition 2.2:

Definition 2.7 (Spectral Scheme: Concrete Definition)**.** A *nonconnective spectral scheme* is a spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ such that there exists a collection of objects $U_\alpha \in \mathcal{X}$ satisfying the following conditions:

(*i*) The objects U_α cover $\mathcal{X}$. That is, the canonical map $\coprod_\alpha U_\alpha \to \mathbf{1}$ is an effective epimorphism, where $\mathbf{1}$ denotes the final object of $\mathcal{X}$.

(*ii*) For each index α, there exists an $\mathbb{E}_\infty$-ring R_α and an equivalence of spectrally ringed ∞-topoi

$$(\mathcal{X}_{/U_\alpha}, \mathcal{O}_\mathcal{X} | U_\alpha) \simeq \mathrm{Spec}^{\mathrm{Z}}(R_\alpha).$$

We let $\mathrm{SpSch}^{\mathrm{nc}}$ denote the full subcategory of $\mathrm{RingTop}_{\mathrm{Zar}}$ spanned by the nonconnective spectral schemes (note that if $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a spectral scheme, then $\mathcal{O}_\mathcal{X}$ is automatically a local sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$).

A *spectral scheme* is a nonconnective spectral scheme $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ such that $\mathcal{O}_\mathcal{X}$ is connective. We let SpSch denote the full subcategory of $\mathrm{SpSch}^{\mathrm{nc}}$ spanned by the spectral schemes.

To make sense of Definition 2.7, we need to understand the functor $\mathrm{Spec}^{\mathrm{Z}} : \mathrm{CAlg}^{op} \to \mathrm{RingTop}_{\mathrm{Zar}}$ whose existence is asserted by Proposition 2.6. This functor can be described very concretely. We will see below that if R is an $\mathbb{E}_\infty$-ring, then $\mathrm{Spec}^{\mathrm{Z}}(R)$ can be identified with the spectrally ringed ∞-topos $(\mathcal{S}\mathrm{hv}(\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)), \mathcal{O})$. Here $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$ denotes the Zariski spectrum of the commutative ring $\pi_0 R$: that is, the topological space whose points are prime ideals $\mathfrak{p} \subseteq \pi_0 R$. This topological space has a basis of open sets $U_f = \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : f \notin \mathfrak{p}\}$, where f ranges over the elements of $\pi_0 R$. The structure sheaf $\mathcal{O}$ is described by the formula $\mathcal{O}(U_f) = R[\frac{1}{f}]$ (see Definition 2.8 below).

It is possible to verify Proposition 2.6 directly by fleshing out the description of $\mathrm{Spec}^{\mathrm{Z}}(R)$ given above. However, we prefer to deduce Proposition 2.6 from the more general results of [42] (in particular, the description of $\mathrm{Spec}^{\mathrm{Z}}(R)$ given above will be deduced from Theorem V.2.2.12). For this, we need to recast the discussion of local sheaves of $\mathbb{E}_\infty$-rings using the language of *geometries*. This has the unfortunate effect of burdening our exposition with an additional layer of abstraction. However, it will be convenient later, when we discuss the relationship between the theory of spectral schemes and other related constructions (see §9).

Definition 2.8. Let $f : A \to B$ be a map of $\mathbb{E}_\infty$-rings. We will say that f *exhibits B as a localization of A by $a \in \pi_0 A$* if the map f is étale and f induces an isomorphism of commutative rings $(\pi_0 A)[a^{-1}] \simeq \pi_0 B$. In this case, we will denote B by $A[a^{-1}]$.

Remark 2.9. Let A be an $\mathbb{E}_\infty$-ring and $a \in \pi_0 A$ an element. Theorem A.7.5.0.6 guarantees that $A[a^{-1}]$ exists and is well-defined up to equivalence (in fact, up to a contractible space of choices). The localization map $A \to A[a^{-1}]$ can be characterized by either of the following conditions (see Corollary A.7.5.4.6):

(1) The map $A \to A[a^{-1}]$ induces an isomorphism of graded rings $(\pi_* A)[a^{-1}] \to (\pi_* A[a^{-1}])$.

(2) For every $\mathbb{E}_\infty$-ring B, the map $\mathrm{Map}_{\mathrm{CAlg}}(A[a^{-1}], B) \to \mathrm{Map}_{\mathrm{CAlg}}(A, B)$ is fully faithful, and its essential image consists of those maps $A \to B$ which carry $a \in \pi_0 A$ to an invertible element of $\pi_0 B$.

Definition 2.10. Let k be an $\mathbb{E}_\infty$-ring. We define a geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ as follows:

(1) On the level of ∞-categories, we have $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k) = \mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k)$ (see Definition 1.25). That is, $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ is the opposite of the ∞-category of compact object of CAlg_k. If A is a compact object of CAlg_k, we let $\mathrm{Spec}\, A$ denote the corresponding object of $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$.

(2) A morphism $f : \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ is *admissible* if and only if there exists an element $b \in \pi_0 B$ such that f carries b to an invertible element in $\pi_0 A$, and the induced map $B[\frac{1}{b}] \to A$ is an equivalence.

(3) A collection of admissible morphisms $\{\mathrm{Spec}\, B[\frac{1}{b_\alpha}] \to \mathrm{Spec}\, B\}$ generates a covering sieve on B if and only if the elements b_α generate the unit ideal in the commutative ring $\pi_0 B$.

If the $\mathbb{E}_\infty$-ring k is connective, we let $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ denote the full category of $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ spanned by objects of the form $\mathrm{Spec}\, A$, where A is a connective k-algebra; we regard $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ as a geometry by taking the admissible morphisms and admissible coverings in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$. When k is the sphere spectrum (regarded as an initial object of CAlg), we will denote $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ and $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ by $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$ and $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}$, respectively.

Remark 2.11. To check that Definition 2.10 describes a geometry, it is necessary to observe that the collection of k-algebra morphisms of the form $A \mapsto A[a^{-1}]$ is stable under retracts. To prove this, let us consider a diagram

$$\begin{array}{ccccc} A & \longrightarrow & A' & \xrightarrow{\phi} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & A'[a^{-1}] & \longrightarrow & B, \end{array}$$

where the horizontal compositions are the identity maps. We will prove that f is admissible by showing that f induces isomorphisms $\theta : (\pi_n A)[\frac{1}{\phi(a)}] \to \pi_n B$ for every integer n. Since $\phi(a)$ clearly acts invertibly on $\pi_n B$, it suffices to show that the action of $\phi(a)$ is locally nilpotent on the kernel and cokernel of the map θ. Since θ is a retract of the map $\theta' : \pi_n A' \to (\pi_n A')[\frac{1}{a}]$, we have surjective maps $\ker(\theta') \to \ker(\theta)$ and $\mathrm{coker}(\theta') \to \mathrm{coker}(\theta)$; it therefore suffices to observe that the action of a is locally nilpotent on the abelian groups $\ker(\theta')$ and $\mathrm{coker}(\theta')$.

Notation 2.12. Let k be an $\mathbb{E}_\infty$-ring. Let Sym^*_k denote the left adjoint to the forgetful functor $\mathrm{CAlg}_k \to \mathrm{Mod}_k$. We will denote the algebra $\mathrm{Sym}^*_k(k^n)$ by $k\{x_1, \ldots, x_n\}$. We will say that a k-algebra A is *polynomial* if it is equivalent to $k\{x_1, \ldots, x_n\}$ for some $n \geq 0$. We let $\mathbf{A}^n$ denote the object $\mathrm{Spec}\, k\{x_1, \ldots, x_n\}$ of $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$.

We will say that a k-algebra A is *localized polynomial* if it is equivalent to $k\{x_1, \ldots, x_n\}[\frac{1}{f}]$ for some $f \in \pi_0 k\{x_1, \ldots, x_n\} \simeq (\pi_0 k)[x_1, \ldots, x_n]$. We let $\mathbf{G}_m$ denote the object $\mathrm{Spec}\, k\{x\}[\frac{1}{x}]$ of $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$. Note that the canonical map $k\{x\} \to k\{x\}[\frac{1}{x}]$ induces a morphism $\mathbf{G}_m \to \mathbf{A}^1$ in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$.

Remark 2.13. Let k be an $\mathbb{E}_\infty$-ring. For any k-algebra A and any element $a \in \pi_0 A$, we have a pushout diagram of k-algebras

$$\begin{array}{ccc} k\{x\} & \longrightarrow & k\{x\}[\frac{1}{x}] \\ \downarrow & & \downarrow \\ A & \longrightarrow & A[\frac{1}{a}]. \end{array}$$

It follows that the collection of of admissible morphisms in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ is generated (under the formation of pullbacks) by the admissible map $\mathbf{G}_m \to \mathbf{A}^1$ appearing in Notation 2.12

Example 2.14. Let k be an $\mathbb{E}_\infty$-ring. The pair of admissible morphisms

$$\operatorname{Spec} k\{x\}[\frac{1}{1-x}] \xrightarrow{\alpha} \operatorname{Spec} k\{x\} \xleftarrow{\beta} \operatorname{Spec} k\{x\}[\frac{1}{x}]$$

is an admissible covering in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$. In fact, this admissible covering, together with the empty covering of the initial object $\operatorname{Spec} 0$, generates the family of admissible coverings in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$. To prove this, let $\mathcal{G}$ be another geometry with the same underlying ∞-category as $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ and the same admissible morphisms, such that α and β generate a covering sieve on $\operatorname{Spec} k\{x\} \in \mathcal{G}$, and the empty sieve is a covering of $\operatorname{Spec} 0 \in \mathcal{G}$. We will show that $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k) \to \mathcal{G}$ is a transformation of geometries.

Let R be a compact k-algebra, and let $\{x_\alpha\}_{\alpha \in A}$ be a collection of elements of $\pi_0 R$ which generate the unit ideal. We wish to show that the maps $\{\operatorname{Spec} R[\frac{1}{x_\alpha}] \to \operatorname{Spec} R\}_{\alpha \in A}$ generate a $\mathcal{G}$-covering sieve S on R. Without loss of generality, we may suppose that $A = \{1, \ldots, n\}$ for some nonnegative integer n; we work by induction on n. Write $1 = x_1 y_1 + \ldots + x_n y_n$ in the commutative ring $\pi_0 R$. Replacing each x_i by the product $x_i y_i$, we may suppose that $1 = x_1 + \ldots + x_n$. If $n = 0$, then $R \simeq 0$ and S is a covering sieve by hypothesis. If $n = 1$, then S contains an isomorphism and therefore generates a covering sieve. If $n = 2$, we have a map $\phi : \operatorname{Spec} R \to \operatorname{Spec} k\{x\}$ given by $x \mapsto x_1$. Then S is obtained from the admissible covering $\{\alpha, \beta\}$ by pullback along ϕ, and therefore generates a covering sieve. Suppose finally that $n > 2$, and set $y = x_2 + \ldots + x_n$. The inductive hypothesis implies that the maps $\operatorname{Spec} R[\frac{1}{y}] \to \operatorname{Spec} R \leftarrow \operatorname{Spec} R[\frac{1}{x_1}]$ generate a $\mathcal{G}$-covering sieve on R. It will therefore suffice to show that S generates a covering sieve after pullback to either $\operatorname{Spec} R[\frac{1}{y}]$ or $\operatorname{Spec} R[\frac{1}{x_1}]$; in either case, this follows from the inductive hypothesis.

Remark 2.15. Let k be a connective $\mathbb{E}_\infty$-ring. Using Remark 2.13 and Example 2.14, we deduce that the collection of admissible morphisms and admissible coverings in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ is generated by admissible morphisms and admissible coverings in $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$. In particular, if $\mathcal{X}$ is an ∞-topos, then a left-exact functor $\mathcal{O} : \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k) \to \mathcal{X}$ is a $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$-structure if and only if the restriction $\mathcal{O} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ is a $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$-structure; a natural transformation $\alpha : \mathcal{O} \to \mathcal{O}'$ of $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$-structures is local if and only if the induced map $\mathcal{O} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{O}' \,|\, \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ is local.

Remark 2.16. Let $\phi : k \to k'$ be a map of $\mathbb{E}_\infty$-rings. Then the construction $\operatorname{Spec} A \mapsto \operatorname{Spec}(A \otimes_k k')$ determines a transformation of geometries $\theta : \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k) \to \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k')$. It follows from Remark 2.13 and Example 2.14 that the collection of admissible morphisms and admissible morphisms in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k')$ is generated (under the formation of pullbacks) by the images under θ of admissible coverings and admissible pullbacks in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$. Consequently, if $\mathcal{X}$ is an ∞-topos, then a left-exact functor $\mathcal{O} : \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k') \to \mathcal{X}$ is a $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k')$-structure on $\mathcal{X}$ if and only if the composite map

$$\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k) \xrightarrow{\theta} \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \xrightarrow{\mathcal{O}} \mathcal{X}$$

is a $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$-structure on $\mathcal{X}$. Similarly, if $\alpha : \mathcal{O} \to \mathcal{O}'$ is a natural transformation between $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k')$-structures, then α is local if and only if the induced natural transformation $\mathcal{O} \circ \theta \to \mathcal{O}' \circ \theta$ is local.

If k and k' are connective, then θ restricts to a transformation of geometries $\theta_0 : \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k')$. The same reasoning shows that a left-exact functor $\mathcal{O} : \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k') \to \mathcal{X}$ is a $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k')$-structure if and only if $\mathcal{O} \circ \theta_0$ is a $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$-structure, and that a natural transformation $\alpha : \mathcal{O} \to \mathcal{O}'$ of $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k')$-structures on $\mathcal{X}$ is local if and only if the induced map $\mathcal{O} \circ \theta_0 \to \mathcal{O}' \circ \theta_0$ is local.

Remark 2.17. Let k denote the sphere spectrum, regarded as an initial object of the ∞-category CAlg of $\mathbb{E}_\infty$-rings. We will denote the geometries $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ and $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ by $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$ and $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}$, respectively. Let $\mathcal{X}$ be an ∞-topos. According to Remark 1.26, we can identify the ∞-category of left-exact functors $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k) \to \mathcal{X}$ with the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}_k}(\mathcal{X})$. Similarly, we can identify the ∞-category of left-exact functors $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{X}$ with the ∞-category of connective sheaves of $\mathbb{E}_\infty$-rings $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}_k}(\mathcal{X})_{\geq 0}$. Under these identifications, the restriction functor $\mathcal{O} \mapsto \mathcal{O} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ is given by passage to the connective cover $\tau_{\geq 0} : \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}_k}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}_k}(\mathcal{X})_{\geq 0}$.

If k is a connective $\mathbb{E}_\infty$-ring, then $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ can be realized as the geometric envelope of a pregeometry.

Definition 2.18. Let k be a connective $\mathbb{E}_\infty$-ring. We let $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ denote the full subcategory of $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ spanned by objects of the form $\operatorname{Spec} A$, where A is a localized polynomial algebra over k. We regard $\mathcal{T}_{\mathrm{Zar}}(k)$ as a pregeometry as follows:

- A morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is admissible if and only if the underlying map $B \to A$ exhibits A as a localization $A \simeq B[\frac{1}{b}]$, for some $b \in \pi_0 B$.
- A collection of admissible morphisms $\{\operatorname{Spec} B[\frac{1}{b_\alpha}] \to \operatorname{Spec} B\}$ generates a covering sieve on $\operatorname{Spec} B \in \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ if and only if the elements $\{b_\alpha\}$ generate the unit ideal of $\pi_0 B$.

In the special case where k is the sphere spectrum (regarded as an initial object of CAlg), we will denote the ∞-category $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ by $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}$.

Remark 2.19. As in Remark V.4.2.1, it is possible to make several variants on Definition 2.18 without changing the underlying theory of $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$-structures.

We have the following analogue of Proposition V.4.2.3:

Proposition 2.20. *Let k be a connective $\mathbb{E}_\infty$-ring. Then the inclusion $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \subseteq \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ exhibits $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ as a geometric envelope of $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$.*

The proof follows the lines of the proof of Proposition V.4.2.3. Let $\mathbf{A}^1$ denote the *affine line* $\operatorname{Spec} k\{x\}$. Note that $\mathbf{A}^1$ has the structure of a homotopy associative monoid in the ∞-category $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$, given by the multiplication map $\mathbf{A}^1 \times \mathbf{A}^1 \simeq \operatorname{Spec} k\{x_0, x_1\} \to \operatorname{Spec} k\{x\} \simeq \mathbf{A}^1$ determined by the map $k\{x\} \to k\{x_0, x_1\}$ given by $x \mapsto x_0 x_1$. Consequently, if $\mathcal{C}$ is any ∞-category which admits finite products and $f : \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{C}$ is a functor which preserves finite products, then $f(\mathbf{A}^1) \in \mathcal{C}$ inherits the structure of a homotopy associative monoid object of $\mathcal{C}$.

Lemma 2.21. *Let $\mathcal{C}$ be an ∞-category which admits finite limits, let k be a connective $\mathbb{E}_\infty$-ring, and let $f : \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{C}$ be a functor which belongs to $\operatorname{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k), \mathcal{C})$ (see Definition V.3.4.1). Then the induced map*

$$\alpha : f(\operatorname{Spec} k\{x\}[\frac{1}{x}]) \to f(\mathbf{A}^1)$$

is a unit subobject of the homotopy associative monoid object $f(\mathbf{A}^1) \in \mathcal{C}$ (see § V.4.2).

Proof. To simplify the notation, we let $X = f(\mathbf{A}^1) \in \mathcal{C}$ and let $X_0 = f(\operatorname{Spec} k\{x\}[\frac{1}{x}]) \in \mathcal{C}$. We have a pullback diagram in $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$

$$\begin{array}{ccc} \operatorname{Spec} k\{x\}[\frac{1}{x}] & \xrightarrow{\mathrm{id}} & \operatorname{Spec} k\{x\}[\frac{1}{x}] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ \operatorname{Spec} k\{x\}[\frac{1}{x}] & \longrightarrow & \operatorname{Spec} k\{x\}, \end{array}$$

where the vertical arrows are admissible. Since f belongs to $\operatorname{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k), \mathcal{C})$, the induced diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\mathrm{id}} & X_0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ X_0 & \longrightarrow & X \end{array}$$

is a pullback square in $\mathcal{C}$. This proves that α is a monomorphism.

We observe that the homotopy associative monoid structure on $\mathbf{A}^1$ described above determines a homotopy associative monoid structure on the subobject $\operatorname{Spec} k\{x\}[\frac{1}{x}]$. Moreover, this homotopy associative monoid structure is actually a homotopy associative *group* structure: the inverse is given induced by the map $k\{x\}[\frac{1}{x}] \mapsto k\{x\}[\frac{1}{x}]$ carrying x to $\frac{1}{x}$ (which is unique up to homotopy). Since f preserves finite products, we conclude that X_0 inherits the structure of a homotopy associative group object of $\mathcal{C}$, and that α is compatible with the homotopy associative monoid structure. It follows that if $p : C \to X$ is a morphism in $\mathcal{C}$ which factors through X_0 up to homotopy, then p determines an invertible element of the monoid $\pi_0 \operatorname{Map}_{\mathcal{C}}(C, X)$. To complete the proof, we need to establish the converse of this result. Let us therefore assume that $p : C \to X$ is a morphism in $\mathcal{C}$ which determines an invertible element of $\pi_0 \operatorname{Map}_{\mathcal{C}}(C, X)$; we wish to show that p factors (up to homotopy) through α.

Let $p' : C \to X$ represent a multiplicative inverse to p in $\pi_0 \operatorname{Map}_{\mathcal{C}}(C, X)$. We wish to show that the product map $(p, p') : C \times C \to X \times X$ factors (up to homotopy) through the monomorphism $\alpha \times \alpha : X_0 \times X_0 \to X \times X$. We observe that the multiplication map $\mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ fits into a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} k\{x\}[\frac{1}{x}] \times \operatorname{Spec} k\{x\}[\frac{1}{x}] & \longrightarrow & \mathbf{A}^1 \times \mathbf{A}^1 \\ \downarrow & & \downarrow \\ \operatorname{Spec} k\{x\}[\frac{1}{x}] & \longrightarrow & \mathbf{A}^1 . \end{array}$$

(In concrete terms, this amounts to the observation that for any $\mathbb{E}_\infty$-ring R, a product $x_0 x_1 \in \pi_0 R$ is invertible if and only if x_0 and x_1 are both invertible). Since the horizontal morphisms are admissible and $f \in \operatorname{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k), \mathcal{C})$, we conclude that the induced diagram

$$\begin{array}{ccc} X_0 \times X_0 & \longrightarrow & X \times X \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X \end{array}$$

is a pullback square in $\mathcal{C}$, where the vertical arrows are given by multiplication. It will therefore suffice to show that the product map $pp' : C \times C \to X$ factors (up to homotopy) through X_0. By construction, this product map is homotopic to the composition

$$C \times C \to 1_{\mathcal{C}} \xrightarrow{u} X,$$

where $u : 1_{\mathcal{C}} \to X$ is the unit map. It therefore suffices to show that u factors through α. The desired factorization is an immediate consequence of the commutativity of the following diagram in $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$:

$$\begin{array}{ccccc} \operatorname{Spec} k & & \xrightarrow{x \mapsto 1} & & \operatorname{Spec} k\{x\}[\frac{1}{x}] \\ & \searrow^{x \mapsto 1} & & \swarrow_{x \mapsto x} & \\ & & \operatorname{Spec} k\{x\}. & & \end{array}$$

□

Proof of Proposition 2.20. Let $\mathcal{T}_0$ denote the full subcategory of $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ spanned by objects of the form $\operatorname{Spec} k\{x_1, \ldots, x_n\}$. It will suffice to show that the inclusion $\mathcal{T}_0 \subseteq \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ satisfies conditions (1) through (6) of Proposition V.3.4.5. Conditions (1) and (2) are obvious, and (3) follows from Remark V.3.4.6, since $\mathcal{T}_0$ forms a set of compact projective generators for the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_k$ (see Proposition A.7.2.5.27). Assertion (4) follows from the observation that every admissible morphism $\operatorname{Spec} A[\frac{1}{a}] \to \operatorname{Spec} A$ in $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$

fits into a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} A[\frac{1}{a}] & \longrightarrow & \operatorname{Spec} k\{x\}[\frac{1}{x}] \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \xrightarrow{f} & \operatorname{Spec} k\{x\}, \end{array}$$

where f is determined up to homotopy by the requirement that it carries $x \in \pi_0 k\{x\}$ to $a \in \pi_0 A$.

We now prove (5). Consider an admissible covering $\{f_i : \operatorname{Spec} A[\frac{1}{a_i}] \to \operatorname{Spec} A\}_{1 \leq i \leq n}$ in $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$, where the elements $a_i \in \pi_0 A$ generate the unit ideal in $\pi_0 A$. We have an equation of the form

$$a_1 b_1 + \ldots + a_n b_n = 1$$

in the commutative ring $\pi_0 A$. Let $B = k\{x_1, \ldots, x_n, y_1, \ldots, y_n\}[\frac{1}{x_1 y_1 + \ldots + x_n y_n}]$. There is a morphism $\phi : B \to A$ carrying each $x_i \in \pi_0 B$ to $a_i \in \pi_0 A$, and each $y_i \in \pi_0 B$ to $b_i \in \pi_0 A$ (in fact, ϕ is uniquely determined up to homotopy). Each map f_i fits into a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} A[\frac{1}{a_i}] & \xrightarrow{f_i} & \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec} B[\frac{1}{x_i}] & \xrightarrow{g_i} & \operatorname{Spec} B. \end{array}$$

It now suffices to observe that the maps $\{g_i : \operatorname{Spec} B[\frac{1}{x_i}] \to \operatorname{Spec} B\}$ determine an admissible covering of $\operatorname{Spec} B$ in $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$.

It remains to verify condition (6). Let $\mathcal{C}$ be an idempotent-complete ∞-category which admits finite limits, and let $\alpha : f \to f'$ be a natural transformation between admissible functors $f, f' : \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{C}$ such that α induces an equivalence $f|\mathcal{T}_0 \simeq f'|\mathcal{T}_0$. We wish to prove that α is an equivalence. Fix an arbitrary object $\operatorname{Spec} R \in \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$, where $R = kx_1, \ldots, x_n[\frac{1}{q}]$; here $q \in \pi_0(k\{x_1, \ldots, x_n\}) \simeq (\pi_0 k)[x_1, \ldots, x_n]$ is classified by some map $k\{y\} \to k\{x_1, \ldots, x_n\}$. We have a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} k\{y\}[\frac{1}{y}] \\ \downarrow & & \downarrow \\ \operatorname{Spec} k\{x_1, \ldots, x_n\} & \longrightarrow & \operatorname{Spec} k\{y\} \end{array}$$

where the vertical arrows are admissible. Since f and f' belong to $\operatorname{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k), \mathcal{C})$, the map α_R will be an equivalence provided that $\alpha_{\operatorname{Spec} k\{x_1, \ldots, x_n\}}$, $\alpha_{\operatorname{Spec} k\{y\}}$, and $\alpha_{\operatorname{Spec} k\{y\}[\frac{1}{y}]}$ are equivalences. The first two cases are evident, and the third follows from Lemma 2.21. □

Corollary 2.22. *Let k be a connective $\mathbb{E}_\infty$-ring. For each $n \geq 0$, let $\mathcal{G}^{\mathrm{Sp}, \leq n}_{\mathrm{Zar}}(k)$ denote the opposite of the ∞-category of compact objects in the ∞-category $(\mathrm{CAlg}^{\mathrm{cn}}_k)_{\leq n}$ of connective, n-truncated k-algebras. Then the composite functor*

$$\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \subseteq \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \stackrel{\tau_{\leq n}}{\to} \mathcal{G}^{\mathrm{Sp}, \leq n}_{\mathrm{Zar}}(k)$$

exhibits $\mathcal{G}^{\mathrm{Sp}, \leq n}_{\mathrm{Zar}}(k)$ as an n-truncated geometric envelope of $\mathcal{T}_{\mathrm{Zar}}(k)$.

Proof. Combine Proposition 2.20 with Lemma V.3.4.11 and the proof of Proposition V.1.5.11. □

Remark 2.23. Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{O} \in \operatorname{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}, \mathcal{X}) \simeq \operatorname{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}, \mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$. The analysis of Example 2.14 shows that $\mathcal{O}$ is a $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$-structure on $\mathcal{X}$ if and only if the following conditions are satisfied:

(a) The object $\mathcal{O}(0)$ is initial in $\mathcal{X}$.

(b) The map $\mathcal{O}(k\{x\}[\frac{1}{x}]) \coprod \mathcal{O}(k\{x\}[\frac{1}{1-x}]) \to \mathcal{O}(k\{x\})$ is an effective epimorphism.

In other words, we can regard the theory of $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}$-structures on $\mathcal{X}$ as providing an ∞-categorical analogue of Definition 2.4.

Remark 2.24. Let k be a connective $\mathbb{E}_\infty$-ring. Every admissible morphism in $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}(k)$ is a monomorphism. It follows from Proposition V.3.3.5 that the pregeometry $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}(k)$ is compatible with n-truncations, for each $0 \leq n \leq \infty$. In particular, to every $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}(k)$-structure $\mathcal{O}_{\mathcal{X}}$ on an ∞-topos $\mathcal{X}$ we can associate an n-truncated $\mathcal{T}_{\mathrm{Zar}}(k)$-structure $\tau_{\leq n}\, \mathcal{O}_{\mathcal{X}}$. Evaluating on $k\{x\}$, we deduce that this truncation construction is compatible with the truncation construction on $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\geq 0}$.

We are now in a position to compare the theory of $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{nSp}}$-structures with Definition 2.5.

Corollary 2.25. *Let $\mathcal{X}$ be an ∞-topos. Then:*

(1) *Let $F : \mathcal{G}_{\mathrm{Zar}}^{\mathrm{Sp}} \to \mathcal{X}$ be a left exact functor, and let $\mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\geq 0}$ be the associated connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ (see Remark 2.17). Then $\mathcal{O}$ is local (in the sense of Definition 2.5) if and only if F is a $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{Sp}}$-structure on $\mathcal{X}$.*

(2) *Let $\alpha : F \to F'$ be a natural transformation between functors $F, F' \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}^{\mathrm{Sp}}}(\mathcal{X})$, and let $\beta : \mathcal{O} \to \mathcal{O}'$ be the induced morphism between local objects of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\geq 0}$. Then β is local (in the sense of Definition 2.5) if and only if α is a morphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}^{\mathrm{Sp}}}(\mathcal{X})$.*

(3) *Let $F : \mathcal{G}_{\mathrm{Zar}}^{\mathrm{nSp}} \to \mathcal{X}$ be a left exact functor, and let $\mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$ be the associated sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then $\mathcal{O}$ is local (in the sense of Definition 2.5) if and only if F is an $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{nSp}}$-structure on $\mathcal{X}$.*

(4) *Let $\alpha : F \to F'$ be a natural transformation between functors $F, F' \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}^{\mathrm{nSp}}}(\mathcal{X})$, and let $\beta : \mathcal{O} \to \mathcal{O}'$ be the induced morphism between local objects of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})$. Then β is local (in the sense of Definition 2.5) if and only if α is a morphism in $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}_{\mathrm{Zar}}^{\mathrm{nSp}}}(\mathcal{X})$.*

Proof. In view of Remark 2.15, assertions (3) and (4) will follow from (1) and (2). We first prove (1). Let $F : \mathcal{G}_{\mathrm{Zar}}^{\mathrm{Sp}} \to \mathcal{X}$ be a left exact functor, and let $F_0 = F|\, \mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}$. Using Remark 2.24, we see that the sheaf of $\mathbb{E}_\infty$-rings $\tau_{\leq 0}\, \mathcal{O}$ is encoded by the composition

$$\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}} \xrightarrow{F_0} \mathcal{X} \xrightarrow{\tau_{\leq 0}} \mathcal{X}\,.$$

Using Remark 2.23, we see that $\mathcal{O}$ is local (in the sense of Definition 2.5) if and only if $\tau_{\leq 0}F_0$ is a $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}$-structure on $\mathcal{X}$. On the other hand, Proposition 2.20 shows that F is a $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{Sp}}$-structure on $\mathcal{X}$ if and only if F_0 is a $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}$-structure on $\mathcal{X}$. To complete the proof of (1), it suffices to show that F_0 is a $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}$-structure on $\mathcal{X}$ if and only if $\tau_{\leq 0}F_0$ is a $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}$-structure on $\mathcal{X}$. The "only if" direction follows from Remark 2.24, and the converse follows from Proposition T.7.2.1.14.

The proof of (2) is similar. Let F_0 and F_0' be the restrictions of F and F' to $\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}$, so that we have a commutative diagram

$$\begin{array}{ccc} F_0 & \xrightarrow{\alpha} & F_0' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ \tau_{\leq 0}F_0 & \xrightarrow{\gamma} & \tau_{\leq 0}F_0' \end{array}$$

in the ∞-category $\mathrm{Fun}(\mathcal{T}_{\mathrm{Zar}}^{\mathrm{Sp}}, \mathcal{X})$. It is easy to see that β is local if and only if γ is local. Proposition V.3.3.3 shows that γ is local if and only if $\gamma \circ \phi \simeq \phi' \circ \alpha$ is local. It therefore suffices to show that $\phi' \circ \alpha$ is local if and only if α is local. This follows from the observation that ϕ' is local (by Proposition V.3.3.3). $\square$

Corollary 2.26. *If k is an $\mathbb{E}_\infty$-ring, then we have an equivalence of ∞-categories*

$$\mathrm{RingTop}(k)_{\mathrm{Zar}} \simeq {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k))^{op}.$$

If k is connective, then we have an equivalence of ∞-categories

$$\mathrm{RingTop}(k)^{\mathrm{cn}}_{\mathrm{Zar}} \simeq {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k))^{op}.$$

Here the subcategories $\mathrm{RingTop}(k)_{\mathrm{Zar}}, \mathrm{RingTop}(k)^{\mathrm{cn}}_{\mathrm{Zar}} \subseteq \mathrm{RingTop}(k)$ *are as in Definition 2.5.*

Definition 2.27 (Spectral Schemes: Abstract Definition)**.** Let k be an $\mathbb{E}_\infty$-ring. A *nonconnective spectral k-scheme* is a pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, where $\mathcal{X}$ is an ∞-topos, $\mathcal{O}_\mathcal{X} : \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k) \to \mathcal{X}$ is an $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$-structure on $\mathcal{X}$, and the pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$-scheme in the sense of Definition V.2.3.9. In this case, we will often abuse notation and identify $\mathcal{O}_\mathcal{X}$ with a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ (Remark 2.17), which we refer to as the *structure sheaf* of $(\mathcal{X}, \mathcal{O}_\mathcal{X})$. We let $\mathrm{SpSch}(k)^{\mathrm{nc}}$ denote the full subcategory of $\mathrm{RingTop}(k)_{\mathrm{Zar}}$ spanned by the nonconnective spectral k-schemes.

If k is connective, we define a *spectral k-scheme* to be a pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$, where $\mathcal{X}$ is an ∞-topos, $\mathcal{O}_\mathcal{X} : \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{X}$ is a $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$-structure on $\mathcal{X}$, and the pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$-scheme. Again, we will generally abuse notation and identify $\mathcal{O}_\mathcal{X}$ with an object of the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X})_{\geq 0}$ of connective sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We let $\mathrm{SpSch}(k)$ denote the full subcategory of $\mathrm{RingTop}(k)^{\mathrm{cn}}_{\mathrm{Zar}}$ spanned by the connective spectral k-schemes.

In the special case where k is the sphere spectrum, we will refer to a (nonconnective) spectral k-scheme simply as a (nonconnective) *spectral scheme.*

Remark 2.28. It follows from Corollary 2.26 that Definitions 2.7 and 2.27 agree (when we take the $\mathbb{E}_\infty$-ring k in Definition 2.27 to be the sphere spectrum S). In particular, we have isomorphisms of ∞-categories

$$\mathrm{SpSch} \simeq \mathrm{SpSch}(S) \qquad \mathrm{SpSch}^{\mathrm{nc}} \simeq \mathrm{SpSch}(S)^{\mathrm{nc}}.$$

Remark 2.29. In §9, we will see that the ∞-category of spectral k-schemes $\mathrm{SpSch}(k)$ can be identified with the ∞-category $\mathrm{SpSch}_{/\,\mathrm{Spec}^{\mathrm{Z}}(k)}$ of spectral schemes $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ equipped with a map $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to \mathrm{Spec}^{\mathrm{Z}}(k)$.

We now discuss the relationship between the theories of spectral schemes and nonconnective spectral schemes.

Proposition 2.30. *Let $i : \mathcal{G} \to \mathcal{G}'$ be a fully faithful transformation of geometries. Let $\mathcal{X}$ be an ∞-topos, let $\mathcal{O} : \mathcal{G} \to \mathcal{X}$ be a $\mathcal{G}$-structure on $\mathcal{X}$, and let $\mathcal{O}' : \mathcal{G}' \to \mathcal{X}$ be a left Kan extension of $\mathcal{O}$. Assume that:*

- (a) *The collection of admissible morphisms in $\mathcal{G}'$ is generated (under the formation of pullbacks) by morphisms of the form $i(\alpha)$, where α is an admissible morphism in $\mathcal{G}$.*
- (b) *For every covering sieve $(\mathcal{G}')^{(0)}_{/X} \subseteq \mathcal{G}'_{/X}$ of an object $X \in \mathcal{G}'$, there exists a morphism $X \to i(Y)$ and an admissible covering $\{V_\beta \to Y\}$ in $\mathcal{G}$, such that each of the pullback maps $i(V_\beta) \times_{i(Y)} X$ belongs to $\mathcal{G}'^{(0)}_{/X}$.*

Then $\mathcal{O}'$ is a $\mathcal{G}'$-structure on $\mathcal{X}$. Moreover, the identity map $(\mathcal{X}, \mathcal{O}) \simeq (\mathcal{X}, \mathcal{O}' \,|\, \mathcal{G})$ exhibits $(\mathcal{X}, \mathcal{O}')$ as a relative spectrum of $(\mathcal{X}, \mathcal{O})$ (see the discussion following Definition V.2.1.2). In particular, if $(\mathcal{X}, \mathcal{O})$ is a $\mathcal{G}$-scheme, then $(\mathcal{X}, \mathcal{O}')$ is a $\mathcal{G}'$-scheme.

Proof. We first show that the functor $\mathcal{O}'$ is left exact. Using Theorem T.5.1.5.6, we can assume that $\mathcal{O}$ factors as a composition

$$\mathcal{G} \xrightarrow{j} \mathcal{P}(\mathcal{G}) \xrightarrow{F} \mathcal{X},$$

where F is a functor which preserves small colimits. Since $\mathcal{O}'$ is a left Kan extension of $\mathcal{O}$, it is equivalent to the composition

$$\mathcal{G}' \to \mathcal{P}(\mathcal{G}') \xrightarrow{\circ i} \mathcal{P}(\mathcal{G}) \xrightarrow{F} \mathcal{O}\,.$$

Since the inclusion $\mathcal{G}' \to \mathcal{P}(\mathcal{G}')$ is left exact (Proposition T.5.1.3.2 and composition with i obviously gives a left exact functor from $\mathcal{P}(\mathcal{G}')$ to $\mathcal{P}(\mathcal{G})$, it suffices to show that F is left exact, which follows from Proposition T.6.1.5.2.

Conditions (a) and (b) guarantee that an arbitrary left exact functor $\mathcal{F} : \mathcal{G}' \to \mathcal{X}$ is a $\mathcal{G}'$-structure on $\mathcal{X}$ if and only if $\mathcal{F} \,|\, \mathcal{G}$ is a $\mathcal{G}$-structure on $\mathcal{X}$. Since $\mathcal{O}' \,|\, \mathcal{G} = \mathcal{O}$ is a $\mathcal{G}$-structure by assumption, we deduce that $\mathcal{O}'$ is a $\mathcal{G}$-structure on $\mathcal{X}$.

We complete the proof by showing that $(\mathcal{X}, \mathcal{O}')$ is a relative spectrum of $(\mathcal{X}, \mathcal{O})$. Let $(\mathcal{Y}, \mathcal{O}'')$ be an arbitrary object of ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}')$. We have a diagram

$$\begin{array}{ccc} \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}')}((\mathcal{X}, \mathcal{O}'), (\mathcal{Y}, \mathcal{O}'')) & \xrightarrow{\theta} & \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})}((\mathcal{X}, \mathcal{O}), (\mathcal{Y}, \mathcal{O}'' \,|\, \mathcal{G})) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})}(\mathcal{X}, \mathcal{Y}), & \end{array}$$

and we wish to show that θ is a homotopy equivalence. It will suffice to show that θ induces a homotopy equivalence after passing to the homotopy fiber over a geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$. Equivalently, we wish to show that the map

$$\theta' : \mathrm{Map}_{\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}'}(\mathcal{Y})}(f^* \, \mathcal{O}', \mathcal{O}'') \to \mathrm{Map}_{\mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{Y})}(f^* \, \mathcal{O}, \mathcal{O}'' \,|\, \mathcal{G})$$

is a homotopy equivalence. Using (a), we deduce that θ' is a homotopy pullback of the restriction functor

$$\theta'' : \mathrm{Map}_{\mathrm{Fun}(\mathcal{G}', \mathcal{Y})}(f^* \, \mathcal{O}', \mathcal{O}'') \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{G}, \mathcal{Y})}(f^* \, \mathcal{O}, \mathcal{O}'' \,|\, \mathcal{G}).$$

To prove this, it suffices to show that $f^* \, \mathcal{O}'$ is a left Kan extension of $f^* \, \mathcal{O}$. This is clear, since $\mathcal{O}'$ is a left Kan extension of $\mathcal{O}$ and the functor f^* preserves small colimits. □

Corollary 2.31. *Let $i : \mathcal{G} \to \mathcal{G}'$ be a fully faithful transformation of geometries satisfying conditions (a) and (b) of Proposition 2.30. Then composition with i induces a colocalization functor $L' : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}') \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})$: that is, L' admits a fully faithful left adjoint, given by the relative spectrum functor $\mathrm{Spec}_{\mathcal{G}}^{\mathcal{G}'}$.*

Corollary 2.32. *Let k be a connective $\mathbb{E}_\infty$-ring. Let U denote the relative spectrum functor associated to the inclusion of geometries $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$. Then U induces a fully faithful embedding*

$$\mathrm{RingTop}(k)^{\mathrm{cn}}_{\mathrm{Zar}} \simeq {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)) \simeq \mathrm{RingTop}(k)_{\mathrm{Zar}}$$

whose essential image consists of those pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{O}_{\mathcal{X}}$ is a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then $\mathcal{O}$ can be encoded in several ways:

(i) As an object of the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{X}) \subseteq \mathrm{Fun}(\mathcal{X}^{op}, \mathrm{CAlg})$.

(ii) As a left-exact functor $\mathcal{F} : \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}} \to \mathcal{X}$.

(iii) As a left-exact functor $\mathcal{F}' : \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}} \to \mathcal{X}$.

Here we can identify $\mathcal{F}'$ with the restriction $\mathcal{F} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}$. Moreover, the condition that $\mathcal{O}$ be connective translates into the requirement that $\mathcal{F}$ be a left Kan extension of $\mathcal{F}'$. Proposition 2.30 implies that if $(\mathcal{X}, \mathcal{F}')$ is a connective spectral scheme, then $(\mathcal{X}, \mathcal{F})$ is a spectral scheme. We have the following strong converse:

Proposition 2.33. *Let k be a connective $\mathbb{E}_\infty$-ring, and suppose that $(\mathcal{X}, \mathcal{O})$ is a nonconnective spectral k-scheme. Then $(\mathcal{X}, \mathcal{O} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k))$ is a spectral k-scheme.*

Corollary 2.34. *Let k be a connective $\mathbb{E}_\infty$-ring, and let $U : \mathrm{RingTop}(k)^{\mathrm{cn}}_{\mathrm{Zar}} \to \mathrm{RingTop}(k)_{\mathrm{Zar}}$ be the relative spectrum functor associated to the inclusion of geometries $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \hookrightarrow \mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$. Then U induces a fully faithful embedding $\mathrm{SpSch}(k) \to \mathrm{SpSch}(k)^{\mathrm{nc}}$, whose essential image consists of those nonconnective spectral k-schemes $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ whose structure sheaf $\mathcal{O}_\mathcal{X}$ is connective.*

Proof. Combine Corollary 2.32 with Proposition 2.33. □

Proof of Proposition 2.33. Let $\mathrm{Spec} : \mathrm{CAlg}_k \to \mathrm{RingTop}(k)_{\mathrm{Zar}}$ denote the absolute spectrum functor associated to the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$, and let $\mathrm{Spec}^{\mathrm{cn}} : \mathrm{CAlg}^{\mathrm{cn}}_k \to \mathrm{RingTop}(k)^{\mathrm{cn}}_{\mathrm{Zar}}$ be the absolute spectrum functor associated to $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$. The assertion is local on $\mathcal{X}$, so we may assume that $(\mathcal{X}, \mathcal{O})$ is an affine nonconnective spectral scheme of the form $\mathrm{Spec}(A)$, where $A \in \mathrm{CAlg}_k$. We will prove that $(\mathcal{X}, \mathcal{O} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k))$ can be identified with the absolute spectrum $\mathrm{Spec}^{\mathrm{cn}}(B)$, where $B = \tau_{\geq 0} A$ is the connective cover of A.

Let $\mathcal{F} : \mathcal{X}^{op} \to \mathrm{CAlg}_k$ correspond to the structure sheaf $\mathcal{O}$ under the equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k), \mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}_k}(\mathcal{X}^{op})$. Similarly, we let $\mathcal{F}_0 : \mathcal{X}^{op} \to (\mathrm{CAlg}_k)_{\geq 0}$ be the functor corresponding to $\mathcal{O} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$, so that $\mathcal{F}_0 \simeq \tau_{\geq 0} \circ \mathcal{F}$.

We now appeal to the construction of the absolute spectrum $\mathrm{Spec}(A)$ given in §V.2.2. Let $\mathcal{C}$ denote the full subcategory of $(\mathrm{CAlg}_A)^{op}$ spanned by those A-algebras of the form $A[\frac{1}{a}]$, where $a \in \pi_0 A$. Using Theorem V.2.2.12, we can identify $\mathcal{X}$ with $\mathcal{S}\mathrm{hv}(\mathcal{C})$ (where we take sheaves with respect to the topology on $\mathcal{C}$ determined by the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$), and $\mathcal{F}$ with the sheafification of the presheaf given by the composition $\mathcal{C}^{op} \hookrightarrow \mathrm{CAlg}_A \to \mathrm{CAlg}_k$. Then $\mathcal{F}_0$ is the sheafification of the composite functor

$$\mathcal{C}^{op} \hookrightarrow \mathrm{CAlg}_A \to \mathrm{CAlg}_k \overset{\tau_{\geq 0}}{\to} (\mathrm{CAlg}_k)_{\geq 0}.$$

Let $\mathcal{C}_0$ be the full subcategory of $(\mathrm{CAlg}_{B/})^{op}$ spanned by those B-algebras of the form $B[\frac{1}{a}]$, where $a \in \pi_0 B \simeq \pi_0 A$. Proposition A.7.2.2.24 implies that the functor $R \mapsto \tau_{\geq 0} R$ determines an equivalence of ∞-categories $\mathcal{C}_0 \to \mathcal{C}$ (the inverse equivalence is given by the formula $R \mapsto R \otimes_A B$). It follows that we can identify $\mathcal{F}_0$ with the sheafification of the composition $\mathcal{C}_0^{op} \hookrightarrow (\mathrm{CAlg}_B)_{\geq 0} \hookrightarrow (\mathrm{CAlg}_k)_{\geq 0}$. Invoking Theorem V.2.2.12 again, we deduce that $(\mathcal{X}, \mathcal{O} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)) \simeq \mathrm{Spec}^{\mathrm{cn}}(B)$ as required. □

Remark 2.35. In the proof of Proposition 2.33, the sheafification is not needed: Theorem 5.14 guarantees that the functors

$$\mathcal{C}^{op} \hookrightarrow \mathrm{CAlg}_A \to \mathrm{CAlg}_k$$

$$\mathcal{C}_0^{op} \hookrightarrow (\mathrm{CAlg}_B)_{\geq 0} \to (\mathrm{CAlg}_k)_{\geq 0}$$

are already sheaves with respect to the Zariski topologies on $\mathcal{C}$ and $\mathcal{C}_0$.

Definition 2.36. Let k be an $\mathbb{E}_\infty$-ring. We will say that a spectral k-scheme $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is *n-localic* if the ∞-topos $\mathcal{X}$ is n-localic (Definition T.6.4.5.8). If k is connective and $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a connective spectral k-scheme, then we will say that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is *n-truncated* if $\mathcal{O}_\mathcal{X}$ is n-truncated, when regarded as a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Combining Corollary 2.22 with Theorem V.2.5.16, we obtain the following relationship between classical and spectral algebraic geometry:

Proposition 2.37. *Let k be a connective $\mathbb{E}_\infty$-ring, and let $\mathrm{Sch}^{\leq 0}_{\leq 0}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k))$ denote the full subcategory of $\mathrm{Sch}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k))$ spanned by the 0-localic, 0-truncated connective spectral k-schemes. Then $\mathrm{Sch}^{\leq 0}_{\leq 0}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k))$ is canonically equivalent to (the nerve of) the category of schemes over the commutative ring $\pi_0 k$, in the sense of classical algebraic geometry.*

Remark 2.38. Let k be a connective $\mathbb{E}_\infty$-ring and A a connective k-algebra, and consider the spectral k-scheme $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ given by the spectrum of A. The underlying ∞-topos $\mathcal{X}$ can be identified with $\mathcal{S}\mathrm{hv}(X)$, where X is the Zariski spectrum of the ordinary commutative ring $\pi_0 A$. This follows from Proposition V.3.4.15, but can also be deduced from the explicit construction provided by Theorem V.2.2.12, since the ∞-category of admissible A-algebras in CAlg_k is equivalent to the nerve of the ordinary category of commutative $(\pi_0 A)$-algebras having the form $(\pi_0 A)[\frac{1}{a}]$. In other words, the local topology of spectral schemes is no more complicated than the local topology of ordinary schemes.

Remark 2.39. Let k be a connective $\mathbb{E}_\infty$-ring, and let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral k-scheme; we will abuse notation by identify $\mathcal{O}_\mathcal{X}$ with the underlying CAlg_k-valued sheaf on $\mathcal{X}$. It follows from Proposition 2.33 that the pair $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_\mathcal{X})$ is a connective spectral k-scheme; we will refer to $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_\mathcal{X})$ as the *underlying connective spectral k-scheme* of $(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Using Proposition V.3.4.15, we conclude that each $(\mathcal{X}, \tau_{\leq n} \tau_{\geq 0} \mathcal{O}_\mathcal{X})$ is an n-truncated connective spectral k-scheme. In particular, if $\mathcal{X}$ is 0-localic and we take $n = 0$, then we obtain a 0-localic, 0-truncated connective spectral k-scheme $(\mathcal{X}, \pi_0 \mathcal{O}_\mathcal{X})$, which we can identify with an ordinary $(\pi_0 k)$-scheme (Proposition 2.37). We will refer to $(\mathcal{X}, \pi_0 \mathcal{O}_\mathcal{X})$ as the *underlying ordinary scheme* of $(\mathcal{X}, \mathcal{O}_\mathcal{X})$.

We conclude this section with a characterization of the class of 0-localic spectral schemes, which establishes the equivalence of Definitions 2.27 and 2.2.

Theorem 2.40. *Let k be an $\mathbb{E}_\infty$-ring, let $\mathcal{X}$ be a 0-localic ∞-topos, $\mathcal{O}_\mathcal{X}$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, which we will identify with a $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$-structure on $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a nonconnective spectral scheme if and only if the following conditions are satisfied:*

(1) *The truncation $(\mathcal{X}, \pi_0 \mathcal{O}_\mathcal{X})$ is a 0-localic, 0-truncated connective spectral k-scheme, corresponding to an ordinary scheme $(X, \mathcal{O}_X)$ (see Proposition 2.37).*

(2) *For every integer i, $\pi_i \mathcal{O}_\mathcal{X}$ determines a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X.*

(3) *The 0th space of the structure sheaf $\mathcal{O}_\mathcal{X}$ is hypercomplete, when viewed as an object of $\mathcal{X}$ (see §T.6.5.2).*

Moreover, $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is affine if and only if $(X, \mathcal{O}_X)$ is an affine scheme.

Proof. First suppose that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a spectral scheme. We will prove that (1), (2), and (3) are satisfied. Assertion (1) follows immediately from Remark 2.39. The remaining assertions are local on X (for assertion (3), this follows from Remark T.6.5.2.22), so we may assume without loss of generality that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an affine spectral scheme, given by the spectrum of an $\mathbb{E}_\infty$-ring A. Then $\mathcal{X}$ can be identified with the ∞-topos $\mathcal{S}\mathrm{hv}(X)$, where X is the set of prime ideals in the commutative ring $\pi_0 A$, with a basis of open sets given by $U_f = \{\mathfrak{p} \subseteq \pi_0 A : f \notin \mathfrak{p}\}$, where f ranges over the collection of elements of $\pi_0 A$ (Remark 2.38). Using Theorem V.2.2.12 and Proposition V.4.3.23, we can identify $\mathcal{O}_\mathcal{X}$ with the CAlg-valued sheaf described by the formula $U_f \mapsto A[\frac{1}{f}]$ (Theorem 5.14 guarantees that this prescription is already a sheaf with respect to the Zariski topology on X). In particular, $\pi_i \mathcal{O}_\mathcal{X}$ is the sheafification of the presheaf of $\pi_0 A$-modules described by the formula $U_f \mapsto (\pi_i A)[\frac{1}{f}]$, which is the quasi-coherent sheaf associated to the module $\pi_i A$; this proves (2). To prove (3), we are free to replace A by its connective cover (since this does not change the 0th space of $\mathcal{O}_\mathcal{X}$). Choose a Postnikov tower

$$\ldots \to \tau_{\leq 2} A \to \tau_{\leq 1} A \to \tau_{\leq 0} A,$$

for A, and let

$$\ldots \to \mathcal{O}^{\leq 2}_\mathcal{X} \to \mathcal{O}^{\leq 1}_\mathcal{X} \to \mathcal{O}^{\leq 0}_\mathcal{X}$$

be the associated $\mathrm{CAlg}_{\geq 0}$-valued sheaves on $\mathcal{X}$. Using the formula above, we conclude that the canonical map $\mathcal{O}_\mathcal{X} \to \varprojlim \{\mathcal{O}^{\leq n}_\mathcal{X}\}$ is an equivalence. To prove (3), it will therefore suffice to show that each $\mathcal{O}^{\leq n}_\mathcal{X}$ has a hypercomplete 0th space, which is clear (since each $\mathcal{O}^{\leq n}_\mathcal{X}$ is n-truncated).

We now prove the converse. Suppose that (1), (2), and (3) are satisfied; we wish to prove that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a spectral scheme. The assertion is local on X, so we may assume without loss of generality that $(X, \pi_0 \mathcal{O}_\mathcal{X})$ is an affine scheme, given by the spectrum of a commutative ring R. We will show that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an affine spectral scheme. We begin by treating the case where the structure sheaf $\mathcal{O}_\mathcal{X}$ is connective (as a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$).

Applying (2), we conclude that each $\pi_i \mathcal{O}_{\mathcal{X}}$ is the quasi-coherent sheaf associated to an R-module M_i. For each $n \geq 0$, let $A_{\leq n} \in \mathrm{CAlg}$ denote the global sections $\Gamma(\mathcal{X}; \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$. There is a convergent spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(X; \pi_q(\tau_{\leq n} \mathcal{O}_{\mathcal{X}})) \Rightarrow \pi_{q-p} A_{\leq n}.$$

Since X is affine, the quasi-coherent sheaves $\pi_i \mathcal{O}_{\mathcal{X}}$ have no cohomology in positive degrees, and the above spectral sequence degenerates to yield isomorphisms

$$\pi_i A_{\leq n} \simeq \begin{cases} M_i & \text{if } i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\pi_0 A_{\leq n} \simeq R$.

Fix $n \geq 0$, and let $(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ be the spectrum of $A_{\leq n}$. The equivalence $A_n \simeq \Gamma(\mathcal{X}; \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ induces a map $\phi_n : (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \to (\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ in $\mathrm{RingTop}_{\mathrm{Zar}}$. The above argument shows that the induced geometric morphism $\phi_n^* : \mathcal{X}_n \to \mathcal{X}$ is an equivalence of ∞-topoi, and that ϕ_n induces an isomorphism of quasi-coherent sheaves $\phi_n^*(\pi_i \mathcal{O}_{\mathcal{X}_n}) \simeq \pi_i \mathcal{O}_{\mathcal{X}}$ for $0 \leq i \leq n$. Since the structure sheaves on both sides are n-truncated, we conclude that ϕ_n is an equivalence.

Let $A \in \mathrm{CAlg}$ be a limit of the tower

$$\ldots \to A_{\leq 2} \to A_{\leq 1} \to A_{\leq 0},$$

so that $\pi_0 A \simeq R$. We can therefore identify the spectrum of A with $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}})$. The first part of the proof shows that $\mathcal{O}'_{\mathcal{X}}$ is the inverse limit of its truncations $\tau_{\leq n} \mathcal{O}'_{\mathcal{X}} \simeq \phi_n^* \mathcal{O}_{\mathcal{X}_n} \simeq \tau_{\leq n} \mathcal{O}_{\mathcal{X}}$. Passing to the inverse limit, we obtain a map $\psi : \mathcal{O}_{\mathcal{X}} \to \lim\{\tau_{\leq n} \mathcal{O}_{\mathcal{X}}\} \simeq \mathcal{O}'_{\mathcal{X}}$. By construction, ψ induces an isomorphism on all (sheaves of) homotopy groups, and is therefore ∞-connective. Since the 0th space of $\mathcal{O}'_{\mathcal{X}}$ is hypercomplete (being the inverse limit of truncated objects of $\mathcal{X}$) and the 0th space of $\mathcal{O}_{\mathcal{X}}$ is hypercomplete by (3), we deduce that ψ is an equivalence, so that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Spec}^{\mathrm{Z}}(A)$ is an affine spectral scheme.

We now treat the general case. The pair $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_{\mathcal{X}})$ satisfies conditions (1), (2), and (3), so the argument above proves that $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Spec}^{\mathrm{Z}}(A)$ for some connective $\mathbb{E}_\infty$-ring A. Let $B \in \mathrm{CAlg}$ be the $\mathbb{E}_\infty$-ring of global sections of $\mathcal{O}_{\mathcal{X}}$. Then $\tau_{\geq 0} B$ is connective cover of the algebra of global sections of $\tau_{\geq 0} \mathcal{O}_{\mathcal{X}}$, and is therefore equivalent to A. In particular, we can identify $\mathrm{Spec}^{\mathrm{Z}}(B)$ with $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}})$, for some sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O}'_{\mathcal{X}}$ on $\mathcal{X}$. To complete the proof, it will suffice to show that the canonical map $\theta : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is an equivalence. Let $\mathcal{F}$ denote the fiber of the map θ, viewed as an object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Since θ induces an equivalence on the level of connective covers, we deduce that $\tau_{\geq 0} \mathcal{F} \simeq 0$. We wish to prove that $\mathcal{F} \simeq 0$. Suppose otherwise. Since $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is right complete (Proposition 1.7), we deduce that there exists an integer n (necessarily positive) such that $\pi_n \mathcal{F}$ is nonzero. We will assume that n is chosen minimal with respect to this property. We have an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$\pi_{1-n} \mathcal{O}'_{\mathcal{X}} \to \pi_{1-n} \mathcal{O}_{\mathcal{X}} \to \pi_{-n} \mathcal{F} \to \pi_{-n} \mathcal{O}'_{\mathcal{X}} \to \pi_{-n} \mathcal{O}_{\mathcal{X}}.$$

The homotopy groups of $\mathcal{O}_{\mathcal{X}}$ are quasi-coherent sheaves on X by (2). Since $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}})$ is a spectral scheme, it also satisfies (2) (by the first part of the proof), so that homotopy groups of $\mathcal{O}'_{\mathcal{X}}$ are also quasi-coherent. It follows that $\pi_{-n} \mathcal{F}$ is a nonzero quasi-coherent sheaf on the affine scheme $(X, \mathcal{O}_X)$, and therefore has a nonvanishing global section. The minimality of n guarantees that $\pi_{-n}\Gamma(X; \mathcal{F}) \simeq \Gamma(X; \pi_{-n} \mathcal{F})$, so that the spectrum $\Gamma(X; \mathcal{F})$ is nonzero. But $\Gamma(X; \mathcal{F})$ can be identified with the fiber of the map of global sections $\Gamma(X; \mathcal{O}'_{\mathcal{X}}) \to \Gamma(X; \mathcal{O}_{\mathcal{X}})$, which is equivalent to the identity map on the $\mathbb{E}_\infty$-ring B. We therefore obtain a contradiction, which completes the proof. □

Remark 2.41. Since the assertion of Theorem 2.40 is local, it generalizes easily to the case where $\mathcal{X}$ is not assumed to be 0-localic; we leave the formulation of this generalization to the reader.

3 Coherent ∞-Topoi

Let $(X, \mathcal{O}_X)$ be a scheme. Recall that $(X, \mathcal{O}_X)$ is said to be *quasi-compact* if the topological space X is quasi-compact: that is, if every covering of X admits a finite subcovering. The scheme $(X, \mathcal{O}_X)$ is said to be *quasi-separated* if, whenever U and V are quasi-compact open subsets of X, the intersection $U \cap V$ is also quasi-compact.

Our goal in this section is to introduce a hierarchy of compactness conditions, generalizing the notions of quasi-compactness and quasi-separatedness to the ∞-categorical setting. Although our main goal is to apply these ideas in the setting of spectral schemes and spectral Deligne-Mumford stacks, we will begin with a general discussion which makes sense in any ∞-topos.

Definition 3.1. Let $\mathcal{X}$ be an ∞-topos. We will say that $\mathcal{X}$ is *quasi-compact* if every covering of $\mathcal{X}$ has a finite subcovering: that is, for every effective epimorphism $\coprod_{i \in I} U_i \to \mathbf{1}$ in $\mathcal{X}$ (where $\mathbf{1}$ is the final object of $\mathcal{X}$), there exists a finite subset $I_0 \subseteq I$ such that the map $\coprod_{i \in I} U_i \to \mathbf{1}$ is also an effective epimorphism. We say that an object $X \in \mathcal{X}$ is *quasi-compact* if the ∞-topos $\mathcal{X}_{/X}$ is quasi-compact.

Let $n \geq 0$ be an integer. We will define the notion of an *n-coherent ∞-topos* by induction on n. We say that an ∞-topos $\mathcal{X}$ is *0-coherent* if it is quasi-compact. Assume that we have defined the notion of an n-coherent ∞-topos for some $n \geq 0$. We will say that an object $U \in \mathcal{X}$ of an ∞-topos $\mathcal{X}$ is *n-coherent* if the ∞-topos $\mathcal{X}_{/U}$ is n-coherent. We say that $\mathcal{X}$ is *locally n-coherent* if, for every object $X \in \mathcal{X}$, there exists an effective epimorphism $\coprod_i U_i \to X$, where each U_i is n-coherent. We say that $\mathcal{X}$ is *$(n+1)$-coherent* if it is locally n-coherent, and the collection of n-coherent objects of $\mathcal{X}$ is closed under finite products.

Remark 3.2. Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is quasi-compact if and only if, for every collection of (-1)-truncated objects $\{U_i \in \mathcal{X}\}_{i \in I}$ such that $\tau_{\leq -1}(\coprod_{i \in I} U_i)$ is a final object of $\mathcal{X}$, there exists a finite subset $I_0 \subseteq I$ such that $\tau_{\leq -1}(\coprod_{i \in I_0} U_i)$ is a final object of $\mathcal{X}$. In particular, the condition that $\mathcal{X}$ is quasi-compact depends only on the underlying locale $\tau_{\leq -1} \mathcal{X}$.

Remark 3.3. Let $\mathcal{X}$ be an n-coherent ∞-topos for $n > 0$. The collection of $(n-1)$-coherent objects of $\mathcal{X}$ is stable under finite products. In particular, the final object of $\mathcal{X}$ is $(n-1)$-coherent, so that $\mathcal{X}$ is $(n-1)$-coherent. It follows that an n-coherent ∞-topos is also m-coherent for each $m \leq n$.

Remark 3.4. Let $\mathcal{X}$ be a locally n-coherent ∞-topos. Then $\mathcal{X}_{/U}$ is locally n-coherent for any object $U \in \mathcal{X}$. In this case, an object $X \in \mathcal{X}$ is $(n+1)$-coherent if and only if it is n-coherent and, for every pullback diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

in $\mathcal{X}$, if U and V are n-coherent, then $U \times_X V$ is also n-coherent.

Remark 3.5. Suppose that $\mathcal{X} = \prod_{1 \leq i \leq k} \mathcal{X}_i$ is a product of finitely many ∞-topoi (corresponding to a *coproduct* in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}$). Then $\mathcal{X}$ is n-coherent if and only if each $\mathcal{X}_i$ is n-coherent. It follows that if $\mathcal{Y}$ is any ∞-topos, then a finite coproduct $U = \coprod_{1 \leq i \leq k} U_i$ in $\mathcal{Y}$ is n-coherent if and only if each U_i is n-coherent.

Remark 3.6. Let $\mathcal{X}$ be a locally n-coherent ∞-topos and let $X \in \mathcal{X}$ be a quasi-compact object. The assumption that $\mathcal{X}$ is locally n-coherent guarantees the existence of an effective epimorphism $\coprod_{i \in I} U_i \to X$, where each U_i is n-coherent. Since X is quasi-compact, we may assume that the index set I is finite. Then $U = \coprod_{i \in I} U_i$ is n-coherent by Remark 3.5. It follows that there exists an effective epimorphism $U \to X$, where U is n-coherent.

Definition 3.7. Let $\mathcal{X}$ be an ∞-topos which is locally n-coherent. We will say that a morphism $f : X' \to X$ in $\mathcal{X}$ is *relatively n-coherent* if, for every n-coherent object $U \in \mathcal{X}$ and every morphism $U \to X$, the fiber product $U \times_X X'$ is also n-coherent.

Example 3.8. Let $\mathcal{X}$ be a locally n-coherent ∞-topos. If $f : X' \to X$ is a morphism such that X' is n-coherent and X is $(n+1)$-coherent, then f is relatively n-coherent.

Proposition 3.9. *Let $n \geq 0$ be an integer and $\mathcal{X}$ an ∞-topos, and let $f : X_0 \to X$ be a morphism in $\mathcal{X}$. Assume that if $n > 0$, then $\mathcal{X}$ is locally $(n-1)$-coherent and that f is relatively $(n-1)$-coherent. Then:*

(1) *The map f is relatively m-coherent for each $m < n$.*

(2) *Assume that f is an effective epimorphism and that X_0 is n-coherent. Then X is n-coherent.*

Proof. The proof proceeds by induction on n. Suppose first that $n = 0$; we must show that if f is an effective epimorphism and X_0 is quasi-compact, then X is quasi-compact. Choose an effective epimorphism $\coprod_{i \in I} X_i \to X$. Then the induced map $\coprod_{i \in I}(X_i \times_X X_0) \to X_0$ is also an effective epimorphism. Since X_0 is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the map $\coprod_{i \in I_0}(X_i \times_X X_0) \to X_0$ is an effective epimorphism. Since f is an effective epimorphism, we conclude that the composite map

$$\coprod_{i \in I_0}(X_i \times_X X_0) \to X_0 \to X$$

is an effective epimorphism. This map factors through $\phi : \coprod_{i \in I_0} X_i \to X$, so that ϕ is an effective epimorphism as desired.

Now suppose that $n > 0$. We begin by proving (1). Choose a morphism $U \to X$, where U is m-coherent; we must show that $U_0 = U \times_X X_0$ is m-coherent. Remark 3.6 guarantees the existence of an effective epimorphism $g : V \to U$, where V is $(n-1)$-coherent. It follows from Example 3.8 that g is relatively $(m-1)$-coherent. Let $V_0 = V \times_X X_0$ and $g_0 : V_0 \to U_0$ the induced map, so that g_0 is also relatively $(m-1)$-coherent. Our assumption that f is relatively $(n-1)$-coherent guarantees that V_0 is $(n-1)$-coherent, and therefore m-coherent (Remark 3.3). Since g_0 is an effective epimorphism, the inductive hypothesis guarantees that U_0 is m-coherent, as desired.

We now prove (2). We will show that X satisfies the criterion for n-coherence described in Remark 3.4. The inductive hypothesis guarantees that X is $(n-1)$-coherent. Choose maps $U \to X$ and $V \to X$, where U and V are $(n-1)$-coherent; we wish to show that $U \times_X V$ is $(n-1)$-coherent. Let $U_0 = U \times_X X_0$ and $V_0 = V \times_X X_0$. Since f is relatively $(n-1)$-coherent, U_0 and V_0 are $(n-1)$-coherent. Since X_0 is n-coherent, we deduce that $U_0 \times_{X_0} V_0$ is $(n-1)$-coherent. The map $f' : U_0 \times_{X_0} V_0 \to U \times_X V$ is a pullback of f and therefore relatively $(n-2)$-coherent by (1). Since f' is an effective epimorphism, the inductive hypothesis guarantees that $U \times_X V$ is $(n-1)$-coherent, as desired. □

Corollary 3.10. *Let $\mathcal{X}$ be an ∞-topos and suppose we are given a full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ with the following properties:*

(a) *Every object $U \in \mathcal{X}_0$ is an n-coherent object of $\mathcal{X}$*

(b) *For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\coprod U_i \to X$, where each U_i belongs to $\mathcal{X}_0$.*

Then:

(1) *A morphism $f : X' \to X$ in $\mathcal{X}$ is relatively n-coherent if and only if, for every morphism $U \to X$ where $U \in \mathcal{X}_0$, the fiber product $U' = U \times_X X'$ is n-coherent.*

(2) *An object $X \in \mathcal{X}$ is $(n+1)$-coherent if and only if it is quasi-compact and, for every pair of maps $U \to X$, $V \to X$ where $U, V \in \mathcal{X}_0$, the fiber product $U \times_X V$ is n-coherent.*

Proof. We first prove (1). The "only if" direction is obvious. For the converse, choose a map $V \to X$ where V is n-coherent; we wish to show that $V' = V \times_X X'$ is n-coherent. Condition (b) and the quasi-compactness of V guarantee the existence of an effective epimorphism $g : \coprod_{i \in I} U_i \to V$, where each U_i belongs to $\mathcal{X}_0$ and the index set I is finite. Let $g' : \coprod_{i \in I}(U_i \times_X X') \to V'$ be the induced map. Using our hypothesis together with Remark 3.5, we see that $\coprod_{i \in I}(U_i \times_X X')$ is n-coherent. The map g is relatively $(n-1)$-coherent by Example 3.8, so that g' is relatively $(n-1)$-coherent. Applying Proposition 3.9, we deduce that V' is n-coherent as desired.

We now prove (2) using induction on n. The "only if" direction is again obvious. Assume therefore that X is quasi-compact and that $U \times_X V$ is n-coherent whenever $U, V \in \mathcal{X}_0$. We note that X is n-coherent: this follows from the inductive hypothesis if $n > 0$, or by assumption if $n = 0$. Using (1), we see that the map $U \to X$ is relatively n-coherent whenever $U \in \mathcal{X}_0$. Consequently, if V is an arbitrary n-coherent object of $\mathcal{X}$ and we are given a map $g : V \to X$, then $U \times_X V$ is n-coherent for each $U \in \mathcal{X}_0$. Applying (1) again, we deduce that g is relatively n-coherent. It follows that the fiber product $U \times_X V$ is n-coherent whenever U and V are n-coherent, so that X is $(n+1)$-coherent by Remark 3.4. □

Corollary 3.11. *Let $\mathcal{X}$ be a locally n-coherent ∞-topos, and let $f : X' \to X$ be a morphism in $\mathcal{X}$. Suppose that there exists an effective epimorphism $U \to X$ such that the induced map $f' : U' \to U$ is relatively n-coherent, where $U' = X' \times_X U$. Then f is relatively n-coherent.*

Proof. Suppose we are given a map $Y \to X$, where Y is n-coherent. We wish to prove that $Y' = X' \times_X Y$ is n-coherent. Replacing X by Y and U by $Y \times_X U$, we are reduced to proving that if X is n-coherent, then X' is also n-coherent.

Since $\mathcal{X}$ is locally n-coherent, there exists an effective epimorphism $\coprod_{i \in I} U_i \to U$, where each U_i is n-coherent. The composite map $\coprod_{i \in I} U_i \to U \to X$ is also an effective epimorphism. Since X is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the map $\coprod_{i \in I_0} U_i \to X$ is an effective epimorphism. The coproduct $\coprod_{i \in I_0} U_i$ is n-coherent (Remark 3.5). Replacing U by $\coprod_i U_i$, we can reduce to the case where U is n-coherent. Since f' is relatively n-coherent, we deduce that U' is n-coherent. Since X is n-coherent and U is $(n-1)$-coherent, the map $U \to X$ is relatively $(n-1)$-coherent (if $n > 0$), so the induced map $U' \to X'$ is an effective epimorphism which is $(n-1)$-coherent (if $n > 0$). Proposition 3.9 now implies that X' is n-coherent as desired. □

Definition 3.12. Let $\mathcal{X}$ be an ∞-topos. We will say that $\mathcal{X}$ is *coherent* if it is n-coherent for every integer n. We will say that an object $U \in \mathcal{X}$ is *coherent* if the ∞-topos $\mathcal{X}_{/U}$ is coherent. We will say that $\mathcal{X}$ is *locally coherent* if, for every object $X \in \mathcal{X}$, there exists an effective epimorphism $\coprod_i U_i \to X$ where each U_i is coherent.

Example 3.13. Let $\mathcal{X} = \mathcal{S}$ be the ∞-category of spaces. Then $\mathcal{X}$ is coherent and locally coherent. An object $X \in \mathcal{X}$ is n-coherent if and only if the homotopy sets $\pi_i(X, x)$ are finite for every point $x \in X$ and all $i \leq n$.

Remark 3.14. Let $\mathcal{X}$ be an ∞-topos. The collection of coherent objects of $\mathcal{X}$ is closed under the formation of pullbacks.

Lemma 3.15. *Let $\mathcal{X}$ be an ∞-topos and $f^* : \mathcal{X} \to \mathcal{Y}$ a geometric morphism, which exhibits $\mathcal{Y}$ as a cotopological localization of $\mathcal{X}$ (see Definition T.6.5.2.17). Let $n \geq 0$ be an integer, and assume that $\mathcal{X}$ is locally $(n-1)$-coherent if $n > 0$.*

1. *An object $X \in \mathcal{X}$ is n-coherent if and only if $f^* X \in \mathcal{Y}$ is n-coherent.*
2. *An object $Y \in \mathcal{Y}$ is n-coherent if and only if $f_* Y \in \mathcal{X}$ is n-coherent.*
3. *If $n > 0$, the ∞-topos $\mathcal{Y}$ is locally $(n-1)$-coherent.*

Proof. Since f^* is a localization functor, the counit map $f^* f_* Y \to Y$ is an equivalence for each $Y \in \mathcal{Y}$. Consequently, assertion (2) follows from (1), applied to $X = f_* Y$. We prove (1) by induction on n. We first note that the inductive hypothesis implies (3). To see this, assume that $n > 0$ and let $Y \in \mathcal{Y}$, so that $Y \simeq f^* X$ for $X = f_* Y \in \mathcal{X}$. Since $\mathcal{X}$ is locally $(n-1)$-coherent, there exists an effective epimorphism $\coprod V_i \to X$ where each V_i is $(n-1)$-coherent. This induces an effective epimorphism $\coprod f^* V_i \to Y$ in $\mathcal{Y}$, and each $f^* V_i$ is $(n-1)$-coherent by the inductive hypothesis.

We now prove (1) in the case $n = 0$. Suppose that $X \in \mathcal{X}$ is quasi-compact; we wish to show that $f^*X \in \mathcal{Y}$ is quasi-compact. Choose an effective epimorphism $u : \coprod_{i \in I} U_i \to f^*X$ in $\mathcal{Y}$. For $i \in I$, let $V_i = f_*U_i \times_{f_*f^*X} X$, so that $u \simeq f^*v$ for some map $v : \coprod_{i \in I} V_i \to X$. Since f^* is a cotopological localization, the map v is an effective epimorphism. Since X is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the induced map $v' : \coprod_{i \in I_0} V_i \to X$ is an effective epimorphism. It follows that $f^*(v') = u' : \coprod_{i \in I_0} U_i \to f^*X$ is an effective epimorphism as well.

Now suppose that f^*X is quasi-compact. We wish to prove that X is quasi-compact. Choose an effective epimorphism $v : \coprod_{i \in I} V_i \to X$, so that $u = f^*v$ is an effective epimorphism $\mathcal{Y}$. Since f^*X is quasi-compact, there exists a finite subset $I_0 \subseteq I$ such that the induced map $\coprod_{i \in I_0} f^*V_i \to f^*X$ is an effective epimorphism. Since f^* is a cotopological localization, we conclude that the map $\coprod_{i \in I_0} V_i \to X$ is an effective epimorphism.

It remains to prove (1) in the case $n > 0$. Suppose first that X is n-coherent. Using the inductive hypothesis, we deduce that f^*X is $(n-1)$-coherent; moreover, we have already seen that $\mathcal{Y}$ is locally $(n-1)$-coherent. To show that f^*X is n-coherent, it suffices to show that for every pair of maps $U \to f^*X$ and $U' \to f^*X$ where $U, U' \in \mathcal{Y}$ are $(n-1)$-coherent, the fiber product $U \times_{f^*X} U'$ is $(n-1)$-coherent (Remark 3.4). Let $V = f_*U \times_{f_*f^*X} X$ and $V' = f_*U' \times_{f_*f^*X} X$. It follows from the inductive hypothesis that V and V' are $(n-1)$-coherent objects of $\mathcal{X}$, so that $V \times_X V'$ is $(n-1)$-coherent. Applying the inductive hypothesis again, we conclude that $U \times_{f^*X} U' \simeq f^*(V \times_X V')$ is $(n-1)$-coherent.

For the converse, suppose that f^*X is n-coherent. Using the inductive hypothesis, we conclude that X is $(n-1)$-coherent. To show that X is n-coherent, it suffices to show that if we are given morphisms $V \to X$, $V' \to X$ where $V, V' \in \mathcal{X}$ are $(n-1)$-coherent, then $V \times_X V'$ is $(n-1)$-coherent. By the inductive hypothesis, it suffices to show that $f^*(V \times_X V') \simeq f^* \times_{f^*X} f^*V'$ is $(n-1)$-coherent, which follows from our assumption that f^*X is n-coherent. □

Proposition 3.16. *Let $\mathcal{X}$ be a locally coherent ∞-topos, and let $f^* : \mathcal{X} \to \mathcal{Y}$ be geometric morphism. Assume that:*

- (a) *The right adjoint to f^* induces an equivalence of hypercompletions $\mathcal{X}^\wedge \simeq \mathcal{Y}^\wedge$.*
- (b) *For every object $Y \in \mathcal{Y}$, there exists an object $X \in \mathcal{X}$ and an effective epimorphism $f^*X \to Y$.*

Then:

1. *An object $X \in \mathcal{X}$ is coherent if and only if $f^*X \in \mathcal{Y}$ is coherent.*
2. *The ∞-topos $\mathcal{Y}$ is locally coherent.*
3. *The ∞-topos $\mathcal{X}$ is coherent if and only if $\mathcal{Y}$ is coherent.*

Proof. Let $\mathcal{Z} = \mathcal{Y}^\wedge$ be the hypercompletion of $\mathcal{Y}$ and $g^* : \mathcal{Y} \to \mathcal{Z}$ a left adjoint to the inclusion $\mathcal{Y}^\wedge \subseteq \mathcal{Y}$. Then g^* exhibits $\mathcal{Z}$ as a cotopological localization of $\mathcal{Y}$, and our assumption (a) guarantees that $g^* \circ f^*$ exhibits $\mathcal{Z}$ as a cotopological localization of $\mathcal{X}$. Assertion (1) now follows from Lemma 3.15. Assertion (2) follows from (1) and (b). Assertion (3) follows by applying (1) to the final object of $\mathcal{X}$. □

We now produce some examples of coherent ∞-topoi.

Definition 3.17. Let $\mathcal{C}$ be an ∞-category which admits finite limits. We will say that a Grothendieck topology on $\mathcal{C}$ is *finitary* if it satisfies the following condition:

- $(*)$ For every object $C \in \mathcal{C}$ and every covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$, there exists a finite collection of morphisms $\{C_i \to C\}_{1 \leq i \leq n}$ in $\mathcal{C}^{(0)}_{/C}$ which generate a covering sieve on C.

Remark 3.18. Let $\mathcal{C}$ be an ∞-category which admits finite limits, and suppose that $\mathcal{C}$ is equipped with an arbitrary Grothendieck topology. Let $\mathcal{D}$ denote the same ∞-category $\mathcal{C}$, and let us say that a sieve $\mathcal{D}^{(0)}_{/D} \subseteq \mathcal{D}_{/D}$ is *covering* if it contains a finite collection of morphisms $\{D_i \to D\}$ which generate a covering sieve in $\mathcal{C}$. This collection of covering sieves determines a Grothendieck topology on $\mathcal{D}$. This Grothendieck topology is the finest finitary topology on $\mathcal{D}$ which is coarser than the given topology on $\mathcal{C}$.

Proposition 3.19. *Let $\mathcal{C}$ be a small ∞-category which admits pullbacks, which is equipped with a finitary Grothendieck topology. Then:*

(1) *Let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the composition of the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ with the sheafification function $\mathcal{P}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$. Then j carries each object $C \in \mathcal{C}$ to a coherent object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$.*

(2) *The ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is locally coherent.*

(3) *If $\mathcal{C}$ has a final object, then $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is coherent.*

Proof. Since $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is generated by $j(\mathcal{C})$ under small colimits, assertion (2) follows immedaitely from (1). Since j preserves finite limits, it carries final objects of $\mathcal{C}$ to final objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$, so assertion (3) also follows from (1). We will prove the following assertions by induction on n:

($1'$) The functor j carries each object $C \in \mathcal{C}$ to an n-coherent object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$.

($2'$) The ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is locally n-coherent.

It is clear that ($1'$) implies ($2'$). To prove ($1'$), let us first assume that $n = 0$. We must show that for $C \in \mathcal{C}$, the object $j(C) \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ is quasi-compact. Choose an effective epimorphism $\coprod_{i \in I} U_i \to j(C)$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})$. It follows that there exists a covering $\{C_\alpha \to C\}$ in $\mathcal{C}$ such that each of the induced maps $j(C_\alpha) \to j(C)$ factors through U_i for some i. Since the topology on $\mathcal{C}$ is finitary, we may assume that this covering is finite; then we may assume that all of this indices $i \in I$ which are used belong to some finite subset $I_0 \subseteq I$, so that $\coprod_{i \in I_0} U_i \to j(C)$ is an effective epimorphism, as desired.

Now suppose that $n > 0$. Using the inductive hypothesis, we may assume that $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is locally $(n-1)$-coherent and that $j(C)$ is $(n-1)$-coherent for $C \in \mathcal{C}$. We wish to show that $j(C)$ is n-coherent. Without loss of generality, we may replace $\mathcal{C}$ by $\mathcal{C}_{/C}$ and $\mathcal{S}\mathrm{hv}(\mathcal{C})$ by $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/j(C)} \simeq \mathcal{S}\mathrm{hv}(\mathcal{C}_{/C})$. We wish to show that the collection of $(n-1)$-coherent objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is closed under finite products. Using Corollary 3.10, we are reduced to showing that $j(C') \times j(C'')$ is $(n-1)$-coherent, for every pair of objects $C', C'' \in \mathcal{C}$. This is clear, since $j(C') \times j(C'') = j(C' \times C'')$. □

We now prove a converse to Proposition 3.19.

Theorem 3.20. *Let $\mathcal{X}$ be an ∞-topos. The following conditions are equivalent:*

(1) *The ∞-topos $\mathcal{X}$ is locally coherent.*

(2) *There exists a small ∞-category $\mathcal{C}$ which admits pullbacks, a finitary Grothendieck topology on $\mathcal{C}$, and a geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X}$ satisfying conditions (a) and (b) of Proposition 3.16.*

If these conditions are satisfied, then we may assume that $\mathcal{C}$ admits finite coproducts and that the topology on $\mathcal{C}$ is subcanonical (that is, for each $C \in \mathcal{C}$, the functor $\mathrm{Map}_{\mathcal{C}}(\bullet, C)$ represented by C belongs to $\mathcal{S}\mathrm{hv}(\mathcal{C})$). Moreover, if $\mathcal{X}$ is coherent, then we may assume that $\mathcal{C}$ has a final object (and therefore admits all finite limits; see Corollary T.4.4.2.4).

Lemma 3.21. *Let $\mathcal{X}$ be an ∞-topos containing a collection of objects $\{X_i\}_{i \in I}$. For every subset $J \subseteq I$, let $X_J \simeq \coprod_{i \in J} X_i$. If $C \in \mathcal{X}$ is a quasi-compact object, then the canonical map*

$$\varinjlim_{J \subseteq I} \mathrm{Map}_{\mathcal{X}}(C, X_J) \to \mathrm{Map}_{\mathcal{X}}(C, X_I)$$

is a homotopy equivalence, where the colimit is taken over all finite subsets $J \subseteq I$.

Proof. Let J be any subset of I, and let $\phi : C \to X_J$ be a morphism in $\mathcal{X}$. Since colimits in $\mathcal{X}$ are universal, this morphism determines a decomposition $C \simeq \coprod_{i \in J} C_i$, where $C_i = C \times_{X_J} X_i$. We define the *support* of ϕ to be the subset of J consisting of those indices $i \in J$ such that C_i is not an initial object of $\mathcal{X}$.

Let $\phi : C \to X_J$ be any morphism. Since C is quasi-compact, there is a finite subset $J_0 \subseteq J$ such that the map $\coprod_{i \in J_0} C_i \to C$ is an effective epimorphism. For $i' \in J$, we have an effective epimorphism $\coprod_{i \in J_0} C_i \times_C C_{i'} \to C_{i'}$. If $i' \notin J_0$, then the left hand side is an initial object of $\mathcal{X}$ (since coproducts in $\mathcal{X}$ are disjoint), so that $C_{i'}$ is likewise initial object of $\mathcal{X}$. It follows that the support of ϕ is contained in J_0, and is therefore finite.

For each $J \subseteq I$, the mapping space $\mathrm{Map}_{\mathcal{X}}(C, X_J)$ decomposes as a coproduct $\coprod_S \mathrm{Map}^S_{\mathcal{X}}(C, X_J)$, where S ranges over finite subsets of I and $\mathrm{Map}^S_{\mathcal{X}}(C, X_J)$ is the summand of $\mathrm{Map}^S_{\mathcal{X}}(C, X_J)$ given by maps $\phi : C \to X_J$ with support S (by convention, this summand is empty unless $S \subseteq J$). It will therefore suffice to prove that for every finite set S, the map

$$\varinjlim_{J \subseteq I} \mathrm{Map}^S_{\mathcal{X}}(C, X_J) \to \mathrm{Map}^S_{\mathcal{X}}(C, X_I)$$

is a homotopy equivalence. To prove this, we observe that $\mathrm{Map}^S_{\mathcal{X}}(C, X_J) \simeq \mathrm{Map}^S_{\mathcal{X}}(C, X_I)$ whenever $S \subseteq J$. □

Proof of Theorem 3.20. The implication (2) ⇒ (1) follows immediately from Propositions 3.19 and 3.16. Conversely, suppose that $\mathcal{X}$ is locally coherent. Choose a small collection of objects $\{X_\alpha\}$ which generates $\mathcal{X}$ under small colimits. Since $\mathcal{X}$ is locally coherent, for each index α we can choose an effective epimorphism $\coprod_\beta U_{\alpha,\beta} \to X_\alpha$ where $U_{\alpha,\beta}$ is coherent. Let $\mathcal{C}$ denote an essentially small full subcategory of $\mathcal{X}$ such that each object of $\mathcal{C}$ is coherent in $\mathcal{X}$, and each $U_{\alpha,\beta}$ belongs to $\mathcal{C}$. Enlarging this collection if necessary, we may assume that it is closed under pullbacks, finite coproducts, and that it contains the a final object of $\mathcal{X}$ if $\mathcal{X}$ is coherent. Endow $\mathcal{C}$ with the canonical topology determined by the inclusion $i : \mathcal{C} \hookrightarrow \mathcal{X}$, so that i induces a geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X}$. To complete the proof, it will suffice to show that f^* satisfies conditions (a) and (b) of Proposition 3.16.

Let $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the Yoneda embedding. By construction, $f^* \circ j$ is equivalent to the inclusion $\mathcal{C} \hookrightarrow \mathcal{X}$. For every object $X \in \mathcal{X}$, there exists an effective epimorphism $\coprod X_i \to X$ where each X_i belongs to $\{X_\alpha\}$; it follows that there exists an effective epimorphism $\coprod C_i \to X$ where each C_i belongs to $\mathcal{C}$. We therefore have an effective epimorphism $f^* \coprod_i j(C_i) \to X$; this proves (b).

The proof of (a) is more elaborate. Since f^* preserves ∞-connective morphisms, its right adjoint f_* preserves hypercompleteness, and therefore restricts to a functor $f'_* : \mathcal{X}^\wedge \to \mathcal{S}\mathrm{hv}(\mathcal{C})^\wedge$. We wish to show that f'_* is an equivalence of ∞-categories. We first show that f'_* is conservative: that is, if $g : X \to Y$ is map between hypercomplete objects of $\mathcal{X}^\wedge$ such that $f_*(g)$ is an equivalence, then g is an equivalence. Since X and Y are hypercomplete, it will suffice to show that g is n-connective for each n. We proceed by induction on n. When $n = 0$, we must show that g is an effective epimorphism. Choose an object $Z \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ and an effective epimorphism $v : f^*Z \to Y$. Then v is adjoint to a map $v' \in \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(Z, f_*Y)$. Since $f_*(u)$ is an equivalence, the map v' factors through f_*X; it follows that v factors as a composition

$$f^*Z \to X \xrightarrow{g} Y$$

so that g is also an effective epimorphism. If $n > 0$, then (since u is an effective epimorphism) we are reduced to proving that the induced map $\beta : X \to X \times_Y X$ is $(n-1)$-connective. This follows from the inductive hypothesis, since $f_*(\beta)$ is also an equivalence. This completes the proof that f'_* is conservative.

We next prove:

$(*)$ For each $n \geq 0$, the functor f_* carries n-connective morphisms in $\mathcal{X}$ to n-connective morphisms in $\mathcal{S}\mathrm{hv}(\mathcal{C})$.

The proof proceeds by induction on n. We begin by treating the case $n = 0$. Fix an effective epimorphism $g : X \to Y$ in $\mathcal{X}$; we wish to show that $f_*(g)$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Unwinding the definitions, we must show that for every object $C \in \mathcal{C}$ and every morphism $\eta : C \to Y$, there exists a covering sieve on $\{C_i \to C\}$ such that each of the composite maps $C_i \to C \to Y$ factors through g. To prove this, it suffices to choose an effective epimorphism $\coprod C_i \to C \times_Y X$, where each $C_i \in \mathcal{C}$; our assumption that g is an effective epimorphism guarantees that the composite map

$$\coprod C_i \to C \times_Y X \to C$$

is also an effective epimorphism, so that the maps $\{C_i \to C\}$ generate a covering sieve in $\mathcal{C}$.

Now suppose $n > 0$ and that $g : X \to Y$ is an n-connective morphism in $\mathcal{X}$; we wish to show that $f_*(g)$ is an n-connective morphism in $\mathcal{S}\mathrm{hv}(\mathcal{C})$. The above argument shows that $f_*(g)$ is an effective epimorphism; it will therefore suffice to show that the diagonal map $f_*X \to f_*X \times_{f_*Y} f_*X = f_*(X \times_Y X)$ is a $(n-1)$-connective. This follows from the inductive hypothesis.

Let $f'^* : \mathcal{S}\mathrm{hv}(\mathcal{C})^\wedge \to \mathcal{X}^\wedge$ be a left adjoint to f'_*, so that f'^* is given by composing f^* with a left adjoint to the inclusion $\mathcal{X}^\wedge \subseteq \mathcal{X}$. To complete the proof that f'_* is an equivalence, it will suffice to show that the unit map $u'_X : X \to f'_* f'^* X$ is an equivalence for each $X \in \mathcal{S}\mathrm{hv}(\mathcal{C})^\wedge$. This map factors as a composition

$$X \stackrel{u_X}{\to} f_* f^* X \stackrel{u''_X}{\to} f'_* f'^* X,$$

where u''_X is ∞-connective by $(*)$. It will therefore suffice to prove that u_X is ∞-connective (then u'_X will be an ∞-connective morphism between hypercomplete objects of $\mathcal{S}\mathrm{hv}(\mathcal{C})$, and therefore an equivalence). We will show:

$(*')$ For every object $X \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ and every $n \geq 0$, the unit map $u_X : X \to f_* f^* X$ is n-connective.

The proof will use the following assertion:

$(*'')$ Let $X \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ be a coproduct of objects belonging to the essential image of $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$. Then the unit map $u_X : X \to f_* f^* X$ is an equivalence.

Assume $(*'')$ for the moment. We will prove $(*')$ using induction on n. We begin with the case $n = 0$. Fix $X \in \mathcal{S}\mathrm{hv}(\mathcal{C})$; we wish to show that the unit map $u_X : X \to f_* f^* X$ is an effective epimorphism. Since $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is generated under colimits by the essential image of $j : \mathcal{C} \to \mathcal{S}\mathrm{hv}(\mathcal{C})$, we can choose an effective epimorphism $v : X' \to X$ in $\mathcal{S}\mathrm{hv}(\mathcal{C})$, where X' is a coproduct of objects belonging to the essential image of j. We have a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & f_* f^* X' \\ \downarrow & & \downarrow \\ X & \longrightarrow & f_* f^* X. \end{array}$$

It will therefore suffice to show that the composite map $X' \to f_* f^* X' \to f_* f^* X$ is an effective epimorphism. Using $(*'')$, we are reduced to proving that $f_* f^*(v)$ is an effective epimorphism. Since v is an effective epimorphism, $f^*(v)$ is an effective epimorphism in $\mathcal{X}$, so that $f_* f^*(v)$ is an effective epimorphism by $(*)$.

Now suppose $n > 0$. We wish to prove that u_X is n-connective. The argument above shows that u_X is an effective epimorphism; it will therefore suffice to show that the diagonal map $\beta : X \to X \times_{f_* f^* X} X$ is $(n-1)$-connective. Let X' be as above so that we have a pullback diagram

$$\begin{array}{ccc} X' \times_X X' & \stackrel{\beta'}{\longrightarrow} & X' \times_{f_* f^* X} X' \\ \downarrow & & \downarrow \\ X & \stackrel{\beta}{\longrightarrow} & X \times_{f_* f^* X} X. \end{array}$$

Since the vertical maps are effective epimorphisms, it will suffice to show that β' is $(n-1)$-connective. Since the evident map $X' \times_{f_* f^* X} X' \to (f_* f^* X') \times_{f_* f^* X} (f_* f^* X')$ is an equivalence by $(*'')$, it will suffice to show that the composition

$$X' \times_X X' \stackrel{\beta'}{\to} X' \times_{f_* f^* X} X' \to (f_* f^* X') \times_{f_* f^* X} (f_* f^* X') \simeq f_* f^*(X' \times_X X')$$

is $(n-1)$-connective. Unwinding the definitions, we see that this composition can be identified with the unit map $u_{X' \times_X X'}$, and is therefore $(n-1)$-connective by the inductive hypothesis.

It remains to prove $(*'')$. Fix a collection of objects $\{C_i\}_{i \in I}$ belonging to $\mathcal{C}$, For every subset $J \subseteq I$, let $X_J \in \mathcal{S}\text{hv}(\mathcal{C})$ denote the coproduct $\coprod_{i \in J} j(C_i)$, and let u_J denote the unit map $X_J \to f_* f^* X_J$. We wish to show that u_I is an equivalence. We first show that u_J is an equivalence when $J \subseteq I$ is finite. Write $C = \coprod_{i \in J} C_i$, so we have equivalences

$$f_* f^* X_J \simeq f_*(f^* \coprod_{i \in J} j(C_i)) \simeq f_*(\coprod_{i \in J} f^* j(C_i)) \simeq f_*(\coprod_{i \in J} C_i) \simeq j(C)$$

(where the last equivalence follows from the fact that our topology on $\mathcal{C}$ is subcanonical). Consequently, we can identify u_J with the canonical map $\coprod_{i \in J} j(C_i) \to j(C)$. Note that the fiber product

$$\coprod_{i \in J} j(C_i) \times_{j(C)} \coprod_{i \in J} j(C_i)$$

is given by $\coprod_{i,j \in J} j(C_i \times_C C_j)$. For $i \neq j$, the fiber product $C_i \times_C C_j$ is an initial object $\emptyset \in \mathcal{C}$. The empty sieve is a covering of $C_i \times_C C_j$, so we have an effective epimorphism from the initial object to $j(C_i \times_C C_j)$ in $\mathcal{S}\text{hv}(\mathcal{C})$ and therefore $j(C_i \times_C C_j)$ is an initial object of $\mathcal{S}\text{hv}(\mathcal{C})$. It follows that $\coprod_{i \in J} j(C_i) \times_{j(C)} \coprod_{i \in J} j(C_i)$ is equivalent to $\coprod_{i \in J} j(C_i \times_C C_i) \simeq \coprod_{i \in J} j(C_i)$: that is, the map u_J becomes an equivalence after pullback along u_J. To complete the proof that u_J is an equivalence, it suffices to show that u_J is an effective epimorphism. This follows from the observation that the collection of maps $\{C_i \to C\}_{i \in J}$ generates a covering sieve.

To complete the proof that u_I is an equivalence, it will suffice to show that the canonical map $\varinjlim_{J \subseteq I} u_J \to u_I$ is an equivalence in $\text{Fun}(\Delta^1, \mathcal{S}\text{hv}(\mathcal{C}))$; here the colimit is taken over all finite subsets $J \subseteq I$. It is easy to see that $X_I \simeq \varinjlim_{J \subseteq I} X_J$ in $\mathcal{S}\text{hv}(\mathcal{C})$. We will complete the proof by showing that $f_* f^* X_I$ is a colimit of the diagram $\{f_* f^* X_J\}_{J \subseteq I}$ in the ∞-category $\mathcal{P}(\mathcal{C})$ (and therefore also in the full subcategory $\mathcal{S}\text{hv}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$). In other words, we claim that for each object $C \in \mathcal{C}$, the canonical map

$$\varinjlim_{J \subseteq I} \text{Map}_{\mathcal{X}}(C, \coprod_{i \in J} C_i) \to \text{Map}_{\mathcal{X}}(C, \coprod_{i \in I} C_i)$$

is a homotopy equivalence. This is a special case of Lemma 3.21. □

Corollary 3.22. *Let $\mathcal{X}$ be an ∞-topos. The following conditions are equivalent:*

(1) *The ∞-topos $\mathcal{X}$ is locally coherent and hypercomplete.*

(2) *There exists a small ∞-category $\mathcal{C}$ which admits finite limits, a finitary Grothendieck topology on $\mathcal{C}$, and an equivalence $\mathcal{X} \simeq \mathcal{S}\text{hv}(\mathcal{C})^{\wedge}$.*

Moreover, if these conditions are satisfied, then we may assume that $\mathcal{C}$ admits finite coproducts and that the topology on $\mathcal{C}$ is subcanonical. If $\mathcal{X}$ is coherent, we may assume that $\mathcal{C}$ admits finite limits.

4 Deligne's Completeness Theorem

A classical result of Deligne asserts that every coherent topos has enough points. Our goal in this section is to prove an ∞-categorical version of this result. We will follow the proof of Deligne's theorem given in [49], with minor modifications. We then give an application (Theorem 4.20) to the theory of hypercoverings, which we will apply in §5.

Theorem 4.1 (∞-Categorical Deligne Completeness Theorem)**.** *Let $\mathcal{X}$ be an ∞-topos which is locally coherent and hypercomplete. Then $\mathcal{X}$ has enough points. In other words, given a morphism $\alpha : X \to Y$ in $\mathcal{X}$ which is not an equivalence, there exists a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}$ such that $f^*(\alpha)$ is not an equivalence.*

Note that Theorem 4.1 recovers the classical version of Deligne's completeness theorem:

Corollary 4.2 (Deligne)**.** *Let $\mathcal{X}$ be a coherent topos. Then $\mathcal{X}$ has enough points.*

Proof. Choose a realization of $\mathcal{X}$ as the category $\mathcal{S}\mathrm{hv}_{\mathcal{S}\mathrm{et}}(\mathcal{C})$ of $\mathcal{S}$et-valued sheaves on a small category $\mathcal{C}$ which admits finite limits, equipped with a finitary Grothendieck topology. Let $\overline{\mathcal{X}}$ be the ∞-topos $\mathcal{S}\mathrm{hv}(\mathrm{N}(\mathcal{C}))$, so that (the nerve of) $\mathcal{X}$ can be identified with the full subcategory of $\overline{\mathcal{X}}$ spanned by the discrete objects. Let $\alpha : X \to Y$ be a morphism in $\mathcal{X}$ which is not an isomorphism. Then α can be regarded as a morphism in $\overline{\mathcal{X}}^{\wedge}$ which is not an equivalence. According to Theorem 4.1, there exists a geometric morphism $\overline{f}^* : \overline{\mathcal{X}}^{\wedge} \to \mathcal{S}$ such that $\overline{f}^*(\alpha)$ is not an equivalence in $\mathcal{S}$. Restricting to discrete objects, we get a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{et}$ such that $f^*(\alpha)$ is not an equivalence. □

We now turn to the proof of Theorem 4.1. We begin by reformulating the condition of having enough points.

Proposition 4.3. *Let $\mathcal{X}$ be an ∞-topos, and suppose we are given a collection of geometric morphisms $\{f^*_\alpha : \mathcal{X} \to \mathcal{X}_\alpha\}$. The following conditions are equivalent:*

(1) *A monomorphism $u : X \to Y$ in $\mathcal{X}$ is an equivalence if and only if each $f^*_\alpha(u)$ is an equivalence in $\mathcal{X}_\alpha$.*

(2) *A morphism $u : X \to Y$ in $\mathcal{X}$ is an effective epimorphism if and only if each $f^*_\alpha(u)$ is an effective epimorphism in $\mathcal{X}_\alpha$.*

(3) *For each $n \geq 0$, a morphism $u : X \to Y$ in $\mathcal{X}$ is n-connective if and only if each $f^*_\alpha(u)$ is n-connective.*

(4) *A morphism $u : X \to Y$ in $\mathcal{X}$ is ∞-connective if and only if each $f^*_\alpha(u)$ is ∞-connective.*

Proof. We will prove that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1). Suppose first that (1) is satisfied, and let $u : X \to Y$ be a morphism in $\mathcal{X}$. Then u factors as a composition $X \stackrel{u'}{\to} X' \stackrel{u''}{\to} Y$, where u' is an effective epimorphism and u'' is a monomorphism. If each $f^*_\alpha(u)$ is an effective epimorphism, then each $f^*_\alpha(u'')$ is an equivalence, so that (1) implies that u'' is an equivalence. It follows that $u \simeq u'$ is an effective epimorphism as desired.

Now suppose that (2) is satisfied; we prove (3) using induction on n, the case $n = -1$ being vacuous. Suppose that $u : X \to Y$ is a morphism in $\mathcal{X}$ such that each $f^*_\alpha(u)$ is n-connective. Let $v : X \to X \times_Y X$ be the diagonal map; then each $f^*_\alpha(v)$ is $(n-1)$-connective. The inductive hypothesis guarantees that v is $(n-1)$-connective, and assumption (2) guarantees that u is an effective epimorphism. It follows that u is n-connective as desired.

The implication (3) ⇒ (4) is obvious, and the implication (4) ⇒ (1) follows from the observation that a monomorphism $u : X \to Y$ is an equivalence if and only if it is ∞-connective. □

Definition 4.4. We will say that a collection of geometric morphisms of ∞-topoi $\{f^*_\alpha : \mathcal{X} \to \mathcal{X}_\alpha\}$ is *jointly surjective* if it satisfies the equivalent conditions of Proposition 4.3. We will say that a geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$ is *surjective* if the one-element collection $\{f^* : \mathcal{X} \to \mathcal{Y}\}$ is jointly surjective.

Example 4.5. Let $\mathcal{X}$ be an ∞-topos, and let $f^* : \mathcal{X} \to \mathcal{X}^{\wedge}$ be a left adjoint to the inclusion. Then f^* is surjective.

Example 4.6. Let $\mathcal{X}$ be an ∞-topos containing an object U. Then the étale geometric morphism $f^* : \mathcal{X} \to \mathcal{X}_{/U}$ is surjective if and only if the object U is 0-connective: that is, if and only if the map $U \to \mathbf{1}$ is an effective epimorphism, where $\mathbf{1}$ denotes a final object of $\mathcal{X}$. If this condition is satisfied, then we will say that $f^* : \mathcal{X} \to \mathcal{X}_{/U}$ is an *étale surjection*.

Remark 4.7. Let $\mathcal{X}$ be an arbitrary ∞-topos. Since the ∞-topos $\mathcal{S}$ is hypercomplete, composition with the localization functor $\mathcal{X} \to \mathcal{X}^{\wedge}$ induces an equivalence between the ∞-category of points of $\mathcal{X}^{\wedge}$ and the ∞-category of points of $\mathcal{X}$. Note that $\mathcal{X}$ is locally coherent if and only if $\mathcal{X}^{\wedge}$ is locally coherent (Proposition 3.16). Consequently, Theorem 4.1 can be reformulated as follows: if $\mathcal{X}$ is a coherent ∞-topos, then there exists a jointly surjective collection of points $\{f^*_\alpha : \mathcal{X} \to \mathcal{S}\}$.

We now construct an ∞-categorical analogue of the Diaconescu cover (see [49]).

Proposition 4.8. *Let $\mathcal{C}$ be an ∞-category equipped with a Grothendieck topology. Then there exists a surjective geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X}$, where $\mathcal{X}$ is a 0-localic ∞-topos.*

Proof. Let $g : \mathcal{D} \to \mathcal{C}$ be a functor between small ∞-categories. We will say that a sieve $\mathcal{D}^{(0)}_{/D} \subseteq \mathcal{D}_{/D}$ on an object $D \in \mathcal{D}$ is *covering* if the following condition is satisfied:

$(*)$ For every morphism $\alpha : D' \to D$ in $\mathcal{D}$, the collection of morphisms $g(\beta) : g(D'') \to g(D')$ such that the composition $(\alpha \circ \beta) : D'' \to D$ belongs to $\mathcal{D}^{(0)}_{/D}$ generates a covering sieve on $g(D') \in \mathcal{C}$.

It is not difficult to see that this defines a Grothendieck topology on $\mathcal{D}$. Let $L_{\mathcal{C}} : \mathcal{P}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ and $L_{\mathcal{D}} : \mathcal{P}(\mathcal{D}) \to \mathcal{S}\mathrm{hv}(\mathcal{D})$ be left adjoints to the inclusions, and consider the composition

$$\overline{f}^* : \mathcal{P}(\mathcal{C}) \stackrel{\circ g}{\to} \mathcal{P}(\mathcal{D}) \stackrel{L_{\mathcal{D}}}{\to} \mathcal{S}\mathrm{hv}(\mathcal{D}).$$

It is clear that $\overline{f}^*$ is a geometric morphism.

We now suppose that the functor g has the following property:

(a) For every object $D \in \mathcal{D}$ and every morphism $\beta : C \to g(D)$ in $\mathcal{D}$, there exists a morphism $\overline{\beta} : \overline{C} \to D$ in $\mathcal{D}$ such that $\beta = g(\overline{\beta})$.

We claim that $\overline{f}^*$ carries $L_{\mathcal{C}}$-equivalences to equivalences in $\mathcal{S}\mathrm{hv}(\mathcal{D})$. To prove this, it suffices to show that if we are given a collection of morphisms $\alpha_i : C_i \to C$ which generate a covering sieve on $C \in \mathcal{C}$, then the induced map $\phi : \coprod \overline{f}^* j(C_i) \to \overline{f}^* j(C)$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{D})$; here $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ denotes the Yoneda embedding (see Proposition T.6.2.3.20).

Let $e : \mathcal{D} \to \mathcal{S}\mathrm{hv}(\mathcal{D})$ be the composition of the Yoneda embedding $\mathcal{D} \to \mathcal{P}(\mathcal{D})$ with the sheafification functor $L_{\mathcal{D}}$. Then $\mathcal{S}\mathrm{hv}(\mathcal{D})$ is generated under colimits by the essential image of e. Consequently, to prove that ϕ is an effective epimorphism, it suffices to show that for every morphism $u : e(D) \to \overline{f}^* j(C)$, the induced map

$$\phi_u : \coprod (\overline{f}^* j(C_i) \times_{\overline{f}^* j(C)} e(D)) \to e(D)$$

is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{D})$. Passing to a covering of D, we may reduce to the case where u is induced by a morphism in $\mathcal{P}(\mathcal{D})$, corresponding to a map $\overline{u} : g(D) \to C$ in $\mathcal{C}$. Let $\mathcal{D}^{(0)}_{/D}$ denote the full subcategory of $\mathcal{D}_{/D}$ spanned by those morphisms $D_0 \to D$ such that the induced map $g(D_0) \to g(D) \stackrel{\overline{u}}{\to} C$ belongs to the sieve generated by the collection of morphisms $\{\alpha_i\}$. It is clear that $\mathcal{D}^{(0)}_{/D}$ is a sieve on D. For every morphism $D_0 \to D$ in $\mathcal{D}^{(0)}_{/D}$, the induced map $e(D_0) \to e(D)$ factors through ϕ_u. Consequently, to show that ϕ_u is an effective epimorphism, it will suffice to show that $\mathcal{D}^{(0)}_{/D}$ is a covering sieve on D: that is, that it satisfies condition $(*)$. Choose a morphism $D' \to D$ in $\mathcal{D}$. Since the collection of covering sieves in $\mathcal{C}$ forms a Grothendieck topology, there exists a collection of morphisms $\beta_j : C'_j \to g(D')$ which generate a covering sieve, each of which fits into a commutative diagram

$$\begin{array}{ccc} C'_j & \xrightarrow{\beta_j} & g(D') \\ \downarrow & & \downarrow \\ C_i & \xrightarrow{\alpha_i} & C. \end{array}$$

Condition (a) guarantees that each β_j can be lifted to a morphism $\overline{\beta}_j : \overline{C}'_j \to D'$ in $\mathcal{D}$, which belongs to the pullback of the sieve $\mathcal{D}^{(0)}_{/D}$. It follows that $\mathcal{D}^{(0)}_{/D}$ satisfies condition $(*)$ and is therefore a covering sieve on D, as required.

Since $\overline{f}^*$ carries $L_{\mathcal{C}}$-equivalences to equivalences in $\mathcal{S}\mathrm{hv}(\mathcal{D})$, it factors up to homotopy as a composition

$$\mathcal{P}(\mathcal{C}) \xrightarrow{L_{\mathcal{C}}} \mathcal{S}\mathrm{hv}(\mathcal{C}) \xrightarrow{f^*} \mathcal{S}\mathrm{hv}(\mathcal{D})$$

where f^* is a colimit-preserving functor which (since it is equivalent to $\overline{f}^*|\,\mathcal{S}\mathrm{hv}(\mathcal{C})$) preserves finite limits. We now make the following additional assumption:

(b) The functor g is surjective on objects.

We claim that condition (b) implies that f^* is surjective in the sense of Definition 4.4. We will show that if $u : X \to Y$ is a morphism $\mathcal{S}\mathrm{hv}(X)$ such that $f^*(u)$ is an effective epimorphism in $\mathcal{S}\mathrm{hv}(\mathcal{D})$, then u is an effective epimorphism in $\mathcal{S}\mathrm{hv}(X)$. Choose an object $C \in \mathcal{C}$ and a point $\eta \in Y(C)$, and let $\mathcal{C}^{(0)}_{/C}$ be the full subcategory of $\mathcal{C}_{/C}$ spanned by those morphisms $C' \to C$ such that the image of η in $\pi_0 Y(C')$ can be lifted to $\pi_0 X(C')$; we wish to prove that $\mathcal{C}^{(0)}_{/C}$ is covering. Assumption (b) implies we can write $C = g(D)$ for some object $D \in \mathcal{D}$. Then η determines a point $\overline{\eta} \in (f^*Y)(D)$. Let $\mathcal{D}^{(0)}_{/D} \subseteq \mathcal{D}_{/D}$ be the sieve consisting of morphisms $D' \to D$ such that the image of $\overline{\eta}$ in $\pi_0(f^*Y)(D')$ lifts to $\pi_0(f^*X)(D)$, and let $\mathcal{D}^{(1)}_{/D} \subseteq \mathcal{D}_{/D}$ be the sieve consisting of morphisms $\beta : D' \to D$ such that the image of η in $\pi_0 Y(g(D'))$ lifts to $\pi_0 X(g(D'))$. The functor g carries $\mathcal{D}^{(1)}_{/D}$ into $\mathcal{C}^{(0)}_{/C}$. Consequently, to prove that $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on $C \in \mathcal{C}$, it suffices to show that $\mathcal{D}^{(1)}_{/D}$ is a covering sieve on $D \in \mathcal{D}$. Since $f^*(u)$ is an effective epimorphism, the sieve $\mathcal{D}^{(0)}_{/D}$ is covering. It therefore suffices to show that for each $\beta : D' \to D$ in $\mathcal{D}^{(0)}_{/D}$, the pullback $\beta^*\,\mathcal{D}^{(1)}_{/D} \subseteq \mathcal{D}_{/D'}$ is a covering sieve on D'. Replacing D by D' (and C by $g(D')$), we may assume that $\overline{\eta}$ lifts to a point $\overline{\eta}' \in (f^*X)(D)$. Note that f^*X is the sheafification of the functor $\mathcal{D} \xrightarrow{g} \mathcal{C} \xrightarrow{X} \mathcal{S}$. It follows that there exists a covering sieve $\mathcal{D}^{(2)}_{/D}$ on D such that for each morphism $D' \to D$ in $\mathcal{D}^{(2)}_{/D}$, the image of $\overline{\eta}'$ in $(f^*X)(D')$ belongs to the image of $X(g(D'))$. We clearly have a containment $\mathcal{D}^{(2)}_{/D} \subseteq \mathcal{D}^{(1)}_{/D}$, so that $\mathcal{D}^{(1)}_{/D}$ is also a covering sieve.

We now add the following additional assumption:

(c) The ∞-category $\mathcal{D}$ is the nerve of a partially ordered set.

In this case, the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{D})$ is 0-localic so that the geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{D})$ satisfies the requirements of Proposition 4.8.

It remains to prove that there exists a functor $g : \mathcal{D} \to \mathcal{C}$ satisfying conditions (a), (b), and (c). For this, we let A denote the partially ordered set of pairs (n, σ), where $n \geq 0$ and $\sigma : \Delta^n \to \mathcal{C}^{op}$ is an n-simplex of $\mathcal{C}^{op}$. We write $(n, \sigma) \leq (n', \sigma')$ if $n \leq n'$ and $\sigma = \sigma' | \Delta^{\{0,\ldots,n\}}$. A k-simplex of the nerve $\mathrm{N}(A)$ consists of a sequence τ :

$$(n_0, \sigma_0) \leq \cdots \leq (n_k, \sigma_k).$$

Let $g(\tau)$ denote the k-simplex of $\mathcal{C}^{op}$ given by the composition

$$\Delta^k \xrightarrow{\gamma} \Delta^{n_k} \xrightarrow{\sigma_k} \mathcal{C}^{op},$$

where γ is given on vertices by the formula $\gamma(i) = n_i$. Then the construction $\tau \mapsto g(\tau)$ determines a map of simplicial sets $g : \mathrm{N}(A)^{op} \to \mathcal{C}$. It is easy to see that this map satisfies conditions (a), (b), and (c). ☐

Lemma 4.9. *Let $\mathcal{X}$ be a 0-localic ∞-topos. Assume that $\mathcal{X}$ is not a contractible Kan complex. Then there exists a nontrivial complete Boolean algebra B and a geometric morphism $f^* : \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(B)$.*

Proof. Let $\mathcal{U}$ be the underlying locale of $\mathcal{X}$: that is, the partially ordered set of subobjects of the unit object $\mathbf{1}_X$. Then $\mathcal{U}$ is a complete lattice: in particular, every set of elements $\{U_\alpha \in \mathcal{U}\}$ has a greatest lower bound $\wedge_\alpha U_\alpha$ and a least upper bound $\vee_\alpha U_\alpha$. In particular, $\mathcal{U}$ has a least element (which we will denote by $\emptyset$) and a greatest element (which we will denote by $\mathbf{1}$). For each element $U \in \mathcal{U}$, we let U' denote the least upper bound of the set $\{V \in \mathcal{U} : U \wedge V = \emptyset\}$. Let $B = \{U \in \mathcal{U} : U = U''\}$. We will prove:

(*a*) The map $U \mapsto U''$ determines a retraction from $\mathcal{U}$ onto B, which commutes with finite meets and infinite joins.

(*b*) As a partially ordered set, B is a complete Boolean algebra.

Assertion (a) implies that B is a left exact localization of $\mathcal{U}$, and is therefore itself a locale; moreover, the proof of Proposition T.6.4.5.7 gives a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(B)$. We begin by proving (a). Note that the construction $U \mapsto U'$ is order-reversing. It follows that $U \leq V$ implies that $U'' \leq V''$. Moreover, we have an evident inequality $U \leq U''$ which guarantees that $U''' = U'$. In particular, $U' \in B$ for each $U \in \mathcal{U}$. We next claim that the construction $U \mapsto U''$ is a left adjoint to the inclusion $B \subseteq \mathcal{U}$. In other words, we claim that for $V \in \mathcal{U}$, we have $U \leq V$ if and only if $U'' \leq V$. The "if" direction is clear (since $U \leq U''$), and the "only if" direction follows from the implications

$$(U \leq V) \Rightarrow (V' \leq U') \Rightarrow (U'' \leq V'') \Rightarrow (U'' \leq V),$$

since $V = V''$. It follows immediately that $U \mapsto U''$ is a retraction onto B which preserves infinite joins.

We now show that the construction $U \mapsto U''$ preserves finite meets (note that, since the inclusion $B \hookrightarrow \mathcal{U}$ admits a left adjoint, B is closed under meets in $\mathcal{U}$). The inequality $U \leq U''$ shows that $\mathbf{1} = \mathbf{1}''$. It therefore suffices to show that $U \mapsto U''$ preserves pairwise meets. The construction $U \mapsto U'$ is an order-reversing bijection from B to itself, and therefore carries finite joins in B to finite meets in B. It will therefore suffice to show that the construction $U \mapsto U'$ carries pairwise meets in $\mathcal{U}$ to pairwise joins in B. In other words, we must show that for $U, V \in \mathcal{U}$, the element $(U \wedge V)'$ is a join of U' and V' in B. It is clear that $U', V' \leq (U \wedge V)'$; it therefore suffices to show that if $W = W''$ is any upper bound for U' and V' in B, then $(U \wedge V)' \leq W = W''$. In other words, we must show that $(U \wedge V)' \wedge W' = \emptyset$: that is, if $X \in \mathcal{U}$ is any object such that $X \wedge W = \emptyset$ and $X \wedge (U \wedge V) = \emptyset$, then $X = \emptyset$. We have $X \wedge U \leq V' \leq W'' = W$, so that $(X \wedge U) \leq X \wedge W = \emptyset$. This shows that $X \leq U' \leq W$, so that $X = X \wedge W = \emptyset$ as desired. This completes the proof of (a).

The proof of (a) shows that B is a locale; in particular, it is a distributive lattice. To prove (b), it suffices to show that B is complemented: that is, for every $U \in B$ there exists $V \in B$ such that $U \wedge V = \emptyset$ and $U \vee V = \mathbf{1}$. For this, we take $V = U'$, so that the equation $U \wedge V = \emptyset$ is obvious. To prove $U \vee V = \mathbf{1}$, it suffices to show that if U and U' are bounded by an element $W \in B$, then $W = \mathbf{1}$. In fact, the inequalities $U \leq W$ and $U' \leq W$ guarantee that $W' \leq U' \wedge U'' = \emptyset$, so that $W = W'' = \emptyset' = \mathbf{1}$. □

Lemma 4.10. *Let $\mathcal{X}$ be a 0-localic ∞-topos. Then there exists a surjective geometric morphism $f^*_\alpha : \mathcal{X} \to \mathcal{S}\mathrm{hv}(B)$, where B is a complete Boolean algebra.*

Proof. Let $\mathcal{U}$ be the locale of equivalence classes of (-1)-truncated objects of $\mathcal{X}$. For every proper inclusion $U \subset V$ in $\mathcal{U}$, there exists a nontrivial complete Boolean algebra $B_{U,V}$ and a left exact, join-preserving map $f_{U,V} : \mathcal{U} \to B_{U,V}$. Let B be the product of the Boolean algebras $B_{U,V}$, and let $f : \mathcal{U} \to B$ be the product functor; then f induces a geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(B)$. We claim that this geometric morphism is surjective.

Let $u : X \to Y$ be a monomorphism in $\mathcal{X}$ such that $f^*(u)$ is an equivalence; we wish to prove that u is an equivalence. For each $V \in \mathcal{U}$, let $\chi_V \in \mathcal{S}\mathrm{hv}(U)$ be the sheaf given by the formula

$$\chi_V(W) = \begin{cases} \Delta^0 & \text{if } W \subseteq V \\ \emptyset & \text{otherwise.} \end{cases}$$

The ∞-category $\mathcal{S}\mathrm{hv}(\mathcal{U})$ is generated under colimits by the objects χ_V. In particular, there exists an effective epimorphism $\coprod_\alpha \chi_{V_\alpha} \to Y$. It therefore suffices to show that the induced map

$$(\coprod_\alpha \chi_{V_\alpha}) \times_Y X \to \coprod_\alpha \chi_{V_\alpha}$$

is an equivalence. This map is a coproduct of morphisms

$$u_\alpha : \chi_{V_\alpha} \times_Y X \to \chi_{V_\alpha}.$$

To complete the proof, it suffices to show that each u_α is an equivalence. We may therefore replace u by u_α, and thereby reduce to the case where Y has the form χ_V for some object $V \in \mathcal{U}$.

Since u is a monomorphism, we can identify X with χ_U for some $U \subseteq V$. We wish to show that $U = V$. Suppose otherwise, so that the geometric morphism $f^*_{U,V} : \mathcal{X} \to \mathcal{S}\mathrm{hv}(B) \to \mathcal{S}\mathrm{hv}(B_{U,V})$ is well-defined. We note that the image of χ_U in $\mathcal{S}\mathrm{hv}(B_{U,V})$ is the initial object, while the image of χ_V in $\mathcal{S}\mathrm{hv}(B_{U,V})$ is the final object. Consequently, $f^*_{U,V}(u)$ is an equivalence in $\mathcal{S}\mathrm{hv}(B_{U,V})$ between the initial and final objects, contradicting the nontriviality of $B_{U,V}$. □

Proposition 4.11. *Let B be a complete Boolean algebra. Then the ∞-topos $\mathcal{S}\mathrm{hv}(B)$ has homotopy dimension ≤ 0: that is, every 0-connective object $X \in \mathcal{S}\mathrm{hv}(B)$ admits a global section.*

Proof. Let $X \in \mathcal{S}\mathrm{hv}(B)$ be a 0-connective object which does not admit a global section. For every ordinal α, we let (α) denote the well-ordered set of ordinals $\{\beta : \beta < \alpha\}$. We will construct a compatible sequence of functors $\phi_\alpha : \mathrm{N}(\alpha) \to \mathcal{X}_{/X}$ with the following property:

- $(*)$ The composite functor $\mathrm{N}(\alpha) \stackrel{\phi_\alpha}{\to} \mathcal{X}_{/X} \to \mathcal{X}$ takes values in the full subcategory of $\mathcal{X}$ spanned by the (-1)-truncated objects, and determines a strictly increasing map $[\alpha] \to B$.

This leads to a contradiction for α sufficiently large (namely, for any ordinal α such that $[\alpha]$ has cardinality greater than that of B).

The construction of the maps ϕ_α proceeds by induction on α. If α is a limit ordinal, we let ϕ_α be the amalgamation of the functors $\{\phi_\beta\}_{\beta<\alpha}$. To complete the construction, it suffices to show that every map $\phi_\alpha : \mathrm{N}(\alpha) \to \mathcal{X}_{/X}$ can be extended to a map $\phi_{\alpha+1} : \mathrm{N}(\alpha+1) \to \mathcal{X}_{/X}$ satisfying $(*)$. The colimit of ϕ_α can be identified with a map $\psi : U \to X$ in $\mathcal{X}$, where U is (-1)-truncated. Let us identify U with an element of the Boolean algebra B, and ψ with a point of the space $X(U)$. Since X does not admit a global section, U is not a maximal element of B. Because B is a Boolean algebra, the object U has a complement $U' \in B$, which is not a minimal element of B. Since $X \in \mathcal{S}\mathrm{hv}(B)$ is 0-connective, the object U' can be written as a join $\bigvee U'_i$ where each $X(U'_i)$ is nonempty. For some index i, the element $U'_i \in B$ is nontrivial. Since $U'_i \wedge U = \emptyset$, the canonical map $X(U'_i \vee U) \to X(U'_i) \to X(U)$ is a homotopy equivalence; it follows that ψ can be lifted (up to homotopy) to a point of $X(U'_i \vee U)$. This point gives an extension $\phi_{\alpha+1}$ of ϕ_α, with $\phi_{\alpha+1}(\alpha)$ given by a map $V \to X$ where V is a (-1)-truncated object corresponding to the element $U'_i \vee U$. of B. □

Corollary 4.12. *Let B be a complete Boolean algebra. Then the ∞-topos $\mathcal{S}\mathrm{hv}(B)$ is locally of homotopy dimension ≤ 0.*

Proof. For each $U \in B$, let $\chi_U \in \mathcal{S}\mathrm{hv}(B)$ be the sheaf given by the formula

$$\chi_U(V) = \begin{cases} \Delta^0 & \text{if } V \leq U \\ \emptyset & \text{otherwise.} \end{cases}$$

The objects χ_U generate $\mathcal{S}\mathrm{hv}(B)$ under colimits. Consequently, it suffices to show that each of the ∞-topoi $\mathcal{S}\mathrm{hv}(B)_{/\chi_U}$ has homotopy dimension ≤ 0. We complete the proof by observing that $\mathcal{S}\mathrm{hv}(B)_{/\chi_U}$ is equivalent to $\mathcal{S}\mathrm{hv}(B_U)$, where B_U denotes the complete Boolean algebra $\{V \in B : V \leq U\}$, and therefore has homotopy dimension ≤ 0 by Proposition 4.11. □

Corollary 4.13. *Let B be a complete Boolean algebra. Then the ∞-topos $\mathcal{S}\mathrm{hv}(B)$ is hypercomplete.*

Proof. Combine Corollary 4.12 with Corollary T.7.2.1.12. □

Corollary 4.14. *Let $\mathcal{X}$ be an ∞-topos. Then there exists a complete Boolean algebra B and a surjective geometric morphism $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(B)$*

Proof. Using Proposition T.6.5.2.19, we deduce that there exists a small ∞-category $\mathcal{C}$ equipped with a Grothendieck topology such that $\mathcal{X}$ is a cotopological localization of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Proposition 4.8 gives a surjective geometric morphism $g^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{Y}$, where $\mathcal{Y}$ is 0-localic. Lemma 4.10 guarantees a surjective geometric morphism $h^* : \mathcal{Y} \to \mathcal{S}\mathrm{hv}(B)$, where $\mathcal{S}\mathrm{hv}(B)$ is a complete Boolean algebra. Since $\mathcal{S}\mathrm{hv}(B)$ is hypercomplete (Corollary 4.13), the functor $h^* \circ g^*$ carries ∞-connective morphisms in $\mathcal{S}\mathrm{hv}(\mathcal{C})$ to equivalences in $\mathcal{S}\mathrm{hv}(B)$, and therefore factors as a composition

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X} \xrightarrow{f^*} \mathcal{S}\mathrm{hv}(B)$$

for some surjective geometric morphism f^*. □

Let B be a Boolean algebra. An *ultrafilter* on B is a homomorphism of Boolean algebras $f : B \to [1]$, where $[1]$ denotes the linearly ordered set $\{0 < 1\}$ (that is, a map $f : B \to [1]$ which preserves finite meets and finite joins). The collection of all ultrafilters on B is called the *Stone space* of B, and will be denoted by $St(B)$. We regard $St(B)$ as a closed subspace of the product $[1]^B$. A basis for the topology of $St(B)$ is given by the collection of open sets $U_b = \{f \in St(B) : f(b) = 1\}$, where $b \in B$. The construction $b \mapsto U_b$ determines an isomorphism of B with the collection of all open-closed subsets of $St(B)$.

Let $\mathcal{U}(St(B))$ denote the collection of *all* open subsets of the Stone space of a Boolean algebra B. If B is complete, there is a canonical map $\phi : \mathcal{U}(St(B)) \to B$, given by the formula $\phi(U) = \bigvee_{U_b \subseteq U} U$. It is easy to see that this map preserves finite meets and arbitrary joins, and can therefore be regarded as a morphism of locales. In particular, we get a geometric morphism of ∞-topoi $f^* : \mathcal{S}\mathrm{hv}(St(B)) \to \mathcal{S}\mathrm{hv}(B)$.

Lemma 4.15. *Let B be a complete Boolean algebra, and let $f^* : \mathcal{S}\mathrm{hv}(St(B)) \to \mathcal{S}\mathrm{hv}(B)$ be the morphism constructed above. Then:*

(1) *The right adjoint f_* to f^* is fully faithful. In other words, the composition $f^* f_*$ is equivalent to the identity on $\mathcal{S}\mathrm{hv}(B)$.*

(2) *For every finite collection of objects $\{X_i\}_{1 \leq i \leq n}$ and effective epimorphism $\coprod X_i \to Y$ in $\mathcal{S}\mathrm{hv}(B)$, the induced map $\coprod f_* X_i \to f_* Y$ is an effective epimorphsim in $\mathcal{S}\mathrm{hv}(St(B))$.*

Proof. The construction $b \mapsto U_b$ determines an injective map of partially ordered sets $i : B \to \mathcal{U}(St(B))$. The functor $f_* : \mathcal{S}\mathrm{hv}(B) \to \mathcal{S}\mathrm{hv}(St(B))$ is given by right Kan extension along the inclusion i, and is fully faithful by Proposition T.4.3.2.15. This proves (1). To prove (2), we note that $\mathcal{S}\mathrm{hv}(B)$ is generated under colimits by objects of the form $\{\chi_U\}_{U \in B}$, as in the proof of Corollary 4.12; consequently, we may suppose that Y has the form χ_U. Each of the maps $X_i \to Y$ factors as a composition

$$X_i \xrightarrow{u_i} \chi_{U_i} \to \chi_U,$$

where u_i is an effective epimorphism. Applying Proposition 4.11 (to the complete Boolean algebra $\{V \in B : V \leq U_i\}$), we deduce that u_i admits a section s_i. We have a commutative diagram

$$\begin{array}{ccccc} & & \coprod f_* X_i & & \\ & \nearrow^{\coprod s_i} & & \searrow & \\ \coprod f_* \chi_{U_i} & & \xrightarrow{\quad \psi \quad} & & f_* \chi_U . \end{array}$$

Consequently, to prove (2), it suffices to show that ψ is an effective epimorphism. For this, it suffices to observe that for each $V \in B$, the functor f_* carries χ_V to the sheaf represented by the open set $i(V) \subseteq St(B)$, and the map $V \mapsto i(V)$ preserves finite joins. □

Lemma 4.16. *Let $\mathcal{C}$ be a small ∞-category which admits finite limits, equipped with a finitary Grothendieck topology. Let B be a complete Boolean algebra, and let $g^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(B)$ be a geometric morphism. Then g^* is homotopic to a composition*

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \xrightarrow{h^*} \mathcal{S}\mathrm{hv}(St(B)) \xrightarrow{f^*} \mathcal{S}\mathrm{hv}(B),$$

where f^* is the geometric morphism of Lemma 4.15.

Proof. For every ∞-topos $\mathcal{Y}$, the ∞-category of geometric morphisms from $\mathcal{S}\mathrm{hv}(\mathcal{C})$ to $\mathcal{Y}$ can be identified with the ∞-category of left-exact functors $u : \mathcal{C} \to \mathcal{Y}$ with the following property: for every every collection of morphisms $\{C_i \to C\}$ which generate a covering sieve on an object $C \in \mathcal{C}$, the induced map $\coprod u(C_i) \to u(C)$ is an effective epimorphism in $\mathcal{Y}$ (Proposition T.6.2.3.20). In particular, g^* is classified by a functor $u : \mathcal{C} \to \mathcal{S}\mathrm{hv}(B)$. Let f_* denote a right adjoint to f^*, and let $u' : \mathcal{C} \to \mathcal{S}\mathrm{hv}(St(B))$ be the composition $f_* \circ u$. It follows from Lemma 4.15 that u' determines a geometric morphism $h^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(St(B))$ such that $f^* \circ h^* \simeq g^*$. □

Lemma 4.17. *Let $\mathcal{C}$ be a small ∞-category which admits finite limits, equipped with a finitary Grothendieck topology. Then there exists a surjective geometric morphism $h^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(X)$, where X is a compact, totally disconnected Hausdorff space.*

Proof. Corollary 4.14 guarantees the existence of a surjective geometric morphism $g^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(B)$, where B is a complete Boolean algebra. Let X be the Stone space of B. Lemma 4.16 guarantees that g^* factors through a geometric morphism $h^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(X)$, which is clearly surjective. □

Proof of Theorem 4.1. We may assume without loss of generality that the ∞-topos $\mathcal{X}$ is coherent and hypercomplete. Using Corollary 3.22, we can assume that $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{C})^\wedge$ for some small ∞-category $\mathcal{C}$ which admits finite limits and is equipped with a finitary Grothendieck topology. Choose a surjective geometric morphism $h^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(X)$ as in Lemma 4.17. For each point $x \in X$, let f_x^* denote the composite map

$$\mathcal{S}\mathrm{hv}(\mathcal{C}) \xrightarrow{h^*} \mathcal{S}\mathrm{hv}(X) \to \mathcal{S}\mathrm{hv}(\{x\}) \simeq \mathcal{S}\,.$$

It is easy to see that the collection of geometric morphisms $\{f_x^*\}_{x \in X}$ is jointly surjective, so that $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathcal{C})^\wedge$ has enough points as desired (see Remark 4.7). □

We close this section with an application of Theorem 4.1 to the theory of hypercoverings. We begin by reviewing some definitions.

Notation 4.18. Let Δ_s denote the subcategory of Δ whose objects are finite linearly ordered sets of the form $[n] = \{0, \ldots, n\}$, and whose morphisms are *injective* monotone maps $[m] \to [n]$. Let $\mathcal{C}$ be an ∞-category. A *semisimplicial object* of $\mathcal{C}$ is a functor $X_\bullet : \mathrm{N}(\mathbf{\Delta}_s)^{op} \to \mathcal{C}$. Assume that $\mathcal{C}$ admits finite limits. For each $n \geq 0$, we let $M_n(X)$ denote the *nth matching object* of $X_\bullet$: that is, $M_n(X)$ is the limit

$$\varprojlim_{f : [m] \to [n]} X_m,$$

where f ranges over all injective monotone maps $[m] \to [n]$ such that $m < n$.

Definition 4.19. Let $\mathcal{X}$ be an ∞-topos. We will say that a semisimplicial object $X_\bullet$ of $\mathcal{X}$ is a *hypercovering* if, for each $n \geq 0$, the canonical map $X_n \to M_n(X)$ is an effective epimorphism.

The following result generalizes Lemma T.6.5.3.11:

Theorem 4.20. *Let $\mathcal{X}$ be an ∞-topos, and let $X_\bullet : \mathrm{N}(\mathbf{\Delta}_s)^{op} \to \mathcal{X}$ be a hypercovering. Then the colimit of $X_\bullet$ is an ∞-connective object of $\mathcal{X}$.*

We will give a proof of Theorem 4.20 which is substantially simpler than the proof given in [40]. The idea is to use Deligne's completeness theorem to reduce to the case where $\mathcal{X} = \mathcal{S}$, where the result admits an elementary proof using the combinatorics of simplicial sets. We need some preliminaries.

Lemma 4.21. *Let $X_\bullet$ be a semisimplicial set. Suppose that, for each $n \geq 0$, the canonical map $\phi_n : X_n \to M_n(X)$ is surjective. Then $X_\bullet$ is the restriction of a simplicial set.*

Proof. For each $n \geq 0$, let $\mathbf{\Delta}^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the objects $\{[m]\}_{m \leq n}$, let $\mathbf{\Delta}_s$ be the subcategory of $\mathbf{\Delta}$ spanned by all objects and all injective morphisms between them, and let $\mathbf{\Delta}_s^{\leq n} = \mathbf{\Delta}^{\leq n} \cap \mathbf{\Delta}_s$. We regard $X_\bullet$ as a functor $\mathbf{\Delta}_s^{op} \to \mathcal{S}\mathrm{et}$. We will construct a compatible sequence of functors $Y(n) : (\mathbf{\Delta}^{\leq n})^{op} \to \mathcal{S}\mathrm{et}$ extending the functors $X(m) = X_\bullet|(\mathbf{\Delta}_s^{\leq n})^{op}$, using induction on n. In the case $n = 0$, we take $Y(0) = X(0)$. Assume now that $Y(n)$ has been constructed. Let LY and MY denote the $(n+1)$st latching and matching objects determined by $Y(n)$, respectively, so that we have a canonical map $f : LY \to MY$. Moreover, since $Y(n)|(\mathbf{\Delta}_s^{\leq n})^{op} \simeq X(n)$, we can identify MY with the matching object $M_{n+1}(X)$ (see the proof of Lemma T.6.5.3.8). According to Corollary T.A.2.9.15, giving an extension $Y(n+1)$ of $Y(n)$ is equivalent to giving a commutative diagram

$$\begin{array}{ccc} & Y_{n+1} & \\ {}^{f'}\nearrow & & \searrow^{f''} \\ LY & \xrightarrow{f} & MY. \end{array}$$

To guarantee that $Y(n+1)$ extends $X(n+1)$, we choose $Y_{n+1} = X_{n+1}$ and $f'' = \phi_{n+1}$. The existence of f' now follows from our assumption that ϕ_{n+1} is surjective. $\square$

Lemma 4.22. *Let $X_\bullet$ be a semisimplicial set. Suppose that, for each $n \geq 0$, the canonical map $\phi_n : X_n \to M_n(X)$ is surjective. Then the colimit of the diagram*

$$\mathrm{N}(\mathbf{\Delta}_s)^{op} \xrightarrow{X_\bullet} \mathrm{N}(\mathcal{S}\mathrm{et}) \subseteq \mathcal{S}$$

is contractible.

Proof. According to Lemma 4.21, we may assume without loss of generality that $X_\bullet$ extends to a simplicial set $Y_\bullet : \mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$. In view of Lemma T.6.5.3.7, it suffices to show that the composite diagram

$$\mathrm{N}(\mathbf{\Delta})^{op} \xrightarrow{Y_\bullet} \mathrm{N}(\mathcal{S}\mathrm{et}) \to \mathcal{S}$$

has contractible colimit. Example T.A.2.9.31 shows that this colimit can be identified with $Y_\bullet$ itself. We conclude by observing that $Y_\bullet$ is a contractible Kan complex. $\square$

Lemma 4.23. *Let $X_\bullet$ be a semisimplicial object of $\mathcal{S}$. Suppose that, for each $n \geq 0$, the canonical map $\pi_0 X_n \to \pi_0 M_n(X)$ is surjective. Then the colimit of the diagram*

$$\mathrm{N}(\mathbf{\Delta}_s)^{op} \xrightarrow{X_\bullet} \mathcal{S}$$

is contractible.

Proof. Using Proposition T.4.2.4.4, we may assume without loss of generality that $X_\bullet$ is obtained from a semisimplicial object $\overline{X}_\bullet$ in the (ordinary) category of simplicial sets. Moreover, we may assume that $\overline{X}_\bullet$ is fibrant and cofibrant with respect to the injective model structure on $(\mathcal{S}\mathrm{et}_\Delta)^{\mathbf{\Delta}_s^{op}}$. It follows in particular that each of the matching objects $M_n(\overline{X})$ is a model for the space $M_n(X)$, and that the maps $\phi_n : \overline{X}_n \to M_n(\overline{X})$ are Kan fibrations of simplicial sets. We may identify $\overline{X}_\bullet$ with a diagram $\overline{Y} : \mathbf{\Delta}^{op} \to \mathrm{Fun}(\mathbf{\Delta}_s^{op}, \mathcal{S}\mathrm{et})$, which determines in turn a diagram $Y : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_s)^{op}, \mathcal{S})$. It follows from Example T.A.2.9.31 that X is a colimit of the diagram of the diagram Y, so that $\varinjlim(X)$ is the geometric realization of the simplicial space $[m] \mapsto \varinjlim Y([m])$. It will therefore suffice to show that each of the spaces $\varinjlim Y([m])$ is contractible. We claim that each of the semisimplicial sets $Y([m])$ satisfies the hypotheses of Lemma 4.22. In other words, we claim that each of the maps $\phi_n : \overline{X}_n \to M_n(\overline{X})$ is surjective on m-simplices. This is clear, since ϕ_n is a Kan fibration which is surjective on connected components. $\square$

Proof of Theorem 4.20. Let $\mathcal{C}$ be an ∞-category freely generated by $\mathrm{N}(\mathbf{\Delta}_s)^{op}$ under finite limits. More precisely, we choose a functor $Y_\bullet : \mathrm{N}(\mathbf{\Delta}_s)^{op} \to \mathcal{C}$, where $\mathcal{C}$ admits finite limits, with the following universal property: for any ∞-category $\mathcal{D}$ which admits finite limits, composition with $Y_\bullet$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_s)^{op}, \mathcal{D})$, where $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve finite limits. The existence of f follows from Remark T.5.3.5.9.

We can regard $Y_\bullet$ as a semisimplicial object of $\mathcal{C}$. Since $\mathcal{C}$ admits finite limits, the matching objects $M_n(Y)$ are well-defined. Let us regard $\mathcal{C}$ as endowed with the coarsest Grothendieck topology such that, for each $n \geq 0$, the single map $\{Y_n \to M_n(Y)\}$ generates a covering sieve on $M_n(Y)$. It follows from Remark 3.18 that this Grothendieck topology is finitary, so that $\mathcal{S}\mathrm{hv}(\mathcal{C})$ is a locally coherent ∞-topos (Proposition 3.19). Using Proposition T.6.2.3.20, we deduce the existence of a geometric morphism $f^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{X}$ such that $X_\bullet$ is equivalent to $f^* Y_\bullet$. Consequently, to prove that $\varinjlim X_\bullet$ is an ∞-connective object of $\mathcal{X}$, it suffices to show that $\varinjlim Y_\bullet$ is an ∞-connective object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. According to Theorem 4.1, it suffices to show that for every geometric morphism $g^* : \mathcal{S}\mathrm{hv}(\mathcal{C}) \to \mathcal{S}$, the space $g^*(\varinjlim Y_\bullet) \simeq \varinjlim (g^* Y_\bullet)$ is contractible. This follows immediately from Lemma 4.23. $\square$

5 Flat Descent

In this section, we will introduce the *flat topology* on the ∞-category of $\mathbb{E}_\infty$-rings. We will then show that for every 0-localic spectral scheme $\mathfrak{X}$, the functor represented by $\mathfrak{X}$ is a sheaf with respect to the flat topology (Theorem 5.15).

We begin by introducing a general construction of Grothendieck topologies.

Proposition 5.1. *Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms in $\mathcal{C}$. Assume that:*

(*a*) *The collection of morphisms S is contains all equivalences and is stable under composition (in particular, if $f, g : C \to D$ are homotopic morphisms in $\mathcal{C}$, then $f \in S$ if and only if $g \in S$).*

(*b*) *The ∞-category $\mathcal{C}$ admits pullbacks. Moreover, the class of morphisms S is stable under pullback: for every pullback diagram*

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ D' & \longrightarrow & D \end{array}$$

such that $f \in S$, the morphism f' also belongs to S

(*c*) *The ∞-category $\mathcal{C}$ admits finite coproducts. Moreover, the collection of morphisms S is stable under finite coproducts: if $f_i : C_i \to D_i$ is a finite collection of morphisms in $\mathcal{C}$ which belong to S, then the induced map $\coprod_i C_i \to \coprod_i D_i$ also belongs to S.*

(*d*) *Finite coproducts in $\mathcal{C}$ are universal. That is, given a diagram $\coprod_{1 \leq i \leq n} C_i \to D \leftarrow D'$, the canonical map $\coprod_{1 \leq i \leq n} (C_i \times_D D') \to (\coprod_{1 \leq i \leq n} C_i) \times_D D'$ is an equivalence in $\mathcal{C}$.*

Then there exists a Grothendieck topology on $\mathcal{C}$ which can be described as follows: a sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ on an object $C \in \mathcal{C}$ is covering if and only if it contains a finite collection of morphisms $\{C_i \to C\}_{1 \leq i \leq n}$ such that the induced map $\coprod C_i \to C$ belongs to S.

Proof. We show that the collection of covering sieves satisfies the conditions of Definition T.6.2.2.1:

(1) For every object $C \in \mathcal{C}$, the sieve $\mathcal{C}_{/C}$ covers C. This is clear, since $\mathcal{C}_{/C}$ contains the identity map $\mathrm{id}_C : C \to C$, which belongs to S by (*a*).

(2) If $\mathcal{C}^{(0)}_{/C}$ is a covering sieve on an object $C \in \mathcal{C}$ and $f : C' \to C$ is a morphism in $\mathcal{C}$, then the pullback sieve $f^* \mathcal{C}^{(0)}_{/C}$ covers C'. To prove this, we observe that there exists a finite collection of morphisms $C_i \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$ such that the induced map $\coprod_i C_i \to C$ belongs to S. Assumption (b) guarantees that the induced map $(\coprod_i C_i) \times_C C' \to C'$ also belongs to S, and assumption (d) gives an identification $(\coprod_i C_i) \times_C C' \simeq \coprod_i (C_i \times_C C')$. It now suffices to observe that each of the morphisms $C_i \times_C C' \to C'$ belongs to the sieve $f^* \mathcal{C}^{(0)}_{/C}$.

(3) Let $\mathcal{C}^{(0)}_{/C}$ be a covering sieve on an object $C \in \mathcal{C}$, and let $\mathcal{C}^{(1)}_{/C}$ be an arbitrary sieve on C. Suppose that, for each morphism $f : C' \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$, the pullback sieve $f^* \mathcal{C}^{(1)}_{/C}$ covers C'. We must show that $\mathcal{C}^{(1)}_{/C}$ covers C. Since $\mathcal{C}^{(0)}_{/C}$ is a covering sieve, there exists a finite collection of morphisms $f_i : C_i \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$ such that the induced map $\coprod_i C_i \to C$ belongs to S. Each $f_i^* \mathcal{C}^{(1)}_{/C}$ is a covering sieve on C_i, so there exists a finite collection of morphisms $C_{i,j} \to C_i$ belonging to $f^* \mathcal{C}^{(1)}_{/C}$ such that the induced map $\coprod_j C_{i,j} \to C_i$ belongs to S. It follows that each of the composite maps $C_{i,j} \to C_i \to C$ belongs to the sieve $\mathcal{C}^{(1)}_{/C}$. To prove that $\mathcal{C}^{(1)}_{/C}$ is covering, it suffices to show that the map $g : \coprod_{i,j} C_{i,j} \to C$ belongs to S. To prove this, we factor g as a composition

$$\coprod_{i,j} C_{i,j} \xrightarrow{g'} \coprod_i C_i \xrightarrow{g''} C.$$

The map g'' belongs to S by assumption, and the map g' is a finite coproduct of maps belonging to S and therefore belongs to S by virtue of (c). It follows from (a) that $g \simeq g'' \circ g'$ belongs to S, as required.

□

We now illustrate Proposition 5.1 by means of an example.

Definition 5.2. Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. We will say that f is *faithfully flat* if it satisfies the following conditions:

(i) The underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is faithfully flat, in the sense of classical commutative algebra.

(ii) For every integer n, the map $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i A) \to \pi_i B$ is an isomorphism.

Remark 5.3. Let $f : A \to B$ be a faithfully flat morphism of $\mathbb{E}_\infty$-rings. A morphism $M \to N$ of A-modules is an equivalence if and only if the induced map $M \otimes_A B \to N \otimes_A B$ is an equivalence. This follows immediately from Corollary A.7.2.1.22.

Proposition 5.4. *Let* Aff *denote the opposite of the* ∞*-category* CAlg *of* $\mathbb{E}_\infty$*-rings; if* A *is an* $\mathbb{E}_\infty$*-ring, we denote the corresponding object of* $\mathcal{C}$ *by* $\operatorname{Spec} A$*. Let* S *denote the collection of all morphisms in* $\mathcal{C}$ *which correspond to faithfully flat maps of* $\mathbb{E}_\infty$*-rings (Definition 5.2). Then* S *satisfies the hypotheses of Proposition 5.1, and therefore determines a Grothendieck topology on* Aff*.*

Remark 5.5. We will refer to the Grothendieck topology of Proposition 5.4 as the *flat topology* on Aff.

Warning 5.6. The ∞-category Aff is not small. Consequently, though it makes sense to consider the ∞-category $\mathcal{S}\mathrm{hv}(\mathrm{Aff}) \subseteq \mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$ of sheaves of spaces on Aff, it is not clear that $\mathcal{S}\mathrm{hv}(\mathrm{Aff})$ is a localization of $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$. In concrete terms, the trouble is that the process of sheafification with respect to the flat topology may produce spaces which are not essentially small (since there does not exist any small, cofinal collection of flat coverings of a given $\mathbb{E}_\infty$-ring). However, this issue will not concern us in this section.

Proof of Proposition 5.4. We consider each condition of Proposition 5.1 in turn:

(*a*) The collection of faithfully flat morphisms in CAlg contains all equivalences and is stable under composition. The first assertion is obvious. To prove the second, consider a pair of faithfully flat morphisms $A \xrightarrow{f} B \xrightarrow{g} C$; we wish to prove that $g \circ f$ is faithfully flat. The underlying map $\pi_0 A \to \pi_0 C$ is a composition of faithfully flat morphisms of commutative rings, and therefore faithfully flat. The map $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 C, \pi_i A) \to \pi_i C$ factors as a composition

$$\begin{aligned} \mathrm{Tor}_0^{\pi_0 A}(\pi_0 C, \pi_i A) &\simeq \mathrm{Tor}_0^{\pi_0 B}(\pi_0 C, \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i A)) \\ &\xrightarrow{\alpha} \mathrm{Tor}_0^{\pi_0 B}(\pi_0 C, \pi_i B) \\ &\xrightarrow{\beta} \pi_i C. \end{aligned}$$

The map α is an isomorphism because f is faithfully flat, and the map β is an isomorphism because g is faithfully flat.

(*b*) It is clear that the ∞-category $\mathcal{C}$ admits pullbacks (the ∞-category CAlg of $\mathbb{E}_\infty$-rings is presentable and therefore admits all small limits and colimits). It therefore suffices to show that if we are given a diagram $A' \xleftarrow{g} A \xrightarrow{f} B$, where f is faithfully flat, then the induced map $A' \to B \otimes_A A'$ is faithfully flat. Since B is flat over A, Proposition A.7.2.2.13 guarantees that the canonical maps $\gamma_i : \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i A') \to \pi_i(B \otimes_A A')$ is an isomorphism. Taking $i = 0$, we deduce that $\pi_0(B \otimes_A A')$ is a pushout of $\pi_0 B$ and $\pi_0 A'$ over $\pi_0 A$, and therefore faithfully flat over $\pi_0 A'$. Moreover, the canonical map $\mathrm{Tor}_0^{\pi_0 A'}(\pi_0(B \otimes_A A'), \pi_i A') \to \pi_i(B \otimes_A A')$ factors as a composition

$$\begin{aligned} \mathrm{Tor}_0^{\pi_0 A'}(\pi_0(B \otimes_A A'), \pi_i A') &\overset{\gamma_0^{-1}}{\simeq} \mathrm{Tor}_0^{\pi_0 A'}(\mathrm{Tor}_0^{\pi_0 A}(\pi_0 A', \pi_0 B), \pi_i A') \\ &\simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_i A') \\ &\xrightarrow{\gamma_i} \pi_i(B \otimes_A A'). \end{aligned}$$

and is therefore an isomorphism.

(*c*) It is clear that the category $\mathcal{C}$ admits pushouts, which are given by products of the corresponding $\mathbb{E}_\infty$-rings. We must show that if we are given a finite collection of faithfully flat morphisms $A_i \to B_i$ and set $A = \prod_i A_i$ and $B = \prod_i B_i$, then the induced map $A \to B$ is also faithfully flat. We have $\pi_0 A = \prod_i \pi_0 A_i$ and $\pi_0 B \simeq \prod_i \pi_0 B_i$. Since a product of faithfully flat morphisms of commutative rings is faithfully flat, we deduce that the map $\pi_0 A \to \pi_0 B$ is faithfully flat. For higher homotopy groups, we have

$$\begin{aligned} \pi_n B &\simeq \prod_i (\pi_n B_i) \\ &\simeq \prod_i \mathrm{Tor}_0^{\pi_0 A_i}(\pi_0 B_i, \pi_n A_i) \\ &\simeq \mathrm{Tor}_0^{\prod_i \pi_0 A_i}(\prod_i \pi_0 B_i, \prod_i \pi_n A_i) \\ &\simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_n A) \end{aligned}$$

as required.

(*d*) Given a finite collection of morphisms $A \to A_i$ and a morphism $A \to B$ in CAlg, we must show that the canonical map

$$(\prod_i A_i) \otimes_A B \to \prod_i (A_i \otimes_A B)$$

is an equivalence of $\mathbb{E}_\infty$-rings. We will show that this map is an equivalence in the ∞-category of B-modules. For this, it suffices to observe that the functor $F : \mathrm{Mod}_A \to \mathrm{Mod}_B$ given by $M \mapsto M \otimes_A B$ preserves finite limits. The functor F evidently preserves small colimits, and therefore also finite limits because the ∞-categories Mod_A and Mod_B are stable (Proposition A.1.1.4.1).

□

We wish to study sheaves with respect to the flat topology on CAlg^{op}. To this end, it is useful to have the following characterization of sheaves:

Proposition 5.7. *Let $\mathcal{C}$ be an ∞-category and S a collection of morphisms in $\mathcal{C}$. Assume that $\mathcal{C}$ and S satisfy the conditions of Proposition 5.1, together with the following additional condition:*

- *(e) Coproducts in the ∞-category $\mathcal{C}$ are disjoint. That is, if C and C' are objects of $\mathcal{C}$, then the fiber product $C \times_{C \coprod C'} C'$ is an initial object of $\mathcal{C}$ (see § T.6.1.1).*

Let $\mathcal{D}$ be an arbitrary ∞-category and let $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{D}$ be a functor. Then $\mathcal{F}$ is a $\mathcal{D}$-valued sheaf on $\mathcal{C}$ if and only if the following conditions are satisfied:

- *(1) The functor $\mathcal{F}$ preserves finite products.*
- *(2) Let $f : U_0 \to X$ be a morphism in $\mathcal{C}$ which belongs to S and let $U_\bullet$ be a Čech nerve of f (see § T.6.1.2), regarded as an augmented simplicial object of $\mathcal{C}$. Then the composite map*

$$\mathrm{N}(\mathbf{\Delta}_+) \stackrel{U_\bullet}{\to} \mathcal{C}^{op} \stackrel{\mathcal{F}}{\to} \mathcal{D}$$

is a limit diagram. In other words, $\mathcal{F}$ exhibits $\mathcal{F}(X)$ as a totalization of the cosimplicial object $[n] \mapsto \mathcal{F}(U_n)$.

Proof. For every object $D \in \mathcal{D}$, let $h_D : \mathcal{D} \to \mathcal{S}$ be the functor corepresented by D. Using Proposition T.5.1.3.2, we deduce that $\mathcal{F}$ is a sheaf $\mathcal{D}$-valued sheaf on $\mathcal{C}$ if and only if each composite map $h_D \circ \mathcal{F}$ is a $\mathcal{S}$-valued sheaf on $\mathcal{C}$, and that $\mathcal{F}$ satisfies conditions (1) and (2) if and only if each $h_D \circ \mathcal{F}$ satisfies the same condition. We may therefore replace $\mathcal{F}$ by $h_D \circ \mathcal{F}$ and thereby reduce to the case where $\mathcal{D} = \mathcal{S}$.

Suppose first that $\mathcal{F}$ is a sheaf; we will prove that $\mathcal{F}$ satisfies conditions (1) and (2). We begin with (1). Let $\{C_i\}_{1 \leq i \leq n}$ be a finite collection of objects in $\mathcal{C}$ and let $C = \coprod_i C_i$ be their coproduct. We wish to prove that the canonical map $\mathcal{F}(C) \to \prod_i \mathcal{F}(C_i)$ is an equivalence. The proof proceeds by induction on n. If $n = 0$, then C is an initial object of $\mathcal{C}$ so that the empty sieve is a covering of C. Since $\mathcal{F}$ is a sheaf, we deduce that $\mathcal{F}(C)$ is a final object of $\mathcal{S}$, as required. If $n = 1$, there is nothing to prove. If $n > 2$, we let $D = \coprod_{1 \leq i < n} C_i$, so that $C = D \amalg C_n$. The natural map $\mathcal{F}(C) \to \prod_i \mathcal{F}(C_i)$ then factors as a composition of maps

$$\mathcal{F}(C) \to \mathcal{F}(D) \times \mathcal{F}(C_n) \to (\prod_{1 \leq i < n} \mathcal{F}(C_i)) \times \mathcal{F}(C_n) \simeq \prod_{1 \leq i \leq n} \mathcal{F}(C_n),$$

each of which is an equivalence by the inductive hypothesis. It remains to treat the case $n = 2$. Let $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ be the sieve generated by C_1 and C_2. This sieve is evidently a covering of C, so that $\mathcal{F}(C) \simeq \varprojlim \mathcal{F} | (\mathcal{C}^{(0)}_{/C})^{op}$. To complete the proof, it suffices to show that the canonical map $\varprojlim \mathcal{F} | (\mathcal{C}^{(0)}_{/C})^{op} \to \mathcal{F}(C_1) \times \mathcal{F}(C_2)$ is an equivalence. Let $p : \Lambda^2_0 \to \mathcal{C}^{(0)}_{/C}$ be the map corresponding to the pullback diagram

$$\begin{array}{ccc} C_1 \times_C C_2 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ C_2 & \longrightarrow & C \end{array}$$

in $\mathcal{C}$. Since $C_1 \times_C C_2$ is initial in $\mathcal{C}$, the above argument shows that $\mathcal{F}(C_1 \times_C C_2)$ is final in $\mathcal{D}$: that is, $\mathcal{F}|(\Lambda^2_0)^{op}$ is a right Kan extension of $\mathcal{F}|\{1,2\}^{op}$, so that $\varprojlim \mathcal{F}|(\Lambda^2_0)^{op} \simeq \mathcal{F}(C_1) \times \mathcal{F}(C_2)$ by Lemma T.4.3.2.7. To complete the proof of (1), we will show that p is left cofinal. According to Theorem T.4.1.3.1, it will suffice to show that for every object $(f : D \to C) \in \mathcal{C}^{(0)}_{/C}$, the ∞-category $S = \Lambda^2_0 \times_{\mathcal{C}^{(0)}_{/C}} (\mathcal{C}^{(0)}_{/C})_{f/}$ is weakly contractible. If $D \in \mathcal{C}$ is initial, then the projection map $S \to \Lambda^2_0$ is a trivial Kan fibration and the result is obvious. If D is not initial, then condition (d) guarantees that there do not exist any maps from D to an initial object of $\mathcal{C}$. Using (e), we deduce that there do not exist any maps from D into $C_1 \times_C C_2$. It follows that f factors through either the map $C_1 \to C$ or $C_2 \to C$, but not both. Without loss of generality, we may assume that f factors through $C_1 \to C$. In this case, we can identify S with the simplicial set $\{C_1\} \times_{\mathcal{C}_{/C}} \mathcal{C}_{D//C}$, which is the homotopy fiber of the composition map $q : \mathrm{Map}_{\mathcal{C}}(D, C_1) \to \mathrm{Map}_{\mathcal{C}}(D, C)$ over f. We wish to show that this homotopy fiber is contractible. By assumption, it is nonempty; it will therefore suffice to show that the morphism q is (-1)-truncated. To prove this, we need only verify that $C_1 \to C$ is a monomorphism; that is, that the diagonal map $C_1 \to C_1 \times_C C_1$ is an equivalence. Using (d), we obtain equivalences

$$C_1 \simeq C_1 \times_C C \simeq C_1 \times_C (C_1 \coprod C_2) \simeq (C_1 \times_C C_1) \coprod (C_1 \times_C C_2),$$

and the first summand maps by an equivalence to $C_1 \times_C C_1$. The second summand is trivial, by virtue of (e).

We now prove (2). Let $f : U_0 \to X$ be a morphism of S and let f be its Čech nerve, so that f generates a covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$. We can regard $U_\bullet$ as determining a simplicial object $V : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}^{(0)}_{/X}$. Our assumption that $\mathcal{F}$ is a sheaf guarantees that $\mathcal{F}(X) \simeq \varprojlim \mathcal{F}|(\mathcal{C}^{(0)}_{/X})^{op}$. To prove (2), it suffices to prove that the map V is left cofinal. According to Theorem T.4.1.3.1, it suffices to show that for every map $f : X' \to X$ belonging to $\mathcal{C}^{(0)}_{/X}$, the ∞-category $\mathcal{X} = \mathrm{N}(\mathbf{\Delta})^{op} \times_{\mathcal{C}^{(0)}_{/X}} (\mathcal{C}^{(0)}_{/X})_{f/}$ is weakly contractible. The projection map $\mathcal{X} \to \mathrm{N}(\mathbf{\Delta})^{op}$ is a left fibration, classified by a functor $\chi : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{S}$. According to Proposition T.3.3.4.5, it will suffice to show that $\varinjlim(\chi)$ is contractible. Note that χ can be identified with the underlying simplicial object of the Čech nerve of the map of spaces $q : \mathrm{Map}_{\mathcal{C}_{/X}}(X', U_0) \to \Delta^0$. Since f belongs to the sieve $\mathcal{C}^{(0)}_{/X}$, the space $\mathrm{Map}_{\mathcal{C}_{/X}}(X', U_0)$ is nonempty so that q is an effective epimorphism. Since $\mathcal{S}$ is an ∞-topos, we conclude that $\varinjlim(\chi) \simeq \Delta^0$ as required.

Now suppose that $\mathcal{F}$ satisfies (1) and (2); we will show that $\mathcal{F}$ is a sheaf on $\mathcal{C}$. Choose an object $X \in \mathcal{C}$ and a covering sieve $\mathcal{C}^{(0)}_{/X}$; we wish to prove that $\mathcal{F}(X) \simeq \varprojlim \mathcal{F}|(\mathcal{C}^{(0)}_{/X})^{op}$. We first treat the case where $\mathcal{C}^{(0)}_{/X}$ is generated by a single morphism $f : U_0 \to X$ which belongs to S. Let $U_\bullet$ be a Čech nerve of f, so that $\mathcal{F}(X)$ can be identified with the totalization of the cosimplicial space $[n] \mapsto \mathcal{F}(U_n)$ by virtue of (2). To complete the proof, we invoke the fact (established above) that $U_\bullet$ determines a left cofinal map $\mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}^{(0)}_{/X}$.

Now suppose that $\mathcal{C}^{(0)}_{/X}$ is generated by a finite collection of morphisms $\{C_i \to X\}_{1 \leq i \leq n}$ such that the induced map $\coprod C_i \to X$ belongs to S. Let $C = \coprod_i C_i$ and let $\mathcal{C}^{(1)}_{/X}$ denote the sieve generated by the induced map $C \to X$. Then $\mathcal{C}^{(1)}_{/X}$ contains $\mathcal{C}^{(0)}_{/X}$ and is therefore a covering sieve; the above argument shows that $\mathcal{F}(X) \simeq \varprojlim \mathcal{F}|(\mathcal{C}^{(1)}_{/X})^{op}$. To complete the proof in this case, it will suffice to show that $\mathcal{F}|(\mathcal{C}^{(1)}_{/X})^{op}$ is a right Kan extension of $\mathcal{F}|(\mathcal{C}^{(0)}_{/X})^{op}$.

Fix an object $f : U \to X$ of the sieve $\mathcal{C}^{(1)}_{/X}$, and let $\mathcal{E}$ denote the full subcategory of $(\mathcal{C}^{(1)}_{/X})_{/U} \simeq \mathcal{C}_{/U}$ spanned by those objects whose image in $\mathcal{C}_{/X}$ belongs to $\mathcal{C}^{(0)}_{/X}$. We wish to prove that the canonical map $\mathcal{F}(U) \to \varprojlim \mathcal{F}|\mathcal{E}^{op}$ is an equivalence. By construction, the map f factors through some map $f_0 : U \to C$. Invoking (b), we have $U \simeq U \times_C C \simeq \coprod_i U \times_C C_i$, so that U can be obtained as a coproduct of objects U_i belonging to $\mathcal{C}^{(0)}_{/X}$. Let $T \subseteq \{1, \ldots, n\}$ denote the collection of indices for which U_i is not initial. We let $\mathcal{E}' \subseteq \mathcal{E}$ denote the full subcategory spanned by morphisms $U' \to U$ which factor through some U_i and such that $U' \in \mathcal{C}$ is not initial. For $i \neq j$, the fiber product $U_i \times_U U_j$ is initial (by (e)) and therefore receives no morphisms from non-initial objects of $\mathcal{C}$ (by (d)); it follows that $\mathcal{E}'$ can be decomposed as a disjoint union $\coprod_{i \in T} \mathcal{E}'_i$ where each $\mathcal{E}'_i$ denotes the full subcategory of $\mathcal{E}'$ spanned by those morphisms $U' \to U$ which factor through U_i. Since the map $U_i \to U$ is a monomorphism, each $\mathcal{E}'_i$ contains the map $U_i \to U$ as a final object, so that the inclusion $\{U_i\}_{i \in T} \to \mathcal{E}'$ is left cofinal. Condition (1) implies that $\mathcal{F}(U) \simeq \prod_{i \in T} \mathcal{F}(U_i)$, so that $\mathcal{F}(U)$ is a limit of the diagram $\mathcal{F} | (\mathcal{E}')^{op}$. We will prove that $\mathcal{F} | \mathcal{E}^{op}$ is a right Kan extension of $\mathcal{F} | (\mathcal{E}')^{op}$, so that $\varprojlim \mathcal{F} | \mathcal{E}^{op} \simeq \varprojlim \mathcal{F} | (\mathcal{E}')^{op} \simeq \mathcal{F}(U)$ by Lemma T.4.3.2.7. To see this, choose an object $U' \to U$ in $\mathcal{E}$; we wish to show that $\mathcal{F}(U')$ is a limit of the diagram $\mathcal{F} | (\mathcal{E}'_{/U'})^{op}$. Let $U'_i = U' \times_U U_i$, and let T' be the collection of indices i for which U'_i is not initial. Then $\mathcal{E}'_{/U'}$ decomposes as a disjoint union $\coprod_{i \in T'} (\mathcal{E}'_{/U'})_i$, each of which has a final object (given by the map $U'_i \to U'$). It follows that $\varprojlim \mathcal{F} | (\mathcal{E}'_{/U'})^{op}$ is equivalent to $\prod_{i \in T'} \mathcal{F}(U'_i)$, which is equivalent to $\mathcal{F}(U')$ by virtue of (1).

We now treat the case of a general covering sieve $\mathcal{C}^{(0)}_{/X} \subseteq \mathcal{C}_{/X}$. By definition, there exists a finite collection of morphisms $f_i : C_i \to X$ belonging to $\mathcal{C}^{(0)}_{/X}$ such that the induced map $\coprod_i C_i \to X$ belongs to S. Let $\mathcal{C}^{(1)}_{/X} \subseteq \mathcal{C}^{(0)}_{/X}$ be the sieve generated by the maps f_i. The above argument shows that $\mathcal{F}(X) \simeq \varprojlim \mathcal{F} | (\mathcal{C}^{(1)}_{/X})^{op}$. To prove that $\mathcal{F}(X) \simeq \varprojlim (\mathcal{C}^{(0)})_{/X})^{op}$, it will suffice to show that $\mathcal{F} | (\mathcal{C}^{(0)}_{/X})^{op}$ is a right Kan extension of $\mathcal{F} | (\mathcal{C}^{(1)}_{/X})^{op}$ (Lemma T.4.3.2.7). Unwinding the definitions, we must show that for every $f : U \to X$ belonging to the sieve $\mathcal{C}^{(0)}_{/X}$, we have $\mathcal{F}(U) \simeq \varprojlim \mathcal{F} | (f^* \mathcal{C}^{(1)}_{/X})$. This is clear, since $f^* \mathcal{C}^{(1)}_{/X}$ is generated by the pullback maps $C_i \times_X U \to U$, and the induced map $\coprod_i (C_i \times_X U) \to U$ factors as a composition

$$\coprod_i (C_i \times_X U) \xrightarrow{\alpha} (\coprod_i C_i) \times_X U \xrightarrow{\beta} U,$$

where α is an equivalence by assumption (d) and the map β belongs to S by assumption (b). □

Example 5.8. Let $\mathrm{Aff} = \mathrm{CAlg}^{op}$ and let S be the collection of faithfully flat morphisms in Aff. Then (Aff, S) satisfies the hypotheses of Proposition 5.7. To prove (e), we must show that if A and B are $\mathbb{E}_\infty$-rings, then the fiber product $A \times_{A \times B} B$ is trivial. To prove this, we observe that the identity element of $\pi_0(A \times B) \simeq \pi_0 A \times \pi_0 B$ can be written as a sum $e + e'$, where $e = (1, 0)$ and $e' = (0, 1)$. The image of e is trivial in $\pi_0 B$, and the image of e' is trivial in $\pi_0 A$. It follows that e and e' both have trivial image in the commutative ring $R = \pi_0(A \times_{A \times B} B)$, so that $1 = 0$ in R. Since every homotopy group of $A \times_{A \times B} B$ is a module over R, each of these groups is trivial.

Here is a more classical example of a sheaf with respect to the flat topology:

Proposition 5.9. *For every $\mathbb{E}_\infty$-ring A, let $\mathrm{Spec}^{\mathrm{Z}} A$ be the Zariski spectrum of the commutative ring $\pi_0 A$, and let $\mathcal{U}(A)$ be the collection set of open subsets of $\mathrm{Spec}^{\mathrm{Z}} A$. Then $\mathcal{U}$ determines a functor $\mathcal{U} : \mathrm{CAlg} \to \mathrm{N}(\mathcal{S}\mathrm{et})$ satisfying the hypotheses of Proposition 5.7 (with respect to the flat topology), and can therefore be regarded as a sheaf (of sets) on $\mathrm{Aff} = \mathrm{CAlg}^{op}$.*

Remark 5.10. The sheaf $\mathcal{U} : \mathrm{CAlg} \to \mathrm{N}(\mathcal{S}\mathrm{et})$ of Proposition 5.9 can be regarded as a discrete object in the ∞-category of $\mathcal{S}$-valued sheaves on CAlg^{op}. As such, it is automatically hypercomplete.

Proof of Proposition 5.9. To verify (1), we must show that for every finite collection of $\mathbb{E}_\infty$-rings A_i, the map $\mathcal{U}(\prod_i A_i) \to \prod_i \mathcal{U}(A_i)$ is bijective. This follows from the observation that there is a canonical homeomorphism $\mathrm{Spec}^{\mathrm{Z}}(\prod_i A_i) \simeq \coprod_i \mathrm{Spec}^{\mathrm{Z}} A_i$.

We now prove (2). Let $f : A \to B$ be a faithfully flat morphism of $\mathbb{E}_\infty$-rings; we wish to prove that

$$\mathcal{U}(A) \longrightarrow \mathcal{U}(B) \rightrightarrows \mathcal{U}(B \otimes_A B)$$

is an equalizer diagram in the category of sets. We can divide this assertion into two parts:

(a) The map $\mathcal{U}(A) \to \mathcal{U}(B)$ is injective. To prove this, we must show that an open subset $U \subseteq \mathrm{Spec}^{\mathrm{Z}} A$ is determined by its inverse image in $\mathrm{Spec}^{\mathrm{Z}} B$. This is clear, since the assumption that $A \to B$ is faithfully flat guarantees that $\psi : \mathrm{Spec}^{\mathrm{Z}} B \to \mathrm{Spec}^{\mathrm{Z}} A$ is a surjection.

(b) Let $\phi_0, \phi_1 : \mathrm{Spec}^{\mathrm{Z}}(B \otimes_A B) \to \mathrm{Spec}^{\mathrm{Z}} B$ be the two projection maps. We claim that if $Z \subseteq \mathrm{Spec}^{\mathrm{Z}} B$ is a closed subset with $\phi_0^{-1} Z = \phi_1^{-1} Z$, then $Z = \phi^{-1} V$ for some closed subset $V \subseteq \mathrm{Spec}^{\mathrm{Z}} A$. Choose an ideal $I \subseteq \pi_0 B$ such that $Z = \{\mathfrak{p} \subseteq \pi_0 B : I \subseteq \mathfrak{p}\}$, and let $J = f^{-1} I \subseteq \pi_0 A$. Set $V = \{\mathfrak{q} \subseteq \pi_0 A : J \subseteq \mathfrak{q}\}$. Then $\phi^{-1} V = \{\mathfrak{p} \subseteq \pi_0 B : f(J)\pi_0 B \subseteq \mathfrak{p}\}$. To prove that $\phi^{-1} V = Z$, it suffices to show that $f(J)\pi_0 B$ and I have the same nilradical. Let R denote the commutative ring $\pi_0 A / J$ and R' the commutative ring $\pi_0 B / J \pi_0 B$, and let I' denote the image of I in R'. Then $R \to R'$ is faithfully flat and the composite map $R \to R' \to R'/I'$ is injective; we wish to prove that I' is a radical ideal. In other words, we wish to show that every element $x \in I'$ is nilpotent. Since $\phi_0^{-1} Z = \phi_1^{-1} Z$, we deduce that the ideals $I' \otimes_R R'$ and $R' \otimes_R I'$ have the same radical in $R' \otimes_R R'$. Consequently, since $x \otimes 1$ belongs to $I' \otimes_R R'$, some power $x^n \otimes 1$ belongs to $R' \otimes_R I'$. It follows that the image of x^n is trivial in $R' \otimes_R R'/I'$. Since R' is flat over R, the injection $R \to R'/I'$ induces an injection $R' \to R' \otimes_R R'/I'$; it follows that $x^n = 0$ in R', as desired.

□

There is an analogue of Proposition 5.7 which describes the class of *hypercomplete* sheaves on an ∞-category $\mathcal{C}$. To state it, we first need to introduce a variation on Definition 4.19.

Definition 5.11. Let $\mathbf{\Delta}_{s,+}$ be the subcategory of $\mathbf{\Delta}_+$ whose morphisms are injective maps of linearly ordered sets $[m] \to [n]$. If $\mathcal{C}$ is an ∞-category, we will refer to a functor $X_\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+})^{op} \to \mathcal{C}$ as an *augmented semisimplicial object* of $\mathcal{C}$. If $\mathcal{C}$ admits finite limits, then for each $n \geq 0$ we can associate to $X_\bullet$ an nth matching object $M_n(X) = \varprojlim_{[m] \to [n]} X_m$, where the limit is taken over all injective maps $[m] \to [n]$ such that $m < n$.

Let S be a collection of morphisms in $\mathcal{C}$. We will say that an augmented semisimplicial object $X_\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+})^{op} \to \mathcal{C}$ is an *S-hypercovering* if, for each $n \geq 0$, the canonical map $X_n \to M_n(X)$ belongs to S.

Proposition 5.12. *Let $\mathcal{C}$ be an ∞-category and S a collection of morphisms in $\mathcal{C}$. Assume that $\mathcal{C}$ and S satisfy the conditions of Proposition 5.1 and condition* (e) *of Proposition 5.7. Let $\mathcal{D}$ be an arbitrary ∞-category and $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{D}$ a functor. Then $\mathcal{F}$ is a hypercomplete $\mathcal{D}$-valued sheaf on $\mathcal{C}$ if and only if the following conditions are satisfied:*

(1) *The functor $\mathcal{F}$ preserves finite products.*

(2) *Let $X_\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+})^{op} \to \mathcal{C}$ be an S-hypercovering. Then the composite map*

$$\mathrm{N}(\mathbf{\Delta}_{s,+}) \xrightarrow{X_\bullet} \mathcal{C}^{op} \xrightarrow{\mathcal{F}} \mathcal{D}$$

is a limit diagram.

Proof. As in the proof of Proposition 5.7, we may assume without loss of generality that $\mathcal{D} = \mathcal{S}$. We first prove the "only if" direction. Assume that $\mathcal{F}$ is a hypercomplete sheaf. Condition (1) follows from Proposition 5.7. To prove (2), let $F : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ denote the composition of the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ with the sheafification functor $\mathcal{P}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C})$, and let $L : \mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C})^\wedge$ be a left adjoint to the inclusion. It will suffice to show that $L \circ F \circ X_\bullet$ is a colimit diagram in $\mathrm{Shv}(\mathcal{C})^\wedge$: in other words, that $X_\bullet$ exhibits $F(X_{-1})$ as a colimit of the diagram $\{F X_n\}_{n \geq 0}$. This follows immediately from Theorem 4.20, applied in the ∞-topos $\mathrm{Shv}(\mathcal{C})_{/F X_{-1}}$.

Now suppose that (1) and (2) are satisfied. Proposition 5.7 guarantees that $\mathcal{F}$ is a sheaf on $\mathcal{C}$; we wish to prove that $\mathcal{C}$ is hypercomplete. Choose an ∞-connective morphism $\alpha : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Shv}(\mathcal{C})$, where $\mathcal{G}$ is hypercomplete (and therefore satisfies conditions (1) and (2)). We wish to show that α is an equivalence. To prove this, it will suffice to verify the following:

($*$) Let $\alpha : \mathcal{F} \to \mathcal{G}$ be an ∞-connective morphism in $\mathcal{S}\mathrm{hv}(\mathcal{C})$, where $\mathcal{F}$ and $\mathcal{G}$ both satisfy (2). Then α is an equivalence.

To prove ($*$), we will show that for every object $C \in \mathcal{C}$ and each $n \geq 0$, the map of spaces $\alpha_C : \mathcal{F}(C) \to \mathcal{G}(C)$ is n-connective. The proof proceeds by induction on n. If $n > 0$, then the inductive hypothesis guarantees that α_C is 0-connective; it therefore suffices to show that the diagonal map $\mathcal{F}(C) \to \mathcal{F}(C) \times_{\mathcal{G}(C)} \mathcal{F}(C)$ is $(n-1)$-connective, which also follows from the inductive hypothesis. It therefore suffices to treat the case $n = 0$: that is, we must show that the map $\mathcal{F}(C) \to \mathcal{G}(C)$ is surjective on connected components. Replacing $\mathcal{C}$ by $\mathcal{C}_{/C}$, we may assume that C is a final object of $\mathcal{C}$, so that a point $\eta \in \mathcal{G}(C)$ determines a map $\mathbf{1} \to \mathcal{G}$, where $\mathbf{1}$ denotes the final object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Replacing $\mathcal{F}$ by $\mathcal{F} \times_{\mathcal{G}} \mathbf{1}$, we are reduced to proving the following:

($*'$) Let $\mathcal{F}$ be an ∞-connective object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ satisfying condition (2), and let $C \in \mathcal{C}$ be a final object. Then $\mathcal{F}(C)$ is nonempty.

To prove ($*'$), let $\widetilde{\mathcal{C}} \to \mathcal{C}$ be the right fibration classified by the functor $\mathcal{F} : \mathcal{C}^{op} \to \mathcal{S}$. We wish to show that $\widetilde{\mathcal{C}} \times_{\mathcal{C}} \{C\}$ is nonempty. We will construct an S-hypercovering $X_\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+})^{op} \to \mathcal{C}$ with $X_{-1} = C$ together with a lifting $Y_\bullet : \mathrm{N}(\mathbf{\Delta}_s)^{op} \to \widetilde{\mathcal{C}}$ of $X_\bullet | \mathrm{N}(\mathbf{\Delta}_s)^{op}$. Condition (2) and Corollary T.3.3.3.3 guarantee that $Y_\bullet$ extends (in an essentially unique fashion) to a map $\overline{Y}_\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+})^{op} \to \widetilde{\mathcal{C}}$ lifting $X_\bullet$, so that $\overline{Y}_{-1}$ is the required point of $\widetilde{\mathcal{C}} \times_{\mathcal{C}} \{C\}$.

The construction of $X_\bullet$ and $Y_\bullet$ proceeds in stages: we define $X_\bullet^{\leq m} : \mathrm{N}(\mathbf{\Delta}_{s,+}^{\leq m})^{op} \to \mathcal{C}$ and $Y_\bullet^{\leq m} : \mathrm{N}(\mathbf{\Delta}_s^{\leq m})^{op} \to \widetilde{\mathcal{C}}$ by induction on m, the case $m = -1$ being trivial. Assuming that $X_\bullet^{\leq m-1}$ has been defined, we can define the matching object $M_n(X) \in \mathcal{C}$. The lifting $Y_\bullet^{\leq m-1}$ determines a map $\partial \Delta^m \to \mathcal{F}(M_n(X))$. Since $\mathcal{F}$ is ∞-connective, there exists a collection of morphisms $\{D_i \to M_n(X)\}$ which generate a covering sieve, such that each composite map $\partial \Delta^m \to \mathcal{F}(M_n(X)) \to \mathcal{F}(D_i)$ is nullhomotopic. Without loss of generality, we may assume that the set of indices D_i is finite, and that the map $\coprod D_i \to M_n(X)$ belongs to S. Let $D = \coprod D_i$. Using condition (1), we see that the composite map $\gamma : \partial \Delta^m \to \mathcal{F}(M_n(X)) \to \mathcal{F}(D)$ is nullhomotopic. We can now define the extension $X_\bullet^{\leq m}$ by setting $X_m = D$, and the extension $Y_\bullet^{\leq m}$ using the nullhomotopy γ. □

Lemma 5.13. *Let $R^\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{CAlg}$ be a flat hypercovering. Then $R^\bullet$ is a limit diagram in* CAlg.

Proof. This is an immediate consequence of Corollary 6.14, which will be proven in §6. □

Theorem 5.14. *The identity functor* CAlg $\to$ CAlg *is a hypercomplete* CAlg*-valued sheaf on* CAlg^{op} *(with respect to the flat topology).*

Proof. Combine Proposition 5.12 with Lemma 5.13. □

We next show that if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a 0-localic spectral scheme, then the functor that it represents is a (hypercomplete) sheaf with respect to the flat topology on the category $\mathrm{Aff} = \mathrm{CAlg}^{op}$. This is a special case of the following more general result:

Theorem 5.15. *Let $\mathcal{X}$ be a 0-localic ∞-topos and let $\mathcal{O}_{\mathcal{X}}$ be a local sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Let* Spec : $\mathrm{CAlg}^{op} \to \mathrm{RingTop}_{\mathrm{Zar}}$ *denote the spectrum functor associated to the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$, and let $X : \mathrm{CAlg} \to \mathcal{S}$ be the functor represented by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, given by $\mathrm{Map}_{\mathrm{RingTop}_{\mathrm{Zar}}}(\mathrm{Spec}\, R, (\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$. Then X is a hypercomplete sheaf with respect to the flat topology on $\mathrm{Aff} = \mathrm{CAlg}^{op}$.*

Let $\mathrm{RingTop}^{\leq 0}_{\mathrm{Zar}}$ denote the full subcategory of $\mathrm{RingTop}_{\mathrm{Zar}}$ spanned by those pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is 0-localic. Theorem 5.15 is an immediate consequence of Proposition 5.12 together with the following result:

Proposition 5.16. *Let* Spec : $\mathrm{CAlg}^{op} \to \mathrm{RingTop}_{\mathrm{Zar}}$ *be the spectrum functor associated to the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$. Then:*

(1) *For every $\mathbb{E}_\infty$-ring R, the underlying ∞-topos of* $\mathrm{Spec}\, R$ *is 0-localic. Consequently,* Spec *can be viewed as a functor from* CAlg^{op} *into the full subcategory* $\mathrm{RingTop}^{\leq 0}_{\mathrm{Zar}} \subseteq \mathrm{RingTop}_{\mathrm{Zar}}$.

(2) *The functor* $\mathrm{Spec} : \mathrm{CAlg}^{op} \to \mathrm{RingTop}_{\mathrm{Zar}}$ *preserves finite coproducts.*

(3) *Let* $R^\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+})^{op} \to \mathrm{CAlg}$ *be a flat hypercovering. Then* $\mathrm{Spec}\, R^\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{RingTop}^{\leq 0}_{\mathrm{Zar}}$ *is a colimit diagram.*

To prove Proposition 5.16, we need a criterion for verifying the descent properties of a $\mathcal{C}\mathrm{at}_\infty$-valued functor.

Lemma 5.17. *Let* $\mathcal{C}$ *be an* ∞*-category and* $\chi : \mathcal{C}^{\triangleleft} \to \mathcal{C}\mathrm{at}_\infty$ *a functor, classified by a coCartesian fibration* $q : \mathcal{D} \to \mathcal{C}^{\triangleleft}$*. Then* χ *is a limit diagram if and only if the following conditions are satisfied:*

(a) *Let* v *denote the cone point of* $\mathcal{C}^{\triangleleft}$*, and for each object* $C \in \mathcal{C}$ *let* $e_C : \mathcal{D}_v \to \mathcal{D}_C$ *be the functor induced by the unique morphism* $f_C : v \to C$ *in* $\mathcal{C}^{\triangleleft}$*. Then the functors* e_C *are jointly conservative: that is, if* α *is a morphism in* $\mathcal{D}_v$ *such that each* $e_C(\alpha)$ *is an equivalence in* $\mathcal{D}_C$*, then* α *is an equivalence in* $\mathcal{D}_v$*.*

(b) *Let* $X \in \mathrm{Fun}_{\mathcal{C}^{\triangleleft}}(\mathcal{C}, \mathcal{D})$ *be a functor which carries each morphism in* $\mathcal{C}$ *to a* q*-coCartesian morphism in* $\mathcal{D}$*. Then* X *can be extended to a* q*-limit diagram* $\overline{X} \in \mathrm{Fun}_{\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \mathcal{D})$*. Moreover,* $\overline{X}$ *carries each* f_C *to a* q*-coCartesian morphism in* $\mathcal{D}$*.*

Moreover, if these conditions are satisfied, then a diagram $\overline{X} \in \mathrm{Fun}_{\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \mathcal{D})$ *is a* q*-limit diagram, provided that it carries each morphism in* $\mathcal{C}^{\triangleleft}$ *to a* q*-coCartesian morphism in* $\mathcal{D}$*.*

Proof. Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}_{\mathcal{C}^{\triangleleft}}(\mathcal{C}, \mathcal{D})$ spanned by those functors which carry each morphism in $\mathcal{C}$ to a q-coCartesian morphism in $\mathcal{D}$, let $\overline{\mathcal{E}}$ be the full subcategory of $\mathrm{Fun}_{\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \mathcal{D})$ spanned by those functors which carry each morphism in $\mathcal{C}^{\triangleleft}$ to a q-coCartesian morphism in $\mathcal{D}$, and let $\overline{\mathcal{E}}'$ be the full subcategory of $\mathrm{Fun}_{\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \mathcal{D})$ spanned by those functors $\overline{X}$ which are q-limit diagrams having the property that $\overline{X}|\,\mathcal{C}$ belongs to $\mathcal{E}$. Using Proposition T.3.3.3.1, we see that χ is a limit diagram if and only if the restriction functor $r : \overline{\mathcal{E}} \to \mathcal{E}$ is an equivalence of ∞-categories. Suppose first that this condition is satisfied. Assertion (a) is then obvious (it is equivalent to the requirement that the functor r is conservative). We will show that the last assertion is satisfied: that is, we have an inclusion $\overline{\mathcal{E}} \subseteq \overline{\mathcal{E}}'$. It follows that every $X \in \mathcal{E}$ can be extended to a q-limit diagram, so that (by Proposition T.4.3.2.15) the restriction functor $\overline{\mathcal{E}}' \to \mathcal{E}$ is a trivial Kan fibration. A two-out-of-three argument then shows that the inclusion $\overline{\mathcal{E}} \subseteq \overline{\mathcal{E}}'$ is an equivalence of ∞-categories, so that $\overline{\mathcal{E}} = \overline{\mathcal{E}}'$. This proves ($b$).

To prove that $\overline{\mathcal{E}} \subseteq \overline{\mathcal{E}}'$, consider an arbitrary diagram $\overline{X} \in \overline{\mathcal{E}}$ and let $X = \overline{X}|\,\mathcal{C}$. To show that $\overline{X}$ is a q-limit diagram, it suffices to show that for every object $D \in \mathcal{D}_v$ the canonical map $\phi : \{D\} \times_{\mathcal{D}} \mathcal{D}_{/\overline{X}} \to \{D\} \times_{\mathcal{D}} \mathcal{D}_{/X}$ is a homotopy equivalence of Kan complexes. Choose a diagram $\overline{Y} \in \overline{\mathcal{E}}$ with $\overline{Y}(v) = D$ (such a diagram exists and is essentially unique, by virtue of Proposition T.4.3.2.15), and let $Y = \overline{Y}|\,\mathcal{C}$. Then ϕ is equivalent to the restriction map

$$\mathrm{Map}_{\overline{\mathcal{E}}}(\overline{Y}, \overline{X}) \to \mathrm{Map}_{\mathcal{E}}(Y, X),$$

which is a homotopy equivalence by virtue of our assumption that the functor r is fully faithful.

Now suppose that conditions (a) and (b) are satisfied; we wish to prove that r is an equivalence of ∞-categories. Condition (b) guarantees that $\overline{\mathcal{E}}' \subseteq \overline{\mathcal{E}}$ and, by virtue of Proposition T.4.3.2.15, that $r|\overline{\mathcal{E}}'$ is a trivial Kan fibration. To complete the proof, it suffices to show that the reverse inclusion $\overline{\mathcal{E}} \subseteq \overline{\mathcal{E}}'$ holds. Fix $\overline{X} \in \overline{\mathcal{E}}$, let $X = \overline{X}|\,\mathcal{C}$, and let $\overline{X}' \in \overline{\mathcal{E}}'$ be a q-limit of the diagram X. We have a canonical map $\alpha : \overline{X} \to \overline{X}'$ which induces the identity map $\mathrm{id}_X : X \to X$ in $\mathcal{E}$. To complete the proof, it suffices to show that α is an equivalence; that is, that the map $\alpha_v : \overline{X}(v) \to \overline{X}'(v)$ is an equivalence in the ∞-category $\mathcal{D}_v$. This is an immediate consequence of assumption (a). □

We conclude this section with the proof of Proposition 5.16.

Proof of Proposition 5.16. Assertion (1) follows from the observation that for every $\mathbb{E}_\infty$-ring R, the underlying ∞-topos of $\mathrm{Spec}\, R$ can be identified with $\mathcal{S}\mathrm{hv}(\mathrm{Spec}^{\mathrm{Z}} R)$ (Remark 2.38). Let ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq 0}$ denote the ∞-category whose objects are 0-localic ∞-topoi, and whose morphisms are geometric morphisms $f_* : \mathcal{X} \to \mathcal{Y}$.

Let $\mathrm{RingTop}^{\leq 0}$ denote the full subcategory of RingTop spanned by those pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is 0-localic. Consider the functors

$$\mathrm{RingTop}_{\mathrm{Zar}}^{\leq 0} \xrightarrow{j} \mathrm{RingTop}^{\leq 0} \xrightarrow{q} {}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq 0}.$$

Here j is the inclusion of a subcategory.

In view of Proposition T.4.3.1.5, assertion (2) will follow from the following three claims:

$(2')$ The functor $q \circ j \circ \mathrm{Spec} : \mathrm{CAlg}^{op} \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq 0}$ preserves finite coproducts.

$(2'')$ The functor $j \circ \mathrm{Spec} : \mathrm{CAlg}^{op} \to \mathrm{RingTop}^{\leq 0}$ carries finite coproducts to q-coproducts.

$(2''')$ The functor $\mathrm{Spec} : \mathrm{CAlg}^{op} \to \mathrm{RingTop}_{\mathrm{Zar}}^{\leq 0}$ carries finite coproducts to j-coproducts.

To prove these claims, let $\{R_i\}_{1 \leq i \leq n}$ be a finite collection of $\mathbb{E}_\infty$-rings having product R. Let $X_i = \mathrm{Spec}^{\mathrm{Z}} R_i$ and let $X = \mathrm{Spec}^{\mathrm{Z}} R$, so that we have identifications $\mathrm{Spec}\, R_i = (\mathcal{S}\mathrm{hv}(X_i), \mathcal{O}_i)$ and $\mathrm{Spec}\, R = (\mathcal{S}\mathrm{hv}(X), \mathcal{O})$. For each index i, let $\phi_i : X_i \to X$ denote the map induced by the projection $R \to R_i$. Assertion $(2')$ follows from the observation that the maps ϕ_i induce a homeomorphism $\coprod X_i \to X$. In view of Proposition T.4.3.1.9, assertion $(2'')$ is equivalent to the requirement that the canonical map $\mathcal{O} \to \prod_i (\phi_i)_* \mathcal{O}_i$ is an equivalence of sheaves of $\mathbb{E}_\infty$-rings on X. Note that X has a basis of open sets of the form $U_f = \{\mathfrak{p} \subset \pi_0 R : f \notin \mathfrak{p}\}$, where $f = (f_1, \ldots, f_n)$ ranges over the elements of $\pi_0 R \simeq \pi_0 R_1 \times \cdots \times \pi_0 R_n$. Since this basis is stable under finite intersections, it suffices to observe that the canonical map

$$R[\frac{1}{f}] \simeq \mathcal{O}(U_f) \to (\prod_i (\phi_i)_* \mathcal{O}_i)(U_f) \simeq \prod_i \mathcal{O}_i(U_f \times_X X_i) \simeq \prod_i R_i[\frac{1}{f_i}]$$

is an equivalence of $\mathbb{E}_\infty$-rings.

Unwinding the definitions, we can formulate assertion $(2''')$ as follows: a morphism $\alpha : (X, \mathcal{O}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in $\mathrm{RingTop}^{\leq 0}$ belongs to $\mathrm{RingTop}_{\mathrm{Zar}}^{\leq 0}$ if and only if, for $1 \leq i \leq n$, the induced map $\alpha_i : (X_i, \mathcal{O}_i) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ belongs to $\mathrm{RingTop}_{\mathrm{Zar}}^{\leq 0}$. This follows immediately from Corollary 2.25, since a map of sheaves of local commutative rings on X is local if and only if it is local when restricted to each X_i.

We now prove (3). Let $R^\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{CAlg}$ be a flat hypercovering. Reasoning as above, we are reduced to proving the following three assertions:

The same reasoning reduces us to the following trio of assertions:

$(3')$ The composition $q \circ j \circ \mathrm{Spec} \circ R^\bullet$ is a colimit diagram in the ∞-category ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}^{\leq 0}$.

$(3'')$ The composition $j \circ \mathrm{Spec} \circ R^\bullet$ is a q-colimit diagram in the ∞-category $\mathrm{RingTop}^{\leq 0}$.

$(3''')$ The composition $\mathrm{Spec} \circ R^\bullet$ is a j-colimit diagram in the ∞-category $\mathrm{RingTop}_{\mathrm{Zar}}^{\leq 0}$.

In view of $(2')$ and Proposition 5.12, assertion $(3')$ will follow if we know that that $q \circ j \circ \mathrm{Spec} : \mathrm{CAlg} \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}^{\leq 0}$ is a hypercomplete sheaf with respect to the flat topology. Since the ∞-category ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}^{\leq 0}$ is equivalent to the nerve of an ordinary category, we need only show that $q \circ j \circ \mathrm{Spec}$ is a sheaf with respect to the flat topology, which follows from Proposition 5.9 (it is here that we use in an essential way the fact that we consider only 0-localic ∞-topoi).

We now prove $(3'')$. Let $X = \mathrm{Spec}^{\mathrm{Z}} R^{-1}$, so that $\mathrm{Spec}\, R^{-1}$ can be identified with a pair $(\mathcal{S}\mathrm{hv}(X), \mathcal{O})$. For every nonnegative integer n let $X_n = \mathrm{Spec}^{\mathrm{Z}} R^n$, so we have an equivalence $\mathrm{Spec}\, R^n = (\mathcal{S}\mathrm{hv}(X_n), \mathcal{O}_n)$; let $\mathcal{F}^n$ denote the pushforward of $\mathcal{O}_n$ along the evident map $X_n \to X$. The construction $[n] \mapsto \mathcal{F}^n$ determines a cosemisimplicial object in the ∞-category of sheaves of $\mathbb{E}_\infty$-rings on X. In view of Proposition T.4.3.1.9, condition $(3'')$ is equivalent to the requirement that the canonical map $\alpha : \mathcal{O} \to \varprojlim \mathcal{F}^n$ is an equivalence. We again note that X has a basis of open sets of the form $U_f = \{\mathfrak{p} \subset \pi_0 R^{-1} : f \notin \mathfrak{p}\}$. Since this collection is stable under finite intersection, to prove that α is an equivalence it suffices to show that α induces an equivalence of $\mathbb{E}_\infty$-rings $\mathcal{O}(U_f) \to \varprojlim \mathcal{F}^n(U_f)$, for each $f \in \pi_0 R^{-1}$. Replacing R^{-1} by $R^{-1}[\frac{1}{f}]$, we can reduce to the case where $U_f = X$. In this case, we need to show that the map

$$R^{-1} \simeq \mathcal{O}(X) \to \varprojlim \mathcal{F}^n(X) \simeq \varprojlim \mathcal{O}_n(X_n) \simeq \varprojlim R^n$$

is an equivalence of $\mathbb{E}_\infty$-rings, which follows from Theorem 5.14.

It remains to prove (3′′′). Unwinding the definitions, we must show that if $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is an object of $\mathrm{RingTop}^{\leq 0}$, then a map $\alpha : \operatorname{Spec} R^{-1} \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ in $\mathrm{RingTop}^{\leq 0}$ belongs to $\mathrm{RingTop}^{\leq 0}_{\mathrm{Zar}}$ if and only if the induced map $\beta : (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \to \operatorname{Spec} R^0$ belongs to $\mathrm{RingTop}^{\leq 0}_{\mathrm{Zar}}$. Let $f^* : \mathcal{Y} \to \mathcal{S}\mathrm{hv}(X)$ be the underlying geometric morphism and let $\mathcal{O}' = f^* \mathcal{O}_\mathcal{Y}$. In view of Corollary 2.25, we are reduced to proving the following: a map between sheaves of local rings $\pi_0 \mathcal{O}' \to \pi_0 \mathcal{O}$ on X is local if and only if the composite map $\phi_0^* \pi_0 \mathcal{O}' \to \phi_0^* \pi_0 \mathcal{O} \to \pi_0 \mathcal{O}_0$ is a local map (between sheaves of local rings on X_0). This follows immediately from the observation that the map $\phi_0 : X_0 \to X$ is surjective (since the underlying map of commutative rings $\pi_0 R^{-1} \to \pi_0 R^0$ is assumed to be faithfully flat). □

6 Flat Descent for Modules

Let $f : A \to B$ be a faithfully flat map of commutative rings. A classical theorem of Grothendieck asserts that the category of A-modules is equivalent to the category $\mathcal{C}$ whose objects are pairs (M, η), where M is a B-module and η is a "descent datum" for M: that is, an automorphism of $B \otimes_A M$ which is compatible with the evident involution on $B \otimes_A B$ and satisfies a suitable cocycle condition. More abstractly, Grothendieck's theorem asserts that the category of pairs (A, M), where A is a commutative ring and M a (discrete) A-module, is a *stack* with respect to the flat topology on the category of commutative rings. Our goal in this section is to prove the following ∞-categorical analogue of Grothendieck's result:

Theorem 6.1. *The construction $A \mapsto \mathrm{Mod}_A(\mathrm{Sp})$ determines a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which is a hypercomplete sheaf with respect to the flat topology on CAlg^{op} (see §5).*

For later use, it will be convenient to prove a somewhat more general form of this result. Let us restrict our attention to the ∞-category CAlg_R of $\mathbb{E}_\infty$-algebras over R, where R is some fixed $\mathbb{E}_\infty$-ring. Rather than than assigning to each object $A \in \mathrm{CAlg}_R$ the ∞-category $\mathrm{Mod}_A = \mathrm{Mod}_A(\mathrm{Sp})$ of A-module spectra, we can assign to A the ∞-category $\mathrm{Mod}_A(\mathcal{C})$ where $\mathcal{C}$ is an arbitrary R-linear ∞-category (see Definition 6.2). Under some mild assumptions on $\mathcal{C}$, we will show that the construction $A \mapsto \mathrm{Mod}_A(\mathcal{C})$ satisfies descent with respect to the flat topology (Theorem 6.27). We begin by introducing some definitions.

Definition 6.2. Let R be an $\mathbb{E}_2$-ring, so that the ∞-category LMod_R of left A-modules is equipped with a monoidal structure. An *R-linear ∞-category* is a presentable ∞-category $\mathcal{C}$ which is tensored over the monoidal ∞-category LMod_R of left R-modules, such that the tensor product $\otimes : \mathrm{LMod}_R \otimes \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable.

Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category of presentable ∞-categories and colimit-preserving functors, endowed with the symmetric monoidal structure described in §A.6.3.1. Then LMod_R can be identified with an associative algebra object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. We let $\mathrm{LinCat}_R = \mathrm{LMod}_{\mathrm{LMod}_R}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. We will refer to LinCat_R as the *∞-category of R-linear ∞-categories.*

Example 6.3. Let S denote the sphere spectrum, regarded as an initial object of the ∞-category $\mathrm{Alg}^{(2)}$ of $\mathbb{E}_2$-rings. Then the forgetful functor $\mathrm{LMod}_S = \mathrm{LMod}_S(\mathrm{Sp}) \to \mathrm{Sp}$ is an equivalence of ∞-categories. It follows that LinCat_S can be identified with the ∞-category $\mathrm{LMod}_{\mathrm{Sp}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. Using Example A.6.3.1.22, we can identify LinCat_S with the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by the presentable stable ∞-categories.

Remark 6.4. If R is a discrete commutative ring, then our theory of R-linear ∞-categories is closely related to the theory of *differential graded categories* over R.

Remark 6.5. Let $f : R' \to R$ be a map of $\mathbb{E}_2$-rings. Then f induces a monoidal functor $\mathrm{LMod}_{R'} \to \mathrm{LMod}_R$. We may therefore view any R-linear ∞-category as an R'-linear ∞-category. In particular, every R-linear ∞-category $\mathcal{C}$ can be regarded as an S-linear ∞-category, and is therefore stable (Example 6.3).

Remark 6.6. Let R be an $\mathbb{E}_2$-ring, and let $\mathcal{C}$ and $\mathcal{C}'$ be R-linear ∞-categories. We will refer to the morphisms from $\mathcal{C}$ to $\mathcal{C}'$ in LinCat_R as *R-linear functors* from $\mathcal{C}$ to $\mathcal{C}'$. Every R-linear functor from $\mathcal{C}$ to $\mathcal{C}'$ determines a colimit-preserving functor between the underlying (presentable) ∞-categories of $\mathcal{C}$ and $\mathcal{C}'$, which therefore admits a right adjoint G (Corollary T.5.5.2.9). For every R-module M and every object $C' \in \mathcal{C}'$, the counit map $(F \circ G)(C') \to C'$ induces a map

$$F(M \otimes G(C')) \to M \otimes (F \circ G)(C') \to M \otimes C',$$

which is adjoint to a morphism $\theta_M : M \otimes G(C') \to G(M \otimes C')$ in $\mathcal{C}$. The left R-modules M for which θ_M is an equivalence span a stable subcategory $\mathcal{X} \subseteq \mathrm{LMod}_R$ which contains R. If G commutes with filtered colimits, then $\mathcal{X}$ is closed under filtered colimits and therefore coincides with LMod_R: that is, θ_M is an equivalence for every left R-module M (and every object $C' \in \mathcal{C}$). It then follows from Remark A.7.3.2.9 that we can regard G as an R-linear functor from $\mathcal{C}'$ to $\mathcal{C}$.

In this paper, we will confine our attention to the study of linear ∞-categories over $\mathbb{E}_\infty$-rings. If $R \in \mathrm{CAlg}$ is an $\mathbb{E}_\infty$-ring, we will generally abuse notation by identifying R with its image in the ∞-category $\mathrm{Alg}^{(2)}$ of $\mathbb{E}_2$-rings, and we let LinCat_R denote the ∞-category of linear ∞-categories over the underlying $\mathbb{E}_2$-ring of R.

Remark 6.7. Let CAlg denote the ∞-category of $\mathbb{E}_\infty$-rings. In §5, we introduced the *flat topology* on the ∞-category CAlg^{op}. If A is an $\mathbb{E}_\infty$-ring, then a sieve on A is covering with respect to the flat topology if and only if it contains a finite collection of maps $\{\phi_\alpha : A \to A_\alpha\}$ which induces a faithfully flat morphism $A \to \prod_\alpha A_\alpha$.

For every $\mathbb{E}_\infty$-ring R, the flat topology on CAlg^{op} determines a Grothendieck topology on the ∞-category CAlg_R^{op} of $\mathbb{E}_\infty$-algebras over R. If R is connective, we also obtain a Grothendieck topology on the ∞-category $(\mathrm{CAlg}_R^{\mathrm{cn}})^{op}$ of connective $\mathbb{E}_\infty$-algebras over R. We will refer to both of these topologies as the *flat topology*.

Definition 6.8. Fix an $\mathbb{E}_\infty$-ring R and an R-linear ∞-category $\mathcal{C}$. We let $\mathrm{Mod}(\mathcal{C})$ denote the fiber product $\mathrm{LMod}(\mathcal{C}) \times_{\mathrm{Alg}(\mathrm{Mod}_A)} \mathrm{CAlg}(\mathrm{Mod}_R)$ whose objects are pairs (A', M), where $A' \in \mathrm{CAlg}(\mathrm{Mod}_A) \simeq \mathrm{CAlg}_A$ is an $\mathbb{E}_\infty$-algebra over A and M is a left A'-module object of $\mathcal{C}$. We will denote the fiber of $\mathrm{Mod}(\mathcal{C})$ over an object $A \in \mathrm{CAlg}_R$ by $\mathrm{Mod}_A(\mathcal{C})$.

The coCartesian fibration $q : \mathrm{Mod}(\mathcal{C}) \to \mathrm{CAlg}(\mathrm{Mod}_A)$ is classified by a functor $\chi : \mathrm{CAlg}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We will say that $\mathcal{C}$ *satisfies flat descent* if the functor χ is a sheaf with respect to the flat topology on CAlg_A^{op}. We will say that $\mathcal{C}$ *satisfies flat hyperdescent* if $\mathcal{C}$ is a hypercomplete sheaf with respect to the flat topology.

We now study some examples of linear ∞-categories which satisfy flat descent.

Definition 6.9. Let $\mathcal{C}$ be a stable ∞-category. We will say that a t-structure on $\mathcal{C}$ is *excellent* if the following conditions are satisfied:

(1) The ∞-category $\mathcal{C}$ is presentable.

(2) The t-structure on $\mathcal{C}$ is compatible with filtered colimits: that is, the full subcategory $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is closed under filtered colimits (in particular, the t-structure on $\mathcal{C}$ is accessible: see Proposition A.1.4.5.13).

(3) The t-structure on $\mathcal{C}$ is both right and left complete.

Example 6.10. If A is a connective $\mathbb{E}_\infty$-ring, then the usual t-structure on Mod_A is excellent. In particular, the usual t-structure on the ∞-category Sp of spectra is excellent.

Remark 6.11. Let $\{\mathcal{C}(i)\}$ be a finite collection of presentable stable ∞-categories, having product $\mathcal{C} = \prod_i \mathcal{C}(i)$. Giving a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on $\mathcal{C}$ is equivalent to giving a t-structure $(\mathcal{C}(i)_{\geq 0}, \mathcal{C}(i)_{\leq 0})$ on each $\mathcal{C}(i)$. Moreover, the t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is excellent if and only if each $(\mathcal{C}(i)_{\geq 0}, \mathcal{C}(i)_{\leq 0})$ is excellent.

Theorem 6.1 is an immediate consequence of Example 6.10 and the following result, which we will prove at the end of this section:

Theorem 6.12. *Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be an A-linear ∞-category. If $\mathcal{C}$ admits an excellent t-structure, then $\mathcal{C}$ satisfies flat hyperdescent.*

Corollary 6.13. *Let A be an $\mathbb{E}_\infty$-ring. Then the A-linear ∞-category Mod_A satisfies flat hyperdescent.*

Proof. Without loss of generality, we may assume that A is the sphere spectrum. In particular, A is connective; the desired result now follows from Example 6.10 and Theorem 6.12. □

Before stating the next consequence of Theorem 6.12, let us introduce a bit of terminology. Let $R^\bullet$ be an augmented cosemisimplicial object of CAlg. We will say that $R^\bullet$ is a *flat hypercovering* if it determines an S-hypercovering in the ∞-category CAlg^{op} in the sense of Definition 2.5, where S is the collection of faithfully flat morphisms in CAlg. In other words, $R^\bullet$ is a flat hypercovering if each of the maps $L_n(R^\bullet) \to R^n$ is faithfully flat, where $L_n(R^\bullet)$ denotes the nth latching object of $R^\bullet$. We will say that an augmented cosemisimplicial commutative ring $R^\bullet$ is a *flat hypercovering* if it determines a flat hypercovering when regarded as an augmented cosemisimplicial object of CAlg.

Corollary 6.14. *Let $R^\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{CAlg}$ be a flat hypercovering of an $\mathbb{E}_\infty$-ring $R = R^{-1}$, let M be an R-module spectrum, and let $M^\bullet$ be the cosemisimplicial $(R^\bullet|\,\mathrm{N}(\mathbf{\Delta}_s))$-module spectrum given informally by the formula $M^n = M \otimes_R R^n$. Then the canonical map $M \to \varprojlim M^\bullet$ is an equivalence.*

Proof. Combine Proposition 5.12, Lemma 5.17, and Corollary 6.13. □

As a first step towards a proof of Theorem 6.27, we observe that it suffices to restrict our attention to the study of modules over connective $\mathbb{E}_\infty$-rings.

Lemma 6.15. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category, let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}$, and suppose we are given a pushout diagram of commutative algebra objects of $\mathcal{C}$:*

$$\begin{array}{ccc} A & \longleftarrow & B \\ \uparrow & & \uparrow \\ A' & \longleftarrow & B' \end{array}$$

Then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{LMod}_A(\mathcal{M}) & \longrightarrow & \mathrm{LMod}_B(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{LMod}_{A'}(\mathcal{M}) & \longrightarrow & \mathrm{LMod}_{B'}(\mathcal{M}) \end{array}$$

is right adjointable.

Proof. This follows immediately from Proposition A.4.3.7.14. □

Lemma 6.16. *Let R be an $\mathbb{E}_2$-ring and let $\mathcal{C}$ be an R-linear ∞-category. Then the construction $A \mapsto \mathrm{LMod}_A(\mathcal{C})$ commutes with finite products (when regarded as a functor $\mathrm{Alg}_R^{(1)} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$).*

Proof. Let $\{A_i\}_{1 \leq i \leq n}$ be a finite collection of $\mathbb{E}_1$-algebras over R and let $A = \prod_{1 \leq i \leq n} A_i$. We wish to show that the canonical functor

$$\theta : \mathrm{LMod}_A(\mathcal{C}) \to \prod_{1 \leq i \leq n} \mathrm{LMod}_{A_i}(\mathcal{C})$$

is an equivalence of ∞-categories. In view of Lemma 5.17, it will suffice to verify the following assertions:

(a) The functor θ is conservative. That is, if $\alpha : M \to N$ is a morphism in $\mathrm{LMod}_A(\mathcal{C})$ is such that each of the induced maps $\alpha_i : A_i \otimes_A M \to A_i \otimes_A M$ is an equivalence, then α is an equivalence. It suffices to show that the image of α is an equivalence in the ∞-category $\mathcal{C}$. This is clear, since α is equivalent to the product of the morphisms α_i in the ∞-category $\mathcal{C}$.

(b) Suppose we are given objects $M_i \in \mathrm{LMod}_{A_i}(\mathcal{C})$, and let $M \simeq \prod_{1 \leq i \leq n} M_i$ (regarded as an A-module). Then the canonical map $\phi : A_i \otimes_A M \to M_i$ is an equivalence for $1 \leq i \leq n$. To prove this, we see that the domain of ϕ is given by the product $\prod_{1 \leq j \leq n}(A_i \otimes_A A_j) \otimes_{A_j} M_j$. To prove that ϕ is an equivalence, it suffices to show that $A_i \otimes_A A_j \simeq 0$ for $i \neq j$, and that the canonical map $A_i \to A_i \otimes_A A_i$ is an equivalence. Since each A_j is flat as a left A-module, we have

$$\pi_*(A_i \otimes_A A_j) \simeq (\pi_* A_i) \otimes_{\pi_* A} \pi_* A_j,$$

so the desired result follows from a simple algebraic calculation.

$\square$

Lemma 6.17. *Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be an A-linear ∞-category, and let $\chi : \mathrm{CAlg}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be as in Definition 6.8. Then:*

(1) *The A-linear ∞-category $\mathcal{C}$ has flat descent if and only if the restriction $\chi | \mathrm{CAlg}_A^{\mathrm{cn}}$ is a sheaf with respect to the flat topology.*

(2) *The A-linear ∞-category $\mathcal{C}$ has flat hyperdescent if and only if the restriction $\chi | \mathrm{CAlg}_A^{\mathrm{cn}}$ is a hypercomplete sheaf with respect to the flat topology.*

Proof. We will prove (1); the proof of (2) is similar. The "only if" direction is obvious. Conversely, suppose that $\chi | \mathrm{CAlg}_A^{\mathrm{cn}}$ is a sheaf with respect to the flat topology on $\mathrm{CAlg}_A^{\mathrm{cn}}$. We wish to show that χ is a sheaf with respect to the flat topology. Using Proposition 5.7 and Lemma 6.16, we are reduced to proving the following:

$(*)$ Let $f : B \to B^0$ be a faithfully flat morphism of A-algebras, and let $B^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}_A$ be the Čech nerve of f (regarded as a morphism in $(\mathrm{CAlg}_A)^{op}$). Then $\chi(B^\bullet)$ is a limit diagram in $\widehat{\mathcal{C}\mathrm{at}}_\infty$.

According to Lemma 5.17, it will suffice to verify the following:

(a) The functor $\phi : \mathrm{Mod}_B(\mathcal{C}) \to \mathrm{Mod}_{B^0}(\mathcal{C})$ is conservative. To prove this, we let $\tau_{\geq 0} B$ and $\tau_{\geq 0} B^0$ be the connective covers of B and B^0, respectively. Since f is flat, the canonical map $B \otimes_{\tau_{\geq 0} B} \tau_{\geq 0} B^0 \to B^0$ is an equivalence. It follows from Lemma 6.15 that ϕ fits into a homotopy commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Mod}_B(\mathcal{C}) & \xrightarrow{\phi} & \mathrm{Mod}_B(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{\tau_{\geq 0} B}(\mathcal{C}) & \xrightarrow{\phi_0} & \mathrm{Mod}_{\tau_{\geq 0} B^0}(\mathcal{C}). \end{array}$$

Here the vertical maps are the evident forgetful functors (and therefore conservative). Consequently, to show that ϕ is conservative it suffices to show that ϕ_0 is conservative, which follows from our assumption that $\chi | \mathrm{CAlg}_A^{\mathrm{cn}}$ is a sheaf with respect to the flat topology.

(b) Let $M^\bullet$ be a cosimplicial object of $\mathcal{C}$ which is a module over the underlying cosimplicial algebra of $B^\bullet$ such that each of the maps $B^p \otimes_{B^q} M^q \to M^p$ is an equivalence. Let $M = \varprojlim M^\bullet$, regarded as a B-module object of $\mathcal{C}$. Then we must show that the canonical map $B^p \otimes_B M \to M^p$ is an equivalence for each $p \geq 0$. To prove this, we note that since f is flat, the map $\tau_{\geq 0} B^p \otimes_{\tau_{\geq 0} B^q} B^q \to B^p$ is an equivalence for every morphism $[p] \to [q]$ in $\mathbf{\Delta}_+$. Let us regard $M^\bullet$ as a cosimplicial module over the underlying cosimplicial algebra of $\tau_{\geq 0} B^\bullet$. Using Lemma 6.15, we conclude that each of the maps $\tau_{\geq 0} B^p \otimes_{\tau_{\geq 0} B^q} M^q \to M^p$ is an equivalence. Using our assumption that $\chi | \operatorname{CAlg}^{\mathrm{cn}}_A$ is a sheaf with respect to the flat topology and Lemma 5.17, we conclude that each of the maps $\tau_{\geq 0} B^p \otimes_{\tau_{\geq 0} B} M \to M^p$ is an equivalence for $p \geq 0$. The desired result now follows from Lemma 6.15.

□

Using the Barr-Beck theorem, we can obtain a very concrete criterion for flat descent.

Proposition 6.18. *Let A be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be an A-linear ∞-category. Then $\mathcal{C}$ satisfies flat descent if and only if, for every faithfully flat map of A-algebras $B \to B^0$, the induced functor $F : \operatorname{Mod}_B(\mathcal{C}) \to \operatorname{Mod}_{B^0}(\mathcal{C})$ has the following property:*

$(*)$ *The functor F is conservative, and preserves totalizations of F-split cosimplicial objects.*

Moreover, if A is connective, then it is sufficient to verify this condition in the case where B and B^0 are connective.

Proof. Using Proposition 5.7 and Lemma 6.16, we see that $\mathcal{C}$ satisfies descent if and only if, for every faithfully flat morphism of A-algebras $f : B \to B^0$, the following condition is satisfied:

$(*')$ Let $B^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \operatorname{CAlg}_A$ be the Čech nerve of f (regarded as a morphism in $\operatorname{CAlg}_A^{op}$, and let $\mathcal{C}^\bullet$ be the augmented cosimplicial ∞-category given by the formula $\mathcal{C}^\bullet = \operatorname{Mod}_{B^\bullet}(\mathcal{C})$. Then $\mathcal{C}^\bullet$ is a limit diagram in $\widehat{\mathcal{C}\mathrm{at}}_\infty$.

Moreover, if A is connective, Lemma 6.17 shows that it suffices to verify $(*')$ in the case where $B \to B^0$ is a faithfully flat map of connective A-algebras.

For every morphism $[m] \to [n]$ in $\mathbf{\Delta}_+$, the induced diagram

$$\begin{array}{ccc} B^m & \longrightarrow & B^{m+1} \\ \downarrow & & \downarrow \\ B^n & \longrightarrow & B^{n+1} \end{array}$$

is a pushout square of $\mathbb{E}_\infty$-rings. It follows from Lemma 6.15 that the diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C}^m & \longrightarrow & \mathcal{C}^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \longrightarrow & \mathcal{C}^{n+1} \end{array}$$

is right adjointable (that is, after passing to opposite ∞-categories, it gives a commutative diagram which is left adjointable in the sense of §A.6.2.4). The equivalence of $(*)$ and $(*')$ now follows from Theorem A.6.2.4.2 and Corollary A.6.2.4.3. □

The proof of Theorem 6.27 will require some permanence properties of the class of excellent t-structures. We begin with a general observation.

Remark 6.19. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$, and suppose that the inclusion $\mathcal{C}_0 \to \mathcal{C}$ admits a left adjoint L. Let T be a monad on $\mathcal{C}$, and assume that T carries L-equivalences to L-equivalences. Let $\operatorname{Fun}_0(\mathcal{C}, \mathcal{C})$ be the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{C})$ spanned by those functors U such that U carries L-equivalences to L-equivalences. Then $\operatorname{Fun}_0(\mathcal{C}, \mathcal{C})$ is stable under composition, and therefore inherits a monoidal structure from the monoidal structure on $\operatorname{Fun}(\mathcal{C}, \mathcal{C})$ (see §A.2.2.1). The left action of $\operatorname{Fun}_0(\mathcal{C}, \mathcal{C})$ on $\mathcal{C}$ is encoded by a coCartesian fibration of ∞-operads $\mathcal{E}^\otimes \to \mathcal{LM}^\otimes$ (see §A.4.2.1). Applying Proposition A.2.2.1.9 to the full subcategories

$$\operatorname{Fun}_0(\mathcal{C}, \mathcal{C}) \subseteq \operatorname{Fun}_0(\mathcal{C}, \mathcal{C}) \qquad \mathcal{C}_0 \subseteq \mathcal{C},$$

we obtain a full subcategory $\mathcal{E}_0^{\otimes} \subseteq \mathcal{E}^{\otimes}$ for which the restriction $\mathcal{E}_0^{\otimes} \to \mathcal{LM}^{\otimes}$ is a coCartesian fibration of ∞-operads, which exhibits $\mathcal{C}_0$ as left tensored over $\mathrm{Fun}_0(\mathcal{C}, \mathcal{C})$. This action is classified by a monoidal functor $\mathrm{Fun}_0(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{C}_0)$, so that T determines a monad on $\mathcal{C}_0$ which we will denote by T_0. Unwinding the definitions, we can identify $\mathrm{Mod}_{T_0}(\mathcal{C}_0)$ with the full subcategory $\mathcal{C}_0 \times_{\mathcal{C}} \mathrm{Mod}_T(\mathcal{C}) \subseteq \mathrm{Mod}_T(\mathcal{C})$. It follows from Proposition A.2.2.1.9 that the inclusion $\mathcal{E}_0^{\otimes} \subseteq \mathcal{E}^{\otimes}$ admits an $\mathcal{LM}$-monoidal left adjoint. It follows that the inclusion $\mathrm{Mod}_T(\mathcal{C}_0) \subseteq \mathrm{Mod}_{T_0}(\mathcal{C})$ admits a left adjoint L', and that the diagram

$$\begin{array}{ccc} \mathrm{Mod}_T(\mathcal{C}) & \xrightarrow{L'} & \mathrm{Mod}_T(\mathcal{C}_0) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{L} & \mathcal{C}_0 \end{array}$$

commutes up to canonical homotopy.

Proposition 6.20. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure and let T be a monad on $\mathcal{C}$. Assume that the underlying functor $\mathcal{C} \to \mathcal{C}$ is exact and carries $\mathcal{C}_{\geq 0}$ into $\mathcal{C}_{\geq 0}$. Then:*

(1) *The ∞-category $\mathrm{LMod}_T(\mathcal{C})$ is stable and the forgetful functor $\theta : \mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$ is exact.*

(2) *Let $\mathrm{LMod}_T(\mathcal{C})_{\geq 0}, \mathrm{LMod}_T(\mathcal{C})_{\leq 0} \subseteq \mathrm{LMod}_T(\mathcal{C})$ be the inverse images of the full subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ under the functor θ. Then $(\mathrm{LMod}_T(\mathcal{C})_{\geq 0}, \mathrm{LMod}_T(\mathcal{C})_{\leq 0})$ determines a t-structure on $\mathrm{LMod}_T(\mathcal{C})$.*

(3) *If $\mathcal{C}$ is left complete, then $\mathrm{LMod}_T(\mathcal{C})$ is also left complete.*

(4) *Assume that $\mathcal{C}$ is presentable and that the functor T preserves small colimits. Then $\mathrm{LMod}_T(\mathcal{C})$ is presentable and the forgetful functor θ preserves small colimits. Moreover, if the t-structure on $\mathcal{C}$ is accessible, then the t-structure on $\mathrm{LMod}_T(\mathcal{C})$ is accessible.*

(5) *Assume that $\mathcal{C}$ is presentable, that the t-structure on $\mathcal{C}$ is excellent, and that the functor T preserves small colimits. Then the t-structure on $\mathrm{LMod}_T(\mathcal{C})$ is also excellent.*

(6) *Assume that T carries $\mathcal{C}_{\leq 0}$ into $\mathcal{C}_{\leq 0}$. If $\mathcal{C}$ is right complete, then $\mathrm{LMod}_T(\mathcal{C})$ is also right complete.*

Proof. Note that T can be regarded as a monad on the ∞-category $\mathcal{C}_{\geq n}$ for every integer n. According to Example A.7.3.2.10, each of the inclusions $\mathrm{LMod}_T(\mathcal{C}_{\geq n}) \hookrightarrow \mathrm{LMod}_T(\mathcal{C})$ admits a right adjoint $\tau^T_{\geq n}$ such that $\theta \circ \tau^T_{\geq n} \simeq \tau_{\geq n} \circ \theta$. We next claim the following:

($*$) If $f : X \to Y$ is a morphism in $\mathcal{C}$ which induces an equivalence $\tau_{\leq n} X \to \tau_{\leq n} Y$, then the induced map $TX \to TY$ has the same property.

To prove ($*$), we let $Z = \tau_{\leq n} Y$. We have a commutative diagram

$$\begin{array}{ccc} \tau_{\leq n} TX & \xrightarrow{\alpha} & \tau_{\leq n} TY \\ & {\scriptstyle\gamma} \searrow \quad \swarrow {\scriptstyle\beta} & \\ & \tau_{\leq n} TZ. & \end{array}$$

To prove that α is an equivalence, it suffices to show that β and γ are equivalences. In other words, it will suffice to prove ($*$) in the special case where $f : X \to Y$ exhibits Y as an n-truncation of X. To show that the map $\tau_{\leq n} T(f)$ is an equivalence, it suffices to show that $\mathrm{fib}\, T(f) \simeq T(\mathrm{fib}(f))$ belongs to $\mathcal{C}_{\geq n+1}$; this follows from the right t-exactness of T, since $\mathrm{fib}(f) \in \mathcal{C}_{\geq n+1}$.

Combining assertion ($*$) with Proposition 6.19, we deduce that T determines a monad T^n on the ∞-category $\mathcal{C}_{\leq n}$ (characterized informally by the formula $T^n(\tau_{\leq n} X) \simeq \tau_{\leq n} T(X)$) such that $\mathrm{LMod}_{T^n}(\mathcal{C}_{\leq n})$ can be identified with $\mathrm{LMod}_T(\mathcal{C})_{\leq n}$. Moreover, the inclusion $\mathrm{LMod}_T(\mathcal{C})_{\leq n} \hookrightarrow \mathrm{LMod}_T(\mathcal{C})$ admits a left adjoint $\tau^T_{\leq n}$ such that $\theta \circ \tau^T_{\leq n} \simeq \tau_{\leq n} \circ \theta$.

Assertion (1) is a special case of Proposition A.4.2.3.4. We now prove (2). We first show that for every object $X \in \mathrm{LMod}_T(\mathcal{C})_{\geq 1}$ and every object $Y \in \mathrm{LMod}_T(\mathcal{C})_{\leq 0}$, the mapping space $\mathrm{Map}_{\mathrm{LMod}_T(\mathcal{C})}(X, Y)$ is contractible. This is clear, since $\mathrm{Map}_{\mathrm{LMod}_T(\mathcal{C})}(X, Y) \simeq \mathrm{Map}_{\mathrm{LMod}_T(\mathcal{C})}(\tau^T_{\leq 0} X, Y)$ and $\tau^T_{\leq 0} X \simeq 0$ (since $\tau_{\leq 0}\theta(X) \simeq 0$).

To complete the proof of (2), it suffices to show that for every object $X \in \mathrm{LMod}_T(\mathcal{C})$, there exists a fiber sequence

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

with $X' \in \mathrm{LMod}_T(\mathcal{C})_{\geq 1}$ and $X'' \in \mathrm{LMod}_T(\mathcal{C})_{\leq 0}$. Set $X' = \tau^T_{\geq 1} X$ and $X'' = \tau^T_{\leq 0} X$, with f and g defined in the obvious way. The preceding argument shows that $g \circ f$ is nullhomotopic, so that we get a commutative diagram σ :

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & X'' \end{array}$$

in $\mathrm{LMod}_T(\mathcal{C})$. Note that $\theta(\sigma)$ is a pullback square in $\mathcal{C}$. Since θ is an exact, conservative functor, we conclude that σ is a pullback square in $\mathrm{LMod}_T(\mathcal{C})$, thereby completing the proof of (2).

We next prove (3). Assume that $\mathcal{C}$ is left complete. Let Y be an object of $\varprojlim_n \mathrm{LMod}_T(\mathcal{C})_{\leq n}$, corresponding to a compatible sequence of objects $Y_n \in \mathrm{LMod}_T(\mathcal{C})_{\leq n}$. Since $\mathcal{C}$ is left complete, the diagram

$$\cdots \to \theta(Y_2) \to \theta(Y_1) \to \theta(Y_0) \to \cdots$$

admits a limit in $\mathcal{C}$. Using Proposition A.4.2.3.1, we conclude that the diagram

$$\cdots \to Y_2 \to Y_1 \to Y_0 \to \cdots$$

admits a limit in $\mathrm{LMod}_T(\mathcal{C})$; let us denote this limit by $G(Y)$. The construction $Y \mapsto G(Y)$ is a right adjoint to the evident functor $F : \mathrm{LMod}_T(\mathcal{C}) \to \varprojlim \mathrm{LMod}_T(\mathcal{C})_{\leq n}$. We claim that for $Y \in \varprojlim_n \mathrm{LMod}_T(\mathcal{C})_{\leq n}$ is as above, the counit map $v : (F \circ G)(Y) \to Y$ is an equivalence. To prove this, it suffices to show that v induces an equivalence $v_n : \tau^T_{\leq n}(F \circ G)(Y) \to Y_n$ for every integer n. This follows from from the fact that θ is conservative, since $\theta(v_n)$ can be identified with the map

$$\tau_{\leq n} \varprojlim_m \theta(Y_m) \to \theta(Y_n)$$

which is an equivalence by virtue of our assumption that $\mathcal{C}$ is left complete. This proves that G is fully faithful; to show that G is an equivalence, it will suffice to show that the functor F is conservative. This is clear: if $f : X \to Y$ is a morphism in $\mathrm{LMod}_T(\mathcal{C})$ such that $F(f)$ is an equivalence, then f induces an equivalence $\tau_{\leq n}\theta(X) \to \tau_{\leq n}\theta(Y)$ for every integer n. Since $\mathcal{C}$ is left complete, this implies that $\theta(f)$ is an equivalence, so that f is an equivalence as desired.

We now prove (4). Assume that $\mathcal{C}$ is presentable and that T preserves small colimits. It follows from Proposition A.4.2.3.4 that $\mathrm{LMod}_T(\mathcal{C})$ is presentable and that θ preserves small colimits. If the t-structure on $\mathcal{C}$ is accessible, then Proposition T.5.5.3.12 implies that $\mathrm{LMod}_T(\mathcal{C})_{\geq 0} \simeq \mathrm{LMod}_T(\mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{\geq 0}$ is presentable, so that the t-structure on $\mathrm{LMod}_T(\mathcal{C})$ is accessible. Suppose, in addition, that the t-structure on $\mathcal{C}$ is excellent. It follows from (3) that $\mathrm{LMod}_T(\mathcal{C})$ is left complete. Since θ preserves small filtered colimits and $\mathcal{C}_{\leq 0}$ is closed under small filtered colimits, the full subcategory $\mathrm{LMod}_T(\mathcal{C})_{\leq 0} = \theta^{-1}\,\mathcal{C}_{\leq 0} \subseteq \mathrm{LMod}_T(\mathcal{C})$ is closed under small filtered colimits. In particular, $\mathrm{LMod}_T(\mathcal{C})_{\leq 0}$ is closed under countable coproducts. Since $\mathcal{C}$ is right complete, the intersection $\bigcap_n \mathcal{C}_{\leq -n}$ consists only of zero objects of $\mathcal{C}$. Because θ is conservative, we deduce that $\theta^{-1} \bigcap_n \mathcal{C}_{\leq -n} = \bigcap_n \mathrm{LMod}_T(\mathcal{C})_{\leq -n}$ consists only of zero objects of $\mathrm{LMod}_T(\mathcal{C})$. Applying Proposition A.1.2.1.19, we conclude that $\mathrm{LMod}_T(\mathcal{C})$ is right complete. This proves (5).

It remains to prove (6). Let $\mathcal{E}$ be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ spanned by exact functors which are t-exact. Since $\mathcal{E}$ is closed under composition, it is a monoidal subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. We have a diagram of ∞-categories

$$\cdots \to \mathcal{C}_{\geq -2} \overset{\tau_{\geq 1}}{\to} \mathcal{C}_{\geq -1} \overset{\tau_{\geq 0}}{\to} \mathcal{C}_{\geq 0}$$

of ∞-categories acted on by $\mathcal{E}$. If $\mathcal{C}$ is right complete, then it can be identified with the limit of this tower; it follows that $\mathrm{LMod}_T(\mathcal{C}) \simeq \varprojlim \mathrm{LMod}_T(\mathcal{C}_{\geq -n}) \simeq \varprojlim \mathrm{LMod}_T(\mathcal{C})_{\geq -n}$ so that $\mathrm{LMod}_T(\mathcal{C})$ is right complete. □

Definition 6.21. Let $\mathcal{C}$ be a presentable stable ∞-category equipped with an t-structure. We say that a monad T on $\mathcal{C}$ is *faithfully flat* if the underlying functor $T : \mathcal{C} \to \mathcal{C}$ is right t-exact and preserves small colimits, and for every $X \in \mathcal{C}_{\leq 0}$, the cofiber of the unit map $X \to TX$ also belongs to $\mathcal{C}_{\leq 0}$.

Lemma 6.22. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings. The following conditions are equivalent:*

(1) *The map f is faithfully flat.*

(2) *The cofiber* $\mathrm{cofib}(f)$ *is flat when regarded as an A-module.*

Proof. Suppose first that (2) is satisfied. We have a fiber sequence of A-modules

$$A \to B \to \mathrm{cofib}(f).$$

Since A and $\mathrm{cofib}(f)$ are flat over A, Theorem A.7.2.2.15 implies that B is flat over A. To prove that f is faithfully flat, we must show that if M is a discrete $\pi_0 A$-module such that $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M) \simeq 0$, then $M \simeq 0$. We can identify $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$ with the discrete B-module $B \otimes_A M$, which fits into a fiber sequence

$$M \to B \otimes_A M \to \mathrm{cofib}(f) \otimes_A .$$

Condition (2) implies that $\mathrm{cofib}(f) \otimes_A M$ is discrete, so we get a short exact sequence of discrete $\pi_0 A$-modules

$$0 \to M \to B \otimes_A M \to \mathrm{cofib}(f) \otimes_A M \to 0$$

which proves that $M \simeq 0$.

Now suppose that (1) is satisfied. Since A and B are connective, $\mathrm{cofib}(f)$ is connective. According to Theorem A.7.2.2.15, it will suffice to show that for every discrete A-module M, the relative tensor product $\mathrm{cofib}(f) \otimes_A M$ is discrete. We have a fiber sequence

$$M \to B \otimes_A M \to \mathrm{cofib}(f) \otimes_A M.$$

Since B is flat over A, $B \otimes_A M$ is discrete. Consequently, to prove that $\mathrm{cofib}(f) \otimes_A M$ is discrete, it suffices to show that the map $\theta : \pi_0 M \to \pi_0(B \otimes_A M) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$ is a monomorphism. Let $K \subseteq \pi_0 M$ denote the kernel of θ. Since $\pi_0 B$ is flat over $\pi_0 A$, we can identify $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, K)$ with a submodule of $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$. This submodule is generated by $\theta(K) = 0$, and therefore vanishes. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, we conclude that $K \simeq 0$ so that θ is injective as desired. □

Remark 6.23. Let A be an $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be an A-linear ∞-category. For every A-algebra B, we have a pair of adjoint functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Mod}_B(\mathcal{C})$$

which determines a monad $T \simeq G \circ F$ on $\mathcal{C}$. In concrete terms, this monad is given by the formula $M \mapsto B \otimes_A M$. Since the functor G is conservative and preserves small colimits, Theorem A.6.2.2.5 implies that $\mathrm{Mod}_B(\mathcal{C})$ can be identified with the ∞-category $\mathrm{Mod}_T(\mathcal{C})$ of T-modules in $\mathcal{C}$.

Lemma 6.24. *Let A be a connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ an A-linear ∞-category equipped with an excellent t-structure. Let B be a faithfully flat A-algebra. Then the monad T of Remark 6.23 is faithfully flat.*

Proof. Since B is connective, the functor $M \mapsto TM \simeq B \otimes_A M$ is right t-exact. Let B/A denote the cofiber of the map of A-modules $A \to B$, so that the cofiber of the map $M \to TM$ can be identified with $B/A \otimes_A M$. Suppose that $M \in \mathcal{C}_{\leq 0}$, and let let $\mathcal{X}$ be the full subcategory of Mod_A spanned by those A-modules N for which $N \otimes_A M \in \mathcal{C}_{\leq 0}$. We wish to prove that $B/A \in \mathcal{X}$. By virtue of Lemma 6.22, it will suffice to show that $\mathcal{X}$ contains all flat A-modules. Since the t-structure on $\mathcal{C}$ is excellent, we see that $\mathcal{X}$ is stable under filtered colimits in Mod_A. Using Theorem A.7.2.2.15, we are reduced to proving that M contains all free A-modules of finite rank, which is clear. □

Proposition 6.25. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with an excellent t-structure, and let T be a faithfully flat monad on $\mathcal{C}$. Let $F : \mathcal{C} \to \mathrm{Mod}_T(\mathcal{C})$ be a left adjoint to the forgetful functor $G : \mathrm{Mod}_T(\mathcal{C}) \to \mathcal{C}$. Then F is conservative and preserves totalizations of F-split cosimplicial objects of $\mathcal{C}$.*

Remark 6.26. According to Theorem A.6.2.2.5, Proposition 6.25 is equivalent to the assertion that $\mathcal{C}$ can be recovered as the ∞-category of algebras over the comonad $F \circ G$ on $\mathrm{Mod}_T(\mathcal{C})$.

Proof of Proposition 6.25. We first show that F is conservative. Let $\alpha : X \to Y$ be a morphism in $\mathcal{C}$ such that $F(\alpha)$ is an equivalence. Then $T\,\mathrm{cofib}(\alpha) \simeq GF(\mathrm{cofib}(\alpha)) \simeq G\,\mathrm{cofib}(F(\alpha)) \simeq 0$. Our assumption that T is faithfully flat guarantees the existence of a monomorphism

$$\pi_i \,\mathrm{cofib}(\alpha) \to \pi_i T(\mathrm{cofib}(\alpha)) \simeq 0$$

in the abelian category $\mathcal{C}^{\heartsuit}$, which proves that $\pi_i \,\mathrm{cofib}(\alpha) \simeq 0$. Since the t-structure on $\mathcal{C}$ is right and left complete, we deduce that $\mathrm{cofib}(\alpha) \simeq 0$ so that α is an equivalence.

Now suppose that $X^{\bullet}$ is an F-split cosimplicial object of $\mathcal{C}$, having a limit $X \in \mathcal{C}$. We wish to prove that the canonical map $FX \to \varprojlim FX^{\bullet}$ is an equivalence in $\mathrm{Mod}_T(\mathcal{C})$, or equivalently that the map $TX \to \varprojlim TX^{\bullet}$ is an equivalence in $\mathcal{C}$.

Since the monad T is faithfully flat, T defines an exact functor T_0 from the abelian category $\mathcal{C}^{\heartsuit}$ to itself. Moreover, the unit of T induces a monomorphism of functors $\mathrm{id}_{\mathcal{C}^{\heartsuit}} \hookrightarrow T_0$, so that T_0 is conservative. For every object $Y \in \mathcal{C}$, we have canonical isomorphisms $\pi_n TY \simeq T_0 \pi_n Y$. Since $X^{\bullet}$ is F-split, it follows that for every integer n, $T_0 \pi_n X^{\bullet}$ is a split cosimplicial object of $\mathcal{C}^{\heartsuit}$. Let

$$A(n)^0 \xrightarrow{d(n)} A(n)^1 \longrightarrow A(n)^2 \longrightarrow \cdots$$

be the unnormalized chain complex (in $\mathcal{C}^{\heartsuit}$) associated to $\pi_n X^{\bullet}$. It follows that $T_0(A(n)^{\bullet})$ is split exact: in particular, we have an exact sequence

$$0 \to K \to T_0 A(n)^0 \to T_0 A(n)^1 \to \cdots .$$

Since T_0 is exact, we can write $K = \ker(T_0 d(n)) \simeq T_0 \ker d(n)$. Since T_0 is exact and conservative, the exactness of the sequence

$$0 \to T_0 \ker d(n) \to T_0 A(n)^0 \to T_0 A(n)^1 \to \cdots$$

implies the exactness of the sequence

$$0 \to \ker d(n) \to A(n)^0 \to A(n)^1 \to \cdots .$$

Using Corollary A.1.2.4.10, we deduce that the map $X \to X^0$ induces an isomorphism $\pi_n X \simeq \ker d(n)$ for every integer n. Using the exactness of T_0 and the identification $T_0 \pi_n X \simeq \pi_n TX$, we see that the map $\alpha : TX \to \varprojlim TX^{\bullet}$ induces an exact sequence

$$0 \to \pi_n TX \to T_0 A(n)^0 \to T_0 A(n)^1 \to \cdots$$

which implies that α is an equivalence (Corollary A.1.2.4.10). □

We are now ready to prove a weaker version Theorem 6.12.

Theorem 6.27. *Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C}$ be an A-linear ∞-category which admits an excellent t-structure. Then $\mathcal{C}$ satisfies flat descent.*

Proof. According to Proposition 6.18, it will suffice to show that for every faithfully flat morphism $B \to B'$ of connective A-algebras, the induced functor $F : \mathrm{Mod}_B(\mathcal{C}) \to \mathrm{Mod}_{B'}(\mathcal{C})$ is conservative and preserves totalizations of F-split cosimplicial objects. Using Proposition 6.20, we can replace $\mathcal{C}$ by $\mathrm{Mod}_B(\mathcal{C})$ and thereby reduce to the case where $B = A$. The desired result now follows from Lemma 6.24 and Proposition 6.25. □

We would like to use Theorem 6.27 to prove Theorem 6.12. The first step is to observe that, in the situation of Theorem 6.27, the excellent t-structure on $\mathcal{C}$ also satisfies flat descent:

Lemma 6.28. *Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be an A-linear ∞-category equipped with an excellent t-structure, let $f : B \to B'$ be a map of connective A-algebras, and let $F : \mathrm{Mod}_B(\mathcal{C}) \to \mathrm{Mod}_{B'}(\mathcal{C})$ be the induced functor. Then:*

(1) *The functor F carries $\mathrm{Mod}_B(\mathcal{C})_{\geq 0}$ into $\mathrm{Mod}_{B'}(\mathcal{C})_{\geq 0}$.*

(2) *If f is flat, then F carries $\mathrm{Mod}_B(\mathcal{C})_{\leq 0}$ into $\mathrm{Mod}_{B'}(\mathcal{C})_{\leq 0}$.*

(3) *If f is faithfully flat, then we have*

$$\mathrm{Mod}_B(\mathcal{C})_{\geq 0} = F^{-1}\,\mathrm{Mod}_{B'}(\mathcal{C})_{\geq 0} \qquad \mathrm{Mod}_B(\mathcal{C})_{\leq 0} = F^{-1}\,\mathrm{Mod}_{B'}(\mathcal{C})_{\leq 0}.$$

Proof. To prove (1), we let $\mathcal{D}$ denote the full subcategory of Mod_B spanned by those B-modules N for which the functor $N \otimes_B \bullet$ carries $\mathrm{Mod}_B(\mathcal{C})_{\geq 0}$ to $\mathcal{C}_{\geq 0}$. We wish to show that $B' \in \mathcal{D}$. Since B' is connective, it will suffice to show that $\mathcal{D}$ contains all connective B-modules. This is clear, since $B \in \mathcal{D}$ and $\mathcal{D}$ is closed under small colimits.

We now prove (2). Let $\mathcal{D}'$ be the full subcategory of Mod_B spanned by those B-modules N for which the functor $N \otimes_B \bullet$ carries $\mathrm{Mod}_B(\mathcal{C})_{\leq 0}$ into $\mathcal{C}_{\leq 0}$; we wish to show that $B' \in \mathcal{D}'$. Since the t-structure on $\mathcal{C}$ is excellent, it is clear that $\mathcal{D}'$ is closed under filtered colimits. Theorem A.7.2.2.15 implies that B' is a filtered colimit of B-modules of the form B^k; it therefore suffices to show that each $B^k \in \mathcal{D}'$, which is obvious.

We now prove (3). We will prove that $F^{-1}\,\mathrm{Mod}_{B'}(\mathcal{C})_{\geq 0} = \mathrm{Mod}_B(\mathcal{C})_{\geq 0}$; the proof that

$$F^{-1}\,\mathrm{Mod}_{B'}(\mathcal{C})_{\leq 0} = \mathrm{Mod}_B(\mathcal{C})_{\leq 0}$$

is similar. Let $M \in \mathrm{Mod}_B(\mathcal{C})$ be such that $B' \otimes_B M \in \mathrm{Mod}_{B'}(\mathcal{C})_{\geq 0}$; we wish to prove that $M \in \mathrm{Mod}_B(\mathcal{C})_{\geq 0}$. Since $\mathrm{Mod}_B(\mathcal{C})$ is right complete (Proposition 6.20), it will suffice to show that $\pi_k M \simeq 0$ for $k < 0$. Since the functor $B' \otimes_B \bullet$ is t-exact (by (1) and (2)), we have $B' \otimes_B \pi_k M \simeq \pi_k(B' \otimes_B M) \simeq 0$. The faithful flatness of $B \to B'$ implies that the cofiber B'/B is a flat B-module, so the proof of (2) gives $B'/B \otimes_B \pi_k M \in \mathcal{C}_{\leq 0}$. It follows that the cofiber sequence

$$\pi_k M \to B' \otimes_B \pi_k M \to (B'/B) \otimes_B \pi_k M$$

induces a monomorphism $\pi_k M \simeq \pi_0(B' \otimes_B \pi_k M) \simeq 0$ in the abelian category $\mathcal{C}^\heartsuit$, so that $\pi_k M \simeq 0$ as desired. □

Construction 6.29. Fix a connective $\mathbb{E}_\infty$-ring A and an A-linear ∞-category $\mathcal{C}$ equipped with an excellent t-structure. Given $-\infty \leq m, n \leq \infty$, we let $\mathcal{N}_{\geq m}$ denote the full subcategory of

$$\mathrm{Mod}(\mathcal{C}) = \mathrm{LMod}(\mathcal{C}) \times_{\mathrm{Alg}_A} \mathrm{CAlg}_A^{\mathrm{cn}}$$

spanned by those pairs (B, M) where B is a connective $\mathbb{E}_\infty$-algebra over A, and $M \in \mathrm{Mod}_B(\mathcal{C})_{\geq m}$. Let $\mathcal{N}_{\geq m}^{\leq n} \subseteq \mathcal{N}_{\geq m}$ the full subcategory spanned by those pairs $(B, M) \in \mathcal{N}_{\geq m}$ such that $M \in \mathrm{Mod}_B(\mathcal{C})_{\leq n}$. It is easy to see that the forgetful functor $p : \mathcal{N}_{\geq m} \to \mathrm{CAlg}_A^{\mathrm{cn}}$ is a coCartesian fibration. For each connective

A-algebra B, the fiber $(\mathcal{N}^{\leq n}_{\geq m})_B \subseteq (\mathcal{N}_{\geq m})_B$ is a the essential image of a localization functor $L_B = \tau_{\leq n}$. If $f : B \to B'$ is a map of connective A-algebras, the base change functor $M \mapsto B' \otimes_B M$ carries L_B-equivalences to $L_{B'}$-equivalences. It follows that the forgetful functor $q : \mathcal{N}^{\leq n}_{\geq m} \to \mathrm{CAlg}(\mathrm{Mod}^{\mathrm{cn}}_A)$ is also a coCartesian fibration (Lemma A.2.2.4.11). Moreover, if $f : B \to B'$ is flat, then the base change functor $M \mapsto B' \otimes_B M$ is t-exact and therefore carries $(\mathcal{N}^{\leq n}_{\geq m})_B$ into $(\mathcal{N}^{\leq n}_{\geq m})_{B'}$ (Lemma 6.28), so that a morphism in $\mathcal{N}^{\leq n}_{\geq m}$ lifting f is q-coCartesian if and only if it is p-coCartesian.

Proposition 6.30. *Let A be a connective $\mathbb{E}_\infty$-ring, let $\mathcal{C}$ be an A-linear ∞-category equipped with an excellent t-structure. Let $-\infty \leq m, n \leq \infty$, and let $q : \mathcal{N}^{\leq n}_{\geq m} \to \mathrm{CAlg}^{\mathrm{cn}}_A$ be defined as in Construction 6.29. Then, as a coCartesian fibration, q is classified by a functor $\chi : \mathrm{CAlg}^{\mathrm{cn}}_A \to \widehat{\mathcal{C}\mathrm{at}_\infty}$ which is a sheaf with respect to the flat topology.*

Proof. We first show that χ preserves finite products. In view of Lemma 6.16, it suffices to observe the following: given a finite collection of connective A-algebras B_i with product $B = \prod_i B_i$ and module objects $M_i \in \mathrm{Mod}_{B_i}(\mathcal{C})$, the product $M = \prod_i M_i$ belongs to $\mathrm{Mod}_B(\mathcal{C})_{\leq n}$ if and only if each $M_i \in \mathrm{Mod}_{B_i}(\mathcal{C})_{\leq n}$, and $M \in \mathrm{Mod}_B(\mathcal{C})_{\geq m}$ if and only if each $M_i \in \mathrm{Mod}_{B_i}(\mathcal{C})_{\geq m}$.

According to Proposition 5.7, it remains to show that if $B \to B^0$ is a faithfully flat morphism of connective A-algebras having Čech nerve $B^\bullet$ in CAlg^{op}_A, then the composite diagram

$$\mathrm{N}(\mathbf{\Delta}_+) \stackrel{B^\bullet}{\to} \mathrm{CAlg}^{\mathrm{cn}}_A \stackrel{\chi}{\to} \widehat{\mathcal{C}\mathrm{at}_\infty}$$

is a limit diagram. Since the inclusion $\mathrm{N}(\mathbf{\Delta}_s) \hookrightarrow \mathrm{N}(\mathbf{\Delta})$ is right cofinal (Lemma T.6.5.3.7), it will suffice to show that

$$\mathrm{N}(\mathbf{\Delta}_{s,+}) \hookrightarrow \mathrm{N}(\mathbf{\Delta}_+) \stackrel{B^\bullet}{\to} \mathrm{CAlg}^{\mathrm{cn}}_A \stackrel{\chi}{\to} \widehat{\mathcal{C}\mathrm{at}_\infty}$$

is a limit diagram. By virtue of Theorem 6.27, this reduces to the following concrete assertion: an object $M \in \mathrm{Mod}_B(\mathcal{C})$ belongs to $\mathrm{Mod}_B(\mathcal{C})_{\geq m} \cap \mathrm{Mod}_B(\mathcal{C})_{\leq n}$ if and only if $B^p \otimes_B M \in \mathrm{LMod}_{B^p}(\mathcal{C})_{\geq m} \cap \mathrm{LMod}_{B^p}(\mathcal{C})_{\leq n}$ for all integers p. This assertion is a special case of Lemma 6.28. □

Lemma 6.31. *Let A be a connective $\mathbb{E}_\infty$-ring and $\mathcal{C}$ an A-linear ∞-category equipped with an excellent t-structure. Let $B^\bullet$ be an augmented cosemisimplicial of $\mathrm{CAlg}^{\mathrm{cn}}_A$ which is a flat hypercovering of $B = B^{-1}$. Let $M^\bullet$ be a coCartesian cosemisimplicial discrete $(B^\bullet | \mathrm{N}(\mathbf{\Delta}_s))$-module object of $\mathcal{C}$: that is, $M^\bullet$ supplies an object $M^p \in \mathrm{Mod}_{B^p}(\mathcal{C})^\heartsuit$ for each $p \geq 0$, and an equivalence $B^{p'} \otimes_{B^p} M^p \to M^{p'}$ for each injection $[p] \to [p']$. Then:*

(1) *The unnormalized cochain complex*

$$M^0 \stackrel{\phi}{\to} M^1 \to M^2 \to \cdots$$

is acyclic in positive degrees (in the abelian category $\mathcal{C}^\heartsuit$).

(2) *The canonical map $B^0 \otimes_B \ker(\phi) \to M^0$ is an isomorphism in the abelian category $\mathcal{C}^\heartsuit$.*

Proof. Fix $n \geq 0$, and let $q : \mathcal{N}^{\leq 0}_{\geq -n} \to \mathrm{CAlg}^{\mathrm{cn}}_A$ and $p : \mathcal{N}^{\leq 0}_{\geq -\infty} \to \mathrm{CAlg}^{\mathrm{cn}}_A$ be defined as in Construction 6.29. We regard $B^\bullet$ as a diagram $\mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{CAlg}^{\mathrm{cn}}_A$, and $M^\bullet$ as a diagram $\mathrm{N}(\mathbf{\Delta}_s) \to \mathcal{N}^{\leq 0}_{\geq -n}$ lifting $B^\bullet | \mathrm{N}(\mathbf{\Delta}_s)$. Let M denote a p-limit of the diagram $M^\bullet$ (lying over $B^\bullet$). Let M' denote a q-limit of the diagram $M^\bullet$ (also lying over $B^\bullet$), so that $M' \simeq \tau_{\geq -n} M$ as objects of $\mathrm{Mod}_B(\mathcal{C})$. It follows from Example A.1.2.4.8 that the homotopy groups of M are given (as objects of the abelian category $\mathcal{C}^\heartsuit$) by the cohomology groups of the unnormalized cochain complex

$$M^0 \to M^1 \to M^2 \to \cdots,$$

and $M' \simeq \tau_{\geq -n} M$. Proposition 5.12 implies that the pullback map $q' : \mathcal{N}^{\leq 0}_{\geq -n} \times_{\mathrm{N}(\mathcal{C})} \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{N}(\mathbf{\Delta}_{s,+})$ is classified by limit diagram $\chi : \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \widehat{\mathcal{C}\mathrm{at}_\infty}$. Combining this observation with Lemma 5.17, we conclude that the natural map $B^0 \otimes_B M' \to M^0$ is an equivalence. It follows from Lemma 6.28 that M' is discrete, so that the homotopy groups $\pi_k M$ vanish for every nonzero $k \geq -n$. Moreover, we have

$$M^0 \simeq \pi_0 M^0 \simeq \pi_0(B^0 \otimes_B M') \simeq B^0 \otimes_B (\pi_0 M') \simeq B^0 \otimes_B \ker(\phi),$$

which proves (2). Assertion (1) follows, since n can be chosen arbitrarily large. □

Proof of Theorem 6.12. Let $\chi : \mathrm{CAlg}_A \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classify the coCartesian fibration $q : \mathrm{Mod}(\mathcal{C}) \to \mathrm{CAlg}_A$. According to Lemma 6.17, it will suffice to show that the restriction $\chi | \mathrm{CAlg}_A^{\mathrm{cn}}$ is a hypercomplete sheaf with respect to the flat topology. Using Proposition 5.12 and Lemma 6.16, we are reduced to proving the following: for every flat hypercovering $B^\bullet : \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{CAlg}_A$ of a connective A-algebra $B = B^{-1}$, the coCartesian fibration $q' : \mathrm{Mod}(\mathcal{C}) \times_{\mathrm{CAlg}_A} \mathrm{N}(\mathbf{\Delta}_{s,+}) \to \mathrm{N}(\mathbf{\Delta}_{s,+})$ is classified by a limit diagram $\mathrm{N}(\mathbf{\Delta}_{s,+}) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. To prove this, we will verify that q' satisfies the hypotheses of Lemma 5.17:

(*a*) Let $\alpha : M \to N$ be a morphism of B-module objects of $\mathcal{C}$, and suppose that the induced map $B^0 \otimes_B M \to B^0 \otimes_B N$ is an equivalence. We wish to prove that α is an equivalence. This follows immediately from Lemma 6.28.

(*b*) Let $M^\bullet : \mathrm{N}(\mathbf{\Delta}_s) \to \mathrm{Mod}(\mathcal{C})$ be a q-coCartesian lifting of $B^\bullet | \mathrm{N}(\mathbf{\Delta}_s)$, and let $M \in \mathrm{LMod}_B(\mathcal{C})$ be a q-limit of $M^\bullet$ (lying over $B^\bullet$). We wish to prove that, for every $p \geq 0$, the canonical map $\alpha : B^p \otimes_B M \to M^p$ is an equivalence. It clearly suffices to treat the case $p = 0$. For every integer q, the cosemisimplicial object $\pi_q M^\bullet$ of $\mathcal{C}^\heartsuit$ satisfies the hypotheses of Lemma 6.31. Using Lemma 6.31 and Corollary A.1.2.4.10, we deduce that the canonical map $B^0 \otimes_B \pi_q M \to \pi_q M^0$ is an equivalence. Combining this with Lemma 6.28, we deduce that the map α induces an equivalence $\pi_q(B^0 \otimes_B M) \to \pi_q M^0$ for every integer q. Since $\mathcal{C}$ is right and left complete, it follows that α is an equivalence.

□

Remark 6.32. Let A be a connective $\mathbb{E}_\infty$-ring, $\mathcal{C}$ an A-linear ∞-category equipped with an excellent t-structure, and fix $-\infty \leq m, n \leq \infty$. Let $q : \mathcal{N}_{\geq m}^{\leq n} \to \mathrm{CAlg}_A$ be defined as in Construction 6.29, and let $\chi : \mathrm{CAlg}_A^{\mathrm{cn}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classify the coCartesian fibration q. Then χ is a hypercomplete sheaf with respect to the flat topology on CAlg_A. This follows immediately from Theorem 6.12 and Lemma 6.28, as in the proof of Proposition 6.30.

7 Digression: Henselian Rings

In this section, we will review some basic facts about Henselian rings which will be needed in the study of spectral algebraic geometry. For a more detailed exposition, we refer the reader to [56].

Definition 7.1. Let R be a commutative ring. We will say that R is *Henselian* if it is a local ring with maximal ideal $\mathfrak{m}$ which satisfies the following condition: for every étale R-algebra R', every map of R-algebras $R' \to R/\mathfrak{m}$ can be lifted to a map of R-algebras $R' \to R$.

Warning 7.2. Our terminology is not completely standard; some authors do not require locality in the definition of a Henselian ring.

Notation 7.3. If R is a commutative ring and we are given a pair of commutative R-algebras A and B, we let $\mathrm{Hom}_R(A, B)$ denote the set of R-algebra homomorphisms from A to B.

Proposition 7.4. *Let R be a Henselian local ring with maximal ideal $\mathfrak{m}$ and let R' be an étale R-algebra. Then the reduction map $\theta_{R'} : \mathrm{Hom}_R(R', R) \to \mathrm{Hom}_R(R', R/\mathfrak{m})$ is bijective.*

Proof. The definition of a Henselian local ring guarantees that $\theta_{R'}$ is surjective. For injectivity, suppose we are given two R-algebra maps $f, g : R' \to R$ with $\theta(f) = \theta(g)$. Since R' is étale over R, the multiplication map $m : R' \otimes_R R' \to R'$ induces an isomorphism $(R' \otimes_R R')[\frac{1}{e}] \simeq R'$ for some idempotent element $e \in R' \otimes_R R'$. The maps f and g determine a map $u : R' \otimes_R R' \to R$. Since $\theta(f) = \theta(g)$, the composite map $u' : R' \otimes_R R' \to R \to R/\mathfrak{m}$ factors through m, so that $u'(e)$ is invertible in $R/\mathfrak{m}$. Since R is local, we conclude that $u(e) \in R$ is invertible, so that u also factors through m; this proves that $f = g$. □

Proposition 7.5. *Let $R \to A$ be a finite étale map between local commutative rings. If R is Henselian, then A is also Henselian.*

Remark 7.6. Proposition 7.11 can be generalized: if R is a Henselian local ring and A is a local R-algebra which is finitely generated as an R-module, then A is also Henselian. We refer the reader to [56] for a proof.

Lemma 7.7. *Let $f : R \to R'$ be an étale map of commutative rings which exhibits R' as a projective R-module of rank n. Then there exists a faithfully flat finite étale morphism $R \to A$ such that $R' \otimes_R A \simeq A^n$.*

Proof. We proceed by induction on n. If $n = 0$, we can take $A = R$. Assume $n > 0$. Then f is faithfully flat. Replacing R by R', we can assume that f admits a left inverse $g : R' \to R$. Since f is étale , the map g determines a decomposition $R' \simeq R \times R''$. Then R'' is finite étale of rank $(n-1)$ over R. By the inductive hypothesis, we can choose a faithfully flat finite étale map $R \to A$ such that $R'' \otimes_R A \simeq A^{n-1}$. It follows that $R' \otimes_R A \simeq A^n$ as desired. □

Lemma 7.8. *Let $R \to A$ be a finite étale map of commutative rings, and let $\phi : A \to A'$ be a ring homomorphism. Then there exists a ring homomorphism $R \to R'$ and a map $\psi : A' \to R' \otimes_R A$ with the following universal property: for every commutative R-algebra B, composition with ψ induces a bijection*

$$\mathrm{Hom}_R(R', B) \to \mathrm{Hom}_A(A', B \otimes_R A).$$

Moreover, if A' is étale over A, then R' is étale over R.

Proof. The assertion is local on R (with respect to the étale topology, say). We may therefore reduce to the case where the finite étale map $R \to A$ splits, so that $A \simeq R^n$ (Lemma 7.7). Then A' is isomorphic to a product $A'_1 \times \cdots \times A'_n$ of R-algebras. Let $R' = A'_1 \otimes \cdots \otimes A'_n$, and let $\psi : A' \to R' \otimes_R A \simeq R'^n$ be the product of the evident maps $A'_i \to R'$. It is easy to see that ψ has the desired property. If A' is étale over A, then each A'_i is étale over R, so that R' is also étale over R. □

Proof of Proposition 7.11. Let $\mathfrak{m}$ denote the maximal ideal of R; since A is local, $\mathfrak{m}A$ is the unique maximal ideal of A. Choose an étale map $A \to A'$ and an A-algebra map $\phi_0 : A' \to A/\mathfrak{m}A$. We wish to prove that ϕ_0 lifts to a map $A' \to A$. Choose an étale map $R \to R'$ as in Lemma 7.8. Then ϕ_0 is classified by a map $\psi_0 : R' \to R/\mathfrak{m}$. Since R is Henselian, we can lift ψ_0 to a map $\psi : R' \to R$, which classifies a lifting $\phi : A' \to A$ of ϕ_0. □

Corollary 7.9. *Let R be a Henselian commutative ring. Suppose we are given a faithfully flat étale map $R \to R'$. Then there exists an idempotent element $e \in R'$ such that $R'[\frac{1}{e}]$ is local, faithfully flat over R, and finitely generated as an R-module.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R and set $k = R/\mathfrak{m}$. Since R' is faithfully flat over R, the quotient $R'/\mathfrak{m}R'$ is a nontrivial étale k-algebra. We can therefore choose a finite separable extension k' of k and a surjective k-algebra map $\phi_0 : R'/\mathfrak{m}R' \to k'$. Choose a filtration

$$k = k_0 \hookrightarrow k_1 \hookrightarrow \cdots \hookrightarrow k_n = k'$$

where each k_{i+1} has the form $k_i[x_i]/(f_i(x_i))$ for some monic polynomial f_i (in fact, we may assume that $n = 1$, by the primitive element theorem, but we will not need to know this). We lift this to a sequence of algebra extensions

$$R = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n$$

where $A_{i+1} = A_i[x_i]/(\overline{f}_i(x_i))$ for some monic polynomial $\overline{f}_i$ lifting f_i. Since k' is separable over k, each derivative $\frac{\partial f_i(x_i)}{\partial x_i}$ is invertible in k_{i+1}. It follows that $\frac{\partial \overline{f}_i(x_i)}{\partial x_i}$ is invertible in A_{i+1}, so that each A_{i+1} is a finite étale extension of A_i. Set $A = A_n$, so that A is a finite étale extension of R. Note that A is a local ring with maximal ideal $\mathfrak{m}A$ and residue field $A/\mathfrak{m}A = k'$. The map ϕ_0 together with the quotient map $A \to A/\mathfrak{m}A$ amalgamate to give an A-algebra map $\psi_0 : A \otimes_R R' \to A/\mathfrak{m}A$. Since A is Henselian (Proposition 7.11), the map ψ_0 lifts to an A-algebra map $\psi : A \otimes_R R' \to A$, which we can identify with a map $\phi : R' \to A$ lifting ϕ_0. Since ϕ_0 is surjective, the map ϕ is surjective modulo $\mathfrak{m}$ and therefore surjective by Nakayama's lemma (since A is a finitely generated R-module). Since R' and A are both étale over R, the map ϕ is an étale surjection. It follows that $A \simeq R'[\frac{1}{e}]$ for some idempotent element $e \in R'$. □

Definition 7.10. Let $\phi : R \to A$ be a map of commutative rings. We will say that ϕ is *quasi-finite* if it exhibits A as a finitely generated R-algebra and, for every map $R \to k$ where k is a field, $\mathrm{Tor}_0^R(A, k)$ is a finite-dimensional vector space over k.

We will need the following nontrivial fact about quasi-finite morphisms of commutative rings (for a proof, we refer the reader to [56]):

Theorem 7.11 (Zariski's Main Theorem)**.** *Let $\phi : R \to A$ be a quasi-finite map of commutative rings. Then ϕ factors as a composition $R \xrightarrow{\phi'} B \xrightarrow{\phi''} A$, where ϕ' exhibits B as a finitely generated R-module and ϕ'' induces an open immersion of schemes $\mathrm{Spec}^{\mathrm{Z}} A \to \mathrm{Spec}^{\mathrm{Z}} B$.*

Proposition 7.12. *Let $\phi : R \to A$ be a map of commutative rings which exhibits A as a finitely generated R-module. Let κ be the residue field of R at some point $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R$, let $A_0 = \mathrm{Tor}_0^R(\kappa, A)$, and suppose we are given a decomposition of commutative rings $A_0 = A_0' \times A_0''$. Then the map $R \to \kappa$ factors as a composition*

$$R \xrightarrow{\psi} R_1 \to \kappa$$

where R_1 is an étale R-algebra and a decomposition of $A_1 = \mathrm{Tor}_0^R(R_1, A)$ as a product $A_1 = A_1' \times A_1''$ with $A_0' \simeq \mathrm{Tor}_0^{R'}(\kappa, A_1')$ and $A_0'' \simeq \mathrm{Tor}_0^{R'}(\kappa, A_1'')$.

Proof. Since A is a finitely generated R-module, A_0 is a finite-dimensional vector space over κ, and therefore admits a basis $\{\overline{x}_1, \ldots, \overline{x}_n\}$. Replacing R by a localization if necessary, we may assume that each $\overline{x}_i$ lifts to an element $x_i \in A$. Using Nakayama's lemma, we may assume (after replacing R by a suitable localization) that the elements x_i generate A as an R-module. Consequently, we have

$$x_i x_j = \sum_k r_{i,j}^k x_k$$

for some elements $\{r_{i,j}^k\}_{1 \leq i,j,k \leq n}$. Let $R' = R[y_1, \ldots, y_n]/(f_1, \ldots, f_n)$ with

$$f_k(y_1, \ldots, y_n) = y_k - \sum_{i,j} r_{i,j}^k y_i y_j,$$

and let $\Delta \in R'$ be the image of the determinant of the Jacobian matrix $[\frac{\partial f_j}{\partial y_i}]$.

For every κ-algebra B, the set of κ-algebra homomorphisms $\mathrm{Tor}_0^R(\kappa, R') \to B$ can be identified with the collection of elements with the set of elements $e = \sum_{1 \leq i \leq n} b_i \overline{x}_i$ in $B \otimes_\kappa A_0$ such that

$$e^2 = \sum_{i,j} r_{i,j}^k b_i b_j \overline{x}_k = e.$$

If $I \subseteq B$ is a nilpotent ideal, then the Zariski spectra of $B \otimes_\kappa A_0$ and $(B/I) \otimes_\kappa A_0$ are homeomorphic, so that reduction moduli I induces a bijection between idempotent elements of $B \otimes_\kappa A_0$ and $(B/I) \otimes_\kappa A_0$. It follows that $\mathrm{Tor}_0^R(\kappa, R')$ is an étale κ-algebra, so that the image of Δ is invertible in $\mathrm{Tor}_0^R(\kappa, R')$.

The decomposition $A_0 = A'_0 \times A''_0$ is determined by an idempotent element $e_0 \in A_0$, which is classified by a map of R-algebras $\phi_0 : R' \to \kappa$. The above argument shows that this map factors through $R_1 = R'[\frac{1}{\Delta}]$. By construction, R_1 is étale over R. Let $A_1 = R_1 \otimes_R A$. By construction, $e = \sum y_i \otimes x_i$ is an idempotent element of A_1, which determines a decomposition $A_1 \simeq A'_1 \times A''_1$ having the desired properties. □

Corollary 7.13. *Let R be a Henselian local ring with maximal ideal $\mathfrak{m}$ and let A be a commutative R-algebra which is finitely generated as an R-module. Then every decomposition of $A/\mathfrak{m}A$ as a product of two rings $(A/\mathfrak{m}A)_0 \times (A/\mathfrak{m}A)_1$ can be lifted to a product decomposition $A \simeq A_0 \times A_1$.*

Proposition 7.14. *Let $\phi : R \to A$ be a quasi-finite map of commutative rings and let κ denote a residue field of R at some point $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{Z}} R$. Then the canonical map $R \to \kappa$ factors as a composition $R \to R_1 \to \kappa$ where R_1 is an étale R-algebra, and $A_1 = R_1 \otimes_R A$ factors as a product $A'_1 \times A''_1$, where A'_1 is a finitely generated R_1-module and $\operatorname{Tor}_0^{R_1}(\kappa, A''_1) \simeq 0$.*

Proof. Using Theorem 7.11, we deduce that the map $\phi : R \to A$ factors as a composition $R \stackrel{\phi'}{\to} B \stackrel{\phi''}{\to} A$ where ϕ' exhibits B as a finitely generated R-module and ϕ'' induces an open immersion of schemes. Let $B_0 = \operatorname{Tor}_0^R(\kappa, B)$ and $A_0 = \operatorname{Tor}_0^R(\kappa, A)$. Since B_0 is a finite dimensional algebra over κ, its spectrum $\operatorname{Spec}^{\mathrm{Z}} B_0$ is discrete. It follows that the open subscheme $\operatorname{Spec}^{\mathrm{Z}} A_0$ is a union of components of $\operatorname{Spec}^{\mathrm{Z}} B_0$: that is, we have a decomposition $B_0 \simeq A_0 \times B''_0$. Using Proposition 7.12, we deduce that there is a factorization $R \to R_1 \to \kappa$ where R_1 is étale over R and $B_1 = R_1 \otimes_R B$ factors as a product $B'_1 \times B''_1$, with $A_0 = \operatorname{Tor}_0^{R_1}(\kappa, B'_1)$ and $B''_0 = \operatorname{Tor}_0^{R_1}(\kappa, B''_1)$. This factorization determines a prime ideal $\mathfrak{q} \in \operatorname{Spec}^{\mathrm{Z}} R_1$ lying over the prime $\mathfrak{p} \in \operatorname{Spec}^{\mathrm{Z}} R$.

Let $A'_1 = A \otimes_B B'_1$ and $A''_1 = A \otimes_B B''_1$. It is clear that $\operatorname{Tor}_0^{R_1}(\kappa, A''_1) \simeq 0$. The map $\operatorname{Spec}^{\mathrm{Z}} A'_1 \to \operatorname{Spec}^{\mathrm{Z}} B'_1$ is an open immersion, and the compement of its image is a closed subset $K \subseteq \operatorname{Spec}^{\mathrm{Z}} B'_1$. Since B'_1 is a direct factor of B_1, it is finitely generated as an R_1-module, so that the image of K in $\operatorname{Spec}^{\mathrm{Z}} R_1$ is closed. By construction, this image does not contain $\mathfrak{q}$. Replacing R_1 by a localization if necessary, we may assume that $K = \emptyset$ so that $A'_1 \simeq B'_1$ is a finitely generated R_1-module. □

Corollary 7.15. *Let R be a Henselian ring with maximal ideal $\mathfrak{m}$ and let $\phi : R \to A$ be a quasi-finite morphism of commutative rings. Then there is a decomposition $A \simeq A' \times A''$ where A' is a finitely generated R-module and $A''/\mathfrak{m}A'' \simeq 0$.*

Proposition 7.16 (Hensel's Lemma)**.** *Let R be a local Noetherian ring which is complete with respect to its maximal ideal $\mathfrak{m}$. Then R is Henselian.*

Proof. Let R' be an étale R-algebra. The structure theory of étale morphisms implies that we can write $R' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)[\Delta^{-1}]$, where Δ denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial x_j}]_{1 \leq i,j \leq n}$ (see Proposition 8.10). We wish to show that every R-algebra homomorphism $\phi_0 : R' \to R/\mathfrak{m}$ can be lifted to a ring homomorphism $\phi : R' \to R$. Since R is complete, it will suffice to construct a compatible sequence of R-algebra homomorphisms $\phi_a : R' \to R/\mathfrak{m}^{a+1}$. Assume that ϕ_a has already been constructed, and choose elements $\{y_j \in R\}_{1 \leq j \leq n}$ such that $\phi_a(x_j) \cong y_j$ modulo $\mathfrak{m}^{a+1}$. Since ϕ_a is a ring homomorphism, we have $f_i(\vec{y}) \in \mathfrak{m}^{a+1}$ for $1 \leq i \leq n$. Let $\Delta[\vec{x}]$ denote the determinant of the Jacobian matrix $M(\vec{x}) = [\frac{\partial f_i}{\partial x_j}]_{1 \leq i,j \leq n}$. Then $\Delta[\vec{y}]$ is invertible modulo $\mathfrak{m}^{a+1}$ and therefore invertible (since R is local). It follows that $M(\vec{y})$ is an invertible matrix over R, so we can define $\vec{y}' = \vec{y} + M(\vec{y})^{-1}\vec{f}(\vec{y})$. A simple calculation gives shows that $f_i(\vec{y}') \in \mathfrak{m}^{2(a+1)}$, so that the assignment $x_i \mapsto y'_i$ determines a ring homomorphism $\phi_{a+1} : R' \to R/\mathfrak{a}^{a+2}$ compatible with ϕ_a. □

Definition 7.17. Let R be a commutative ring. We say that R is *strictly Henselian* if R is Henselian and the residue field $R/\mathfrak{m}$ is separably closed.

Proposition 7.18. *Let R be a commutative ring. The following conditions are equivalent:*

(1) *The ring R is strictly Henselian.*

(2) *For every finite collection of étale maps $\{\phi_\alpha : R \to R_\alpha\}$ such that the induced map $R \to \prod_\alpha R_\alpha$ is faithfully flat, one of the maps ϕ_α admits a left inverse.*

Proof. Suppose first that condition (1) is satisfied, and let $\mathfrak{m}$ denote the maximal ideal of R. Let $\{\phi_\alpha : R \to R_\alpha\}$ be as in (2). Since the map $R \to \prod_\alpha R_\alpha$ is faithfully flat, there exists an index α such that $R_\alpha/\mathfrak{m}R_\alpha$ is nonzero. Since R_α is étale over R, $R_\alpha/\mathfrak{m}R_\alpha$ is a product of separable field extensions of $k = R/\mathfrak{m}$. Since k is separably closed, we can choose a map of R-algebras $\theta : R_\alpha/\mathfrak{m} \to R/\mathfrak{m}$. The assumption that R is Henselian implies that θ lifts to a map of R-algebras $R_\alpha \to R$, which is left inverse to ϕ_α.

Now suppose that (2) is satisfied; we wish to prove that R is strictly Henselian. We first observe that R is nonzero (otherwise the map from R to an empty product is faithfully flat, contradicting (2)). For every element $x \in R$, the map $R \to R[\frac{1}{x}] \times R[\frac{1}{1-x}]$ is faithfully flat, so condition (2) implies that either x or $1-x$ is invertible in R: that is, R is a local ring.

We now claim that R is Henselian. Let R' be an étale R-algebra and choose a map of R-algebras $\theta : R' \to R/\mathfrak{m}$. We wish to prove that θ can be lifted to an R-algebra map $R' \to R$. Let $k = R/\mathfrak{m}$, so that $R'/\mathfrak{m}R'$ is a product of finite separable extensions of k. We proceed by induction on the dimension n of $R'/\mathfrak{m}R'$ as a k-vector space. Note that $n > 0$, since θ induces a surjection $R'/\mathfrak{m}R' \to k$. It follows that R' is faithfully flat over R, so condition (2) implies that there is a map of R-algebras $\phi : R' \to R$. Since R' is étale over R, the kernel of the map ϕ is generated by an idempotent element $e \in R'$. If $\theta(e) = 0$, then θ factors as a composition $R' \xrightarrow{\phi} R \to R/\mathfrak{m}$ so that ϕ is the desired lifting of θ. Assume otherwise. Then $\theta(e) = 1$ (since e is idempotent and k is a field), so that θ factors through the quotient $R'' = R'/(1-e)$ of R'. The inductive hypothesis then implies that the induced map $R'' \to R/\mathfrak{m}$ lifts to a map of R-algebras $R'' \to R$, so that the composite map $R' \to R'' \to R$ is the desired lifting of θ.

To complete the proof, we must show that the field $k = R/\mathfrak{m}$ is separably closed. Assume otherwise. Then we can choose a nontrivial finite separable extension field k' of k. Without loss of generality, k' is generated by a single element; we may therefore write $k' = k[x]/(f(x))$ for some monic polynomial f with coefficients in k. Let $\overline{f}(x)$ be a monic polynomial with coefficients in R which lifts f (and has the same degree as f), and let $R' = R[x]/(\overline{f}(x))$. Then R' is finite as an R-module. The derivative of $\overline{f}(x)$ is invertible in $R'/\mathfrak{m}R'$, and therefore (by Nakayama's lemma) invertible in R'. It follows that R' is faithfully flat and étale over R. Using condition (2), we deduce that there is a map of R-algebras $R' \to R$. Reducing modulo $\mathfrak{m}$, we obtain a map of k-algebras $k' \to k$, contradicting our assumption that k' is a proper extension of k. ∎

Corollary 7.19. *Let $\phi : R \to A$ be a map of commutative rings. The following conditions are equivalent:*

(1) *The commutative ring A is strictly Henselian.*

(2) *For every finitely presented R-algebra R' and every finite collection of étale maps $\{R' \to R'_\alpha\}$ which induce a faithfully flat map $R' \to \prod_\alpha R'_\alpha$, every R-algebra map $R' \to A$ factors through some R_α.*

Proof. Assume that (1) is satisfied, and let $\{R' \to R'_\alpha\}$ be as in (2). For any map $R' \to A$, we obtain a finite collection of étale maps $\{\phi_\alpha : A \to R'_\alpha \otimes_R A\}$ which induce a faithfully flat map $A \to \prod_\alpha (R'_\alpha \otimes_R A)$. Proposition 7.18 implies that one of the maps ϕ_α admits a left inverse, which determines a map of R'-algebras from R'_α into A.

Now suppose that (2) is satisfied. We will show that A satisfies the criterion of Proposition 7.18. Choose a finite collection of étale maps $\{A \to A_\alpha\}$ which induce a faithfully flat map $A \to \prod_\alpha A_\alpha$. Using the structure theory for étale morphisms (see Proposition 8.10 below), we may assume that there exists a finitely presented R-algebra R' and étale maps $R' \to R'_\alpha$ such that $A_\alpha \simeq R'_\alpha \otimes_R A$. Replacing R' by a product of localizations if necessary, we may suppose that the map $R' \to \prod_\alpha R'_\alpha$ is faithfully flat. Condition (2) then guarantees the existence of a map of R'-algebras ∎

8 Spectral Deligne-Mumford Stacks

In §2 we introduced the definition of a *spectral scheme*: that is, a spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ which is locally equivalent to the spectrum of a connective $\mathbb{E}_\infty$-ring. Here the definition of spectrum is taken with respect to the Zariski topology, which is formally encoded in the geometry $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{Sp}}$ of Definition 2.18. In this section, we will introduce a variation on the theory of spectral schemes: the theory of *spectral Deligne-Mumford stacks*.

We begin by generalizing the theory of strictly Henselian rings to an arbitrary topos.

Definition 8.1. Let $\mathcal{X}$ be a topos and let $\mathcal{O}_{\mathcal{X}}$ be a commutative ring object of $\mathcal{X}$. For every finitely generated commutative ring R, let $\mathcal{S}\mathrm{ol}_R(\mathcal{O}_{\mathcal{X}}) \in \mathcal{X}$ be an object having the following universal property: for every object $U \in \mathcal{X}$, there is a canonical bijection

$$\mathrm{Hom}_{\mathcal{X}}(U, \mathcal{S}\mathrm{ol}_R(\mathcal{O}_{\mathcal{X}})) \simeq \mathrm{Hom}_{\mathrm{Ring}}(R, \mathrm{Hom}_{\mathcal{X}}(U, \mathcal{O}_{\mathcal{X}})).$$

We will say that $\mathcal{O}_{\mathcal{X}}$ is *strictly Henselian* if the following condition is satisfied:

$(*)$ For every finitely generated commutative ring R and every finite collection of étale maps $R \to R_\alpha$ which induce a faithfully flat map $R \to \prod_\alpha R_\alpha$, the induced map

$$\coprod_\alpha \mathcal{S}\mathrm{ol}_{R_\alpha}(\mathcal{O}_{\mathcal{X}}) \to \mathcal{S}\mathrm{ol}_R(\mathcal{O}_{\mathcal{X}})$$

is an effective epimorphism.

Example 8.2. If $\mathcal{X}$ is the topos of sets, then we can identify commutative ring objects of $\mathcal{X}$ with commutative rings. Under this identification, a commutative ring object of $\mathcal{X}$ is strictly Henselian in the sense of Definition 8.1 if and only if it is strictly Henselian in the sense of Definition 7.17 (the equivalence follows from Corollary 7.19).

Definition 8.3. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O}_{\mathcal{X}}$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We will say that $\mathcal{O}_{\mathcal{X}}$ is *strictly Henselian* if $\pi_0 \mathcal{O}_{\mathcal{X}}$ is a strictly Henselian commutative ring object of the topos of discrete objects of $\mathcal{X}$. Note that if $\mathcal{O}_{\mathcal{X}}$ is strictly Henselian, then it is local (in the sense of Definition 2.5). We let $\mathrm{RingTop}_{\text{ét}}$ denote the full subcategory of $\mathrm{RingTop}_{\mathrm{Zar}}$ spanned by the locally spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that $\mathcal{O}_{\mathcal{X}}$ is strictly Henselian. We will say that a spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *strictly Henselian* if it belongs to $\mathrm{RingTop}_{\text{ét}}$.

The starting point for our definition of spectral Deligne-Mumford stacks is the following analogue of Proposition 2.6:

Proposition 8.4. *The functor* $\Gamma| \mathrm{RingTop}_{\acute{e}t} : \mathrm{RingTop}_{\acute{e}t} \to \mathrm{CAlg}^{op}$ *admits a right adjoint.*

Proposition 8.4 asserts that for every $\mathbb{E}_\infty$-ring R, there exists a strictly Henselian spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and a map $\theta : R \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ with the following universal property: for every strictly Henselian spectrally ringed ∞-topos $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, composition with θ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{RingTop}_{\mathrm{Zar}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \to \mathrm{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{Y}; \mathcal{O}_{\mathcal{Y}})).$$

The spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is uniquely determined up to equivalence; and we will denote it by $\mathrm{Spec}^{\text{ét}}(R)$. We will refer to $\mathrm{Spec}^{\text{ét}}(R)$ as the *spectrum of R with respect to the étale topology.*

Definition 8.5 (Spectral Deligne-Mumford Stack: Concrete Definition)**.** A *nonconnective spectral Deligne-Mumford stack* is a spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that there exists a collection of objects $U_\alpha \in \mathcal{X}$ satisfying the following conditions:

(i) The objects U_α cover $\mathcal{X}$. That is, the canonical map $\coprod_\alpha U_\alpha \to \mathbf{1}$ is an effective epimorphism, where $\mathbf{1}$ denotes the final object of $\mathcal{X}$.

(*ii*) For each index α, there exists an $\mathbb{E}_\infty$-ring R_α and an equivalence of spectrally ringed ∞-topoi

$$(\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathcal{X}} | U_\alpha) \simeq \mathrm{Spec}^{\text{ét}}(R_\alpha).$$

We let $\mathrm{Stk}^{\mathrm{nc}}$ denote the full subcategory of $\mathrm{RingTop}_{\mathrm{Zar}}$ spanned by the nonconnective spectral Deligne-Mumford stacks.

A *spectral Deligne-Mumford stack* is a nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ such that $\mathcal{O}_{\mathcal{X}}$ is connective. We let Stk denote the full subcategory of $\mathrm{Stk}^{\mathrm{nc}}$ spanned by the spectral Deligne-Mumford stacks.

In order to process Definition 8.5, we need to understand the functor $\mathrm{Spec}^{\text{ét}} : \mathrm{CAlg}^{op} \to \mathrm{RingTop}_{\text{ét}}$ whose existence is asserted by Proposition 8.4. As in §2, we will construct this functor by appealing to the general results of [42]. For this, we will introduce a geometry $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}$, which "controls" the theory of strictly Henselian sheaves of $\mathbb{E}_\infty$-rings. This will require a few preliminary concerning étale morphisms between $\mathbb{E}_\infty$-rings.

Recall that a morphism $f : A \to B$ of $\mathbb{E}_\infty$-rings is said to be *étale* if the underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is étale, and the canonical map $(\pi_i A) \otimes_{\pi_0 A} (\pi_0 B) \to \pi_i B$ is an isomorphism for every integer i. We refer the reader to §A.7.5 for an extensive discussion of the theory of étale maps between $\mathbb{E}_\infty$-rings.

If $f : A \to B$ is an étale map between $\mathbb{E}_\infty$-rings, then the relative cotangent complex $L_{B/A}$ vanishes (Corollary A.7.5.4.5). We now establish some partial converses.

Lemma 8.6. *Let $f : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. The following conditions are equivalent:*

(1) *The abelian group $\pi_0 L_{B/A}$ vanishes, and $\pi_0 B$ is finitely generated as an algebra over $\pi_0 A$.*

(2) *There exist finitely many elements $x_1, \ldots, x_n \in \pi_0 B$ which generate the unit ideal, such that each of the induced maps $A \to B[\frac{1}{x_i}]$ factors as a composition*

$$A \xrightarrow{f'} A_i \xrightarrow{f''} B[\frac{1}{x_i}]$$

where f' is étale and f'' induces a surjection $\pi_0 A_i \to \pi_0 B[\frac{1}{x_i}]$.

Proof. Suppose first that (2) is satisfied. Each of the commutative rings $\pi_0 B[\frac{1}{x_i}]$ is a quotient of an étale $\pi_0 A$-algebra, and therefore finitely generated over $\pi_0 A$. Let $B_0 \subseteq \pi_0 B$ be a finitely generated $\pi_0 A$-subalgebra containing each x_i, such that $B_0[\frac{1}{x_i}] \to \pi_0 B[\frac{1}{x_i}]$ is surjective for each i. Since the x_i generate the unit ideal in B, we deduce that $\pi_0 B = B_0$ is finitely generated over $\pi_0 A$.

It remains to prove that $\pi_0 L_{B/A} \simeq 0$. Since the elements x_i generate the unit ideal, it will suffice to show that $(\pi_0 L_{B/A})[\frac{1}{x_i}] \simeq \pi_0(L_{B/A} \otimes_B B[\frac{1}{x_i}]) \simeq \pi_0 L_{B[\frac{1}{x_i}]/A}$ vanishes for each index i. Choose a factorization

$$A \xrightarrow{f'} A_i \xrightarrow{f''} B[\frac{1}{x_i}]$$

as in (2). We have a short exact sequence of abelian groups

$$\pi_0(B[\frac{1}{x_i}] \otimes_{A'} L_{A_i/A}) \to \pi_0 L_{B[\frac{1}{x_i}]/A} \to \pi_0 L_{B[\frac{1}{x_i}]/A_i}.$$

Here $L_{A_i/A}$ vanishes since f' is étale (Corollary A.7.5.4.5) and $\pi_0 L_{B[\frac{1}{x_i}]/A_i}$ can be identified with the relative Kähler differentials $\Omega_{\pi_0 B[\frac{1}{x_i}]/\pi_0 A_i}$ (Proposition A.7.4.3.9), which vanishes because f'' is surjective on connected components. It follows that $\pi_0 L_{B[\frac{1}{x_i}]/A} \simeq 0$ as desired.

Now suppose that (1) is satisfied. Let $R = \pi_0 B$. Since R is finitely generated over $\pi_0 A$, we can choose a presentation $R \simeq (\pi_0 A)[x_1, \ldots, x_n]/I$ for some ideal $I \subseteq (\pi_0 A)[x_1, \ldots, x_n]$. Then $\pi_0 L_{B/A}$ is the module of Kähler differentials of R over $\pi_0 A$ (Proposition A.7.4.3.9). That is, $\pi_0 L_{B/A}$ is the quotient of the free R-module generated by elements $\{dx_i\}_{1 \leq i \leq n}$ by the submodule generated by elements $\{df\}_{f \in I}$. Since $\pi_0 L_{B/A} \simeq 0$, we can choose a finite collection of elements $\{f_j \in I\}_{1 \leq j \leq m}$ such that the Jacobian matrix $M = \{\frac{\partial f_j}{\partial x_i}\}$ has rank n. Let $\{a_k\}$ be the collection of determinants of n-by-n submatrices of the matrix M, so that the elements a_k generate the unit ideal in R. We will prove that each of the composite maps $A \rightarrow B[\frac{1}{a_k}]$ factors as a composition

$$A \xrightarrow{f'} A_k \xrightarrow{f''} B[\frac{1}{a_k}],$$

where f' is étale and f'' is surjective on connected components. Reordering the f_j if necessary, we may suppose that $m \geq n$ and that x_k is the determinant of the matrix $\{\frac{\partial f_j}{\partial x_i}\}_{1 \leq i,j \leq n}$. Set

$$R' = (\pi_0 R)[x_1, \ldots, x_n, \frac{1}{a_k}]/(f_1, \ldots, f_m),$$

so that R is an étale algebra over $\pi_0 A$. It follows from Theorem A.7.5.0.6 that R' can be lifted (in an essentially unique fashion) to an étale A-algebra A_k. Moreover, Corollary A.7.5.4.6 implies that the surjective map $R' \rightarrow R[\frac{1}{a_k}] = \pi_0 B[\frac{1}{a_k}]$ lifts to a map $A_k \rightarrow B[\frac{1}{a_k}]$, thereby giving us the desired factorization. □

Lemma 8.7. *Let $f : A \rightarrow B$ be a map of connective $\mathbb{E}_\infty$-rings. Assume that:*

(1) *The map f induces a surjection $f_0 : \pi_0 A \rightarrow \pi_0 B$.*

(2) *The commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$ (that is, the kernel of f_0 is a finitely generated ideal in $\pi_0 A$).*

(3) *The abelian group $\pi_1 L_{B/A}$ vanishes.*

Then there exists an element $D \in \pi_0 A$ such that $\pi_0 B \simeq (\pi_0 A)[\frac{1}{D}]$.

Proof. Let I denote the kernel of f_0, and let $R = (\pi_0 A)/I^2$. It follows from Corollary A.7.4.1.27 that, in the ∞-category CAlg_A, we have a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \pi_0 B \\ \downarrow & & \downarrow \\ \pi_0 B & \longrightarrow & (\pi_0) B \oplus (I/I^2)[1]. \end{array}$$

Since $\pi_i L_{B/A} \simeq 0$ for $i \leq 1$, the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, \pi_0 B) \rightarrow \mathrm{Map}_{\mathrm{CAlg}_A}(B, \pi_0 B \oplus (I/I^2)[1])$$

is a homotopy equivalence, so that $\mathrm{Map}_{\mathrm{CAlg}_R}(B, R) \rightarrow \mathrm{Map}_{\mathrm{CAlg}_k}(B, \pi_0 B)$ is also a homotopy equivalence. In particular, the truncation map $B \rightarrow \pi_0 B$ lifts (in an essentially unique fashion) to a map $B \rightarrow R$. Passing to connected components, we deduce that the quotient map of commutative algebras $\phi : (\pi_0 A)/I^2 \rightarrow (\pi_0 A)/I$ admits a section as a map of $\pi_0 A$-algebras. This implies that ϕ is an isomorphism: that is, that $I = I^2$.

Because $\pi_0 B$ is finitely presented over $\pi_0 A$, the ideal I is generated by finitely elements $y_1, \ldots, y_m$. Since $I = I^2$, we can write $y_i = \sum_j z_{i,j} y_j$ for some elements $z_{i,j} \in I$. Let Z denote the matrix $\{z_{i,j}\}_{1 \leq i,j \leq m}$. Then $\mathrm{id} - Z$ annihilates the vector $(y_1, \ldots, y_m) \in (\pi_0 A)^p$. Let $D \in \pi_0 A$ denote the determinant of $\mathrm{id} - Z$. Since the entries of Z belong to I, D is congruent to 1 modulo I and is therefore invertible in $\pi_0 B$. It follows that we have a canonical map $g : A[\frac{1}{D}] \rightarrow B$. We claim that g is an isomorphism on connected components. The surjectivity of g is clear, and the injectivity follows from the observation that multiplication by D annihilates every element of I. □

Lemma 8.8. *Let $f : A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings and let $1 \leq n \leq \infty$ be an integer. The following conditions are equivalent:*

(1) *The commutative ring $\pi_0 B$ is finitely presented over $\pi_0 A$, and $\pi_i L_{B/A} \simeq 0$ for $i \leq n$.*

(2) *The map f factors as a composition*
$$A \xrightarrow{f'} A' \xrightarrow{f''} B$$
where f' is étale, f'' induces an isomorphism $\pi_i A \to \pi_i B$ for $i < n$, and f'' induces a surjection $\pi_n A \to \pi_n B$.

Proof. Suppose first that (2) is satisfied. Then $\pi_0 B \simeq \pi_0 A'$ is étale over $\pi_0 A$, and therefore finitely presented as a $\pi_0 A$-algebra. We have a fiber sequence

$$B \otimes_{A'} L_{A'/A} \to L_{B/A} \to L_{B/A'}.$$

Since A' is étale over A, we deduce that $L_{A'/A} \simeq 0$. Since f'' is n-connective, Corollary A.7.4.3.2 implies that $L_{B/A} \simeq L_{B/A'}$ is $(n+1)$-connective, thereby completing the proof of (1).

Now assume that condition (1) holds. We first prove that $\pi_0 B$ is étale over $\pi_0 A$. Using Lemma 8.6, we can choose a finite collection of elements $x_i \in \pi_0 B$ generating the unit ideal such that each of the induced maps $A \to B[\frac{1}{x_i}]$ factors as a composition

$$A \xrightarrow{g'} A_i \xrightarrow{g''} B[\frac{1}{x_i}]$$

where g' is étale and g'' is surjective on connected components. Note that $\pi_1 L_{B[\frac{1}{x_i}]/A_i} \simeq (\pi_1 L_{B/A})[\frac{1}{x_i}] \simeq 0$. Using Lemma 8.7, we deduce that $\pi_0 B[\frac{1}{x_i}]$ is étale over $\pi_0 A_i$ and therefore over $\pi_0 A$, from which it follows that $\pi_0 B$ is étale over $\pi_0 A$.

Using Theorem A.7.5.0.6, we can choose an étale A-algebra A' and an isomorphism of $\pi_0 A$-algebras $\alpha : \pi_0 A' \simeq \pi_0 B$. Theorem A.7.5.4.2 implies that we can lift α to a map of A-algebras $f'' : A' \to B$. To complete the proof, it will suffice to show that f'' is n-connective; this follows from Corollary A.7.4.3.2. □

Lemma 8.9. *Let $A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings, and assume that the relative cotangent complex $L_{B/A}$ vanishes. The following conditions are equivalent:*

(1) *The commutative algebra $\pi_0 B$ is finitely presented over $\pi_0 A$.*

(2) *The algebra B is of finite presentation over A.*

(3) *The algebra B is almost of finite presentation over A.*

(4) *The map $A \to B$ is étale.*

Proof. The implication (4) ⇒ (1) is obvious, the equivalences (1) ⇔ (2) ⇔ (3) follow from Theorem A.7.4.3.18, and the implication (1) ⇒ (4) is a special case of Lemma 8.8. □

Using Lemma 8.9, we can prove an analogue of the usual structure theorem for étale morphisms between commutative rings.

Proposition 8.10. *Let k be a connective $\mathbb{E}_\infty$-ring, and let $\phi : A \to B$ be a morphism between connective k-algebras. The following conditions are equivalent:*

(1) *The map ϕ is étale.*

(2) *There exists a pushout diagram of k-algebras*

$$\begin{array}{ccc} k\{x_1,\ldots,x_n\} & \longrightarrow & A \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi} \\ k\{y_1,\ldots,y_n\}[\frac{1}{\Delta}] & \longrightarrow & B, \end{array}$$

where $\phi_0(x_i) = f_i(y_1,\ldots,y_n) \in (\pi_0 k)[y_1,\ldots,y_n]$ and $\Delta \in (\pi_0 k)[y_1,\ldots,y_n]$ denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1\leq i,j\leq n}$.

Proof. To prove that (2) implies (1), it suffices to observe that the map ϕ_0 appearing in the diagram is étale. Note that the relative cotangent complex of ϕ_0 can be identified with the cofiber of the map

$$L_{k\{x_1,\ldots,x_n\}/k} \otimes_{k\{x_1,\ldots,x_n\}} k\{y_1,\ldots,y_n\}[\frac{1}{\Delta}] \to L_{k\{y_1,\ldots,y_n\}[\frac{1}{\Delta}]/k}.$$

This is a map of free modules of rank n, which is given on π_0 by the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1\leq i,j\leq n}$. Since this matrix is invertible in $\pi_0 k\{y_1,\ldots,y_n\}[\frac{1}{\Delta}]$, we deduce that the relative cotangent complex of ϕ_0 vanishes, so that ϕ_0 is étale by Lemma 8.9.

We now prove that (1) $\Rightarrow$ (2). Suppose that ϕ is étale. The structure theorem for étale morphisms of ordinary commutative rings implies the existence of an isomorphism

$$\pi_0 B \simeq (\pi_0 A)[y_1,\ldots,y_m]/(f_1,\ldots,f_m),$$

such that the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1\leq i,j\leq m}$ is invertible in $\pi_0 B$. Let $\{a_i \in \pi_0 A\}_{1\leq i\leq k}$ be the nonzero coefficients appearing in the polynomials f_i. Choose a commutative diagram

$$\begin{array}{ccc} k\{x_1,\ldots,x_k\} & \xrightarrow{g_0} & A \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ k\{x_1,\ldots,x_k,y_1,\ldots,y_m\} & \xrightarrow{g_1} & B \end{array}$$

where g_0 carries each x_i to $a_i \in \pi_0 A$. For each $1 \leq i \leq m$, choose a polynomial

$$\overline{f}_i \in (\pi_0 k)[y_1,\ldots,y_m,x_1,\ldots,x_k]$$

lifting f_i, so that $g_1(\overline{f}_i) = 0 \in \pi_0 B$. Let $\Delta \in (\pi_0 k)[y_1,\ldots,y_m,x_1,\ldots,x_k]$ be the determinant of the Jacobian matrix $[\frac{\partial \overline{f}_i}{\partial y_j}]_{1\leq i,j\leq n}$. It follows that there exists a commutative diagram Using Corollary A.7.5.4.6, we deduce the existence of a commutative diagram

$$\begin{array}{ccccc} k\{x_1,\ldots,x_k,z_1,\ldots,z_m\} & \xrightarrow{\epsilon} & k\{x_1,\ldots,x_k\} & \longrightarrow & A \\ \downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle \phi} \\ k\{x_1,\ldots,x_k,y_1,\ldots,y_m,\}[\frac{1}{\Delta}] & & \xrightarrow{\quad g_1 \quad} & & B \end{array}$$

where $h(z_i) = \overline{f}_i$ and $\epsilon(z_i) = 0$ for $1 \leq i \leq m$. We claim that the outer square appearing in this diagram is a pushout. To see this, form a pushout diagram

$$\begin{array}{ccc} k\{x_1,\ldots,x_k,z_1,\ldots,z_m\} & \longrightarrow & A \\ \downarrow{\scriptstyle \phi_0} & & \downarrow \\ k\{x_1,\ldots,x_k,y_1,\ldots,y_m\}[\frac{1}{\Delta}] & \longrightarrow & B' \end{array}$$

so that we have a canonical map $\psi : B' \to B$; we wish to show that ψ is an equivalence. By construction, $\psi : B' \to B$ induces an isomorphism on connected components. The first part of the proof shows that B' is étale over A, so that $L_{B'/A} \simeq 0$. Since $L_{B/A} \simeq 0$, we conclude that $L_{B/B'} \simeq 0$, so that $B \simeq B'$ by Corollary A.7.4.3.2. □

We are now ready to introduce the analogues of the geometries $\mathcal{G}_{\text{Zar}}^{\text{Sp}}$ and $\mathcal{G}_{\text{Zar}}^{\text{nSp}}$ of §2.

Definition 8.11. Let k be an $\mathbb{E}_\infty$-ring. We define a geometry $\mathcal{G}_{\text{ét}}^{\text{nSp}}(k)$ as follows:

(1) On the level of ∞-categories, we have $\mathcal{G}_{\text{ét}}^{\text{nSp}}(k) = \mathcal{G}_{\text{Zar}}^{\text{nSp}}(k) = \mathcal{G}_{\text{disc}}^{\text{nSp}}(k)$: that is, $\mathcal{G}_{\text{ét}}^{\text{nSp}}(k)$ is the opposite of the ∞-category of compact k-algebras. If A is a compact k-algebra, we let $\operatorname{Spec} A$ denote the corresponding object of $\mathcal{G}_{\text{ét}}^{\text{nSp}}(k)$.

(2) A morphism $f : \operatorname{Spec} A \to \operatorname{Spec} B$ in $\mathcal{G}_{\text{ét}}^{\text{nSp}}(k)$ is *admissible* if the underlying map of $\mathbb{E}_\infty$-rings $B \to A$ is étale.

(3) A collection of admissible morphisms $\{\operatorname{Spec} A_\alpha \to \operatorname{Spec} A\}$ generates a covering sieve on $\operatorname{Spec} A$ if and only if, for every prime ideal $\mathfrak{p} \subseteq \pi_0 A$, there exists an index α such that $\mathfrak{p}(\pi_0 A_\alpha) \neq \pi_0 A_\alpha$.

If k is the sphere spectrum (regarded as an initial object of CAlg), then we will denote the geometry $\mathcal{G}_{\text{ét}}^{\text{nSp}}(k)$ by $\mathcal{G}_{\text{ét}}^{\text{nSp}}$.

Remark 8.12. The condition appearing in (3) of Definition 8.11 is equivalent to the requirement that the map $\coprod \operatorname{Spec}(\pi_0 A_\alpha) \to \coprod \operatorname{Spec}(\pi_0 A)$ is a surjection of topological spaces (endowed with the Zariski topology). It is also equivalent to the requirement that there exist a finite collection of indices $\{\alpha_1, \ldots, \alpha_n\}$ such that the product map $\pi_0 A \to \prod_{1 \leq i \leq n} \pi_0 A_{\alpha_i}$ is a faithfully flat map of commutative rings.

The role of the geometry $\mathcal{G}_{\text{ét}}^{\text{nSp}}$ is summarized by the following result, which we will prove at the end of this section:

Proposition 8.13. *Let $\mathcal{X}$ be an ∞-topos and let*

$$\theta : \operatorname{Fun}^{\text{lex}}(\mathcal{G}_{\acute{e}t}^{\text{nSp}}(k), \mathcal{X}) \simeq \mathcal{S}\text{hv}_{\text{CAlg}_k}(\mathcal{X})$$

be the equivalence of Remark 2.17. A left exact functor $\mathcal{O} : \mathcal{G}_{\acute{e}t}^{\text{nSp}}(k) \to \mathcal{X}$ is a $\mathcal{G}_{\acute{e}t}^{\text{nSp}}(k)$-structure on $\mathcal{X}$ if and only if the underlying sheaf of $\mathbb{E}_\infty$-rings of $\theta(\mathcal{O})$ is strictly Henselian, in the sense of Definition 8.1. A natural transformation $\alpha : \mathcal{O} \to \mathcal{O}'$ in $\operatorname{Fun}^{\text{lex}}(\mathcal{G}_{\acute{e}t}^{\text{nSp}}(k), \mathcal{X})$ is local if and only if $\theta(\alpha)$ determines a local map between sheaves of $\mathbb{E}_\infty$-rings, in the sense of Definition 2.5. In particular, we have an equivalence of ∞-categories

$$\operatorname{RingTop}_{\acute{e}t} \simeq {}^{\text{L}}\mathcal{T}\text{op}(\mathcal{G}_{\acute{e}t}^{\text{nSp}})^{op}.$$

Proposition 8.13 implies that if $\mathcal{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on an ∞-topos $\mathcal{X}$, then the condition that the corresponding left exact functor $\mathcal{G}_{\text{ét}}^{\text{nSp}} \to \mathcal{X}$ be a $\mathcal{G}_{\text{ét}}^{\text{nSp}}$-structure depends only on the underlying sheaf of commutative rings $\pi_0 \mathcal{O}$. In particular, it is insensitive to operations like replacing $\mathcal{O}$ by its connective cover, and by replacing that connective cover by its truncations. To study these operations, we introduce a variant of the geometry $\mathcal{G}_{\text{ét}}^{\text{nSp}}$.

Definition 8.14. Let k be a connective $\mathbb{E}_\infty$-ring. We let $\mathcal{G}_{\text{ét}}^{\text{Sp}}(k)$ denote the full category of $\mathcal{G}_{\text{ét}}^{\text{nSp}}(k)$ spanned by objects of the form $\operatorname{Spec} A$, where A is a connective compact k-algebra. We regard $\mathcal{G}_{\text{ét}}^{\text{Sp}}(k)$ as a geometry, where a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is admissible if the underlying map of $\mathbb{E}_\infty$-algebras $B \to A$ is étale, and a collection of admissible morphisms $\{\operatorname{Spec} A_\alpha \to \operatorname{Spec} A\}$ generates a covering sieve on $\operatorname{Spec} A$ in $\mathcal{G}_{\text{ét}}^{\text{Sp}}(k)$ if and only if it generates a covering sieve in $\mathcal{G}_{\text{ét}}^{\text{nSp}}(k)$. If k is the sphere spectrum (regarded as an initial object of CAlg), then we will denote the geometry $\mathcal{G}_{\text{ét}}^{\text{Sp}}(k)$ by $\mathcal{G}_{\text{ét}}^{\text{Sp}}$.

Remark 8.15. Let k be an $\mathbb{E}_\infty$-ring. Every admissible morphism (admissible cover) in $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ is also an admissible morphism (admissible cover) in $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$. In other words, the identity map $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k) \to \mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$ is a transformation of geometries. If k is connective, then the analogous assertions hold for $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ and $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$.

Remark 8.16. Let k be a connective $\mathbb{E}_\infty$-ring, and let $\mathcal{G}_{\text{ét}}(\pi_0 k)$ be the geometry introduced in §V.2.6. The truncation functor $A \mapsto \pi_0 A$ determines a transformation of geometries $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k) \to \mathcal{G}_{\text{ét}}(\pi_0 k)$, which exhibits $\mathcal{G}_{\text{ét}}(\pi_0 k)$ as a 0-stub of $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$.

We now compare the geometries $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$ and $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$.

Proposition 8.17. *Let $\phi : k \to k'$ be a map of $\mathbb{E}_\infty$-rings, where k is connective, so that base change along ϕ induces a transformation of geometries $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k) \to \mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k')$. Then:*

(1) *Every admissible morphism in $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k')$ is a pullback of the image of an admissible morphism in $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$.*

(2) *The Grothendieck topology on $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k')$ is generated by the Grothendieck topology on $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$ (for a more precise statement, see the proof below).*

Proof. Assertion (1) follows immediately from Proposition 8.10. To prove (2), let $\mathcal{G}$ denote the ∞-category $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}(k')$, endowed with an arbitrary Grothendieck topology. Suppose that every admissible covering in $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$ determines a covering sieve in $\mathcal{G}$. We wish to show that every admissible covering $\{f_\alpha : \operatorname{Spec} A_\alpha \to \operatorname{Spec} A\}$ generates a covering sieve in $\mathcal{G}$. We may assume without loss of generality that the set of indices α is finite. Using (1), we deduce that each of the maps f_α fits into a pullback diagram

$$\begin{array}{ccc} \operatorname{Spec} A_\alpha & \xrightarrow{f_\alpha} & \operatorname{Spec} A \\ \downarrow & & \downarrow{\scriptstyle g_\alpha} \\ \operatorname{Spec} B'_\alpha & \xrightarrow{f'_\alpha} & \operatorname{Spec} B_\alpha, \end{array}$$

where f'_α is an admissible morphism in $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$. Replacing each B_α with the tensor product $B = \otimes_\alpha B_\alpha$ (taken over the $\mathbb{E}_\infty$-ring k), we may assume that the underlying map $g_\alpha : B_\alpha \to A$ is independent of α. Let $g_0 : \pi_0 B \to \pi_0 A$ be the induced map of commutative rings.

Let $X = \operatorname{Spec}^{\mathrm{Z}} B$. Each of the maps $\pi_0 B \to \pi_0 B'_\alpha$ is étale, so that $\operatorname{Spec}^{\mathrm{Z}} B'_\alpha$ has open image $U_\alpha \subseteq X$ (Proposition 0.2). Since the maps f_α cover A, we deduce that the Zariski spectrum of A has image contained in $\bigcup_\alpha U_\alpha$. It follows that there exists a finite collection of elements $b_1, \ldots, b_n \in \pi_0 B$ with the following properties:

(i) The images $a_i = g_0(b_i)$ generate the unit ideal in $\pi_0 A$.

(ii) For $1 \leq i \leq n$, there exists an index α_i such that the basic open set $V_i = \{\mathfrak{p} \subseteq \pi_0 B : b_i \notin \mathfrak{p}\}$ is contained in U_{α_i}.

Condition (i) implies that the morphisms $\{\operatorname{Spec} A[\frac{1}{a_i}] \to \operatorname{Spec} A\}$ form an admissible covering with respect to the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k')$, and therefore generate a covering sieve with respect to the topology on $\mathcal{G}$ (see Remarks 2.15 and 2.16). Consequently, it will suffice to show that, for each index $1 \leq i \leq n$, the pullback maps $\{f^i_\alpha : \operatorname{Spec} A_\alpha[\frac{1}{a_i}] \to \operatorname{Spec} A[\frac{1}{a_i}]\}$ generate a $\mathcal{G}$-covering sieve on $\operatorname{Spec} A[\frac{1}{a_i}]$ (here we abuse notation by identifying $a_i \in \pi_0 A$ with its image in $\pi_0 A_\alpha$). Let α_i be as in (ii). We claim that the single morphism $f^i_{\alpha_i} : \operatorname{Spec} A_{\alpha_i}[\frac{1}{a_i}] \to \operatorname{Spec} A[\frac{1}{a_i}]$ generates a $\mathcal{G}$-covering sieve. To prove this, we observe that $f^i_{\alpha_i}$ is a pullback of the map $\operatorname{Spec} B'_{\alpha_i}[\frac{1}{b_i}] \to \operatorname{Spec} B[\frac{1}{b_i}]$, which is a $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$-covering by construction. □

Corollary 8.18. *Let k be a connective $\mathbb{E}_\infty$-ring, and let $\mathcal{X}$ be an ∞-topos. Then:*

(1) *A left-exact functor $\mathcal{O} : \mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k) \to \mathcal{X}$ is a $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$-structure on $\mathcal{X}$ if and only if $\mathcal{O} \,|\, \mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$ is a $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$-structure on $\mathcal{X}$.*

(2) *Let* $\mathcal{O}, \mathcal{O}' : \mathcal{G}_{ét}^{\mathrm{nSp}}(k) \to \mathcal{X}$ *be* $\mathcal{G}_{ét}^{\mathrm{nSp}}(k)$*-structures on* $\mathcal{X}$*. A natural transformation* $\alpha : \mathcal{O} \to \mathcal{O}'$ *is local if and only if the induced map* $\mathcal{O} \,|\, \mathcal{G}_{ét}^{\mathrm{Sp}}(k) \to \mathcal{O} \,|\, \mathcal{G}_{ét}^{\mathrm{Sp}}(k)$ *is local.*

Corollary 8.19. *Let* k *be an* $\mathbb{E}_\infty$*-ring, and let* $f : \mathcal{G}_{ét}^{\mathrm{nSp}} \to \mathcal{G}_{ét}^{\mathrm{nSp}}(k)$ *be the transformation of geometries induced by the map* $S \to k$*, where* S *denotes the sphere spectrum. Then:*

(1) *A left-exact functor* $\mathcal{O} : \mathcal{G}_{ét}^{\mathrm{nSp}}(k) \to \mathcal{X}$ *is a* $\mathcal{G}_{ét}^{\mathrm{nSp}}(k)$*-structure on* $\mathcal{X}$ *if and only if* $\mathcal{O} \circ f$ *is a* $\mathcal{G}_{ét}^{\mathrm{nSp}}$*-structure on* $\mathcal{X}$*.*

(2) *Let* $\mathcal{O}, \mathcal{O}' : \mathcal{G}_{ét}^{\mathrm{nSp}}(k) \to \mathcal{X}$ *be* $\mathcal{G}_{ét}^{\mathrm{nSp}}(k)$*-structures on* $\mathcal{X}$*. A natural transformation* $\alpha : \mathcal{O} \to \mathcal{O}'$ *is local if and only if the induced map* $\mathcal{O} \circ f \to \mathcal{O}' \circ f$ *is local.*

Remark 8.20. If k is connective, then f restricts to a transformation of geometries $f_0 : \mathcal{G}_{\text{ét}}^{\mathrm{Sp}} \to \mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$. We have the following analogous results:

(1′) A left-exact functor $\mathcal{O} : \mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k) \to \mathcal{X}$ is a $\mathcal{G}_{\mathrm{Zar}}^{\mathrm{nSp}}(k)$-structure on $\mathcal{X}$ if and only if $\mathcal{O} \circ f_0$ is a $\mathcal{G}_{\text{ét}}^{\mathrm{Sp}}$-structure on $\mathcal{X}$.

(2′) Let $\mathcal{O}, \mathcal{O}' : \mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k) \to \mathcal{X}$ be $\mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$-structures on $\mathcal{X}$. A natural transformation $\alpha : \mathcal{O} \to \mathcal{O}'$ is local if and only if the induced map $\mathcal{O} \circ f_0 \to \mathcal{O}' \circ f_0$ is local.

Our next goal is to show that when k is connective, the geometry $\mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$ can be obtained as the geometric envelope of a pregeometry $\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k)$. To define this pregeometry, we need a few remarks about the notion of a *smooth* morphism between $\mathbb{E}_\infty$-rings.

Definition 8.21. Let k be a connective $\mathbb{E}_\infty$-ring. Let $\phi : \widetilde{B} \to B$ be a map of connective k-algebras. We will say that ϕ is a *nilpotent thickening* if the underlying map of commutative rings $(\pi_0 \phi) : \pi_0 \widetilde{B} \to \pi_0 B$ is surjective, and the ideal $I = \ker(\pi_0 \phi)$ is nilpotent.

Let $F : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ be a (space-valued) functor on the ∞-category of connective k-algebras. We will say that F is *formally smooth* if, for every nilpotent thickening $\widetilde{B} \to B$, the induced map $F(\widetilde{B}) \to F(B)$ is surjective.

Proposition 8.22. *Let* $k \to A$ *be a morphism of connective* $\mathbb{E}_\infty$*-algebras, and let* $F : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ *be the functor corepresented by* A *(given informally by* $B \mapsto \mathrm{Map}_{\mathrm{CAlg}_k}(A, B)$*. The following conditions are equivalent:*

(1) *The functor* F *is formally smooth, in the sense of Definition 8.21.*

(2) *The relative cotangent complex* $L_{A/k}$ *is a projective* A*-module (see Definition A.7.2.2.4).*

Proof. Assume first that F is formally smooth; we wish to show that $L_{A/k}$ is projective. In view of Proposition A.7.2.2.6, it will suffice to show that for every cofiber sequence $N' \to N \to N''$ of connective A-modules, the induced map $\phi : \mathrm{Map}_{\mathrm{Mod}_A}(L_{A/k}, N) \to \mathrm{Map}_{\mathrm{Mod}_A}(L_{A/k}, N'')$ is surjective on connected components: in other words, we wish to show that the homotopy fibers of ϕ are nonempty. Fix a map $L_{A/k} \to N''$, corresponding to a section of s the projection map $A \oplus N'' \to A$. Invoking the definition of $L_{A/k}$, we see that the homotopy fiber of ϕ over s can be identified with the homotopy fiber of the map $\phi' : F(A \oplus N) \to F(A \oplus N'')$. We now observe that the map $A \oplus N \to A \oplus N''$ is a nilpotent thickening, so that the homotopy fibers of ϕ' are nonempty by virtue of (1).

Now suppose that $L_{A/k}$ is projective. We wish to prove that F is formally smooth. Let $\widetilde{B} \to B$ be a nilpotent thickening and let $\eta \in F(B)$; we wish to show that η can be lifted to a point in $F(\widetilde{B})$. We define a tower of $\widetilde{B}$-algebras

$$\ldots \to B(2) \to B(1) \to B(0) = B$$

Assume that $B(i)$ has been constructed, and let $M(i) = L_{B(i)/\widetilde{B}}$. By construction, we have a derivation $d : B(i) \to B(i) \oplus M(i)$, fitting into a commutative diagram

$$\begin{array}{ccc} \widetilde{B} & \longrightarrow & B(i) \\ \downarrow & & \downarrow {\scriptstyle d_0} \\ B(i) & \xrightarrow{d} & B(i) \oplus M(i); \end{array}$$

here d_0 denotes the trivial derivation. We now define $B(i+1)$ to be the fiber product $B(i) \times_{B(i)\oplus M(i)} B(i)$.

Let $I \subseteq \pi_0 \widetilde{B}$ be the kernel of the surjection $\pi_0 \widetilde{B} \to \pi_0 B$. We next claim:

- $(*)$ For every integer $n \geq 0$, the algebra $B(n)$ is connective. Moreover, the map $\pi_0 \widetilde{B} \to \pi_0 B(n)$ is a surjection, whose kernel is the ideal I^{2^n}.

The proof of $(*)$ proceeds by induction on n. Assume that $(*)$ holds for $B(n)$, and let K denote the fiber of the map $\widetilde{B} \to B(n)$. Condition $(*)$ guarantees that K is connective, and that the image of the map $\pi_0 K \to \pi_0 \widetilde{B}$ is the ideal $J = I^{2^n}$. We have a map of fiber sequences

$$\begin{array}{ccccc} K & \longrightarrow & \widetilde{B} & \longrightarrow & B(n) \\ \downarrow & & \downarrow & & \downarrow \\ M(n)[-1] & \longrightarrow & B(n+1) & \longrightarrow & B(n), \end{array}$$

so the fiber K' of the map $\widetilde{B} \to B(n+1)$ can be identified with the fiber of the composition

$$K \xrightarrow{\beta} K \otimes_{\widetilde{B}} B(n) \xrightarrow{\alpha} M(n)[-1].$$

To prove $(*)$, it will suffice to show that K' is connective and the image of the map $\pi_0 K' \to \pi_0 \widetilde{B}$ is J^2. We have a fiber sequence

$$\mathrm{fib}(\beta) \to K' \to \mathrm{fib}(\alpha).$$

Since K is connective, Theorem A.7.4.3.1 guarantees that $\mathrm{fib}(\alpha)$ is 1-connective. It follows that the maps $\pi_i \,\mathrm{fib}(\beta) \to \pi_i K'$ are surjective for $i \leq 0$. To complete the proof, it will therefore suffice to show that $\mathrm{fib}(\beta)$ is connective and the map $\pi_0 \,\mathrm{fib}(\beta) \to \pi_0 \widetilde{B}$ has image J^2. This follows from the observation that $\mathrm{fib}(\beta) \simeq K \times_{\widetilde{B}} K$, so that $\pi_0 \,\mathrm{fib}(\beta) \simeq \mathrm{Tor}_0^{\pi_0 \widetilde{B}}(\pi_0 K, \pi_0 K)$. Under this identification, the map $\pi_0 \,\mathrm{fib}(\beta) \to \pi_0 \widetilde{B}$ corresponds to the bilinear multiplication map

$$\pi_0 K \times \pi_0 K \to J \times J \to \pi_0 \widetilde{B},$$

whose image generates the ideal $J^2 \subseteq \pi_0 \widetilde{B}$.

Choose any map of k-algebras $A \to B(n)$. Since $L_{A/k}$ is projective, the space $\mathrm{Map}_{\mathrm{Mod}_A}(L_{A/k}, M(n))$ is connected. It follows that the homotopy fibers of the projection map $F(B(n) \oplus M(n)) \to F(B(n))$ are connected. Consequently, for any derivation $d : B(n) \to B(n) \oplus M(n)$, the homotopy fibers of the induced section $F(B(n)) \to F(B(n) \oplus M(n))$ are nonempty. It follows that the homotopy fibers of the pullback map $F(B(n+1)) \to F(B(n))$ are also nonempty: in other words, every point of $\eta_n \in F(B(n))$ can be lifted to a point $\eta_{n+1} \in F(B(n+1))$. Consequently, we are free to replace the pair (B, η_0) with $(B(n), \eta_n)$ for any $n \geq 0$. Since I is nilpotent, condition $(*)$ implies that the map $\pi_0 \widetilde{B} \to \pi_0 B(n)$ is bijective for $n \gg 0$. Replacing B by $B(n)$, we can reduce to the case where $\pi_0 \widetilde{B} \to \pi_0 B$ is an isomorphism.

Let $M = \tau_{\leq 1} L_{B/\widetilde{B}}$, let $d : B \to B \oplus M$ be the canonical derivation, and form a pullback diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & & \downarrow {\scriptstyle d} \\ B & \xrightarrow{d_0} & B \oplus M. \end{array}$$

Repeating the above arguments, we deduce that M is 1-connective, so that $F(B') \to F(B)$ is surjective. We claim that the canonical map $\widetilde{B} \to B'$ is 1-connective. Let N denote the kernel of the projection $\widetilde{B} \to B$, so that we have a map of fiber sequences

$$\begin{array}{ccccc} N & \longrightarrow & \widetilde{B} & \longrightarrow & B \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\ M[-1] & \longrightarrow & B' & \longrightarrow & B. \end{array}$$

To show that $\widetilde{B} \to B'$ is 1-connective, it will suffice to show that f is 1-connective; that is, f induces a bijection $\pi_0 N \to \pi_1 M$ and a surjection $\pi_1 N \to \pi_2 M$. The second assertion is clear, since $\pi_2 M \simeq 0$ by construction. For the first, we factor $\pi_0 f$ as a composition

$$\pi_0 N \xrightarrow{\beta} \pi_0(N \otimes_{\widetilde{B}} B) \simeq \pi_0 N \otimes_{\pi_0 \widetilde{B}} (\pi_0 B) \xrightarrow{\beta'} \pi_1 L_{B/\widetilde{B}} \xrightarrow{\beta''} \pi_1 M.$$

The map β is an isomorphism because $\pi_0 \widetilde{B} \simeq \pi_0 B$, the map β' is an isomorphism by Theorem A.7.4.3.1, and the map β'' is an isomorphism by construction. Replacing B by B', we can assume that $\widetilde{B} \to B$ is 1-connective.

We now repeat the original construction of the tower

$$\ldots \to B(2) \to B(1) \to B(0)$$

and prove the following strengthening of $(*)$:

$(*')$ For $n \geq 0$, the map $\widetilde{B} \to B(n)$ is 2^n-connective.

The proof of $(*')$ proceeds by induction on n, the case $n = 1$ being obvious. Assume therefore that $\widetilde{B} \to B(n)$ is 2^n-connective, and let K and K' be as in the proof of $(*)$. We wish to prove that K' is 2^{n+1}-connective. As before, we have a fiber sequence

$$\mathrm{fib}(\alpha) \to K' \to \mathrm{fib}(\beta).$$

Here $\mathrm{fib}(\beta) \simeq K \otimes_{\widetilde{B}} K$, and is therefore 2^{n+1}-connective since K is 2^n-connective by the inductive hypothesis. The map α is $(2^{n+1}+1)$-connective by Theorem A.7.4.3.1.

As before, each of the maps $F(B(n+1)) \to F(B(n))$ is surjective on connected components, so we can lift η to a point of $\varprojlim F(B(n)) \simeq F(\varprojlim B(n))$. To complete, the proof, it suffices to show that the canonical map $\widetilde{B} \to \varprojlim B(n)$ is an equivalence. This follows from $(*)$, since Postnikov towers of connective k-algebras are convergent (Proposition A.7.1.3.19). $\square$

Proposition 8.23. *Let $k \to A$ be a morphism of connective $\mathbb{E}_\infty$-rings. Assume that $\pi_0 A$ is a finitely generated algebra over $\pi_0 k$. The following conditions are equivalent:*

1. *The functor $\mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ corepresented by A is formally smooth.*
2. *The relative cotangent complex $L_{A/k}$ is a projective A-module.*
3. *The relative cotangent complex $L_{A/k}$ is a finitely generated projective A-module.*
4. *There exist elements $a_1, \ldots, a_n \in \pi_0 A$ which generate the unit ideal and a collection of étale maps $k\{x_1, \ldots, x_{m_i}\} \to A[\frac{1}{a_i}]$.*

Moreover, if these conditions are satisfied, then A satisfies the following:

5. *The algebra A is locally of finite presentation over k.*

(6) *The underlying commutative ring $\pi_0 A$ is a smooth algebra over $\pi_0 k$, in the sense of classical commutative algebra.*

Proof. The implication (3) $\Rightarrow$ (2) is obvious; the converse follows from the observation that $\pi_0 L_{A/k}$ is the module of Kähler differentials of $\pi_0 A$ over $\pi_0 k$ (Proposition A.7.4.3.9) and therefore finitely presented over $\pi_0 A$. The equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 8.22. The implication (1) $\Rightarrow$ (6) is obvious, and the implication (3) $\Rightarrow$ (5) follows from Theorem A.7.4.3.18. We will complete the proof by showing that (4) $\Leftrightarrow$ (3).

Assume first that (4) is satisfied. Note that $L_{A/k}$ is a finitely generated projective A-module if and only if $\pi_0 L_{A/k}$ is a finitely generated projective module over $\pi_0 A$, and each of the induced maps

$$\mathrm{Tor}_0^{\pi_0 A}(\pi_0 L_{A/k}, \pi_j A) \to \pi_j L_{A/k}$$

is an isomorphism. Since the elements $a_i \in \pi_0 A$ generate the unit ideal, it will suffice to show that each $(\pi_0 L_{A/k})[\frac{1}{a_i}]$ is a finitely generated projective module over $(\pi_0 A)[\frac{1}{a_i}]$, and that the induced maps

$$\mathrm{Tor}_0^{(\pi_0 A)[\frac{1}{a_i}]}((\pi_0 A L_{A/k})[\frac{1}{a_i}], (\pi_j A)[\frac{1}{a_i}]) \to (\pi_j L_{A/k})[\frac{1}{a_i}].$$

Each of the algebras $A[\frac{1}{a_i}]$ is étale over A, so that $L_{A[\frac{1}{a_i}]/k} \simeq L_{A/k} \otimes_A A[\frac{1}{a_i}]$. Consequently, we may replace A by $A[\frac{1}{a_i}]$ and thereby reduce to proving (3) in the case where we have an étale map $k\{x_1, \ldots, x_m\} \to A$. In this case, we have

$$L_{A/k} \simeq L_{k\{x_1,\ldots,x_m\}/k} \otimes_{k\{x_1,\ldots,x_m\}} A \simeq A^m$$

and the result is obvious.

Converely, suppose that (3) is satisfied; we will prove (4). The module $\pi_0 L_{A/k}$ is projective and of finite rank over $\pi_0 A$. Consequently, there exist elements $a_1, \ldots, a_n$ generating the unit ideal in $\pi_0 A$ such that each of the modules $(\pi_0 L_{A/k})[\frac{1}{a_i}]$ is a free module of some rank m_i over $(\pi_0 A)[\frac{1}{a_i}]$. Replacing A by $A[\frac{1}{a_i}]$, we may suppose that $\pi_0 L_{A/k}$ is a free module of some rank m. Proposition A.7.4.3.9 allows us to identify $\pi_0 L_{A/k}$ with the module of Kähler differentials of $\pi_0 A$ over $\pi_0 k$. In particular, $\pi_0 L_{A/k}$ is generated (as an A-module) by finitely many differentials $\{dx_p\}_{1 \leq p \leq q}$. The identification $\pi_0 L_{A/k} \simeq (\pi_0 A)^m$ allows us to view the differentials $\{dx_q\}_{1 \leq p \leq q}$ as an m-by-q matrix M. Let $\{b_j\}$ be the collection of all determinants of m-by-m square submatrices appearing in M. Since the elements $\{dx_p\}_{1 \leq p \leq q}$ generate $(\pi_0 A)^m$, the matrix M has rank m so that the elements b_j generate the unit ideal in A. It therefore suffices to prove that (4) is satisfied by each of the algebras $A[\frac{1}{b_j}]$. We may therefore assume (after discarding some of the elements x_i) that $q = m$ and that $\pi_0 L_{A/k}$ is freely generated by the elements dx_i. The choice of elements $x_1, \ldots, x_m \in \pi_0 A$ determines a map $k\{x_1, \ldots, x_m\} \to A$. The fiber sequence

$$L_{k\{x_1,\ldots,x_m\}/k} \otimes_{k\{x_1,\ldots,x_m\}} A \to L_{A/k} \to L_{A/k\{x_1,\ldots,x_m\}}$$

shows that the relative cotangent complex $L_{A/k\{x_1,\ldots,x_m\}}$ vanishes, so that A is étale over $k\{x_1, \ldots, x_m\}$ by Lemma 8.9. □

Definition 8.24. Let k be a connective $\mathbb{E}_\infty$-ring. We will say that a k-algebra A is *smooth over k* if it satisfies the equivalent conditions of Proposition 8.23: that is, if A is connective, formally smooth over k, and $\pi_0 A$ is finitely generated over $\pi_0 k$.

Warning 8.25. A smooth morphism of connective $\mathbb{E}_\infty$-rings $k \to A$ need not be flat (in contrast with the situation in classical algebraic geometry).

We now organize the smooth k-algebras into a pregeometry.

Definition 8.26. Let k be a connective $\mathbb{E}_\infty$-ring. We let $\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k)$ denote the full subcategory of $\mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$ spanned by the objects of the form $\mathrm{Spec}\, A$ for which there exists an étale morphism of k-algebras

$$k\{x_1, \ldots, x_n\} \to A,$$

for some $n \geq 0$. We regard $\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k)$ as a pregeometry as follows:

(1) A morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ in $\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k)$ is admissible if and only if the corresponding map of k-algebras $B \to A$ is étale.

(2) A collection of admissible morphisms $\{\operatorname{Spec} A_\alpha \to \operatorname{Spec} A\}$ in $\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k)$ generates a covering sieve on $\operatorname{Spec} A$ if and only if it generates a covering sieve in $\mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$: that is, if and only if there exists a finite set of indices $\{\alpha_i\}_{1 \leq i \leq n}$ such that the induced map $A \to \prod_{1 \leq i \leq n} A_{\alpha_i}$ is faithfully flat.

If k is the sphere spectrum (regarded as an initial object of CAlg), then we will denote the pregeometry $\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k)$ by $\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}$.

Variant 8.27. If k is a connective $\mathbb{E}_\infty$-ring, we let $\mathcal{T}_{\text{ét}}'^{\mathrm{Sp}}(k)$ denote the full subcategory of $\mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$ spanned by those objects of the form $\operatorname{Spec} A$ where A is a smooth k-algebra. We regard $\mathcal{T}_{\text{ét}}'^{\mathrm{Sp}}(k)$ as a pregeometry, using the admissible morphisms and admissible coverings in $\mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$, as in Definition 8.26. Using condition (4) of Proposition 8.23 and Proposition V.3.2.8, we deduce that the inclusion $\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k) \subseteq \mathcal{T}_{\text{ét}}'^{\mathrm{Sp}}(k)$ is a Morita equivalence of pregeometries. In particular, for any ∞-topos $\mathcal{X}$, the restriction map

$$\operatorname{Str}^{\mathrm{loc}}_{\mathcal{T}_{\text{ét}}'^{\mathrm{Sp}}(k)}(\mathcal{X}) \to \operatorname{Str}^{\mathrm{loc}}_{\mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k)}(\mathcal{X})$$

is an equivalence of ∞-categories.

Proposition 8.28. *Let k be a connective $\mathbb{E}_\infty$-ring. The inclusion $\mathcal{T}_{\acute{e}t}^{\mathrm{Sp}}(k) \subseteq \mathcal{G}_{\acute{e}t}^{\mathrm{Sp}}(k)$ exhibits $\mathcal{G}_{\acute{e}t}^{\mathrm{Sp}}(k)$ as a geometric envelope of $\mathcal{T}_{\acute{e}t}^{\mathrm{Sp}}(k)$.*

Proof. As in the proof of Proposition 2.20, we let $\mathcal{T}_0$ denote the full subcategory of $\mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$ spanned by objects of the form $k\{x_1, \ldots, x_n\}$; we will show that the inclusion $\mathcal{T}_0 \subseteq \mathcal{G}_{\text{ét}}^{\mathrm{Sp}}(k)$ satisfies conditions (1) through (6) of Proposition V.3.4.5. Conditions (1), (2) and (3) follow as in the proof of Proposition 2.20, and assertions (4) and (5) are proven as in Proposition 8.17. To verify (6), let us suppose that $\mathcal{C}$ is an idempotent complete ∞-category which admits finite limits and that $\alpha : f \to f'$ is a natural transformation between admissible functors $f, f' : \mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k) \to \mathcal{C}$ which induces an equivalence $f|\,\mathcal{T}_0 \simeq f'|\,\mathcal{T}_0$. We wish to prove that α is an equivalence.

Fix an object of $\operatorname{Spec} A \in \mathcal{T}_{\text{ét}}^{\mathrm{Sp}}(k)$ corresponding to a smooth k-algebra A for which there exists an étale map $k\{z_1, \ldots, z_m\} \to A$. Using Proposition 8.10, we deduce the existence of a pushout diagram of k-algebras

$$\begin{array}{ccc} k\{x_1, \ldots, x_n\} & \longrightarrow & k\{z_1, \ldots, z_m\} \\ \downarrow{\scriptstyle\phi} & & \downarrow \\ k\{y_1, \ldots, y_n\}[\frac{1}{\Delta}] & \longrightarrow & A \end{array}$$

where $\phi(x_i) = f_i(y_1, \ldots, y_n)$ and Δ is the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leq i,j \leq n}$. We wish to show that α is an equivalence on A. Since $f, f' : \mathcal{T}_{\text{ét}}(k)$ both preserve pullbacks by étale morphisms, it will suffice to show that α is an equivalence on $k\{x_1, \ldots, x_n\}$, $k\{z_1, \ldots, z_m\}$, and $k\{y_1, \ldots, y_n\}[\frac{1}{\Delta}]$. In the first two cases, this is clear (since $f|\,\mathcal{E} = f'|\,\mathcal{E}$); in the third case, it follows from the proof of Proposition 2.20. □

Corollary 8.29. *Let k be a connective $\mathbb{E}_\infty$-ring. For each $n \geq 0$, let $\mathcal{G}_{\acute{e}t}^{\mathrm{Sp}}(k)^{\leq n}$ denote the opposite of the ∞-category of compact objects in the ∞-category $\tau_{\leq n} \mathrm{CAlg}_k^{\mathrm{cn}}$ of connective, n-truncated $\mathbb{E}_\infty$-algebras over k. The composite functor*

$$\mathcal{T}_{\acute{e}t}^{\mathrm{Sp}}(k) \subseteq \mathcal{G}_{\acute{e}t}^{\mathrm{Sp}}(k) \xrightarrow{\tau_{\leq n}} \mathcal{G}_{\acute{e}t}^{\mathrm{Sp}}(k)^{\leq n}$$

exhibits $\mathcal{G}_{\acute{e}t}^{\mathrm{Sp}}(k)^{\leq n}$ as an n-truncated geometric envelope of $\mathcal{T}_{\acute{e}t}(k)$. In particular, the functor $A \mapsto \pi_0 A$ exhibits $\mathcal{G}_{\acute{e}t}(\pi_0 k)$ as a 0-truncated geometric envelope of $\mathcal{T}_{\acute{e}t}^{\mathrm{Sp}}(k)$.

Proof. Combine Proposition 8.28, Lemma V.3.4.11, and Remark 8.16. □

Remark 8.30. Let k be a connective $\mathbb{E}_\infty$-ring and $\mathcal{X}$ an ∞-topos. The proofs of Propositions 2.20 and 8.28 imply that the restriction functors

$$\begin{aligned}\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X}) &\to \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X}) \\ &\to \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X}) \\ &\to \mathrm{Fun}^{\pi}(\mathcal{T}_0, \mathcal{X})\end{aligned}$$

are equivalences of ∞-categories (here $\mathcal{T}_0$ denotes the full subcategory of $\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k)$ spanned by those objects of the form $\operatorname{Spec} k\{x_1, \ldots, x_n\}$, and $\mathrm{Fun}^{\pi}(\mathcal{T}_0, \mathcal{X})$ the full subcategory of $\mathrm{Fun}(\mathcal{T}_0, \mathcal{X})$ spanned by those functors which preserve finite products). Remarks V.1.1.5 and V.1.1.6 allow us to identify the ∞-category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X})$ with the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}^{\mathrm{cn}}_k}(\mathcal{X})$ of sheaves of connective k-algebras on $\mathcal{X}$. In particular, for each $n \geq 0$, we have a truncation functor $\tau_{\leq n} : \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X})$. This induces truncation functors

$$\tau_{\leq n} : \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X}) \to \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X})$$
$$\tau_{\leq n} : \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k), \mathcal{X}) \to \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k), \mathcal{X})$$
$$\tau_{\leq n} : \mathrm{Fun}^{\pi}(\mathcal{T}_0, \mathcal{X}) \to \mathrm{Fun}^{\pi}(\mathcal{T}_0, \mathcal{X}).$$

We claim that each of these truncation functors is simply given by composition with the truncation functor $\tau^{\mathcal{X}}_{\leq n}$ on $\mathcal{X}$. Unwinding the definitions, this amounts to the following assertion:

($*$) Let $\mathcal{O} : \mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k) \to \mathcal{X}$ be a left exact functor, and $\mathcal{O}'$ its n-truncation in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X})$. Then, for every $A \in \mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k)$, the induced map $\mathcal{O}(A) \to \mathcal{O}'(A)$ exhibits $\mathcal{O}'(A)$ as an n-truncation of $\mathcal{O}(A)$ in $\mathcal{X}$.

Note that if $\pi^* : \mathcal{Y} \to \mathcal{X}$ is a geometric morphism and $\mathcal{O} \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{Y})$ satisfies ($*$), then $\pi^* \mathcal{O}$ also satisfies ($*$) (because the induced map $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{Y}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X})$ commutes with n-truncation, by Proposition T.5.5.6.28).

Without loss of generality, we may suppose that $\mathcal{X}$ arises as a left-exact localization of a presheaf ∞-category $\mathcal{P}(\mathcal{C})$. Let $\pi^* : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ be the localization functor, and $\pi_* : \mathcal{X} \to \mathcal{P}(\mathcal{C})$ its right adjoint. Then, for each $\mathcal{O} \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k), \mathcal{X})$, the counit map $\pi^*\pi_* \mathcal{O} \to \mathcal{O}$ is an equivalence. In view of the above remark, it will suffice to prove that $(\mathcal{P}(\mathcal{C}), \pi_* \mathcal{O})$ satisfies ($*$). In particular, we may assume that $\mathcal{X}$ has enough points (given by evaluation at objects of $\mathcal{C}$), and can therefore reduce to the case $\mathcal{X} = \mathcal{S}$. In this case, we can identify $\mathcal{O}$ with a connective k-algebra R, and assertion ($*$) can be reformulated as follows:

($*'$) Let R be a connective k-algebra and let A be a smooth k-algebra. Then the map

$$\mathrm{Map}_{\mathrm{CAlg}_k}(A, R) \to \mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R)$$

exhibits $\mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R)$ as an n-truncation of the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$.

Assertion ($*'$) is equivalent to the requirement that the diagram $\{\mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R)\}$ is a Postnikov tower for $\mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$. Since R is the limit of its Postnikov tower (Proposition A.7.1.3.19), we deduce that $\mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$ is the limit of $\{\mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R)\}$; it therefore suffices to show that the tower $\{\mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R)\}$ *is* a Postnikov tower. In other words, it suffices to show that each of the maps $\phi : \mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n+1} R) \to \mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R)$ exhibits $\mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R)$ as an n-truncation of $\mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n+1} R)$. Since $\mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R)$ is evidently n-truncated, it suffices to show that the homotopy fibers of ϕ are $(n+1)$-connective. According to Theorem A.7.4.1.26, there is a pullback diagram of k-algebras

$$\begin{array}{ccc}\tau_{\leq n+1} R & \longrightarrow & \tau_{\leq n} R \\ \downarrow & & \downarrow \\ \tau_{\leq n} R & \longrightarrow & (\tau_{\leq n} R) \oplus (\pi_{n+1} R)[n+2].\end{array}$$

Consequently, it suffices to show that the homotopy fibers of the map

$$\phi' : \mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R) \to \mathrm{Map}_{\mathrm{CAlg}_k}(A, (\tau_{\leq n} R) \oplus (\pi_{n+1} R)[n+2]).$$

The map ϕ' is a section of the projection map

$$\psi : \mathrm{Map}_{\mathrm{CAlg}_k}(A, (\tau_{\leq n} R) \oplus (\pi_{n+1} R)[n+2]) \to \mathrm{Map}_{\mathrm{CAlg}_k}(A, \tau_{\leq n} R).$$

To complete the proof, we show that ψ has $(n+2)$-connective homotopy fibers. Note that the homotopy fiber of ψ over an algebra map $A \to \tau_{\leq n} R$ is given by the mapping space $\mathrm{Map}_{\mathrm{Mod}_A}(L_{A/k}, (\pi_{n+1} R)[n+2])$, which is $(n+2)$-connective because $L_{A/k}$ is a projective A-module.

We are now ready to give the proof of Proposition 8.13.

Proof of Proposition 8.13. Let $\mathcal{X}$ be an ∞-topos and k an $\mathbb{E}_\infty$-ring. We must prove two assertions:

- (a) Let $\mathcal{O} : \mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k) \to \mathcal{X}$ be a left exact functor, and $\mathcal{O}_0$ the corresponding sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then $\mathcal{O}$ is a $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}$-structure on $\mathcal{X}$ if and only if $\mathcal{O}_0$ is strictly Henselian.
- (b) Let $\alpha : \mathcal{O} \to \mathcal{O}'$ be a map of $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$-structures on $\mathcal{X}$, and $\alpha_0 : \mathcal{O}_0 \to \mathcal{O}'_0$ the corresponding map between sheaves of $\mathbb{E}_\infty$-rings. Then α is a local transformation of $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$-structures if and only if α_0 is a local map of sheaves of $\mathbb{E}_\infty$-rings (in the sense of Definition 2.5).

Let us first prove (a). Using Corollary 8.19, we can reduce to the case where k is the sphere spectrum. In particular, k is connective. Using Corollary 8.18, we can replace $\mathcal{O}_0$ by its connective cover and $\mathcal{O}$ by its restriction to $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}} \subseteq \mathcal{G}^{\mathrm{nSp}}_{\text{ét}}$. Using Proposition 8.28 and Remark 8.30, we can replace $\mathcal{O}_0$ by its 0-truncation $\pi_0 \mathcal{O}_0$, in which case the desired result follows immediately from the definitions. The proof of (b) is similar: using the same arguments, we can replace $\mathcal{O}_0$ and $\mathcal{O}'_0$ by $\pi_0 \mathcal{O}_0$ and $\pi_0 \mathcal{O}'_0$, in which case the desired result follows from Proposition V.2.6.16. □

Remark 8.31. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectrally ringed ∞-topos. If the underlying topos X of discrete objects of $\mathcal{X}$ has enough points, then we can give an even more concrete criterion: the sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O}_{\mathcal{X}}$ is strictly Henselian if and only if, for every point x of the topos X, the stalk $(\pi_0 \mathcal{O}_{\mathcal{X}})_x$ is a strictly Henselian local ring. Moreover, a map of strictly Henselian $\mathbb{E}_\infty$-rings $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}'_{\mathcal{X}}$ is local if and only if, for each point $x \in X$, the induced map $(\pi_0 \mathcal{O}_{\mathcal{X}})_x \to (\pi_0 \mathcal{O}'_{\mathcal{X}})_x$ is a local homomorphism between local commutative rings.

Using Proposition 8.13, we see that Proposition 8.4 is a special case of Theorem V.2.1.1. We can also recast Definition 8.5 in the language of geometries:

Definition 8.32 (Spectral Deligne-Mumford Stack: Abstract Definition)**.** Let k be an $\mathbb{E}_\infty$-ring. A *nonconnective spectral Deligne-Mumford stack* over k is a $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$-scheme (see Definition V.2.3.9), where $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$ is the geometry of Definition 8.11.

If k is connective, a *spectral Deligne-Mumford stack* over k is a $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$-scheme, where $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}$ is the geometry of Definition 8.14.

Remark 8.33. In the special case where k is the sphere spectrum (regarded as an initial object of CAlg), the notion of (nonconnective) spectral Deligne-Mumford stack over k reduces to the notion of (nonconnective) spectral Deligne-Mumford stack introduced in Definition 2.7.

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack over an $\mathbb{E}_\infty$-ring k. We can think of $\mathcal{O}_{\mathcal{X}}$ either as a sheaf of $\mathbb{E}_\infty$-algebras over k, or as a left exact functor $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k) \to \mathcal{X}$. We will generally abuse notation by not distinguishing between these two avatars of $\mathcal{O}_{\mathcal{X}}$: we will whichever point of view is more convenient for the problem at hand.

Definition 8.34. Let k be a connective $\mathbb{E}_\infty$-ring, and let $n \geq 0$. We will say that a connective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *n-truncated* if $\mathcal{O}_{\mathcal{X}}$ is n-truncated, when regarded as a sheaf of spectra on $\mathcal{X}$.

Equivalently, $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is n-truncated if the restriction $\mathcal{O}_{\mathcal{X}} \mid \mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k)$ takes values in the full subcategory of $\mathcal{X}$ spanned by the n-truncated objects.

Definition 8.35. Let k be an $\mathbb{E}_\infty$-ring, and let $n \geq 0$. We will say that a spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is *n-localic* if the ∞-topos $\mathcal{X}$ is n-localic, in the sense of Definition T.6.4.5.8.

The following result shows that the theory of derived Deligne-Mumford stacks really does generalize the classical theory of Deligne-Mumford stacks:

Proposition 8.36. *Let k be a connective $\mathbb{E}_\infty$-ring, and let $\mathrm{Sch}^{\leq 0}_{\leq 1}(\mathcal{G}^{\mathrm{Sp}}_{ét}(k))$ denote the full subcategory of $\mathrm{Sch}(\mathcal{G}^{\mathrm{Sp}}_{ét}(k))$ spanned by those $\mathcal{G}^{\mathrm{Sp}}_{ét}(k)$-schemes which are 0-truncated and 1-localic. Then $\mathrm{Sch}^{\leq 0}_{\leq 1}(\mathcal{G}^{\mathrm{Sp}}_{ét}(k))$ is canonically equivalent to the ∞-category of Deligne-Mumford stacks over the commutative ring $\pi_0 k$ (see Definition V.2.6.9).*

Proof. Combine Corollary 8.29 with Theorem V.2.6.18. □

Our next goal is to compare the theories of connective and nonconnective spectral Deligne-Mumford stacks. The following assertion is an immediate consequence of Proposition 2.30:

Proposition 8.37. *Let k be a connective $\mathbb{E}_\infty$-ring, and let $U : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{ét}(k)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nSp}}_{ét}(k))$ be the relative spectrum functor associated to the inclusion of geometries $\mathcal{G}^{\mathrm{Sp}}_{ét}(k) \hookrightarrow \mathcal{G}^{\mathrm{nSp}}_{ét}(k)$. Then U is a fully faithful embedding, whose essential image consists of those pairs $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{O}_{\mathcal{X}}$ determines a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.*

The proof of Proposition 2.33 immediately yields the following analogue for the étale topology:

Proposition 8.38. *Let k be a connective $\mathbb{E}_\infty$-ring, and let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral Deligne-Mumford stack over k. Then $(\mathcal{X}, \mathcal{O} \mid \mathcal{G}^{\mathrm{Sp}}_{ét}(k))$ is a spectral Deligne-Mumford stack over k.*

Combining Propositions 8.37 and 8.38, we obtain the following result:

Corollary 8.39. *Let k be a connective $\mathbb{E}_\infty$-ring and let U be as in Proposition 8.37. Then U induces a fully faithful embedding $\mathrm{Sch}(\mathcal{G}^{\mathrm{Sp}}_{ét}(k)) \to \mathrm{Sch}(\mathcal{G}^{\mathrm{nSp}}_{ét}(k))$, whose essential image consists of those spectral Deligne-Mumford stacks $(\mathcal{X}, \mathcal{O})$ such that $\mathcal{O}$ determines a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.*

We now discuss the operation of truncation for structure sheaves of spectral Deligne-Mumford stacks.

Proposition 8.40. *Let k be a connective $\mathbb{E}_\infty$-ring. For each $n \geq 0$, the pregeometry $\mathcal{T}^{\mathrm{Sp}}_{ét}(k)$ is compatible with n-truncations.*

Proof. This is a consequence of the corresponding result for the Nisnevich topology which we will prove in a sequel to this paper. □

Remark 8.41. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. We will abuse notation by identifying $\mathcal{O}_{\mathcal{X}}$ with the underlying CAlg-valued sheaf on $\mathcal{X}$. It follows from Proposition 8.38 that the pair $(\mathcal{X}, \tau_{\geq 0}\, \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack; we will refer to $(\mathcal{X}, \tau_{\geq 0}\, \mathcal{O}_{\mathcal{X}})$ as the *underlying spectral Deligne-Mumford stack of* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Using Propositions 8.40 and V.3.4.15, we conclude that for each $n \geq 0$, the pair $(\mathcal{X}, \tau_{\leq n}\tau_{\leq 0}\, \mathcal{O}_{\mathcal{X}})$ is an n-truncated spectral Deligne-Mumford stack. In particular, if $\mathcal{X}$ is 1-localic and we take $n = 0$, then we obtain a 1-localic, 0-truncated Deligne-Mumford stack $(\mathcal{X}, \pi_0\, \mathcal{O}_{\mathcal{X}})$, which we can identify with an ordinary Deligne-Mumford stack (Proposition 8.36). We will refer to $(\mathcal{X}, \pi_0\, \mathcal{O}_{\mathcal{X}})$ as the *underlying ordinary Deligne-Mumford stack* of $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

We conclude this section with a concrete characterization of the class of spectral Deligne-Mumford stacks, analogous the the description of spectral schemes given in Definition 2.2.

Theorem 8.42. *Let $\mathcal{X}$ an ∞-topos, and $\mathcal{O}_{\mathcal{X}}$ a sheaf of $\mathbb{E}_\infty$-algebras on $\mathcal{X}$. Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack if and only if the following conditions are satisfied:*

(1) *Let $\phi_* : \mathcal{X} \to \mathcal{X}'$ be a geometric morphism of ∞-topoi, where $\mathcal{X}'$ is 1-localic and ϕ_* is an equivalence on discrete objects (so that ϕ_* exhibits $\mathcal{X}'$ as the 1-localic reflection of $\mathcal{X}$). Let $\mathcal{O}_{\mathcal{X}'}$ be the commutative ring object in the underlying topos of $\mathcal{X}'$ corresponding to $\pi_0 \mathcal{O}_{\mathcal{X}}$. Then $(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is a spectral Deligne-Mumford stack over k (which is 1-localic and 0-truncated, and can therefore be identified with an ordinary Deligne-Mumford stack X by Proposition 8.36).*

(2) *For every integer i, $\pi_i \mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf on X.*

(3) *The object $\Omega^\infty \mathcal{O}_{\mathcal{X}} \in \mathcal{X}$ is hypercomplete.*

Proof. The proof follows the same lines as that of Theorem 2.40, but is slightly more complicated (because assertions (1) and (2) are not local on $\mathcal{X}$). In what follows, we will abuse notation by identifying $\mathcal{O}_{\mathcal{X}}$ with the corresponding left exact functor $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}} \to \mathcal{X}$. Suppose first that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack. We will prove that (1), (2), and (3) are satisfied. Remark 8.41 implies that $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack, and Corollary 8.29 allows us to identify $\pi_0 \mathcal{O}_{\mathcal{X}}$ with a $\mathcal{G}_{\text{ét}}$-structure on $\mathcal{X}$. Let $\phi_* : \mathcal{X} \to \mathcal{X}'$ be the 1-localic reflection of $\mathcal{X}$; then Theorem V.2.3.13 implies that $(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is again a spectral Deligne-Mumford stack, and that the map $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{X}', _{\mathcal{X}'})$ is étale. This proves (1).

Assertion (3) is local on the ∞-topos $\mathcal{X}$. Consequently, to prove that (3) holds, we may assume that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is affine; the proof then proceeds exactly as in the proof of Theorem 2.40. To prove (2), we consider a collection of objects $\{U_\alpha \in \mathcal{X}\}$ such that $\coprod U_\alpha \to 1_{\mathcal{X}}$ is an effective epimorphism, and each of the $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$-schemes $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_{\mathcal{X}} | U_\alpha)$ is affine, equivalent to $\operatorname{Spec}^{\text{ét}} A_\alpha$ for some $A_\alpha \in \mathrm{CAlg}_k$. The composite geometric morphisms

$$\mathcal{X}_{/U_\alpha} \to \mathcal{X} \to \mathcal{X}'$$

are étale and cover $\mathcal{X}'$. Since assertion (2) is local on $\mathcal{X}'$, it is sufficient to show that the restriction of each $\pi_i \mathcal{O}_{\mathcal{X}}$ to $\mathcal{X}_{/U_\alpha}$ is a quasi-coherent sheaf on the ordinary Deligne-Mumford stack given by $(\mathcal{X}_{/U_\alpha}, \pi_0(\mathcal{O}_{\mathcal{X}} | U_\alpha))$ (in other words, the affine scheme $\operatorname{Spec}(\pi_0 A_\alpha)$. This follows immediately from Theorem V.2.2.12: the restriction of $\pi_i \mathcal{O}_{\mathcal{X}}$ is the quasi-coherent sheaf associated to $\pi_i A_\alpha$, viewed as a module over the commutative ring $\pi_0 A_\alpha$.

We now prove the converse. Suppose that (1), (2), and (3) are satisfied; we wish to prove that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a nonconnective spectral Deligne-Mumford stack. The assertion is local on $\mathcal{X}'$. The étale geometric morphism $\mathcal{X} \to \mathcal{X}'$ determines an equivalence $\mathcal{X} \simeq \mathcal{X}'_{/U}$, for some 2-connective object U in $\mathcal{X}'$. Passing to a cover of $\mathcal{X}'$, we may assume without loss of generality that U admits a global section $s : 1_{\mathcal{X}} \to U$; since U is 1-connective, this map is an effective epimorphism. This section determines a geometric morphism of ∞-topoi $s_* : \mathcal{X}' \to \mathcal{X}$. In view of Proposition V.2.3.10, it will suffice to show that $(\mathcal{X}', s^* \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack over k. Replacing $\mathcal{X}$ by $\mathcal{X}'$, we are reduced to the case where $\mathcal{X}$ is 1-localic and $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ is a spectral Deligne-Mumford stack. Passing to a cover of $\mathcal{X}$ again if necessary, we may suppose that $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ is the spectrum of a (discrete) commutative ring R.

We now proceed as in the proof of Theorem 2.40. Assume first that the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is connective (as a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$). Applying (2), we conclude that each $\pi_i \mathcal{O}_{\mathcal{X}}$ is the quasi-coherent sheaf associated to an R-module M_i. We have isomorphisms

$$\mathrm{H}^n(\mathcal{X}; \pi_i \mathcal{O}_{\mathcal{X}}) \simeq \begin{cases} M_i & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(see §T.7.2.2 for a discussion of the cohomology of an ∞-topos, and Remark T.7.2.2.17 for a comparison with the usual theory of sheaf cohomology.) For each $n \geq 0$, let $A_{\leq n} \in \mathrm{CAlg}$ denote the $\mathbb{E}_\infty$-ring of global section $\Gamma(\mathcal{X}; \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$. There is a convergent spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathcal{X}; \pi_q(\tau_{\leq n} \mathcal{O}_{\mathcal{X}})) \Rightarrow \pi_{q-p} A_{\leq n}.$$

It follows that this spectral sequence degenerates to yield isomorphisms

$$\pi_i A_{\leq n} \simeq \begin{cases} M_i & \text{if } 0 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\pi_0 A_{\leq n} \simeq R$.

Fix $n \geq 0$, and let $(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ be the spectrum of $A_{\leq n}$. The equivalence $A_n \simeq \Gamma(\mathcal{X}; \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ induces a map $\phi_n : (\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}) \to (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nSp}}_{\text{ét}})$. Since $\pi_0 A_n \simeq R$, the geometric morphism $\phi_n^* : \mathcal{X}_n \to \mathcal{X}$ is an equivalence of ∞-topoi, and ϕ_n induces an isomorphism of quasi-coherent sheaves $\phi_n^*(\pi_i \mathcal{O}_{\mathcal{X}_n}) \simeq \pi_i \mathcal{O}_{\mathcal{X}}$ for $0 \leq i \leq n$. Since the structure sheaves on both sides are n-truncated, we conclude that ϕ_n is an equivalence.

Let A denote the inverse limit of the tower of $\mathbb{E}_\infty$-rings

$$\ldots \to A_{\leq 2} \to A_{\leq 1} \to A_{\leq 0},$$

so that $\pi_0 A \simeq R$. We can therefore identify the spectrum of A with $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}})$. As in the proof of Theorem 2.40, we see that $\mathcal{O}'_{\mathcal{X}}$ is the inverse limit of its truncations

$$\tau_{\leq n} \mathcal{O}'_{\mathcal{X}} \simeq \phi_n^* \mathcal{O}_{\mathcal{X}_n} \simeq \tau_{\leq n} \mathcal{O}_{\mathcal{X}}.$$

Passing to the inverse limit, we obtain a map

$$\psi : \mathcal{O}_{\mathcal{X}} \to \lim\{\tau_{\leq n} \mathcal{O}_{\mathcal{X}}\} \simeq \mathcal{O}'_{\mathcal{X}}.$$

By construction, ψ induces an isomorphism on all (sheaves of) homotopy groups, and is therefore ∞-connective. The 0th space of $\mathcal{O}'_{\mathcal{X}}$ is a hypercomplete object of $\mathcal{X}$ (since it is an inverse limit of truncated objects of $\mathcal{X}$), and the 0th space of $\mathcal{O}_{\mathcal{X}}$ is hypercomplete by assumption (3). It follows that ψ is an equivalence, so that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Spec}^{\text{ét}} A$ is a spectral Deligne-Mumford stack as desired..

We now treat the case where the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is not assumed to be connective. The pair $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_{\mathcal{X}})$ satisfies conditions (1), (2), and (3), so the argument above proves that $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Spec}^{\text{ét}}(A)$ for some connective $\mathbb{E}_\infty$-ring A. Let $B \in \mathrm{CAlg}$ be the $\mathbb{E}_\infty$-ring of global sections of $\mathcal{O}_{\mathcal{X}}$. Then $\tau_{\geq 0} B$ is connective cover of the algebra of global sections of $\tau_{\geq 0} \mathcal{O}_{\mathcal{X}}$, and is therefore equivalent to A. In particular, we can identify $\mathrm{Spec}^{\text{ét}}(B)$ with $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}})$, for some sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O}'_{\mathcal{X}}$ on $\mathcal{X}$. To complete the proof, it will suffice to show that the canonical map $\theta : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is an equivalence. Let $\mathcal{F}$ denote the fiber of the map θ, viewed as an object of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Since θ induces an equivalence on the level of connective covers, we deduce that $\tau_{\geq 0} \mathcal{F} \simeq 0$. We wish to prove that $\mathcal{F} \simeq 0$. Suppose otherwise. Since $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is right complete (Proposition 1.7), we deduce that there exists an integer n (necessarily positive) such that $\pi_n \mathcal{F}$ is nonzero. We will assume that n is chosen minimal with respect to this property. We have an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$\pi_{1-n} \mathcal{O}'_{\mathcal{X}} \to \pi_{1-n} \mathcal{O}_{\mathcal{X}} \to \pi_{-n} \mathcal{F} \to \pi_{-n} \mathcal{O}'_{\mathcal{X}} \to \pi_{-n} \mathcal{O}_{\mathcal{X}}.$$

The homotopy groups of $\mathcal{O}_{\mathcal{X}}$ are quasi-coherent sheaves on X by (2). Since $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}})$ is a spectral Deligne-Mumford stack, it also satisfies (2) (by the first part of the proof), so that homotopy groups of $\mathcal{O}'_{\mathcal{X}}$ are also quasi-coherent sheaves on the ordinary Deligne-Mumford stack X. It follows that $\pi_{-n} \mathcal{F}$ is a nonzero quasi-coherent sheaf on the ordinary Deligne-Mumford stack X. Since X is the spectrum of the commutative ring R, we conclude that $\pi_{-n} \mathcal{F}$ has a nonvanishing global section. The minimality of n guarantees that $\pi_{-n}\Gamma(X; \mathcal{F}) \simeq \Gamma(X; \pi_{-n} \mathcal{F})$, so that the spectrum $\Gamma(X; \mathcal{F})$ is nonzero. But $\Gamma(X; \mathcal{F})$ can be identified with the fiber of the map of global sections $\Gamma(X; \mathcal{O}'_{\mathcal{X}}) \to \Gamma(X; \mathcal{O}_{\mathcal{X}})$, which is equivalent to the identity map on the $\mathbb{E}_\infty$-ring B. We therefore obtain a contradiction, which completes the proof. □

Remark 8.43. In the situation of Theorem 8.42, the spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is affine if and only if $\mathcal{X} \simeq \mathcal{X}'$ and the Deligne-Mumford stack $(\mathcal{X}', \phi_* \pi_0 \mathcal{O}_{\mathcal{X}})$ is affine. The "if" direction follows immediately from the proof of Theorem 8.42. To prove the "only if" direction, we may assume that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Spec}^{\text{ét}}(A)$ for some $\mathbb{E}_\infty$-ring A. Then the results of §V.2.2 show that $\mathcal{X}$ can be identified with the ∞-category of sheaves on the ∞-category of étale A-algebras, which is equivalent (by Theorem A.7.5.0.6) to the ∞-category of sheaves on the ordinary category of étale $\pi_0 A$, and therefore 1-localic. It follows that $(\mathcal{X}', \phi_* \pi_0 \mathcal{O}_{\mathcal{X}}) \simeq (\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ which can be identified with the spectrum of the discrete commutative ring $\pi_0 A$.

9 Comparison Results

In this section, we will discuss how some of the definitions given earlier in this paper are related to one another, and to some of the ideas introduced in [42]. We can summarize our main results as follows:

(A) In §2, we introduced the notion of a (nonconnective) *spectral scheme*, which can be interpreted as a scheme with respect to the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$, which encodes the Zariski topology on the ∞-category CAlg of $\mathbb{E}_\infty$-rings (see Definition 2.10). Replacing the Zariski topology by the étale topology, we obtain another geometry $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}$ whose schemes are the (nonconnective) *spectral Deligne-Mumford stacks* of §8. There is an evident transformation of geometries $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}} \to \mathcal{G}^{\mathrm{nSp}}_{\text{ét}}$, which is the identity functor at the level of underlying ∞-categories. This transformation determines a relative spectrum functor $\mathrm{Spec}^{\text{ét}}_{\mathrm{Z}} : \mathrm{SpSch}^{\mathrm{nc}} \to \mathrm{Stk}^{\mathrm{nc}}$. We will show that this functor is fully faithful when restricted to 0-localic spectral schemes (Theorem 9.1).

(B) Let k be an $\mathbb{E}_\infty$-ring. The geometries $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$ and $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}$ have relative versions $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ and $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}(k)$, which control the Zariski and étale topologies on the ∞-category of $\mathbb{E}_\infty$-algebras over k. However, the role of k is inessential. For example, the ∞-category of (nonconnective) spectral k-schemes is equivalent to the ∞-category of (nonconnective) spectral schemes $\mathfrak{X}$ equipped with a map $\mathfrak{X} \to \mathrm{Spec}^{\mathrm{Z}}(k)$. We will deduce this from a general relativization statement (Proposition 9.17).

(C) Let k be an ordinary commutative ring, regarded as a discrete $\mathbb{E}_\infty$-ring. The theory of spectral k-schemes is closely related to the theory of *derived k-schemes* introduced in [42]. More precisely, there is a forgetful functor from derived k-schemes to spectral k-schemes, which an equivalence when k contains the field $\mathbf{Q}$ of rational numbers (Corollary 9.28).

We begin by giving a precise formulation of (A):

Theorem 9.1. *Let* $\mathrm{Spec}^{\acute{e}t}_{\mathrm{Z}} : \mathrm{RingTop}_{\mathrm{Zar}} \to \mathrm{RingTop}_{\acute{e}t}$ *be the relative spectrum functor associated to the transformation of geometries* $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}} \to \mathcal{G}^{\mathrm{nSp}}_{\acute{e}t}$*. Then* $\mathrm{Spec}^{\acute{e}t}_{\mathrm{Z}}$ *induces a fully faithful functor from the ∞-category* $\mathrm{SpSch}^{\mathrm{nc}}_{\leq 0}$ *of 0-localic nonconnective spectral schemes to the ∞-category* $\mathrm{Stk}^{\mathrm{nc}}_{\leq 1}$ *of 1-localic nonconnective spectral Deligne-Mumford stacks.*

We will prove Theorem 9.1 at the end of this section.

Remark 9.2. Since the relative spectrum functor $\mathrm{Spec}^{\text{ét}}_{\mathrm{Z}}$ carries $\mathrm{Spec}^{\mathrm{Z}}(R)$ into $\mathrm{Spec}^{\text{ét}}(R)$ for any $\mathbb{E}_\infty$-ring R, it preserves various local properties of sheaves of $\mathbb{E}_\infty$-rings, such as the property of being connective or n-truncated. In particular, $\mathrm{Spec}^{\text{ét}}_{\mathrm{Z}}$ determines a fully faithful embedding from the ∞-category of 0-localic, 0-truncated connective spectral schemes into the ∞-category of 1-localic, 0-truncated connective Deligne-Mumford stacks. This fully faithful embedding can be identified with the usual embedding of the category of schemes into the 2-category of Deligne-Mumford stacks (see Propositions 2.37 and 8.36).

Warning 9.3. The relative spectrum functor $\mathrm{Spec}^{\text{ét}}_{\mathrm{Z}}$ is not fully faithful in general. This is a reflection of the fact that the theory of spectral schemes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is ill-behaved if we do not require $\mathcal{X}$ to be 0-localic.

Definition 9.4. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. We will say that $\mathfrak{X}$ is *schematic* if it belongs to the essential image of the fully faithful functor of Theorem 9.1. That is, $\mathfrak{X}$ is schematic if it has the form $\mathrm{Spec}^{\text{ét}}_{\mathrm{Zar}} \mathfrak{Y}$ for some 0-localic nonconnective spectral scheme $\mathfrak{Y}$.

It is not difficult to characterize the class of schematic spectral Deligne-Mumford stacks. For this, we need to introduce a bit of terminology.

Definition 9.5. Suppose that $j : \mathfrak{U} \to \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a map of nonconnective spectral Deligne-Mumford stacks (nonconnective spectral schemes). We will say that j is an *open immersion* if it factors as a composition

$$\mathfrak{U} \stackrel{j'}{\to} (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U) \stackrel{j''}{\to} (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

where j' is an equivalence and j'' is the étale morphism associated to a (-1)-truncated object $U \in \mathcal{X}$. In this case, we will also say that $\mathfrak{U}$ is an *open substack* (*open subscheme*) of $\mathfrak{X}$.

Proposition 9.6. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *There exists a collection of open immersions $\{j_\alpha : \mathfrak{U}_\alpha \to \mathfrak{X}\}$ which determine a covering of $\mathfrak{X}$, where each $\mathfrak{U}_\alpha$ is affine.*

(2) *Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi which exhibits $\mathcal{Y}$ as a 0-localic reflection of $\mathcal{X}$ (so that $\mathcal{Y}$ can be identified with the ∞-topos of sheaves on the underlying locale of (-1)-truncated objects of $\mathcal{X}$). Then the pair $\mathfrak{Y} = (\mathcal{Y}, f_* \mathcal{O}_{\mathcal{X}})$ is a nonconnective spectral scheme, and the geometric morphism f_* induces an equivalence $\mathfrak{X} \simeq \mathrm{Spec}^{\acute{e}t}_{\mathrm{Zar}} \mathfrak{Y}$.*

(3) *The nonconnective spectral Deligne-Mumford stack $\mathfrak{X}$ is schematic.*

Lemma 9.7. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral scheme, let $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathrm{Spec}^{\acute{e}t}_{\mathrm{Zar}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be the nonconnective associated spectral Deligne-Mumford stack, and let $f^* : \mathcal{X} \to \mathcal{Y}$ be the associated geometric morphism. Then f^* induces an equivalence $\tau_{\leq -1} \mathcal{X} \to \tau_{\leq -1} \mathcal{Y}$ between the underlying locales of $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. The assertion is local on $\mathcal{X}$; we may therefore assume without loss of generality that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \simeq \mathrm{Spec}^{\mathrm{Zar}} A$ is the affine nonconnective spectral scheme associated to an $\mathbb{E}_\infty$-ring A. Then $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(X)$, where X is the Zariski spectrum of the commutative ring $\pi_0 A$ (see Lemma V.2.5.18). That is, X is the collection of prime ideals $\mathfrak{p} \subseteq \pi_0 A$. As a topological space, X has a basis of open sets given by $U_a = \{\mathfrak{p} \in X : a \notin \mathfrak{p}\}$, where a ranges over elements of the commutative ring $\pi_0 A$. The ∞-category $\tau_{\leq -1} \mathcal{X}$ is equivalent to the nerve of the partially ordered set $\mathcal{U}(X)$ of open subsets of X.

Let $\mathcal{C} = (\mathrm{CAlg}^{\acute{e}t}_{A/})^{op}$ denote the opposite of the ∞-category of étale A-algebras. As explained in §V.2.2, we can identify $\mathcal{Y}$ with the ∞-category $\mathcal{S}\mathrm{hv}(\mathcal{C})$. In particular, $\tau_{\leq -1} \mathcal{Y}$ is equivalent to the nerve of the partially ordered set P consisting of sieves $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ which are *saturated* in the following sense: if A' is an étale A-algebra, and there exists a covering family $\{A' \to A'_i\}$ such that each A'_i belongs to $\mathcal{C}^{(0)}$, then A' belongs to $\mathcal{C}^{(0)}$.

The pullback functor $f^* : \tau_{\leq -1} \mathcal{X} \to \tau_{\leq -1} \mathcal{Y}$ determines a map of partially ordered sets $\lambda : \mathcal{U}(X) \to P$. Unwinding the definitions, we see that λ carries an open subset $U \subseteq X$ to the smallest saturated sieve $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ which contains $A[a^{-1}]$ whenever $U_a \subseteq U$. To complete the proof, it will suffice to show that λ is an isomorphism of partially ordered sets.

For every open set $U \subseteq X$, let $\lambda'(U)$ denote the full subcategory of $\mathcal{C}$ spanned by those étale A-algebras A' such that the map $\mathrm{Spec}^{\mathrm{Z}} A' \to \mathrm{Spec}^{\mathrm{Z}} A = X$ factors through U, where $\mathrm{Spec}^{\mathrm{Z}}$ denotes the Zariski spectrum. We claim that $\lambda'(U) = \lambda(U)$. Since $\lambda'(U)$ is a saturated sieve which contains $A[a^{-1}]$ whenever $U_a \subseteq U$, we immediately deduce that $\lambda(U) \subseteq \lambda'(U)$. Conversely, suppose that A' is an étale A-algebra belonging to $\lambda'(U)$. We wish to prove that $A' \in \lambda(U)$. The map $\mathrm{Spec}^{\mathrm{Z}} A' \to \mathrm{Spec}^{\mathrm{Z}} A$ is open (Proposition 0.2), so its image is a quasi-compact open subset $V \subseteq U \subseteq X$. We can therefore write $V = \bigcup_{1 \leq i \leq n} U_{a_i}$ for some finite sequence of elements $a_1, \ldots, a_n \in \pi_0 A$. For $1 \leq i \leq n$, let a'_i denote the image of a_i in $\pi_0 A'$. Since the inverse images of the open subsets $U_a \subseteq X$ cover $\mathrm{Spec}^{\mathrm{Z}} A'$, the map $A' \to \prod_{1 \leq i \leq n} A'[a'^{-1}_i]$ is étale and faithfully flat. Since $\lambda(U)$ is saturated, it will suffice to show that $A'[a'^{-1}_i] \in \lambda(U)$; this follows immediately, since $A[a_i^{-1}] \in \lambda(U)$ by construction.

We next claim that if U and V are open subsets of X such that $\lambda(U) \subseteq \lambda(V)$, then $U \subseteq V$. Since U is the union of basic open sets of the form U_a, we may assume that $U = U_a$ for some $a \in \pi_0 A$. Then $A[a^{-1}] \in \lambda(U) \subseteq \lambda(V) = \lambda'(V)$, so that V contains the image of the map $\mathrm{Spec}^{\mathrm{Z}} A[a^{-1}] \to \mathrm{Spec}^{\mathrm{Z}} A = X$.

The above argument shows that λ is an isomorphism of $\mathcal{U}(X)$ onto a partially ordered subset of P. To complete the proof, it will suffice to show that λ is surjective. To this end, choose a saturated sieve $\mathcal{C}^{(0)} \subseteq \mathcal{C}$; we wish to show that $\mathcal{C}^{(0)}$ lies in the image of λ. Let U be the smallest open subset of X which contains the (open) image $U_{A'}$ of the map $\mathrm{Spec}^{\mathrm{Z}} A' \to \mathrm{Spec}^{\mathrm{Z}} A$ whenever $A' \in \mathcal{C}^{(0)}$. By construction, we have $\mathcal{C}^{(0)} \subseteq \lambda'(U) = \lambda(U)$. To complete the proof, it suffices to show that this inclusion is an equality. That is, we must show that if B is an étale A-algebra such that the image of the map $\theta : \mathrm{Spec}^{\mathrm{Z}} B \to \mathrm{Spec}^{\mathrm{Z}} A = X$ is contained in U, then $B \in \mathcal{C}^{(0)}$. Since the image of θ is quasi-compact, it is contained in a finite union of $\bigcup_{1 \leq i \leq n} U_{A_i}$, where each $A_i \in \mathcal{C}^{(0)}$. It follows that the map $B \to \prod_{1 \leq i \leq n}(A_i \otimes_A B)$ is étale and faithfully flat. Since $\mathcal{C}^{(0)}$ is a saturated sieve containing each A_i, it must also contain B. □

Remark 9.8. In the situation of Lemma 9.7, if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a 0-localic spectral scheme, then the geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$ exhibits $\mathcal{X}$ as the 0-localic ∞-topos associated to $\mathcal{Y}$; see §T.6.4.5.

Proof of Proposition 9.6. The implication (2) ⇒ (3) is obvious. We next prove that (3) ⇒ (1). If (3) is satisfied, then $\mathfrak{X} \simeq \mathrm{Spec}^{\text{ét}}_{\mathrm{Zar}} \mathfrak{Y}$ for some 0-localic nonconnective spectral scheme $\mathfrak{Y}$. The proof of Theorem 2.40 shows that $\mathfrak{Y}$ admits a covering by open immersions $\{\mathfrak{V}_\alpha \to \mathfrak{Y}\}$, where each $\mathfrak{V}_\alpha$ is an affine nonconnective spectral scheme. It follows that $\mathfrak{X}$ admits a covering by open immersions $\{\mathrm{Spec}^{\text{ét}}_{\mathrm{Z}} \mathfrak{V}_\alpha \to \mathfrak{X}\}$, and each $\mathrm{Spec}^{\text{ét}}_{\mathrm{Z}} \mathfrak{V}_\alpha$ is an affine nonconnective spectral Deligne-Mumford stack.

We complete the proof by showing that (1) ⇒ (2). The content of assertion (2) is local on $\mathfrak{Y}$. We are therefore free to replace $\mathfrak{X}$ by one of the open substacks $\mathfrak{U}_\alpha$ (note that since $\mathcal{Y}$ is the 0-localic reflection of $\mathcal{X}$, the pullback functor $f^* : \mathcal{Y} \to \mathcal{X}$ induces an equivalence on (-1)-truncated objects) and thereby reduce to the case where $\mathfrak{X} = \mathrm{Spec}^{\text{ét}} R$ is affine. Using Lemma 9.7, we can identify $\mathfrak{Y}$ with the ∞-topos $\mathcal{S}\mathrm{hv}(\mathrm{Spec}^{\mathrm{Z}} R)$ of sheaves on the topological space $\mathrm{Spec}^{\mathrm{Z}} R$. Unwinding the definitions, we obtain an identification $\mathfrak{Y} \simeq \mathrm{Spec}^{\mathrm{Z}} R$, from which assertion (2) follows immediately. □

Corollary 9.9. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. Then $\mathfrak{X}$ is schematic if and only if the 0-truncated spectral Deligne-Mumford stack $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$ is schematic.*

Proof. This follows from the criterion of Proposition 9.6, since a nonconnective spectral Deligne-Mumford stack $(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ is affine if and only if $(\mathcal{U}, \pi_0 \mathcal{O}_{\mathcal{U}})$ is affine (see the proof of Theorem 2.40). □

The proof of Theorem 9.1 involves some formal arguments which require a bit of a digression. Let $\mathcal{G}$ be a geometry and let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}$-scheme. Then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ represents a functor $X : \mathrm{Ind}(\mathcal{G}^{op}) \to \mathcal{S}$, given informally by the formula $R \mapsto \mathrm{Map}_{\mathrm{Sch}(\mathcal{G})}(\mathrm{Spec}^{\mathcal{G}} R, (\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$. According to Theorem V.2.4.1, the $\mathcal{G}$-scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is determined up to canonical equivalence by the functor X. In particular, for any ∞-topos $\mathcal{Y}$ and any $\mathcal{G}$-structure $\mathcal{O}_{\mathcal{Y}} \in \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{Y})$, the mapping space $\mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$ can be recovered from the functor X. Our first goal is to make this recovery somewhat explicit. To this end, let us identify $\mathcal{O}_{\mathcal{Y}}$ with an $\mathrm{Ind}(\mathcal{G}^{op})$-valued sheaf on $\mathcal{Y}$. Then the composition

$$\mathcal{Y}^{op} \xrightarrow{\mathcal{O}_{\mathcal{Y}}} \mathrm{Ind}(\mathcal{G}^{op}) \xrightarrow{X} \mathcal{S}.$$

can be regarded as a presheaf of spaces on $\mathcal{Y}$. The main technical ingredient in the proof of Theorem 9.1 is the following:

Theorem 9.10. *Let $\mathcal{G}$ be a geometry, let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}$-scheme representing the functor $X : \mathrm{Ind}(\mathcal{G}^{op}) \to \mathcal{S}$, and let $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be an arbitrary object of ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})$. Then the functor*

$$(U \in \mathcal{Y}) \mapsto \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), (\mathcal{Y}^{/U}, \mathcal{O}_{\mathcal{Y}} | U))$$

can be identified with the sheafification of the presheaf given by the composition $\mathcal{Y}^{op} \xrightarrow{\mathcal{O}_{\mathcal{Y}}} \mathrm{Ind}(\mathcal{G}^{op}) \xrightarrow{X} \mathcal{S}$.

Remark 9.11. Theorem 9.10 plays an important role in mediating between two different pictures of a $\mathcal{G}$-scheme:

- (*i*) A $\mathcal{G}$-scheme can be thought of as a pair $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, where $\mathcal{Y}$ is an ∞-topos and $\mathcal{O}_{\mathcal{Y}}$ is a $\mathcal{G}$-structure on $\mathcal{Y}$.
- (*ii*) A $\mathcal{G}$-scheme can be thought of as representing a functor $X : \mathrm{Ind}(\mathcal{G}^{op}) \to \mathcal{S}$.

Both of these points of view are valuable, because they suggest two *different* generalizations of the notion of a $\mathcal{G}$-scheme. In case (i), we can consider arbitrary $\mathcal{G}$-structures on ∞-topoi (instead of considering only those which are locally of the form $\mathrm{Spec}^{\mathcal{G}} R$), and in case (ii) one can consider arbitrary functors $X : \mathrm{Ind}(\mathcal{G}^{op}) \to \mathcal{S}$ (not only representable functors). However, these two generalizations can be related as follows: given a $\mathcal{G}$-structured ∞-topos $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and a functor $X : \mathrm{Ind}(\mathcal{G}^{op}) \to \mathcal{S}$, one can define a mapping space $\mathrm{Map}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), X)$ to be the global sections of the sheafification of the presheaf given by $(U \in \mathcal{Y}) \mapsto X(\mathcal{O}_{\mathcal{Y}}(U))$. Theorem 9.10 asserts that this definition is sensible: that is, it recovers the usual mapping spaces in ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})^{op}$ in cases where X is representable by a scheme (a somewhat easier argument shows that it also recovers the usual mapping spaces in $\mathrm{Fun}(\mathrm{Ind}(\mathcal{G}^{op}), \mathcal{S})$ in cases where $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a $\mathcal{G}$-scheme).

Before giving the proof of Theorem 9.10, let us record some consequences.

Corollary 9.12. *Let $\alpha : \mathcal{G} \to \mathcal{G}'$ be a transformation of geometries which determines an equivalence between the underlying ∞-categories, and let $\mathcal{C} = \mathrm{Ind}(\mathcal{G}^{op}) \simeq \mathrm{Ind}(\mathcal{G}'^{op})$. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a $\mathcal{G}$-scheme representing a functor $X : \mathcal{C} \to \mathcal{S}$, and let $X' : \mathcal{C} \to \mathcal{S}$ be the functor represented by the $\mathcal{G}'$-scheme $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Regard $\mathcal{C}^{op}$ as endowed with the Grothendieck topology determined by the geometry $\mathcal{G}'$. Assume that*

- $(*)$ *The topology on $\mathcal{C}^{op}$ is subcanonical: that is, the identity functor $\mathcal{C} \to \mathcal{C}$ is a $\mathcal{C}$-valued sheaf on $\mathcal{C}^{op}$.*

Then the evident map $X \to X'$ exhibits X' as a sheafification of X (with respect to the Grothendieck topology on $\mathcal{C}$).

Proof. The functor X' is given informally by the formulas

$$X'(R) = \mathrm{Map}_{\mathrm{Sch}(\mathcal{G}')}(\mathrm{Spec}^{\mathcal{G}'} R, \mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \simeq \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), \mathrm{Spec}^{\mathcal{G}'} R).$$

Fix an object $R \in \mathcal{C}$, and let $\mathcal{D}$ denote the full subcategory of $(\mathcal{C}_{R/})^{op}$ spanned by those morphisms $R \to R'$ in $\mathcal{C}$ which are pushouts of admissible morphisms in $\mathcal{G}'$, endowed with the Grothendieck topology determined by the geometry $\mathcal{G}'$. The results of §V.2.2 show that we can identify $\mathrm{Spec}^{\mathcal{G}'} R$ with the pair $(\mathcal{S}\mathrm{hv}(\mathcal{D}), \mathcal{O})$, where $\mathcal{O}$ can be identified with the sheafification of the $\mathcal{C}$-valued presheaf on $\mathcal{D}$ given by the composition $\mathcal{O}' : \mathcal{D}^{op} \subseteq \mathcal{C}_{R/} \to \mathcal{C}$. Let $Y : \mathcal{D}^{op} \to \mathcal{S}$ be the sheafification of the presheaf $X \circ \mathcal{O}$. Theorem 9.10 gives an identification $X'(R) \simeq Y(R)$. Assumption $(*)$ implies that $\mathcal{O} \simeq \mathcal{O}'$, so that Y is the sheafification of the presheaf $X \circ \mathcal{O}'$. It follows that X' is the sheafification of X (see the proof of Proposition V.2.4.4). $\square$

Proof of Theorem 9.1. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a 0-localic spectral scheme. We will prove that for every spectral scheme $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, the map

$$\phi_{\mathcal{Y}} : \mathrm{Map}_{\mathrm{SpSch}^{\mathrm{nc}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \to \mathrm{Map}_{\mathrm{Stk}^{\mathrm{nc}}}(\mathrm{Spec}^{\text{ét}}_{\mathrm{Z}}(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \mathrm{Spec}^{\text{ét}}_{\mathrm{Z}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$$

is a homotopy equivalence. Using Lemma V.2.3.11, we can reduce to the case where $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is affine. Let $X : \mathrm{CAlg} \to \mathcal{S}$ be the functor represented by $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and let $X' : \mathrm{CAlg} \to \mathcal{S}$ be the functor represented by $\mathrm{Spec}^{\text{ét}}_{\mathrm{Z}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. We wish to prove that the evident natural transformation $\alpha : X \to X'$ is an equivalence. The étale topology on the ∞-category CAlg is subcanonical (Theorem 5.14), so Corollary 9.12 implies that α exhibits X' as a sheafification of X with respect to the étale topology on CAlg. To complete the proof, it suffices to observe that Theorem 5.15 guarantees that X is already a sheaf with respect to the étale topology on CAlg. $\square$

We now turn to the proof of Theorem 9.10. The proof relies on the following:

Lemma 9.13. *Let $\mathcal{G}$ be a geometry, and suppose we are given a pair of objects $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \in {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})$. Let $F : \mathcal{Y}^{op} \times \mathcal{X} \to \widehat{\mathcal{S}}$ be the functor given informally by the formula*

$$F(U, V) = \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})}((\mathcal{X}_{/V}, \mathcal{O}_{\mathcal{X}} |V), (\mathcal{Y}_{U/}, \mathcal{O}_{\mathcal{Y}} |U)).$$

Then F determines a colimit-preserving functor $\mathcal{X} \to \mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}(\mathcal{Y})$.

Proof. Proposition V.2.3.5 shows that F preserves limits in the first variable, and can therefore be identified with a functor $f : \mathcal{X} \to \mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}(\mathcal{Y})$. We wish to show that f preserves small colimits. Fix a small diagram $\{V_\alpha\}$ in $\mathcal{X}$ having colimit $V \in \mathcal{X}$; we need to prove that the canonical map $\varinjlim f(V_\alpha) \to f(V)$ is an equivalence in $\mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}(\mathcal{Y})$. For each object $U \in \mathcal{Y}$, let $e_U : \mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}(\mathcal{Y}) \subseteq \mathrm{Fun}(\mathcal{Y}^{op}, \widehat{\mathcal{S}})$ be the functor represented by U. The objects e_U generate $\mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}(\mathcal{Y})$ under (non-small) colimits. It therefore suffices to prove that for every map of the form $\alpha : e_U \to f(V)$, the pullback map

$$(\varinjlim f(V_\alpha)) \times_{f(V)} e_U \simeq \varinjlim (f(V_\alpha) \times_{f(V)} e_U) \to e_U$$

is an equivalence. We can identify α with a point $\eta \in F(U, V)$, which determines a geometry morphism $\alpha^* : \mathcal{X}_{/V} \to \mathcal{Y}_{/U}$. Remark V.2.3.4 gives canonical identifications $f(V_\alpha) \times_{f(V)} e_U \simeq e_{\alpha^* V_\alpha}$. Since the functor α^* preserves small colimits, we are reduced to proving that the Yoneda embedding e preserves small colimits. Note that e can be identified with the inclusion $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathcal{X}) \subseteq \mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}(\mathcal{X})$, which preserves small colimits. □

Proof of Theorem 9.10. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a $\mathcal{G}$-scheme. For each $V \in \mathcal{X}$, let $\overline{X}_V : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}) \to \widehat{\mathcal{S}}$ denote the functor corepresented by $(\mathcal{X}_{/V}, \mathcal{O}_\mathcal{X} | V)$, so that $\overline{X}_V \circ \mathrm{Spec}^\mathcal{G}$ can be identified with the functor $X_V : \mathrm{Ind}(\mathcal{G}^{op}) \to \mathcal{S}$ determined by the $\mathcal{G}$-scheme $(\mathcal{X}_{/V}, \mathcal{O}_\mathcal{X} | V)$. Let $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ be an arbitrary $\mathcal{G}$-structured ∞-topos, and let $\chi : \mathcal{Y}\,\mathcal{Y} \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})^{op}$ denote the functor given informally by the formula $U \mapsto (\mathcal{Y}_{/U}, \mathcal{O}_\mathcal{Y} | U)$. Let Q be a left adjoint to the inclusion of the ∞-category of affine $\mathcal{G}$-schemes into ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})^{op}$, given informally by the formula $(\mathcal{Z}, \mathcal{O}_\mathcal{Z}) \mapsto \mathrm{Spec}^\mathcal{G} \Gamma(\mathcal{O}_\mathcal{Z})$. We will prove that, for every object $V \in \mathcal{X}$, the natural transformation

$$\theta_V : X_V \circ \mathcal{O}_\mathcal{Y} \simeq \overline{X}_V \circ Q \circ \chi \to \overline{X}_V \circ \chi$$

exhibits $\overline{X}_V \circ \chi$ as a sheafification of the presheaf $X_V \circ \mathcal{O}_\mathcal{Y} : \mathcal{Y}^{op} \to \mathcal{S}$.

Let $L : \mathrm{Fun}(\mathcal{Y}^{op}, \widehat{\mathcal{S}}) \to \mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}(\mathcal{Y})$ be a left adjoint to the inclusion. Let $\mathcal{X}^0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects for which $L(\theta_V)$ is an equivalence; we wish to prove that $\mathcal{X}^0 \subseteq \mathcal{X}$. If $(\mathcal{X}_{/V}, \mathcal{O}_\mathcal{X} | V)$ is affine, then θ_V is an equivalence of presheaves so that $V \in \mathcal{X}^0$. In view of Lemma V.2.3.11, it will suffice to show that $\mathcal{X}^0$ is stable under small colimits. Lemma 9.13 shows that the functor $V \mapsto \overline{X}_V \circ \chi$ preserves small colimits (as a functor from $\mathcal{X}$ to $\mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}(\mathcal{Y})$). To complete the proof, it will suffice to show that the functor $V \mapsto L(X_V \circ \mathcal{O}_\mathcal{Y})$ also preserves small colimits. In other words, we must show that if $V \in \mathcal{X}$ is a colimit of a diagram $\{V_\alpha\}$ in $\mathcal{X}$ and $\mathcal{F} : \mathcal{Y}^{op} \to \widehat{\mathcal{S}}$ is a sheaf, then the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{Y}^{op}, \widehat{\mathcal{S}})}(X_V \circ \mathcal{O}_\mathcal{Y}, \mathcal{F}) \to \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathcal{Y}^{op}, \widehat{\mathcal{S}})}(X_{V_\alpha} \circ \mathcal{O}_\mathcal{Y}, \mathcal{F})$$

is a homotopy equivalence.

Let $\mathcal{C}$ denote the full subcategory of the fiber product $\mathcal{Y}^{op} \times_{\mathrm{Fun}(\{0\}, \mathrm{Ind}(\mathcal{G}^{op}))} \mathrm{Fun}(\Delta^1, \mathrm{Ind}(\mathcal{G}^{op}))$ spanned by pairs $(U, f : \mathcal{O}_\mathcal{Y}(U) \to A)$, where $U \in \mathcal{Y}$ and f is an admissible morphism in $\mathrm{Ind}(\mathcal{G}^{op})$. Let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $(U, f : \mathcal{O}_\mathcal{Y}(U) \to A)$ where f is an equivalence, so that the projection map $\mathcal{C}_0 \to \mathcal{Y}^{op}$ is an equivalence. Note that the inclusion $i : \mathcal{C}_0 \subseteq \mathcal{C}$ admits a left adjoint f. Let $\pi : \mathcal{C} \to \mathrm{Ind}(\mathcal{G}^{op})$ be the functor given by the formula $(U, f : \mathcal{O}_\mathcal{Y}(U) \to A) \mapsto A$. Let $\mathcal{F}'$ denote the composition $\mathcal{C} \xrightarrow{f} \mathcal{C}_0 \to \mathcal{Y}^{op} \xrightarrow{\mathcal{F}} \widehat{\mathcal{S}}$, so that $\mathcal{F}'$ is a right Kan extension of its restriction $\mathcal{F}' | \mathcal{C}_0$. It follows that, for every object $V \in \mathcal{X}$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})}(X_V \circ \pi, \mathcal{F}') \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_0, \widehat{\mathcal{S}})}(X_V \circ \pi \circ i, \mathcal{F}' \circ i) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{Y}^{op}, \widehat{\mathcal{S}})}(X_V \circ \mathcal{O}_\mathcal{Y}, \mathcal{F}).$$

We are thereby reduced to proving that if V is the colimit of a diagram $\{V_\alpha\}$ in $\mathcal{X}$, then the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})}(X_V \circ \pi, \mathcal{F}') \to \varprojlim \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})}(X_{V_\alpha} \circ \pi, \mathcal{F}')$$

is a homotopy equivalence.

For each object $U \in \mathcal{Y}$, let $\mathcal{C}_U$ denote the fiber product $\mathcal{C} \times_{\mathcal{Y}^{op}} \{U\}$: that is, the full subcategory of $\mathrm{Ind}(\mathcal{G}^{op})^{\mathcal{O}_\mathcal{Y}(U)/}$ spanned by the admissible morphisms $\mathcal{O}_\mathcal{Y}(U) \to A$. Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})$ spanned by those functors $F : \mathcal{C} \to \widehat{\mathcal{S}}$ with the property that each restriction $F | \mathcal{C}_U$ is a sheaf (with respect to the Grothendieck topology determined by the collection of admissible coverings in $\mathcal{G}$). Using Lemma V.2.4.9, we deduce that the inclusion $\mathcal{E} \subseteq \mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{S}})$ admits a left adjoint L', which is characterized by the requirement $(L'F) | \mathcal{C}_U$ is a sheafification of $F | \mathcal{C}_U$ for each $V \in \mathcal{Y}$. We observe that $\mathcal{F}' \in \mathcal{E}$. It therefore suffices to show that if V is the colimit of a diagram $\{V_\alpha\}$ in $\mathcal{X}$, then the canonical map $L' \varinjlim (X_{V_\alpha} \circ \pi) \to L'(X_V \circ \pi)$ is an equivalence. In other words, we must show that for each $U \in \mathcal{Y}$, the presheaves $\varinjlim X_{V_\alpha} | \mathcal{C}_U$ and $X_V | \mathcal{C}_U$ have the same sheafification. This follows from Lemma 9.13, applied to the affine $\mathcal{G}$-scheme $\mathrm{Spec}^{\mathcal{G}} \mathcal{O}_{\mathcal{Y}}(U)$. □

We now discuss the dependence of the theory of spectral k-schemes on the choice of $\mathbb{E}_\infty$-ring k. We begin with some general remarks.

Proposition 9.14. *Let $\mathcal{C}$ be a compactly generated presentable ∞-category, and let $\mathcal{G}$ be the full subcategory of $\mathcal{C}^{op}$ spanned by the compact objects. Suppose that $\mathcal{G}$ is equipped with the structure of a finitary geometry (Definition V.1.2.5 and Remark V.2.2.8). Then:*

- (a) *For each object $X \in \mathcal{C}$, the ∞-category $\mathcal{C}_{X/}$ is compactly generated; we let $\mathcal{G}(X)$ denote the full subcategory of $(\mathcal{C}_{X/})^{op}$ spanned by the compact objects of $\mathcal{C}_{X/}$.*

Given an object $Z \in \mathcal{G}(X)$, we will say that a sieve $\mathcal{G}(X)^{(0)}_{/Z} \subseteq \mathcal{G}(X)_{/Z}$ is covering if there exists a morphism $Z \to Z_0$ in $\mathcal{C}^{op}$ where $Z_0 \in \mathcal{G}$, and an admissible covering $\{Y_\alpha \to Z_0\}$ of Z_0 such that each of the induced maps $Y_\alpha \times_{Z_0} Z \to Z$ belongs to the sieve $\mathcal{G}(X)^{(0)}_{/Z}$.

- (b) *For each object $X \in \mathcal{C}$, the collection of covering sieves determines a Grothendieck topology on the ∞-category $\mathcal{G}(X)$.*

We will say that a morphism $f : Y \to Z$ in $\mathcal{C}^{op}$ is admissible if there exists a pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \downarrow & & \downarrow \\ Y_0 & \xrightarrow{f_0} & Z_0 \end{array}$$

in $\mathcal{C}^{op}$, where f_0 is an admissible morphism in $\mathcal{G}$. Suppose that the collection of admissible morphisms in $\mathcal{C}^{op}$ is stable under retracts. Then:

- (c) *For each object $X \in \mathcal{C}$, the collection of admissible morphisms together with the Grothendieck topology on $\mathcal{G}(X)$ exhibit $\mathcal{G}(X)$ as a finitary geometry.*

Proof. We first prove (a). Proposition T.5.5.3.11 guarantees that $\mathcal{C}_{X/}$ is presentable. Let $\mathcal{G}(X)$ denote the full subcategory of $(\mathcal{C}_{/X})^{op}$ spanned by those objects which are compact in $\mathcal{C}_{/X}$. The inclusion $\mathcal{G}(X)^{op} \subseteq \mathcal{C}_{/X}$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathcal{G}(X)^{op}) \to \mathcal{C}_{/X}$ (Proposition T.5.3.5.11) which preserves small colimits (Proposition T.5.5.1.9). It follows that F admits a right adjoint G (Corollary T.5.5.2.9). To complete the proof of (a), it will suffice to show that F is essentially surjective; equivalently, we must show that G is conservative. To this end, let $\alpha : Y \to Z$ be a morphism in $\mathcal{C}_{X/}$ such that $G(\alpha)$ is an equivalence. We wish to show that α is an equivalence. Since $\mathcal{C}$ is compactly generated, this is equivalent to the requirement that that for every compact object $K \in \mathcal{C}$, composition with α induces a homotopy equivalence $\theta : \mathrm{Map}_{\mathcal{C}}(K, Y) \to \mathrm{Map}_{\mathcal{C}}(K, Z)$. Let $u : \mathcal{C} \to \mathcal{C}_{X/}$ denote a left adjoint to the forgetful functor, given informally by the formula $u(C) \simeq C \coprod X$. We can identify θ with the map $\mathrm{Map}_{\mathcal{C}_{X/}}(u(K), Y) \to \mathrm{Map}_{\mathcal{C}_{X/}}(u(K), Z)$. Since $G(\alpha)$ is an equivalence by assumption, it will suffice to show that $u(K)$ is a compact object of $\mathcal{C}_{X/}$. We now complete the proof by observing that because the forgetful functor $\mathcal{C}_{X/} \to \mathcal{C}$ preserves filtered colimits (Proposition T.4.4.2.9), the functor u preserves compact objects (Proposition T.5.5.7.2).

We now prove (b). It will suffice to show that the collection of covering sieves in $\mathcal{G}(X)$ satisfies conditions (1), (2), and (3) of Definition T.6.2.2.1:

(1) If Z is an object of $\mathcal{G}(X)$, then $\mathcal{G}(X)_{/Z}$ is a covering sieve on Z. This is clear, since $\mathcal{G}(X)_{/Z}$ contains the pullback of the admissible covering sieve $\{\mathbf{1} \to \mathbf{1}\}$ in $\mathcal{G}$, where $\mathbf{1}$ denotes the final object of $\mathcal{G}$.

(2) If $f : Y \to Z$ is a morphism in $\mathcal{G}(X)$ and $\mathcal{G}(X)^{(0)}_{/Z}$ is a covering sieve on Z, then the pullback sieve $f^* \mathcal{G}(X)^{(0)}_{/Y} \subseteq \mathcal{G}(X)_{/Y}$ is a covering sieve on Y. This follows immediately from the definition.

(3) Let $Z \in \mathcal{G}(X)$ be an object, let $\mathcal{G}(X)^{(0)}_{/Z} \subseteq \mathcal{G}(X)_{/Z}$ be a covering sieve on Z, and let $\mathcal{G}(X)^{(1)}_{/Z} \subseteq \mathcal{G}(X)_{/Z}$ be another sieve on Z. Suppose that, for every morphism $f : Y \to Z$ belonging to $\mathcal{G}(X)^{(0)}_{/Z}$, the pullback sieve $f^* \mathcal{G}(X)^{(1)}_{/Z}$ is a covering sieve on Y. We must show that $\mathcal{G}(X)^{(1)}_{/Z}$ is a covering sieve on Z. Invoking our assumption that $\mathcal{G}(X)^{(0)}_{/Z}$ is covering, we deduce the existence of a map $Z \to Z_0$ for $Z_0 \in \mathcal{G}$ and an admissible covering $\{Y_\alpha \to Z_0\}_{\alpha \in A}$ in $\mathcal{G}$ such that each of the maps $f_\alpha : Y_\alpha \times_{Z_0} Z \to Z$ belongs to $\mathcal{G}(X)^{(0)}_{/Z}$. Since $\mathcal{G}$ is finitary, we may assume that the collection of indices A is finite. For each $\alpha \in A$, there exists a morphism $Y_\alpha \times_{Z_0} Z \to W_\alpha$, where $W_\alpha \in \mathcal{G}$, and an admissible covering $\{V_{\alpha\beta} \to W_\alpha\}$ such that each of the pullback maps $V_{\alpha\beta} \times_{W_\alpha} (Y_\alpha \times_{Z_0} Z) \to (Y_\alpha \times_{Z_0} Z)$ belongs to $f^* \mathcal{G}(X)^{(1)}_{/Z}$. Let $Z'_0 = Z_0 \times \prod_\alpha W_\alpha \in \mathcal{G}$, so that the collection of products $\{Y_\alpha \times V_{\alpha\beta} \to Z'_0\}$ forms an admissible covering of Z'_0. We observe that each of the pullback maps $(Y_\alpha \times V_{\alpha\beta}) \times_{Z'_0} Z \to Z$ belongs to $\mathcal{G}^{(1)}_{/Z}$, so that $\mathcal{G}^{(1)}_{/Z}$ is covering as desired.

We now prove (c). We first claim that the collection of admissible morphisms in $\mathcal{G}(X)$ is stable under composition (note that this collection clearly contains all equivalences in $\mathcal{G}(X)$). Suppose we are given admissible morphisms $f : U \to V$ and $g : V \to W$ in $\mathcal{G}(X)$. We wish to prove that $g \circ f$ is an admissible morphism. Since f and g are admissible, there exist a pullback diagrams

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ U_0 & \xrightarrow{f_0} & V_0 \end{array} \qquad \begin{array}{ccc} V & \longrightarrow & W \\ \downarrow & & \downarrow \\ V_1 & \xrightarrow{g_0} & W_1 \end{array}$$

in $\mathcal{C}^{op}$, where f_0 and g_0 are admissible morphisms in $\mathcal{G}$. Write W as a filtered limit $\varprojlim\{W_\alpha\}$ of compact objects of $(\mathcal{C}_{W_1/})^{op}$, so that $V \simeq V_1 \times_{W_1} W$ is the limit of the filtered diagram $\{V_1 \times_{W_1} W_\alpha\}$. Since V_0 is a compact object of $\mathcal{C}$, the projection $V \to V_0$ factors as a composition

$$V \to V_1 \times_{W_1} W_\alpha \to V_0$$

for some index α. Replacing W_1 by W_α (and V_1 by $V_1 \times_{W_1} W_\alpha$), we may suppose that the map $V \to V_0$ factors through V_1. Replacing V_0 by V_1 (and U_0 by $U_0 \times_{V_0} V_1$), we may suppose that $V_0 = V_1$ (as objects of $\mathcal{C}_{/V}$). Then $g \circ f$ is a pullback of $g_0 \circ f_0$, and therefore admissible as desired.

It is clear that the Grothendieck topology on $\mathcal{G}(X)$ is generated by a Grothendieck topology on the subcategory of $\mathcal{G}(X)$ spanned by the admissible morphisms, and that this Grothendieck topology is finitary. To complete the proof of (c), it will suffice to show that the collection of admissible morphisms satisfies conditions (i) through (iii) of Definition V.1.2.1. Condition (i) is obvious, and condition (iii) follows from our assumption that the class of admissible morphisms is stable under retracts. To prove (ii), suppose we are given a diagram

$$\begin{array}{ccccc} & & V & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ U & & \xrightarrow{h} & & W \end{array}$$

in $\mathcal{G}(X)$, where g and h are admissible. We must show that f is admissible. As above, we can choose pullback diagrams

$$\begin{array}{ccc} U & \xrightarrow{h} & W \\ \downarrow & & \downarrow \\ U_0 & \xrightarrow{h_0} & W_0 \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{g} & W \\ \downarrow & & \downarrow \\ V_1 & \xrightarrow{g_0} & W_1 \end{array}$$

in $\mathcal{C}^{op}$, where g_0 and h_0 are admissible morphisms in $\mathcal{G}$. Replacing W_0 and W_1 by the product $W_0 \times W_1$, we can assume that $W_0 = W_1$. Write W as a filtered limit $\varinjlim\{W_\alpha\}$ of compact objects of $(\mathcal{C}_{W_0/})^{op}$. Then the map $\varprojlim\{U_0 \times_{W_0} W_\alpha\} \simeq U \xrightarrow{f} V \to V_1$ factors through $U_0 \times_{W_0} W_\alpha$ for some index α. Replacing W_0 by W_α, we reduce to the case where there exists a map $f_0 : U_0 \to V_1$ such that f is a pullback of f_0. Since g_0 and h_0 are admissible, the map f_0 is admissible (since $\mathcal{G}$ is assumed to be a geometry); it follows that f is an admissible morphism in $\mathcal{G}(X)$. □

Remark 9.15. In the situation of Proposition 9.14, suppose that we are given a morphism $f : X \to Y$ in the ∞-category $\mathcal{C}$. Then the left adjoint to the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ preserves compact objects, and therefore induces a functor $f^* : \mathcal{G}(X) \to \mathcal{G}(Y)$. It is easy to see that this functor is a transformation of geometries.

Remark 9.16. Let $\mathcal{C}$ be as in Proposition 9.14, let $f : A \to B$ be a morphism in $\mathcal{C}$, and let $f^* : \mathcal{G}(A) \to \mathcal{G}(B)$ be the transformation of geometries of Remark 9.15. Let $\mathcal{X}$ be an ∞-topos, and let $\mathcal{O}_{\mathcal{X}} : \mathcal{G}(B) \to \mathcal{X}$ be a left-exact functor, which we will identify with a $\mathcal{C}_{B/}$-valued sheaf $\mathcal{F}$ on $\mathcal{X}$. Then $f^* \circ \mathcal{O}_{\mathcal{X}}$ is a left-exact functor from $\mathcal{G}(A)$ to $\mathcal{X}$¡ which we can identify with the $\mathcal{C}_{A/}$-valued sheaf on $\mathcal{X}$ obtained by the composition

$$\mathcal{X}^{op} \xrightarrow{\mathcal{F}} \mathcal{C}_{B/} \to \mathcal{C}_{A/}.$$

Note that the collection of admissible morphisms and admissible coverings in $\mathcal{G}(B)$ is generated by the f^*-images of admissible morphisms and admissible coverings in $\mathcal{G}(A)$. Consequently, the sheaf $\mathcal{O}_{\mathcal{X}}$ is $\mathcal{G}(B)$-local if and only if $f^* \mathcal{O}_{\mathcal{X}}$ is $\mathcal{G}(A)$-local. Similarly, a morphism $\alpha : \mathcal{O}_{\mathcal{X}} \to \mathcal{O}'_{\mathcal{X}}$ between $\mathcal{G}(B)$-local sheaves is $\mathcal{G}(B)$-local if and only if it induces a $\mathcal{G}(A)$-local morphism $f^* \circ \mathcal{O}_{\mathcal{X}} \to f^* \circ \mathcal{O}'_{\mathcal{X}}$.

Proposition 9.17. *Let $\mathcal{C}$ be as in Proposition 9.14, let $f : A \to B$ be a morphism in $\mathcal{C}$, and let $\theta : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}(B)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}(A))$ be the functor given by composition with induced transformation of geometries $f^* : \mathcal{G}(A) \to \mathcal{G}(B)$. Then:*

(1) *For every object $R \in \mathcal{C}_{B/} \simeq \mathrm{Ind}(\mathcal{G}(B)^{op})$, the functor θ carries $\mathrm{Spec}^{\mathcal{G}(B)} R$ to $\mathrm{Spec}^{\mathcal{G}(A)} R$ (here we abuse notation by identifying R with its image in $\mathcal{C}_{A/} \simeq \mathrm{Ind}(\mathcal{G}(A)^{op})$).*

(2) *The functor θ induces an equivalence of ∞-categories*

$$ {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}(B)) \simeq {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}(B))_{\mathrm{Spec}^{\mathcal{G}(B)} B/} \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}(A))_{\theta(\mathrm{Spec}^{\mathcal{G}(B)} B)/} \simeq {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}(A))_{\mathrm{Spec}^{\mathcal{G}(A)} B/}.$$

(3) *The functor θ carries $\mathcal{G}(B)$-schemes to $\mathcal{G}(A)$-schemes, and induces an equivalence of ∞-categories*

$$\mathrm{Sch}(\mathcal{G}(B)) \to \mathrm{Sch}(\mathcal{G}(A))_{/\,\mathrm{Spec}^{\mathcal{G}}(A) B}.$$

In particular, for every object $B \in \mathcal{C}$ we have a categorical equivalence $\mathrm{Sch}(\mathcal{G}(B)) \simeq \mathrm{Sch}(\mathcal{G})_{/\,\mathrm{Spec}^{\mathcal{G}} B}$.

Proof. We first prove (1). Let $\mathcal{D}$ denote the full subcategory of $(\mathcal{C}_{R/})^{op}$ spanned by the admissible morphisms $R' \to R$ in $\mathcal{C}^{op}$, endowed with the Grothendieck topology determined by the geometry structure on $\mathcal{G}(R)$. Using the explicit construction of spectra described in §V.2.2, we see that $\mathrm{Spec}^{\mathcal{G}(A)} R$ can be identified with the pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, where $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{D})$ and $\mathcal{O}_{\mathcal{X}}$ corresponds to the $\mathcal{C}_{A/}$-valued sheaf given by sheafifying the presheaf given by the composition

$$\mathcal{F}_A : \mathcal{D}^{op} \subseteq \mathcal{C}_{R/} \to \mathcal{C}_{A/}.$$

Similarly, $\mathrm{Spec}^{\mathcal{G}(B)} R$ can be identified with the pair $(\mathcal{X}, \mathcal{O}'_{\mathcal{X}})$, where $\mathcal{O}'_{\mathcal{X}}$ is obtained by sheafifying the presheaf given by the composition

$$\mathcal{F}_B : \mathcal{D}^{op} \to \mathcal{C}_{R/} \to \mathcal{C}_{B/} .$$

If $\phi : \mathcal{C}_{B/} \to \mathcal{C}_{A/}$ denotes the forgetful functor, then we have an equivalence $\mathcal{F}_A \simeq \phi \mathcal{F}_B$, so that

$$\theta(\mathcal{X}, \mathcal{O}'_{\mathcal{X}}) = (\mathcal{X}, \phi\, \mathcal{O}'_{\mathcal{X}}) \simeq (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

as required.

We now prove (2). Replacing $\mathcal{C}$ by $\mathcal{C}_{A/}$, we are reduced to proving that θ induces an equivalence $\phi : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}(B)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})_{\mathrm{Spec}^{\mathcal{G}} B/}$. The map ϕ fits into a commutative diagram

$$\begin{array}{ccc} {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}(B)) & \xrightarrow{\phi} & {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G})_{\mathrm{Spec}^{\mathcal{G}} B/} \\ & \searrow \quad \swarrow & \\ & {}^{\mathrm{L}}\mathcal{T}\mathrm{op} & \end{array}$$

where the vertical maps are coCartesian fibrations, and the map ϕ preserves coCartesian morphisms. It therefore suffices to show that ϕ induces an equivalence of ∞-categories after passing to the fiber over any object $\mathcal{X} \in {}^{\mathrm{L}}\mathcal{T}\mathrm{op}$ (Corollary T.2.4.4.4). Unwinding the definitions, we are reduced to proving that homotopy coherent diagram

$$\begin{array}{ccc} \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}(B)}(\mathcal{X}) & \longrightarrow & \mathrm{Str}^{\mathrm{loc}}_{\mathcal{G}}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathcal{C}_{B/} & \longrightarrow & \mathcal{C} \end{array}$$

is a homotopy pullback square, where the vertical maps are given by the formation of global sections. Using Remark 9.16, we can reduce to proving the analogous assertion in the case where $\mathcal{G}$ is a discrete geometry. In this case, the above square is equivalent to the diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\mathcal{C}^{B/}}(\mathcal{X}) & \longrightarrow & \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \\ \downarrow & & \downarrow \\ \mathcal{C}^{B/} & \longrightarrow & \mathcal{C}, \end{array}$$

where the vertical maps are given by evaluation on the final object of $\mathcal{X}$. Since horizontal maps in this diagram are left fibrations, we can reduce (using Corollary T.2.4.4.4 again) to proving that the left vertical map induces a homotopy equivalence

$$\chi : \mathrm{Map}_{\mathrm{Fun}(\mathcal{X}^{op}, \mathcal{C})}(\mathcal{F}, \mathcal{G}) \to \mathrm{Map}_{\mathcal{C}}(B, \mathcal{G}(\mathbf{1})).$$

Here $\mathcal{F}$ denotes the constant functor $\mathcal{X}^{op} \to \mathcal{C}$ taking the value B, $\mathcal{G}$ denotes an arbitrary $\mathcal{C}$-valued sheaf on $\mathcal{X}$, and χ is given by evaluation on the final object $\mathbf{1}$ in $\mathcal{X}$. The desired result now follows from the observation that $\mathcal{F}$ is a left Kan extension of the constant functor $\{\mathbf{1}\} \to \{B\} \hookrightarrow \mathcal{C}$ along the inclusion $\{\mathbf{1}\} \hookrightarrow \mathcal{X}^{op}$.

Assertion (3) follows immediately from (1) and (2). □

Example 9.18. Let $\mathcal{C}$ denote the ∞-category CAlg of $\mathbb{E}_\infty$-rings, and let us identify the full subcategory $\mathcal{G} \subseteq \mathcal{C}^{op}$ with the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$ of Definition 2.10. We note that this example satisfies the hypothesis of Proposition 9.14: namely, the admissible morphisms in $\mathcal{C}$ are precisely those maps of $\mathbb{E}_\infty$-algebras of the form $A \mapsto A[\frac{1}{a}]$, where $a \in \pi_0 A$; this collection is stable under retracts by virtue of Remark 2.11. For every $\mathbb{E}_\infty$-ring k, the geometry $\mathcal{G}(k)$ described in Proposition 9.14 agrees with the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}(k)$ of Definition

2.10: this follows immediately from Remark 2.16. Proposition 9.17 then provides a justification for the terminology of Definition 2.27: the ∞-category of nonconnective spectral k-schemes is equivalent to the ∞-category of nonconnective spectral schemes lying over $\mathrm{Spec}^{\mathbf{Z}} k$. Similar reasoning applies if we replace the geometry $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$ by $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{\acute{e}t}}$ (using Proposition 8.17 in place of Remark 2.16; in this case, the admissible morphisms in $\mathcal{C}$ are precisely the étale morphisms of $\mathbb{E}_\infty$-rings).

Example 9.19. Let $\mathcal{C}$ denote the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings, and let us identify the full subcategory $\mathcal{G} \subseteq \mathcal{C}^{op}$ with the geometry $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}$ of Definition 2.10. This example also satisfies the hypothesis of Proposition 9.14. For every connective $\mathbb{E}_\infty$-ring k, the geometry $\mathcal{G}(k)$ described in Proposition 9.14 agrees with the geometry $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ of Definition 2.10: this follows again from Remark 2.16. It follows that the ∞-category of spectral k-schemes is equivalent to the ∞-category of spectral schemes lying over $\mathrm{Spec}^{\mathbf{Z}} k$. The same reasoning applies to spectral Deligne-Mumford stacks (using Proposition 8.17 again).

We can also apply Proposition 9.17 to the theory of derived schemes introduced in [42]. First, we need to recall a bit of notation. Let SCR denote the ∞-category of simplicial commutative rings, so that full subcategory $\mathrm{SCR}_{\leq 0} \subseteq \mathrm{SCR}$ spanned by the discrete objects can be identified with the nerve of the category of ordinary commutative rings. For every commutative ring k, we let Poly_k denote the category of polynomial algebras $k[x_1, \ldots, x_n]$, regarded as a full subcategory of the category of commutative k-algebras. There is an evident fully faithful embedding

$$f : \mathrm{N}(\mathrm{Poly}_k) \hookrightarrow (\mathrm{SCR}_{\leq 0})_{k/} \hookrightarrow \mathrm{SCR}_{k/}\,.$$

Using Proposition T.5.5.8.15, we deduce that this inclusion can be extended (in an essentially unique fashion) to a functor $F : \mathrm{SCR}_k = \mathcal{P}_\Sigma(\mathrm{N}(\mathrm{Poly}_k)) \to \mathrm{SCR}_{k/}$ which preserves sifted colimits.

Proposition 9.20. *For every commutative ring k, the functor $F : \mathrm{SCR}_k \to \mathrm{SCR}_{k/}$ defined above is an equivalence of ∞-categories.*

Proof. We first show that f preserves finite coproducts. Since F clearly preserves initial objects, it suffices to show that F preserves pairwise coproducts: in other words, we must show that for every pair of nonnegative integers m and n, the diagram

$$\begin{array}{ccc} k & \longrightarrow & k[x_1, \ldots, x_m] \\ \downarrow & & \downarrow \\ k[y_1, \ldots, y_n] & \longrightarrow & k[x_1, \ldots, x_m, y_1, \ldots, y_n] \end{array}$$

is a pushout square in the ∞-category SCR. In view of Proposition V.4.1.11, it suffices to show that the image of this diagram is a pushout square in the ∞-category CAlg of $\mathbb{E}_\infty$-rings. In other words, we must show that the canonical map

$$k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

is an equivalence of k-module spectra; this follows from Proposition A.7.2.2.13.

Using Proposition T.5.5.8.15, we deduce that the functor $F : \mathrm{SCR}_k \to \mathrm{SCR}_{k/}$ preserves small colimits. We next claim that F is fully faithful. In view of Proposition T.5.5.8.22, it suffices to show that every polynomial ring $R : k[x_1, \ldots, x_n]$ is a compact projective object of $\mathrm{SCR}_{k/}$. Let $e : \mathrm{SCR}_{k/}$ be the functor corepresented by R; we wish to show that e preserves sifted colimits. Since R is the coproduct of k with $\mathbf{Z}[x_1, \ldots, x_n]$ in SCR, we can identify e with the composition

$$\mathrm{SCR}_{k/} \xrightarrow{\theta} \mathrm{SCR} \xrightarrow{e'} \mathcal{S},$$

where e' is the functor corepresented by $\mathbf{Z}[x_1, \ldots, x_n]$ and θ is the evident forgetful functor. The functor θ preserves all contractible colimits (Proposition T.4.4.2.9) and $\mathbf{Z}[x_1, \ldots, x_n]$ is a compact projective object of SCR (Proposition T.5.5.8.22), we conclude that e preserves sifted colimits as desired.

Corollary T.5.5.2.9 guarantees that the functor F admits a right adjoint G. To complete the proof that the fully faithful functor F is an equivalence of ∞-categories, it will suffice to show that G is conservative. In other words, we must show that if $\alpha : R \to R'$ is a morphism in $\mathrm{SCR}_{k/}$ such that $G(\alpha)$ is an equivalence, then α is an equivalence. This follows from the existence of a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{SCR}}(\mathbf{Z}[x], R) & \longrightarrow & \mathrm{Map}_{\mathrm{SCR}}(\mathbf{Z}[x], R') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{SCR}_k}(k[x], G(R)) & \longrightarrow & \mathrm{Map}_{\mathrm{SCR}\,k}(k[x], R') \end{array}$$

where the vertical maps are homotopy equivalences. □

Example 9.21. Let $\mathcal{G}$ denote the full subcategory of SCR^{op} spanned by those objects which are compact in SCR, and let us identify $\mathcal{G}$ with the geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}$ introduced in §V.4.2. The hypothesis of Proposition 9.14 are satisfied, so that for every object $k \in \mathrm{SCR}$ we get an induced geometry $\mathcal{G}(k)$. If k is a discrete commutative ring, then we can identify $\mathcal{G}(k)$ with the geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$ of §V.4.2. It follows from Proposition 9.17 that the ∞-category of derived k-schemes is equivalent to the ∞-category of derived schemes lying over $\mathrm{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}}(k)$. Similarly reasoning shows that the ∞-category of derived Deligne-Mumford stacks over k is equivalent to the ∞-category of derived Deligne-Mumford stacks lying over $\mathrm{Spec}^{\mathcal{G}^{\mathrm{der}}_{\text{ét}}}(k)$.

Our final goal in this section is to explain the connection between the theory of spectral algebraic geometry developed in this paper with the theory of derived algebraic geometry introduced in [42]. Fix a commutative ring k; we will abuse notation by identifying k with a discrete $\mathbb{E}_\infty$-ring (see Proposition A.7.1.3.18). Let SCR_k denote the ∞-category of simplicial commutative k-algebras (see §V.4.1) and $\mathrm{CAlg}^{\mathrm{cn}}_k$ the ∞-category of connective $\mathbb{E}_\infty$-algebras over k. Proposition V.4.1.11 furnishes a forgetful functor $\theta : \mathrm{SCR}_k \to \mathrm{CAlg}^{\mathrm{cn}}_k$, which admits both right and left adjoints. We let $\Psi : \mathrm{CAlg}^{\mathrm{cn}}_k \to \mathrm{SCR}_k$ denote a left adjoint to θ.

Example 9.22. The functor Ψ carries free $\mathbb{E}_\infty$-algebras to polynomial algebras; that is, we have canonical equivalences $\Psi(k\{x_1, \ldots, x_n\}) \simeq k[x_1, \ldots, x_n]$.

Since the forgetful functor θ preserves small colimits (Proposition V.4.1.11), the functor Ψ preserves compact objects (Proposition T.5.5.7.2), and therefore induces left-exact functors

$$\Psi_{\mathrm{Zar}} : \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k) \qquad \Psi_{\text{ét}} : \mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k) \to \mathcal{G}^{\mathrm{der}}_{\text{ét}}(k).$$

Our first goal in this section is to prove the following:

Proposition 9.23. *The functors $\Psi_{\mathrm{Zar}} : \mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$ and $\Psi_{\text{ét}} : \mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k) \to \mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$ are transformations of geometries.*

To prove Proposition 9.23, we need to understand the functor Ψ a bit better. Note that the forgetful functor $\theta : \mathrm{SCR}_k \to \mathrm{CAlg}^{\mathrm{cn}}_k$ is compatible with the formation of the underlying spaces. In particular, for every object $A \in \mathrm{SCR}_k$, we have a canonical isomorphism $\pi_0 A \simeq \pi_0 \theta(A)$ of commutative k-algebras.

Lemma 9.24. *Let A be a connective $\mathbb{E}_\infty$-algebra over k. Then the unit map $A \to \theta(\Psi(A))$ induces an isomorphism of commutative rings*

$$\phi : \pi_0 A \to \pi_0 \theta(\Psi(A)) \simeq \pi_0 \Psi(A).$$

Proof. Let R be a (discrete) commutative k-algebra. We wish to show that composition with ϕ induces a bijection $\psi : \mathrm{Hom}(\pi_0 \Psi(A), R) \to \mathrm{Hom}(\pi_0 A, R)$. Regard R as a discrete object of SCR_k (so that $\theta(R)$ is a discrete object of SCR_k), we can identify ψ with the homotopy equivalence $\mathrm{Map}_{\mathrm{SCR}_k}(\Psi(A), R) \to \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{cn}}_k}(A, \theta(R))$ resulting from the adjunction between θ and Ψ. □

Lemma 9.25. *Let $f : A \to B$ be an étale morphism in $\mathrm{CAlg}_k^{\mathrm{cn}}$. Then the induced map $\Psi(A) \to \Psi(B)$ is an étale morphism in SCR_k.*

Proof. The morphism f induces an étale map of commutative rings $\pi_0 A \to \pi_0 B$. Using Lemma 9.24, we can identify $\pi_0 B$ with an étale algebra over the commutative ring $\pi_0 \Psi(A)$. Corollary V.4.3.12 ensures the existence of an (essentially unique) étale morphism $f' : \Psi(A) \to B'$ in SCR_k with $\pi_0 B' \simeq \pi_0 B$. The map f' is adjoint to a map of $\mathbb{E}_\infty$-algebras $g : A \to \theta(B')$, and $\pi_0 g : \pi_0 A \to \pi_0 \theta(B') \simeq \pi_0 B'$ lifts to an isomorphism $\pi_0 B \simeq \pi_0 B'$. Applying Corollary A.7.5.4.6, we deduce that g factors as a composition $A \xrightarrow{f} B \to \theta(B')$, so that f' factors as a composition

$$\Psi(A) \xrightarrow{\Psi(f)} \Psi(B) \xrightarrow{\gamma} B'.$$

Since f' is étale, it will suffice to show that γ is an equivalence. To this end, choose an arbitrary morphism $\Psi(A) \to R$ in SCR_k; we will show that composition with γ induces a homotopy equivalence

$$\mathrm{Map}_{(\mathrm{SCR}_k)_{\Psi(A)/}}(B', R) \to \mathrm{Map}_{(\mathrm{SCR}_k)_{\Psi(A)/}}(\Psi(B), R) \simeq \mathrm{Map}_{\mathrm{CAlg}_{A/}}(B, \theta(R)).$$

This is clear, since Propositions A.7.5.4.6 and V.4.3.11 allow us to identify both sides with the discrete space $\mathrm{Hom}_{\pi_0 A}(\pi_0 B, \pi_0 R)$. □

Example 9.26. Let A be a connective k-algebra, and let $a \in \pi_0 A \simeq \pi_0 \Psi(A)$. Lemmas 9.24 and 9.25 imply that the functor Ψ carries $A[\frac{1}{a}]$ to an étale $\Psi(A)$-algebra R with $\pi_0 R \simeq (\pi_0 A)[\frac{1}{a}]$. In other words, the functor Ψ commutes with localization of algebras: we have canonical equivalences $\Psi(A[\frac{1}{a}]) \simeq \Psi(A)[\frac{1}{a}]$.

Proof of Proposition 9.23. Lemma 9.25 shows that $\Psi_{\text{ét}}$ carries admissible morphisms in $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$ to admissible morphisms in $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$, and Example 9.26 shows that Ψ_{Zar} carries admissible morphisms in $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ to admissible morphisms in $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$. For each of the geometries under consideration, a collection of admissible morphisms $\{\mathrm{Spec}\, A_\alpha \to \mathrm{Spec}\, A\}$ is a covering if and only if there exists a finite collection of indices $\{\alpha_1, \ldots, \alpha_n\}$ such that the underlying map of commutative rings $\pi_0 A \to \prod_{1 \leq i \leq n} \pi_0 A_{\alpha_i}$. Using Lemma 9.24, we deduce that the functors Ψ_{Zar} and $\Psi_{\text{ét}}$ preserve admissible coverings. □

Using Proposition 9.23, we deduce that composition with Ψ_{Zar} and $\Psi_{\text{ét}}$ yields functors

$$\Theta_{\mathrm{Zar}} : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)) \qquad \Theta_{\text{ét}} : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)).$$

These functors can be described concretely as follows. Let $\mathcal{X}$ be an ∞-topos and $\mathcal{O}_{\mathcal{X}}$ a sheaf on $\mathcal{X}$ with values in SCR_k. Composition with the forgetful functor $\theta : \mathrm{SCR}_k \to \mathrm{CAlg}_k^{\mathrm{cn}}$ determines a sheaf of connective $\mathbb{E}_\infty$-algebras on $\mathcal{X}$. If $\mathcal{O}_{\mathcal{X}}$ is local or strictly Henselian, then $\theta(\mathcal{O}_{\mathcal{X}})$ has the same property; the functors Θ_{Zar} and $\Theta_{\text{ét}}$ are both given by the construction

$$(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \mapsto (\mathcal{X}, \theta\, \mathcal{O}_{\mathcal{X}}).$$

Proposition 9.27. *The functor Θ_{Zar} carries derived schemes over k to spectral schemes over k, and the functor $\Theta_{ét}$ carries derived Deligne-Mumford stacks over k to spectral Deligne-Mumford stacks over k.*

Proof. The assertion is local. It therefore suffices to show that for every object $A \in \mathrm{SCR}_k$, we have $\Theta_{\mathrm{Zar}} \mathrm{Spec}^{\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)} A \simeq \mathrm{Spec}^{\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)} \theta(A)$ and $\Theta_{\text{ét}} \mathrm{Spec}^{\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)} A \simeq \mathrm{Spec}^{\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)} \theta(A)$. Since the forgetful functor θ induces an equivalence from the ∞-category of étale A-algebras in SCR_k to the ∞-category of étale $\theta(A)$-algebras in CAlg_k (both are equivalent to the nerve of the ordinary category of étale $\pi_0 A$-algebras), this follows from the explicit construction of spectra given in §V.2.2. Alternatively, one can deduce the first assertion by combining Theorems 2.40 and V.4.2.15 (after reducing to the affine case), and the second assertion by combining Theorems 8.42 and V.4.3.32. □

In what follows, we let ${}_{\mathrm{Z}}\mathrm{Spec}^{\mathrm{der}}_{\mathrm{Sp}}$ and ${}_{\text{ét}}\mathrm{Spec}^{\mathrm{der}}_{\mathrm{Sp}}$ denote the relative spectrum functors associated to the transformations of geometries

$$\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k) \to \mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k) \qquad \mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k) \to \mathcal{G}^{\mathrm{der}}_{\text{ét}}(k).$$

Corollary 9.28. *The adjoint functors* $\Theta_{\mathrm{Zar}} : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k))$ *and*

$$_{\mathrm{Z}}\,\mathrm{Spec}^{\mathrm{der}}_{\mathrm{Sp}} : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k))$$

restrict to determine an adjunction

$$\mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)) \underset{}{\overset{\Theta_{\mathrm{Zar}}}{\rightleftarrows}} \mathrm{Sch}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)).$$

The adjoint functors

$$\Theta_{\acute{e}t} : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\acute{e}t}(k))$$

and

$$_{\acute{e}t}\,\mathrm{Spec}^{\mathrm{der}}_{\mathrm{Sp}} : {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\acute{e}t}(k)) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k))$$

restrict to determine an adjunction

$$\mathrm{Sch}(\mathcal{G}^{\mathrm{der}}_{\acute{e}t}(k)) \underset{}{\overset{\Theta_{\acute{e}t}}{\rightleftarrows}} \mathrm{Sch}(\mathcal{G}^{\mathrm{Sp}}_{\acute{e}t}(k)).$$

If k is an algebra over the ring **Q** *of natural numbers, then each of these functors is an equivalence of ∞-categories.*

Proof. The first two assertions follow from Proposition 9.27 (note that a relative spectrum functor $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}$ always carries $\mathcal{G}$-schemes to $\mathcal{G}'$-schemes). The final assertion follows from the observation that if k is a **Q**-algebra, then the forgetful functor $\theta : \mathrm{SCR}_k \to \mathrm{CAlg}^{\mathrm{cn}}_k$ is an equivalence of ∞-categories (Proposition V.4.1.11). □

We conclude this section by giving an explicit description of the relative spectrum functors $_{\mathrm{Z}}\,\mathrm{Spec}^{\mathrm{der}}_{\mathrm{Sp}}$ and $_{\acute{\mathrm{e}}\mathrm{t}}\,\mathrm{Spec}^{\mathrm{der}}_{\mathrm{Sp}}$ appearing in the statement of Corollary 9.28. Fix an ∞-topos $\mathcal{X}$. Let $\mathcal{O}_{\mathcal{X}}$ be a connective sheaf of $\mathbb{E}_\infty$-algebras over k on $\mathcal{X}$, which we view as a functor $\mathcal{X}^{op} \to \mathrm{CAlg}^{\mathrm{cn}}_k$. The composite functor $\Psi \circ \mathcal{O}_{\mathcal{X}} : \mathcal{X}^{op} \to \mathrm{SCR}_k$ need not be a SCR_k-valued sheaf on $\mathcal{X}$. However, it admits a sheafification, which we will denote by $\mathcal{O}^{\Psi}_{\mathcal{X}}$ (see Lemma 1.12).

Proposition 9.29. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an object of ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k))$. Then the canonical map*

$$\phi : \mathcal{O}_{\mathcal{X}} \to (\theta \circ \Psi)\,\mathcal{O}_{\mathcal{X}} \to \theta(\mathcal{O}^{\Psi}_{\mathcal{X}})$$

determines an equivalence

$$(\mathcal{X}, \mathcal{O}^{\Psi}_{\mathcal{X}}) \simeq_{\mathrm{Z}} \mathrm{Spec}^{\mathrm{der}}_{\mathrm{Sp}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

If $\mathcal{O}_{\mathcal{X}}$ is strictly Henselian, then ϕ determines an equivalence $(\mathcal{X}, \mathcal{O}^{\Psi}_{\mathcal{X}}) \simeq_{\acute{e}t} \mathrm{Spec}^{\mathrm{der}}_{\mathrm{Sp}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Proof. Using Lemma 9.24, we deduce that the canonical map $\pi_0\,\mathcal{O}_{\mathcal{X}} \to \pi_0\,\mathcal{O}^{\Psi}_{\mathcal{X}}$ is an isomorphism. Using Corollary 2.25 and Remark V.4.2.13, we deduce that $\mathcal{O}^{\Psi}_{\mathcal{X}}$ is local (and can therefore be identified with a $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$-structure on $\mathcal{X}$); similarly, if $\mathcal{O}_{\mathcal{X}}$ is strictly Henselian, then Remark V.4.3.18 guarantees that $\mathcal{O}^{\Psi}_{\mathcal{X}}$ is strictly Henselian (and can therefore be identified with a $\mathcal{G}^{\mathrm{der}}_{\acute{\mathrm{e}}\mathrm{t}}(k)$-structure on $\mathcal{X}$). Let $\mathcal{Y}$ be any other ∞-topos and $\mathcal{O}_{\mathcal{Y}}$ any SCR_k-valued sheaf on $\mathcal{Y}$. It is easy to see that ϕ induces a homotopy equivalence

$$\mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{der}}_{\mathrm{disc}}(k))}((\mathcal{X}, \mathcal{O}^{\Psi}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})) \to \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}(k))}((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \theta\,\mathcal{O}_{\mathcal{Y}})).$$

To complete the proof, it will suffice to show the following:

(1) If $\mathcal{O}_{\mathcal{Y}}$ is local and $f^* : \mathcal{X} \to \mathcal{Y}$ is a geometric morphism, then a morphism $\alpha : f^*\,\mathcal{O}_{\mathcal{X}} \to \theta\,\mathcal{O}_{\mathcal{Y}}$ in $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}_k}(\mathcal{Y})_{\geq 0}$ is local (with respect to the geometry $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$) if and only if the adjoint morphism $\beta : f^*(\mathcal{O}^{\Psi}_{\mathcal{X}}) \to \mathcal{O}_{\mathcal{Y}}$ is local (with respect to the geometry $\mathcal{G}^{\mathrm{der}}_{\mathrm{Zar}}(k)$).

(2) If $\mathcal{O}_{\mathcal{Y}}$ is strictly Henselian, then α is local (with respect to the geometry $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}(k)$) if and only if β is local (with respect to the geometry $\mathcal{G}^{\mathrm{der}}_{\text{ét}}(k)$).

To prove (1), we observe that both conditions are equivalent to the locality of the induced map $\pi_0 f^* \mathcal{O}_{\mathcal{X}} \simeq \pi_0 f^* \mathcal{O}^{\Psi}_{\mathcal{X}} \to \pi_0 \mathcal{O}_{\mathcal{Y}} \simeq \pi_0 \theta \mathcal{O}_{\mathcal{Y}}$, since the pregeometries $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}(k)$ and $\mathcal{T}_{\mathrm{Zar}}(k)$ are compatible with 0-truncations (Corollary 2.24 and Remark V.4.2.11). Similarly, assertion (2) follows since the pregeometries $\mathcal{T}^{\mathrm{Sp}}_{\text{ét}}(k)$ and $\mathcal{T}_{\text{ét}}(k)$ are compatible with 0-truncations (Propositions V.4.3.28 and 8.40). □

Remark 9.30. Proposition 9.29 asserts that if $\mathcal{O}_{\mathcal{X}}$ is a local sheaf of $\mathbb{E}_\infty$-algebras over k on an ∞-topos $\mathcal{X}$, then its relative spectrum has the same underlying ∞-topos, with structure sheaf given by the sheafification of the presheaf $(U \in \mathcal{X}) \mapsto \Psi \mathcal{O}_{\mathcal{X}}(U)$. If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a connective spectral scheme over k (or a connective spectral Deligne-Mumford stack over k), then we can be even more explicit. If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is the spectrum of a connective k-algebra A, then the results of §V.2.2 show that $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathcal{C})$, where $\mathcal{C}$ is the opposite of the ∞-category of étale A-algebras. Similarly, the spectrum of $\Psi(A)$ has the form $(\mathcal{X}', \mathcal{O}'_{\mathcal{X}'})$, where $\mathcal{X}' = \mathcal{S}\mathrm{hv}(\mathcal{C}')$ for $\mathcal{C}'$ the opposite of the ∞-category of étale $\Psi(A)$-algebras in SCR_k. Lemmas 9.24 and 9.25 show that the functor Ψ determines an equivalence from $\mathcal{C}$ to $\mathcal{C}'$, thereby giving an identification of $\mathcal{X}$ with $\mathcal{X}'$. The composition of $\mathcal{O}_{\mathcal{X}}$ with the canonical map $\phi : \mathcal{C}^{op} \to \mathcal{P}(\mathcal{C})^{op} \to \mathcal{S}\mathrm{hv}(\mathcal{C})^{op}$ can be identified with the composition $\mathcal{C}^{op} \subseteq \mathrm{CAlg}_{A/} \to \mathrm{CAlg}_k$, and the restriction $\mathcal{O}'_{\mathcal{X}'} \,|\, \mathcal{C}'^{op}$ admits a similar description. It follows that the canonical map $\Psi \mathcal{O}_{\mathcal{X}}(U) \to \mathcal{O}'_{\mathcal{X}'}(U)$ is an equivalence whenever U lies in the essential image of ϕ. Extrapolating to the non-affine case, we arrive at the following conclusion: the sheafification of the presheaf formulate our observation as follows: the sheafification of the presheaf $(U \in \mathcal{X}) \mapsto \Psi \mathcal{O}_{\mathcal{X}}(U)$ does not change the values of that presheaf on any object $U \in \mathcal{X}$ for which $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} |U)$ is affine.

References

[1] Atiyah, M. and I. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ontario.

[2] Behrend, K. and B. Fantechi. *The intrinsic normal cone.* Inventiones Mathematicae 128 (1997) no. 1, 45-88.

[3] Ben-Zvi, D., Francis, J., and D. Nadler. *Integral Transforms and and Drinfeld Centers in Derived Algebraic Geometry.*

[4] Bergner, J. *Three models for the homotopy theory of homotopy theories.* Topology 46 (2007), no. 4, 397–436.

[5] Boardman, J. and R. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces.* Lecture Notes in Mathematics, 347, Springer-Verlag (1973).

[6] Crane, L. and D.N. Yetter. *Deformations of (bi)tensor categories.* Cahiers Topologie Geom. Differentielle Categ. 39 (1998), no. 3, 163–180.

[7] Deligne, P. *Catégories tannakiennes.* The Grothendieck Festschrift, Vol. II, 111195, Progr. Math., 87, Birkhuser Boston, Boston, MA, 1990.

[8] Grothendieck, A. *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : III. Étude cohomologique des faisceaux cohérents, Première partie.* Publications Mathématiques de l'IHÉS, 11 (1961), p. 5-167

Demazure, M. and Grothendieck, A., eds. *Schémas en groupes vol. 1-3.* Séminaire de Géométrie Algbrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Mathematics, Vol. 151-153 Springer-Verlag, Berlin-New York.

[9] Demazure, M. and Grothendieck, A., eds. *Schémas en groupes vol. 1-3.* Séminaire de Géométrie Algbrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Mathematics, Vol. 151-153 Springer-Verlag, Berlin-New York.

[10] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. I: General Theory.* Adv. Math. 222 (2009), no. 2, 359–401.

[11] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. II: Pro-representability of the deformation functor.* Available at arXiv:math/0702839v3 .

[12] Efimov, A., Lunts, V., and D. Orlov. *Deformation theory of objects in homotopy and derived categories. III: Abelian categories.* Available as arXiv:math/0702840v3 .

[13] Elmendorf, A.D., Kriz, I. , Mandell, M.A., and J.P. May. *Rings, modules and algebras in stable homotopy theory.* Mathematical Surveys and Monographs 47, American Mathematical Society, 1997.

[14] Etingof, P., Nikshych, D., and V. Ostrik. *On fusion categories.* Ann. of Math. (2) 162 (2005), no. 2, 581–642.

[15] Francis, J. *Derived Algebraic Geometry over $\mathcal{E}_n$-Rings.* Unpublished MIT PhD dissertation.

[16] Frenkel, E., Gaitsgory, D., and K. Vilonen. *Whittaker patterns in the geometry of moduli spaces of bundles on curves.* Ann. of Math. (2) 153 (2001), no. 3, 699–748.

[17] Fresse, B. *Koszul duality of E_n-operads.* Available as arXiv:0904.3123v6 .

[18] Fukuyama, H. and I. Iwanari. *Monoidal Infinity Category of Complexes from Tannakian Viewpoint.* Available at http://arxiv.org/abs/1004.3087.

[19] Fulton, W. *Algebraic curves.* W.A. Benjamin, Inc., New York, 1969.

[20] Fulton, W. and R. Pandharipande. *Notes on stable maps and quantum cohomology.* Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[21] Getzler, E. *Lie theory for L_∞-algebras.* Ann. of Math. (2) 170 (2009), no. 1, 271–301.

[22] Goldman, W. and J. Millson. *Deformations of flat bundles over Kähler manifolds.* Geometry and Topology, Manifolds, Varieties and Knots (C. McCrory and T. Shifrin, eds.), Lecture Notes in Pure and Applied Math., vol. 105, Marcel Dekker, New York/Basel, 1987, pp. 129-145.

[23] Goldman, W. and J. Millson. *The deformation theory of representations of fundamental groups of compact Kähler manifolds.* Bull. Amer. Math. Soc. (N.S.), Volume 18, Number 2 (1988), 153-158.

[24] Grauert, H. and R. Remmert. *Coherent Analytic Sheaves.* Springer, 1984.

[25] Grothendieck, A. *Sur quelques points d'algebra homologique.* Tohoku Math. J. 9, 1957, 119–221.

[26] Hinich, V. *DG coalgebras as formal stacks.* J. Pure Appl. Algebra, 162 (2001), 209-250.

[27] Hinich, V. *Deformations of homotopy algebras.* Communication in Algebra, 32 (2004), 473-494.

[28] Johnstone, P. *Stone Spaces.* Cambridge University Press, Cambridge, 1982.

[29] Kapranov, M. and V. Ginzburg. *Koszul duality for Operads.* Duke Math. J., 1994.

[30] Keller, B. and W. Lowen. *On Hochschild cohomology and Morita deformations.* Int. Math. Res. Not. IMRN 2009, no. 17, 3221–3235.

[31] Knutson, D. *Algebraic spaces.* Lecture Notes in Mathematics 203, Springer-Verlag, 1971.

[32] Kontsevich, M. and Y. Soibelman. *Deformations of algebras over operads and the Deligne conjecture.* Conference Moshe Flato 1999, Vol. I (Dijon), 255–307, Math. Phys. Stud., 21, Kluwer Acad. Publ., Dordrecht, 2000.

[33] Kontsevich, M. and Y. Soibelman. *Deformation Theory.* Unpublished book available at http://www.math.ksu.edu/šoibel/Book-vol1.ps .

[34] Kontsevich, M. and Y. Soibelman. *Notes on A-infinity algebras, A-infinity categories and noncommutative geometry.* Available as math.RA/060624.

[35] Laplaza, M. *Coherence for distributivity.* Coherence in categories, 29-65. Lecture Notes in Mathematics 281, Springer-Verlag, 1972.

[36] Laumon, G. and L. Moret-Bailly. *Champs algebriques.* Springer-Verlag, 2000.

[37] Lowen, W. *Obstruction theory for objects in abelian and derived categories.* Comm. Algebra 33 (2005), no. 9, 3195–3223.

[38] Lowen, W. *Hochschild cohomology, the characteristic morphism, and derived deformations.* Compos. Math. 144 (2008), no. 6, 1557–1580.

[39] Lurie, J. *Tannaka Duality for Geometric Stacks.* Available for download at http://www.math.harvard.edu/ lurie/ .

[40] Lurie, J. *Higher Topos Theory.* Available for download at http://www.math.harvard.edu/ lurie/ .

[41] Lurie, J. *Higher Algebra.* Available for download.

[42] Lurie, J. *Derived Algebraic Geometry V: Structured Spaces.* Available for download.

[43] Lurie, J. *Derived Algebraic Geometry VII: Spectral Schemes.* Available for download.

[44] Lurie, J. *Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems.* Available for download.

[45] Lurie, J. *Derived Algebraic Geometry IX: Closed Immersions.* Available for download.

[46] Lurie, J. *Derived Algebraic Geometry X: Formal Moduli Problems.* Available for download.

[47] Lurie, J. *Derived Algebraic Geometry XI: Descent Theorems*

[48] Lurie, J. *Derived Algebraic Geometry XIV: Representability Theorems.*

[49] MacLane, S., and I. Moerdijk. *Sheaves in Geometry and Logic.* Springer-Verlag, Berlin and New York, 1992.

[50] Manetti, M. *Extended deformation functors.* Int. Math. Res. Not. 2002, no. 14, 719–756.

[51] Matsumura, H. *Commutative ring theory.* Cambridge University Press, 1986.

[52] May, P. *The Geometry of Iterated Loop Spaces.* Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972. viii+175 pp.

[53] Nisnevich, Y. *The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory.* NATO Advanced Study Institute Series, Ser. C, v. 279, Kluwer, 1989, pp. 241-342.

[54] Pridham, J. *Unifying derived deformation theories.* Adv. Math. 224 (2010), no.3, 772-826.

[55] Quillen, D. *Homotopical Algebra.* Lectures Notes in Mathematics 43, SpringerVerlag, Berlin, 1967.

[56] Raynaud, M. *Anneaux locaux henséliens.* Lecture Notes in Mathematics, 169, Berlin-New York: Springer-Verlag.

[57] Rognes, J. *Galois extensions of structured ring spectra. Stably dualizable groups.* Mem. Amer. Math. Soc. 192 (2008), no. 898, viii+137 pp.

[58] Schlessinger, M. and J. Stasheff. *The Lie algebra structure of tangent cohomology and deformation theory.* Journal of Pure and Applied Algebra 38 (1985), 313-322.

[59] Schlessinger, M. *Functors of Artin Rings.* Trans. Amer. Math. Soc. 130 1968 208–222.

[60] Serre, Jean-Pierre. *Local algebra.* Springer-Verlag, 2000.

[61] Thomason, R. W. and T. Trobough. *Higher Algebraic K-theory of schemes and of derived categories.* The Grothendieck Festschrift III, Progress in Math., vol. 88, Birkhuser, 1990, pp. 247435.

[62] Tits, J. *Sur les analogues algbriques des groupes semi-simples complexes.* Colloque dalgebre superieure, tenu a Bruxelles du 19 au 22 decembre 1956, Centre Belge de Recherches Mathematiques Etablissements Ceuterick, Louvain, Paris: Librairie Gauthier-Villars, pp. 261289.

[63] Toën, B. *Champs affines.* Selecta mathematica, new series, 12, 2006.

[64] Toën, B. *Finitude homotopique des dg-algèbres propres et lisses.* Available at math.AT/0609762. Finitude homotopique des dg-algebras

[65] Toën, B. *The homotopy theory of dg-categories and derived Morita theory.* Invent. Math. 167 (2007), no. 3, 615–667.

[66] Toën, B. *Derived Azumaya's algebras and generators for twisted derived categories.*

[67] Toën, B., and M. Vaquié. *Moduli of objects in dg-categories.* Ann. Sci. cole Norm. Sup. (4) 40 (2007), no. 3, 387–444.

[68] Toën, B. and G. Vezzosi. *Brave new algebraic geometry and global derived moduli spaces of ring spectra.* Elliptic cohomology, 325359, London Math. Soc. Lecture Note Ser., 342, Cambridge Univ. Press, Cambridge, 2007.

[69] Toën, B. and G. Vezzosi. *Homotopical algebraic geometry. I. Topos theory.* Adv. Math. 193 (2005), no. 2, 257372.

[70] Toën, B. and G. Vezzosi. *Homotopical algebraic geometry. II. Geometric stacks and applications.* Mem. Amer. Math. Soc. 193 (2008), no. 902, x+224 pp.

[71] Toën, B. and G. Vezzosi. *From HAG to DAG: derived moduli stacks.* Axiomatic, enriched and motivic homotopy theory, 173216, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer Acad. Publ., Dordrecht, 2004.

[72] Toën, B. and G. Vezzosi. *Algebraic geometry over model categories.* Available for download: math.AG/0110109.

[73] Quillen, D. *Rational homotopy theory.* Ann. of Math. (2) 90 1969 205–295.

[74] Wallbridge, J. *Higher Tannaka duality.* Doctoral dissertation.

[75] Yetter, D. *Braided deformations of monoidal categories and Vassiliev invariants.* Higher category theory (Evanston, IL, 1997), 117–134, Contemp. Math., 230, Amer. Math. Soc., Providence, RI, 1998.