REVISITING THE DE RHAM-WITT COMPLEX

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Abstract. The goal of this paper is to offer a new construction of the de Rham-Witt complex of smooth varieties over perfect fields of characteristic p.

We introduce a category of cochain complexes equipped with an endomorphism F (of underlying graded abelian groups) satisfying $dF = pFd$, whose homological algebra we study in detail. To any such object satisfying an abstract analog of the Cartier isomorphism, an elementary homological process associates a generalization of the de Rham-Witt construction. Abstractly, the homological algebra can be viewed as a calculation of the fixed points of the Berthelot-Ogus operator $L\eta_p$ on the p-complete derived category. We give various applications of this approach, including a simplification of the crystalline comparison in $A\Omega$-cohomology theory.

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1. Introduction

1.1. **The de Rham-Witt complex.** Let X be a smooth algebraic variety defined over a field k. The *algebraic de Rham cohomology* $\mathrm{H}^*_{\mathrm{dR}}(X)$ is defined as the hypercohomology of the de Rham complex

$$\Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots.$$

When $k = \mathbf{C}$ is the field of complex numbers, Grothendieck [20] showed that the algebraic de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(X)$ is isomorphic to the usual de Rham cohomology of the underlying complex manifold $X(\mathbf{C})$ (and therefore also to the singular cohomology of the topological space $X(\mathbf{C})$, with complex coefficients). However, over fields of characteristic $p > 0$, algebraic de Rham cohomology is a less satisfactory invariant. This is due in part to the fact that it takes values in the category of vector spaces over k, and therefore consists entirely of p-torsion. To address this point, Berthelot [5] and Grothendieck [21] introduced the theory of *crystalline cohomology.* To every smooth algebraic variety over a perfect field k of characteristic p, this theory associates cohomology groups $\mathrm{H}^*_{\mathrm{crys}}(X)$ which are modules over the ring of Witt vectors $W(k)$, and can be regarded as integral versions of the de Rham cohomology groups $\mathrm{H}^*_{\mathrm{dR}}(X)$.

The crystalline cohomology groups of a variety X were originally defined as the cohomology groups of the structure sheaf of a certain ringed topos, called the *crystalline topos* of X. However, Bloch [14] (in the case of small dimension) and Deligne-Illusie [28] later gave an alternative description of crystalline cohomology, which is closer in spirit to the definition of algebraic de Rham cohomology. More precisely, they showed that the crystalline cohomology of a smooth variety X can be realized as the hypercohomology of a complex of sheaves

$$W\Omega^0_X \xrightarrow{d} W\Omega^1_X \xrightarrow{d} W\Omega^2_X \xrightarrow{d} \cdots$$

called the *de Rham-Witt complex of* X. The complex $W\Omega^*_X$ is related to the usual de Rham complex by the following pair of results:

Theorem 1.1.1. *Let X be a smooth algebraic variety defined over a perfect field k. Then there is a canonical map of cochain complexes of sheaves $W\Omega^*_X \to \Omega^*_X$, which induces a quasi-isomorphism $W\Omega^*_X/pW\Omega^*_X \to \Omega^*_X$.*

Theorem 1.1.2. *Let $\mathfrak{X}$ be a smooth formal scheme over* $\mathrm{Spf}(W(k))$ *with central fiber $X = \mathrm{Spec}(k)\times_{\mathrm{Spf}(W(k))}\mathfrak{X}$. Moreover, suppose that the Frobenius map $\varphi_X\colon X \to X$ extends to a map of formal schemes $\varphi_{\mathfrak{X}}\colon \mathfrak{X} \to \mathfrak{X}$. Then there is a natural quasi-isomorphism $\Omega^*_{\mathfrak{X}} \to W\Omega^*_X$ of cochain complexes of abelian sheaves on the underlying topological space of X (which is the same as the underlying topological space of $\mathfrak{X}$). Here $\Omega^*_{\mathfrak{X}}$ denotes the de Rham complex of the formal scheme $\mathfrak{X}$ relative to $W(k)$.*

Warning 1.1.3. In the situation of Theorem 1.1.2, the map of cochain complexes $\Omega^*_{\mathfrak{X}} \to W\Omega^*_X$ depends on the choice of map $\varphi_{\mathfrak{X}}$, though the induced map on cohomology groups (or even in the derived category) is independent of that choice.

The proofs of Theorems 1.1.1 and 1.1.2 which appear in [28] depend on some rather laborious calculations. Our aim in this paper is to present an alternate construction of the complex $W\Omega^*_X$ (which agrees with the construction of Deligne and Illusie for smooth varieties; see Theorem 4.4.12) which can be used to prove Theorems 1.1.1 and 1.1.2 in an essentially calculation-free way: the only input we will need is the Cartier isomorphism (which we recall as Theorem 3.3.6) and some elementary homological algebra.

The de Rham-Witt complexes constructed in this paper agree with those of [28] in the smooth case, but differ in general. For this reason, we distinguish the two constructions notationally: we write $W\Omega^*_X$ for the construction from [28] and refer to it as the *classical de Rham-Witt complex*, while we write $\mathcal{W}\Omega^*_X$ for our construction and refer to it as the *saturated de Rham-Witt complex*.

1.2. **Overview of the Construction.** Assume for simplicity that k is a perfect field and that $X = \operatorname{Spec}(R)$ is the spectrum of a smooth k-algebra R. In this case, we can identify $\mathrm{H}^*_{\mathrm{dR}}(X)$ with the cohomology of a cochain complex of k-vector spaces (Ω^*_R, d). This complex admits both concrete and abstract descriptions:

(a) For each $n \geq 0$, Ω^n_R is given by the nth exterior power of the module of Kähler differentials $\Omega_{R/k}$, which is a projective R-module of finite rank. The de Rham differential $d\colon \Omega^n_R \to \Omega^{n+1}_R$ is then given concretely by the formula

$$d(a_0(da_1 \wedge da_2 \wedge \cdots \wedge da_n)) = da_0 \wedge da_1 \wedge \cdots \wedge da_n$$

for $a_0, a_1, \ldots, a_n \in R$.

(b) The de Rham complex (Ω^*_R, d) has the structure of a commutative differential graded algebra over $\mathbf{F}_p$ (see §3.1). Moreover, it is characterized by the following universal property: if (A^*, d) is any commutative differential graded algebra over $\mathbf{F}_p$, then every ring homomorphism $R \to A^0$ extends uniquely to a map of commutative differential graded algebras $\Omega^*_R \to A^*$.

Both of these descriptions have analogues for our saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$. Let us begin with (a). Fix an isomorphism $R \simeq \widetilde{R}/p\widetilde{R}$, where $\widetilde{R}$ is a $W(k)$-algebra which is p-adically complete and p-torsion-free (such an isomorphism always exists, by virtue our assumption that R is smooth over k). Let $\widehat{\Omega}^*_{\widetilde{R}}$ denote the p-adic completion of the de Rham complex of $\widetilde{R}$ (or, equivalently, of the de Rham complex of $\widetilde{R}$ relative to $W(k)$). Then $\widehat{\Omega}^*_{\widetilde{R}}$ is a commutative differential graded algebra over $W(k)$, and we have a canonical isomorphism $\Omega^*_R \simeq \widehat{\Omega}^*_{\widetilde{R}}/p\widehat{\Omega}^*_{\widetilde{R}}$. Beware that the cochain complex $\widehat{\Omega}^*_{\widetilde{R}}$ depends on the choice of $\widetilde{R}$, and does not depend functorially on R.

We now make a further choice: a ring homomorphism $\varphi: \widetilde{R} \to \widetilde{R}$ which is compatible with the Frobenius endomorphism of R (so that $\varphi(x) \equiv x^p \pmod{p}$). Then φ^* is divisible by p^n on $\Omega^n_{\widetilde{R}}$, so we obtain a homomorphism of graded algebras

$$F: \widehat{\Omega}^*_{\widetilde{R}} \to \widehat{\Omega}^*_{\widetilde{R}} \tag{1}$$

by the formula

$$\begin{aligned} F(a_0 da_1 \wedge da_2 \wedge \cdots \wedge da_n) &:= \text{“}\frac{1}{p^n} \cdot \varphi^*(a_0 da_1 \wedge da_2 \wedge \cdots \wedge da_n)\text{”} \\ &:= \varphi(a_0) \frac{d\varphi(a_1)}{p} \wedge \cdots \wedge \frac{d\varphi(a_n)}{p}. \end{aligned}$$

The homomorphism F is not a map of *differential* graded algebras because it does not commute with the differential d: instead, we have the identity $dF(\omega) = pF(d\omega)$ for each $\omega \in \widehat{\Omega}^*_{\widetilde{R}}$.

Motivated by the preceding observation, we define a *Dieudonné complex* to be a cochain complex of abelian groups (M^*, d) equipped with a map of graded abelian groups $F: M^* \to M^*$ satisfying $dF = pFd$ (Definition 2.1.1). We devote §2 of this paper to a general study of the homological algebra of Dieudonné complexes.

For any Dieudonné complex $M^* = (M^*, d, F)$, the Frobenius map F determines a map of cochain complexes

$$(M^*/pM^*, 0) \to (M^*/pM^*, d). \tag{2}$$

In the case where $M^* = \widehat{\Omega}^*_{\widetilde{R}}$ is the completed de Rham complex of $\widetilde{R}$, this map turns out to be a quasi-isomorphism: that is, it induces a (Frobenius-semilinear) isomorphism between the de Rham *complex* Ω^*_R and the de Rham *cohomology* $\mathrm{H}^*_{\mathrm{dR}}(\mathrm{Spec}(R))$. This is (the inverse of) the classical *Cartier isomorphism*. In general, we say that a Dieudonné complex M^* is of *Cartier type* if it is p-torsion-free and the map (2) is a quasi-isomorphism (Definition 2.4.1).

Given an arbitrary Dieudonné complex M^*, we will construct a new (generally much larger) Dieudonné complex $\mathcal{W}\mathrm{Sat}(M^*)$, which we call the *completed saturation* of M^*. This is a combination of two simpler constructions: a saturation process which produces a Dieudonné complex with an additional *Verschiebung* operator V, and a completion process which ensures completeness with respect to V. When M^* is of Cartier type, we will show that this procedure does not change the cohomology of M^*/pM^* (Theorem 2.4.2).

Our saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ can be described as the completed saturation of $\widehat{\Omega}^*_{\widetilde{R}}$. We will see that it is equipped with an algebra structure which is (in a suitable sense) compatible with its structure as a Dieudonné complex, endowing it with the structure of what we will call a *strict Dieudonné algebra* (Definition 3.5.6). The Frobenius map on R induces a ring homomorphism

$$R \to \mathrm{H}^0(\mathcal{W}\Omega^*_R/p),$$

which is analogous to observing that the de Rham differential on Ω^*_R is linear over the subring of p-th powers in R. In the setting of strict Dieudonné algebras, we can use this map to characterize $\mathcal{W}\Omega^*_R$ by a universal property in the spirit of (b) (see Definition 4.1.1 and Proposition 2.7.1):

$(*)$ If A^* is any strict Dieudonné algebra, then every ring homomorphism $R \to \mathrm{H}^0(A^*/pA^*)$ can be extended uniquely to a map of strict Dieudonné algebras $\mathcal{W}\Omega^*_R \to A^*$.

Consequently, the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ depends functorially on R, and is independent of the choice of the $W(k)$-algebra $\widetilde{R}$ lifting R or the map $\varphi: \widetilde{R} \to \widetilde{R}$ lifting the Frobenius.

Remark 1.2.1 (Extension to all $\mathbf{F}_p$-algebras)**.** Using the characterization of $\mathcal{W}\Omega^*_R$ by the universal property mentioned above, we can extend the definition of $\mathcal{W}\Omega^*_R$ to the case where R is an arbitrary $\mathbf{F}_p$-algebra, not necessarily smooth over a perfect field k. In the general case, we will see that the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ differs from the classical one. For instance, $\mathcal{W}\Omega^*_R$ only depends on the reduction R^{red} of R (see Lemma 3.6.1), while the analogous statement fails for the classical de Rham-Witt complex $W\Omega^*_R$. In §6, we will prove a stronger result: $\mathcal{W}\Omega^*_R$ depends only on the seminormalization R^{sn} of R. Moreover, we can identify R^{sn} with the 0th cohomology group $\mathrm{H}^0(\mathcal{W}\Omega^*_R/p)$. We are not aware of a similar description for the higher cohomology groups of $\mathcal{W}\Omega^*_R/p$.

Remark 1.2.2 (Frobenius untwisted characterization)**.** It is possible to give a formulation of the universal property of $\mathcal{W}\Omega^*_R$ which is closer in spirit to (b). Any saturated Dieudonné complex (M^*, d, F) carries a unique graded Verschiebung map $V: M^* \to M^*$, determined by the formula $FV = p$ (Proposition 2.2.4). If (M^*, d, F) is saturated and $M^i = 0$ for $i < 0$, then F induces an isomorphism $M^0/VM^0 \to \mathrm{H}^0(M^*/pM^*)$ (Proposition 2.7.1). It follows the Frobenius map $R \to \mathrm{H}^0(\mathcal{W}\Omega^*_R/p)$ factors uniquely as a composition $R \to \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R \xrightarrow{F} \mathrm{H}^0(\mathcal{W}\Omega^0_R/p)$. The universal property $(*)$ then translates to the following:

- If A^* is a strict Dieudonné algebra, then every ring homomorphism $R \to A^0/VA^0$ can be extended uniquely to a map of strict Dieudonné algebras $\mathcal{W}\Omega^*_R \to A^*$.

Here we can think of the map $\gamma: R \to \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R$ as an analogue of the identification $R \simeq W(R)/VW(R)$ for the ring of Witt vectors $W(R)$. Beware, however, that the map γ need not be an isomorphism when R is not regular: in fact, it exhibits $W\Omega^0_R/VW\Omega^0_R$ with the seminormalization R^{sn} (Theorem 6.5.3).

Remark 1.2.3 (Differences with the classical theory)**.** Let us briefly contrast our definition of the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ with the classical de Rham-Witt complex $W\Omega^*_R$ of [28]. Recall that $W\Omega^*_R$ is defined as the inverse limit of a tower of differential graded algebras $\{W_r\Omega^*_R\}_{r\geq 0}$, which is defined to be an initial object of the category of V*-pro-complexes* (see §4.4 for a review of the definitions). Our approach differs in several respects:

- Our saturated de Rham-Witt complex can also be described as the inverse limit of a tower $\{\mathcal{W}_r\Omega^*_R\}_{r\geq 0}$, but for most purposes we find it more convenient to work directly with the inverse limit $\mathcal{W}\Omega^*_R = \varprojlim_r \mathcal{W}_r\Omega^*_R$. This allows us to sidestep certain complications: for example, the limit $\mathcal{W}\Omega^*_R$ is p-torsion free, while the individual terms $\mathcal{W}_r\Omega^*_R$ are not (and are not even flat over $\mathbf{Z}/p^r\mathbf{Z}$).
- In the construction of the classical de Rham-Witt pro-complex $\{W_r\Omega^*_R\}_{r\geq 0}$, the Verschiebung operator V is fundamental and the Frobenius operator F plays an ancillary role. In our presentation, these roles are reversed: we regard the Frobenius operator as an essential part of the structure $\mathcal{W}\Omega^*_R$, while the Verschiebung operator is determined by requiring $FV = p = VF$.
- The notion of a V-pro-complex introduced in [28] is essentially non-linear: the axioms make sense only for differential graded algebras, rather than for general cochain complexes. By contrast, our Dieudonné complexes form an additive category $\mathbf{DC}$, and our Dieudonné algebras can be viewed as commutative algebras in the category $\mathbf{DC}$ (satisfying a mild additional condition).

Remark 1.2.4. The theory of the de Rham-Witt complex has taken many forms since its original introduction in [28], and our approach is restricted to the classical case of algebras over $\mathbf{F}_p$. In particular, we do not discuss the relative de Rham-Witt complexes of Langer-Zink [35], or the absolute de Rham-Witt complexes considered by Hesselholt-Madsen [23] in relation to topological Hochschild homology.

1.3. **Motivation via the $L\eta$ Functor.** In this section, we briefly discuss the original (and somewhat more highbrow) motivation that led us to the notion of strict Dieudonné complexes, and some related results. To avoid a proliferation of Frobenius twists, we work over the field $\mathbf{F}_p$ (instead of an arbitrary perfect field k of characteristic p) for the rest of this section.

Let R be a smooth $\mathbf{F}_p$-algebra and let Ω^*_R be the algebraic de Rham complex of R. Then Ω^*_R is a cochain complex of vector spaces over $\mathbf{F}_p$, and can therefore be viewed as an object of the derived category $D(\mathbf{F}_p)$. The Berthelot-Grothendieck theory of crystalline cohomology provides an object $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ of the derived category $D(\mathbf{Z})$, which plays the role of a "characteristic zero lift" of Ω^*_R in the sense that there is a canonical isomorphism

$$\alpha\colon \mathbf{F}_p \otimes^L_{\mathbf{Z}} \mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \to \Omega^*_R$$

in the category $D(\mathbf{F}_p)$. Explicitly, the object $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \in D(\mathbf{Z})$ can be realized as the global sections of $\mathscr{I}^*$, where $\mathscr{I}^*$ is any injective resolution of the structure sheaf on the crystalline site of $\mathrm{Spec}(R)$. Beware that, when regarded as a cochain complex, $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ depends on the choice of the injective resolution $\mathscr{I}^*$: different choices of injective resolutions generally yield nonisomorphic cochain complexes, which are nevertheless (canonically) isomorphic as objects of the derived category $D(\mathbf{Z})$. From this perspective, it is somewhat surprising that the object $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \in D(\mathbf{Z})$ admits a canonical representative at the level of cochain complexes (given by the de Rham-Witt complex $W\Omega^*_R$).

One of our goals in this paper is to give an explanation for this surprise: the cochain complex $W\Omega^*_R$ can actually be functorially recovered from its image $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \in D(\mathbf{Z})$, together with an additional structure which can also be formulated intrinsically at the level of the derived category.

To formulate this additional piece of structure, recall that the derived category of abelian groups $D(\mathbf{Z})$ is equipped with an additive (but non-exact) endofunctor $L\eta_p: D(\mathbf{Z}) \to D(\mathbf{Z})$ (see Construction 2.1.3 for the definition). This operation was discovered in early work on crystalline cohomology stemming from Mazur's resolution [40] of a conjecture of Katz, giving a relationship between the Newton and Hodge polygons associated to a smooth projective variety X. In [6], Ogus showed (following a suggestion of Deligne) that Mazur's result can be obtained from the following stronger local assertion: the Frobenius endomorphism of R enables one to endow the crystalline cochain complex $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ of a smooth affine variety $\mathrm{Spec}(R)$ with a canonical isomorphism

$$\widetilde{\varphi}_R: R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \simeq L\eta_p R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \tag{3}$$

in the derived category $D(\mathbf{Z})$. We may thus regard the pair $(R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)), \widetilde{\varphi}_R)$ as a fixed point of the $L\eta_p$ operator acting on $D(\mathbf{Z})$.

On the other hand, it was observed in [11] that for each $K \in D(\mathbf{Z})$, the object $(\mathbf{Z}/p^n\mathbf{Z}) \otimes^L_Z (L\eta_p)^n K$ of the derived category $D(\mathbf{Z}/p^n\mathbf{Z})$ admits a *canonical* representative in the category of cochain complexes, given by the Bockstein complex $(\mathrm{H}^*(\mathbf{Z}/p^n\mathbf{Z} \otimes^L_{\mathbf{Z}} K), \mathrm{Bock}_{p^n})$ Consequently, if the object $K \in D(\mathbf{Z})$ is equipped with an isomorphism $\alpha: K \simeq L\eta_p K$ (in the derived category), then each tensor product

$$(\mathbf{Z}/p^n\mathbf{Z}) \otimes^L_{\mathbf{Z}} K \simeq (\mathbf{Z}/p^n\mathbf{Z}) \otimes^L_{\mathbf{Z}} (L\eta_p)^n K$$

also has a canonical representative by cochain complex of $\mathbf{Z}/p^n\mathbf{Z}$-modules. One might therefore hope that these canonical representatives can be amalgamated to obtain a representative for K itself (at least up to p-adic completion). We will verify this heuristic expectation in Theorem 7.3.4: our category $\mathbf{DC}_{\mathrm{str}}$ of *strict Dieudonne complexes* (Definition 2.5.4) is equivalent to the category $\widehat{D(\mathbf{Z})}_p^{L\eta_p}$ of fixed points for the endofunctor $L\eta_p$ on the p-completed derived category of abelian groups (see Definition 7.1.5). From this optic, Ogus's isomorphism $\widetilde{\varphi}$ in (3) above guarantees that $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ admits a canonical presentation by a strict Dieudonne complex, which can then be identified with the de Rham-Witt complex $W\Omega_R^*$ of Deligne-Illusie (this observation traces back to the work of Katz, cf. [29, §III.1.5]).

We give two applications of the equivalence of categories $\mathbf{DC}_{\mathrm{str}} \simeq \widehat{D(\mathbf{Z})}_p^{L\eta_p}$ in this paper. The first is internal to the considerations of this paper: in §9, we give an alternative construction of the saturated de Rham-Witt complex $\mathcal{W}\Omega_R^*$ as a *saturation* of the derived de Rham-Witt complex $LW\Omega_R$ (for an arbitrary $\mathbf{F}_p$-algebra R). That is, we observe that the derived de Rham-Witt complex $LW\Omega_R$ is equipped with a natural map $\alpha_R\colon LW\Omega_R \to L\eta_p(LW\Omega_R)$ and we have the formula (in the p-complete derived category $\widehat{D(\mathbf{Z})}_p$)

$$\mathcal{W}\Omega_R^* \simeq \varinjlim \left(LW\Omega_R \xrightarrow{\alpha_R} L\eta_p(LW\Omega_R) \xrightarrow{L\eta_p(\alpha_R)} L\eta_{p^2}(LW\Omega_R) \to \dots \right)_p^{\widehat{}} .$$

As a result, we deduce an analog of the Berthelot-Ogus isogeny theorem for $\mathcal{W}\Omega_R^*$. Our second application concerns integral p-adic Hodge theory: in §10, we apply the universal property of saturated de Rham-Witt complexes to give a relatively simple construction of the crystalline comparison map for the $A\Omega$-complexes introduced in [11].

Remark 1.3.1. One particularly striking feature of the de Rham-Witt complex $W\Omega_R^*$ is that it provides a cochain-level model for the cup product on crystalline cohomology, where the associativity and (graded)-commutativity are visible at the level of cochains: that is, $W\Omega_R^*$ is a commutative differential graded algebra. From a homotopy-theoretic point of view, this is a very strong condition. For example, there is no analogous realization for the $\mathbf{F}_\ell$-étale cohomology of algebraic varieties (or for the singular cohomology of topological spaces), due to the potential existence of nontrivial Steenrod operations.

1.4. **Notation and Terminology.** Throughout this paper, we let p denote a fixed prime number. We say that an abelian group M is *p-torsion-free* if the map $p\colon M \to M$ is a monomorphism. A cochain complex (M^*, d) is *p-torsion-free* if each M^i is so.

Given a cochain complex (M^*, d) and an integer $k \in \mathbf{Z}$, we write $\tau^{\leq k} M^*$ for the canonical truncation in degrees $\leq k$, i.e., $\tau^{\leq k} M^*$ is the subcomplex

$$\left\{ \dots \to M^{k-2} \to M^{k-1} \to \ker(d\colon M^k \to M^{k+1}) \to 0 \to \dots \right\}$$

of M^*; similarly for $\tau^{\geq k} M^*$.

For a commutative ring R, we write $D(R)$ for the (unbounded) derived category of R-modules, viewed as a symmetric monoidal category by the usual tensor product of complexes. As it is convenient to use the language of ∞-categories in the later sections of this paper (especially when discussing the comparison with derived crystalline cohomology), we write $\mathcal{D}(R)$ for the derived ∞-category of R-modules, so $D(R)$ is the homotopy category of $\mathcal{D}(R)$.

Given a cochain complex (M^*, d), we sometimes also need to regard it as an object of a derived category. To distinguish the two notationally, we often drop the superscript when passing to the derived category, i.e., we write M for the corresponding object of the derived category. Thus, $\mathrm{H}^i(M)$ is identical to $\mathrm{H}^i((M^*, d))$.

1.5. **Prerequisites.** A primary goal of this paper is to give an account of the theory of the de Rham-Witt complex for $\mathbf{F}_p$-algebras which is as elementary and accessible as possible. In particular, part 1 (comprising sections 1 through 6 of the paper), which contains the construction of $\mathcal{W}\Omega^*_R$, presupposes only a knowledge of elementary commutative and homological algebra.

By contrast, part 2 will require additional prerequisites, and we will freely use the language of ∞-categories, for instance. In section 7, we use the theory of the derived category, and its ∞-categorical enhancement. In section 9, we will use the theory of nonabelian derived functors, such as the cotangent complex and derived de Rham and crystalline cohomology for $\mathbf{F}_p$-algebras. Finally, in section 10, we will use techniques from the paper [11].

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Part 1. Construction of $\mathcal{W}\Omega^*_R$

2. Dieudonné Complexes

Our goal in this section is to study the category **DC** of Dieudonné complexes. In particular, we define the classes of *saturated* and *strict* Dieudonné complexes, and the operations of saturation and completed saturation. We also introduce the class of Dieudonné complexes of *Cartier type*, for which the operations of saturation and completed saturation do not change the mod p homology (Theorem 2.4.2).

2.1. The Category DC.

Definition 2.1.1. A *Dieudonné complex* is a triple (M^*, d, F), where (M^*, d) is a cochain complex of abelian groups and $F: M^* \to M^*$ is a map of graded abelian groups which satisfies the identity $dF(x) = pF(dx)$.

We let **DC** denote the category whose objects are Dieudonné complexes, where a morphism from (M^*, d, F) to (M'^*, d', F') in **DC** is a map of graded abelian groups $f: M^* \to M'^*$ satisfying $d'f(x) = f(dx)$ and $F'f(x) = f(Fx)$ for each $x \in M^*$.

Remark 2.1.2. Let (M^*, d, F) be a Dieudonné complex. We will generally abuse terminology by simply referring to the underlying graded abelian group M^* as a *Dieudonné complex*; in this case we implicitly assume that the data of d and F have also been specified. We will refer to F as the *Frobenius map* on the complex M^*.

In the situation of Definition 2.1.1, it will be convenient to interpret F as a map of cochain complexes.

Construction 2.1.3. Let (M^*, d) be a p-torsion-free cochain complex of abelian groups, which we identify with a subcomplex of the localization $M^*[p^{-1}]$. We define another subcomplex $(\eta_p M)^* \subseteq M^*[p^{-1}]$ as follows: for each integer n, we set $(\eta_p M)^n = \{x \in p^n M^n : dx \in p^{n+1} M^{n+1}\}$.

Remark 2.1.4. Let (M^*, d, F) be a Dieudonné complex, and suppose that M^* is p-torsion-free. Then F determines a map of cochain complexes $\alpha_F: M^* \to (\eta_p M)^*$, given by the formula $\alpha_F(x) = p^n F(x)$ for $x \in M^n$. Conversely, if (M^*, d) is a cochain complex which is p-torsion-free and $\alpha: M^* \to (\eta_p M)^*$ is a map of cochain complexes, then we can form a Dieudonné complex (M^*, d, F) by setting $F(x) = p^{-n}\alpha(x)$ for $x \in M^n$. These constructions are inverse to one another.

Remark 2.1.5 (Tensor Products)**.** Let M^* and N^* be Dieudonné complexes. Then the tensor product complex $M^* \otimes N^*$ admits the structure of a Dieudonné complex, with Frobenius given by $F(x \otimes y) = Fx \otimes Fy$. This tensor product operation (together with the usual commutativity and associativity isomorphisms) endows the category **DC** of Dieudonné complexes with the structure of a symmetric monoidal category.

2.2. **Saturated Dieudonné Complexes.**

Definition 2.2.1. Let (M^*, d, F) be a Dieudonné complex. We will say that (M^*, d, F) is *saturated* if the following pair of conditions is satisfied:

- (i) The graded abelian group M^* is p-torsion-free.
- (ii) For each integer n, the map F induces an isomorphism of abelian groups
$$M^n \to \{x \in M^n : dx \in pM^{n+1}\}.$$

We let $\mathbf{DC}_{\mathrm{sat}}$ denote the full subcategory of **DC** spanned by the saturated Dieudonné complexes.

Remark 2.2.2. Let (M^*, d, F) be a Dieudonné complex, and suppose that M^* is p-torsion-free. Then (M^*, d, F) is a saturated Dieudonné complex if and only if the map $\alpha_F : M^* \to (\eta_p M)^*$ is an isomorphism (see Remark 2.1.4).

Remark 2.2.3. Let (M^*, d, F) be a saturated Dieudonné complex. Then the map $F : M^* \to M^*$ is injective. Moreover, the image of F contains the subcomplex pM^*. It follows that for each element $x \in M^n$, there exists a unique element $Vx \in M^n$ satisfying $F(Vx) = px$. We will refer to the map $V : M^* \to M^*$ as the *Verschiebung*.

Proposition 2.2.4. *Let (M^*, d, F) be a saturated Dieudonné complex. Then the Verschiebung $V : M^* \to M^*$ is an injective map which satisfies the identities*

$$F \circ V = V \circ F = p(\mathrm{id}) \tag{4}$$

$$F \circ d \circ V = d \tag{5}$$

$$p(d \circ V) = V \circ d. \tag{6}$$

Proof. The relation $F \circ V = p(\mathrm{id})$ follows from the definition, and immediately implies that V is injective (since M^* is p-torsion-free). Precomposing with F, we obtain $F \circ V \circ F = pF$. Since F is injective, it follows that $V \circ F = p(\mathrm{id})$. We have $pd = (d \circ F \circ V) = p(F \circ d \circ V)$, so that $d = F \circ d \circ V$. Postcomposition with V then yields the identity $V \circ d = V \circ F \circ d \circ V = p(d \circ V)$. □

Proposition 2.2.5. *Let (M^*, d, F) be a saturated Dieudonné complex and let $r \geq 0$ be an integer. Then the map F^r induces an isomorphism of graded abelian groups*

$$M^* \to \{x \in M^* : dx \in p^r M^{*+1}\}.$$

Proof. It is clear that F^r is injective (since F is injective), and the inclusion $F^r M^* \subseteq \{x \in M^* : dx \in p^r M^{*+1}\}$ follows from the calculation $d(F^r y) = p^r F^r(dy)$. We will prove the reverse inclusion by induction on r. Assume that $r > 0$ and that $x \in M^*$ satisfies $dx = p^r y$ for some $y \in M^{*+1}$. Then $d(p^r y) = 0$, so our assumption that M^* is p-torsion-free guarantees that $dy = 0$. Applying our assumption that M^* is a saturated Dieudonné complex to x and y, we can write $x = F(x')$ and $y = F(y')$ for some $x' \in M^*$ and $y' \in M^{*+1}$. We then compute

$$pF(dx') = d(F(x')) = dx = p^r y = p^r F(y').$$

Canceling F and p, we obtain $dx' = p^{r-1}y'$. It follows from our inductive hypothesis that $x' \in \mathrm{im}(F^{r-1})$, so that $x = F(x') \in \mathrm{im}(F^r)$. □

Remark 2.2.6. Let (M^*, d, F) be a saturated Dieudonné complex. It follows from Proposition 2.2.5 that every cycle $z \in M^*$ is infinitely divisible by F: that is, it belongs to $F^r M^*$ for each $r \geq 0$.

2.3. **Saturation of Dieudonné Complexes.** Let $f\colon M^* \to N^*$ be a morphism of Dieudonné complexes. We will say that f *exhibits N^* as a saturation of M^** if N^* is saturated and, for every saturated Dieudonné complex K^*, composition with f induces a bijection

$$\mathrm{Hom}_{\mathbf{DC}}(N^*, K^*) \to \mathrm{Hom}_{\mathbf{DC}}(M^*, K^*). \tag{7}$$

In this case, the Dieudonné complex N^* (and the morphism f) are determined by M^* up to unique isomorphism; we will refer to N^* as the *saturation* of M^* and denote it by $\mathrm{Sat}(M^*)$.

Proposition 2.3.1. *Let M^* be a Dieudonné complex. Then M^* admits a saturation* $\mathrm{Sat}(M^*)$.

Proof. Let $T^* \subseteq M^*$ be the graded subgroup consisting of elements $x \in M^*$ satisfying $p^n x = 0$ for $n \gg 0$. Replacing M^* by the quotient M^*/T^*, we can reduce to the case where M^* is p-torsion-free. In this case, Remark 2.2.2 implies that the direct limit of the sequence

$$M^* \xrightarrow{\alpha_F} (\eta_p M)^* \xrightarrow{\eta_p(\alpha_F)} (\eta_p\eta_p M)^* \xrightarrow{\eta_p(\eta_p(\alpha_F))} (\eta_p\eta_p\eta_p M)^* \to \cdots$$

is a saturation of M^*. □

Corollary 2.3.2. *The inclusion functor* $\mathbf{DC}_{\mathrm{sat}} \hookrightarrow \mathbf{DC}$ *admits a left adjoint, given by the saturation construction $M^* \mapsto \mathrm{Sat}(M^*)$.*

Remark 2.3.3. Let (M^*, d, F) be a Dieudonné complex, and assume that M^* is p-torsion-free. Then we can identify M^* with a subgroup of the graded abelian group $M^*[F^{-1}]$, defined as the colimit $M^* \xrightarrow{F} M^* \xrightarrow{F} \ldots$, whose elements can be written as formal expressions of the form $F^n x$ for $x \in M^*$ and $n \in \mathbf{Z}$. Unwinding the definitions, we see that the saturation $\mathrm{Sat}(M^*)$ can be described more concretely as follows:

- As a graded abelian group, $\mathrm{Sat}(M^*)$ can be identified with the subgroup of $M^*[F^{-1}]$ consisting of those elements x satisfying $d(F^n(x)) \in p^n M^*$ for $n \gg 0$ (note that if this condition is satisfied for some nonnegative integer n, then it is also satisfied for every larger integer).

- The differential on $\mathrm{Sat}(M^*)$ is given by the construction $x \mapsto F^{-n}p^{-n}d(F^n x)$ for $n \gg 0$.
- The Frobenius map on $\mathrm{Sat}(M^*)$ is given by the restriction of the map $F\colon M^*[F^{-1}] \to M^*[F^{-1}]$.

Remark 2.3.4. For future reference, we record a slight variant of Remark 2.3.3. Let (M^*, d, F) be a p-torsion-free Dieudonné complex on which F is injective, and consider the localization $M^*[F^{-1}]$ defined above. The operator $d\colon M^* \to M^{*+1}$ extends to a map

$$d\colon M^*[F^{-1}] \to M^{*+1}[F^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[1/p]$$

given by the formula

$$d(F^{-n}x) = p^{-n}F^{-n}dx.$$

Then $\mathrm{Sat}(M^*) \subseteq M^*[F^{-1}]$ can be identified with the graded subgroup consisting of those elements $y = F^{-n}x \in M^*[F^{-1}]$ for which $dy \in M^*[F^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[1/p]$ belongs to $M^*[F^{-1}]$.

To see this, suppose that we can write $d(F^{-n}x) = F^{-m}z$ in $M^*[F^{-1}]$ for $x, z \in M^*$. Then $F^{-n}dx = p^n F^{-m}z$ and $F^{r-n}dx = p^n F^{r-m}dz$ for $r \gg 0$. It follows that $d(F^{r-n}x) = d(F^r F^{-n}x)$ is divisible by p^r for $r \gg 0$. It follows that $F^{-n}x$ belongs to $\mathrm{Sat}(M^*)$ as in the formula of Remark 2.3.3. Conversely, if $F^{-n}x$ belongs to $\mathrm{Sat}(M^*)$ as in the formula of Remark 2.3.3, then it is easy to see that $d(F^{-n}x) \in M^*[F^{-1}]$ as desired.

2.4. **The Cartier Criterion.** Let M^* be a Dieudonné complex. For each $x \in M^*$, the equation $dFx = pF(dx)$ shows that the image of Fx in the quotient complex M^*/pM^* is a cycle, and therefore represents an element in the cohomology $\mathrm{H}^*(M/pM)$. This construction determines a map of graded abelian groups $M^* \to \mathrm{H}^*(M/pM)$, which factors uniquely through the quotient M^*/pM^*.

Definition 2.4.1. We say that the Dieudonné complex M^* is of *Cartier type* if

- (i) The complex M^* is p-torsion-free.
- (ii) The Frobenius map F induces an isomorphism of graded abelian groups $M^*/pM^* \to \mathrm{H}^*(M/pM)$.

Theorem 2.4.2 (Cartier Criterion)**.** *Let M^* be a Dieudonné complex which is of Cartier type. Then the canonical map $M^* \to \mathrm{Sat}(M^*)$ induces a quasi-isomorphism of cochain complexes $M^*/pM^* \to \mathrm{Sat}(M^*)/p\,\mathrm{Sat}(M^*)$.*

To prove Theorem 2.4.2, we will need to review some properties of the construction $M^* \mapsto (\eta_p M)^*$.

Construction 2.4.3. Let M^* be a cochain complex which is p-torsion-free. For each element $x \in (\eta_p M)^k$, we let $\overline{\gamma}(x) \in \mathrm{H}^k(M/pM)$ denote the cohomology class represented by $p^{-k}x$ (which is a cycle in M^*/pM^*, by virtue of our assumption that $x \in (\eta_p M)^k$). This construction determines a map of graded abelian groups

$\overline{\gamma}\colon (\eta_p M)^* \to \mathrm{H}^*(M/pM)$. Since the codomain of $\overline{\gamma}$ is a p-torsion group, the map $\overline{\gamma}$ factors (uniquely) as a composition

$$(\eta_p M)^* \to (\eta_p M)^*/p(\eta_p M)^* \xrightarrow{\gamma} \mathrm{H}^*(M/pM).$$

Proposition 2.4.4 (Compare [11, Prop. 6.12])**.** *Let M^* be a cochain complex of abelian groups which is p-torsion-free. Then Construction 2.4.3 determines a quasi-isomorphism of cochain complexes*

$$\gamma\colon (\eta_p M)^*/p(\eta_p M)^* \to \mathrm{H}^*(M/pM).$$

Here we regard $\mathrm{H}^*(M/pM)$ as a cochain complex with respect to the Bockstein map $\beta\colon \mathrm{H}^*(M/pM) \to \mathrm{H}^{*+1}(M/pM)$ determined the short exact sequence of cochain complexes $0 \to M^*/pM^* \xrightarrow{p} M^*/p^2M^* \to M^*/pM^* \to 0$.

Proof. For each element $x \in M^k$ satisfying $dx \in pM^{k+1}$, let $[x] \in \mathrm{H}^k(M/pM)$ denote the cohomology class represented by x. Unwinding the definitions, we see that the map β satisfies the identity $\beta([x]) = [p^{-1}dx]$. It follows that for each $y \in (\eta_p M)^k$, we have

$$\beta(\overline{\gamma}(y)) = \beta([p^{-k}y]) = [p^{-k-1}dy] = \overline{\gamma}(dy),$$

so that $\overline{\gamma}\colon (\eta_p M)^* \to \mathrm{H}^*(M/pM)$ is a map of cochain complexes. It follows that γ is also a map of cochain complexes. It follows immediately from the definition of $(\eta_p M)^*$ that the map γ is surjective. Let K^* denote the kernel of γ. We will complete the proof by showing that the cochain complex K^* is acyclic. Unwinding the definitions, we can identify K^* with the subquotient of $M^*[p^{-1}]$ described by the formula

$$K^n = (p^{n+1}M^n + d(p^nM^{n-1})/(p^{n+1}M^n \cap d^{-1}(p^{n+2}M^{n+1})).$$

Suppose that $e \in K^n$ is a cocycle, represented by $p^{n+1}x + d(p^n y)$ for some $x \in M^n$ and $y \in M^{n-1}$. The equation $de = 0 \in K^{n+1}$ implies that $d(p^{n+1}x) \in p^{n+2}M^{n+1}$. It follows that e is also represented by $d(p^n y)$, and is therefore a coboundary in K^n. $\square$

Corollary 2.4.5. *Let $f\colon M^* \to N^*$ be a map of cochain complexes of abelian groups. Assume that M^* and N^* are p-torsion-free. If f induces a quasi-isomorphism $M^*/pM^* \to N^*/pN^*$, then the induced map*

$$(\eta_p M)^*/p(\eta_p M)^* \to (\eta_p N)^*/p(\eta_p N)^*$$

is also a quasi-isomorphism.

Proof. By virtue of Proposition 2.4.4, it suffices to show that f induces a quasi-isomorphism $\mathrm{H}^*(M/pM) \to \mathrm{H}^*(N/pN)$ (where both are regarded as chain complexes with differential given by the Bockstein operator). In fact, this map is even an isomorphism (by virtue of our assumption that f induces a quasi-isomorphism $M^*/pM^* \to N^*/pN^*$). $\square$

Proof of Theorem 2.4.2. Let (M^*, d, F) be a Dieudonné complex which is p-torsion-free. Then the saturation $\mathrm{Sat}(M^*)$ can be identified with the colimit of the sequence

$$M^* \xrightarrow{\alpha_F} (\eta_p M)^* \xrightarrow{\eta_p(\alpha_F)} (\eta_p \eta_p M)^* \xrightarrow{\eta_p(\eta_p(\alpha_F))} (\eta_p \eta_p \eta_p M)^* \to \cdots$$

Consequently, to show that the canonical map $M^*/pM^* \to \mathrm{Sat}(M^*)/p\,\mathrm{Sat}(M^*)$ is a quasi-isomorphism, it will suffice to show that each of the maps

$$(\eta_p^k M)^*/p(\eta_p^k M)^* \to (\eta_p^{k+1} M)^*/p(\eta_p^{k+1} M)^*$$

is a quasi-isomorphism. By virtue of Corollary 2.4.5, it suffices to verify this when $k = 0$. In this case, we have a commutative diagram of cochain complexes

$$\begin{tikzcd} M^*/pM^* \arrow[rr, "\alpha_F"] \arrow[dr] & & (\eta_p M)^*/p(\eta_p M)^* \arrow[dl, "\gamma"] \\ & \mathrm{H}^*(M/pM), & \end{tikzcd}$$

where the map $M^*/pM^* \to \mathrm{H}^*(M/p)$ is induced by the Frobenius. Since M^* is assumed to be of Cartier type, this map is an isomorphism. Since γ is a quasi-isomorphism by virtue of Proposition 2.4.4, it follows that the map $\alpha_F \colon M^*/pM^* \to (\eta_p M)^*/p(\eta_p M)^*$ is also a quasi-isomorphism as desired. □

2.5. **Strict Dieudonné Complexes.** We now introduce a special class of saturated Dieudonné complexes, characterized by the requirement that they are complete with respect to the filtration determined by the Verschiebung operator V.

Construction 2.5.1 (Completion)**.** Let M^* be a saturated Dieudonné complex. For each integer $r \geq 0$, we let $\mathcal{W}_r(M)^*$ denote the quotient of M^* by the subcomplex $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$, where V is the Verschiebung map of Remark 2.2.3. We denote the natural quotient maps $\mathcal{W}_{r+1}(M)^* \to \mathcal{W}_r(M)^*$ by $\mathrm{Res} \colon \mathcal{W}_{r+1}(M)^* \to \mathcal{W}_r(M)^*$. We let $\mathcal{W}(M)^*$ denote the inverse limit of the tower of cochain complexes

$$\to \cdots \to \mathcal{W}_3(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_2(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_1(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_0(M)^* = 0.$$

We will refer to $\mathcal{W}(M)^*$ as the *completion* of the saturated Dieudonné complex M^*.

Remark 2.5.2 (Frobenius and Verschiebung at Finite Levels)**.** Let M^* be a saturated Dieudonné complex. Using the identities $F \circ d \circ V = d$ and $F \circ V = p\,\mathrm{id}$, we deduce that the Frobenius map $F \colon M^* \to M^*$ carries $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$ into

$\mathrm{im}(V^{r-1}) + \mathrm{im}(dV^{r-1})$. It follows that there is a unique map of graded abelian groups $F: \mathcal{W}_r(M)^* \to \mathcal{W}_{r-1}(M)^*$ for which the diagram

$$\begin{tikzcd} M^* \arrow[r, "F"] \arrow[d] & M^* \arrow[d] \\ \mathcal{W}_r(M)^* \arrow[r, "F"] & \mathcal{W}_{r-1}(M)^* \end{tikzcd}$$

commutes. Passing to the inverse limit over r, we obtain a map $F: \mathcal{W}(M)^* \to \mathcal{W}(M)^*$ which endows the completion $\mathcal{W}(M)^*$ with the structure of a Dieudonné complex.

Similarly, the identity $V \circ d = p(d \circ V)$ implies that V carries $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$ into $\mathrm{im}(V^{r+1}) + \mathrm{im}(dV^{r+1})$, so that there is a unique map $V: \mathcal{W}_r(M)^* \to \mathcal{W}_{r+1}(M)^*$ for which the diagram

$$\begin{tikzcd} M^* \arrow[r, "V"] \arrow[d] & M^* \arrow[d] \\ \mathcal{W}_r(M)^* \arrow[r, "V"] & \mathcal{W}_{r+1}(M)^* \end{tikzcd}$$

commutes.

Remark 2.5.3. The formation of completions is functorial: if $f: M^* \to N^*$ is a morphism of saturated Dieudonné complexes, then f induces a morphism $\mathcal{W}(f): \mathcal{W}(M)^* \to \mathcal{W}(N)^*$, given by the inverse limit of the tautological maps

$$M^*/(\mathrm{im}(V^r) + \mathrm{im}(dV^r)) \xrightarrow{f} N^*/(\mathrm{im}(V^r) + \mathrm{im}(dV^r)).$$

For every saturated Dieudonné complex M^*, the tower of cochain complexes

$$M^* \to \cdots \to \mathcal{W}_3(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_2(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_1(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_0(M)^* = 0.$$

determines a tautological map $\rho_M: M^* \to \mathcal{W}(M)^*$. It follows immediately from the definitions that ρ_M is a map of (saturated) Dieudonné complexes, which depends functorially on M^*.

Definition 2.5.4. Let M^* be a Dieudonné complex. We will say that M^* is *strict* if it is saturated and the map $\rho_M: M^* \to \mathcal{W}(M)^*$ is an isomorphism. We let $\mathbf{DC}_{\mathrm{str}}$ denote the full subcategory of $\mathbf{DC}$ spanned by the strict Dieudonné complexes.

Remark 2.5.5. Let M^* be a strict Dieudonné complex. Then each M^n is a torsion-free abelian group: it is p-torsion free by virtue of the assumption that M^* is saturated, and ℓ-torsion free for $\ell \neq p$ because M^n is p-adically complete.

Example 2.5.6. Let A be an abelian group and let

$$M^* = \begin{cases} A & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

denote the chain complex consisting of the abelian group A in degree zero. Then any choice of endomorphism $F: A \to A$ endows M^* with the structure of a Dieudonné complex. The Dieudonné complex M^* is saturated if and only if A is p-torsion-free and the map F is an automorphism. If these conditions are satisfied, then M^* is strict if and only if A is p-adically complete: that is, if and only if the canonical map $A \to \varprojlim A/p^r A$ is an isomorphism.

Example 2.5.7 (Free Strict Dieudonné Complexes)**.** Let M^0 denote the abelian group consisting of formal expressions of the form

$$\sum_{m \geq 0} a_m F^m x + \sum_{n > 0} b_n V^n x$$

where the coefficients a_m and b_n are p-adic integers with the property that the sequence $\{a_m\}_{m \geq 0}$ converges to zero (with respect to the p-adic topology on $\mathbf{Z}_p$). Let M^1 denote the abelian group of formal expressions of the form

$$\sum_{m \geq 0} c_m F^m dx + \sum_{n > 0} d_n V^n x$$

where the coefficients c_m and d_n are p-adic integers with the property that the sequence $\{c_m\}_{m \geq 0}$ converges to zero (also with respect to the p-adic topology). Then we can form a strict Dieudonné complex

$$\cdots \to 0 \to 0 \to M^0 \xrightarrow{d} M^1 \to 0 \to 0 \to \cdots,$$

where the differential d is given by the formula

$$d(\sum_{m \geq 0} a_m F^m x + \sum_{n > 0} b_n V^n x) = \sum_{m \geq 0} p^m a_m F^m dx + \sum_{n > 0} b_n dV^n x$$

Moreover, the strict Dieudonné complex M^* is freely generated by $x \in M^0$ in the following sense: for any strict Dieudonné complex N^*, evaluation on x induces a bijection

$$\mathrm{Hom}_{\mathbf{DC}}(M^*, N^*) \simeq N^0.$$

2.6. **Strict Dieudonné Towers.** For later use, it will be convenient to axiomatize some of the features of the complexes $\mathcal{W}_r(M)^*$ arising from Construction 2.5.1.

Definition 2.6.1. A **strict Dieudonné tower** is an inverse system of cochain complexes

$$\cdots \to X_3^* \xrightarrow{R} X_2^* \xrightarrow{R} X_1^* \xrightarrow{R} X_0^*$$

which is equipped with maps of graded abelian groups $F: X_{r+1}^* \to X_r^*$ and $V: X_r^* \to X_{r+1}^*$ which satisfy the following axioms:

(1) The chain complex X_0^* vanishes.
(2) The map $R: X_{r+1}^* \to X_r^*$ is surjective for each $r \geq 0$.
(3) The operator $F: X_{r+1}^* \to X_r^*$ satisfies $dF = pFd$ for each $r \geq 0$.
(4) The operators F, R, and V commute with each other.

(5) We have $F(V(x)) = px = V(F(x))$ for each $x \in X_r^*$.
(6) Given any $x \in X_r^*$ such that dx is divisible by p, x belongs to the image of $F : X_{r+1}^* \to X_r^*$.
(7) The kernel of the map $R\colon X_{r+1}^* \to X_r^*$ coincides with the submodule $X_{r+1}^*[p] \subseteq X_{r+1}^*$ consisting of those elements $x \in X_{r+1}^*$ satisfying $px = 0$.
(8) The kernel of $R\colon X_{r+1}^* \to X_r^*$ is the span of the images of $V^r, dV^r\colon X_1^* \to X_{r+1}^*$.

A morphism of strict Dieudonné towers is a morphism of towers of cochain complexes which are compatible with the F, V operators. We let **TD** denote the category of strict Dieudonné towers.

We now show that Construction 2.5.1 gives rise to strict Dieudonné towers:

Proposition 2.6.2. *Let M^* be a saturated Dieudonné complex. Then the diagram*

$$\cdots \to \mathcal{W}_3(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_2(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_1(M)^* \xrightarrow{\mathrm{Res}} \mathcal{W}_0(M)^*$$

is a strict Dieudonné tower, when endowed with the Frobenius and Verschiebung operators of Remark 2.5.2.

The proof of Proposition 2.6.2 will require some preliminaries.

Lemma 2.6.3. *Let M^* be a saturated Dieudonné complex and let $x \in M^*$ satisfy $d(V^r x) \in pM^{*+1}$. Then $x \in \operatorname{im}(F)$.*

Proof. We have $dx = F^r d(V^r x) \in F^r pM^{*+1} \subseteq pM^{*+1}$, so the desired result follows from our assumption that M^* is saturated. □

Lemma 2.6.4. *Let M^* be a saturated Dieudonné complex and let $x \in M^*$. If the image of px vanishes in $\mathcal{W}_r(M)^*$, then the image of x vanishes in $\mathcal{W}_{r-1}(M)^*$.*

Proof. Write $px = V^{r+1}a + dV^{r+1}b$ for some $a, b \in M^*$. Then $dV^{r+1}a = p(dx)$ is divisible by p, so that Lemma 2.6.3 allows us to write $a = F\widetilde{a}$ for some $\widetilde{a} \in M^*$. This guarantees that $dV^{r+1}b = p(x - V^r\widetilde{a})$ is divisible by p, so that $b = F\widetilde{b}$ for some $\widetilde{b} \in M^*$ (Lemma 2.6.3). Then we can write

$$px = pV^r\widetilde{a} + dV^r(p\widetilde{b}),$$

so that $x = V^r\widetilde{a} + dV^r\widetilde{b} \in \operatorname{im}(V^r) + \operatorname{im}(dV^r)$ as desired. □

Proof of Proposition 2.6.2. Let M^* be a saturated Dieudonné complex; we wish to show that the inverse system $\{\mathcal{W}_r(M)^*\}_{r\geq 0}$ of Construction 2.5.1 is a strict Dieudonné tower (when equipped with the Frobenius and Verschiebung operators of Remark 2.5.2). It is clear that this system satisfies axioms (1) through (5) and (8) of Definition 2.6.1. To verify (6), suppose that we are given an element $\overline{x} \in \mathcal{W}_r(M)^*$ such that $d\overline{x}$ is divisible by p. Choose an element $x \in M^*$ representing $\overline{x}$, so that have an identity of the form $dx = py + V^r a + dV^r b$ for some $y, a, b \in M^*$.

Then $d(V^r a)$ is divisible by p, so Lemma 2.6.3 implies that we can write $a = F\widetilde{a}$ for some $\widetilde{a} \in M^*$. We then have

$$d(x - V^r b) = p(y + V^{r-1}\widetilde{a}),$$

so our assumption that M^* is saturated guarantees that we can write $x - V^r b = Fz$ for some $z \in M^*$. It follows that $\overline{x} = F\overline{z}$, where $\overline{z} \in \mathcal{W}_{r+1}M^*$ denotes the image of z.

We now verify (7). We first note that if x belongs to $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$, then px belongs to

$$\mathrm{im}(pV^r) + \mathrm{im}(pdV^r) = \mathrm{im}(V^{r+1}F) + \mathrm{im}(dV^{r+1}F) \subseteq \mathrm{im}(V^{r+1}) + \mathrm{im}(dV^{r+1}).$$

It follows that the kernel of the restriction map $\mathrm{Res}\colon \mathcal{W}_{r+1}(M)^* \to \mathcal{W}_r(M)^*$ is contained in $(\mathcal{W}_{r+1}(M)^*)[p]$. The reverse inclusion follows from Lemma 2.6.4. □

We next establish a sort of converse to Proposition 2.6.2: from any strict Dieudonné tower, we can construct a saturated Dieudonné complex.

Proposition 2.6.5. *Let $\{X_r^*\}$ be a strict Dieudonné tower and let X^* denote the cochain complex given by the inverse limit $\varprojlim_r X_r^*$. Let $F\colon X^* \to X^*$ denote the map of graded abelian groups given by the inverse limit of the Frobenius operators $F\colon X_{r+1}^* \to X_r^*$. Then (X^*, F) is a saturated Dieudonné complex.*

Proof. It follows from axiom (3) of Definition 2.6.1 that the Frobenius map $F\colon X^* \to X^*$ satisfies $dF = pFd$, so that (X^*, F) is a Dieudonné complex. We claim that the graded abelian group X^* is p-torsion-free. Choose any element $x \in X^*$, given by a compatible system of elements $\{x_r \in X_r^*\}_{r \geq 0}$. If $px = 0$, then each x_r satisfies $px_r = 0$ in X_r^*, so that $x_{r-1} = 0$ by virtue of axiom (7) of Definition 2.6.1. Since r is arbitrary, it follows that $x = 0$.

To complete the proof that X^* is saturated, it suffices now to show that if $x = \{x_r\}_{r \geq 0}$ is an element of X^* and dx is divisible by p, then x belongs to the image of F. Using axiom (6) of Definition 2.6.1, we can choose $y_{r+1} \in X_{r+1}^*$ such that $Fy_{r+1} = x_r$. Beware that we do not necessarily have $R(y_{r+1}) = y_r$ for $r > 0$. However, the difference $R(y_{r+1}) - y_r \in X_r^*$ is annihilated by F, hence also by $p = VF$, hence also by the restriction map R (by virtue of axiom (7) of Definition 2.6.1). It follows that the sequence $\{R(y_{r+1}) \in X_r^*\}$ is determines an element $y \in X^*$ satisfying $Fy = x$. □

2.7. **The Completion of a Saturated Dieudonné Complex.** Let M^* be a saturated Dieudonné complex. It follows from Propositions 2.6.2 and 2.6.5 that the completion $\mathcal{W}(M)^*$ is also a saturated Dieudonné complex. Our next goal is to show that the completion $\mathcal{W}(M)^*$ is strict (Corollary 2.7.6), and is *universal* among strict Dieudonné complexes receiving a map from M^* (Proposition 2.7.7). We begin by studying the homological properties of Construction 2.5.1.

Proposition 2.7.1. *Let M^* be a saturated Dieudonné complex and let r be a nonnegative integer. Then the map $F^r: M^* \to M^*$ induces an isomorphism of graded abelian groups $\mathcal{W}_r(M)^* \to \mathrm{H}^*(M^*/p^r M^*)$.*

Proof. The equality $d \circ F^r = p^r(F^r \circ d)$ guarantees that F^r carries each element of M^* to a cycle in the quotient complex $M^*/p^r M^*$. Moreover, the identities

$$F^r V^r(x) = p^r x \qquad F^r dV^r(y) = dy$$

imply that F^r carries the subgroup $\mathrm{im}(V^r) + \mathrm{im}(dV^r) \subseteq M^*$ to the set of boundaries in the cochain complex $M^*/p^r M^*$. We therefore obtain a well-defined homomorphism $\theta: \mathcal{W}_r(M)^* \to \mathrm{H}^*(M^*/p^r M^*)$. The surjectivity of θ follows from Proposition 2.2.5. To prove injectivity, suppose that $x \in M^*$ has the property that $F^r x$ represents a boundary in the quotient complex $M^*/p^r M^*$, so that we can write $F^r x = p^r y + dz$ for some $y \in M^*$ and $z \in M^{*-1}$. Then

$$p^r x = V^r F^r x = V^r(p^r y + dz) = p^r V^r(y) + p^r dV^r(z),$$

so that $x = V^r(y) + dV^r(z) \in \mathrm{im}(V^r) + \mathrm{im}(dV^r)$. □

Corollary 2.7.2. *Let (M^*, d, F) be a saturated Dieudonné complex. Then the quotient map $u: M^*/pM^* \to \mathcal{W}_1(M)^*$ is a quasi-isomorphism of cochain complexes.*

Proof. Regard $\mathrm{H}^*(M/pM)$ as a cochain complex with respect to the Bockstein operator, as in Proposition 2.4.4. We have a commutative diagram of cochain complexes

$$\begin{tikzcd} M^*/pM^* \arrow[r, "\alpha_F"] \arrow[d, "u"] & (\eta_p M)^*/p(\eta_p M)^* \arrow[d] \\ \mathcal{W}_1(M)^* \arrow[r] & \mathrm{H}^*(M/pM), \end{tikzcd}$$

where the bottom horizontal map is the isomorphism of Proposition 2.7.1 (which one easily checks to be a map of cochain complexes as in Proposition 2.4.4), the right vertical map is the quasi-isomorphism of Proposition 2.4.4, and α_F is an isomorphism by virtue of our assumption that M^* is saturated. It follows that u is also a quasi-isomorphism. □

Remark 2.7.3. More generally, if M^* is a saturated Dieudonné complex, then the quotient map $M^*/p^r M^* \to \mathcal{W}_r(M)^*$ is a quasi-isomorphism for each $r \geq 0$. This can be proven by essentially the same argument, using a slight generalization of Proposition 2.4.4.

Corollary 2.7.4. *Let $f: M^* \to N^*$ be a morphism of saturated Dieudonné complexes. The following conditions are equivalent:*

(1) *The induced map $M^*/pM^* \to N^*/pN^*$ is a quasi-isomorphism of cochain complexes.*

(2) *The induced map* $\mathcal{W}_1(M)^* \to \mathcal{W}_1(N)^*$ *is an isomorphism of cochain complexes.*
(3) *For each* $r \geq 0$, *the map* $M^*/p^r M^* \to N^*/p^r N^*$ *is a quasi-isomorphism of cochain complexes.*
(4) *For each* $r \geq 0$, *the map* $\mathcal{W}_r(M)^* \to \mathcal{W}_r(N)^*$ *is an isomorphism of cochain complexes.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) follow from Proposition 2.7.1. The implication (3) $\Rightarrow$ (1) is obvious, and the reverse implication follows by induction on r. □

Proposition 2.7.5. *Let* M^* *be a saturated Dieudonné complex. Then the map* $\rho_M: M^* \to \mathcal{W}(M)^*$ *induces isomorphisms* $\mathcal{W}_r(M)^* \to \mathcal{W}_r(\mathcal{W}(M))^*$ *for* $r \geq 0$.

Proof. By virtue of Corollary 2.7.4, it will suffice to treat the case $r = 1$. Let $\gamma: \mathcal{W}_1(M)^* \to \mathcal{W}_1(\mathcal{W}(M))^*$ be the map determined by ρ_M. We first show that γ is injective. Choose an element $\overline{x} \in \mathcal{W}_1(M)^*$ satisfying $\gamma(\overline{x}) = 0$; we wish to show that $\overline{x} = 0$. Represent $\overline{x}$ by an element $x \in M^*$. The vanishing of $\gamma(\overline{x})$ then implies that we can write $\rho_M(x) = Vy + dVz$ for some $y \in \mathcal{W}(M)^*$ and $z \in \mathcal{W}(M)^{*-1}$. Then we can identify y and z with compatible sequences $\overline{y}_m \in \mathcal{W}_m(M)^*$ and $\overline{z}_m \in \mathcal{W}_m(M)^{*-1}$. The equality $\rho_M(x) = Vy + dVz$ then yields $\overline{x} = V(\overline{y}_0) + dV(\overline{z}_0) = 0$.

We now prove that γ is surjective. Choose an element $e \in \mathcal{W}_1(\mathcal{W}(M))^*$; we wish to show that e belongs to the image of γ. Let $x \in \mathcal{W}(M)^*$ be an element representing e, which we can identify with a compatible sequence of elements $\overline{x}_m \in \mathcal{W}_m(M)^*$ represented by elements $x_m \in M^*$. The compatibility of the sequence $\{\overline{x}_m\}_{m\geq 0}$ guarantees that we can write $x_{m+1} = x_m + V^m y_m + dV^m z_m$ for some elements $y_m \in M^*$ and $z_m \in M^{*-1}$. We then have the identity

$$x = \rho_M(x_1) + V(\sum_{m\geq 0} V^m y_{m+1}) + dV(\sum_{m\geq 0} V^m z_{m+1})$$

in $\mathcal{W}(M)^*$, which yields the identity $e = \gamma(\overline{x}_1)$ in $\mathcal{W}_1(\mathcal{W}(M))^*$. □

Corollary 2.7.6. *Let* M^* *be a saturated Dieudonné complex. Then the completion* $\mathcal{W}(M)^*$ *is a strict Dieudonné complex.*

Proof. It follows from Propositions 2.6.2 and 2.6.5 that N^* is a saturated Dieudonné complex. Let $\rho_M: M^* \to \mathcal{W}(M)^*$ be the canonical map, so that we have a pair of maps of Dieudonné complexes

$$\rho_{\mathcal{W}(M)}, \mathcal{W}(\rho_M): \mathcal{W}(M)^* \to \mathcal{W}(\mathcal{W}(M))^*.$$

We wish to show that $\rho_{\mathcal{W}(M)}$ is an isomorphism. It follows from Proposition 2.7.5 that the map $\mathcal{W}(\rho_M)$ is an isomorphism. It will therefore suffice to show that $\rho_{\mathcal{W}(M)}$ and $\mathcal{W}(\rho_M)$ coincide. Fix an element $x \in \mathcal{W}(M)^*$; we wish to show that $\rho_{\mathcal{W}(M)}(x)$ and $\mathcal{W}(\rho_M)(x)$ have the same image in $\mathcal{W}_r(\mathcal{W}(M))^*$ for each $r \geq 0$.

Replacing x by an element of the form $x + V^r y + dV^r z$, we can assume without loss of generality that $x = \rho_M(x')$ for some $x' \in M^*$. In this case, the desired result follows from the commutativity of the diagram

$$\begin{tikzcd} M^* \arrow[r, "\rho_M"] \arrow[d, "\rho_M"] & \mathcal{W}(M)^* \arrow[d] \\ \mathcal{W}_r(M)^* \arrow[r] & \mathcal{W}_r(\mathcal{W}(M))^*. \end{tikzcd}$$

□

Proposition 2.7.7. *Let M^* and N^* be saturated Dieudonné complexes, where N^* is strict. Then composition with the map ρ_M induces a bijection*

$$\theta: \mathrm{Hom}_{\mathbf{DC}}(\mathcal{W}(M)^*, N^*) \to \mathrm{Hom}_{\mathbf{DC}}(M^*, N^*).$$

Proof. We first show that θ is injective. Let $f: \mathcal{W}(M)^* \to N^*$ be a morphism of Dieudonné complexes such that $\theta(f) = f \circ \rho_M$ vanishes; we wish to show that f vanishes. For each $r \geq 0$, we have a commutative diagram of cochain complexes

$$\begin{tikzcd} M^* \arrow[r, "\rho_M"] \arrow[d] & \mathcal{W}(M)^* \arrow[r, "f"] \arrow[d] & N^* \arrow[d] \\ \mathcal{W}_r(M)^* \arrow[r, "\mathcal{W}_r(\rho_M)"] & \mathcal{W}_r(\mathcal{W}(M))^* \arrow[r, "\mathcal{W}_r(f)"] & \mathcal{W}_r(N)^* \end{tikzcd}$$

Since $\mathcal{W}_r(f) \circ \mathcal{W}_r(\rho_M) = \mathcal{W}_r(f \circ \rho_M)$ vanishes and $\mathcal{W}_r(\rho_M)$ is an isomorphism (Proposition 2.7.5), we conclude that $\mathcal{W}_r(f) = 0$. It follows that the composite map $\mathcal{W}(M)^* \xrightarrow{f} N^* \to \mathcal{W}_r(N)^*$ vanishes for all r. Invoking the strictness of N^*, we conclude that $f = 0$.

We now argue that θ is surjective. Suppose we are given a map of Dieudonné complexes $f_0: M^* \to N^*$; we wish to show that f_0 belongs to the image of θ. We have a commutative diagram of Dieudonné complexes

$$\begin{tikzcd} M^* \arrow[r, "f_0"] \arrow[d, "\rho_M"] & N^* \arrow[d, "\rho_N"] \\ \mathcal{W}(M)^* \arrow[r, "\mathcal{W}(f_0)"] & \mathcal{W}(N)^*, \end{tikzcd}$$

where the right vertical map is an isomorphism (since N^* is complete). It follows that $f_0 = \theta(\rho_N^{-1} \circ \mathcal{W} f_0)$ belongs to the image of θ, as desired. □

Corollary 2.7.8. *The inclusion functor $\mathbf{DC}_{\mathrm{str}} \hookrightarrow \mathbf{DC}_{\mathrm{sat}}$ admits a left adjoint, given by the completion functor $M^* \mapsto \mathcal{W}(M)^*$.*

2.8. **Comparison of M^* with $\mathcal{W}(M)^*$.** We now record a few homological consequences of the results of §2.7.

Proposition 2.8.1. *Let M^* be a saturated Dieudonné complex and let $\mathcal{W}(M)^*$ be its completion. Then the tautological map $\rho_M: M^* \to \mathcal{W}(M)^*$ induces a quasi-isomorphism $M^*/p^r M^* \to \mathcal{W}(M)^*/p^r \mathcal{W}(M)^*$ for every nonnegative integer r.*

Proof. Combine Proposition 2.7.5 with Corollary 2.7.4. □

Corollary 2.8.2. *Let M^* be a saturated Dieudonné complex and let $\mathcal{W}(M)^*$ be its completion. Then the tautological map $\rho_M: M^* \to \mathcal{W}(M)^*$ exhibits $\mathcal{W}(M)^*$ as a p-completion of M^* in the derived category of abelian groups (see § 7.1).*

Proof. By virtue of Proposition 2.8.1, it will suffice to show that $\mathcal{W}(M)^*$ is a p-complete object of the derived category. In fact, something stronger is true: each term in the cochain complex $\mathcal{W}(M)^*$ is a p-complete abelian group, since $\mathcal{W}(M)^n$ is given by an inverse limit $\varprojlim \mathcal{W}_r(M)^n$, where $\mathcal{W}_r(M)^n$ is annihilated by p^r. □

Corollary 2.8.3. *Let M^* be a saturated Dieudonné complex and let $\mathcal{W}(M)^*$ be its completion. The following conditions are equivalent:*

(1) *The tautological map $\rho_M: M^* \to \mathcal{W}(M)^*$ is a quasi-isomorphism.*
(2) *The underlying cochain complex M^* is a p-complete object of the derived category of abelian groups.*

We now combine Corollary 2.8.3 with our study of the saturation construction $M^* \mapsto \mathrm{Sat}(M^*)$.

Notation 2.8.4. Let M^* be a Dieudonné complex. We let $\mathcal{W}\mathrm{Sat}(M^*)$ denote the completion of the saturation $\mathrm{Sat}(M^*)$. Note that the construction $M^* \mapsto \mathcal{W}\mathrm{Sat}(M^*)$ is left adjoint to the inclusion functor $\mathbf{DC}_{\mathrm{str}} \hookrightarrow \mathbf{DC}$ (see Corollaries 2.3.2 and 2.8.3).

Corollary 2.8.5. *Let M^* be a Dieudonné complex of Cartier type, and suppose that each of the abelian groups M^n is p-adically complete. Then the canonical map $u: M^* \to \mathcal{W}\mathrm{Sat}(M^*)$ is a quasi-isomorphism.*

Proof. Note that the domain and codomain of u are cochain complexes of abelian groups which are p-complete and p-torsion-free. Consequently, to show that u is a quasi-isomorphism, it will suffice to show that the induced map $M^*/pM^* \to \mathcal{W}\mathrm{Sat}(M^*)/p\mathcal{W}\mathrm{Sat}(M^*)$ is a quasi-isomorphism. This map factors as a composition

$$M^*/pM^* \xrightarrow{v} \mathrm{Sat}(M^*)/p\,\mathrm{Sat}(M^*) \xrightarrow{v'} \mathcal{W}\mathrm{Sat}(M^*)/p\mathcal{W}\mathrm{Sat}(M^*).$$

The map v is a quasi-isomorphism by virtue of Theorem 2.4.2, and the map v' is a quasi-isomorphism by Proposition 2.8.1. □

2.9. **More on Strict Dieudonné Towers.** We close this section by showing that the category $\mathbf{DC}_{\mathrm{str}}$ of strict Dieudonné complexes is equivalent to the category $\mathbf{TD}$ of strict Dieudonné towers (Corollary 2.9.4).

Proposition 2.9.1. *Let $\{X_r^*\}_{r\geq 0}$ be a strict Dieudonné tower and let $X^* = \varprojlim_r X_r^*$ be its inverse limit, which we regard as a Dieudonné complex as in Proposition 2.6.5. Then, for each $r \geq 0$, the canonical map $X^* \to X_r^*$ induces an isomorphism of cochain complexes*

$$\mathcal{W}_r(X)^* = X^*/(\mathrm{im}(V^r) + \mathrm{im}(dV^r)) \to X_r^*.$$

Proof. Axiom (2) of Definition 2.6.1 guarantees that the restriction maps in the tower $\{X_k^*\}_{k\geq r}$ are surjective, so that the induced map $\theta\colon X^* \to X_r^*$ is surjective. Moreover, the vanishing of X_0^* guarantees that $\ker(\theta)$ contains $\mathrm{im}(V^r)+\mathrm{im}(dV^r)$. We will complete the proof by verifying the reverse inclusion. Fix an element $x \in \ker(\theta)$, which we can identify with a compatible sequence of elements $\{x_k \in X_{k+r}^*\}_{k\geq 0}$ satisfying $x_0 = 0$. We wish to show that x can be written as a sum $V^r y + dV^r z$ for some $y, z \in X^*$. Equivalently, we wish to show that we can choose compatible sequences of elements $\{y_k, z_k \in X_k^*\}$ satisfying $x_k = V^r y_k + dV^r z_k$. We proceed by induction on k, the case $k = 0$ being trivial. To carry out the inductive step, let us assume that $k > 0$ and that we have chosen $y_{k-1}, z_{k-1} \in X_{k-1}^*$ satisfying $x_{k-1} = V^r y_{k-1} + dV^r z_{k-1}$. Using Axiom (2) of Definition 2.6.1, we can lift y_{k-1} and z_{k-1} to elements $\overline{y}_k, \overline{z}_k$ in X_k^*. We then have

$$x_k = V^r \overline{y}_k + dV^r \overline{z}_k + e$$

where e belongs to the kernel of the restriction map $X_{r+k}^* \to X_{r+k-1}^*$. Applying axiom (8) of Definition 2.6.1, we can write $e = V^{r+k-1}e' + dV^{r+k-1}e''$ for some $e', e'' \in X_1^*$. We conclude the proof by setting $y_k = \overline{y}_k + V^{k-1}e'$ and $z_k = \overline{z}_k + V^{k-1}e''$. □

Corollary 2.9.2. *Let $\{X_r^*\}_{r\geq 0}$ be a strict Dieudonné tower. Then there exists a saturated Dieudonné complex M^* and an isomorphism of strict Dieudonné towers $\{X_r^*\}_{r\geq 0} \simeq \{\mathcal{W}_r(M)^*\}_{r\geq 0}$.*

Proof. Combine Proposition 2.9.1 with Proposition 2.6.5. □

Corollary 2.9.3. *Let $\{X_r^*\}_{r\geq 0}$ be a strict Dieudonné tower. Then the inverse limit $X^* = \varprojlim_r X_r^*$ is a strict Dieudonné complex.*

Proof. By virtue of Corollary 2.9.2, we can assume that $\{X_r^*\}_{r\geq 0} = \{\mathcal{W}_r(M)^*\}_{r\geq 0}$, where M^* is a saturated Dieudonné complex. In this case, the desired result follows from Corollary 2.7.6. □

Corollary 2.9.4. *The composite functor $\mathbf{DC}_{\mathrm{str}} \hookrightarrow \mathbf{DC} \to \mathbf{TD}$ sending a strict Dieudonné complex M^* to the strict Dieudonné tower $\{\mathcal{W}(M)^*\}$ is an equivalence of categories, with inverse equivalence given by $\{X_r^*\}_{r\geq 0} \mapsto \varprojlim_r X_r^*$.*

Proof. For any saturated Dieudonné complex M^* and every strict Dieudonné tower $\{X_r^*\}_{r \geq 0}$, we have canonical maps

$$M^* \to \varprojlim \mathcal{W}_r(M)^* \qquad \mathcal{W}_{r'}(\varprojlim_r X_r^*) \to X_{r'}^*.$$

The first of these maps is an isomorphism when M^* is strict (by definition), and the second for any strict Dieudonné tower (by virtue of Proposition 2.9.1). $\square$

3. Dieudonné Algebras

In this section, we introduce the notion of a *Dieudonné algebra*. Roughly speaking, a Dieudonné algebra is a Dieudonné complex (A^*, d, F) which is equipped with a ring structure which is suitably compatible with the differential d and the Frobenius operator F (see Definition 3.1.2). We show that if A is a p-torsion-free ring equipped with a lift of the Frobenius $\varphi_{A/pA}: A/pA \to A/pA$, then the absolute de Rham complex Ω^*_A inherits the structure of a Dieudonné algebra (Proposition 3.2.1). We also show that the saturation and completed saturation constructions of §2 have nonlinear counterparts in the setting of Dieudonné algebras.

3.1. The Category DA.

Recall that a *commutative differential graded algebra* is a cochain complex (A^*, d) which is equipped with the structure of a graded ring satisfying the following conditions:

(a) The multiplication on A^* is graded-commutative: that is, for every pair of elements $x \in A^m$ and $y \in A^n$, we have $xy = (-1)^{mn} yx$ in A^{m+n}.

(b) If $x \in A^n$ is a homogeneous element of odd degree, then x^2 vanishes in A^{2n}.

(c) The differential d satisfies the Leibniz rule $d(xy) = (dx)y + (-1)^m x dy$ for $x \in A^m$.

Remark 3.1.1. For each homogeneous element $x \in A^n$ of odd degree, assumption (a) implies that $x^2 = -x^2$. Consequently, assumption (b) is automatic if A^* is 2-torsion free.

Definition 3.1.2. A *Dieudonné algebra* is a triple (A^*, d, F), where (A^*, d) is a commutative differential graded algebra and $F: A^* \to A^*$ is a homomorphism of graded rings satisfying the following additional conditions:

(i) For each $x \in A^*$, we have $dF(x) = pF(dx)$ for $x \in A^*$.

(ii) The groups A^n vanish for $n < 0$.

(iii) For $x \in A^0$, we have $Fx \equiv x^p \pmod{p}$.

Let (A^*, d, F) and (A'^*, d', F') be Dieudonné algebras. A *morphism of Dieudonné algebras* from (A^*, d, F) to (A'^*, d', F') is a graded ring homomorphism $f: A^* \to A'^*$ satisfying $d' \circ f = f \circ d$ and $F' \circ f = f \circ F$. We let **DA** denote the category of whose objects are Dieudonné algebras and whose morphisms are Dieudonné algebra morphisms.

Remark 3.1.3. In the situation of Definition 3.1.2, we will generally abuse terminology and simply refer to A^* as a Dieudonné algebra.

Remark 3.1.4. There is an evident forgetful functor **DA** $\to$ **DC**, which carries a Dieudonné algebra A^* to its underlying Dieudonné complex (obtained by neglecting the multiplication on A^*).

Remark 3.1.5. Let us regard the category of $\mathbf{DC}$ of Dieudonné complexes as a symmetric monoidal category with respect to the tensor product described in Remark 2.1.5. Then we can identify the category $\mathbf{DA}$ of Dieudonné algebras with a full subcategory of the category $\mathrm{CAlg}(\mathbf{DC})$ of commutative algebra objects of $\mathbf{DC}$. A commutative algebra object A^* of $\mathbf{DC}$ is a Dieudonné algebra if and only if it satisfies the following conditions:

(1) The groups A^n vanish for $n < 0$.
(2) For $x \in A^0$, we have $Fx \equiv x^p \pmod p$.
(3) Every homogeneous element $x \in A^*$ of odd degree satisfies $x^2 = 0$.

Note that condition (3) is automatic if the Dieudonné complex A^* is strict, since it is 2-torsion free by virtue of Remark 2.5.5.

Remark 3.1.6. Let A^* be a commutative differential graded algebra, and suppose that A^* is p-torsion-free. Let $\eta_p A^* \subseteq A^*[p^{-1}]$ denote the subcomplex introduced in Remark 2.2.2. Then $\eta_p A^*$ is closed under multiplication, and therefore inherits the structure of a commutative differential graded algebra. Moreover, the dictionary of Remark 2.2.2 establishes a bijection between the following data:

- Maps of differential graded algebras $\alpha\colon A^* \to \eta_p A^*$
- Graded ring homomorphisms $F\colon A^* \to A^*$ satisfying $d \circ F = p(F \circ d)$.

Remark 3.1.7. Let (A^*, d, F) be a Dieudonné algebra, and suppose that A^* is p-torsion-free. Then we can regard $\eta_p A^*$ as a subalgebra of A^*, which is closed under the action of F. We claim that $(\eta_p A^*, d|_{\eta_p A^*}, F|_{\eta_p A^*})$ is also a Dieudonné algebra. The only nontrivial point is to check condition (iii) of Definition 3.1.2. Suppose we are given an element $x \in (\eta_p A)^0$: that is, an element $x \in A^0$ such that $dx = py$ for some $y \in A^1$. Since A^* satisfies condition (iii) of Definition 3.1.2, we can write $Fx = x^p + pz$ for some element $z \in A^0$. We wish to show that z also belongs to $(\eta_p A)^0$: that is, $dz \in pA^1$. This follows from the calculation

$$p(dz) = d(pz) = d(Fx - x^p) = pF(dx) - px^{p-1}dx = p^2F(y) - p^2x^{p-1}y,$$

since A^1 is p-torsion-free.

Example 3.1.8. Let R be a commutative $\mathbf{F}_p$-algebra and let $W(R)$ denote the ring of Witt vectors of R, regarded as a commutative differential graded algebra which is concentrated in degree zero. Then the Witt vector Frobenius $F : W(R) \to W(R)$ exhibits $W(R)$ as a Dieudonné algebra, in the sense of Definition 3.1.2. The Dieudonné algebra $W(R)$ is saturated (as a Dieudonné complex) if and only if the $\mathbf{F}_p$-algebra R is perfect. If this condition is satisfied, then $W(R)$ is also strict (as a Dieudonné complex).

3.2. **Example: The de Rham Complex.** We now consider an important special class of Dieudonné algebras.

Proposition 3.2.1. *Let R be a commutative ring which is p-torsion-free, and let $\varphi: R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod p$. Then there is a unique ring homomorphism $F: \Omega^*_R \to \Omega^*_R$ with the following properties:*

(1) *For each element $x \in R = \Omega^0_R$, we have $F(x) = \varphi(x)$.*
(2) *For each element $x \in R$, we have $F(dx) = x^{p-1}dx + d(\frac{\varphi(x)-x^p}{p})$*

Moreover, the triple (Ω^*_R, d, F) is a Dieudonné algebra.

Remark 3.2.2. In the situation of Proposition 3.2.1, suppose that the de Rham complex Ω^*_R is p-torsion-free, and regard $\eta_p\Omega^*_R$ as a (differential graded) subalgebra of Ω^*_R. The ring homomorphism φ extends uniquely to a map of commutative differential graded algebras $\alpha: \Omega^*_R \to \Omega^*_R$. The calculation

$$\alpha(dx) = d\alpha(x) = d\varphi(x) \equiv dx^p = px^{p-1}dx \equiv 0 \pmod p$$

shows that α carries Ω^n_R into $p^n\Omega^n_R$, and can therefore be regarded as a map of differential graded algebras $\Omega^*_R \to \eta_p\Omega^*_R$. Applying Remark 3.1.6, we see that there is a unique ring homomorphism $F: \Omega^*_R \to \Omega^*_R$ which satisfies condition (1) of Proposition 3.2.1 and endows Ω^*_R with the structure of a Dieudonné algebra. Moreover, the map F automatically satisfies condition (2) of Proposition 3.2.1: this follows from the calculation

$$\begin{aligned} pF(dx) &= d(Fx) \\ &= d(\varphi(x)) \\ &= d(x^p + p\frac{\varphi(x) - x^p}{p}) \\ &= p(x^{p-1}dx + d\frac{\varphi(x) - x^p}{p}). \end{aligned}$$

Proof of Proposition 3.2.1. The uniqueness of the homomorphism F is clear, since the de Rham complex Ω^*_R is generated (as a graded ring) by elements of the form x and dx, where $x \in R$. We now prove existence. Define $\theta: R \to R$ by the formula $\theta(x) = \frac{\varphi(x)-x^p}{p}$. A simple calculation then gives

$$\theta(x+y) = \theta(x) + \theta(y) - \sum_{0<i<p} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i} \tag{8}$$

$$\theta(xy) = \varphi(x)\theta(y) + \theta(x)\varphi(y) - p\theta(x)\theta(y). \tag{9}$$

Consider the map $\rho\colon R \to \Omega^1 R$ given by the formula $\rho(x) = x^{p-1}dx + d\theta(x)$. We first claim that ρ is a group homomorphism. This follows from the calculation

$$\begin{aligned}
\rho(x+y) &= (x+y)^{p-1}d(x+y) + d\theta(x+y) \\
&= (x+y)^{p-1}d(x+y) + d\theta(x) + d\theta(y) - d\Big(\sum_{0<i<p} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}\Big) \\
&= \rho(x) + \rho(y) + \Big((x+y)^{p-1} - x^{p-1} - \sum_{0<i<p} \frac{(p-1)!}{(i-1)!(p-i)!} x^{i-1} y^{p-i}\Big)dx \\
&\quad + \Big((x+y)^{p-1} - y^{p-1} - \sum_{0<i<p} \frac{(p-1)!}{i!(p-i-1)!} x^i y^{p-i-1}\Big)dy \\
&= \rho(x) + \rho(y).
\end{aligned}$$

We next claim that ρ is a φ-linear derivation from R into Ω^1_R: that is, it satisfies the identity $\rho(xy) = \varphi(y)\rho(x) + \varphi(x)\rho(y)$. This follows from the calculation

$$\begin{aligned}
\rho(xy) &= (xy)^{p-1}d(xy) + d\theta(xy) \\
&= (xy)^{p-1}d(xy) + d(\varphi(x)\theta(y) + \theta(x)\varphi(y) - p\theta(x)\theta(y)) \\
&= (x^{p-1}y^p dx + \theta(y)d\varphi(x) + \varphi(y)d\theta(x) - p\theta(y)d\theta(x)) \\
&\quad + (x^p y^{p-1} dy + \varphi(x)d\theta(y) + \theta(x)d\varphi(y) - p\theta(x)d\theta(y)) \\
&= ((y^p + p\theta(y))x^{p-1}dx + \varphi(y)d\theta(x)) + ((x^p + p\theta(x))y^{p-1}dy + \varphi(x)d\theta(y)) \\
&= \varphi(y)(x^{p-1}dx + d\theta(x)) + \varphi(x)(y^{p-1}dy + d\theta(y)) \\
&= \varphi(y)\rho(x) + \varphi(x)\rho(y).
\end{aligned}$$

Invoking the universal property of the module of Kähler differentials Ω^1_R, we deduce that there is a unique φ-semilinear map $F\colon \Omega^1_R \to \Omega^1_R$ satisfying $F(dx) = \lambda(x)$ for $x \in R$. For every element $\omega \in \Omega^1_R$, we have $F(\omega)^2 = 0$ in Ω^*_R (since $F(\omega)$ is a 1-form), so F extends to a ring homomorphism $\Omega^*_R \to \Omega^*_R$ satisfying (1) and (2).

To complete the proof, it will suffice to show that F endows Ω^*_R with the structure of a Dieudonné algebra. It follows (1) that $F(x) \equiv x^p \pmod p$ for $x \in \Omega^0_R$. It will therefore suffice to show that $d \circ F = p(F \circ d)$. Let $A \subseteq \Omega^*_R$ be the subset consisting of those elements ω which satisfy $dF(\omega) = pF(d\omega)$. It is clear that A is a graded subgroup of Ω^*_R. Moreover, A is closed under multiplication: if $\omega \in \Omega^m_R$ and $\omega' \in \Omega^n_R$ belong to A, then we compute

$$\begin{aligned}
dF(\omega \wedge \omega') &= d(F(\omega) \wedge F(\omega')) \\
&= ((dF\omega) \wedge F\omega') + (-1)^m(F\omega \wedge (dF\omega')) \\
&= p((Fd\omega) \wedge F\omega') + (-1)^m(F\omega \wedge (Fd\omega')) \\
&= pF(d\omega \wedge \omega' + (-1)^m \omega \wedge d\omega') \\
&= pFd(\omega \wedge \omega').
\end{aligned}$$

Consequently, to show that $A = \Omega^*_R$, it will suffice to show that A contains x and dx for each $x \in R$. The inclusion $dx \in A$ is clear (note that $Fddx = 0 = dF(dx)$, since $F(x) = x^{p-1}dx + d\theta(x))$ is a closed 1-form). The inclusion $x \in A$ follows from the calculation

$$\begin{aligned} dFx &= d\varphi(x) \\ &= d(x^p + p\theta(x)) \\ &= px^{p-1}dx + pd\theta(x) \\ &= p(x^{p-1}dx + d\theta(x)) \\ &= pF(dx). \end{aligned}$$

□

The Dieudonné algebra of Proposition 3.2.1 enjoys the following universal property:

Proposition 3.2.3. *Let R be a commutative ring which is p-torsion-free, let $\varphi: R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod p$, and regard the de Rham complex Ω^*_R as equipped with the Dieudonné algebra structure of Proposition 3.2.1. Let A^* be a Dieudonné algebra which is p-torsion-free. Then the restriction map*

$$\mathrm{Hom}_{\mathbf{DA}}(\Omega^*_R, A^*) \to \mathrm{Hom}(R, A^0)$$

is injective, and its image consists of those ring homomorphisms $f: R \to A^0$ satisfying $f(\varphi(x)) = F(f(x))$ for $x \in R$.

Proof. Let $f: R \to A^0$ be a ring homomorphism. Invoking the universal property of the de Rham complex Ω^*_R, we see that f extends uniquely to a homomorphism of differential graded algebras $\overline{f}: \Omega^*_R \to A^*$. To complete the proof, it will suffice to show that $\overline{f}$ is a map of Dieudonné algebras if and only if f satisfies the identity $f(\varphi(x)) = F(f(x))$ for each $x \in R$. The "only if" direction is trivial. Conversely, suppose that $f(\varphi(x)) = F(f(x))$ for $x \in R$; we wish to show that $\overline{f}(F\omega) = F\overline{f}(\omega)$ for each $\omega \in \Omega^*_R$. The collection of those elements $\omega \in \Omega^*_R$ which satisfy this identity form a subring of Ω^*_R. Consequently, we may assume without loss of generality that $\omega = x$ or $\omega = dx$, for some $x \in R$. In the first case, the desired result follows from our assumption. To handle the second, we compute

$$\begin{aligned}
p\overline{f}(Fdx) &= p\overline{f}(x^{p-1}dx + d\frac{\varphi(x)-x^p}{p}) \\
&= \overline{f}(px^{p-1}dx + d(\varphi(x)-x^p)) \\
&= \overline{f}(d\varphi(x)) \\
&= d\overline{f}(\varphi(x)) \\
&= dFf(x) \\
&= pFd(f(x)).
\end{aligned}$$

Since A^1 is p-torsion-free, it follows that $\overline{f}(Fdx) = Fd\overline{f}(x)$, as desired. □

3.3. **The Cartier Isomorphism.** We will need a variant of Proposition 3.2.1 for *completed* de Rham complexes.

Variant 3.3.1 (Completed de Rham Complexes)**.** Let R be a commutative ring. We let $\widehat{\Omega}^*_R$ denote the inverse limit

$$\varprojlim_n \Omega^*_R/p^n\Omega^*_R \simeq \varprojlim_n \Omega^*_{R/p^nR}.$$

We will refer to $\widehat{\Omega}^*_R$ as the *completed de Rham complex* of R. Note that there is a unique multiplication on $\widehat{\Omega}^*_R$ for which the tautological map $\Omega^*_R \to \widehat{\Omega}^*_R$ is a morphism of differential graded algebras.

Now suppose that R is p-torsion-free, and that we are given a ring homomorphism $\varphi\colon R \to R$ satisfying $\varphi(x) \equiv x^p \pmod p$. Then the homomorphism $F\colon \Omega^*_R \to \Omega^*_R$ of Proposition 3.2.1 induces a map $\widehat{\Omega}^*_R \to \widehat{\Omega}^*_R$, which we will also denote by F, and which endows $\widehat{\Omega}^*_R$ with the structure of a Dieudonné algebra. Moreover, this Dieudonné algebra enjoys the following universal property: if A^* is any Dieudonné algebra which is p-adically complete and p-torsion-free, then the restriction map

$$\mathrm{Hom}_{\mathbf{DA}}(\widehat{\Omega}^*_R, A^*) \to \mathrm{Hom}(R, A^0)$$

is an injection, whose image is the collection of ring homomorphisms $f\colon R \to A^0$ satisfying $f \circ \varphi = F \circ f$. This follows immediately from Proposition 3.2.3.

Remark 3.3.2. Let R be a commutative ring. Then the quotient $\widehat{\Omega}^*_R/p^n\widehat{\Omega}^*_R$ can be identified with the de Rham complex Ω^*_{R/p^nR}.

Remark 3.3.3. Let R be a commutative ring which is p-torsion-free. Suppose that there exists a perfect $\mathbf{F}_p$-algebra k such that R/pR is a smooth k-algebra. Then each R/p^nR is also a smooth $W_n(k)$-algebra, and our assumption that k is perfect guarantees that the quotient $\widehat{\Omega}^*_R/p^n\widehat{\Omega}^*_R$ can be identified with the de

Rham complex of R/p^nR *relative* to $W_n(k)$. It follows that each $\widehat{\Omega}^i_R/p^n\widehat{\Omega}^i_R$ is a projective R/p^nR-module of finite rank, and that the transition maps

$$(\widehat{\Omega}^i_R/p^n\widehat{\Omega}^i_R) \otimes_{\mathbf{Z}/p^n\mathbf{Z}} (\mathbf{Z}/p^{n-1}\mathbf{Z}) \to (\widehat{\Omega}^i_R/p^{n-1}\widehat{\Omega}^i_R)$$

are isomorphisms. In this case, we conclude that each $\widehat{\Omega}^i_R$ is a projective module of finite rank over the completion $\widehat{R} = \varprojlim R/p^nR$. In particular, $\widehat{\Omega}^*_R$ is p-torsion-free.

We now recall the classical *Cartier isomorphism*, which will be useful for studying the completed de Rham complexes of Variant 3.3.1.

Proposition 3.3.4. *Let A be a commutative $\mathbf{F}_p$-algebra. Then there is a unique homomorphism of graded algebras* $\mathrm{Cart}\colon \Omega^*_A \to \mathrm{H}^*(\Omega^*_A)$ *satisfying*

$$\mathrm{Cart}(x) = x^p \qquad \mathrm{Cart}(dy) = [y^{p-1}dy],$$

where $[y^{p-1}dy]$ denotes the cohomology class of the cocycle $y^{p-1}dy \in \Omega^1_A$.

Proof. It will suffice to show that the construction $(y \in A) \mapsto [y^{p-1}dy] \in \mathrm{H}^1(\Omega^*_A)$ is a derivation with respect to the Frobenius map $A \to \mathrm{H}^0(\Omega^*_A)$. This follows from the calculations

$$\begin{aligned}(xy)^{p-1}d(xy) &= x^p(y^{p-1}dy) + y^p(x^{p-1}dx)\\ (x+y)^{p-1}d(x+y) &= x^{p-1}dx + y^{p-1}dy + d\Big(\sum_{0<i<p} \frac{(p-1)!}{i!(p-i)!}x^i y^{p-i}\Big).\end{aligned}$$

□

We will refer to the homomorphism Cart of Proposition 3.3.4 as the *Cartier map*.

Example 3.3.5. Let R be a commutative ring which is p-torsion-free, let $\varphi\colon R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod p$, and regard $\widehat{\Omega}^*_R$ as a Dieudonné algebra as in Variant 3.3.1. Then the Frobenius map $F\colon \widehat{\Omega}^*_R \to \widehat{\Omega}^*_R$ induces a map of graded rings

$$\Omega^*_{R/pR} \simeq \widehat{\Omega}^*_R/p\widehat{\Omega}^*_R \to \mathrm{H}^*(\widehat{\Omega}^*_R/p\widehat{\Omega}^*_R) \simeq \mathrm{H}^*(\Omega^*_{R/p}).$$

Using the formulae of Proposition 3.2.1, we see that this map is given concretely by the formula

$$x_0 dx_1 \wedge \cdots dx_n \mapsto [x_0^p(x_1^{p-1}dx_1) \wedge \cdots \wedge (x_n^{p-1}dx_n)],$$

and therefore coincides with the Cartier map of Proposition 3.3.4. In particular, it does not depend on the choice of φ.

We refer the reader to [32, Th. 7.2] for a proof of the following:

Theorem 3.3.6 (Cartier Isomorphism)**.** *Let k be a perfect $\mathbf{F}_p$-algebra and let A be a smooth k-algebra. Then the Cartier map* $\mathrm{Cart}\colon \Omega^*_A \to \mathrm{H}^*(\Omega^*_A)$ *is an isomorphism.*

Remark 3.3.7. In §9.5, we will use Theorem 3.3.6 to show that the Cartier map $\mathrm{Cart}: \Omega^*_A \to \mathrm{H}^*(\Omega^*_A)$ is an isomorphism whenever A is a regular Noetherian $\mathbf{F}_p$-algebra (Theorem 9.5.1).

Corollary 3.3.8. *Let R be a commutative ring which is p-torsion-free and let $\varphi: R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$. Suppose that there exists a perfect $\mathbf{F}_p$-algebra k such that R/pR is a smooth algebra over k. Then:*

(1) *When regarded as a Dieudonné complex, the completed de Rham complex $\widehat{\Omega}^*_R$ is of Cartier type (in the sense of Definition 2.4.1).*
(2) *The canonical map $\widehat{\Omega}^*_R \to \mathcal{W}\mathrm{Sat}(\widehat{\Omega}^*_R)$ is a quasi-isomorphism of chain complexes.*

Proof. Assertion (1) follows from Theorem 3.3.6 and Example 3.3.5. Assertion (2) follows from (1) and Corollary 2.8.5. □

3.4. **Saturated Dieudonné Algebras.**

Definition 3.4.1. Let A^* be a Dieudonné algebra. We will say that A^* is *saturated* if it is saturated when regarded as a Dieudonné complex: that is, if it is p-torsion-free and the map $F: A^* \to \{x \in A^* : dx \in pA^*\}$ is a bijection. We let $\mathbf{DA}_{\mathrm{sat}}$ denote the full subcategory of $\mathbf{DA}$ spanned by the saturated Dieudonné algebras.

In the setting of saturated Dieudonné algebras, axiom (iii) of Definition 3.1.2 can be slightly weakened:

Proposition 3.4.2. *Let A^* be a saturated Dieudonné complex satisfying $A^* = 0$ for $* < 0$. Suppose that A^* is equipped with the structure of a commutative differential graded algebra for which the Frobenius map $F: A^* \to A^*$ is a ring homomorphism. The following conditions are equivalent:*

(a) *The complex A^* is a Dieudonné algebra: that is, each element $x \in A^0$ satisfies $Fx \equiv x^p \pmod{p}$.*
(b) *Each element $x \in A^0$ satisfies $Fx \equiv x^p \pmod{VA^0}$.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. For the converse, suppose that (b) is satisfied and let $x \in A^0$. Then we can write $Fx = x^p + Vy$ for some $y \in A^0$. Applying the differential d, we obtain $d(Vy) = d(Fx - x^p) = p(F(dx) - x^{p-1}dx) \in pA^1$. Invoking Lemma 2.6.3, we can write $y = Fz$ for some $z \in A^0$, so that $Fx = x^p + Vy = x^p + VFz = x^p + pz \equiv x^p \pmod{p}$. □

Let $f: A^* \to B^*$ be a morphism of Dieudonné algebras. We will say that f *exhibits B^* as a saturation of A^** if B^* is saturated and, for every saturated Dieudonné algebra C^*, composition with f induces a bijection

$$\mathrm{Hom}_{\mathbf{DA}}(B^*, C^*) \to \mathrm{Hom}_{\mathbf{DA}}(A^*, C^*).$$

Proposition 3.4.3. *Let A^* be a Dieudonné algebra. Then there exists a map of Dieudonné algebras $f\colon A^* \to B^*$ which exhibits B^* as a saturation of A^*. Moreover, f induces an isomorphism of Dieudonné complexes* $\mathrm{Sat}(A^*) \to B^*$, *where* $\mathrm{Sat}(A^*)$ *is the saturation of §2.3.*

Remark 3.4.4. We can summarize Proposition 3.4.3 more informally as follows: if A^* is a Dieudonné algebra and $\mathrm{Sat}(A^*)$ is the saturation of A^* in the category of Dieudonné complexes, then $\mathrm{Sat}(A^*)$ inherits the structure of a Dieudonné algebra, and is also a saturation of A^* in the category of Dieudonné algebras.

Proof of Proposition 3.4.3. Replacing A^* by the quotient $A^*/A^*[p^\infty]$, we can reduce to the case where A^* is p-torsion-free. In this case, the Frobenius on A determines a map of Dieudonné algebras $\alpha_F\colon A^* \to (\eta_p A)^*$ (Remark 3.1.7), and A^* is saturated if and only if α_F is an isomorphism. The saturation of A can then be described as the direct limit of the sequence

$$A^* \xrightarrow{\alpha_F} (\eta_p A)^* \xrightarrow{\eta_p(\alpha_F)} (\eta_p\eta_p A)^* \xrightarrow{\eta_p(\eta_p(\alpha_F))} (\eta_p\eta_p\eta_p A)^* \to \cdots,$$

which is also the saturation of A^* as a Dieudonné complex. □

3.5. Completions of Saturated Dieudonné Algebras.

Proposition 3.5.1. *Let A^* be a saturated Dieudonné algebra, and let $V\colon A^* \to A^*$ be the Verschiebung map of Remark 2.2.3. Then:*

- (i) *The map V satisfies the projection formula $xV(y) = V(F(x)y)$.*
- (ii) *For each $r \geq 0$, the sum $\mathrm{im}(V^r) + \mathrm{im}(dV^r) \subseteq A^*$ is a (differential graded) ideal.*

Proof. We first prove (i). Given $x, y \in A^*$, we compute

$$F(xV(y)) = F(x)F(V(y)) = pF(x)y = F(V(F(x)y).$$

Since the map F is injective, we conclude that $xV(y) = V(F(x)y)$.

We now prove (ii). Let x be an element of A^*; we wish to show that multiplication by x carries $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$ into itself. This follows from the identities

$$xV^r(y) = V^r(F^r(x)y)$$
$$\pm xd(V^r y) = d(xV^r(y)) - (dx)V^r(y) = dV^r(F^r(x)y) - V^r(F^r(dx)y).$$

□

Corollary 3.5.2. *Let A^* be a saturated Dieudonné algebra. For each $r \geq 0$, there is a unique ring structure on the quotient $\mathcal{W}_r(A)^*$ for which the projection map $A^* \to \mathcal{W}_r(A)^*$ is a ring homomorphism. Moreover, this ring structure exhibits $\mathcal{W}_r(A)^*$ as a commutative differential graded algebra.*

Remark 3.5.3. Let A^* be a saturated Dieudonné algebra. Since $A^{-1} = 0$, we have $\mathcal{W}_r(A)^0 = A^0/V^r A^0$. In particular, we have $\mathcal{W}_1(A)^0 = A^0/VA^0$.

Construction 3.5.4. Let A^* be a saturated Dieudonné algebra. We let $\mathcal{W}(A)^*$ denote the completion of A^* as a (saturated) Dieudonné complex, given by the inverse limit of the tower

$$\cdots \to \mathcal{W}_3(A)^* \to \mathcal{W}_2(A)^* \to \mathcal{W}_1(A)^* \to \mathcal{W}_0(A)^* \simeq 0.$$

Since each $\mathcal{W}_n(A)^*$ has the structure of a commutative differential graded algebra, the inverse limit $\mathcal{W}(A)^*$ inherits the structure of a commutative differential graded algebra.

Proposition 3.5.5. *Let A^* be a saturated Dieudonné algebra. Then the completion $\mathcal{W}(A)^*$ is also a saturated Dieudonné algebra.*

Proof. It follows from Corollary 2.7.6 that $\mathcal{W}(A)^*$ is a saturated Dieudonné complex. Moreover, the Frobenius map $F: \mathcal{W}(A)^* \to \mathcal{W}(A)^*$ is an inverse limit of the maps $F: \mathcal{W}_r(A)^* \to \mathcal{W}_{r-1}(A)^*$ appearing in Remark 2.5.2, each of which is a ring homomorphism (since the Frobenius on A^* is a ring homomorphism). It follows that $F: \mathcal{W}(A)^* \to \mathcal{W}(A)^*$ is a ring homomorphism. The vanishing of $\mathcal{W}(A)^*$ for $* < 0$ follows immediately from the analogous property of A^*. To complete the proof, it will suffice to show that each element $x \in \mathcal{W}(A)^0$ satisfies $Fx \equiv x^p \pmod{V\,\mathcal{W}(A)^0}$ (Proposition 3.4.2). In other words, we must show that F induces the usual Frobenius map on the $\mathbf{F}_p$-algebra $\mathcal{W}_1(\mathcal{W}(A)^0) = \mathcal{W}(A)^0 / V\,\mathcal{W}(A)^0$. This follows from the analogous property for A^*, since the tautological map $\mathcal{W}_1(A)^* \to \mathcal{W}_1(\mathcal{W}(A))^*$ is an isomorphism by Proposition 2.7.5. $\square$

Let A^* be a saturated Dieudonné algebra. Then the tautological map $\rho_A: A^* \to \mathcal{W}(A)^*$ of §2.5 is a morphism of (saturated) Dieudonné algebras.

Definition 3.5.6. Let A^* be a saturated Dieudonné algebra. We will say that A^* is *strict* if the map $\rho_A: A^* \to \mathcal{W}(A)^*$ is an isomorphism of Dieudonné algebras. We let $\mathbf{DA}_{\mathrm{str}}$ denote the full subcategory of $\mathbf{DA}_{\mathrm{sat}}$ spanned by the strict Dieudonné algebras.

Remark 3.5.7. A saturated Dieudonné algebra A^* is strict if and only if it is strict when regarded as a saturated Dieudonné complex.

We have the following nonlinear version of Proposition 2.7.7:

Proposition 3.5.8. *Let A^* and B^* be saturated Dieudonné algebras, where B^* is strict. Then composition with the map ρ_A induces a bijection*

$$\theta: \mathrm{Hom}_{\mathbf{DA}}(\mathcal{W}(A)^*, B^*) \to \mathrm{Hom}_{\mathbf{DA}}(A^*, B^*).$$

Proof. By virtue of Proposition 2.7.7, it will suffice to show that if $f: \mathcal{W}(A)^* \to B^*$ is a map of Dieudonné complexes for which $f \circ \rho_A$ is a ring homomorphism, then f is a ring homomorphism. Since B^* is complete, it will suffice to show that each of the composite maps $\mathcal{W}(A)^* \to B^* \to \mathcal{W}_r(B)^*$ is a ring homomorphism. This follows by inspecting the commutative diagram

$$\begin{tikzcd} A^* \arrow[r, "\rho_A"] \arrow[d] & \mathcal{W}(A)^* \arrow[r, "f"] \arrow[d] & B^* \arrow[d] \\ \mathcal{W}_r(A)^* \arrow[r, "\beta"] & \mathcal{W}_r(\mathcal{W}(A))^* \arrow[r] & \mathcal{W}_r(B)^*, \end{tikzcd}$$

since the map β bijective (Proposition 2.7.5). □

Corollary 3.5.9. *The inclusion functor $\mathbf{DA}_{\mathrm{str}} \hookrightarrow \mathbf{DA}_{\mathrm{sat}}$ admits a left adjoint, given by the formation of completions $A^* \to \mathcal{W}(A)^*$.*

Corollary 3.5.10. *The inclusion functor $\mathbf{DA}_{\mathrm{str}} \hookrightarrow \mathbf{DA}$ admits a left adjoint, given by the completed saturation functor $\mathcal{W}\,\mathrm{Sat}\colon \mathbf{DA} \to \mathbf{DA}_{\mathrm{str}}$.*

Proof. Combine Proposition 3.4.3 with Corollary 3.5.9. □

3.6. **Comparison with Witt Vectors.**

Lemma 3.6.1. *Let A^* be a saturated Dieudonné algebra. Then the quotient $R = A^0/VA^0$ is a reduced $\mathbf{F}_p$-algebra.*

Proof. It follows from Remark 3.5.3 that VA^0 is an ideal in A^0, so that we can regard R as a commutative ring. Note that the ideal VA^0 contains $V(1) = V(F(1)) = p$, so that R is an $\mathbf{F}_p$-algebra. To show that R is reduced, it will suffice to show that if $\overline{x}$ is an element of R satisfying $\overline{x}^p = 0$, then $\overline{x} = 0$. Choose an element $x \in A^0$ representing $\overline{x}$. The condition $\overline{x}^p = 0$ implies that $x^p \in VA^0$, so that $F(x) \equiv x^p \pmod p$ also belongs to VA^0. We can therefore write $Fx = Vy$ for some $y \in A^0$. Applying the differential d, we obtain $d(Vy) = d(Fx) = pF(dx) \in pA^1$. Invoking Lemma 2.6.3, we can write $y = Fz$ for some element $z \in A^0$. Then $Fx = VFz = FVz$, so that $x = Vz \in VA^0$ and therefore $\overline{x} = 0$, as desired. □

Proposition 3.6.2. *Let A^* be a strict Dieudonné algebra and let $R = A^0/VA^0$. Then there is a unique ring isomorphism $u\colon A^0 \to W(R)$ with the following properties:*

(*i*) *The diagram*

$$\begin{tikzcd} A^0 \arrow[r, "u"] \arrow[d] & W(R) \arrow[d] \\ A^0/VA^0 \arrow[r, "\mathrm{id}"] & R \end{tikzcd}$$

commutes (where the vertical maps are the natural projections).

(*ii*) *The diagram*

$$\begin{tikzcd} A^0 \arrow[r, "u"] \arrow[d, "F"] & W(R) \arrow[d, "F"] \\ A^0 \arrow[r, "u"] & W(R) \end{tikzcd}$$

commutes (where the right vertical map is the Witt vector Frobenius).

Proof. Since the ring R is reduced (Lemma 3.6.1), the ring of Witt vectors $W(R)$ is p-torsion free as it embeds inside $W(R^{1/p^\infty})$ where R^{1/p^∞} denotes the *perfection*

$$\varinjlim (R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} R \to \cdots)$$

Applying the universal property of $W(R)$ (see [30]), we deduce that there exists a unique ring homomorphism $u: A^0 \to W(R)$ which satisfies conditions (i) and (ii). To complete the proof, it will suffice to show that u is an isomorphism. We first note that u satisfies the identity $u(Vx) = Vu(x)$ for each $x \in A^0$ (where the second V denotes the usual Witt vector Verschiebung). To prove this, we begin with the identity

$$F(u(Vx)) = u(FVx) = u(px) = pu(x) = FVu(x)$$

and then invoke the fact that the Frobenius map $F: W(R) \to W(R)$ is injective (by virtue of the fact that R is reduced). It follows that u carries $V^r A^0$ into $V^r W(R)$, and therefore induces a ring homomorphism $u_r: \mathcal{W}_r(A)^0 \to W_r(R)$ for every nonnegative integer r. Since A^* is strict, we can identify u with the inverse limit of the tower of maps $\{u_r\}_{r \geq 0}$. Consequently, to show that u is an isomorphism, it will suffice to show that each u_r is an isomorphism. We now proceed by induction on r, the case $r = 0$ being trivial. We have a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A^0/VA^0 \arrow[r, "V^{r-1}"] \arrow[d, "u_1"] & A^0/V^r A^0 \arrow[r] \arrow[d, "u_r"] & A^0/V^{r-1}A^0 \arrow[r] \arrow[d, "u_{r-1}"] & 0 \\ 0 \arrow[r] & R \arrow[r, "V^{r-1}"] & W_r(R) \arrow[r] & W_{r-1}(R) \arrow[r] & 0 \end{tikzcd}$$

where u_1 is the identity map from $A^0/VA^0 = R$ to itself, and u_{r-1} is an isomorphism by the inductive hypothesis. It follows that u_r is also an isomorphism, as desired. □

Combining Proposition 3.6.2 with the universal property of the Witt vectors $W(R)$, we obtain the following result:

Corollary 3.6.3. *Let B be a commutative ring which is p-torsion-free, and let $\varphi: B \to B$ be a ring homomorphism satisfying $\varphi(b) \equiv b^p \pmod{p}$ for each $b \in B$. Let A^* be a strict Dieudonné algebra. Then every ring homomorphism $f_0: B \to$*

A^0/VA^0 admits an essentially unique lift to a ring homomorphism $f: B \to A^0$ which satisfies $f \circ \varphi = F \circ f$.

We will also need a slight variation on Corollary 3.6.3.

Proposition 3.6.4. *Let A^* be a strict Dieudonné algebra and let R be a commutative $\mathbf{F}_p$-algebra. For every ring homomorphism $f_0: R \to A^0/VA^0$, there is a unique ring homomorphism $f: W(R) \to A^0$ satisfying the following pair of conditions:*

- (i) *The diagram*

$$\begin{tikzcd} W(R) \arrow[r, "f"] \arrow[d] & A^0 \arrow[d] \\ R \arrow[r, "f_0"] & A^0/VA^0 \end{tikzcd}$$

commutes (where the vertical maps are the natural projections).

- (ii) *The diagram*

$$\begin{tikzcd} W(R) \arrow[r, "f"] \arrow[d, "F"] & A^0 \arrow[d, "F"] \\ W(R) \arrow[r, "f"] & A^0 \end{tikzcd}$$

commutes (where the left vertical map is the Witt vector Frobenius).

Remark 3.6.5. In the special case where R is reduced, the ring of Witt vectors $W(R)$ is p-torsion-free, so that Proposition 3.6.4 follows from Corollary 3.6.3.

In the situation of Proposition 3.6.4, we can apply Proposition 3.6.2 to identify A^0 with the ring of Witt vectors $W(A^0/VA^0)$. Consequently, the *existence* of the map $f: W(R) \to A^0$ is clear: we simply apply the Witt vector functor to the ring homomorphism $f_0: R \to A^0/VA^0$. The uniqueness of f is a consequence of the following elementary assertion:

Lemma 3.6.6. *Let $f: R \to R'$ be a homomorphism of commutative $\mathbf{F}_p$-algebras, and let $g: W(R) \to W(R')$ be a ring homomorphism. Suppose that the diagrams*

$$\begin{tikzcd} W(R) \arrow[r, "g"] \arrow[d] & W(R') \arrow[d] \\ R \arrow[r, "f"] & R' \end{tikzcd} \qquad \begin{tikzcd} W(R) \arrow[r, "g"] \arrow[d, "F"] & W(R') \arrow[d, "F"] \\ W(R) \arrow[r, "g"] & W(R') \end{tikzcd}$$

commute. If R' is reduced, then $g = W(f)$ coincides with the map determined by f (and the functoriality of the construction $A \mapsto W(A)$).

Proof. For each element $a \in R$, let $[a] \in W(R)$ denote its Teichmüller representative, and define $[b] \in W(R')$ for $b \in R'$ similarly. We first claim that g is compatible with the formation of Teichmüller representatives: that is, we have $g([a]) = [f(a)]$ for each $a \in R$. For this, it will suffice to show that $g([a]) \equiv [f(a)] \pmod{V^r}$ for each $r \geq 0$. We proceed by induction on r. The case $r = 1$ follows from our compatibility assumption concerning f and g. To carry out the inductive step, let us assume that we have an identity of the form $g([a]) = [f(a)] + V^r b$ for some $x \in W(R')$. We then compute

$$\begin{aligned} Fg([a]) &= g(F[a]) \\ &= g([a]^p) \\ &= g([a])^p \\ &= ([f(a)] + V^r b)^p \\ &\equiv [f(a)]^p \pmod{pV^r} \\ &\equiv F[f(a)] \pmod{FV^{r+1}}. \end{aligned}$$

Since R' is reduced, the Frobenius map $F\colon W(R') \to W(R')$ is injective, so we deduce that $g([a]) \equiv [f(a)] \pmod{V^{r+1}}$, thereby completing the induction.

For each $x \in W(R)$, we have the identity

$$F(g(Vx)) = g(FVx) = g(px) = pg(x) = FVg(x).$$

Using the injectivity of F again, we conclude that $g(Vx) = Vg(x)$: that is, the map g commutes with Verschiebung.

Combining these observations, we see that for any element $x \in W(R)$ with Teichmüller expansion $\sum_{n \geq 0} V^n [a_n]$, we have a congruence

$$\begin{aligned} g(x) &\equiv g([a_0] + V[a_1] + \cdots + V^n[a_n]) \pmod{V^{n+1}} \\ &\equiv g([a_0]) + Vg([a_1]) + \cdots + V^n g([a_n]) \pmod{V^{n+1}} \\ &\equiv [f(a_0)] + V[f(a_1)] + \cdots + V^n[f(a_n)] \pmod{V^{n+1}} \\ &\equiv W(f)(x) \pmod{V^{n+1}}. \end{aligned}$$

Allowing n to vary, we deduce that $g(x) = W(f)(x)$, as desired. □

3.7. **Aside: Rings with p-Torsion.** We now describe a generalization of Proposition 3.2.1 which allows us to relax the assumption that R is p-torsion free. This material will not be used in the sequel, and can be safely skipped without loss of continuity. We begin by recalling the notion of a λ_p-*ring* or θ-*algebra*, introduced by Joyal in [30] and further developed (using the language of *plethories*) in [16].

Definition 3.7.1. A λ_p-*ring* is a commutative ring R equipped with a function $\theta\colon R \to R$ satisfying the identities

$$\theta(x + y) = \theta(x) + \theta(y) - \sum_{0<i<p} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}$$

$$\theta(xy) = x^p\theta(y) + \theta(x)y^p + p\theta(x)\theta(y).$$

$$\theta(1) = \theta(0) = 0.$$

Remark 3.7.2. Let (R, θ) be a λ_p-ring, and define $\varphi: R \to R$ by the formula $\varphi(x) = x^p + p\theta(x)$. Then φ is a ring homomorphism satisfying the condition $\varphi(x) \equiv x^p \pmod{p}$. Conversely, if R is a p-torsion-free ring equipped with a ring homomorphism $\varphi: R \to R$ satisfying $\varphi(x) \equiv x^p \pmod{p}$, then the construction

$$x \mapsto \frac{\varphi(x) - x^p}{p}$$

determines a map $\theta: R \to R$ which endows R with the structure of a λ_p-ring (as noted in the proof of Proposition 3.2.1). This observation does not extend to the case where R has p-torsion. For example, if R is an $\mathbf{F}_p$-algebra, then there is a unique ring homomorphism $\varphi: R \to R$ satisfying $\varphi(x) \equiv x^p \pmod{p}$, but R never admits the structure of a λ_p-ring (except in the trivial case $R \simeq 0$).

Remark 3.7.3. The free λ_p-ring on a single generator x is given as a commutative ring by $S = \mathbf{Z}[x, \theta(x), \theta^2(x), \dots]$. More generally, the free λ_p-ring on a set of generators $\{x_i\}_{i \in I}$ can be identified with an I-fold tensor product of copies of S, and is therefore isomorphic (as a commutative ring) to a polynomial algebra over $\mathbf{Z}$ on infinitely many generators.

Definition 3.7.4. A *λ_p-cdga* is a commutative differential graded algebra (A^*, d) together with a map of graded rings $F: A^* \to A^*$ and a map of sets $\theta: A^0 \to A^0$ satisfying the following conditions:

(1) The group A^n vanishes for $n < 0$.
(2) The map $\theta: A^0 \to A^0$ endows A^0 with the structure of a λ_p-ring.
(3) For each $x \in A^0$, we have $F(x) = x^p + p\theta(x)$.
(4) For each $x \in A^*$, we have $dF(x) = pF(dx)$.
(5) For each $x \in A^0$, we have $F(dx) = x^{p-1}dx + d\theta(x)$ (note that this condition is automatic if A^1 is p-torsion-free, since it holds after multiplying by p).

Remark 3.7.5. Let (A^*, d, F, θ) be a λ_p-cdga. Then the triple (A^*, d, F) is a Dieudonné algebra. Conversely, if (A^*, d, F) is a p-torsion-free Dieudonné algebra, then there is a unique map $\theta: A^0 \to A^0$ for which (A^*, d, F, θ) is a λ_p-cdga.

Proposition 3.7.6. *Let (R, θ) be a λ_p-ring. Then there is a unique map $F: \Omega^*_R \to \Omega^*_R$ which endows the de Rham complex Ω^*_R with the structure of a λ_p-cdga. Moreover, for any λ_p-cdga A^*, we have a canonical bijection*

$$\{\lambda_p - \text{cdga maps } \Omega^*_R \to A^*\} \simeq \left\{\lambda_p - \text{ring maps } R \to A^0\right\}.$$

In other words, the construction $R \mapsto \Omega^*_R$ is the left adjoint of the forgetful functor $A^* \mapsto A^0$ from λ_p-cdgas to λ_p-rings.

Proof. Assume first that R is p-torsion free. The existence and uniqueness of F follows from Proposition 3.2.1. Let A^* be any λ_p-cdga. Using the universal property of the de Rham complex Ω^*_R, we see that every morphism of λ_p-rings $f\colon R \to A^0$ extends uniquely to a map of differential graded algebras $\overline{f}\colon \Omega^*_R \to A^*$. It remains to check that $\overline{f}$ is a map of λ_p-cdgas: that is, that it commutes with the Frobenius operator F. Since Ω^*_R is generated by R together with elements of the form dx, we are reduced to showing that $\overline{f}(F dx) = F\overline{f}(dx)$ for each $x \in R$, which follows from the identity $F(dy) = y^{p-1}dy + d\theta(y)$ for $y = f(x) \in A^0$.

To treat the general case, we note that any λ_p-ring R can be presented as a reflexive coequalizer $\varinjlim(F' \rightrightarrows F)$, where F and F' are free λ_p-rings. In this case, F and F' are p-torsion-free (Remark 3.7.3). It follows from the preceding argument that the de Rham complexes Ω^*_F and $\Omega^*_{F'}$ can be regarded as λ_p-cdgas. Since reflexive coequalizers in the category of λ_p-cdgas can be computed at the level of the underlying abelian groups, it follows that $\Omega^*_R \simeq \varinjlim(\Omega^*_{F'} \rightrightarrows \Omega^*_F)$ inherits the structure of a λ_p-algebra with the desired universal property. $\square$

4. The Saturated de Rham-Witt Complex

In this section, we introduce the *saturated de Rham-Witt complex* $\mathcal{W}\Omega^*_R$ of a commutative $\mathbf{F}_p$-algebra R (Definition 4.1.1). We define $\mathcal{W}\Omega^*_R$ to be a strict Dieudonné algebra satisfying a particular universal property, analogous to the characterization of the usual de Rham complex Ω^*_R as the commutative differential graded algebra generated by R. In the case where R is a smooth algebra over a perfect ring k, we show that $\mathcal{W}\Omega^*_R$ is quasi-isomorphic to the completed de Rham complex of any $W(k)$-lift of R (Theorem 4.2.4). In the case where R is a regular Noetherian ring, we show that the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ agrees with the classical de Rham-Witt complex $W\Omega^*_R$ constructed in [28] (Theorem 4.2.4).

4.1. Construction of $\mathcal{W}\Omega^*_R$.

Definition 4.1.1. Let A^* be a strict Dieudonné algebra and suppose we are given an $\mathbf{F}_p$-algebra homomorphism $f : R \to A^0/VA^0$. We will say that f *exhibits* A^* *as a saturated de Rham-Witt complex of* R if it satisfies the following universal property: for every strict Dieudonné algebra B^*, composition with f induces a bijection

$$\mathrm{Hom}_{\mathbf{DA}}(A^*, B^*) \to \mathrm{Hom}(R, B^0/VB^0).$$

Notation 4.1.2. Let R be an $\mathbf{F}_p$-algebra. It follows immediately from the definitions that if there exists a strict Dieudonné algebra A^* and a map $f : R \to A^0/VA^0$ which exhibits A^* as a de Rham-Witt complex of R, then A^* (and the map f) are determined up to unique isomorphism. We will indicate this dependence by denoting A^* as $\mathcal{W}\Omega^*_R$ and referring to A^* as *the saturated de Rham-Witt complex of* R.

Warning 4.1.3. The saturated de Rham Witt complex $\mathcal{W}\Omega^*_R$ of Notation 4.1.2 is generally not the same as the de Rham-Witt complex defined in [28] (though they agree for a large class of $\mathbf{F}_p$-algebras: see Theorem 4.4.12). For this reason, we use the term *classical de Rham-Witt complex* to refer to complex $W\Omega^*_R$ constructed in [28] (see Definition 4.4.6 for a review).

For example, it follows from Lemma 3.6.1 that the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ agrees with the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{R^{\mathrm{red}}}$, where R^{red} denotes the quotient of R by its nilradical; the classical de Rham-Witt complex does not have this property. For a reduced example of this phenomenon, see Proposition 6.3.1.

For existence, we have the following:

Proposition 4.1.4. *Let R be an $\mathbf{F}_p$-algebra. Then there exists strict Dieudonné algebra A^* and a map $f : R \to A^0/VA^0$ which exhibits A^* as a saturated de Rham-Witt complex of R.*

Proof. By virtue of Lemma 3.6.1, we may assume without loss of generality that R is reduced. Let $W(R)$ denote the ring of Witt vectors of R, and let $\varphi: W(R) \to W(R)$ be the Witt vector Frobenius. Since R is reduced, $W(R)$ is p-torsion-free. Using Proposition 3.2.1, we can endow the de Rham complex $\Omega^*_{W(R)}$ with the structure of a Dieudonné algebra. Let A^* be the completed saturation $\mathcal{W}\mathrm{Sat}(\Omega^*_{W(R)})$. Combining the universal properties described in Proposition 3.5.8, Proposition 3.4.3, Proposition 3.2.3, and Proposition 3.6.4, we see that for every strict Dieudonné algebra B^*, we have natural bijections

$$\begin{aligned}\mathrm{Hom}_{\mathbf{DA}_{\mathrm{str}}}(A^*, B^*) &\simeq \mathrm{Hom}_{\mathbf{DA}_{\mathrm{sat}}}(\mathrm{Sat}(\Omega^*_{W(R)}), B^*) \\ &\simeq \mathrm{Hom}_{\mathbf{DA}}(\Omega^*_{W(R)}, B^*) \\ &\simeq \mathrm{Hom}_F(W(R), B) \\ &\simeq \mathrm{Hom}(R, B^0/VB^0);\end{aligned}$$

here $\mathrm{Hom}_F(W(R), B)$ denotes the set of ring homomorphisms from $f: W(R) \to B$ satisfying $f \circ \varphi = F \circ f$. Taking $B^* = A^*$, the image of the identity map id_{A^*} under the composite bijection determines a ring homomorphism $R \to A^0/VA^0$ with the desired property. $\square$

Let $\mathrm{CAlg}_{\mathbf{F}_p}$ denote the category of commutative $\mathbf{F}_p$-algebras. Then the construction $A^* \mapsto A^0/VA^0$ determines a functor $\mathbf{DA}_{\mathrm{str}} \to \mathrm{CAlg}_{\mathbf{F}_p}$.

Corollary 4.1.5. *The functor $A^* \mapsto A^0/VA^0$ described above admits a left adjoint $\mathcal{W}\Omega^*: \mathrm{CAlg}_{\mathbf{F}_p} \to \mathbf{DA}_{\mathrm{str}}$, given by the formation of saturated de Rham-Witt complexes $R \mapsto \mathcal{W}\Omega^*_R$.*

Remark 4.1.6. Corollary 4.1.5 can also be proved by applying the adjoint functor theorem [1, Th. 1.66]. The category $\mathbf{DA}_{\mathrm{str}}$ is a locally presentable category, and it is a straightforward consequence of Proposition 3.6.2 that the construction $A^* \mapsto A^0/VA^0$ commutes with arbitrary limits (which are computed at the level of underlying graded abelian groups) and filtered colimits (see Proposition 4.3.3). We leave the details to the reader.

4.2. **Comparison with a Smooth Lift.**

Proposition 4.2.1. *Let R be a commutative ring which is p-torsion-free, and let $\varphi: R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$ for $x \in R$. Let $\widehat{\Omega}^*_R$ be the completed de Rham complex of Variant 3.3.1, and let A^* be a strict Dieudonné algebra. Then the evident restriction map*

$$\mathrm{Hom}_{\mathbf{DA}}(\widehat{\Omega}^*_R, A^*) \to \mathrm{Hom}(R, A^0/VA^0)$$

is bijective. In other words, every ring homomorphism $R \to A^0/VA^0$ can be lifted uniquely to a map of Dieudonné algebras $\widehat{\Omega}^*_R \to A^*$.

Proof. Combine the universal properties of Variant 3.3.1 and Corollary 3.6.3 (noting that if A^* is a strict Dieudonné algebra, then each of the abelian groups A^n is p-adically complete). □

Notation 4.2.2. Let R be an $\mathbf{F}_p$-algebra. For each $r \geq 0$, we let $\mathcal{W}_r \Omega^*_R$ denote the quotient $\mathcal{W}\Omega^*_R / (\mathrm{im}(V^r) + \mathrm{im}(dV^r))$. By construction, we have a tautological map $e: R \to \mathcal{W}_1 \Omega^*_R$.

Corollary 4.2.3. *Let R be a commutative ring which is p-torsion-free, and let $\varphi: R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$ for $x \in R$. Then there is a unique map of Dieudonné algebras $\mu: \widehat{\Omega}^*_R \to \mathcal{W}\Omega^*_{R/pR}$ for which the diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \widehat{\Omega}^0_R \arrow[r, "\mu"] & \mathcal{W}\Omega^0_{R/pR} \arrow[d] \\ R/pR \arrow[rr, "e"] & & \mathcal{W}_1\Omega^0_{R/pR} \end{tikzcd}$$

commutes (here e is the map appearing in Notation 4.2.2). Moreover, μ induces an isomorphism of Dieudonné algebras $\mathcal{W}\mathrm{Sat}(\widehat{\Omega}^*_R) \to \mathcal{W}\Omega^*_{R/pR}$.

Proof. The existence and uniqueness of μ follow from Proposition 4.2.1. To prove the last assertion, it will suffice to show that for every strict Dieudonné algebra A^*, composition with μ induces an isomorphism $\theta: \mathrm{Hom}_{\mathbf{DA}}(\mathcal{W}\Omega^*_{R/pR}, A^*) \to \mathrm{Hom}_{\mathbf{DA}}(\widehat{\Omega}^*_R, A^*)$. Using Proposition 4.2.1 and the definition of the saturated de Rham-Witt complex, we can identify θ with the evident map $\mathrm{Hom}(R/pR, A^0/VA^0) \to \mathrm{Hom}(R, A^0/VA^0)$. This map is bijective, since A^0/VA^0 is an $\mathbf{F}_p$-algebra. □

Theorem 4.2.4. *Let R be a commutative ring which is p-torsion-free, and let $\varphi: R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$. Suppose that there exists a perfect ring k of characteristic p such that R/pR is a smooth k-algebra. Then the map $\mu: \widehat{\Omega}^*_R \to \mathcal{W}\Omega^*_{R/pR}$ of Corollary 4.2.3 is a quasi-isomorphism.*

Proof. By virtue of Corollary 4.2.3, it will suffice to show that the tautological map $\widehat{\Omega}^*_R \to \mathcal{W}\mathrm{Sat}(\widehat{\Omega}^*_R)$ is a quasi-isomorphism, which follows from Corollary 3.3.8. □

Example 4.2.5 (Powers of $\mathbf{G}_m$)**.** Let $R = \mathbf{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial ring on variables $x_1, \ldots, x_n$. Then the de Rham complex Ω^*_R is isomorphic to an exterior algebra over R on generators $d\log x_i = \frac{dx_i}{x_i}$ for $1 \leq i \leq n$, with differential given by $d(x_i^n) = n x_i^n d\log x_i$. Using Proposition 3.2.1, we see that there is a unique Dieudonné algebra structure on Ω^*_R which satisfies $F(x_i) = x_i^p$ for $1 \leq i \leq n$; note that we then have $F(d\log x_i) = d\log x_i$ for $1 \leq i \leq n$.

Set $R_\infty = \mathbf{Z}[x_1^{\pm 1/p^\infty}, \ldots, x_n^{\pm 1/p^\infty}]$, so that the localization $\Omega^*_R[F^{-1}]$ can be identified with the exterior algebra over R_∞ on generators $d\log x_i$ for $1 \leq i \leq n$, equipped with a differential $d: \Omega^*_R[F^{-1}] \to \Omega^*_R[F^{-1}][p^{-1}]$ given by $d(x_i^{a/p^n}) = (a/p^n)x_i^{a/p^n}$. Using Remark 2.3.4, we see that the saturation $\mathrm{Sat}(\Omega^*_R)$ can be identified with subalgebra of $\Omega^*_R[F^{-1}]$ given by those differential forms ω for which $d\omega$ also belongs to $\Omega^*_R[F^{-1}]$: that is, which have integral coefficients when expanded in terms of x_i and $d\log x_i$. Furthermore, the saturated de Rham-Witt complex of $\mathbf{F}_p[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is obtained as the completion $\mathcal{W}\mathrm{Sat}(\Omega^*_R)$. This recovers the explicit description of the classical de Rham-Witt complex $W\Omega^*_R$ via the *integral forms* in [28, Sec. I.2].

4.3. **Comparison with the de Rham Complex.** Let R be a commutative $\mathbf{F}_p$-algebra, let $\mathcal{W}\Omega^*_R$ be the saturated de Rham-Witt complex of R, and let $\mathcal{W}_1\Omega^*_R$ be the quotient of $\mathcal{W}\Omega^*_R$ of Notation 4.2.2. Then the tautological ring homomorphism $e: R \to \mathcal{W}_1\Omega^0_R$ admits a unique extension to a map of differential graded algebras $\nu: \Omega^*_R \to \mathcal{W}_1\Omega^*_R$.

Theorem 4.3.1. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the map $\nu: \Omega^*_R \to \mathcal{W}_1\Omega^*_R$ is an isomorphism.*

We first treat the special case where R is a smooth algebra over a perfect field k. In fact, our argument works more generally if k is a perfect ring:

Proposition 4.3.2. *Let k be a perfect $\mathbf{F}_p$-algebra and let R be a smooth algebra over k. Then the map $\nu: \Omega^*_R \to \mathcal{W}_1\Omega^*_R$ is an isomorphism.*

Proof. Let A be a smooth $W(k)$-algebra equipped with an isomorphism $A/pA \simeq R$, and let $\widehat{A}$ denote the p-adic completion of A. We note that by a result of Elkik [19] and (in greater generality) Arabia [2], one can always find such an A; see also [47, Tag 07M8] for an exposition. Alternatively, for the following argument, one only needs to work with $\widehat{A}$, which can be constructed more easily (by inductively lifting R to $W_n(k)$ for each n).

Using the smoothness of A, we see that the ring homomorphism

$$A \to A/pA \simeq R \xrightarrow{x \mapsto x^p} R$$

can be lifted to a map $A \to \widehat{A}$. Passing to the completion, we obtain a ring homomorphism $\varphi: \widehat{A} \to \widehat{A}$ satisfying $\varphi(x) \equiv x^p \pmod{p}$. Applying Corollary 4.2.3, we see that there is a unique morphism of Dieudonné algebras $\mu: \widehat{\Omega}^*_{\widehat{A}} \to \mathcal{W}\Omega^*_R$ for which the diagram

$$\begin{tikzcd}
\widehat{A} \arrow[r, "\sim"] \arrow[d] & \widehat{\Omega}^0_{\widehat{A}} \arrow[r, "\mu"] & \mathcal{W}\Omega^0_R \arrow[d] \\
\widehat{A}/p\widehat{A} \arrow[r, "\sim"] & R \arrow[r, "e"] & \mathcal{W}_1\Omega^0_R
\end{tikzcd}$$

commutes. From this commutativity, we see that ν can be identified with the composition

$$\Omega^*_R \simeq \widehat{\Omega}^*_{\widetilde{A}}/p\widehat{\Omega}^*_{\widetilde{A}} \xrightarrow{\mu} \mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R \to \mathcal{W}_1\Omega^*_R.$$

We now have a commutative diagram of graded abelian groups

$$\begin{tikzcd}
\widehat{\Omega}^*_{\widetilde{A}}/p\widehat{\Omega}^*_{\widetilde{A}} \arrow[rr, "\mathrm{Cart}"] \arrow[d] \arrow[dr, "\nu"] & & \mathrm{H}^*(\widehat{\Omega}^*_{\widetilde{A}}/p\widehat{\Omega}^*_{\widetilde{A}}) \arrow[d] \\
\mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R \arrow[r] & \mathcal{W}_1\Omega^*_R \arrow[r] & \mathrm{H}^*(\mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R),
\end{tikzcd}$$

where the vertical maps are induced by μ, and the top horizontal and bottom right horizontal maps are induced by the Frobenius on $\widehat{\Omega}^*_{\widetilde{A}}$ and $\mathcal{W}\Omega^*_R$, respectively. We now observe that the top horizontal map is the Cartier isomorphism (Theorem 3.3.6), the right vertical map is an isomorphism by virtue of Theorem 4.2.4, and the bottom right horizontal map is an isomorphism by Proposition 2.7.1. It follows that ν is also an isomorphism. □

To handle the general case, we will need the following:

Proposition 4.3.3. *The category $\mathbf{DA}_{\mathrm{str}}$ admits small filtered colimits, which are preserved by the functor $A^* \mapsto \mathcal{W}_r(A)^* = A^*/(\mathrm{im}(V^r) + \mathrm{im}(dV^r))$.*

Proof. We first observe that the category $\mathbf{DA}$ of Dieudonné algebras admits filtered colimits, which can be computed at the level of the underlying graded abelian groups. Moreover, the subcategory $\mathbf{DA}_{\mathrm{sat}} \subseteq \mathbf{DA}$ of saturated Dieudonné algebras is closed under filtered colimits. In particular, the construction $A^* \mapsto \mathcal{W}_r(A)^*$ preserves filtered colimits when regarded as a functor from the category $\mathbf{DA}_{\mathrm{sat}}$ to the category of graded abelian groups. It follows from Corollary 3.5.9 that the category $\mathbf{DA}_{\mathrm{str}}$ also admits small filtered colimits, which are computed by first taking a colimit in the larger category $\mathbf{DA}_{\mathrm{sat}}$ and then applying the completion construction $A^* \mapsto \mathcal{W}(A)^*$. Consequently, to show that restriction of $\mathcal{W}_r$ to $\mathbf{DA}_{\mathrm{str}}$ commutes with filtered colimits, it suffices to show that the canonical map $\mathcal{W}_r(\varinjlim A_\alpha)^* \to \mathcal{W}_r(\mathcal{W}(\varinjlim A_\alpha))^*$ is an isomorphism for every filtered diagram $\{A^*_\alpha\}$ in $\mathbf{DA}_{\mathrm{str}}$. This is a special case of Proposition 2.7.5. □

Remark 4.3.4. In the statement and proof of Proposition 4.3.3, we can replace Dieudonné algebras by Dieudonné complexes.

Corollary 4.3.5. *The construction $R \mapsto \mathcal{W}_r\Omega^*_R$ commutes with filtered colimits (when regarded as a functor from the category of $\mathbf{F}_p$-algebras to the category of graded abelian groups).*

Proof. Combine Proposition 4.3.3 with the observation that the functor $R \mapsto \mathcal{W}\Omega^*_R$ preserves colimits (since it is defined as the left adjoint to the functor $A^* \mapsto A^0/VA^0$). □

We now prove Theorem 4.3.1 using Popescu's smoothing theorem. In §9.6, we will give an alternative proof using the theory of derived de Rham(-Witt) cohomology, which avoids the use of Popescu's theorem (see Theorem 9.6.1).

Proof of Theorem 4.3.1. Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Applying Popescu's smoothing theorem [47, Tag 07GB], we can write R as a filtered colimit $\varinjlim R_\alpha$, where each R_α is a smooth $\mathbf{F}_p$-algebra. It follows from Corollary 4.3.5 that the canonical map $\nu\colon \Omega^*_R \to \mathcal{W}_1\,\Omega^*_R$ is a filtered colimit of maps $\nu_\alpha\colon \Omega^*_{R_\alpha} \to \mathcal{W}_1\,\Omega^*_{R_\alpha}$. It will therefore suffice to show that each of the maps ν_α is an isomorphism, which follows from Proposition 4.3.2. □

Remark 4.3.6. Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Composing the tautological map $\mathcal{W}\Omega^*_R \to \mathcal{W}_1\,\Omega^*_R$ with the inverse of the isomorphism $\nu\colon \Omega^*_R \to \mathcal{W}_1\,\Omega^*_R$, we obtain a surjective map of commutative differential graded algebras $\mathcal{W}\Omega^*_R \to \Omega^*_R$. It follows from Theorem 4.3.1 and Corollary 2.7.2 that the induced map $\mathcal{W}\Omega^*_R/p\mathcal{W}\Omega^*_R \to \Omega^*_R$ is a quasi-isomorphism.

4.4. **The Classical de Rham-Witt Complex.** We now consider the relationship between the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of Notation 4.1.2 and the classical de Rham-Witt complex $W\Omega^*_R$ of Bloch-Deligne-Illusie. We begin with some definitions.

Definition 4.4.1. Let R be a commutative $\mathbf{F}_p$-algebra. An *R-framed V-procomplex* consists of the following data:

(1) An inverse system
$$\cdots \to A^*_4 \to A^*_3 \to A^*_2 \to A^*_1 \to A^*_0$$
of commutative differential graded algebras. We will denote each of the transition maps in this inverse system by $\mathrm{Res}\colon A^*_{r+1} \to A^*_r$, and refer to them as *restriction maps*. We let A^*_∞ denote the inverse limit $\varprojlim_r A^*_r$.

(2) A collection of maps $V\colon A^*_r \to A^*_{r+1}$ in the category of graded abelian groups, which we will refer to as *Verschiebung maps*.

(3) A ring homomorphism $\beta\colon W(R) \to A^0_\infty$. For each $r \geq 0$, we let β_r denote the composite map $W(R) \xrightarrow{\beta} A^0_\infty \to A^0_r$.

These data are required to satisfy the following axioms:

(a) The groups A^*_r vanish for $* < 0$ and for $r = 0$.

(b) The Verschiebung and restriction maps are compatible with one another: that is, for every $r > 0$, we have a commutative diagram

$$\begin{tikzcd} A^*_r \arrow[r, "V"] \arrow[d, "\mathrm{Res}"] & A^*_{r+1} \arrow[d, "\mathrm{Res}"] \\ A^*_{r-1} \arrow[r, "V"] & A^*_r. \end{tikzcd}$$

It follows that the Verschiebung maps induce a homomorphism of graded abelian groups $V: A^*_\infty \to A^*_\infty$, which we will also denote by V.

(c) The map $\beta: W(R) \to A^0_\infty$ is compatible with Verschiebung: that is, we have a commutative diagram

$$\begin{tikzcd} W(R) \arrow[r, "\beta"] \arrow[d, "V"] & A^0_\infty \arrow[d, "V"] \\ W(R) \arrow[r, "\beta"] & A^0_\infty. \end{tikzcd}$$

(d) The Verschiebung maps $V: A^*_r \to A^*_{r+1}$ satisfy the identity $V(xdy) = V(x)dV(y)$.

(e) Let λ be an element of R and let $[\lambda] \in W(R)$ denote its Teichmüller representative. Then, for each $x \in A^*_r$, we have an identity

$$(Vx)d\beta_{r+1}([\lambda]) = V(\beta_r([\lambda])^{p-1}xd\beta_r[\lambda])$$

in A^*_{r+1}.

Let $(\{A^*_r\}_{r\geq 0}, V, \beta)$ and $(\{A'^*_r\}_{r\geq 0}, V', \beta')$ be R-framed V-pro-complexes. A *morphism* of R-framed V-pro-complexes from $(\{A^*_r\}_{r\geq 0}, V, \beta)$ to $(\{A'^*_r\}_{r\geq 0}, V', \beta')$ is a collection of differential graded algebra homomorphisms $f_r: A^*_r \to A'^*_r$ for which the diagrams

$$\begin{tikzcd} A^*_r \arrow[r, "f_r"] \arrow[d, "\mathrm{Res}"] & A'^*_r \arrow[d, "\mathrm{Res}"] \\ A^*_{r-1} \arrow[r, "f_{r-1}"] & A'^*_{r-1} \end{tikzcd} \qquad \begin{tikzcd} A^*_r \arrow[r, "f_r"] \arrow[d, "V"] & A'^*_r \arrow[d, "V'"] \\ A^*_{r+1} \arrow[r] & A'^*_{r+1} \end{tikzcd}$$

$$\begin{tikzcd} & W(R) \arrow[dl, "\beta_r"'] \arrow[dr, "\beta'_r"] & \\ A^0_r \arrow[rr, "f_r"] & & A'^0_r \end{tikzcd}$$

are commutative. We let $\mathbf{VPC}_R$ denote the category whose objects are R-framed V-pro-complexes and whose morphisms are morphisms of R-framed V-pro-complexes.

Remark 4.4.2. In the situation of Definition 4.4.1, we will generally abuse terminology by simply referring to the inverse system $\{A^*_r\}_{r\geq 0}$ as an *R-framed V-pro-complex*; in this case, we are implicitly assuming that Verschiebung maps $V: A^*_r \to A^*_{r+1}$ and a map $\beta: W(R) \to A^0_\infty$ have also been specified.

Remark 4.4.3. Let R be a commutative $\mathbf{F}_p$-algebra and let $\{A^*_r\}$ be an R-framed V-pro-complex. It follows from conditions (a) and (b) of Definition 4.4.1 that, for each $r \geq 0$, the composite map

$$A^*_r \xrightarrow{V^r} A^*_{2r} \xrightarrow{\mathrm{Res}^r} A^*_r$$

vanishes. Consequently, condition (c) of Definition 4.4.1 implies that the map $\beta_r: W(R) \to A_r^0$ annihilates the subgroup $V^r W(R) \subseteq W(R)$, and therefore factors through the quotient $W_r(R) \simeq W(R)/V^r W(R)$.

Proposition 4.4.4. *Let R be a commutative $\mathbf{F}_p$-algebra. Then the category $\mathbf{VPC}_R$ has an initial object.*

Proof Sketch. For each $r \geq 0$, let $\Omega^*_{W_r(R)}$ denote the de Rham complex of $W_r(R)$ (relative to $\mathbf{Z}$). We will say that a collection of differential graded ideals $\{I^*_r \subseteq \Omega^*_{W_r(R)}\}_{r\geq 0}$ is *good* if the following conditions are satisfied:

- (i) Each of the restriction maps $W_{r+1}(R) \to W_r(R)$ determines a map of de Rham complexes $\Omega^*_{W_{r+1}(R)} \to \Omega^*_{W_r(R)}$ which carries I^*_{r+1} into I^*_r, and therefore induces a map of differential graded algebras $\Omega^*_{W_{r+1}(R)}/I^*_{r+1} \to \Omega^*_{W_r(R)}/I^*_r$.
- (ii) For each $r \geq 0$, there exists a map $V: \Omega^*_{W_r(R)}/I^*_r \to \Omega^*_{W_{r+1}(R)}/I^*_{r+1}$ which satisfies the identity
$$V(x_0 dx_1 \wedge \cdots \wedge dx_n) = V(x_0) dV(x_1) \wedge \cdots \wedge dV(x_n);$$
here we abuse notation by identifying an element of $\Omega^*_{W_r(R)}$ with its image in the quotient $\Omega^*_{W_r(R)}/I^*_r$. Note that such a map V is automatically unique.
- (iii) For each $x \in \Omega^*_{W_r(R)}/I^*_r$ and each $\lambda \in R$, the difference
$$(Vx)d[\lambda] - V([\lambda]^{p-1}x)dV[\lambda]$$
belongs to I^{*+1}_{r+1} (here we abuse notation by identifying the Teichmüller representative $[\lambda]$ with its image each $\Omega^*_{W_s(R)}/I^*_s$).

It is not difficult to see that there exists a smallest good collection of differential graded ideals $\{I^*_r\}_{r\geq 0}$, and that the inverse system $\{\Omega^*_{W_r(R)}/I^*_r\}_{r\geq 0}$ (together with with the Verschiebung maps defined in (ii) and the evident structural map $\beta: W(R) \to \varprojlim_r \Omega^0_{W_r(R)}/I^0_r$) is an initial object of $\mathbf{VPC}_R$. $\square$

Remark 4.4.5. In the situation of the proof of Proposition 4.4.4, the differential graded ideal I^*_1 vanishes: that is, the tautological map $\Omega^*_R \to W_1\Omega^*_R$ is always an isomorphism.

Definition 4.4.6. Let R be a commutative $\mathbf{F}_p$-algebra and let $\{W_r\Omega^*_R\}_{r\geq 0}$ denote an initial object of the category of $\mathbf{VPC}_R$. We let $W\Omega^*_R$ denote the inverse limit $\varprojlim_r W_r\Omega^*_R$. We will refer to $W\Omega^*_R$ as the *classical de Rham-Witt complex of R*.

Remark 4.4.7. The proof of Proposition 4.4.4 provides an explicit model for the inverse system $\{W_r\Omega^*_R\}$: each $W_r\Omega^*_R$ can be regarded as a quotient of the absolute de Rham complex $\Omega^*_{W_r(R)}$ by a differential graded ideal I^*_r, which is dictated by the axiomatics of Definition 4.4.1.

Warning 4.4.8. Our definition of the classical de Rham-Witt complex $W\Omega^*_R$ differs slightly from the definition appearing in [28]. In [28], the inverse system $\{W_r\Omega^*_R\}_{r\geq 0}$ is defined to be an initial object of a category $\mathbf{VPC}'_R$, which can be identified with the full subcategory of $\mathbf{VPC}_R$ spanned by those R-framed V-pro-complexes $\{A^*_r\}$ for which $\beta: W(R) \to A^0_\infty$ induces isomorphisms $W_r(R) \simeq A^0_r$ for $r \geq 0$. However, it is not difficult to see that the initial object of $\mathbf{VPC}_R$ belongs to this subcategory (this follows by inspecting the proof of Proposition 4.4.4), so that $W\Omega^*_R$ is isomorphic to the de Rham-Witt complex which appears in [28].

Remark 4.4.9. Let R be a commutative $\mathbf{F}_p$-algebra and let $W\Omega^*_R$ be the classical de Rham-Witt complex of R. The proof of Proposition 4.4.4 shows that the restriction maps $W_{r+1}\Omega^*_R \to W_r\Omega^*_R$ are surjective. It follows that each $W_r\Omega^*_R$ can be viewed as a quotient of $W\Omega^*_R$.

Our next goal is to compare the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ with the classical de Rham-Witt complex $W\Omega^*_R$. We begin with the following observation:

Proposition 4.4.10. *Let R be a commutative $\mathbf{F}_p$-algebra, let A^* be a strict Dieudonné algebra, and let $\beta_1: R \to A^0/VA^0$ be a ring homomorphism. Then:*

(1) *The map β_1 admits a unique lift to a ring homomorphism $\beta: W(R) \to A^0$ satisfying $\beta \circ F = F \circ \beta$.*
(2) *The inverse system $\{\mathcal{W}_r(A)^*\}_{r\geq 0}$ is an R-framed V-pro-complex (when equipped with the Verschiebung maps $V: \mathcal{W}_r(A)^* \to \mathcal{W}_{r+1}(A)^*$ of Remark 2.5.2 and the map β described in* (1)*).*

Proof. Assertion (1) is the content of Proposition 3.6.4. To prove (2), we must verify that β and V satisfy axioms (a) through (e) of Definition 4.4.1. Axioms (a) and (b) are obvious. To prove (c), we compute

$$F\beta(Vx) = \beta(FVx) = \beta(px) = p\beta(x) = FV\beta(x),$$

so that $\beta(Vx) = V\beta(x)$ by virtue of the fact that the Frobenius map $F: A^0 \to A^0$ is injective. The verification of (d) is similar: for $x, y \in A^*$, we have

$$FV(xdy) = (px)(dy) = (FVx)(FdVy) = F((Vx)(dVy));$$

invoking the injectivity of F again, we obtain $V(xdy) = (Vx)(dVy)$. To prove (e), we compute

$$\begin{aligned} F((Vx)d\beta([\lambda])) &= F(Vx)F(d\beta([\lambda])) \\ &= pxF(d\beta([\lambda)]) \\ &= xd(F\beta([\lambda])) \\ &= xd(\beta(F[\lambda])) \\ &= xd\beta([\lambda]^p) \\ &= xd\beta([\lambda])^p \\ &= px\beta([\lambda])^{p-1}d\beta([\lambda]) \\ &= FV(x\beta([\lambda])^{p-1}d\beta([\lambda]). \end{aligned}$$

Canceling F, we obtain the desired identity

$$(Vx)d\beta([\lambda]) = V(x\beta([\lambda])^{p-1}d\beta([\lambda])).$$

$\square$

Applying Proposition 4.4.10 in the case $A^* = \mathcal{W}\Omega^*_R$ and invoking the universal property of the classical de Rham-Witt complex, we obtain the following:

Corollary 4.4.11. *Let R be a commutative $\mathbf{F}_p$-algebra. Then there is a unique homomorphism of differential graded algebras $\gamma: W\Omega^*_R \to \mathcal{W}\Omega^*_R$ with the following properties:*

(*i*) *For each $r \geq 0$, the composite map*

$$W\Omega^*_R \xrightarrow{\gamma} \mathcal{W}\Omega^*_R \to \mathcal{W}_r\Omega^*_R$$

factors through $W_r\Omega^*_R$ (necessarily uniquely, by virtue of Remark 4.4.9); consequently, γ can be realized as the inverse limit of a tower of differential graded algebra homomorphisms $\gamma_r: W_r\Omega^*_R \to \mathcal{W}_r\Omega^*_R$.

(*ii*) *The diagrams*

$$\begin{tikzcd} W\Omega^*_R \arrow[r, "V"] \arrow[d, "\gamma"] & W\Omega^*_R \arrow[d, "\gamma"] \\ \mathcal{W}\Omega^*_R \arrow[r, "V"] & \mathcal{W}\Omega^*_R \end{tikzcd} \qquad \begin{tikzcd} W(R) \arrow[r] \arrow[d] & W\Omega^0_R \arrow[d, "\gamma"] \\ R \arrow[r] & \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R \end{tikzcd}$$

are commutative.

We can now formulate the main result of this section:

Theorem 4.4.12. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the map $\gamma: W\Omega^*_R \to \mathcal{W}\Omega^*_R$ of Corollary 4.4.11 is an isomorphism.*

Proof. The map γ can be realized as the inverse limit of a tower of maps $\gamma_r\colon W_r\Omega^*_R \to \mathcal{W}_r\,\Omega^*_R$; it will therefore suffice to show that each γ_r is an isomorphism. Since the constructions $R \mapsto W_r\Omega^*_R$ and $R \mapsto \mathcal{W}_r\,\Omega^*_R$ commute with filtered colimits, we can use Popescu's smoothing theorem to reduce to the case where R is a smooth $\mathbf{F}_p$-algebra. Using Theorem I.2.17 and Proposition I.2.18 of [28], we see that there exists a map $F\colon W\Omega^*_R \to W\Omega^*_R$ which endows $W\Omega^*_R$ with the structure of a Dieudonné algebra and satisfies the identities $FV = VF = p$. Since R is smooth over $\mathbf{F}_p$, Remark I.3.21.1 of [28] guarantees that the Dieudonné algebra $W\Omega^*_R$ is saturated. Moreover, Proposition I.3.2 implies that the kernel of the projection map $W\Omega^*_R \to W_r\Omega^*_R$ is equal to $\mathrm{im}(V^r) + \mathrm{im}(dV^r)$, so that

$$W\Omega^*_R \simeq \varprojlim W_r\Omega^*_R \simeq \varprojlim W\Omega^*_R/(\mathrm{im}(V^r) + \mathrm{im}(dV^r))$$

is a strict Dieudonné algebra. For each $x \in W\Omega^*_R$, we have

$$V\gamma(Fx) = \gamma(VFx) = \gamma(px) = p\gamma(x) = VF\gamma(x).$$

Since the map $V\colon \mathcal{W}\,\Omega^*_R \to \mathcal{W}\,\Omega^*_R$ is injective, it follows that $\gamma(Fx) = F\gamma(x)$: that is, γ is a morphism of (strict) Dieudonné algebras. We wish to show $\gamma_r = \mathcal{W}_r(\gamma)$ is an isomorphism for each $r \geq 0$. By virtue of Corollary 2.7.4, it suffices to prove this in the case $r = 1$. Invoking Remark 4.4.5, we are reduced to proving that the tautological map $\Omega^*_R \to \mathcal{W}_1\,\Omega^*_R$ is an isomorphism, which follows from Theorem 4.3.1. $\square$

5. Localizations of Dieudonné Algebras

Our goal in this section is to introduce a global version of the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of Definition 4.1.1. Let X be an $\mathbf{F}_p$-scheme. In §5.2, we show that there is a sheaf of commutative differential graded algebras $\mathcal{W}\Omega^*_X$ with the property that, for every affine open subset $U \subseteq X$, the complex $\mathcal{W}\Omega^*_X(U)$ can be identified with the saturated de Rham-Witt complex of the $\mathbf{F}_p$-algebra $\mathscr{O}_X(U)$ (Theorem 5.2.2). To prove the existence of this sheaf, we will need to analyze the behavior of the saturated de Rham-Witt complex with respect to Zariski localization. In §5.1, we show that if R is an $\mathbf{F}_p$-algebra and $s \in R$ is an element, then the complex $\mathcal{W}\Omega^*_{R[s^{-1}]}$ admits a relatively simple description in terms of $\mathcal{W}\Omega^*_R$: roughly speaking, it can be obtained from $\mathcal{W}\Omega^*_R$ by inverting the Teichmüller representative $[s]$ and then passing to a suitable completion (Corollary 5.1.6). In §5.3, we formulate a more general version of this result, where the localization $R[s^{-1}]$ is replaced by an arbitrary étale R-algebra S (Corollary 5.3.5). The proof requires some general facts about the behavior of the Witt vector functor with respect to étale ring homomorphisms, which we review in §5.4.

5.1. **Localization for the Zariski Topology.**

Let A^* be a graded-commutative ring and let $S \subseteq A^0$ be a multiplicatively closed subset of A^0. We let $A^*[S^{-1}]$ denote the (graded) ring obtained from A^* by formally inverting the elements of S. In this section, we study the construction $A^* \mapsto A^*[S^{-1}]$ in the case where A^* is a Dieudonné algebra:

Proposition 5.1.1. *Let A^* be a Dieudonné algebra and let $S \subseteq A^0$ be a multiplicatively closed subset. Assume that the Frobenius map $F\colon A^* \to A^*$ satisfies $F(S) \subseteq S$. Then the graded ring $A^*[S^{-1}]$ inherits the structure of a Dieudonné algebra, which is uniquely determined by the requirement that the tautological map $A^* \to A^*[S^{-1}]$ is a morphism of Dieudonné algebras.*

Remark 5.1.2. In the situation of Proposition 5.1.1, the Dieudonné algebra $A^*[S^{-1}]$ is characterized by the following universal property: for any Dieudonné algebra B^*, precomposition with the tautological map $A^* \to A^*[S^{-1}]$ induces a monomorphism of sets

$$\mathrm{Hom}_{\mathbf{DA}}(A^*[S^{-1}], B^*) \to \mathrm{Hom}_{\mathbf{DA}}(A^*, B^*),$$

whose image consists of those morphisms of Dieudonné algebras $f\colon A^* \to B^*$ with the property that $f(s)$ is an invertible element of B^0, for each $s \in S$.

Proof of Proposition 5.1.1. We first observe that there is a unique differential on the ring $A^*[S^{-1}]$ for which the map $A^* \to A^*[S^{-1}]$ is a morphism of differential graded algebras, given concretely by the formula

$$d\left(\frac{x}{s}\right) = \frac{dx}{s} - \frac{x\,ds}{s^2}.$$

The assumption that $F(S) \subseteq S$ guarantees that there is a unique homomorphism of graded rings $F\colon A^*[S^{-1}] \to A^*[S^{-1}]$ for which the diagram

$$\begin{tikzcd} A^* \arrow[r] \arrow[d, "F"] & A^*[S^{-1}] \arrow[d, "F"] \\ A^* \arrow[r] & A^*[S^{-1}] \end{tikzcd}$$

is commutative, given concretely by the formula $F(\frac{x}{s}) = \frac{F(x)}{F(s)}$. To complete the proof, it will suffice to show that the triple $(A^*[S^{-1}], d, F)$ is a Dieudonné algebra: that is, that is satisfies the axioms of Definition 3.1.2. Axiom (ii) is immediate. To verify (i), we compute

$$\begin{aligned} dF(\frac{x}{s}) &= d\frac{F(x)}{F(s)} \\ &= \frac{dF(x)}{F(s)} - \frac{F(x)dF(s)}{F(s)^2} \\ &= \frac{pF(dx)}{F(s)} - \frac{pF(x)F(ds)}{F(s)^2} \\ &= pF(\frac{dx}{s} - \frac{xds}{s^2}) \\ &= pF(d\left(\frac{x}{s}\right)). \end{aligned}$$

To prove (iii), choose any $x \in A^0$ and any $s \in S$. Since A^* is a Dieudonné algebra, we have

$$F(x)s^p \equiv x^p s^p \equiv x^p F(s) \pmod{p},$$

so that we can write $F(x)s^p = x^p F(s) + py$ for some $y \in A^0$. It follows that

$$F(\frac{x}{s}) = \frac{F(x)}{F(s)} = \frac{x^p}{s^p} + p\frac{y}{s^p F(s)}$$

in the commutative ring $A^0[S^{-1}]$, so that $F(\frac{x}{s}) \equiv (\frac{x}{s})^p \pmod{p}$. □

Example 5.1.3. Let A^* be a Dieudonné algebra and let s be an element of A^0 which satisfies the equation $F(s) = s^p$. Applying Proposition 5.1.1 to the set $S = \{1, s, s^2, \ldots\}$, we conclude that the localization $A^*[s^{-1}]$ inherits the structure of a Dieudonné algebra.

Proposition 5.1.4. *Let A^* be a Dieudonné algebra, and let $s \in A^0$ be an element satisfying the equation $F(s) = s^p$. If A^* is saturated, then the localization $A^*[s^{-1}]$ is also saturated.*

Proof. We first show that the Frobenius map $F: A^*[s^{-1}] \to A^*[s^{-1}]$ is a monomorphism. Suppose that $F(\frac{x}{s^k}) = 0$ for some $x \in A^*$. Then we have an equality $s^n F(x) = 0$ in A^* for some $n \gg 0$. Enlarging n if necessary, we can assume that $n = pm$ for some integer $m \geq 0$, so that

$$0 = s^n F(x) = F(s^m)F(x) = F(s^m x).$$

Since the Frobenius map F is a monomorphism on A^*, it follows that $s^m x = 0$, so that $\frac{x}{s^k}$ vanishes in $A^*[s^{-1}]$.

Now suppose that we are given an element $\frac{y}{s^n} \in A^*[S^{-1}]$ such that $d(\frac{y}{s^n})$ is divisible by p in $A^*[S^{-1}]$; we wish to show that $\frac{y}{s^n}$ belongs to the image of F. Enlarging n if necessary, we can assume that $n = pm$ for some $m \geq 0$, so that

$$d(\frac{y}{s^n}) = \frac{dy}{s^n} - pm\frac{yds}{s^{n+1}} \equiv \frac{dy}{s^n} \pmod{p}.$$

It follows that we can choose $k \gg 0$ such that $s^{pk}dy$ belongs to pA^*. We then have

$$d(s^{pk}y) = s^{pk}dy + pks^{pk-1}yds \equiv s^{pk}dy \equiv 0 \pmod{p}.$$

Invoking our assumption that A^* is saturated, we can write $s^{pk}y = Fz$ for some $z \in A^*$. It follows that $\frac{y}{s^n} = \frac{s^{pk}y}{s^{pm}} = F(\frac{z}{s^m})$, so that $\frac{y}{s^n}$ belongs to the image of F as desired. □

In the situation of Proposition 5.1.4, the Dieudonné algebra $A^*[s^{-1}]$ is usually not strict, even if A^* is assumed to be strict. However, the completed saturation of $A^*[s^{-1}]$ is easy to describe, by virtue of the following result:

Proposition 5.1.5. *Let A^* be a saturated Dieudonné algebra and let $s \in A^0$ be an element satisfying $F(s) = s^p$. Let $r \geq 0$ and let $\overline{s}$ denote the image of s in the quotient $\mathcal{W}_r(A)^0$. Then the canonical map $\psi: \mathcal{W}_r(A)^*[\overline{s}^{-1}] \to \mathcal{W}_r(A[s^{-1}])^*$ is an isomorphism of differential graded algebras.*

Proof. The surjectivity of ψ is immediate from the definitions. To prove injectivity, we must verify the inclusion

$$(V^r A^* + dV^r A^*)[s^{-1}] \subseteq V^r(A^*[s^{-1}]) + dV^r(A^*[s^{-1}]).$$

Since the left hand side is a subcomplex of $A^*[s^{-1}]$, it will suffice to show that the left hand side contains $V^r(A^*[s^{-1}])$. This follows from the identity

$$V^r(s^{-n}x) = V^r(F^r(s^{-n})s^{n(p^r-1)}x) = s^{-n}V^r(s^{n(p^r-1)}x).$$

□

Corollary 5.1.6. *Let R be a commutative $\mathbf{F}_p$-algebra, let A^* be a Dieudonné algebra, and let $f: R \to A^0/VA^0$ be a ring homomorphism which exhibits A^* as a saturated de Rham-Witt complex of R (in the sense of Definition 4.1.1). Let $s \in R$ be an element, and let us abuse notation by identifying the Teichmüller* *representative* $[s] \in W(R)$ *with its image in* A^0. *Set* $B^* = A^*[[s]^{-1}]$. *Then the map*

$$R[s^{-1}] \xrightarrow{f} (A^0/VA^0)[s^{-1}] \simeq B^0/VB^0 = \mathcal{W}(B)^*/V\mathcal{W}(B)^*$$

exhibits $\mathcal{W}(B)^*$ *as a saturated de Rham-Witt complex of* $R[s^{-1}]$.

Proof. For any strict Dieudonné algebra C^*, we have a commutative diagram of sets

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{DA}}(\mathcal{W}(B)^*, C^*) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{DA}}(A^*, C^*) \arrow[d] \\ \mathrm{Hom}(R[s^{-1}], C^0/VC^0) \arrow[r] & \mathrm{Hom}(R, C^0/VC^0). \end{tikzcd}$$

Note that the right vertical map is a bijection, and we wish to show that the left vertical map is also a bijection. For this, it will suffice to show the diagram is a pullback square. Invoking the universal property of B^* supplied by Remark 5.1.2, we can reformulate this assertion as follows:

($*$) Let $f: A^* \to C^*$ be a morphism of Dieudonné algebras. Then $f([s])$ is an invertible element of C^0 if and only if its image in C^0/VC^0 is invertible.

The "only if" direction is obvious. For the converse, suppose that $f([s])$ admits an inverse in C^0/VC^0, which is represented by an element $y \in C^0$. Then we can write $yf([s]) = 1 - Vz$ for some $z \in C^0$. Since C^0 is V-adically complete, the element $f([s])$ has an inverse in C^0, given by the product $y(1+Vz+(Vz)^2+\cdots)$. □

5.2. **The saturated de Rham-Witt Complex of an $\mathbf{F}_p$-Scheme.** We now apply the ideas of §5.1 to globalize the construction of Definition 4.1.1.

Construction 5.2.1. Let $X = (X, \mathscr{O}_X)$ be an $\mathbf{F}_p$-scheme, and let $\mathcal{U}_{\mathrm{aff}}(X)$ denote the collection of all affine open subsets of X. We define a functor $\mathcal{W}\Omega^*_X: \mathcal{U}_{\mathrm{aff}}(X)^{\mathrm{op}} \to$ **DA** by the formula

$$\mathcal{W}\Omega^*_X(U) = \mathcal{W}\Omega^*_{\mathscr{O}_X(U)}.$$

We regard $\mathcal{W}\Omega^*_X(U)$ as a presheaf on X (defined only on affine subsets of X) with values in the category of Dieudonné algebras.

Theorem 5.2.2. *Let* X *be an* $\mathbf{F}_p$*-scheme. Then the presheaf* $\mathcal{W}\Omega^*_X$ *of Construction 5.2.1 is a sheaf (with respect to the topology on* $\mathcal{U}_{\mathrm{aff}}(X)$ *given by open coverings).*

Proof. We will show that, for every integer $d \geq 0$, the presheaf $\mathcal{W}\Omega^d_X$ is a sheaf of abelian groups on X. Unwinding the definitions, we can write $\mathcal{W}\Omega^d_X$ as the inverse limit of a tower of presheaves

$$\cdots \to \mathcal{W}_3\Omega^d_X \to \mathcal{W}_2\Omega^d_X \to \mathcal{W}_1\Omega^d_X,$$

where $\mathcal{W}_r\Omega^d_X$ is given by the formula $\mathcal{W}_r\Omega^d_X(U) = \mathcal{W}_r\Omega^d_{\mathscr{O}_X(U)}$. It will therefore suffice to show that each $\mathcal{W}_r\Omega^d_X$ is a sheaf of abelian groups. In other words, we must show that for every affine open subset $U \subseteq X$ and every covering of U by affine open subsets U_α, the sequence

$$0 \to \mathcal{W}_r\,\Omega^d_X(U) \to \prod_\alpha \mathcal{W}_r\,\Omega^d_X(U_\alpha) \to \prod_{\alpha,\beta} \mathcal{W}_r\,\Omega^d_X(U_\alpha \cap U_\beta)$$

is exact. Without loss of generality, we may replace X by U and thereby reduce to the case where $X = \operatorname{Spec}(R)$. By a standard argument, we can further reduce to the case where each U_α is the complement of the vanishing locus of some element $s_\alpha \in R$. In this case, we wish to show that the sequence

$$0 \to \mathcal{W}_r\,\Omega^d_R \to \prod_\alpha \mathcal{W}_r\,\Omega^d_{R[s_\alpha^{-1}]} \to \prod_{\alpha,\beta} \mathcal{W}_r\,\Omega^d_{R[s_\alpha^{-1}, s_\beta^{-1}]}$$

is exact. Setting $M = \mathcal{W}_r\,\Omega^d_R$, we can apply Corollary 5.1.6 and Proposition 5.1.5 to rewrite this sequence as

$$0 \to M \to \prod_\alpha M[\overline{s}_\alpha^{-1}] \to \prod_{\alpha,\beta} M[\overline{s}_\alpha^{-1}, \overline{s}_\beta^{-1}],$$

where $\overline{s}_\alpha$ denotes the image in $W_r(R)$ of the Teichmüller representative $[s_\alpha] \in W(R)$. The desired result now follows from the observation that the elements $\overline{s}_\alpha$ generate the unit ideal in $W_r(R)$. $\square$

It follows that from Theorem 5.2.2 that for any $\mathbf{F}_p$-scheme X, the presheaf $\mathcal{W}\,\Omega^*_X$ can be extended uniquely to a sheaf of Dieudonné algebras on the collection of all open subsets of X. We will denote this sheaf also by $\mathcal{W}\,\Omega^*_X$ and refer to it as the *saturated de Rham-Witt complex of X*.

Remark 5.2.3. The proof of Theorem 5.2.2 shows that each $\mathcal{W}_r\,\Omega^d_X$ can be regarded as a *quasi-coherent* sheaf on the $(\mathbf{Z}/p^r\mathbf{Z})$-scheme $(X, W_r(\mathscr{O}_X))$; here $W_r(\mathscr{O}_X)$ denotes the sheaf of commutative rings on X given on affine open sets U by the formula $W_r(\mathscr{O}_X)(U) = W_r(\mathscr{O}_X(U))$.

Proposition 5.2.4. *Let R be a commutative $\mathbf{F}_p$-algebra and set $X = \operatorname{Spec}(R)$. Then, for every integer d, the canonical map $\mathcal{W}\,\Omega^d_R \to \mathrm{H}^0(X; \mathcal{W}\,\Omega^d_X)$ is an isomorphism and the cohomology groups $\mathrm{H}^n(X; \mathcal{W}\,\Omega^d_X)$ vanish for $n > 0$.*

Proof. Let $\mathscr{F}^\bullet$ denote the homotopy limit of the diagram

$$\cdots \to \mathcal{W}_3\,\Omega^d_X \to \mathcal{W}_2\,\Omega^d_X \to \mathcal{W}_1\,\Omega^d_X,$$

in the derived category of abelian sheaves on X. For every affine open subset $U \subseteq X$, the hypercohomology $\mathrm{R}\Gamma(U; \mathscr{F}^\bullet)$ can be identified with the homotopy limit of the diagram

$$\cdots \to \mathrm{R}\Gamma(U; \mathcal{W}_3\,\Omega^d_X) \to \mathrm{R}\Gamma(U; \mathcal{W}_2\,\Omega^d_X) \to \mathrm{R}\Gamma(U; \mathcal{W}_1\,\Omega^d_X)$$

in the derived category of abelian groups. It follows from Remark 5.2.3 that each $\mathcal{W}_r\,\Omega^d_X$ can be regarded as a quasi-coherent sheaf on $(X, W_r(\mathscr{O}_X))$, so we can identify the preceding diagram with the tower of abelian groups

$$\cdots \to \mathcal{W}_3\,\Omega^d_{\mathscr{O}_X(U)} \to \mathcal{W}_2\,\Omega^d_{\mathscr{O}_X(U)} \to \mathcal{W}_1\,\Omega^d_{\mathscr{O}_X(U)}.$$

This diagram has surjective transition maps, so its homotopy limit can be identified with the abelian group $\mathcal{W}\Omega^d_{\mathscr{O}_X(U)}$ (regarded as an abelian group, concentrated in degree zero). It follows that $\mathscr{F}^\bullet$ can be identified with $\mathcal{W}\Omega^d_X$ (regarded as a chain complex concentrated in degree zero), so the preceding calculation gives $\mathrm{R}\Gamma(U;\mathcal{W}\Omega^d_X) \simeq \mathcal{W}\Omega^d_{\mathscr{O}_X(U)}$ for each affine open subset $U \subseteq X$. Proposition 5.2.4 now follows by taking $U = X$. □

5.3. **Localization for the Étale Topology.** Let R be a commutative $\mathbf{F}_p$-algebra. In §5.1, we showed that the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{R[s^{-1}]}$ of any localization $R[s^{-1}]$ can be described explicitly in terms of $\mathcal{W}\Omega^*_R$ (Corollary 5.1.6). In this section, we will formulate a generalization of this result which applies to any étale R-algebra S (Corollary 5.3.5).

Let (A^*, d) be a commutative differential graded algebra. In what follows, we will use the term A^**-algebra* to refer to a commutative differential graded algebra B^* equipped with a morphism of differential graded algebras $A^* \to B^*$.

Definition 5.3.1. Let $f\colon A^* \to B^*$ be a map of commutative differential graded algebras. We will say that f is *étale* (or that B^* is an *étale* A^**-algebra*) if the ring homomorphism $f\colon A^0 \to B^0$ is étale and the map of graded algebras $A^* \otimes_{A^0} B^0 \to B^*$ is an isomorphism.

We observe now that any étale A^0-algebra can be realized as an étale commutative differential graded algebra:

Proposition 5.3.2. *Let A^* be a commutative differential graded algebra. Then:*

(1) *For every étale A^0-algebra R, there exists an étale A^*-algebra B^* and an isomorphism of A^0-algebras $R \simeq B^0$.*
(2) *Let B^* be an étale A^*-algebra. Then, for any A^*-algebra C^*, the canonical map*
$$\mathrm{Hom}_{A^*}(B^*, C^*) \to \mathrm{Hom}_{A^0}(B^0, C^0)$$
is bijective.
(3) *The construction $B^* \mapsto B^0$ induces an equivalence from the category of étale A^*-algebras to the category of étale A^0-algebras.*

Proof. Let R be an étale A^0-algebra. Then the canonical map $R \otimes_{A^0} \Omega^1_{A^0} \to \Omega^1_R$ is an isomorphism. It follows that the de Rham complex Ω^*_S is given, as a graded ring, by the tensor product $R \otimes_{A^0} \Omega^*_{A^0}$. Extending scalars along the map $\Omega^*_{A^0} \to A^*$, we obtain an isomorphism of graded rings

$$R \otimes_{A^0} A^* \simeq \Omega^*_R \otimes_{\Omega^*_{A^0}} A^*.$$

Setting $B^* = \Omega^*_R \otimes_{\Omega^*_{A^0}} A^*$, we conclude that B^* is an étale A^*-algebra with $B^0 \simeq R$, which proves (1). Moreover, B^* has the universal property described in assertion (2). If B'^* is any other étale A^*-algebra equipped with an isomorphism $\alpha\colon R \simeq B'^0$, then α extends uniquely to a map of A^*-algebras $\overline{\alpha}\colon B^* \to B'^*$, which is automatically an isomorphism (since it is an isomorphism in degree zero and the domain and codomain of $\overline{\alpha}$ are both étale over A^*). It follows that B'^* also has the universal property of assertion (2). Assertion (3) is an immediate consequence of (1) and (2). □

We now formulate an analogue of Proposition 5.3.2 in the setting of strict Dieudonné algebras.

Definition 5.3.3. Let $f\colon A^* \to B^*$ be a morphism of strict Dieudonné algebras. We will say that f is *V-adically étale* if, for each n, $\mathcal{W}_n(f)\colon \mathcal{W}_n(A)^* \to \mathcal{W}_n(B)^*$ is étale as a morphism of commutative differential graded algebras.

Theorem 5.3.4. *Let A^* be a strict Dieudonné algebra. Then:*

(1) *For every étale A^0/VA^0-algebra R, there exists an V-adically étale morphism of strict Dieudonné algebras $A^* \to B^*$ and an isomorphism of A^0/VA^0-algebras $B^0/VB^0 \simeq R$.*
(2) *Let $f\colon A^* \to B^*$ be a V-adically étale morphism of strict Dieudonn'e algebras. Then, for every morphism of strict Dieudonné algebras $A^* \to C^*$, the canonical map*

$$\mathrm{Hom}_{\mathbf{DA}_{A^*}}(B^*, C^*) \to \mathrm{Hom}_{A^0/VA^0}(B^0/VB^0, C^0/VC^0)$$

is bijective.
(3) *The construction $B^* \mapsto B^0/VB^0$ induces an equivalence from the category of V-adically étale strict Dieudonné algebras over A^* to the category of étale A^0/VA^0-algebras.*

We will give the proof of Theorem 5.3.4 in §5.5. Note that it implies an analogue of Corollary 5.1.6:

Corollary 5.3.5. *Let $R \to S$ be an étale map of $\mathbf{F}_p$-algebras. Then the map $\mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_S$ of strict Dieudonné complexes is V-adically étale. Furthermore, for each n, we have an isomorphism*

$$\mathcal{W}_n\Omega^*_R \otimes_{W_n(R)} W_n(S) \simeq \mathcal{W}_n\Omega^*_S.$$

Proof. Let A^* be a strict Dieudonné algebra equipped with a map $R \to A^0/VA^0$ which exhibits A^* as a saturated de Rham-Witt complex of R. Set $S' = S \otimes_R (A^0/VA^0)$. By Theorem 5.3.4, there exists a V-adically étale strict Dieudonné algebra B^* over A^* such that $B^0/VB^0 \simeq S'$. For any strict Dieudonné algebra

C^*, we have bijections

$$\begin{aligned} \mathrm{Hom}_{\mathbf{DA}}(B^*, C^*) &\simeq \mathrm{Hom}_{\mathbf{DA}}(A^*, C^*) \times_{\mathrm{Hom}(A^0/VA^0, C^0/VC^0)} \mathrm{Hom}(S', C^0/VC^0) \\ &\simeq \mathrm{Hom}_{\mathbf{DA}}(A^*, C^*) \times_{\mathrm{Hom}(R, C^0/VC^0)} \mathrm{Hom}(S, C^0/VC^0) \\ &\simeq \mathrm{Hom}(S, C^0/VC^0), \end{aligned}$$

so that the canonical map $S \to B^0/VB^0$ exhibits B^* as a saturated de Rham-Witt complex of S. For each $n \geq 0$, we have canonical maps

$$\mathcal{W}_n(A)^* \otimes_{W_n(R)} W_n(S) \xrightarrow{\alpha} \mathcal{W}_n(A)^* \otimes_{W_n(A^0/VA^0)} W_n(S') \xrightarrow{\beta} \mathcal{W}_n(B)^*.$$

Here β is an isomorphism because B^* is V-adically étale over A^*, and α is an isomorphism by virtue of Theorem 5.4.1. It follows that $\beta \circ \alpha$ is an isomorphism. $\square$

Let us now formulate the étale analog of Theorem 5.2.2.

Construction 5.3.6. For any scheme X, write $\mathcal{U}_{\mathrm{aff,ét}}(X)$ denote the category of étale maps $U \to X$ with U affine. When X is an $\mathbf{F}_p$-scheme, the assignment

$$\mathcal{W}\Omega^*_{X_{\mathrm{ét}}}(U \to X) = \mathcal{W}\Omega^*_{\mathcal{O}_U(U)},$$

defines a $\mathbf{DA}$-valued presheaf $\mathcal{W}\Omega^*_{X_{\mathrm{ét}}}(-)$ on $\mathcal{U}_{\mathrm{aff,ét}}(X)$.

Theorem 5.3.7. *Let X be an $\mathbf{F}_p$-scheme. Then the presheaf $\mathcal{W}\Omega^*_{X_{\acute{e}t}}$ of Construction 5.3.6 is a sheaf with respect to the étale topology on $\mathcal{U}_{\mathrm{aff},\acute{e}t}(X)$.*

Proof. Let us begin by recalling a general fact. If $\mathrm{Spec}(A)$ is an affine scheme and M is an A-module, write $(\widetilde{M})_{\mathrm{ét}}$ is presheaf on $\mathcal{U}_{\mathrm{aff,ét}}(\mathrm{Spec}(A))$ determined by the formula $(\mathrm{Spec}(B) \to \mathrm{Spec}(A)) \mapsto M \otimes_A B$. It is a basic fact in descent theory that $(\widetilde{M})_{\mathrm{ét}}$ is an étale sheaf.

To prove Theorem 5.3.7, it will suffice (as in the proof of Theorem 5.2.2) to show that the presheaf $\mathcal{W}_n\Omega^i_{X_{\mathrm{ét}}}(-)$ determined by $(U \to X) \mapsto \mathcal{W}_n\Omega^i_{\mathcal{O}(U)}$ is a sheaf of abelian groups on $\mathcal{U}_{\mathrm{aff,ét}}(X)$ for each $i, n \geq 0$. To prove this, we may assume without loss of generality that $X \simeq \mathrm{Spec}(R)$ is affine. In this case, the functor $S \mapsto W_n(S)$ identifies the category of étale R-algebras with category of étale $W_n(R)$-algebras: this follows by combining Theorem 5.4.1 below with the topological invariance of the étale site (since the restriction map $W_n(R) \to W_1(R) \simeq R$ is surjective with nilpotent kernel). Under the resulting equivalence $\mathcal{U}_{\mathrm{aff,ét}}(\mathrm{Spec}(R)) \simeq \mathcal{U}_{\mathrm{aff,ét}}(\mathrm{Spec}(W_n(R)))$, it follows from Corollary 5.3.5 that the presheaf $\mathcal{W}_n\Omega^i_{X_{\mathrm{ét}}}(-)$ is isomorphic to the sheaf $\widetilde{M}_{\mathrm{ét}}$ associated to the $W_n(R)$-module $M = \mathcal{W}_n\Omega^i_R$. $\square$

5.4. **Digression: Witt Vectors and Étale Morphisms.** Our proof of Theorem 5.3.4 will make use of the following result:

Theorem 5.4.1. *Let $f\colon A \to B$ be an étale morphism of commutative rings. Then, for every integer $n \geq 0$, the induced map $W_n(A) \to W_n(B)$ is also étale. Moreover, the diagram*

$$\begin{tikzcd} W_n(A) \arrow[r] \arrow[d, "R"] & W_n(B) \arrow[d, "R"] \\ W_{n-1}(A) \arrow[r] & W_{n-1}(B) \end{tikzcd}$$

is a pushout square of commutative rings.

Theorem 5.4.1 (in various forms) has been proved by Langer-Zink [35, Prop. A.8, A.11], van der Kallen [50, Th. 2.4], and Borger [15, Th. 9.2]. We will need only the special case where A and B are $\mathbf{F}_p$-algebras which appears in the original work of Illusie [28, Prop.1.5.8].

Proof of Theorem 5.4.1 for $\mathbf{F}_p$-algebras. We proceed by induction on n, the case $n = 1$ being trivial. To carry out the inductive step, let us assume that the map $W_n(A) \to W_n(B)$ is étale and that the diagram

$$\begin{tikzcd} W_n(A) \arrow[r] \arrow[d, "R^{n-1}"] & W_n(B) \arrow[d, "R^{n-1}"] \\ A \arrow[r] & B \end{tikzcd}$$

is a pushout square. Since A is an $\mathbf{F}_p$-algebra, we can regard $W_{n+1}(A)$ as a square-zero extension of $W_n(A)$. Invoking the topological invariance of the étale site, we see that $W_n(f)\colon W_n(A) \to W_n(B)$ can be lifted to an étale map $\overline{f}\colon W_{n+1}(A) \to \widetilde{B}$. Moreover, the infinitesimal lifting property of étale morphisms guarantees that there is a unique $W_{n+1}(A)$-algebra map $\rho\colon \widetilde{B} \to W_{n+1}(B)$ lifting the identity map $\mathrm{id}\colon W_n(B) \to W_n(B)$. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & V^n A \arrow[r] \arrow[d] & W_{n+1}(A) \arrow[r, "R"] \arrow[d] & W_n(A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V^n A \otimes_{W_{n+1}(A)} \widetilde{B} \arrow[r] \arrow[d, "\rho_0"] & \widetilde{B} \arrow[r] \arrow[d, "\rho"] & W_n(B) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V^n B \arrow[r] & W_{n+1}(B) \arrow[r, "R"] & W_n(B) \arrow[r] & 0 \end{tikzcd}$$

where each row is a short exact sequence, and the middle row is obtained from the top row by extending scalars along $\widetilde{f}$. To complete the proof, it will suffice to show that ρ is an isomorphism of commutative rings, or equivalently that ρ_0 is an isomorphism of abelian groups. Note that the action of $W_{n+1}(A)$ on the ideal $V^n A$ factors through the restriction map $R^n\colon W_{n+1}(A) \to A$ (and that A acts on $V^n A \simeq A$ via the iterated Frobenius map $\varphi_A^n : A \to A$). Moreover, the map ρ_0 factors as a composition

$$V^n A \otimes_{W_{n+1}(A)} \widetilde{B} \simeq V^n A \otimes_{W_n(A)} W_n(B) \xrightarrow{\alpha} V^n A \otimes_A B \xrightarrow{\beta} V^n B,$$

where α is an isomorphism by virtue of our inductive hypothesis and β is an isomorphism because the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_A^n"] \arrow[d] & A \arrow[d] \\ B \arrow[r, "\varphi_B^n"] & B \end{tikzcd}$$

is a pushout square (since $A \to B$ is étale; see [47, Tag 0EBS]). It follows that ρ_0 is an isomorphism, as desired. □

Remark 5.4.2. Let $f : A \to B$ be an étale morphism of $\mathbf{F}_p$-algebras. Then, for every pair of integers $n, k \geq 0$, the diagram

$$\begin{tikzcd} W_n(A) \arrow[r, "W_n(f)"] \arrow[d, "F^k"] & W_n(B) \arrow[d, "F^k"] \\ W_n(A) \arrow[r, "W_n(f)"] & W_n(B) \end{tikzcd}$$

is a pushout square of commutative rings. To prove this, we note that the induced map

$$\theta : W_n(A) \otimes_{W_n(A)} W_n(B) \to W_n(B)$$

is a morphism of étale $W_n(A)$-algebras. Consequently, it will suffice to show that θ is an isomorphism after extending scalars along the restriction map $W_n(A) \to A$ (since $\ker(W_n(A) \to A)$ is nilpotent), which allows us to reduce to the case $n = 1$.

Definition 5.4.3. Let A be an $\mathbf{F}_p$-algebra and let M be a $W(A)$-module. We will say that M is *nilpotent* if M is annihilated by the ideal $V^n W(A) \subseteq W(A)$ for some n, i.e., if M is actually a $W_n(A)$-module for some n.

Definition 5.4.4. Let $A \to B$ be an étale map of $\mathbf{F}_p$-algebras. If M is a nilpotent $W(A)$-module, so that M is a $W_n(A)$-module for some n, then we write M_B for the $W(B)$-module $M \otimes_{W_n(A)} W_n(B)$. By Theorem 5.4.1, the construction of M_B does not depend on the choice of n.

Remark 5.4.5. The construction $M \mapsto M_B$ is clearly left adjoint to the forgetful functor from nilpotent $W(B)$-modules to nilpotent $W(A)$-modules. In addition, the functor $M \mapsto M_B$ is exact because $W_n(A) \to W_n(B)$ is étale, hence flat.

Definition 5.4.6. Let A be an $\mathbf{F}_p$-algebra and let M be a $W(A)$-module. For each non-negative integer k, we let $M_{(k)}$ denote the $W(A)$-module obtained by restriction of scalars of M along the Witt vector Frobenius map $F^k: W(A) \to W(A)$.

Remark 5.4.7. If M is a nilpotent $W(A)$-module, then $M_{(k)}$ is also nilpotent. In fact, if M is a $W_n(A)$-module for some integer n, then $M_{(k)}$ is also a $W_n(A)$-module.

Proposition 5.4.8. *Given an étale map $A \to B$ of $\mathbf{F}_p$-algebras and a nilpotent $W(A)$-module M, we have $(M_B)_{(k)} \simeq (M_{(k)})_B$ for each integer k.*

Proof. Choose an integer $n \gg 0$ for which M can be regarded as a $W_n(A)$-module. The desired result then follows from the fact that the diagram

$$\begin{tikzcd} W_n(A) \arrow[r, "W_n(f)"] \arrow[d, "F^k"] & W_n(B) \arrow[d, "F^k"] \\ W_n(A) \arrow[r, "W_n(f)"] & W_n(B) \end{tikzcd}$$

is a pushout (see Remark 5.4.2). □

Definition 5.4.9. Let M, N be nilpotent $W(A)$-modules. We will say that a homomorphism $f: M \to N$ of abelian groups is (F^k, F^ℓ)-linear if for all $x \in W(A)$ and $m \in M$, we have

$$f((F^k x)m) = (F^\ell x) f(x) \in N.$$

Equivalently, f defines a map of $W(A)$-modules $M_{(k)} \to N_{(\ell)}$.

Corollary 5.4.10. *Fix an étale map $A \to B$. Let M be a nilpotent $W(A)$-module and let N be a nilpotent $W(B)$-module, considered as a nilpotent $W(A)$-module via restriction of scalars. Then any (F^k, F^ℓ)-linear map $f: M \to N$ extends uniquely to a (F^k, F^ℓ)-linear map $M_B \to N$.*

Proof. Apply Proposition 5.4.8. □

Remark 5.4.11. In the situation of Corollary 5.4.10, suppose that we are given nilpotent $W(A)$-modules M, N, and P, together with an (F^a, F^b)-lienar map $f: M \to N$ and an (F^c, F^d)-linear map $g: N \to P$. Suppose that the sequence $M \xrightarrow{f} N \xrightarrow{g} P$ is exact in the category of abelian groups. Then $M_B \xrightarrow{f_B} N_B \xrightarrow{g_B} P_B$ is exact as well. To see this, by replacing (a, b) by $(a+k, b+k)$ and (c, d) by $(c+l, d+l)$ for some $k, l \geq 0$, we can assume that $b = c$. We then have an exact sequence of nilpotent $W(A)$-modules $M_{(a)} \to N_{(b)} \to P_{(d)}$, so that extending scalars to $W(B)$ yields an exact sequence $(M_B)_{(a)} \to (N_B)_{(b)} \to (P_B)_{(d)}$ by virtue of Remark 5.4.5.

5.5. **The Proof of Theorem 5.3.4.** We begin by proving the second assertion of Theorem 5.3.4.

Proposition 5.5.1. *Let A^* be a strict Dieudonné algebra, let B^* be a strict Dieudonné algebra which is V-adically étale over A^*, and let C^* be another strict Dieudonné algebra over A^*. Then the restriction map*

$$\mathrm{Hom}_{\mathbf{DA}_{A^*}}(B^*, C^*) \simeq \mathrm{Hom}_{A^0/VA^0}(B^0/VB^0, C^0/VC^0). \tag{10}$$

is bijective.

Proof. Let $\overline{f}: B^0/VB^0 \to C^0/VC^0$ be a morphism of A^0/VA^0-algebras; we wish to show that $\overline{f}$ can be lifted uniquely to a morphism $f: B^* \to C^*$ of strict Dieudonné algebras over A^*. For each $n \geq 0$, we have a commutative diagram

$$\begin{tikzcd} A^0/V^nA^0 \arrow[r] \arrow[d] & C^0/V^nC^0 \arrow[d] \\ B^0/V^nB^0 \arrow[r, "\overline{f}"] \arrow[ur, dashed, "f_n^0"] & C^0/VC^0, \end{tikzcd}$$

where the left vertical map is étale (by virtue of Proposition 3.6.2 and Theorem 5.4.1) and the right vertical map is a surjection with nilpotent kernel. It follows that there is a unique ring homomorphism $f_n^0: B^0/V^nB^0 \to C^0/V^nC^0$ as indicated which makes the diagram commute. Using Proposition 5.3.2, we see that each f_n^0 can be extended uniquely to a map of $\mathcal{W}_n(A)^*$-modules $f_n: \mathcal{W}_n(B)^* \to \mathcal{W}_n(C)^*$. Passing to the inverse limit over n, we obtain a map of differential graded algebras $f: B^* \to C^*$. We will complete the proof by showing that f is a morphism of strict Dieudonné algebras: that is, that it commutes with the Frobenius. For this, it suffices to prove the commutativity of the diagram

$$\begin{tikzcd} \mathcal{W}_n(B)^* \arrow[r, "f_n"] \arrow[d, "F"] & \mathcal{W}_n(C)^* \arrow[d, "F"] \\ \mathcal{W}_{n-1}(B)^* \arrow[r, "f_{n-1}"] & \mathcal{W}_{n-1}(C)^* \end{tikzcd}$$

for each $n \geq 0$. Invoking Proposition 5.3.2 again, we are reduced to proving the commutativity of the diagram of the left square in the diagram of A^0/V^nA^0-algebras

$$\begin{tikzcd} B^0/V^nB^0 \arrow[r, "f_n^0"] \arrow[d, "F"] & C^0/V^nC^0 \arrow[r] \arrow[d, "F"] & C^0/VC^0 \arrow[d, "F"] \\ B^0/V^{n-1}B^0 \arrow[r, "f_{n-1}^0"] & C^0/V^{n-1}C^0 \arrow[r] & C^0/VC^0. \end{tikzcd}$$

Note that the right square in this diagram commutes, and that the horizontal maps on the right are surjections with nilpotent kernel. Since B^0/V^nB^0 is étale over A^0/V^nA^0, we are reduced to showing that the outer rectangle commutes.

This follows from the commutativity of the diagram

$$\begin{tikzcd}
B^0/V^nB^0 \arrow[r] \arrow[d, "F"] & B^0/VB^0 \arrow[r, "\overline{f}"] \arrow[d, "F"] & C^0/VC^0 \arrow[d, "F"] \\
B^0/V^{n-1}B^0 \arrow[r] & B^0/VB^0 \arrow[r, "\overline{f}"] & C^0/VC^0.
\end{tikzcd}$$

□

To prove the first assertion of Theorem 5.3.4, we will need the following:

Lemma 5.5.2. *Let R be a commutative ring. Let M be a $W_n(R)$-module such that $p^k M = 0$ and let $d: W_n(R) \to M$ be a derivation. Then the composite $d \circ F^k: W_{n+k}(R) \to M$ vanishes.*

Proof. Let x be an element of $W_{n+k}(R)$; we wish to show that $d(F^k x)$ vanishes. Without loss of generality, we may assume that $x = V^m[a]$, where $[a] \in W_{n+k-m}(R)$ denotes the image of the Teichmüller representative of some $a \in R$. If $m \geq k$, then $F^k(x) = p^k V^{m-k}([a])$. If $m \leq k$, then $F^k x = p^m [a]^{p^{k-m}}$, so that $d(F^k x) = p^k [a]^{p^{k-m}-1} d[a]$. In either case, the desired result follows from our assumption that $p^k M = 0$. □

Example 5.5.3. Let $A \to B$ be an étale morphism of $\mathbf{F}_p$-algebras. Suppose we are given a commutative differential graded algebra R^* and a map of rings $W(A) \to R^0$ which factors through $W_n(R)$ for some n. Let R^*_B denote the tensor product $R^* \otimes_{W_n(A)} W_n(B)$, formed in the category of graded commutative rings. It follows from Lemma 5.5.2 that the differential $d: R^* \to R^*$ is (F^n, F^n)-linear, in the sense of Definition 5.4.9. It follows that there is a unique differential d on the graded ring R^*_B which is (F^n, F^n)-linear as a map of $W(B)$-modules and compatible with the differential on R^*. Note that R^*_B is étale when regarded as a commutative differential graded algebra over R^* (in the sense of Definition 5.3.1).

Proof of Theorem 5.3.4. Assertion (2) of Theorem 5.3.4 follows from Proposition 5.5.1, and assertion (3) follows formally from (1) and (2) (as in the proof of Proposition 5.3.2). We will prove (1). Let A^* be a strict Dieudonné algebra, set $R = A^0/VA^0$, and let S be an étale R-algebra; we will construct a strict Dieudonné algebra B^* which is V-adically étale over A^* such that B^0/VB^0 is isomorphic to S (as an R-algebra).

For each $n \geq 0$, the commutative ring $W_n(S)$ is étale over $\mathcal{W}_n(A)^0$ (Theorem 5.4.1). Using part (1) of Proposition 5.3.2, we can choose a commutative differential graded algebra B^*_n which is étale over $\mathcal{W}_n(A)^*$ and an isomorphism $B^0_n \simeq W_n(S)$ of $\mathcal{W}_n(A)^0$-algebras. Using part (2) of Proposition 5.3.2, we see that the Frobenius and restriction maps $R, F: W_n(S) \to W_{n-1}(S)$ extend uniquely to maps of commutative differential graded algebras $R, F: B^*_n \to B^*_{n-1}$ for which the diagrams

$$\begin{tikzcd} \mathcal{W}_n(A)^* \arrow[r] \arrow[d] & B_n^* \arrow[d, "R"] \\ \mathcal{W}_{n-1}(A)^* \arrow[r] & B_{n-1}^* \end{tikzcd} \qquad \begin{tikzcd} \mathcal{W}_n(A)^* \arrow[r] \arrow[d, "F"] & B_n^* \arrow[d, "F"] \\ \mathcal{W}_{n-1}(A)^* \arrow[r] & B_{n-1}^* \end{tikzcd}$$

commute. Let B^* denote the inverse limit $\varprojlim B_n^*$ in the category of commutative differential graded algebras, so that the Frobenius maps $F: B_n^* \to B_{n-1}^*$ assemble to a map of graded algebras $F: B^* \to B^*$. We will show that (B^*, F) is a strict Dieudonné algebra.

Note that conditions (ii) and (iii) of Definition 3.1.2 are automatically satisfied by B^* (since $B^0 \simeq W(S)$ by construction). It will therefore suffice to show that F exhibits B^* as a strict Dieudonné complex. By virtue of Proposition 2.9.1, it will suffice to show that we can define Verschiebung maps $V: B_n^* \to B_{n+1}^*$ which endow $\{B_n^*\}_{n\geq 0}$ with the structure of a strict Dieudonné tower, in the sense of Definition 2.6.1. Note that the Verschiebung map $V: \mathcal{W}_n(A)^* \to \mathcal{W}_{n+1}(A)^*$ is an (F^1, F^0)-linear map of nilpotent $W(R)$-modules. Using Corollary 5.4.10, we see that V admits an essentially unique extension to a map $V: B_n^* \to B_{n+1}^*$ which is (F^1, F^0)-linear as a map of $W(S)$-modules. To complete the proof, it suffices to observe that the Frobenius, Verschiebung, and restriction maps on $\{B_n^*\}_{n\geq 0}$ satisfy axioms (1) through (8) appearing in Definition 2.6.1. We consider axioms (6) through (8) (the first five are essentially formal and left to the reader):

(6) For each $n \geq 0$, the sequence $B_{n+1}^* \xrightarrow{F} B_n^* \xrightarrow{d} B_n^{*+1}/pB_n^{*+1}$ is exact. This follows from applying Remark 5.4.11 to the sequence

$$\mathcal{W}_{n+1}(A)^* \xrightarrow{F} \mathcal{W}_n(A)^* \xrightarrow{d} \mathcal{W}_n(A)^{*+1}/p\mathcal{W}_n(A)^{*+1}$$

(which is exact by virtue of Proposition 2.6.2); note that the maps in this sequence are (F^0, F^1)-linear and (F^n, F^n)-linear, respectively (Lemma 5.5.2).

(7) For each $n \geq 0$, we have an exact sequence

$$B_{n+1}^*[p] \xrightarrow{\mathrm{id}} B_{n+1}^* \xrightarrow{R} B_n^*.$$

This follows from from the exactness of the sequence

$$\mathcal{W}_{n+1}(A)^*[p] \xrightarrow{\mathrm{id}} \mathcal{W}_{n+1}(A)^* \xrightarrow{R} \mathcal{W}_n(A)^*$$

by extending scalars along the étale map $W_{n+1}(R) \to W_{n+1}(S)$.

(8) For each $n \geq 0$, we haven an exact sequence

$$B_1^* \oplus B_1^{*-1} \xrightarrow{(V^n, dV^n)} B_{n+1}^* \xrightarrow{R} B_n^*.$$

This follows from applying Remark 5.4.11 to the sequence

$$\mathcal{W}_1(A)^* \oplus \mathcal{W}_1(A)^{*-1} \xrightarrow{(V^n, dV^n)} \mathcal{W}_{n+1}(A)^* \xrightarrow{R} \mathcal{W}_n(A)^*$$

(which is exact by virtue of Proposition 2.6.2); note that the maps in this sequence are (F^{2n}, F^{n})-linear and (F^{0}, F^{0})-linear, respectively (again by Lemma 5.5.2).

$\square$

6. The Case of a Cusp

Let R be a commutative $\mathbf{F}_p$-algebra. If R is smooth over a perfect field k of characteristic p, then Theorem 4.4.12 supplies a canonical isomorphism from the classical de Rham-Witt complex $W\Omega^*_R$ of [28] to the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of Definition 4.1.1. It follows that we can regard $\mathcal{W}\Omega^*_R$ as a representative, in the derived category of abelian groups, for the crystalline cochain complex $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ of the affine scheme $\mathrm{Spec}(R)$.

In this section, we study the effect of introducing a mild singularity, taking $R = \mathbf{F}_p[x,y]/(x^2 - y^3)$ to be the ring of functions of an affine curve with single cusp. Our main result, which we prove in §6.2, asserts that the canonical map from R to its normalization $\widetilde{R} \simeq \mathbf{F}_p[t]$ induces an isomorphism of saturated de Rham-Witt complexes $\mathcal{W}\Omega^*_R \simeq \mathcal{W}\Omega^*_{\widetilde{R}}$ (Proposition 6.2.1). From this, we will deduce several consequences:

- The comparison map $\gamma: W\Omega^*_R \to \mathcal{W}\Omega^*_R$ is not an isomorphism when $R = \mathbf{F}_p[x,y]/(x^2 - y^3)$ (Proposition 6.3.1).
- The cochain complex $\mathcal{W}\Omega^*_R$ is not isomorphic to $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ as an object of the derived category of abelian groups (Corollary 6.4.2).

Recall that the cusp R studied above is, in some sense, the universal example of a commutative ring that fails to be seminormal. Using the results for $\mathcal{W}\Omega^*_R$ mentioned above, we deduce the following:

- Let S be any commutative $\mathbf{F}_p$-algebra, and let $S \to S^{\mathrm{sn}}$ be the seminormalization of S. Then the induced map $\mathcal{W}\Omega^*_S \to \mathcal{W}\Omega^*_{S^{\mathrm{sn}}}$ is an isomorphism (Corollary 6.5.2). Moreover, the unit map
$$S \to \mathcal{W}\Omega^0_S / V\mathcal{W}\Omega^0_S$$
exhibits $\mathcal{W}\Omega^0_S / V\mathcal{W}\Omega^0_S$ as the seminormalization S^{sn} of S (Theorem 6.5.3).

This result is inspired by a result in complex algebraic geometry: if $\underline{\Omega}^*_X$ is the Deligne-du Bois complex of a complex algebraic variety X, then $\mathcal{H}^0(\underline{\Omega}^0_X)$ identifies with the structure sheaf of the seminormalization of X [33, Proposition 7.8].

6.1. Digression: The de Rham Complex of a Graded Ring.

We begin with some elementary remarks. Suppose that $R = \bigoplus_{d \in \mathbf{Z}} R_d$ is a graded ring. Then the de Rham complex of R admits a bigrading $\Omega^*_R \simeq \bigoplus_{d,n} (\Omega^n_R)_d$, characterized by the requirement that for every sequence of homogeneous elements $x_0, x_1, \ldots, x_n \in R$ of degrees $d_0, d_1, \ldots, d_n \in \mathbf{Z}$, the differential form
$$x_0(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n)$$
belongs to the summand $(\Omega^n_R)_{d_0 + \cdots + d_n} \subseteq \Omega^n_R$.

Remark 6.1.1. Let $R = \bigoplus_{d \in \mathbf{Z}} R_d$ be a graded ring which is p-torsion free and let $\varphi: R \to R$ be a ring homomorphism satisfying $\varphi(x) \equiv x^p \pmod{p}$ for $x \in R$, so that the de Rham complex Ω^*_R inherits the structure of a Dieudonné algebra

(Proposition 3.2.1). Suppose that, for every homogeneous element $x \in R_d$, we have $\varphi(x) \in R_{pd}$. Then, for every homogeneous differential form $\omega \in (\Omega^*_R)_d$, we have $F(\omega) \in (\Omega^*_R)_{pd}$.

Proposition 6.1.2. *Let $f: R \to R'$ be a homomorphism of non-negatively graded rings equipped with ring homomorphisms $\varphi: R \to R$ and $\varphi': R' \to R'$. Assume that:*

- *The rings R and R' are p-torsion-free.*
- *The diagram of ring homomorphisms*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\varphi"] & R' \arrow[d, "\varphi'"] \\ R \arrow[r, "f"] & R' \end{tikzcd}$$

  is commutative.
- *The homomorphisms φ and φ' satisfy $\varphi(x) \equiv x^p \pmod p$ and $\varphi(x') \equiv x'^p \pmod p$ for all $x \in R$, $x' \in R'$.*
- *We have $\varphi(R_d) \subseteq R_{pd}$ and $\varphi'(R'_d) \subseteq R'_{pd}$.*
- *The ring homomorphism $f_0: R_0 \to R'_0$ is an isomorphism.*
- *There exists an integer $N \gg 0$ such that f induces an isomorphism $R_d \to R'_d$ for $d \geq N$.*

Then the map of de Rham complexes $\Omega^*_R \to \Omega^*_{R'}$ induces an isomorphism

$$\mathrm{Sat}(\Omega^*_R) \to \mathrm{Sat}(\Omega^*_{R'})$$

of saturated Dieudonné algebras.

Proof. Let $(\Omega^*_R)_{\mathrm{tors}} \subseteq \Omega^*_R$ and $(\Omega^*_{R'})_{\mathrm{tors}} \subseteq \Omega^*_{R'}$ denote the submodules of p-power torsion elements, and let us write F for the Frobenius map on the Dieudonné algebras Ω^*_R and $\Omega^*_{R'}$. Using the description of the saturation given in Remark 2.3.4, we are reduced to showing that the natural map

$$\theta: (\Omega^*_R/(\Omega^*_R)_{\mathrm{tors}})[1/F] \to (\Omega^*_{R'}/(\Omega^*_{R'})_{\mathrm{tors}})[1/F]$$

is an isomorphism of graded rings.

Choose an integer $r \geq 0$ such that $p^r \geq N$. Then the map f induces an isomorphism $R_d \simeq R'_d$ whenever d is divisible by p^r. It follows that there is a unique map $\psi: R' \to R$ for which the diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\varphi^r"] & R' \arrow[d, "\varphi'^r"] \arrow[dl, "\psi"] \\ R \arrow[r, "f"'] & R' \end{tikzcd},$$

is commutative. Consequently, the kernel of the map

$$\Omega^*_R/(\Omega^*_R)_{\mathrm{tors}} \to \Omega^*_{R'}/(\Omega^*_{R'})_{\mathrm{tors}}$$

is annihilated by F^r, since $p^r F^r = (\varphi^r)^*$ on Ω^*_R. It follows that θ is injective.

Note that, since f induces an isomorphism $R[1/F] \to R'[1/F]$, we can identify θ with the natural map

$$((R' \otimes_R \Omega^*_R)/(R' \otimes_R \Omega^*_R)_{\mathrm{tors}})[1/F] \to \Omega^*_{R'}/(\Omega^*_{R'})_{\mathrm{tors}})[1/F].$$

We will complete the proof by showing that this map is surjective. Note that the codomain of θ is generated, as an algebra over $R'[1/F]$, by elements of degree 1. We are therefore reduced to showing that the natural map

$$(R' \otimes_R \Omega^1_R)[1/F] \to \Omega^1_{R'}[1/F]$$

is surjective. For this, it will suffice to show that f induces a surjection $\rho\colon (R' \otimes_R \Omega^1_R)_d \to (\Omega^1_{R'})_d$ for $d = 0$ (which is clear) and for $d \geq 2N$. In the latter case, we note that $(\Omega^1_{R'})_d$ is generated by elements of the form xdy for homogeneous elements $x, y \in R'$, where either x or y has degree $\geq N$. Writing $xdy = d(xy) - ydx$, we can reduce to the case where y has degree $\geq N$ and therefore belongs to the image of f, so that xdy belongs to the image of ρ as desired. □

6.2. **The Saturated de Rham-Witt Complex of a Cusp.** We now apply Proposition 6.1.2 to study the saturated de Rham-Witt complex of a cusp.

Proposition 6.2.1. *Let $R \subseteq \mathbf{F}_p[t]$ be the subring generated by t^2 and t^3. Then the inclusion $R \hookrightarrow \mathbf{F}_p[t]$ induces an isomorphism of saturated de Rham-Witt complexes*

$$\mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_{\mathbf{F}_p[t]}.$$

Proof. Let $\widetilde{R}$ denote the $\mathbf{Z}_p$-subalgebra of the polynomial ring $\mathbf{Z}_p[t]$ generated by t^2 and t^3. Then $\widetilde{R}$ is a p-torsion free ring satisfying $R \simeq \widetilde{R}/p\widetilde{R}$. Let us regard $\mathbf{Z}_p[t]$ as a graded ring in which the element t is homogeneous of degree 1, and let $\varphi\colon \mathbf{Z}_p[t] \to \mathbf{Z}_p[t]$ be the $\mathbf{Z}_p$-algebra homomorphism given by $\varphi(t) = t^p$. Then $\widetilde{R}$ is a graded subring of $\mathbf{Z}_p[t]$ satisfying $\varphi(\widetilde{R}) \subseteq \widetilde{R}$, and the inclusion map $\widetilde{R} \hookrightarrow \mathbf{Z}_p[t]$ is an isomorphism in degrees $\neq 1$. Applying Proposition 6.1.2, we deduce that the associated map of de Rham complexes $\Omega^*_{\widetilde{R}} \to \Omega^*_{\mathbf{Z}_p[t]}$ induces an isomorphism of saturated Dieudonné algebras $\mathrm{Sat}(\Omega^*_{\widetilde{R}}) \to \mathrm{Sat}(\Omega^*_{\mathbf{Z}_p[t]})$. By virtue of Corollary 4.2.3, we have a commutative diagram of Dieudonné algebras

$$\begin{tikzcd}
\mathrm{Sat}(\Omega^*_{\widetilde{R}}) \arrow[r] \arrow[d] & \mathrm{Sat}(\Omega^*_{\mathbf{Z}_p[t]}) \arrow[d] \\
\mathcal{W}\Omega^*_R \arrow[r] & \mathcal{W}\Omega^*_{\mathbf{F}_p[t]},
\end{tikzcd}$$

where the vertical maps exhibit $\mathcal{W}\Omega^*_R$ and $\mathcal{W}\Omega^*_{\mathbf{F}_p[t]}$ as the completions of $\mathrm{Sat}(\Omega^*_{\widetilde{R}})$ and $\mathrm{Sat}(\Omega^*_{\mathbf{Z}_p[t]})$, respectively. It follows that the lower horizontal map is also an isomorphism. □

Remark 6.2.2. The inclusion map $\widetilde{R} \hookrightarrow \mathbf{Z}_p[t]$ appearing in the proof of Proposition 6.2.1 does not induce an isomorphism of de Rham complexes $\rho: \Omega^*_{\widetilde{R}} \to \Omega^*_{\mathbf{Z}_p[t]}$: in fact, the differential form $dt \in \Omega^1_{\mathbf{Z}_p[t]}$ does not belong to the image of ρ. Nevertheless, Proposition 6.1.2 guarantees that $F^n(dt)$ belongs to the image of ρ for some integer n. In fact, we can be more explicit. Set $x = t^3$ and $y = t^2$, so that x and y belong to $\widetilde{R}$. For $p \geq 5$, we have $F(dt) = t^{p-1}dt = \frac{1}{2}xy^{(p-5)/2}dy$. For $p = 3$, we have $F^2(dt) = t^8 dt = \frac{1}{2}xy^2 dy$. For $p = 2$, we have $F^3(dt) = t^7 dt = \frac{1}{3}xydx$.

6.3. **The Classical de Rham-Witt Complex of a Cusp.** Let R be a commutative $\mathbf{F}_p$-algebra. In §4.4, we constructed a comparison map $W\Omega^*_R \to \mathcal{W}\Omega^*_R$ (Corollary 4.4.11) and showed that it is an isomorphism in the case where R is smooth over a perfect field (Theorem 4.4.12). Proposition 6.2.1 shows that it is not an isomorphism in general, even for reduced rings:

Proposition 6.3.1. *Let $R \subseteq \mathbf{F}_p[t]$ be the subalgebra generated by t^2 and t^3. Then the comparison map*

$$c_R: W\Omega^*_R \to \mathcal{W}\Omega^*_R$$

of Corollary 4.4.11 is not an isomorphism.

Proof. Set $x = t^3$ and $y = t^2$, so that we can identify R with the quotient ring $\mathbf{F}_p[x,y]/(x^2 - y^3)$. Then the module of Kähler differentials Ω^1_R is generated by dx and dy, subject only to the relation $2xdx = 3y^2dy$. It follows that dx and dy have linearly independent images in the $\mathbf{F}_p$-vector space $R/(x,y) \otimes_R \Omega^1_R$, so that the differential form $dx \wedge dy \in \Omega^2_R$ is nonzero. Since the natural map $W\Omega^*_R \to \Omega^*_R$ is surjective, we must have $W\Omega^2_R \neq 0$. On the other hand, Proposition 6.2.1 supplies an isomorphism $\mathcal{W}\Omega^*_R \simeq \mathcal{W}\Omega^*_{\mathbf{F}_p[t]}$, so that we can use Corollary 4.2.3 to identify $\mathcal{W}\Omega^*_R$ with the completed saturation of the completed de Rham complex $\widehat{\Omega}^*_{\mathbf{Z}_p[t]}$. It follows that $\mathcal{W}\Omega^2_R$ vanishes, so that the map $c_R: W\Omega^*_R \to \mathcal{W}\Omega^*_R$ cannot be an isomorphism in degree 2. □

Remark 6.3.2. The comparison map $c_R: W\Omega^*_R \to \mathcal{W}\Omega^*_R$ need not even be a quasi-isomorphism. Note that we have a commutative diagram

$$\begin{tikzcd} W\Omega^*_R \arrow[r, "c_R"] \arrow[d, "c'"] & \mathcal{W}\Omega^*_R \arrow[d] \\ W\Omega^*_{\mathbf{F}_p[t]} \arrow[r] & \mathcal{W}\Omega^*_{\mathbf{F}_p[t]}, \end{tikzcd}$$

where the right vertical map is a quasi-isomorphism by virtue of Proposition 6.2.1. Consequently, if c_R is a quasi-isomorphism, then the map c' is also a quasi-isomorphism, and therefore induces an isomorphism $W\Omega^*_R \otimes^L_{\mathbf{Z}} \mathbf{F}_p \to W\Omega^*_{\mathbf{F}_p[t]} \otimes^L_{\mathbf{Z}} \mathbf{F}_p$

in the derived category of abelian groups. Using the commutativity of the diagram

$$\begin{tikzcd} W\Omega^*_R \otimes^L_{\mathbf{Z}} \mathbf{F}_p \arrow[r] \arrow[d] & W\Omega^*_{\mathbf{F}_p[t]} \otimes^L_{\mathbf{Z}} \mathbf{F}_p \arrow[d] \\ \Omega^*_R \arrow[r] & \Omega^*_{\mathbf{F}_p[t]}, \end{tikzcd}$$

and the fact that the right vertical map is an isomorphism (since $\mathbf{F}_p[t]$ is a smooth $\mathbf{F}_p$-algebra), it would follow that that the map of de Rham complexes $\Omega^*_R \to \Omega^*_{\mathbf{F}_p[t]}$ induces a surjection on cohomology. In particular, if c_R is a quasi-isomorphism, then the 0-cocycle $t^p \in \Omega^0_{\mathbf{F}_p[t]}$ can be lifted to a 0-cocycle in Ω^*_R: that is, the differential $d(t^p)$ vanishes as an element of Ω^1_R. This is not true for $p \leq 7$. To see this, write $x = t^3$ and $y = t^2$ as in Proposition 6.3.1, so that $t^p = x^m y^n$ for some $0 \leq m, n < p$. For $p \leq 7$, the canonical map

$$\Big(R \cdot dx \oplus R \cdot dy\Big) \to \Big(R \cdot dx \oplus R \cdot dy\Big)/R \cdot (2x \cdot dx - 3y^2 \cdot dy) \simeq \Omega^1_R$$

is an isomorphism in degree p (with respect to the grading of §6.1), so that $d(t^p) = mt^{p-3} \cdot dx + nt^{p-2} \cdot dy$ does not vanish.

6.4. **The Crystalline Cohomology of a Cusp.** Let R be an $\mathbf{F}_p$-algebra. If R is smooth over a perfect field, then the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ can be identified with the classical de Rham-Witt complex $W\Omega^*_R$, and can therefore be identified with the crystalline cochain complex $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ (as an object of the derived category). This observation does not extend to the non-smooth case:

Proposition 6.4.1. *Let $R \subseteq \mathbf{F}_p[t]$ be the subalgebra generated by t^2 and t^3. Then the cochain complex $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$ has nonvanishing cohomology in degree 2.*

Corollary 6.4.2. *Let R be as in Proposition 6.4.1. Then there does not exist a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \simeq \mathcal{W}\Omega^*_R$.*

Proof. Suppose that there exists a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \simeq \mathcal{W}\Omega^*_R$. Reducing modulo p and using Proposition 6.2.1 and Remark 4.3.6, we obtain quasi-isomorphisms

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p &\simeq \mathcal{W}\Omega^*_R \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p \\ &\simeq \mathcal{W}\Omega^*_R / p\mathcal{W}\Omega^*_R \\ &\simeq \mathcal{W}\Omega^*_{\mathbf{F}_p[t]} / p\mathcal{W}\Omega^*_{\mathbf{F}_p[t]} \\ &\simeq \Omega^*_{\mathbf{F}_p[t]}. \end{aligned}$$

This contradicts Proposition 6.4.1, since $\Omega^*_{\mathbf{F}_p[t]}$ is concentrated in cohomological degrees ≤ 1. $\square$

Remark 6.4.3. We will see in §9 that, for any $\mathbf{F}_p$-algebra R which can be realized as a local complete intersection over a perfect field k, there is a natural comparison map $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \to \mathcal{W}\Omega^*_R$ in the derived category $D(\mathbf{Z}_p)$. By virtue of Corollary 6.4.2, this map cannot be an isomorphism in general.

Proof of Proposition 6.4.1. Since $R \simeq \mathbf{F}_p[x,y]/(x^2 - y^3)$ is a local complete intersection over $\mathbf{F}_p$, we can identify $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$ with the derived de Rham complex $L\Omega_R$ (see [10, Theorem 1.5]). Because R admits a lift to $\mathbf{Z}/(p^2)$ together with a lift of Frobenius, there is a natural identification

$$\bigoplus_i (\bigwedge^i L_{R/\mathbf{F}_p})[-i] \simeq L\Omega^*_R$$

where $\bigwedge^i L_{R/\mathbf{F}_p}$ denotes the ith derived exterior power of the cotangent complex $L_{R/\mathbf{F}_p}$. In particular, the cohomology group $\mathrm{H}^2(L\Omega_R)$ contains the classical exterior power $\mathrm{H}^0(\bigwedge^2 L_{R/\mathbf{F}_p}) \simeq \Omega^2_R$ as a direct summand. This direct summand is nonzero (since $dx \wedge dy$ is a nonvanishing element of Ω^2_R, as noted in the proof of Proposition 6.3.1). □

6.5. **Seminormality.** Recall from that a ring R is *seminormal* if it is reduced and satisfies the following condition: given $x, y \in R$ with $x^2 = y^3$, we can find $t \in R$ with $x = t^3$ and $y = t^2$ (see [48]).

Proposition 6.5.1. *Let A^* be a strict Dieudonné algebra. Then the commutative ring A^0/VA^0 is seminormal.*

Proof. Lemma 3.6.1 asserts that A^0/VA^0 is reduced. Suppose we are given elements $x, y \in A^0/VA^0$ satisfying $x^2 = y^3$, classified by an $\mathbf{F}_p$-algebra homomorphism $u: R = \mathbf{F}_p[x,y]/(x^2 - y^3) \to A^0/VA^0$. Then we can identify f with a map of strict Dieudonné algebras $v: \mathcal{W}\Omega^*_R \to A^*$. By virtue of Proposition 6.2.1, the map v factors uniquely as a composition

$$\mathcal{W}\Omega^*_R \simeq \mathcal{W}\Omega^*_{\mathbf{F}_p[t]} \xrightarrow{v'} A^*.$$

It follows that u factors as a composition $R \hookrightarrow \mathbf{F}_p[t] \xrightarrow{u'} A^0/VA^0$: that is, there exists an element $t \in A^0/VA^0$ satisfying $x = t^3$ and $y = t^2$. □

Any commutative ring R admits a universal map $R \to R^{\mathrm{sn}}$ to a seminormal ring by [48, Theorem 4.1]; we refer to R^{sn} as the *seminormalization* of R.

Corollary 6.5.2. *Let $f: R \to S$ be a map of $\mathbf{F}_p$-algebras which induces an isomorphism of seminormalizations $R^{\mathrm{sn}} \to S^{\mathrm{sn}}$. Then the induced map $\rho: \mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_S$ is an isomorphism of strict Dieudonné algebras.*

Proof. For any strict Dieudonné algebra A^*, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(S^{\mathrm{sn}}, A^0/VA^0) \arrow[r] \arrow[d] & \mathrm{Hom}(R^{\mathrm{sn}}, A^0/VA^0) \arrow[d] \\
\mathrm{Hom}(S, A^0/VA^0) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(R, A^0/VA^0) \arrow[d, "\sim"] \\
\mathrm{Hom}_{\mathbf{DA}}(\mathcal{W}\Omega^*_S, A^*) \arrow[r] & \mathrm{Hom}_{\mathbf{DA}}(\mathcal{W}\Omega^*_R, A^*)
\end{tikzcd}$$

where the top vertical maps are bijective by Proposition 6.5.1 and the top horizontal map is bijective by virtue of our assumption that f induces an isomorphism of seminormalizations. It follows that the bottom horizontal map is also bijective. $\square$

We will show that Corollary 6.5.2 is, in some sense, optimal:

Theorem 6.5.3. *Let R be a commutative $\mathbf{F}_p$-algebra. Then the unit map $R \to \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R$ exhibits $\mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R$ as a seminormalization of R.*

Remark 6.5.4. Recall that we have a pair of adjoint functors

$$\mathrm{CAlg}_{\mathbf{F}_p} \underset{A^0/VA^0 \leftarrow A^*}{\overset{R \mapsto \mathcal{W}\Omega^*_R}{\rightleftarrows}} \mathbf{DA}_{\mathrm{str}}$$

(see Corollary 4.1.5). It follows from formal categorical considerations that this adjunction restricts to an equivalence of categories $\mathcal{C} \simeq \mathcal{D}$, where $\mathcal{C} \subseteq \mathrm{CAlg}_{\mathbf{F}_p}$ is the full subcategory spanned by those commutative $\mathbf{F}_p$-algebras R for which the unit map $R \to \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R$ is an isomorphism, and $\mathcal{D} \subseteq \mathbf{DA}_{\mathrm{str}}$ is the full subcategory spanned by those strict Dieudonné algebras A^* for which the counit map $\mathcal{W}\Omega^*_{A^0/VA^0} \to A^*$ is an isomorphism. It follows from Theorem 6.5.3 that a commutative $\mathbf{F}_p$-algebra belongs to $\mathcal{C}$ if and only if it is seminormal. It then follows from Proposition 6.5.1 that the functor $A^* \mapsto A^0/VA^0$ takes values in $\mathcal{C}$, and from Corollary 6.5.2 that the functor $\mathcal{W}\Omega^*$ takes values in $\mathcal{D}$. In particular, the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{CAlg}_{\mathbf{F}_p}$ admits a left adjoint (given by $R \mapsto \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R \simeq R^{\mathrm{sn}}$), and the inclusion functor $\mathcal{D} \hookrightarrow \mathbf{DA}_{\mathrm{str}}$ admits a right adjoint (given by $A^* \mapsto \mathcal{W}\Omega^*_{A^0/VA^0}$). In other words, the adjunction between $\mathrm{CAlg}_{\mathbf{F}_p}$ and $\mathbf{DA}_{\mathrm{str}}$ is *idempotent.*

Remark 6.5.5. Theorem 6.5.3 gives a clean and intrinsic description of the seminormalization R^{sn} of an $\mathbf{F}_p$-algebra R in terms of the ring of Witt vectors $W(R)$. Assume that R is reduced, so that $W(R)$ is p-torsion-free. For each $n \geq 0$, let $W(R)^{(n)}$ denote the subring of $W(R)$ consisting of those elements f which satisfy

$$df \in p^n\big(\Omega^1_{W(R)}/\mathrm{torsion}\big).$$

Then the Witt vector Frobenius F carries $W(R)^{(n)}$ into $W(R)^{(n+1)}$, and the Verschiebung V carries $W(R)^{(n+1)}$ into $W(R)^{(n)}$. In particular, the colimit

$$W(R)^{(\infty)} := \varinjlim_{n} \left(W(R)^{(0)} \xrightarrow{F} W(R)^{(1)} \xrightarrow{F} W(R)^{(2)} \xrightarrow{F} \dots \right)$$

comes equipped with an endomorphism $V: W(R)^{(\infty)} \to W(R)^{(\infty)}$. Unwinding definitions, we have $W(R)^{(\infty)} \simeq \mathrm{Sat}(\Omega^*_{W(R)})^0$, so that we have an isomorphism $R^{\mathrm{sn}} \simeq W(R)^{(\infty)}/VW(R)^{(\infty)}$ by Theorem 6.5.3 and the proof of Proposition 4.1.4.

6.6. **The Proof of Theorem 6.5.3.** We begin by introducing a (temporary) bit of notation.

Notation 6.6.1. For every commutative $\mathbf{F}_p$-algebra R, we let $\Psi(R)$ denote the commutative $\mathbf{F}_p$-algebra $\mathcal{W}\Omega^0_R / V\mathcal{W}\Omega^0_R$, so that the definition of the saturated de Rham-Witt complex supplies a tautological map $u_R: R \to \Psi(R)$.

It follows from Proposition 6.5.1 that for any commutative $\mathbf{F}_p$-algebra R, the R-algebra $\Psi(R)$ is seminormal. We wish to show that u_R exhibits $\Psi(R)$ as a seminormalization of R. The proof proceeds in several steps.

Remark 6.6.2. By virtue of Proposition 4.3.3, the functor $R \mapsto \Psi(R)$ commutes with filtered colimits.

Remark 6.6.3. Let k be a perfect $\mathbf{F}_p$-algebra. Then, for any smooth k-algebra R, the unit map $u_R: R \to \Psi(R)$ is an isomorphism (Proposition 4.3.2). In particular, if R is perfect, then the map $u_R: R \to \Psi(R)$ is an isomorphism.

Lemma 6.6.4. *Let R be a commutative $\mathbf{F}_p$-algebra and let $\varphi: R \to R$ be the Frobenius endomorphism. Then $\Psi(\varphi): \Psi(R) \to \Psi(R)$ is the Frobenius endomorphism of $\Psi(R)$.*

Proof. For every map of Dieudonné algebras $f: \mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_R$, let $f_0: \Psi(R) \to \Psi(R)$ be the induced map. From the universal property of the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$, we see that there is a unique map of Dieudonné algebras $f: \mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_R$ for which the diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "u_R"] & R \arrow[d, "u_R"] \\ \Psi(R) \arrow[r, "f_0"] & \Psi(R) \end{tikzcd}$$

commutes, and $\Psi(\varphi)$ is then given by f_0. Consequently, to show that $\Psi(\varphi)$ coincides with the Frobenius map on $\Psi(R)$, it will suffice to construct a map of Dieudonné algebras $f: \mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_R$ which induces the Frobenius on $\Psi(R)$. We conclude by observing that such a map exists, given by $f(x) = p^n F(x)$ for $x \in \mathcal{W}\Omega^n_R$. $\square$

Corollary 6.6.5. *Let R be any $\mathbf{F}_p$-algebra. Then the canonical map $u_R: R \to \Psi(R)$ induces an isomorphism of perfections $R^{1/p^\infty} \to \Psi(R)^{1/p^\infty}$.*

Proof. Combine Lemma 6.6.4 with Remarks 6.6.2 and 6.6.3. □

Corollary 6.6.6. *Let R be a commutative $\mathbf{F}_p$-algebra and let S be a reduced R-algebra. If there exists an R-algebra homomorphism $f: \Psi(R) \to S$, then f is unique.*

Proof. Suppose that we have a pair of R-algebra homomorphisms $f, g: \Psi(R) \to S$. Passing to perfections, we obtain a pair of R^{1/p^∞}-algebra homomorphisms $f^{1/p^\infty}, g^{1/p^\infty}: \Psi(R)^{1/p^\infty} \to S^{1/p^\infty}$. It follows from Corollary 6.6.5 that $f^{1/p^\infty} = g^{1/p^\infty}$, so that f and g agree after composition with the map $S \to S^{1/p^\infty}$. Since S is reduced, this map is injective; it follows that $f = g$. □

Proposition 6.6.7. *Let R be a commutative $\mathbf{F}_p$-algebra and let k be a field of characteristic p. Then every ring homomorphism $f: R \to k$ factors uniquely through $u_R: R \to \Psi(R)$.*

Proof. We have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "u_R"] \arrow[d, "f"] & \Psi(R) \arrow[d] \\ k \arrow[r, "u_k"] & \Psi(k). \end{tikzcd}$$

Since k can be written as a filtered colimit of smooth $\mathbf{F}_p$-algebras, the map u_k is an isomorphism (Remarks 6.6.2 and 6.6.3). It follows that f factors through u_R; the uniqueness of the factorization follows from Corollary 6.6.6. □

Proposition 6.6.8. *Let R be a finitely generated reduced $\mathbf{F}_p$-algebra, let $K(R)$ denote the total ring of fractions of R (that is, the product of the residue fields of the generic points of* $\mathrm{Spec}(R)$*), and let $\overline{R}$ be the integral closure of R in $K(R)$. Then the canonical map $R \to \overline{R}$ factors uniquely through u_R.*

Proof. Let x be a generic point of $\mathrm{Spec}(R)$. Using Proposition 6.6.7, we see that there exists an R-algebra homomorphism $f_x: \Psi(R) \to \kappa(x)$. Let $A(x)$ denote the integral closure of R in $\kappa(x)$. Then $A(x)$ is an integrally closed Noetherian domain, and can therefore be written as an intersection $\bigcap_{\mathfrak{p}} A(x)_{\mathfrak{p}}$ where $\mathfrak{p}$ ranges over the height 1 primes of $A(x)$. Note that each $A(x)_{\mathfrak{p}}$ is regular and can therefore be realized as a filtered colimit of smooth $\mathbf{F}_p$-algebras. We have a commutative diagram

$$\begin{tikzcd} R \arrow[r, "u_R"] \arrow[d] & \Psi(R) \arrow[d] \\ A(x)_{\mathfrak{p}} \arrow[r, "u_{A(x)_{\mathfrak{p}}}"] & \Psi(A(x)_{\mathfrak{p}}) \end{tikzcd}$$

where the bottom horizontal map is an isomorphism (Remarks 6.6.2 and 6.6.3). It follows that there exists an R-algebra homomorphism $g\colon \Psi(R) \to A(x)_{\mathfrak{p}}$. Using Corollary 6.6.6, we see that the composition $\Psi(R) \xrightarrow{g} A(x)_{\mathfrak{p}} \hookrightarrow \kappa(x)$ coincides with f_x. It follows that f_x factors through the valuation ring $A(x)_{\mathfrak{p}}$. Allowing $\mathfrak{p}$ to vary, we conclude that f_x factors through the subalgebra $A(x) \subseteq \kappa(x)$. Forming a product as x varies, we obtain an R-algebra homomorphism $\Psi(R) \to \overline{R}$; by virtue of Corollary 6.6.6, this homomorphism is unique. □

Corollary 6.6.9. *Let R be a commutative $\mathbf{F}_p$-algebra. Then the unit map $u_R\colon R \to \Psi(R)$ is integral.*

Proof. Combine Proposition 6.6.8 with Remark 6.6.2. □

Proof of Theorem 6.5.2. Let R be a commutative $\mathbf{F}_p$-algebra; we wish to show that the canonical map $R \to \Psi(R)$ exhibits $\Psi(R)$ as a seminormalization of R. By virtue of Corollary 6.5.2, we can replace R by R^{sn} and thereby reduce to the case where R is seminormal. In particular, R is reduced, so the map u exhibits R as a subring of $\Psi(R)$ (Corollary 6.6.5). It follows from Corollary 6.6.9 and Proposition 6.6.7 that the map $u_R\colon R \to \Psi(R)$ is *subnormal*, in the sense of [48]: that is it is an integral ring homomorphism which induces a bijection $\mathrm{Spec}(\Psi(R))(k) \to \mathrm{Spec}(R)(k)$ for every field k. Since R is seminormal, it follows that the map u_R is an isomorphism ([48, Lemma 2.6]). □

Part 2. Complements and Applications

7. Homological Algebra

For every cochain complex M^* of p-torsion-free abelian groups, let $(\eta_p M)^*$ denote the subcomplex of $M^*[1/p]$ introduced in Construction 2.1.3. Recall that endowing M^* with the structure of a Dieudonné complex is equivalent to choosing a map $\alpha\colon M^* \to (\eta_p M)^*$ (Remark 2.1.4), which is an isomorphism if and only if the Dieudonné complex M^* is saturated. We can reformulate this observation as follows: the category $\mathbf{DC}_{\mathrm{sat}}$ of saturated Dieudonné complexes can be identified with the category of *fixed points* for the functor η_p on p-torsion-free cochain complexes (see Definition 7.3.1 and Example 7.3.3).

In this section, we will study a variant of this construction, where we replace the category of cochain complexes $\mathrm{Chain}(\mathbf{Z})$ by the *derived category* $D(\mathbf{Z})$ obtained by formally inverting quasi-isomorphisms. The construction $M^* \mapsto (\eta_p M)^*$ induces an endofunctor of the derived category $L\eta_p\colon D(\mathbf{Z}) \to D(\mathbf{Z})$, which was originally introduced in the work of Berthelot-Ogus ([6]). Every cochain complex M^* can be regarded as an object of the derived category $D(\mathbf{Z})$, and every saturated Dieudonné complex can be regarded as an object of the category $D(\mathbf{Z})^{L\eta_p}$ of fixed points for the action of $L\eta_p$ on $D(\mathbf{Z})$. The main result of this section is that this construction restricts to an equivalence of categories

$$\{\text{Strict Dieudonné Complexes}\} \simeq \{p\text{-Complete Fixed Points of } L\eta_p\}$$

(see Theorem 7.3.4 for a precise statement; we also prove an ∞-categorical refinement as Theorem 7.4.8). In other words, we show that a p-complete object M of the derived category $D(\mathbf{Z})$ which is equipped with an isomorphism $\alpha\colon M \simeq L\eta_p M$ (in the derived category) a canonical representative M^* at the level of cochain complexes (which is determined up to isomorphism by the requirement that α lifts to an isomorphism of cochain complexes $M^* \simeq (\eta_p M)^*$). In particular, the de Rham-Witt complex $W\Omega^*_R$ of a smooth algebra R over a perfect field k can be functorially reconstructed from the crystalline cochain complex $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$, together with the isomorphism $R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \simeq L\eta_p R\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$. In §10, we will apply this observation to give a simple construction of the crystalline comparison isomorphism for the cohomology theory $A\Omega$ of Bhatt-Morrow-Scholze [11].

7.1. p-Complete Objects of the Derived Category.

We begin with some recollections on the derived category of abelian groups. For a modern textbook reference, we refer the reader to [31, Ch. 13–14].

Definition 7.1.1. The derived category $D(\mathbf{Z})$ of the integers can be defined in two equivalent ways:

(1) Let $\mathrm{Chain}(\mathbf{Z})$ denote the category of cochain complexes of abelian groups. Then $D(\mathbf{Z})$ is the localization $\mathrm{Chain}(\mathbf{Z})[W^{-1}]$, where W is the collection

of all quasi-isomorphisms in $\mathrm{Chain}(\mathbf{Z})$. In other words, $D(\mathbf{Z})$ is obtained from $\mathrm{Chain}(\mathbf{Z})$ by formally adjoining inverses of quasi-isomorphisms.

(2) Let $\mathrm{Chain}(\mathbf{Z})^{\mathrm{free}} \subseteq \mathrm{Chain}(\mathbf{Z})$ denote the full subcategory spanned by the cochain complexes of *free* abelian groups. Then $D(\mathbf{Z})$ is equivalent to the homotopy category of $\mathrm{Chain}(\mathbf{Z})^{\mathrm{free}}$. That is, given a pair of objects $X^*, Y^* \in \mathrm{Chain}(\mathbf{Z})^{\mathrm{free}}$, we can identify $\mathrm{Hom}_{D(\mathbf{Z})}(X, Y)$ with the abelian group of chain homotopy classes of maps of cochain complexes $X^* \to Y^*$ (in fact, this holds more generally if X^* is a cochain complex of free abelian groups and Y^* is arbitrary).

Using description (2), we see that $D(\mathbf{Z})$ admits a symmetric monoidal structure, given by the usual tensor product of cochain complexes (of free abelian groups). We will denote the underlying tensor product functor by

$$\overset{L}{\otimes}: D(\mathbf{Z}) \times D(\mathbf{Z}) \to D(\mathbf{Z}).$$

Variant 7.1.2. In version (1) of Definition 7.1.1, one can replace the category $\mathrm{Chain}(\mathbf{Z})$ of all cochain complexes with various subcategories. For example, $D(\mathbf{Z})$ can be obtained from $\mathrm{Chain}(\mathbf{Z})^{\mathrm{free}}$ by formally inverting all quasi-isomorphisms. It will be helpful for us to know that $D(\mathbf{Z})$ can also be obtained from the intermediate subcategory $\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \subseteq \mathrm{Chain}(\mathbf{Z})$ consisting of cochain complexes of *torsion-free* abelian groups (again by formally inverting quasi-isomorphisms).

Notation 7.1.3. In what follows, it will be convenient to distinguish between a cochain complex of abelian groups $M^* \in \mathrm{Chain}(\mathbf{Z})$ and its image in the derived category $D(\mathbf{Z})$. We will typically denote the latter simply by M (that is, by omitting the superscript $*$).

Remark 7.1.4. In description (2) of Definition 7.1.1, we have implicitly used the fact that the ring $\mathbf{Z}$ of integers has finite projective dimension. If we were to replace $\mathbf{Z}$ by a more general ring R (not assumed to be of finite projective dimension), then we should also replace $\mathrm{Chain}(\mathbf{Z})^{\mathrm{free}}$ by the category of *K-projective* complexes in the sense of [46]. See also [24, Ch. 2] for a presentation as a model category.

Next, we recall the notion of a *p-complete* object in $D(\mathbf{Z})$, as in [18] and going back to ideas of Bousfield [17].

Definition 7.1.5. Let X be an object of $D(\mathbf{Z})$. We will say that X is *p-complete* if for every object $Y \in D(\mathbf{Z})$ such that $p: Y \to Y$ is an isomorphism in $D(\mathbf{Z})$ (i.e., such that Y arises via restriction of scalars from $D(\mathbf{Z}[1/p])$), we have $\mathrm{Hom}_{D(\mathbf{Z})}(Y, X) = 0$. We let $\widehat{D(\mathbf{Z})}_p \subseteq D(\mathbf{Z})$ be the full subcategory spanned by the p-complete objects.

Remark 7.1.6 (Derived p-Completion)**.** The inclusion functor $\widehat{D(\mathbf{Z})}_p \hookrightarrow D(\mathbf{Z})$ admits a left adjoint, which we will denote by $X \mapsto \widehat{X}_p$ and refer to as the *derived*

p-completion functor. Concretely, if X^* is a p-torsion-free cochain complex representing an object $X \in D(\mathbf{Z})$, then the derived p-completion $\widehat{X}_p$ is represented by the cochain complex $\widehat{X}_p^*$, obtained from X^* by levelwise p-adic completion.

Definition 7.1.7. Let X be an abelian group. We say that X is *derived p-complete* if, when considered as an object of $D(\mathbf{Z})$ concentrated in a single degree, it is p-complete.

Remark 7.1.8. Let X be an abelian group. Then X is derived p-complete if and only if

$$\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[1/p], X) \simeq 0 \simeq \mathrm{Ext}_{\mathbf{Z}}(\mathbf{Z}[1/p], X).$$

In other words, X is derived p-complete if and only if every short exact sequence $0 \to X \to M \to \mathbf{Z}[1/p] \to 0$ admits a unique splitting.

Remark 7.1.9. Let X be an object of $D(\mathbf{Z})$. Then X can be written (non-canonically) as a product $\prod_{n \in \mathbf{Z}} \mathrm{H}^n(X)[-n]$. It follows that X is p-complete (in the sense of Definition 7.1.5) if and only if each cohomology group $\mathrm{H}^n(X)$ is derived p-complete (in the sense of Definition 7.1.7). See [18, Prop. 5.2].

Remark 7.1.10. Let M be an abelian group. We say that M is *p-adically separated* if the canonical map $\rho: M \to \varprojlim M/p^n M$ is injective, and *p-adically complete* if ρ is an isomorphism. Then an abelian group M is p-adically complete if and only if it is p-adically separated and derived p-complete. In particular, every p-adically complete abelian group is derived p-complete. However, the converse is not true.

Remark 7.1.11. The category of derived p-complete abelian groups is an abelian category, stable under the formation of kernels, cokernels and extensions in the larger category of all abelian groups. For a detailed discussion of the theory of derived p-complete (also called *L-complete*) abelian groups, we refer the reader to [25, Appendix A].

Example 7.1.12. Let X^* be a cochain complex. Suppose that for each $n \in \mathbf{Z}$, the abelian group X^n is derived p-complete. Then each cohomology group $\mathrm{H}^n(X)$ is derived p-complete (Remark 7.1.11), so that $X \in D(\mathbf{Z})$ is p-complete (Remark 7.1.9).

In particular, if each X^n is a p-adically complete abelian group, then X is a p-complete object of the derived category $D(\mathbf{Z})$.

Proposition 7.1.13. *Let M be an abelian group. The following conditions are equivalent:*

- (a) *The abelian group M is isomorphic to the p-adic completion of a free abelian group.*
- (b) *The abelian group M is derived p-complete and p-torsion free.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. Conversely, suppose that (b) is satisfied. Choose a system of elements $\{x_i\}_{i \in I}$ in M which form a basis for M/pM as a vector space over $\mathbf{F}_p$. Let F denote the free abelian group on the generators $\{x_i\}_{i \in I}$, so that we have a canonical map $u: F \to M$ which induces an isomorphism $F/pF \to M/pM$. It follows from (b) that u induces an isomorphism of M with the p-adic completion of F. □

Definition 7.1.14. We will say that an abelian group M is *pro-free* if it satisfies the equivalent conditions of Proposition 7.1.13.

Example 7.1.15. Let M be a pro-free abelian group and let $M_0 \subseteq M$ be a subgroup. If M_0 is derived p-complete, then it is also pro-free.

We close this section by showing that the p-complete derived category $\widehat{D(\mathbf{Z})}_p$ can be realized as the homotopy category of chain complexes of pro-free abelian groups. Closely related results appear in appear in [45, 49] (where a model category structure is produced) and [39, Sec. 7.3.7]; for convenience of the reader, we spell out the details in this special case.

Proposition 7.1.16. *Let* $\mathrm{Chain}(\mathbf{Z})^{\text{pro-free}} \subseteq \mathrm{Chain}(\mathbf{Z})$ *denote the full subcategory whose objects are cochain complexes* X^* *of pro-free abelian groups. Then the construction* $X^* \to X$ *determines a functor* $\mathrm{Chain}(\mathbf{Z})^{\text{pro-free}} \to \widehat{D(\mathbf{Z})}_p$ *which exhibits* $\widehat{D(\mathbf{Z})}_p$ *as the homotopy category of* $\mathrm{Chain}(\mathbf{Z})^{\text{pro-free}}$. *In other words:*

(a) *An object of* $D(\mathbf{Z})$ *is* p*-complete if and only if it is isomorphic (in* $D(\mathbf{Z})$*) to a cochain complex of pro-free abelian groups.*

(b) *For every pair of objects* $X^*, Y^* \in \mathrm{Chain}(\mathbf{Z})^{\text{pro-free}}$, *the canonical map*

$$\mathrm{Hom}_{\mathrm{Chain}(\mathbf{Z})}(X^*, Y^*) \to \mathrm{Hom}_{D(\mathbf{Z})}(X, Y)$$

is a surjection, and two chain maps $f, g: X^* \to Y^*$ *have the same image in* $\mathrm{Hom}_{D(\mathbf{Z})}(X, Y)$ *if and only if they are chain homotopic.*

Lemma 7.1.17. *Let* X^* *be a cochain complex of free abelian groups. Suppose the underlying object* $X \in D(\mathbf{Z})$ *is* p*-complete. Let* $\widehat{X}^*_p$ *denote the (levelwise)* p*-completion of the cochain complex* X^*. *Then* $X^* \to \widehat{X}^*_p$ *is a quasi-isomorphism.*

Proof. Since both X^* and $\widehat{X}^*_p$ are p-complete objects of $D(\mathbf{Z})$, it suffices to show that the map $X^* \to \widehat{X}^*_p$ becomes a quasi-isomorphism after taking the derived tensor product with $\mathbf{Z}/p\mathbf{Z}$. Since both X^* and $\widehat{X}^*_p$ are p-torsion free, the derived tensor product coincides with the usual tensor product: that is, we are reduced to showing that the map of cochain complexes $X^*/pX^* \to \widehat{X}^*_p/p\widehat{X}^*_p$ is a quasi-isomorphism. In fact, this map is even an isomorphism. □

Lemma 7.1.18. *Let* X^* *be a cochain complex of pro-free abelian groups. Then* X^* *splits (noncanonically) as a direct sum* $\bigoplus_{n \in \mathbf{Z}} X(n)^*$, *where each* $X(n)^*$ *is a cochain complex of pro-free abelian groups concentrated in degrees* n *and* $n-1$.

Proof. For every integer n, let $B^n \subseteq X^n$ denote the image of the differential $d: X^{n-1} \to X^n$. Then B^n is derived p-complete (Remark 7.1.11). Since X^n is pro-free, it follows that B^n is also pro-free (Example 7.1.15). It follows that the surjection $d: X^{n-1} \to B^n$ splits: that is, we can choose a subgroup $Y^{n-1} \subseteq X^{n-1}$ for which the differential d on X^* restricts to an isomorphism $Y^{n-1} \simeq B^{n-1}$. Let $X(n)^*$ denote the subcomplex of X^* given by $Y^{n-1} \oplus \ker(d: X^n \to X^{n+1})$. It is then easy to verify that these subcomplexes determine a splitting $X^* \simeq \bigoplus_{n \in \mathbf{Z}} X(n)^*$. □

Proof of Proposition 7.1.16. We first prove (a). The "if" direction follows from Example 7.1.12. To prove the converse, let X be an object of $\widehat{D(\mathbf{Z})}_p$; we wish to show that X belongs to the essential image of the functor $\mathrm{Chain}(\mathbf{Z})^{\text{pro-free}} \to \widehat{D(\mathbf{Z})}_p$. Without loss of generality, we may assume that X is represented by a cochain complex X^* of free abelian groups. Let $\widehat{X}^*_p$ be the p-adic completion of X^*. Lemma 7.1.17 implies that X is isomorphic to $\widehat{X}_p$ as an object of $D(\mathbf{Z})$, and therefore belongs to the essential image of the functor $\mathrm{Chain}(\mathbf{Z})^{\text{pro-free}} \to D(\mathbf{Z})$.

We now prove (b). Let X^* be a cochain complex of pro-free abelian groups; we wish to show that for every cochain complex of pro-free abelian groups Y^*, the canonical map

$$\rho: \mathrm{Hom}_{\mathrm{hChain}(\mathbf{Z})}(X^*, Y^*) \to \mathrm{Hom}_{D(\mathbf{Z})}(X, Y)$$

is bijective; here $\mathrm{hChain}(\mathbf{Z})$ denote the homotopy category of cochain complexes. Using Lemma 7.1.18, we can reduce to the case where X^* is concentrated in degrees n and $n-1$, for some integer n. Using the (levelwise split) exact sequence of cochain complexes

$$0 \to X^n[-n] \to X^* \to X^{n-1}[-n+1] \to 0,$$

we can further reduce to the case where X^* is concentrated in a single degree n. In this case, we can write X^* as the p-adic completion of a chain complex $\widetilde{X}^*$ of free abelian groups. In this case, ρ can be identified with the restriction map

$$\mathrm{Hom}_{\mathrm{hChain}(\mathbf{Z})}(X^*, Y^*) \to \mathrm{Hom}_{\mathrm{hChain}(\mathbf{Z})}(\widetilde{X}^*, Y^*),$$

which is a bijective by virtue of the fact that each term of Y^* is p-adically complete. □

7.2. **The Functor $L\eta_p$.** For every cochain complex M^* of p-torsion-free abelian groups, we let $\eta_p M^*$ denote the subcomplex of $M^*[1/p]$ defined in Construction 2.1.3, given by $(\eta_p M)^n = \{x \in p^n M^n : dx \in p^{n+1} M^{n+1}\}$. The construction $M^* \mapsto \eta_p M^*$ determines a functor

$$\eta_p: \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \to \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}.$$

In this section, we discuss a corresponding operation on the derived category $D(\mathbf{Z})$, which we will denote by $L\eta_p$. For a more extensive discussion, we refer the reader to [11, Sec. 6].

We begin with an elementary property of the functor η_p:

Proposition 7.2.1. *For every cochain complex (M^*, d) of p-torsion-free abelian groups, then there is a canonical isomorphism of graded abelian groups*

$$\mathrm{H}^*(\eta_p M) \simeq \mathrm{H}^*(M)/\mathrm{H}^*(M)[p],$$

where $\mathrm{H}^*(M)[p]$ denotes the p-torsion subgroup of $\mathrm{H}^*(M)$

Proof. For every integer n, the construction $z \mapsto p^n z$ determines a bijection from the set of cocycles in M^n to the set of cocycles in $(\eta_p M)^n$. Note that if $z = dy$ is a boundary in M^n, then $p^n z = d(p^n y)$ is a boundary in $(\eta_p M)^n$; it follows that the construction $z \mapsto p^n z$ induces a surjection of cohomology groups $\mathrm{H}^n(M) \to \mathrm{H}^n(\eta_p M)$. By definition, the kernel of this map consists of those cohomology classes $[z]$ which are represented by cocycles $z \in M^n$ for which $p^n z = d(p^{n-1}x)$ for some $x \in M^{n-1}$, which is equivalent to the requirement that $p[z] = 0$ in $\mathrm{H}^n(M)$. □

Corollary 7.2.2. *Let $f: M^* \to N^*$ be a map between p-torsion-free cochain complexes of abelian groups. If f is a quasi-isomorphism, then the induced map $(\eta_p M)^* \to (\eta_p N)^*$ is also a quasi-isomorphism.*

Combining Corollary 7.2.2 with Variant 7.1.2, we see that η_p induces an endofunctor of the derived category $D(\mathbf{Z})$:

Corollary 7.2.3. *There is an essentially unique functor $L\eta_p: D(\mathbf{Z}) \to D(\mathbf{Z})$ for which the diagram of categories*

$$\begin{tikzcd} \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \arrow[r, "\eta_p"] \arrow[d] & \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \arrow[d] \\ D(\mathbf{Z}) \arrow[r, "L\eta_p"] & D(\mathbf{Z}) \end{tikzcd}$$

commutes up to isomorphism.

Given an object $X \in D(\mathbf{Z})$, we have $\mathrm{H}^*(L\eta_p X) \simeq \mathrm{H}^*(X)/\mathrm{H}^*(X)[p]$ by Proposition 7.2.1. In particular, $L\eta_p$ is an operator on $D(\mathbf{Z})$ which reduces the p-torsion by a small amount.

We close this section with the following (see [11, Lem. 6.19]):

Proposition 7.2.4. *The functor $L\eta_p$ commutes with the derived p-completion functor of Remark 7.1.6. More precisely, if $M \to N$ is a morphism in $D(\mathbf{Z})$ which exhibits N as a derived p-completion of N, then the induced map $L\eta_p M \to L\eta_p N$ exhibits $L\eta_p N$ as a derived p-completion of $L\eta_p M$. In particular, the functor $L\eta_p$ carries p-complete objects of $D(\mathbf{Z})$ to p-complete objects of $D(\mathbf{Z})$.*

Proof. Let M be an object of $D(\mathbf{Z})$, which we can represent by a cochain complex M^* of free abelian groups. Let $\widehat{M}^*_p$ denote the cochain complex obtained from

M^* by levelwise p-completion, so that the canonical map $M^* \to \widehat{M}_p^*$ exhibits $\widehat{M}_p$ as a derived p-completion of M in $D(\mathbf{Z})$. We wish to show that the induced map $L\eta_p M \to L\eta_p \widehat{M}_p^*$ exhibits $L\eta_p \widehat{M}_p^*$ as a derived p-completion of $L\eta_p M$. Since $(\eta_p M)^*$ is a cochain complex of free abelian groups, it suffices to observe that that the canonical map $(\eta_p M)^* \to (\eta_p \widehat{M}_p)^*$ exhibits $(\eta_p \widehat{M}_p)^*$ as the levelwise p-adic completion of $(\eta_p M)^*$. □

7.3. **Fixed Points of $L\eta_p$: 1–Categorical Version.** In this section, we explain that the category $\mathbf{DC}_{\mathrm{str}}$ of strict Dieudonné complexes can be realized as the *fixed points* for the operator $L\eta_p$ on the p-complete derived category $\widehat{D(\mathbf{Z})}_p$ (Theorem 7.3.4). In particular, this implies that every fixed point of $L\eta_p$ of $\widehat{D(\mathbf{Z})}_p$ admits a canonical representative in the category $\mathrm{Chain}(\mathbf{Z})$ of cochain complexes.

We use the following definition of fixed points.

Definition 7.3.1. Let $\mathcal{C}$ be a category and $T: \mathcal{C} \to \mathcal{C}$ be an endofunctor. The *fixed point category* $\mathcal{C}^T$ is defined as follows:

- The objects of $\mathcal{C}^T$ are pairs (X, φ), where X is an object of $\mathcal{C}$ and $\varphi: X \simeq T(X)$ is an isomorphism.
- A morphism from (X, φ) to (X', φ') in $\mathcal{C}^T$ is a morphism $f: X \to X'$ in the category $\mathcal{C}$ with the property that the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r, "\varphi'"] & T(X') \end{tikzcd}$$

commutes.

Remark 7.3.2 (Functoriality)**.** Let $\mathcal{C}$ and $\mathcal{C}'$ be categories equipped with endofunctors $T: \mathcal{C} \to \mathcal{C}$ and $T': \mathcal{C}' \to \mathcal{C}'$. Suppose we are given a functor $U: \mathcal{C} \to \mathcal{C}'$ for which the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "T"] \arrow[d, "U"] & \mathcal{C} \arrow[d, "U"] \\ \mathcal{C}' \arrow[r, "T'"] & \mathcal{C}' \end{tikzcd}$$

commutes up to (specified) isomorphism. Then U induces a functor of fixed point categories $\mathcal{C}^T \to \mathcal{C}'^{T'}$, given on objects by the construction $(C, \varphi) \mapsto (U(C), U(\varphi))$.

Example 7.3.3. Let $\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}$ denote the category of torsion-free cochain complexes, and let $\eta_p: \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \to \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}$ be the functor of Construction 2.1.3. If M^* is a torsion-free cochain complex, then endowing M^* with the structure of a Dieudonné complex (M^*, F) is equivalent to supplying a map of cochain complexes $\alpha_F: M^* \to \eta_p M^*$ (Remark 2.1.4). Moreover, the Dieudonné complex

(M^*, F) is saturated if and only if the map α_F is an isomorphism. It follows that we can identify the category $\mathbf{DC}_{\mathrm{sat}}$ of saturated Dieudonné complexes with the fixed point category $(\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}})^{\eta_p}$ of Definition 7.3.1).

Combining the identification of Example 7.3.3 with Remark 7.3.2, we obtain a natural map

$$\mathbf{DC}_{\mathrm{sat}} \simeq (\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}})^{\eta_p} \to D(\mathbf{Z})^{L\eta_p}.$$

We can now state the main result of this section:

Theorem 7.3.4. *The map $\mathbf{DC}_{\mathrm{sat}} \to D(\mathbf{Z})^{L\eta_p}$ constructed above restricts to an equivalence of categories*

$$\mathbf{DC}_{\mathrm{str}} \to \widehat{D(\mathbf{Z})}_p^{L\eta_p}.$$

The proof of Theorem 7.3.4 will require some preliminaries.

Definition 7.3.5. Let X^*, Y^* be p-torsion free Dieudonné complexes. Let $f: X^* \to Y^*$ be a map of the underlying cochain complexes. We say that f is *weakly F-compatible* if the diagram of cochain complexes

$$\begin{tikzcd} X^* \arrow[r, "\alpha_F"] \arrow[d, "f"] & \eta_p X^* \arrow[d, "\eta_p f"] \\ Y^* \arrow[r, "\alpha_F"] & \eta_p Y^* \end{tikzcd}$$

commutes up to chain homotopy. Equivalently, if Y^* is saturated, we require that the map $F^{-1} \circ f \circ F$ of cochain complexes $X^* \to Y^*$ is chain homotopic to f.

The main ingredient in the proof of Theorem 7.3.4 is the following:

Proposition 7.3.6. *Let X^* and Y^* be Dieudonné complexes and let $f: X^* \to Y^*$ be a map which is weakly F-compatible. If Y^* is strict, then there is a unique map of Dieudonné complexes $\overline{f}: X^* \to Y^*$ which is chain homotopic to f.*

Proof. Let $u: X^* \to Y^{*-1}$ be an arbitrary map of graded abelian groups. Then, using $FdV = d$, we have

$$F^{-1} \circ (du + ud) \circ F = d(Vu) + (Vu)d.$$

Write $F^{-1} \circ f \circ F = f + dh + hd$ for some map $h: X^* \to Y^{*-1}$. We now set $u = \sum_{n=0}^{\infty} V^n h$ and $\overline{f} = f + du + ud$. It follows that

$$\begin{aligned} F^{-1} \circ \overline{f} \circ F - \overline{f} &= dh + hd + d(Vu) + (Vu)d - (du + ud) \\ &= dh + hd + \sum_{n=0}^{\infty} (dV^{n+1}h + V^{n+1}hd) - (\sum_{n=0}^{\infty} dV^n h + V^n hd) \\ &= 0. \end{aligned}$$

It follows that $\overline{f}: X^* \to Y^*$ is a map of Dieudonné complexes which is chain homotopic to f as a map of chain complexes.

To establish uniqueness, suppose that $g: X^* \to Y^*$ is a map of Dieudonné complexes which is chain homotopic to zero; we can then write $g = dh + hd$ for some map $h: X^* \to Y^{*-1}$. For each $r \geq 0$, we have

$$\begin{aligned} g &= F^{-r} \circ g \circ F^r \\ &= F^{-r}(dh + hd)F^r \\ &= dV^r h F^r + V^r h F^r d \\ &\in dV^r Y^* + V^r Y^*. \end{aligned}$$

It follows that the composite map $X^* \xrightarrow{g} Y^* \to \mathcal{W}_r(Y)^*$ vanishes for each $r \geq 0$, so that g vanishes by virtue of our assumption that Y^* is complete. $\square$

Proof of Theorem 7.3.4. Let X^* and Y^* be strict Dieudonné complexes, and let X and Y denote their images in $D(\mathbf{Z})$. Since X and Y are pro-free (in the sense of Definition 7.1.14), we can identify $\mathrm{Hom}_{D(\mathbf{Z})}(X, Y)$ with the set of chain homotopy equivalence classes of maps from X^* to Y^* (Proposition 7.1.16). It follows that we can identify $\mathrm{Hom}_{D(\mathbf{Z})^{L\eta_p}}$ with the set of chain homotopy equivalence classes of maps $f: X^* \to Y^*$ which are weakly F-compatible. Using Proposition 7.3.6, we deduce that the canonical map

$$\mathrm{Hom}_{\mathbf{DC}_{\mathrm{str}}}(X^*, Y^*) \to \mathrm{Hom}_{D(\mathbf{Z})^{L\eta_p}}(X, Y)$$

is bijective.

To complete the proof, we must show that every p-complete object $(X, \varphi) \in D(\mathbf{Z})^{L\eta_p}$ is isomorphic to a strict Dieudonné complex. Without loss of generality, we may assume that X is represented by a cochain complex X^* of free abelian groups. In this case, we can choose a quasi-isomorphism of cochain complexes $\alpha: X^* \to \eta_p X^*$ representing the isomorphism $\varphi: X \simeq L\eta_p X$. By virtue of Remark 2.1.4, this choice endows X^* with the structure of a Dieudonné complex. Using Corollary 7.2.2, we see that each of the transition maps in the diagram

$$X^* \xrightarrow{\alpha} (\eta_p X)^* \xrightarrow{\eta_p(\alpha)} (\eta_p \eta_p X)^* \xrightarrow{\eta_p(\eta_p(\alpha))} (\eta_p \eta_p \eta_p X)^* \to \cdots$$

is a quasi-isomorphism. It follows that the natural map $X^* \to \mathrm{Sat}(X^*)$ is a quasi-isomorphism. Since X is a p-complete object of $D(\mathbf{Z})$, it follows from Corollary 2.8.3 that the canonical map $\mathrm{Sat}(X^*) \to \mathcal{W}\mathrm{Sat}(X^*)$ is also a quasi-isomorphism. In particular, X is isomorphic to $\mathcal{W}\mathrm{Sat}(X)$ as an object of the fixed point category $\widehat{D(\mathbf{Z})}_p^{L\eta_p}$. $\square$

7.4. **Fixed Points of $L\eta_p$: ∞–Categorical Version.** From a homotopy-theoretic point of view, the definition of the fixed-point category $D(\mathbf{Z})^{L\eta_p}$ has an unnatural feature. Concretely, a morphism in $D(\mathbf{Z})^{L\eta_p}$ can be represented by a diagram of cochain complexes of free abelian groups

$$\begin{tikzcd} X^* \arrow[r, "\varphi"] \arrow[d, "f"] & (\eta_p X)^* \arrow[d, "\eta_p(f)"] \\ Y^* \arrow[r, "\varphi'"] & (\eta_p Y)^* \end{tikzcd}$$

which is required to commute up to chain homotopy, but the chain homotopy itself need not be specified. Typically, constructions which allow ambiguities of this nature will give rise to categories which are badly behaved. One can avoid this problem by contemplating a *homotopy coherent* variant of the fixed point construction, where the datum of a morphism from (X^*, φ) to (Y^*, φ') is require to also supply a homotopy $h: X^* \to (\eta_p Y)^{*-1}$ which witnesses the homotopy-commutativity of the diagram above. Our goal in this section is to show that, if we restrict our attention to the *p-complete* derived category $\widehat{D(\mathbf{Z})}_p$, then this modification is unnecessary: the fixed point category $\widehat{D(\mathbf{Z})}_p^{L\eta_p}$ given by Definition 7.3.1 agrees with its homotopy-coherent refinement. To give a precise formulation of this statement, it will be convenient to use the theory of ∞-categories (see [37] for a general discussion, and [38, Sec. 1.3] for a treatment of the derived ∞-category).

Notation 7.4.1. Given an ∞-category $\mathcal{D}$ and pair of morphisms $f, g: X \to Y$ in $\mathcal{D}$, we define an *equalizer* of f and g to be a limit of the diagram $X \rightrightarrows Y$, indexed by the ∞-category $\Delta^1 \cup_{\partial \Delta^1} \Delta^1$.

Example 7.4.2. Given spectra X, Y and a pair of maps $f, g: X \to Y$, the equalizer $\mathrm{eq}(f, g)$ can be identified with the fiber $\mathrm{fib}(f - g)$.

Definition 7.4.3. Let $\mathcal{C}$ be an ∞-category and let $T: \mathcal{C} \to \mathcal{C}$ be a functor. We define the ∞-category $\mathcal{C}^T$ of *fixed points of* T to be the equalizer of the diagram

$$\mathrm{id}_{\mathcal{C}}, T: \mathcal{C} \rightrightarrows \mathcal{C},$$

formed in the ∞-category of ∞-categories.

Remark 7.4.4. In the situation of Definition 7.4.3, we can use the Grothendieck construction [37, Ch. 2] to identify the pair $(\mathcal{C}, T)$ with a cocartesian fibration $\pi: \widetilde{\mathcal{C}} \to \Delta^1/\partial\Delta^1$, whose fiber over the unique vertex of $\Delta^1/\partial\Delta^1$ is the ∞-category $\mathcal{C}$. The fixed point ∞-category $\mathcal{C}^T$ can then be realized explicitly as the ∞-category of cocartesian sections of π.

We refer to [41, Sec. II.1] for a detailed treatment of fixed point ∞-categories.

Remark 7.4.5. By construction, an object of $\mathcal{C}^T$ is represented by a pair (X, φ_X) where $X \in \mathcal{C}$ and $\varphi_X: X \simeq TX$ is an equivalence. The space of maps between two objects $(X, \varphi_X), (Y, \varphi_Y)$ is can be described concretely as the (homotopy) equalizer of the pair of maps

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(X, TY),$$

given by $f \mapsto \varphi_Y \circ f$ and $f \mapsto F(f) \circ \varphi_X$, respectively (see [41, Prop. II.1.5(ii)]).

Notation 7.4.6. We let $\mathcal{D}(\mathbf{Z})$ denote the derived ∞-category of $\mathbf{Z}$-modules. This ∞-category has many equivalent descriptions, of which we single out the following:

(1) The ∞-category $\mathcal{D}(\mathbf{Z})$ can be obtained from the ordinary category $\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}$ of torsion-free cochain complexes by formally adjoining inverses of quasi-isomorphisms (here we can also replace the category $\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}$ by the larger category $\mathrm{Chain}(\mathbf{Z})$ of all cochain complexes, or the smaller category $\mathrm{Chain}(\mathbf{Z})^{\mathrm{free}}$ of cochain complexes of free abelian groups).
(2) The ∞-category $\mathcal{D}(\mathbf{Z})$ can be realized as the *differential graded nerve* (in the sense of [38, Sec. 1.3.1]) of $\mathrm{Chain}(\mathbf{Z})^{\mathrm{free}}$, regarded as a differential graded category (that is, a category enriched in cochain complexes).

The ∞-category $\mathcal{D}(\mathbf{Z})$ can be regarded as an "enhancement" of the derived category $D(\mathbf{Z})$ of abelian groups. In particular, there is a forgetful functor $\mathcal{D}(\mathbf{Z}) \to D(\mathbf{Z})$, which exhibits $D(\mathbf{Z})$ as the homotopy category of $\mathcal{D}(\mathbf{Z})$. We let $\widehat{\mathcal{D}(\mathbf{Z})}_p \subseteq \mathcal{D}(\mathbf{Z})$ denote the inverse image of the subcategory $\widehat{D(\mathbf{Z})}_p \subseteq D(\mathbf{Z})$ (that is, the full subcategory of $\mathcal{D}(\mathbf{Z})$ spanned by the p-complete objects).

Notation 7.4.7. It follows from Corollary 7.2.2 (and the first characterization of $\mathcal{D}(\mathbf{Z})$ given in Notation 7.4.6) that there is an essentially unique functor $L\eta_p \colon \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$ for which the diagram

$$\begin{tikzcd} \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \arrow[r, "\eta_p"] \arrow[d] & \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \arrow[d] \\ \mathcal{D}(\mathbf{Z}) \arrow[r, "L\eta_p"] & \mathcal{D}(\mathbf{Z}) \end{tikzcd}$$

commutes up to equivalence. Note that after passing to homotopy categories, the functor $L\eta_p \colon \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$ reduces to the functor $L\eta_p \colon D(\mathbf{Z}) \to D(\mathbf{Z})$ of Corollary 7.2.3. Moreover, the functor $L\eta_p$ carries the full subcategory $\widehat{\mathcal{D}(\mathbf{Z})}_p \subseteq \mathcal{D}(\mathbf{Z})$ into itself.

We can now formulate our main result:

Theorem 7.4.8. *The forgetful functor $\widehat{\mathcal{D}(\mathbf{Z})}_p \to \widehat{D(\mathbf{Z})}_p$ induces an equivalence of ∞-categories*

$$\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p} \to \widehat{D(\mathbf{Z})}_p^{L\eta_p}.$$

In particular, $\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$ is (equivalent to) an ordinary category.

Corollary 7.4.9. *The construction $M^* \mapsto M$ induces an equivalence of ∞-categories*

$$\mathbf{DC}_{\mathrm{str}} \to \widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}.$$

Proof. Combine Theorems 7.4.8 and 7.3.4. □

7.5. **The Proof of Theorem 7.4.8.** To prove Theorem 7.4.8, we will need a mechanism for computing spaces of morphisms in the fixed point ∞-category $\mathcal{D}(\mathbf{Z})^{L\eta_p}$. For this, it will be convenient to work with a certain model for $\mathcal{D}(\mathbf{Z})$ at the level of differential graded categories.

Definition 7.5.1. We let $\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})$ be the *differential graded category* whose objects are the cochain complexes of free abelian groups. Given objects $X^*, Y^* \in \mathcal{D}(\mathbf{Z})$, we let $\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*)$ denote the truncation $\tau^{\leq 0}[X^*, Y^*]$, where $[X^*, Y^*]$ denotes the usual cochain complex of maps from X^* to Y^*. More concretely, we have

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*)^r \simeq \begin{cases} 0 & \text{if } r > 0 \\ \mathrm{Hom}_{\mathrm{Chain}(\mathbf{Z})^{\mathrm{free}}}(X^*, Y^*) & \text{if } r = 0 \\ \prod_{n \in \mathbf{Z}} \mathrm{Hom}(X^n, Y^{n+r}) & \text{if } r < 0 \end{cases}.$$

Note that we can identify the derived ∞-category $\mathcal{D}(\mathbf{Z})$ with the differential graded nerve [38, Sec. 1.3.1] of $\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})$. This is a slight variant of the second description appearing in Notation 7.4.6; note that replacing the mapping complexes $[X^*, Y^*]$ by their truncations $\tau^{\leq 0}[X^*, Y^*]$ has no effect on the differential graded nerve.

Construction 7.5.2. We define a functor $\eta_p : \mathcal{D}_{\mathrm{dg}}(\mathbf{Z}) \to \mathcal{D}_{\mathrm{dg}}(\mathbf{Z})$ of differential graded categories defined as follows:

(1) On objects, the functor η_p is given by Construction 2.1.3.
(2) For every pair of complexes $X^*, Y^* \in \mathcal{D}_{\mathrm{dg}}(\mathbf{Z})$, we associate a map of chain complexes

$$\rho : \mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*) \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}((\eta_p X)^*, (\eta_p Y)^*).$$

The map ρ vanishes in positive degrees, and is given in degree zero by the map $\mathrm{Hom}_{\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}}(X^*, Y^*) \to \mathrm{Hom}_{\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}}((\eta_p X)^*, (\eta_p Y)^*)$ induced by the functoriality of Construction 2.1.3. In degrees $r < 0$, it is given by the construction

$$\rho(\{f_i : X^i \to Y^{i+r}\}_{i \in \mathbf{Z}}) = \{f'_i : (\eta_p X)^i \to (\eta_p Y)^i),$$

where f'_i is given by the restriction of the map $f_i[1/p] : X^i[1/p] \to Y^{i+r}[1/p]$. This is well-defined by virtue of the observation that

$$(\eta_p X)^i \subseteq p^i X^i \subseteq f_i[1/p]^{-1}(p^i Y^{i+r}) \subseteq f_i[1/p]^{-1}(\eta_p Y)^{i+r}.$$

Passing to differential graded nerves, we obtain a functor of ∞-categories $L\eta_p\colon \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$. By construction, the diagram

$$\begin{tikzcd} \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \arrow[r, "\eta_p"] \arrow[d] & \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}} \arrow[d] \\ \mathcal{D}(\mathbf{Z}) \arrow[r, "L\eta_p"] & \mathcal{D}(\mathbf{Z}) \end{tikzcd}$$

commutes up to homotopy, so that we recover the functor $L\eta_p$ described in Notation 7.4.7 (up to canonical equivalence).

The crucial feature of η_p needed for the proof of Theorem 7.4.8 is the following divisibility property:

Proposition 7.5.3. *Let X^* and Y^* be cochain complexes of free abelian groups and let r be a positive integer. Then:*

(1) *The canonical map*

$$u\colon \mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*)^{-r} \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)^{-r}$$

is divisible by p^{r-1}.

(2) *The restriction to $(-r)$-cocycles*

$$u_0\colon Z^{-r}\,\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*) \to Z^{-r}\,\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)$$

is divisible by p^r.

(3) *Let*

$$v\colon \mathrm{H}^{-r}\,\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*) \to \mathrm{H}^{-r}\,\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)$$

be the map induced by u. Then we can write $v = p^r v'$ for some $v'\colon \pi_r \mathrm{H}^{-r}\,\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*) \to \mathrm{H}^{-r}\,\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)$.

Proof. By definition, a $(-r)$-cochain of $\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*)$ is given by a system of maps $\{f_n\colon X^n \to Y^{n-r}\}_{n\in\mathbf{Z}}$. Let us abuse notation by identifying each f_n with its extension to a map $X^n[1/p] \to Y^{n-r}[1/p]$. Assertion (1) follows from the inclusions

$$f_n((\eta_p X)^n) \subseteq f_n(p^n X^n) \subseteq p^n Y^{n-r} \subseteq p^{r-1}(\eta_p Y)^{n-r}.$$

If $\{f_n\}_{n\in\mathbf{Z}}$ is a cocycle, then we have $df_n = (-1)^r f_{n+1} d$ for each n. For $x \in (\eta_p X)^n$, it follows that

$$df_n(x) = \pm f_{n+1}(dx) \in p^{n+1} Y^{n+1-r}) \cap \ker(d) \subseteq p^r (\eta_p Y)^{n+1-r}$$

so that $f_n(x) \in p^r(\eta_p Y)^{n-r}$, which proves (2).

We now prove (3). For each cocycle z in a cochain complex M^*, let $[z]$ denote its image in the cohomology $\mathrm{H}^*(M)$. We claim that there is a unique map

$$v'\colon \mathrm{H}^{-r}\,\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*) \to \mathrm{H}^{-r}\,\mathrm{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)$$

satisfying $v'([z]) = [p^{-r}u(z)]$ for each $z \in Z^{-r}\operatorname{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*)$. To show that this construction is well-defined, it suffices to show that if z and z' are cohomologous $(-r)$-cocycles of $\operatorname{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*)$ are cohomologous r-cocycles of $\operatorname{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(\eta_p X^*, \eta_p Y^*)$. Writing $z = z' + dw$ for $w \in \mathrm{H}^{-r-1}\operatorname{Hom}_{\mathcal{D}_{\mathrm{dg}}(\mathbf{Z})}(X^*, Y^*)$, we are reduced to proving that $u(dw) = du(w)$ is divisible by p^r. This is clear, since $u(w)$ is divisible by p^r (by virtue of (1)). □

The next result now follows directly from Proposition 7.5.3 and the construction of $\mathcal{D}(\mathbf{Z})$ as a homotopy coherent nerve.

Corollary 7.5.4. *Let $X, Y \in \mathcal{D}(\mathbf{Z})$. Then, for $r \geq 0$, the canonical map*

$$v\colon \pi_r \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(X, Y) \to \pi_r \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(L\eta_p X, L\eta_p Y)$$

is divisible by p^r. That is, there exists a map

$$v'\colon \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(X, Y) \to \pi_r \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(L\eta_p X, L\eta_p Y)$$

such that $v = p^r v'$.

Remark 7.5.5. The functor $L\eta_p\colon \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$ preserves zero objects, so there is a canonical natural transformation $f\colon \Sigma \circ L\eta_p \to L\eta_p \circ \Sigma$, where $\Sigma\colon \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$ denotes the suspension equivalence. One can show that the natural transformation f is divisible by p, which gives another proof of Corollary 7.5.4.

We will deduce Theorem 7.4.8 by combining Corollary 7.5.4 with the following:

Lemma 7.5.6. *Let A be a derived p-complete abelian group and let $f, g\colon A \to A$ be group homomorphisms. Suppose f is an automorphism of A. Then $f + pg$ is an automorphism of A.*

Proof. Since A is derived p-complete, it suffices to show that $f + pg$ induces an automorphism of $A \otimes^L \mathbf{Z}/p\mathbf{Z} \in D(\mathbf{Z})$. This follows from the observation that f and $f + pg$ induce the same endomorphism of $A \otimes^L \mathbf{Z}/p\mathbf{Z} \in D(\mathbf{Z})$. □

Proof of Theorem 7.4.8. It is clear that the forgetful functor

$$\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p} \to \widehat{D(\mathbf{Z})}_p^{L\eta_p}.$$

is essentially surjective. We will show that it is fully faithful. Fix objects (X, φ_X) and $(Y, \varphi_Y) \in \widehat{D(\mathbf{Z})}_p^{L\eta_p}$, so that $\varphi_X\colon X \simeq L\eta_p X$ and $\varphi_Y\colon Y \simeq L\eta_p Y$ are equivalences in $\mathcal{D}(\mathbf{Z})$. Using Remark 7.4.5, we see that the mapping space

$$\operatorname{Hom}_{\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}}((X, \varphi_X), (Y, \varphi_Y))$$

can be realized as the homotopy fiber

$$\operatorname{fib}(f - g\colon \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(X, Y) \to \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(X, L\eta_p Y)),$$

where f is given by composition with φ_Y and g is the composite of $L\eta_p$ and precomposition with φ_X. Note that f is a homotopy equivalence. Combining

Lemma 7.5.6 with Corollary 7.5.4, we find that $f - g$ induces an isomorphism on $\pi_r \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(X, Y)$ for $r > 0$. It follows that the homotopy fiber of $f - g$ can be identified with the discrete space given by the kernel of the map

$$\pi_0(f) - \pi_0(g): \pi_0 \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(X, Y) \to \pi_0 \operatorname{Hom}_{\mathcal{D}(\mathbf{Z})}(X, L\eta_p Y)),$$

which is the set of homomorphisms from (X, φ_X) to (Y, φ_Y) in the ordinary category $\widehat{D(\mathbf{Z})}_p^{L\eta_p}$. □

7.6. **Tensor Products of Strict Dieudonné Complexes.** The derived ∞-category $\mathcal{D}(\mathbf{Z})$ admits a symmetric monoidal structure, with underlying tensor product

$$\otimes^L: \mathcal{D}(\mathbf{Z}) \times \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z}).$$

It is not difficult to see that for any pair of objects $M, N \in \mathcal{D}(\mathbf{Z})$, the canonical map

$$M \otimes^L N \to \widehat{M}_p \otimes^L \widehat{N}_p$$

induces an isomorphism after derived p-completion. It follows that there is an essentially unique symmetric monoidal structure on the p-complete derived ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p$ for which the derived p-completion functor $\mathcal{D}(\mathbf{Z}) \to \widehat{\mathcal{D}(\mathbf{Z})}_p$ is symmetric monoidal. We will denote the underlying tensor product on $\widehat{\mathcal{D}(\mathbf{Z})}_p$ by $\widehat{\otimes}^L: \widehat{\mathcal{D}(\mathbf{Z})}_p \times \widehat{\mathcal{D}(\mathbf{Z})}_p \to \widehat{\mathcal{D}(\mathbf{Z})}_p$; it is given concretely by the formula

$$M \widehat{\otimes}^L N = \widehat{M \otimes^L N}_p.$$

Proposition 7.6.1. *The functor $L\eta_p: \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$ admits a symmetric monoidal structure. In particular, there are canonical isomorphisms*

$$(L\eta_p)(X) \otimes^L (L\eta_p)(Y) \simeq (L\eta_p)(X \otimes^L Y)$$

in $\mathcal{D}(\mathbf{Z})$.

Proof. We first note that X^* and Y^* are torsion-free cochain complexes of abelian groups, then the canonical isomorphism

$$X^*[1/p] \otimes Y^*[1/p] \simeq (X^* \otimes Y^*)[1/p]$$

carries $(\eta_p X)^* \otimes (\eta_p Y)^*$ into $\eta_p(X^* \otimes Y^*)$. It follows that the functor η_p can be regarded as a *lax* symmetric monoidal functor from $\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}$ to itself. Beware that this functor is not symmetric monoidal: the induced map $\alpha: (\eta_p X)^* \otimes (\eta_p Y)^*$ into $\eta_p(X^* \otimes Y^*)$ is usually not an isomorphism of cochain complexes. The content of Proposition 7.6.1 is that α is nevertheless a *quasi-isomorphism*: that is, it induces an isomorphism $L\eta_p X \otimes^L L\eta_p Y \to L\eta_p(X \otimes^L Y)$ in the ∞-category $\mathcal{D}(\mathbf{Z})$. To prove this, we can use the fact that X and Y split (noncanonically) as a direct sum of their cohomology groups to reduce to the case where X and Y are abelian groups, regarded as cochain complexes concentrated in degree zero. Writing X and Y as filtered colimits of finitely generated abelian groups, we may further assume that X and Y are finitely generated. We can then decompose X and Y as direct sums of cyclic abelian groups, and thereby reduce to the case where X and Y are cyclic. In this case, the desired result follows from an explicit calculation (see [11, Prop. 6.8]). $\square$

Corollary 7.6.2. *The functor $L\eta_p\colon \widehat{\mathcal{D}(\mathbf{Z})}_p \to \widehat{\mathcal{D}(\mathbf{Z})}_p$ admits a symmetric monoidal structure. In particular, there are canonical isomorphisms*

$$(L\eta_p)(X)\widehat{\otimes}^L(L\eta_p)(Y) \simeq (L\eta_p)(X\widehat{\otimes}^L Y)$$

in the ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p$.

Proof. Combine Proposition 7.6.1 with Proposition 7.2.4. $\square$

Let $\mathcal{W}\mathrm{Sat}\colon \mathbf{DC} \to \mathbf{DC}_{\mathrm{str}}$ be the completed saturation functor of Notation 2.8.4. In what follows, we will abuse notation by using Corollary 7.4.9 to identify $\mathcal{W}\mathrm{Sat}$ with a functor from $\mathbf{DC}$ to the fixed point ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$. Note that since $L\eta_p$ is a symmetric monoidal functor from $\widehat{D(\mathbf{Z})}_p$ to itself (Corollary 7.6.2), the fixed point ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$ inherits a symmetric monoidal structure.

Proposition 7.6.3. *The functor $\mathcal{W}\mathrm{Sat}\colon \mathbf{DC} \to \widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}$ is symmetric monoidal, where we regard $\mathbf{DC}$ as equipped with the symmetric monoidal structure given by the tensor product of Dieudonné complexes (Remark 2.1.5).*

Proof. Let $\mathbf{DC}^{\mathrm{tf}}$ denote the category of torsion-free Dieudonné complexes and let $L\colon \mathbf{DC} \to \mathbf{DC}^{\mathrm{tf}}$ be a left adjoint to the inclusion (given by $LM^* = M^*/\{\mathrm{torsion}\}$). Then the functor L is symmetric monoidal, and $\mathcal{W}\mathrm{Sat}$ factors as a composition

$$\mathbf{DC} \xrightarrow{L} \mathbf{DC}^{\mathrm{tf}} \xrightarrow{\mathcal{W}\mathrm{Sat}} \widehat{D(\mathbf{Z})}_p^{L\eta_p}.$$

It will therefore suffice to construct a symmetric monoidal structure on the functor $\mathcal{W}\mathrm{Sat}|_{\mathbf{DC}^{\mathrm{tf}}}$.

For every pair of torsion-free Dieudonné complexes M^* and N^*, the natural maps

$$\rho_n\colon (\eta_p^n M)^* \otimes (\eta_p^n N)^* \to (\eta_p^n(M\otimes N))^*.$$

Passing to the limit over n (and dropping the requirement of torsion-freeness), we obtain comparison maps

$$\rho\colon \mathrm{Sat}(M^*) \otimes \mathrm{Sat}(N^*) \to \mathrm{Sat}(M^* \otimes N^*)$$

which allow us to regard Sat as a *lax* symmetric monoidal functor from the category $\mathbf{DC}^{\mathrm{tf}}$ to $\mathbf{DC}_{\mathrm{sat}} \simeq (\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}})^{\eta_p}$. Composing with the (symmetric monoidal) functors $(\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}})^{\eta_p} \to \mathcal{D}(\mathbf{Z})^{L\eta_p} \to \widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$, we obtain a lax symmetric monoidal structure on the composite functor $\mathcal{W}\mathrm{Sat}\colon \mathbf{DC}^{\mathrm{tf}} \to \widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$.

We will complete the proof by showing that this lax symmetric monoidal structure is actually a symmetric monoidal structure. For this, it will suffice to show that the comparison map ρ is a quasi-isomorphism, for every pair of objects $M^*, N^* \in \mathbf{DC}^{\mathrm{tf}}$. In fact, we claim that each ρ_n is an isomorphism: this follows from Proposition 7.6.1, using induction on n. □

Remark 7.6.4. By virtue of Corollary 7.4.9, there is an essentially unique symmetric monoidal structure on the category $\mathbf{DC}_{\mathrm{str}}$ for which the functor

$$\mathbf{DC}_{\mathrm{str}} \to \widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$$

is symmetric monoidal. Let us denote the underlying tensor product functor by

$$\otimes_{\mathrm{str}} : \mathbf{DC}_{\mathrm{str}} \times \mathbf{DC}_{\mathrm{str}} \to \mathbf{DC}_{\mathrm{str}} .$$

Proposition 7.6.3 can be understood as giving an explicit description of this symmetric monoidal structure: it is uniquely determined by requirement that the localization functor $\mathbf{DC} \to \mathbf{DC}_{\mathrm{str}}$ is symmetric monoidal. In particular, for every pair of strict Dieudonné complexes M^* and N^*, we have a canonical isomorphism

$$M^* \otimes_{\mathrm{str}} N^* \simeq \mathcal{W}\,\mathrm{Sat}(M^* \otimes N^*),$$

where $M^* \otimes N^*$ is the usual tensor product of Dieudonné complexes (Remark 2.1.5).

Proposition 7.6.3 has a number of consequences.

Example 7.6.5. Let R be a commutative ring which is p-complete and p-torsion free. Then we can regard R as a commutative ring object of the ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p$. The ∞-category $\mathrm{Mod}_R(\widehat{\mathcal{D}(\mathbf{Z})}_p)$ can then be identified with the p-complete derived ∞-category $\widehat{\mathcal{D}(R)}_p$: that is, with the full subcategory of $\mathcal{D}(R)$ spanned by the p-complete objects.

Let $\sigma : R \to R$ be an automorphism, and let $\sigma_* : \widehat{\mathcal{D}(R)}_p \to \widehat{\mathcal{D}(R)}_p$ be the equivalence of ∞-categories given by restriction of scalars along σ. We let $L\eta_p^{\sigma} : \widehat{\mathcal{D}(R)}_p \to \widehat{\mathcal{D}(R)}_p$ denote the composite functor $\sigma_* \circ L\eta_p$. Unwinding the definitions, we have an equivalence of ∞-categories

$$\widehat{\mathcal{D}(R)}_p^{L\eta_p^{\sigma}} \simeq \mathrm{Mod}_R(\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}),$$

where we regard R as a (commutative algebra) object of the fixed point ∞-category $\widehat{\mathcal{D}}(\mathbf{Z})_p^{L\eta_p}$ via the isomorphism $\sigma : R \simeq \sigma_* R$. Combining this observation with Remark 7.6.4, we see that $\widehat{\mathcal{D}(R)}_p^{L\eta_p^{\sigma}}$ can be identified with the ordinary category $\mathrm{Mod}_R(\mathbf{DC}_{\mathrm{str}})$ of R-module in the category of strict Dieudonné complexes. Concretely, the objects of $\mathrm{Mod}_R(\mathbf{DC}_{\mathrm{str}})$ are given by triples (M^*, d, F), where

(M^*, d) is a cochain complex of R-modules and $F: M^* \to M^*$ is a map of graded abelian groups satisfying the identities

$$d(Fx) = pF(dx) \qquad F(ax) = \sigma(a)F(x) \text{ for } a \in R,$$

where the underlying Dieudonné complex (obtained by forgetting the R-module structure) is strict.

We can summarize the situation as follows: given a p-complete object M of the derived ∞-category $\mathcal{D}(R)$ equipped with an isomorphism $\alpha: M \simeq \sigma_* L\eta_p(M)$, we can find a canonical representative of M as a cochain complex M^* of (p-torsion free) R-modules, for which α can be lifted to an isomorphism of cochain complexes $M^* \simeq \sigma_* \eta_p(M^*)$.

Remark 7.6.6. In practice, we will be most interested in the special case of Example 7.6.5 where $R = W(k)$ is the ring of Witt vectors of a perfect field k, and $\sigma: W(k) \to W(k)$ is the Witt vector Frobenius map. In this case, we will denote the functor $L\eta_p^\sigma$ by $L\eta_p^\varphi$.

Example 7.6.7. Combining Theorem 7.4.8, Theorem 7.3.4, and Remark 7.6.4, we see that the following are equivalent:

- The category of commutative algebra objects A^* of $\mathbf{DC}$ which are strict as Dieudonné algebras.
- The category of commutative algebra objects A^* of $\mathbf{DC}_{\text{str}}$ (with respect to the tensor product of Remark 7.6.4).
- The category of commutative algebra objects of the fixed point category $\widehat{D(\mathbf{Z})}_p^{L\eta_p}$ (or, equivalently, the fixed points of $L\eta_p$ on the category of commutative algebra objects of $\widehat{D(\mathbf{Z})}_p$).
- The ∞-category of commutative algebra objects of the fixed point ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$ (or, equivalently, the fixed points of $L\eta_p$ on the ∞-category of commutative algebra objects of $\widehat{\mathcal{D}(\mathbf{Z})}_p$).

By virtue of Remark 3.1.5, each of these categories contains the category $\mathbf{DA}_{\text{str}}$ of strict Dieudonné algebras as a full subcategory.

In particular, if A is a p-complete E_∞-algebra over $\mathbf{Z}$ for which there exists an isomorphism of E_∞-algebras $A \simeq L\eta_p(A)$, then A has a canonical representative by a commutative differential graded algebra over $\mathbf{Z}_p$.

Remark 7.6.8. In view of the Nikolaus-Scholze description [41] of the homotopy theory CycSp of *cyclotomic spectra*, we can view the category $\mathbf{DC}_{\text{str}}$ as a toy analog of CycSp (although this is perhaps more appropriate for the *lax* fixed points of $L\eta_p$ discussed in section 9.1). The connection between the de Rham-Witt complex and cyclotomic spectra goes back to the work of Hesselholt [22].

8. The Nygaard Filtration

Let $\widetilde{R}$ be a flat $\mathbf{Z}_p$-algebra whose reduction $R = \widetilde{R}/p\widetilde{R}$ is smooth over $\mathbf{F}_p$, and let $\widehat{\Omega}^*_{\widetilde{R}}$ be the completed de Rham complex of $\widetilde{R}$ (Variant 3.3.1). Then $\widehat{\Omega}^*_{\widetilde{R}}$ admits subcomplexes

$$\mathcal{N}^k\,\widehat{\Omega}^*_{\widetilde{R}} := p^k\widetilde{R} \to p^{k-1}\widehat{\Omega}^1_{\widetilde{R}} \xrightarrow{d} \dots \xrightarrow{d} p\widehat{\Omega}^{k-1}_{\widetilde{R}} \xrightarrow{d} \widehat{\Omega}^k_{\widetilde{R}} \xrightarrow{d} \widehat{\Omega}^{k+1}_{\widetilde{R}} \xrightarrow{d} \cdots. \tag{11}$$

Allowing k to vary, we obtain a descending filtration $\{\mathcal{N}^k\,\widehat{\Omega}^*_{\widetilde{R}}\}_{k\geq 0}$ of $\widehat{\Omega}^*_{\widetilde{R}}$. This filtration has the following features:

(1) Let $\varphi\colon \widetilde{R} \to \widetilde{R}$ be any lift of the Frobenius map on R. Then the pullback map $\varphi^*\colon \widehat{\Omega}^*_{\widetilde{R}} \to \widehat{\Omega}^*_{\widetilde{R}}$ carries $\mathcal{N}^k\,\widehat{\Omega}^*_{\widetilde{R}}$ into $p^k\widehat{\Omega}^*_{\widetilde{R}}$ (since φ^* is divisible by p^i on i-forms).

(2) Let $\varphi\colon \widetilde{R} \to \widetilde{R}$ be any lift of the Frobenius map on R. Then the maps $\mathcal{N}^i\,\widehat{\Omega}^*_{\widetilde{R}} \xrightarrow{\varphi^*} p^i\widehat{\Omega}^*_{\widetilde{R}}$ determined by (1) for $i = k, k+1$ induce a quasi-isomorphism

$$\mathrm{gr}^k_{\mathcal{N}}\widehat{\Omega}^*_{\widetilde{R}} \to \tau^{\leq k}\big(p^k\widehat{\Omega}^*_{\widetilde{R}}/p^{k+1}\widehat{\Omega}^*_{\widetilde{R}}\big) \simeq \tau^{\leq k}\Omega^*_R,$$

where the second isomorphism is obtained by dividing by p^k.

The filtered complex $\{\mathcal{N}^k\,\widehat{\Omega}^*_{\widetilde{R}}\}_{k\geq 0}$ of $\widehat{\Omega}^*_{\widetilde{R}}$ depends on the $\mathbf{Z}_p$-algebra $\widetilde{R}$, and not only on its mod p reduction $R = \widetilde{R}/p\widetilde{R}$. Nevertheless, it was observed by Nygaard [42, §1] (and elaborated by Illusie-Raynaud [29, §III.3]) that the de Rham-Witt complex $W\Omega^*_R$ admits an analogous filtration $\{\mathcal{N}^k\,W\Omega^*_R\}_{k\geq 0}$, which depends only on R. Moreover, if $\varphi\colon \widetilde{R} \to \widetilde{R}$ is a lift of Frobenius, then the quasi-isomorphism $\widehat{\Omega}^*_{\widetilde{R}} \simeq W\Omega^*_R$ of Theorem 4.2.4 is actually a *filtered* quasi-isomorphism. Consequently, when viewed as an object of the filtered derived category, $\{\mathcal{N}^k\,\widehat{\Omega}^*_{\widetilde{R}}\}_{k\geq 0}$ depends only on R.

In this section, we construct *Nygaard filtration* $\{\mathcal{N}^k\,M^*\}_{k\geq 0}$ on any saturated Dieudonné complex M^* (Construction 8.1.1). In §8.2, we show that this filtration always satisfies analogues of properties (1) and (2) above (Proposition 8.2.1). In the special case where M^* is the saturation of a Dieudonné complex N^* of Cartier type, we show that the inclusion $N^* \hookrightarrow M^*$ is a filtered quasi-isomorphism, where M^* is equipped with the Nygaard filtration and N^* is equipped with an analogue of the filtration (11) (Proposition 8.3.3). In §8.4, we use the ideas of §7 to give an alternative description of the Nygaard filtration, using the language of filtered derived categories.

8.1. The Nygaard Filtration of a Saturated Dieudonné Complex.

Throughout this section, we let (M^*, d, F) denote a saturated Dieudonné complex.

Construction 8.1.1 (The Nygaard filtration)**.** For every pair of integers i and k, we define

$$\mathcal{N}^k M^i = \begin{cases} p^{k-i-1} V M^i & \text{if } i < k \\ M^i & \text{if } i \geq k. \end{cases}$$

Note that for $i < k$ and $x = p^{k-i} V y \in \mathcal{N}^k M^{i-1}$, we have

$$\begin{aligned} dx &= d(p^{k-i} V y) \\ &= p^{k-i} d(Vy) \\ &= p^{k-i-1} V(dy) \\ &\in \mathcal{N}^k M^i. \end{aligned}$$

It follows that

$$\mathcal{N}^k M^* := \cdots \xrightarrow{d} pVM^{k-2} \xrightarrow{d} VM^{k-1} \xrightarrow{d} M^k \xrightarrow{d} M^{k+1} \ldots.$$

is a subcomplex of M^*. Note that $\mathcal{N}^{k+1} M^* \subseteq \mathcal{N}^k M^*$, so we can view $\{\mathcal{N}^k M^*\}_{k \in \mathbf{Z}}$ as a descending filtration on M^*. We will refer to this filtration as the *Nygaard filtration* on M^*, and write

$$\mathrm{gr}^k_{\mathcal{N}} M^* := \mathcal{N}^k M^* / \mathcal{N}^{k+1} M^*$$

for the corresponding graded pieces.

Remark 8.1.2. Note that $p^{k-i} M^i \subseteq \mathcal{N}^k M^i \subseteq p^{k-i-1} M^i$ for all $i < k$: the second containment is obvious, while the first follows from the equality $p^{k-i} M^i = p^{k-i-1} V(FM^i)$. It follows that M^* complete for the Nygaard filtration (that is, the map $M^* \to \varprojlim M^* / \mathcal{N}^k M^*$ is an isomorphism) if and only if each M^i is p-adically complete.

Example 8.1.3. Say R is a perfect ring of characteristic p, and let $M = W(R)$, regarded as a Dieudonné complex concentrated in degree zero (Example 3.1.8). Then $\mathcal{N}^k W(R) = p^k W(R)$ for $k \geq 0$, since F is an automorphism of $W(R)$ and $V = F^{-1} p$.

Remark 8.1.4. Let R be a smooth algebra over a perfect field k of characteristic p, so that the de Rham-Witt complex $W\Omega^*_R$ is a saturated Dieudonné complex. The image of $W\Omega^*_R$ in the derived category $D(\mathbf{Z}_p)$ can be described as the absolute crystalline cohomology of $\mathrm{Spec}(R)$: that is, it is given by $\mathrm{R}\Gamma((\mathrm{Spec}(R))_{\mathrm{crys}}, \mathscr{O}_{\mathrm{crys}})$ where $\mathscr{O}_{\mathrm{crys}}$ is the structure sheaf on the crystalline site of $\mathrm{Spec}(R)$. The Nygaard filtration $\mathcal{N}^k W\Omega^*_R$ also admits a crystalline interpretation for $0 \leq k < p$, as explained by Langer-Zink in [36, Theorem 4.6]: the image of $\mathcal{N}^k W\Omega^*_R$ in the derived category $D(\mathbf{Z}_p)$ is identified with $\mathrm{R}\Gamma((\mathrm{Spec}(R))_{\mathrm{crys}}, \mathcal{I}^{[k]}_{\mathrm{crys}})$, where $\mathcal{I}^{[k]}_{\mathrm{crys}} \subseteq \mathscr{O}_{\mathrm{crys}}$ is the k-th divided power of the universal pd-ideal sheaf $\mathcal{I}_{\mathrm{crys}} \subseteq \mathscr{O}_{\mathrm{crys}}$ on the crystalline site. This description fails for $k \geq p$: for example, taking $p = 2$ and $R = \mathbf{F}_2$ perfect gives a counterexample as $(2^{[k]}) = (2)$ as ideals in $W(\mathbf{F}_2)$ when k is a power of 2, while $\mathcal{N}^k W(\mathbf{F}_2) = (2^k)$ for all $k \geq 0$. We are not aware of a crystalline description of the Nygaard filtration for $k \geq p$.

8.2. **The Nygaard Filtration of a Completion.** Let (M^*, F) be a Dieudonné complex, where M^* is p-torsion-free. As explained in Remark 2.1.4, we can identify F with a map of chain complexes $\alpha_F: M^* \to \eta_p M^*$. In what follows, we will identify $\eta_p M^*$ with a subcomplex of $M^*[p^{-1}]$, so that we can also regard α_F as a map from M^* to $M^*[p^{-1}]$ (given by $\alpha_F(x) = p^i F(x)$ for $x \in M^i$).

Proposition 8.2.1. *Let M^* be a saturated Dieudonné complex and let k be an integer. Then:*

(1) *The terms of the complex $\mathrm{gr}^k_{\mathcal{N}} M^*$ vanish in degrees $> k$.*
(2) *We have $\mathcal{N}^k M^* = \alpha_F^{-1}(p^k M^*)$. In particular, α_F determines a map of cochain complexes $\mathcal{N}^k M^* \to p^k M^*$.*
(3) *The map α_F induces an isomorphism of cochain complexes $\mathrm{gr}^k_{\mathcal{N}} M^* \to \tau^{\leq k}(p^k M^*/p^{k+1} M^*)$.*
(4) *The map $x \mapsto p^{-k}\alpha_F(x)$ induces an isomorphism $\mathrm{gr}^k_{\mathcal{N}} M^* \simeq \tau^{\leq k}(M^*/pM^*)$.*

Proof. Assertion (1) follows from the observation that $\mathcal{N}^k M^i = M^i$ for $i \geq k$, and assertion (4) is a formal consequence of (3). We now prove (2). We first claim that $\alpha_M(\mathcal{N}^k M^i) \subseteq p^k M^i$. By definition, we have $\alpha_F(x) = p^i F(x)$ for $x \in M^i$, so the inclusion is trivial when $i \geq k$. For $i < k$, we observe that

$$\alpha_F(p^{k-i-1} V M^i) = p^i F p^{k-i-1} V M^i = p^{k-1} F V M^i = p^k M^i.$$

To complete the proof of (2), we must show that if $x \in M^i$ satisfies $\alpha_F(x) \in p^k M^i$, then x belongs to $\mathcal{N}^k M^i$. This is vacuous for $i \geq k$, so we may assume that $i < k$. Our hypothesis gives

$$p^i F(x) = p^k y = (p^i F)(p^{k-i-1} V y)$$

for some $y \in M^i$. Since $p^i F$ is an injection, we obtain $x = p^{k-i-1} V y \in \mathcal{N}^k M^i$, as desired.

We now prove (3). In degrees $i < k$, the map under consideration is given by

$$p^{k-i-1} V M^i / p^{k-i} V M^i \xrightarrow{p^i F} p^k M^i / p^{k-1} M^i.$$

It is thus sufficient to check that $p^i F$ gives a bijection $p^{k-i-1} V M^i \to p^k M^i$ for $i < k$. Since M^* has no p-torsion, this reduces to checking that F gives a bijection $V M^i \to p M^i$, which is clear: F is injective and $FV = p$.

In degree k, the map under consideration is given by

$$M^k / V M^k \xrightarrow{p^k F} \Big(p^k M^k \cap d^{-1}(p^{k+1} M^{k+1})\Big) / p^{k+1} M^k.$$

To show this is an isomorphism, we may divide by p^k to reduce to checking that the right vertical map in the diagram

$$\begin{tikzcd} 0 \arrow[r] & VM^k \arrow[r] \arrow[d, "F"] & M^k \arrow[r] \arrow[d, "F"] & M^k/VM^k \arrow[r] \arrow[d, "F"] & 0 \\ 0 \arrow[r] & pM^k \arrow[r] & d^{-1}(pM^{k+1}) \arrow[r] & d^{-1}(pM^{k+1})/pM^k \arrow[r] & 0 \end{tikzcd}$$

is an isomorphism. This is clear, since the rows are exact and the maps

$$F: VM^k \to pM^k \qquad F: M^k \to d^{-1}(pM^{k+1})$$

are isomorphisms (by virtue of our assumption that M^* is saturated). □

Corollary 8.2.2. *Let $f: M^* \to N^*$ be a map of saturated Dieudonné complexes. Suppose that f induces a quasi-isomorphism $M^*/pM^* \to N^*/pN^*$. Then, for every integer k, the induced map*

$$M^*/\mathcal{N}^k M^* \to N^*/\mathcal{N}^k N^*$$

is a quasi-isomorphism.

Proof. For each $k \in \mathbf{Z}$, Proposition 8.2.1 supplies a commutative diagram of cochain complexes

$$\begin{tikzcd} \mathrm{gr}^k_{\mathcal{N}} M^* \arrow[r] \arrow[d] & \mathrm{gr}^k_{\mathcal{N}} N^* \arrow[d] \\ \tau^{\leq k}(M^*/pM^*) \arrow[r] & \tau^{\leq k}(N^*/pN^*) \end{tikzcd}$$

where the vertical maps are isomorphisms. It follows that f induces a quasi-isomorphism $\mathrm{gr}^k_{\mathcal{N}} M^* \to \mathrm{gr}_{\mathcal{N}} N^*$. Proceeding by induction, we deduce that the maps

$$\mathcal{N}^{k'} M^*/\mathcal{N}^k M^* \to \mathcal{N}^{k'} N^*/\mathcal{N}^k N^*$$

are quasi-isomorphisms for each $k' \leq k$. The desired result now follows by passing to the direct limit $k' \to -\infty$. □

Corollary 8.2.3. *Let M^* be a saturated Dieudonné complex. Then, for every integer k, the canonical map $M^*/\mathcal{N}^k M^* \to \mathcal{W}(M)^*/\mathcal{N}^k \mathcal{W}(M)^*$ is a quasi-isomorphism.*

Proof. By virtue of Corollary 8.2.2, it will suffice to show that the map $M^*/pM^* \to \mathcal{W}(M)^*/p\mathcal{W}(M)^*$ is a quasi-isomorphism. This follows from Proposition 2.7.5 and Corollary 2.7.2. □

Corollary 8.2.4. *Let $f: M^* \to N^*$ be a quasi-isomorphism of saturated Dieudonné complexes. Then f is a filtered quasi-isomorphism: that is, for every integer k, the induced map $\mathcal{N}^k M^* \to \mathcal{N}^k N^*$ is a quasi-isomorphism.*

Proof. If f is a quasi-isomorphism, then the induced map $M^*/pM^* \to N^*/pN^*$ is also a quasi-isomorphism. We have a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N}^k M^* \arrow[r] \arrow[d] & M^* \arrow[r] \arrow[d, "f"] & M^*/\mathcal{N}^k M^* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{N}^k N^* \arrow[r] & N^* \arrow[r] & N^*/\mathcal{N}^k N^* \arrow[r] & 0 \end{tikzcd}$$

where the vertical map in the middle is a quasi-isomorphism by assumption and the vertical map on the right is a quasi-isomorphism by Corollary 8.2.2, so that the vertical map on the left is also a quasi-isomorphism. □

8.3. **Dieudonné Complexes of Cartier Type.** We now consider a variant of the Nygaard filtration which can be defined for *any* cochain complex.

Construction 8.3.1. Let M^* be any cochain complex of abelian groups. For every integer k, we set $\mathcal{N}_u^k M^* \subseteq M^*$ denote the subcomplex given by

$$\mathcal{N}_u^k M^* := \cdots \to p^3 M^{k-3} \to p^2 M^{k-2} \to pM^{k-1} \to M^k \to M^{k+1} \to \cdots.$$

This defines a descending filtration $\{\mathcal{N}_u^k M^*\}_{k\in\mathbf{Z}}$ of M^* in the category of cochain complexes of abelian groups.

Remark 8.3.2. Let M^* be a saturated Dieudonné complex. Then, for every integer k, we have $\mathcal{N}_u^k M^* \subseteq \mathcal{N}^k M^*$.

Let M^* be a Dieudonné complex and let $\mathrm{Sat}(M^*)$ denote its saturation. Using Remark 8.3.2, we obtain maps of filtered complexes

$$\{\mathcal{N}_u^k M^*\}_{k\in\mathbf{Z}} \to \{\mathcal{N}_u^k \mathrm{Sat}(M^*)\}_{k\in\mathbf{Z}} \to \{\mathcal{N}^k \mathrm{Sat}(M^*)\}_{k\in\mathbf{Z}}.$$

Proposition 8.3.3. *Let M^* be a Dieudonné complex of Cartier type (Definition 2.4.1). Then, for every integer k, the canonical map*

$$M^*/\mathcal{N}_u^k M^* \to \mathrm{Sat}(M^*)/\mathcal{N}^k \mathrm{Sat}(M^*)$$

is a quasi-isomorphism.

Proof. For every pair of integers i and k, we have canonical isomorphisms

$$\mathrm{H}^i(\mathrm{gr}^k_{\mathcal{N}_u} M^*) \simeq (\mathrm{gr}^k_{\mathcal{N}_u} M)^i \simeq \begin{cases} M^i/pM^i & \text{if } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Using Proposition 8.2.1, we obtain isomorphisms

$$\mathrm{H}^i(\mathrm{gr}^k_{\mathcal{N}} \mathrm{Sat}(M^*)) \simeq \begin{cases} \mathrm{H}^i(M^*/pM^* & \text{if } i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Under these isomorphisms, we can identify the canonical map

$$\mathrm{H}^i(\mathrm{gr}^k_{\mathcal{N}_u} M^*) \to \mathrm{H}^i(\mathrm{gr}^k_{\mathcal{N}} \mathrm{Sat}(M^*))$$

with the composition

$$M^i/pM^i \xrightarrow{F} \mathrm{H}^i(M^*/pM^*) \to \mathrm{H}^i(\mathrm{Sat}(M)^*/p\,\mathrm{Sat}(M)^*)$$

for $i \leq k$, and with the zero map otherwise. Here the first map is a quasi-isomorphism by virtue of our assumption that M^* is of Cartier type, and the second by virtue of Theorem 2.4.2.

Proceeding by induction, we deduce that for each $k' \leq k$, the induced map

$$\mathcal{N}_u^{k'} M^*/\mathcal{N}_u^k M^* \to \mathcal{N}^{k'}\,\mathrm{Sat}(M^*)/\mathcal{N}^k\,\mathrm{Sat}(M^*)$$

is a quasi-isomorphism. Passing to the direct limit as $k' \to -\infty$, we obtain the desired quasi-isomorphism $M^*/\mathcal{N}_u^k M^* \to \mathrm{Sat}(M^*)/\mathcal{N}^k\,\mathrm{Sat}(M^*)$. □

Corollary 8.3.4. *Let M^* be a Dieudonné complex of Cartier type. Then, for every integer k, the canonical map*

$$M^*/\mathcal{N}_u^k M^* \to \mathcal{W}\,\mathrm{Sat}(M^*)/\mathcal{N}^k\,\mathcal{W}\,\mathrm{Sat}(M^*)$$

is a quasi-isomorphism.

Proof. Combine Proposition 8.3.3 with Corollary 8.2.3. □

Corollary 8.3.5. *Let M^* be a Dieudonné complex of Cartier type, and suppose that each of the abelian groups M^i is p-adically complete. Then, for every integer k, the canonical map $\mathcal{N}_u^k M^* \to \mathcal{N}^k\,\mathcal{W}\,\mathrm{Sat}(M^*)$ is a quasi-isomorphism. In other words, the map $M^* \to \mathcal{W}\,\mathrm{Sat}(M^*)$ is a filtered quasi-isomorphism, where we equip $\mathcal{W}\,\mathrm{Sat}(M^*)$ with the Nygaard filtration of Construction 8.1.1 and M^* with the filtration of Construction 8.3.1.*

Proof. Combine Corollary 8.3.4 with Corollary 2.8.5. □

Example 8.3.6. Let $\widetilde{R}$ be a flat $\mathbf{Z}_p$-algebra for which the reduction $R = \widetilde{R}/p\widetilde{R}$ is smooth over $\mathbf{F}_p$, and let $\varphi\colon \widetilde{R} \to \widetilde{R}$ be a lift of the Frobenius map on R. Then the completed de Rham complex $\widehat{\Omega}^*_{\widetilde{R}}$ is a Dieudonné complex of Cartier type (Corollary 3.3.8), and the saturated de Rham-Witt complex $\mathcal{W}\,\Omega^*_R$ can be identified with the completed saturation $\mathcal{W}\,\mathrm{Sat}(\widehat{\Omega}^*_{\widetilde{R}})$. It follows from Corollary 8.3.5 that the canonical map $\rho\colon \mathcal{N}_u^k \widehat{\Omega}^*_{\widetilde{R}} \to \mathcal{N}^k\,\mathcal{W}\,\Omega^*_R$ is a quasi-isomorphism for every integer k.

8.4. **The Nygaard Filtration and $L\eta_p$ via Filtered Derived Categories.** Let M^* be a saturated Dieudonné complex and let $\{\mathcal{N}^k M^*\}_{k\in\mathbf{Z}}$ be its Nygaard filtration. The goal of this section is to show that, as an object of the filtered derived category $DF(\mathbf{Z})$ (see Construction 8.4.1 below), the filtered complex $\{\mathcal{N}^k M^*\}_{k\in\mathbf{Z}}$ depends only on the image $M \in D(\mathbf{Z})$ of M^* in the derived category $D(\mathbf{Z})$, together with the isomorphism $M \simeq L\eta_p M$ induced by the Frobenius operator on M^*. In other words, it depends only on the structure of M as an object of the fixed point category $D(\mathbf{Z})^{L\eta_p}$ studied in §7.

Let us first recall some definitions and facts about the filtered derived category $DF(\mathbf{Z})$ and the realization of the $L\eta_p$ operator as a truncation operator. We refer the reader to [12] for details.

Construction 8.4.1 (The Filtered Derived Category)**.** Let us regard the set of integers $\mathbf{Z}$ as a linearly ordered set. For any ∞-category $\mathcal{C}$, we can identify functors $\mathbf{Z}^{\mathrm{op}} \to \mathcal{C}$ with diagrams

$$\cdots \to C^2 \to C^1 \to C^0 \to C^{-1} \to C^{-2} \to \cdots$$

The collection of all such diagrams $\mathcal{C}$ can be organized into an ∞-category $\mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathcal{C})$. We define the *filtered derived category* $DF(\mathbf{Z})$ to be the homotopy category of the ∞-category $\mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathcal{D}(\mathbf{Z}))$. Here $\mathcal{D}(\mathbf{Z})$ denotes the derived ∞-category of abelian groups (Notation 7.4.6).

Example 8.4.2. Let $\{\mathcal{F}^k M^*\}_{k\in\mathbf{Z}}$ be a filtered cochain complex of abelian groups, which we can identify with a diagram

$$\cdots \hookrightarrow \mathcal{F}^k M^* \hookrightarrow \mathcal{F}^1 M^* \hookrightarrow \mathcal{F}^0 M^* \hookrightarrow \mathcal{F}^{-1} M^* \hookrightarrow \mathcal{F}^{-2} M^* \hookrightarrow \cdots$$

in the category $\mathrm{Chain}(\mathbf{Z})$ of cochain complexes. Applying the functor $\mathrm{Chain}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$, we obtain an object of the filtered derived category $DF(\mathbf{Z})$. This construction determines a functor from the category of filtered cochain complexes to the filtered derived category $DF(\mathbf{Z})$, and one can show that this functor exhibits $DF(\mathbf{Z})$ as the localization of the category of filtered cochain complexes with respect to the class of filtered quasi-isomorphisms.

Remark 8.4.3 (The Underlying Complex)**.** For each object $\vec{C} = \{C^k\}_{k\in\mathbf{Z}} \in DF(\mathbf{Z})$, we let C denote the direct limit $\varinjlim_{k\to-\infty} C^k$. We will refer to C as the *underlying complex* of $\vec{C}$. The construction $\vec{C} \mapsto C^{-\infty}$ determines a functor of ∞-categories $\mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathcal{D}(\mathbf{Z})) \to \mathcal{D}(\mathbf{Z})$, hence also a functor of ordinary categories $DF(\mathbf{Z}) \to D(\mathbf{Z})$.

Note that if $\vec{C}$ arises from a filtered cochain complex $\{\mathcal{F}^k M^*\}_{k\in\mathbf{Z}}$ as in Example 8.4.2, then C is the image in the derived category of the usual direct limit $\varinjlim_{k\to\infty} \mathcal{F}^k M^*$.

Remark 8.4.4 (The Associated Graded)**.** For each object $\vec{C} = \{C^k\}_{k\in\mathbf{Z}} \in DF(\mathbf{Z})$ and each integer n, we let $\mathrm{gr}^n(\vec{C})$ denote the cofiber of the map $C^{n+1} \to C^n$. The construction $\vec{C} \mapsto \mathrm{gr}^n(\vec{C})$ determines a functor of ∞-categories $\mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathcal{D}(\mathbf{Z})) \to \mathcal{D}(\mathbf{Z})$, hence also a functor of ordinary categories $DF(\mathbf{Z}) \to D(\mathbf{Z})$.

Note that if $\vec{C}$ arises from a filtered cochain complex $\{\mathcal{F}^k M^*\}_{k\in\mathbf{Z}}$ as in Example 8.4.2, then $\mathrm{gr}^n(\vec{C})$ is the image in the derived category of the quotient $\mathcal{F}^n M^* / \mathcal{F}^{n+1} M^*$.

Example 8.4.5. Let M^* be a cochain complex of p-torsion-free abelian groups. Then we can equip the localization $M^*[1/p]$ with the p-adic filtration, given by the diagram

$$\cdots \hookrightarrow p^2 M^* \hookrightarrow pM^* \hookrightarrow M^* \hookrightarrow p^{-1}M^* \hookrightarrow p^{-2}M^* \hookrightarrow \cdots.$$

This construction determines a filtered cochain complex $\{p^k M^*\}_{k\in\mathbf{Z}}$, hence an object of the filtered derived category $DF(\mathbf{Z})$. Note that the construction $M^* \mapsto \{p^k M^*\}_{k\in\mathbf{Z}}$ carries quasi-isomorphisms in $\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}$ to filtered quasi-isomorphisms. We therefore obtain a functor of derived categories $D(\mathbf{Z}) \to DF(\mathbf{Z})$, which we will denote by $M \mapsto p^\bullet M$.

Remark 8.4.6 (The Beilinson t-Structure)**.** The filtered derived category $DF(\mathbf{Z})$ arises as the homotopy category of a stable ∞-category $\mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathcal{D}(\mathbf{Z}))$, and therefore inherits the structure of a triangulated category. This triangulated category admits a t-structure $(DF^{\le 0}(\mathbf{Z}), DF^{\ge 0})$ which can be described as follows:

- An object $\vec{C} \in DF(\mathbf{Z})$ belongs to $DF^{\le 0}(\mathbf{Z})$ if and only if, for every integer k, the object $\mathrm{gr}^k(\vec{C})$ belongs to $D^{\le k}(\mathbf{Z})$: that is, the cohomology groups $\mathrm{H}^n(\mathrm{gr}^k(\vec{C}))$ vanish for $n > k$.
- An object $\vec{C} = \{C^k\}_{k\in\mathbf{Z}} \in DF(\mathbf{Z})$ belongs to $DF^{\ge 0}(\mathbf{Z})$ if and only if, for every integer k, we have $C^k \in D^{\ge k}(\mathbf{Z})$: that is, the cohomology groups $\mathrm{H}^n(C^k)$ vanish for $n < k$.

We will refer to $(DF^{\le 0}(\mathbf{Z}), DF^{\ge 0})$ as the *Beilinson t-structure on* $DF(\mathbf{Z})$. The heart of this t-structure can be identified with the abelian cateogry $\mathrm{Chain}(\mathbf{Z})$ of cochain complexes of abelian groups.

Example 8.4.7. Let $\{\mathcal{F}^k M^*\}_{k\in\mathbf{Z}}$ be a filtered cochain complex, which we identify with its image in the filtered derived category. Let $\tau_B^{\le 0}\{\mathcal{F}^k M^*\}_{k\in\mathbf{Z}}$ denote its truncation with respect to the Beilinson t-structure. Then $\tau_B^{\le 0}\{\mathcal{F}^k M^*\}_{k\in\mathbf{Z}}$ can be represented explicitly by the filtered subcomplex $\{\mathcal{F}'^k M^*\}_{k\in\mathbf{Z}}$ described by the formula

$$\mathcal{F}'^k M^n = \begin{cases} \mathcal{F}^k M^n & \text{if } n < k \\ \{x \in \mathcal{F}^n M^n : dx \in \mathcal{F}^{n+1}M^{n+1}\} & \text{if } n \ge k. \end{cases}$$

To prove this, it suffices to observe that each quotient $\mathcal{F}'^k M^*/\mathcal{F}'^{k-1}M^*$ can be identified with $\tau^{\le k}M^*$ (so that $\{\mathcal{F}'^k M^*\}_{k\in\mathbf{Z}}$ belongs to $DF^{\le 0}(\mathbf{Z})$), and that each quotient $\mathcal{F}^k M^*/\mathcal{F}'^k M^*$ belongs to $D^{>k}(\mathbf{Z})$ (so that $\{\mathcal{F}^k M^*/\mathcal{F}'^k M^*\}_{k\in\mathbf{Z}}$ belongs to $DF^{>0}(\mathbf{Z})$).

In particular, the underlying cochain complex of $\tau_B^{\le 0}\{\mathcal{F}^k M^*\}_{k\in\mathbf{Z}}$ can be identified with N^*, where $N^k = \{x \in \mathcal{F}^k M^k : dx \in \mathcal{F}^{k+1}M^{k+1}\}$.

Proposition 8.4.8. *The functor $L\eta_p\colon D(\mathbf{Z}) \to D(\mathbf{Z})$ of Corollary 7.2.3 is isomorphic to the composition*

$$D(\mathbf{Z}) \xrightarrow{M \mapsto p^\bullet M} DF(\mathbf{Z}) \xrightarrow{\tau_B^{\le 0}} DF(\mathbf{Z}) \xrightarrow{\vec{M} \mapsto M} D(\mathbf{Z}).$$

Here the first functor is given by the formation of p-adic filtrations (Example 8.4.5), the second by truncation for the Beilinson t-structure (Remark 8.4.6), and the third is given by passing to the underlying cochain complex (Remark 8.4.3).

Proof. Let M^* be a cochain complex of p-torsion-free abelian groups. Using the analysis of Example 8.4.7, we see that the underlying cochain complex of $\tau_B^{\leq 0}(p^\bullet M^*)$ can be identified with $(\eta_p M)^* = \{x \in p^* M^* : dx \in p^{*+1} M^{*+1}\}$. □

It follows from Proposition 8.4.8 that, for any object M of the derived category $D(\mathbf{Z})$, the object $L\eta_p(M)$ comes equipped with a canonical filtration: more precisely, it can be lifted to an object of the filtered derived category $DF(\mathbf{Z})$, given by $\tau_B^{\leq 0}(p^\bullet M)$. In particular, if M is equipped with an isomorphism $\varphi: M \simeq L\eta_p(M)$, then we can lift M itself to an object of $DF(\mathbf{Z})$. In the case where M arises from a saturated Dieudonné complex M^*, we can model this lift explicitly using the Nygaard filtration of Construction 8.1.1:

Proposition 8.4.9. *Let M^* be a saturated Dieudonné complex, so that the Frobenius map F induces an isomorphism $\alpha_F: M^* \simeq (\eta_p M)^*$ (Remark 2.1.4). Let us abuse notation by identifying α_F with an isomorphism $M \simeq L\eta_p(M)$ in the derived category $D(\mathbf{Z})$. Then α_F can be lifted to an isomorphism $\{\mathcal{N}^k M^*\}_{k \in \mathbf{Z}} \simeq \tau_B^{\leq 0}(p^\bullet M)$ in the filtered derived category $DF(\mathbf{Z})$.*

Proof. Using Example 8.4.7, we see that $\tau_B^{\leq 0}(p^\bullet M)$ can be represented explicitly by the filtered cochain complex $\{\mathcal{N}'^k(\eta_p M)^*\}_{k \in \mathbf{Z}}$ given by the formula

$$\mathcal{N}'^k(\eta_p M)^n = \begin{cases} p^k M^n & \text{if } n < k \\ \{x \in p^n M^n : dx \in p^{n+1} M^{n+1}\} & \text{if } n \geq k. \end{cases}$$

It now suffices to observe that each $\mathcal{N}^k M^n$ can be described as the inverse image of $\mathcal{N}'^k(\eta_p M)^n$ under the isomorphism $(p^n F): M^n \to (\eta_p M)^n$ determined by α_F. □

9. Comparison with Derived Crystalline Cohomology

In this section, we provide an alternative description of the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ of an $\mathbf{F}_p$-algebra R using the theory of *derived crystalline cohomology* or equivalently the *derived de Rham-Witt complex* (cf. [9] and [27, Ch. VIII]). Our main result (Theorem 9.3.1) can be summarized informally as follows: the derived de Rham-Witt complex $LW\Omega_R$ of an $\mathbf{F}_p$-algebra R admits a natural "divided Frobenius" map $\alpha_R: LW\Omega_R \to L\eta_p(LW\Omega_R)$, and inverting the map α_R yields the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ studied in this paper.

We give two applications of the preceding description. First, we prove an analog of the Berthelot-Ogus isogeny theorem [4] for the saturated de Rham-Witt complex without any regularity constraints: if R is an $\mathbf{F}_p$-algebra with embedding dimension $\leq d$, then the Frobenius endomorphism of $\mathcal{W}\Omega^*_R$ factors multiplication by p^d (Corollary 9.3.6); in particular, it is a p-isogeny. Second, we give an alternative proof that the comparison map $\Omega^*_R \to \mathcal{W}_1\Omega^*_R$ is an isomorphism for a regular Noetherian $\mathbf{F}_p$-algebra R (Theorem 4.3.1), which avoids the use of Popescu's theorem.

9.1. **Lax Fixed Points.** We begin by introducing a variant of Definition 7.4.3.

Definition 9.1.1. Let $\mathcal{C}$ be an ∞-category and let $T: \mathcal{C} \to \mathcal{C}$ be a functor. Form a pullback diagram

$$\begin{tikzcd} \mathcal{C}^T_{\mathrm{lax}} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d] \\ \mathcal{C} \arrow[r, "(\mathrm{id}, T)"] & \mathcal{C} \times \mathcal{C}. \end{tikzcd}$$

Then $\mathcal{C}^T_{\mathrm{lax}}$ is an ∞-category, whose objects can be identified with pairs (C, φ) where C is an object of $\mathcal{C}$ and $\varphi: C \to TC$ is a morphism in $\mathcal{C}$. We will refer to $\mathcal{C}^T_{\mathrm{lax}}$ as the *∞-category of lax fixed points of T on $\mathcal{C}$.*

Example 9.1.2. Let $\mathbf{DC}^{\mathrm{tf}} \subseteq \mathbf{DC}$ be the subcategory spanned by the torsion-free Dieudonné complexes. Then $\mathbf{DC}^{\mathrm{tf}}$ can be identified with the category of lax fixed points for η_p on the category $\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}$ of torsion-free cochain complexes of abelian groups (see Remark 2.1.4).

Construction 9.1.3. Let $\mathcal{C}$ be an ∞-category and let $T: \mathcal{C} \to \mathcal{C}$ be a functor. Then we can regard the fixed point ∞-category $\mathcal{C}^T$ of Definition 7.4.3 as a full subcategory of the lax fixed point ∞-category $\mathcal{C}^T_{\mathrm{lax}}$ of Definition 9.1.1. If $\mathcal{C}$ admits filtered colimits which are preserved by the functor T, then the inclusion $\mathcal{C}^T \hookrightarrow \mathcal{C}^T_{\mathrm{lax}}$ admits a left adjoint, which carries a pair (C, φ) to the colimit of the diagram

$$C \xrightarrow{\varphi} TC \xrightarrow{T\varphi} T^2C \xrightarrow{T^2\varphi} T^3C \to \cdots$$

Compare the proof of [41, Prop. II.5.3].

Example 9.1.4. In the case where $\mathcal{C} = \mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}}$ and T is the functor η_p, the functor

$$\mathbf{DC}^{\mathrm{tf}} = (\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}})^{\eta_p}_{\mathrm{lax}} \to (\mathrm{Chain}(\mathbf{Z})^{\mathrm{tf}})^{\eta_p} = \mathbf{DC}_{\mathrm{sat}}$$

given in Construction 9.1.3 agrees with the saturation functor on Dieudonné complexes introduced in §2.3.

Notation 9.1.5. Let $\widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}}$ denote the lax fixed points for $L\eta_p$, regarded as a functor from the p-complete derived ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p$ to itself. Note that the functor $L\eta_p \colon \widehat{\mathcal{D}(\mathbf{Z})}_p \to \widehat{\mathcal{D}(\mathbf{Z})}_p$ preserves filtered colimits, since it commutes with filtered colimits on the larger ∞-category $\mathcal{D}(\mathbf{Z})$ and with the operation of p-completion (Proposition 7.2.4). Applying Construction 9.1.3, we see that the inclusion $\widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{p} \hookrightarrow \widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}}$ admits a left adjoint, which we will denote by

$$\widehat{\mathrm{Sat}} \colon \widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}} \to \widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{p}.$$

Remark 9.1.6. The ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}}$ can be described as the *lax equalizer* of the pair of functors $(\mathrm{id}, L\eta_p) \colon \widehat{\mathcal{D}(\mathbf{Z})}_p \to \widehat{\mathcal{D}(\mathbf{Z})}_p$, in the sense of [41, Def. II.1.4]. It follows that $\widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}}$ is a presentable ∞-category and that the forgetful functor $\widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}} \to \widehat{\mathcal{D}(\mathbf{Z})}_p$ preserves small colimits.

Remark 9.1.7. Let M^* be a torsion-free Dieudonné complex having image $M \in \mathcal{D}(\mathbf{Z})$. Then we can regard M as a lax fixed point for the functor $L\eta_p \colon \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$, so that its derived p-completion $\widehat{M}$ is a lax fixed point for the restriction $L\eta_p \colon \widehat{\mathcal{D}(\mathbf{Z})}_p \to \widehat{\mathcal{D}(\mathbf{Z})}_p$. Using Corollary 2.8.2, we see that the saturation functor $\widehat{\mathrm{Sat}}$ of Construction 9.1.3 carries $\widehat{M}$ to the completed saturation $\mathcal{W}\mathrm{Sat}(M^*)$ of Notation 2.8.4. In other words, the diagram of ∞-categories

$$\begin{tikzcd}
\mathbf{DC}^{\mathrm{tf}} \arrow[r, "\mathcal{W}\mathrm{Sat}"] \arrow[d, "M^* \mapsto \widehat{M}"] & \mathbf{DC}_{\mathrm{str}} \arrow[d] \\
\widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}} \arrow[r] & \widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{p}
\end{tikzcd}$$

commutes up to canonical isomorphism. Here the right vertical map is an equivalence by virtue of Corollary 7.4.9.

9.2. **Digression: Nonabelian Derived Functors.** We now give a brief review of the theory of nonabelian derived functors in the setting of ∞-categories, following [37, §5.5.9]. For the sake of concreteness, we will confine our attention to functors which are defined on the ∞-category of simplicial commutative $\mathbf{F}_p$-algebras.

Notation 9.2.1. Let $\mathrm{SCR}_{\mathbf{F}_p}$ denote the ∞-category of simplicial commutative $\mathbf{F}_p$-algebras. More precisely, we define $\mathrm{SCR}_{\mathbf{F}_p}$ to be the ∞-category obtained from the ordinary category of simplicial commutative $\mathbf{F}_p$-algebras (see [44]) by formally inverting quasi-isomorphisms. We refer to [39, Chapter 25] for a detailed treatment of the ∞-category $\mathrm{SCR}_{\mathbf{F}_p}$.

Let $\mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}$ denote the ordinary category whose objects are $\mathbf{F}_p$-algebras of the form $\mathbf{F}_p[x_1, \ldots, x_n]$. In what follows, we will identify $\mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}$ with a full subcategory of the ∞-category $\mathrm{SCR}_{\mathbf{F}_p}$. By virtue of [37, Proposition 5.5.8.15], the ∞-category $\mathrm{SCR}_{\mathbf{F}_p}$ can be characterized by a universal property:

Proposition 9.2.2. *Let $\mathcal{D}$ be any ∞-category which admits small sifted colimits, and let $\mathrm{Fun}'(\mathrm{SCR}_{\mathbf{F}_p}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{SCR}_{\mathbf{F}_p}, \mathcal{D})$ spanned by those functors which preserve small sifted colimits. Then composition with the inclusion functor $\mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p} \hookrightarrow \mathrm{SCR}_{\mathbf{F}_p}$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}'(\mathrm{SCR}_{\mathbf{F}_p}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}, \mathcal{D}).$$

We can summarize Proposition 9.2.2 more informally as follows: any functor $f\colon \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p} \to \mathcal{D}$ admits an essentially unique extension to a functor $F\colon \mathrm{SCR}_{\mathbf{F}_p} \to \mathcal{D}$ which preserves small sifted colimits. In this case, we will refer to F as the *nonabelian derived functor* of f.

Example 9.2.3. Let $\mathcal{D} = \mathcal{D}(\mathbf{F}_p)$ be the derived ∞-category of $\mathbf{F}_p$ and let $f\colon \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p} \to \mathcal{D}(\mathbf{F}_p)$ be the functor given by $f(R) = \Omega^1_{R/\mathbf{F}_p}$. Then the nonabelian derived functor $F\colon \mathrm{SCR}_{\mathbf{F}_p} \to \mathcal{D}$ carries a simplicial commutative $\mathbf{F}_p$-algebra R to the *cotangent complex* $L_{R/\mathbf{F}_p}$ (regarded as an object of $\mathcal{D}(\mathbf{F}_p)$ via restriction of scalars).

Remark 9.2.4. Let $\mathcal{D}$ be an ∞-category which admits small sifted colimits, let $f\colon \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p} \to \mathcal{D}$ be a functor, and let $F\colon \mathrm{SCR}_{\mathbf{F}_p} \to \mathcal{D}$ be its nonabelian left derived functor. Then F can be characterized as the *left Kan extension* of the functor f (see [37, Sec. 4.3.2]). It follows that if $\mathcal{C} \subseteq \mathrm{SCR}_{\mathbf{F}_p}$ is a full subcategory containing $\mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}$ and $G\colon \mathcal{C} \to \mathcal{D}$ is any functor, then the restriction map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F|_{\mathcal{C}}, G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p},\mathcal{D})}(f, G|_{\mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}})$$

is a homotopy equivalence. We will be primarily interested in the special case where $\mathcal{C} = \mathrm{CAlg}_{\mathbf{F}_p}$ is the category of commutative $\mathbf{F}_p$-algebras.

We now restrict our attention to the case of primary interest to us.

Construction 9.2.5 (The derived de Rham-Witt complex)**.** For each polynomial algebra $R \in \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}$, let us abuse notation by identifying the de Rham-Witt complex $W\Omega^*_R \simeq \mathcal{W}\Omega^*_R$ with its image in the (p-complete) derived ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p$. The construction $R \mapsto W\Omega^*_R$ determines a functor $f\colon \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p} \to \widehat{\mathcal{D}(\mathbf{Z})}_p$, which admits a nonabelian left derived functor $F\colon \mathrm{SCR}_{\mathbf{F}_p} \to \widehat{\mathcal{D}(\mathbf{Z})}_p$. Given a simplicial commutative $\mathbf{F}_p$-algebra R, we will denote its image under F by $LW\Omega_R$, and refer to it as the *derived de Rham-Witt complex of* R. We refer the reader to [12] for a more detailed discussion.

Note that for $R \in \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}$, we can regard $W\Omega^*_R$ as a Dieudonné complex. In particular, it can be regarded as a lax fixed point for the functor $L\eta_p$ on $\widehat{\mathcal{D}(\mathbf{Z})}_p$; that is, we can promote f to a functor $\widetilde{f}\colon \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p} \to \widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}}$. Applying Proposition 9.2.2, we see that $\widetilde{f}$ admits a nonabelian left derived functor $\widetilde{F}\colon \mathrm{SCR}_{\mathbf{F}_p} \to \widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}}$. Since the forgetful functor $\widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}} \to \widehat{\mathcal{D}(\mathbf{Z})}_p$ preserves small colimits (Remark 9.1.6), we can regard $\widetilde{F}$ as a lift of the functor F. Given a simplicial commutative $\mathbf{F}_p$-algebra R, we will abuse notation by denoting its image under the functor $\widetilde{F}$ also by $LW\Omega_R$. In other words, we regard $\widetilde{F}$ as supplying a canonical map $\alpha_R\colon LW\Omega_R \to L\eta_p(LW\Omega_R)$, for each simplicial commutative $\mathbf{F}_p$-algebra R.

Variant 9.2.6 (The derived de Rham complex)**.** For any $R \in \mathrm{SCR}_{\mathbf{F}_p}$, write

$$L\Omega_{R/\mathbf{F}_p} := LW\Omega_R \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p \in \mathcal{D}(\mathbf{F}_p).$$

We will refer to $L\Omega_{R/\mathbf{F}_p}$ as the *derived de Rham complex* of R. Using Theorem 4.3.1 and Corollary 2.7.2, we see that the construction $R \mapsto L\Omega_{R/\mathbf{F}_p}$ is the nonabelian left derived functor of the usual de Rham complex functor $\Omega^*_{-/\mathbf{F}_p}\colon \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p} \to \mathcal{D}(\mathbf{F}_p)$.

Remark 9.2.7 (The Conjugate Filtration)**.** For every polynomial algebra $R \in \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}$, let $\tau^{\leq n}\Omega^*_{R/\mathbf{F}_p}$ denote the nth truncation of the de Rham complex $\Omega^*_{R/\mathbf{F}_p}$, which we regard as an object of the derived ∞-category $\mathcal{D}(\mathbf{F}_p)$. The construction $R \mapsto \tau^{\leq n}\Omega^*_{R/\mathbf{F}_p}$ admits a nonabelian left derived functor

$$\mathrm{SCR}_{\mathbf{F}_p} \to \mathcal{D}(\mathbf{F}_p) \qquad R \mapsto \mathrm{Fil}^{\mathrm{conj}}_n L\Omega_{R/\mathbf{F}_p}.$$

Then we can regard $\{\mathrm{Fil}^{\mathrm{conj}}_n L\Omega_{R/\mathbf{F}_p}\}_{n\geq 0}$ as an exhaustive filtration of the derived de Rham complex $L\Omega_{R/\mathbf{F}_p}$, which we refer to as the *conjugate filtration*. Using the Cartier isomorphism for $R \in \mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}$, we obtain canonical isomorphisms

$$\mathrm{gr}^n \mathrm{Fil}^{\mathrm{conj}}_* L\Omega_{R/\mathbf{F}_p} \simeq (\bigwedge^n L_{R^{(1)}/\mathbf{F}_p})[-n]$$

for all $R \in \mathrm{SCR}_{\mathbf{F}_p}$; here $R^{(1)} \to R$ denotes the Frobenius endomorphism of R.

Remark 9.2.8. For any commutative $\mathbf{F}_p$-algebra R, there is a canonical map $L\Omega_{R/\mathbf{F}_p} \to \Omega^*_{R/\mathbf{F}_p}$, which is uniquely determined by the requirement that it depends functorially on R and is the identity when R is a polynomial algebra over $\mathbf{F}_p$. Using the conjugate filtration of Remark 9.2.7, one can show that this map is a quasi-isomorphism when R is a smooth algebra over a perfect field of characteristic p (see [9, Cor. 3.10]).

9.3. **Saturated Derived Crystalline Cohomology.** If R is a polynomial algebra over $\mathbf{F}_p$, then the de Rham-Witt complex $W\Omega^*_R$ is a *strict* Dieudonné algebra. In particular, we can regard $W\Omega^*_R$ as a fixed point (rather than a lax fixed point) for the endofunctor $L\eta_p: \widehat{\mathcal{D}(\mathbf{Z})}_p \to \widehat{\mathcal{D}(\mathbf{Z})}_p$. In general, the derived de Rham-Witt complex $LW\Omega_R$ of Construction 9.2.5 need not have this property: the canonical map $\alpha_R: LW\Omega_R \to L\eta_p(LW\Omega_R)$ need not be an isomorphism, since the functor $L\eta_p$ does not commute with sifted colimits. This can be corrected by passing to the saturation $\widehat{\mathrm{Sat}}(LW\Omega_R)$ in the sense of Notation 9.1.5. This saturation admits a more concrete description:

Theorem 9.3.1. *Let R be a simplicial commutative $\mathbf{F}_p$-algebra. Then there is a canonical map*

$$\eta_R: LW\Omega_R \to \mathcal{W}\Omega^*_{\pi_0(R)}$$

which exhibits the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{\pi_0(R)}$ as a saturation of $LW\Omega_R$, in the sense of Notation 9.1.5.

Remark 9.3.2. Concretely, Theorem 9.3.1 asserts that the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_{\pi_0(R)}$ can be computed as the p-completed direct limit of the diagram

$$LW\Omega_R \xrightarrow{\alpha_R} L\eta_p(LW\Omega_R) \xrightarrow{L\eta_p(\alpha_R)} L\eta_{p^2}(LW\Omega_R) \to \cdots.$$

The proof of Theorem 9.3.1 will make use of the following:

Lemma 9.3.3. *The category $\mathbf{DA}_{\mathrm{str}}$ of strict Dieudonné algebras admits small colimits, and the forgetful functor $\mathbf{DA}_{\mathrm{str}} \to \mathbf{DC}_{\mathrm{str}}$ commutes with sifted colimits.*

Proof. By virtue of Corollary 3.5.10, the colimit of a diagram $\{A^*_\alpha\}$ in the category $\mathbf{DA}_{\mathrm{str}}$ can be computed by applying the completed saturation functor $\mathcal{W}\mathrm{Sat}$ to the colimit $\varinjlim A^*_\alpha$, formed in the larger category $\mathbf{DA}$ of all Dieudonné algebras. It will therefore suffice to observe that the forgetful functor $\mathbf{DA} \to \mathbf{DC}$ preserves sifted colimits. □

Proof of Theorem 9.3.1. We wish to prove that the diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{SCR}_{\mathbf{F}_p} \arrow[r, "\pi_0"] \arrow[d, "LW\Omega_\bullet"] & \mathrm{CAlg}_{\mathbf{F}_p} \arrow[r, "\mathcal{W}\Omega^*_\bullet"] & \mathbf{DA}_{\mathrm{str}} \arrow[d] \\
\widehat{\mathcal{D}(\mathbf{Z})}^{L\eta_p}_{\mathrm{lax}} \arrow[r, "\widehat{\mathrm{Sat}}"] & \widehat{\mathcal{D}}(\mathbf{Z})^{L\eta_p} \arrow[r, "\sim"] & \mathbf{DC}_{\mathrm{str}}
\end{tikzcd}$$

commutes (up to natural isomorphism), where the lower right horizontal map is an inverse to the equivalence of Corollary 7.4.9. Note that the horizontal maps in this diagram admit right adjoints, and therefore preserve all small colimits. The left vertical map preserves sifted colimits by construction, and the right vertical map preserves sifted colimits by Lemma 9.3.3. Using the universal property of $\mathrm{SCR}_{\mathbf{F}_p}$ (Proposition 9.2.2), we are reduced to showing that this diagram commutes when restricted to the full subcategory $\mathrm{CAlg}^{\mathrm{poly}}_{\mathbf{F}_p}$. This follows from the observation that the canonical maps

$$\widehat{\mathrm{Sat}}(LW\Omega_R) \leftarrow LW\Omega_R \rightarrow \mathcal{W}\Omega^*_R$$

are isomorphisms when R is a polynomial algebra over $\mathbf{F}_p$ (on the left because the divided Frobenius $W\Omega^*_R \rightarrow \eta_p W\Omega^*_R$ is an isomorphism, and on the right because of Theorem 4.4.12). □

Remark 9.3.4. For any commutative $\mathbf{F}_p$-algebra R, the derived de Rham-Witt complex $LW\Omega_R$ provides a lift to $\mathbf{Z}_p$ of the derived de Rham complex $L\Omega_R$. In particular, if R is not a local complete intersection, then $LW\Omega_R$ potentially has cohomology in infinitely many negative (cohomological) degrees; for instance, this holds if R itself admits a lift to $\mathbf{Z}_p$ with a lift of Frobenius (by [10, Theorem 1.5]). By virtue of Remark 9.3.2, we can identify the saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$ with the colimit of the diagram

$$LW\Omega_R \xrightarrow{\alpha_R} L\eta_p(LW\Omega_R) \xrightarrow{L\eta_p(\alpha_R)} L\eta_{p^2}(LW\Omega_R) \rightarrow \cdots,$$

computed in the ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p$. It follows that the formation of this direct limit has the effect of killing the cohomology of $LW\Omega_R$ in negative degrees.

Remark 9.3.5 (Comparison with crystalline cohomology)**.** Let k be a perfect field of characteristic p and let R be a commutative k-algebra which is a local complete intersection over k. In this case, one the main theorems of [9] identifies $LW\Omega_R$ with the crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$. In particular, for such R, Theorem 9.3.1 supplies a canonical map

$$\beta_R\colon \mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \rightarrow \mathcal{W}\Omega^*_R$$

in the derived category $D(\mathbf{Z})$ (or even the lax fixed point ∞-category $\mathcal{D}(\mathbf{Z})^{L\eta_p}_{\mathrm{lax}}$). Using Remark 9.2.8, we see that this map is a quasi-isomorphism when R is smooth over k.

Using the description of saturated de Rham-Witt complexes via derived de Rham cohomology, we can prove an analog of the Berthelot-Ogus isogeny theorem [4, Theorem 1.6] in our setting.

Corollary 9.3.6. *Let R be an $\mathbf{F}_p$-algebra. Then the endomorphism $\varphi^*\colon \mathcal{W}\Omega^*_R \rightarrow \mathcal{W}\Omega^*_R$ induced by the Frobenius endomorphism $\varphi\colon R \rightarrow R$ is a p-isogeny: that is, it induces an isomorphism after inverting the prime p. Moreover, if the R-module Ω^1_R can be locally generated by $\leq d$ elements, then multiplication by p^d factors through φ^*.*

Proof. Let us begin with a general remark. For any Dieudonné algebra (A^*, d, F), let $\varphi_{A^*}: A^* \to A^*$ be the map of cochain complexes determined by $\varphi_{A^*} = p^n F$ in degree n for all n: that is, the composition $A^* \xrightarrow{\alpha_F} \eta_p A^* \xrightarrow{i_A} A^*$ where i_A is the natural inclusion. If A^* is a saturated Dieudonné algebra, then $\varphi_{A^*}[1/p]$ is an isomorphism: the map α_F is an isomorphism because A^* is saturated, and the map $i_A[1/p]$ is always an isomorphism. If, in addition, we have $A^i = 0$ for $i > d$, then φ_{A^*} divides p^d: the subcomplex $p^d A^*$ is contained inside $i_A(\eta_p A^*) = \varphi_{A^*}(A^*) \subseteq A^*$.

Applying the preceding observation to $A^* = \mathcal{W}\Omega^*_R$, we deduce that φ^* is a p-isogeny (note that $\varphi_{\mathcal{W}\Omega^*_R} = \varphi^*$ by the universal property of the saturated de Rham-Witt complex). For the last assertion, it will suffice to show that the group $\mathcal{W}\Omega^i_R$ vanishes for $i > d$. By virtue of Theorem 5.2.2, this vanishing can be tested locally on $\operatorname{Spec}(R)$, so we may assume Ω^1_R is generated by d elements. Since the Dieudonné algebra $\mathcal{W}\Omega^*_R$ is strict, it will suffice to show that the groups

$$\mathcal{W}_r\Omega^i_R \simeq \mathrm{H}^i(\mathcal{W}\Omega^*_R/p^r)$$

vanish for $i > d$ and $r \geq 1$. Proceeding by induction on r, we can assume without loss of generality that $r = 1$. By virtue of Remark 9.3.2, it is enough to show that the group $\mathrm{H}^i(\mathbf{F}_p \otimes L\eta_{p^n}(LW\Omega_R))$ vanishes for $i > d$ and all $n \geq 0$: that is, that the object $\mathbf{F}_p \otimes^L_{\mathbf{Z}} L\eta_{p^n}(LW\Omega^*_R)$ belongs to the subcategory $D(\mathbf{F}_p)^{\leq d}$ of the derived category $D(\mathbf{F}_p)$. Because the functor $L\eta_{p^n}$ commutes with truncations, it suffices to check that $\mathbf{F}_p \otimes^L_{\mathbf{Z}} LW\Omega_R$ belongs to $D(\mathbf{F}_p)^{\leq d}$. Note that $\mathbf{F}_p \otimes^L_{\mathbf{Z}} LW\Omega_R$ can be identified with the derived de Rham complex $L\Omega_R$ (see Variant 9.2.6). Using the conjugate filtration (Remark 9.2.7), we are reduced to checking that the derived exterior powers $\bigwedge^i_R L_{R/\mathbf{F}_p}$ belong to $D(R)^{\leq d-i}$ for all $i \geq 0$.

Choose a sequence of elements $x_1, \dots, x_d$ which generate the R-module Ω^1_R, so that the x_i determine a map $f: R^d \to L_{R/\mathbf{F}_p}$ in the derived category $D(R)$. Let K denote the cone of f, so that K belongs to $D(R)^{\leq -1}$. Using the distinguished triangle

$$R^d \to L_{R/\mathbf{F}_p} \to K \to R^d[1],$$

we see that $\bigwedge^i_R L_{R/\mathbf{F}_p}$ admits a finite filtration whose successive quotients have the form $\bigwedge^a_R(R^d) \otimes^L_R \bigwedge^{i-a}_R(K)$ for $0 \leq a \leq i$. Note that each $\bigwedge^a_R(R^d)$ is a projective R-module of finite rank which vanishes for $a > d$. The desired result now follows from Quillen's connectivity estimate on derived exterior powers (Lemma 9.3.7 below). □

Lemma 9.3.7 (Quillen's connectivity estimate)**.** *Let R be a commutative ring, and let $M \in D(R)^{\leq -1}$. Then $\bigwedge^i M \in D(R)^{\leq -i}$ for each $i \geq 0$.*

Proof. See [43, Cor. 7.40] for this as well as additional connectivity assertions on derived symmetric and exterior powers. Alternatively, the result follows from Illusie's formula $\bigwedge^i(N[1]) \simeq (\Gamma^i N)[i]$ for Γ^i the ith divided power functor, cf. [26, Ch. I, Sec. 4.3.2] and the fact that, by construction, Γ^i is defined as a functor $D(R)^{\leq 0} \to D(R)^{\leq 0}$. □

Remark 9.3.8. In contrast with Corollary 9.3.6, the endomorphism

$$\varphi^*: \mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R)) \to \mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$$

induced by the Frobenius map $\varphi: R \to R$ is generally not an p-isogeny if R is not regular. For instance, this phenomenon can already be seen when $R = \mathbf{F}_p[x]/(x^2)$ when p is odd. Let D denote the p-adically completed divided power envelope of the natural surjection $\mathbf{Z}_p[x] \to \mathbf{F}_p[x]/(x^2)$ (where the divided powers are required to be compatible with those on (p)), given explicitly by the formula

$$\begin{aligned} D &\simeq \mathbf{Z}_p\left[x, \{\frac{x^{2k}}{k!}\}_{k\geq 1}\right]^{\wedge} \\ &\simeq \big(\bigoplus_{m\geq 0} \mathbf{Z}_p \cdot \frac{x^{2m}}{m!}\big)^{\wedge} \oplus \big(\bigoplus_{n\geq 0} \mathbf{Z}_p \cdot \frac{x^{2n+1}}{n!}\big)^{\wedge}, \end{aligned}$$

where the completions are p-adic. Standard results in crystalline cohomology (see [6, Theorem 7.23]) allow us to identify $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ with the de Rham complex

$$A^* := \Big(D \xrightarrow{d} D \cdot dx\Big).$$

Under this identification, the Frobenius endomorphism of $\mathrm{R}\Gamma_{\mathrm{crys}}(\mathrm{Spec}(R))$ is induced by the map $x \mapsto x^p$ of D. A simple calculation then gives an isomorphism

$$\mathrm{H}^1(A^*) \simeq \Big(\bigoplus_{n\geq 0} \mathbf{Z}_p/(2n+1) \cdot \frac{x^{2n}}{n!} \cdot dx\Big)^{\wedge} \oplus \Big(\bigoplus_{m\geq 0} \mathbf{Z}_p/(2) \cdot \frac{x^{2m+1}}{m!} \cdot dx\Big)^{\wedge},$$

where again the completions are p-adic. When p is odd, the second summand above vanishes, but the first contains non-torsion elements: for example, it contains the non-torsion $\mathbf{Z}_p$-module

$$\Big(\bigoplus_{k\geq 0} \mathbf{Z}_p/(p^k)\Big)^{\wedge}$$

as a direct summand. In particular, $\mathrm{H}^1_{\mathrm{crys}}(\mathrm{Spec}(R))[\frac{1}{p}] \neq 0$. On the other hand, the Frobenius map $\varphi: \mathbf{F}_p[x]/(x^2) \to \mathbf{F}_p[x]/(x^2)$ factor through the inclusion map $\mathbf{F}_p \hookrightarrow \mathbf{F}_p[x]/(x^2)$, so the induced map $\varphi^*: \mathrm{H}^1_{\mathrm{crys}}(\mathrm{Spec}(R)) \to \mathrm{H}^1_{\mathrm{crys}}(\mathrm{Spec}(R))$ vanishes. In particular, the map $\varphi^*[1/p]$ is not an isomorphism.

Remark 9.3.9. Fix a perfect field k of characteristic p, and let $K = W(k)[1/p]$. The construction $R \mapsto \mathcal{W}\Omega^*_R[1/p]$ determines a functor from category of finite type k-algebras to the derived ∞-category $\mathcal{D}(K)$ of the field K. This presheaf is a sheaf for the étale topology (Theorem 5.3.7) and carries universal homeomorphisms to quasi-isomorphisms (Corollary 9.3.6, because universal homeomorphisms between perfect $\mathbf{F}_p$-schemes are isomorphisms as in, e.g., [13, Lemma 3.8]). It is tempting to guess that this functor is a sheaf for the h-topology. However, we do not even know if it satisfies fppf descent.

9.4. **Digression: The Cotangent Complex of a Regular Ring.** Let R be a commutative ring. We will say that an object of the derived category $D(R)$ is *flat* if it is isomorphic to a flat R-module, regarded as a cochain complex concentrated in degree zero. Our goal in this section is to prove the following result which will be needed in §9.6:

Proposition 9.4.1. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the cotangent complex $L_{R/\mathbf{F}_p}$ is flat.*

Remark 9.4.2. If R is smooth over $\mathbf{F}_p$, then $L_{R/\mathbf{F}_p} \simeq \Omega^1_{R/\mathbf{F}_p}$ is flat; therefore, Proposition 9.4.1 follows from Popescu's theorem since any regular Noetherian $\mathbf{F}_p$-algebra is a filtered colimit of smooth algebras. Our argument will be independent of Popescu's theorem, though.

Our proof will use the following flatness criterion. This is a special case of a more general result on flat dimension in [3, Prop. 5.3F].

Lemma 9.4.3. *Let R be a Noetherian ring and fix $M \in D^{\leq 0}(R)$. Then M is flat if and only if $M \otimes^L_R k$ belongs to $D^{\geq 0}(R)$, for each residue field k of R.*

Proof. We will prove the following:

($*$) For every R-module N, the derived tensor product $M \otimes^L_R N$ belongs to the heart of $D(R)$.

Applying ($*$) in the case $N = R$, we deduce that M belongs to the heart of $D(R)$, in which case ($*$) implies the flatness of M. Since the construction $N \mapsto \mathrm{H}^*(M \otimes^L_R N)$ commutes with filtered colimits, it will suffice to prove assertion ($*$) in the case where N is a finitely generated R-module. Proceeding by Noetherian induction, we may assume that ($*$) is satisfied for every finitely generated R-module N' with $\mathrm{supp}(N') \subsetneq \mathrm{supp}(N)$ (as subsets of $\mathrm{Spec}(R)$). Writing N as a finite extension, we can reduce to the case where $N \simeq R/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subseteq R$. Let k denote the fraction field of $R/\mathfrak{p}$. We have an exact sequence of R-modules

$$0 \to N \to k \to N' \to 0,$$

where N' can be written as a filtered colimit of finitely generated R-modules whose support is a proper subset of $\mathrm{supp}(N)$. We therefore obtain a distinguished triangle

$$M \otimes^L_R N \to M \otimes^L_R k \to M \otimes^L_R N' \to (M \otimes^L_R N)[1],$$

where $M \otimes^L_R N'$ belongs to $D^{\geq 0}(R)$ (by our inductive hypothesis), and $M \otimes^L_R k$ belongs to $D^{\geq 0}(R)$ (by assumption). It follows that $M \otimes^L_R N$ also belongs to $D^{\geq 0}(R)$. Since $D^{\leq 0}(R)$ is closed under derived tensor products, the object $M \otimes^L_R N$ belongs to the heart $D^{\leq 0}(R) \cap D^{\geq 0}(R)$ of $D(R)$. ∎

For later reference, we record the following consequence of Lemma 9.4.3:

Corollary 9.4.4. *Let R be a Noetherian ring and let $f: M \to N$ be a homomorphism of flat R-modules. Assume that, for each residue field k of R, the map $f_k: M \otimes_R k \to N \otimes_R k$ is a monomorphism. Then f is a monomorphism and the cokernel* $\mathrm{coker}(f)$ *is flat. If each f_k is an isomorphism, then f is an isomorphism.*

Proof. The first assertion follows by applying Lemma 9.4.3 to the cone $\mathrm{cn}(f)$ of f (formed in the derived category $D(R)$), and the second assertion from applying Lemma 9.4.3 to the shift $\mathrm{cn}(f)[1]$. $\square$

Proof of Proposition 9.4.1. Let R be a regular Noetherian $\mathbf{F}_p$-algebra. By virtue of Lemma 9.4.3, it will suffice to show that $L_{R/\mathbf{F}_p} \otimes^L_R k$ belongs to $D^{\geq 0}(R)$ for each residue field k of R. As the formation of the cotangent complex commutes with localization, we may assume R is a regular local ring with maximal ideal $\mathfrak{m} \subseteq R$ and residue field $k \simeq R/\mathfrak{m}$. Writing the field k as a filtered colimit of smooth $\mathbf{F}_p$-algebras (using generic smoothness applied to any finitely generated extension of the perfect field $\mathbf{F}_p$), we see that $L_{k/\mathbf{F}_p}$ belongs to the heart of $D(k)$. Since R is regular, the kernel of the map $R \to k$ is generated by a regular sequence, so that $L_{k/R}[-1] \simeq \mathfrak{m}/\mathfrak{m}^2$ also belongs to the heart of $D(k)$. The transitivity triangle for $\mathbf{F}_p \to R \to k$ gives a distinguished triangle

$$L_{k/R}[-1] \to L_{R/\mathbf{F}_p} \otimes^L_R k \to L_{k/\mathbf{F}_p} \to L_{k/R}.$$

which shows that $L_{R/\mathbf{F}_p} \otimes^L_R k$ also belongs to the heart of $D(k)$. $\square$

Corollary 9.4.5. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then, for each $n \geq 0$, the module Ω^n_R is flat over R.*

Proof. For $n = 1$, this follows from Proposition 9.4.1. The general case then follows from the definition $\Omega^n_R = \bigwedge^n(\Omega^1_R)$. $\square$

Corollary 9.4.6. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Assume that R is local with maximal ideal $\mathfrak{m}$, and let $\widehat{R}$ denote the completion of R. Let N be an R-module on which the action of $\mathfrak{m}$ is locally nilpotent (so that N can also be regarded as a $\widehat{R}$-module). Then the canonical map*

$$\rho_m: \Omega^m_R \otimes_R N \to \Omega^m_{\widehat{R}} \otimes_{\widehat{R}} N$$

is an isomorphism for each $m \geq 0$.

Proof. Writing N as a union of finitely generated submodules, we can reduce to the case where N is finitely generated and therefore Artinian. In this case, we can write N as an extension of finitely many copies of the residue field k of R. In this case, we can identify ρ_m with the mth exterior power of ρ_1 (in the category of vector spaces over k). Using Proposition 9.4.1, we can identify ρ_1 with the canoncial map $L_{R/\mathbf{F}_p} \otimes^L_R k \to L_{\widehat{R}/\mathbf{F}_p} \otimes_{\widehat{R}} \mathbf{F}_p$. Using the commutative diagram of transitivity sequences

$$\begin{tikzcd} L_{R/\mathbf{F}_p} \otimes^L_R k \arrow[r] \arrow[d, "\rho_1"] & L_{k/\mathbf{F}_p} \arrow[r] \arrow[d, "\mathrm{id}"] & L_{k/R} \arrow[d, "\rho'"] \\ L_{\widehat{R}/\mathbf{F}_p} \otimes^L_{\widehat{R}} k \arrow[r] & L_{k/\mathbf{F}_p} \arrow[r] & L_{k/\widehat{R}}, \end{tikzcd}$$

we are reduced to showing that the map ρ' is an isomorphism. This follows from the observation that the map $R \to \widehat{R}$ induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \simeq \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$, where $\widehat{\mathfrak{m}}$ denotes the maximal ideal of $\widehat{R}$. $\square$

9.5. **The Cartier Isomorphism for Regular $\mathbf{F}_p$-Algebras.** Let R be a commutative $\mathbf{F}_p$-algebra and let

$$\mathrm{Cart}\colon \Omega^*_R \to \mathrm{H}^*(\Omega^*_R)$$

be the Cartier map (Proposition 3.3.4). It follows from Theorem 3.3.6 that Cart is an isomorphism when R is a smooth algebra over a perfect field k. Our goal in this section is to prove the following more general result:

Theorem 9.5.1. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the Cartier map*

$$\mathrm{Cart}\colon \Omega^*_R \to \mathrm{H}^*(\Omega^*_R)$$

is an isomorphism of graded rings.

Remark 9.5.2. It follows from Popescu's theorem that every regular Noetherian $\mathbf{F}_p$-algebra can be realized as a filtered colimit of smooth $\mathbf{F}_p$-algebras, so that Theorem 9.5.1 is a formal consequence of Theorem 3.3.6. However, we will give a more direct argument which avoids the use of Popescu's theorem.

The proof of Theorem 9.5.1 will require some preliminaries.

Notation 9.5.3. Let R be an $\mathbf{F}_p$-algebra. To avoid confusion, we let $R^{(1)}$ denote the same commutative ring R, where we view R as an $R^{(1)}$-module via the Frobenius map $R^{(1)} \to R$.

For each $n \geq 0$, we let $B^n\Omega^*_R$ denote the n-coboundaries in the de Rham complex Ω^*_R: that is, the image of the de Rham differential $d\colon \Omega^{n-1}_R \to \Omega^n_R$ (by convention, we set $B^0\Omega^*_R = (0)$). We will regard the cohomology group $\mathrm{H}^n(\Omega^*_R)$ as a subgroup of the quotient $\Omega^n_R/B^n\Omega^*_R$, and we define $Q^{n+1}\Omega^*_R$ to be the cokernel of the composite map

$$\Omega^n_{R^{(1)}} \to \mathrm{H}^n(\Omega^*_R) \hookrightarrow \Omega^n_R/B^n\Omega^*_R.$$

By construction, we have short exact sequences of $R^{(1)}$-modules

$$\Omega^n_{R^{(1)}} \xrightarrow{\mathrm{Cart}} \Omega^n_R/B^n\Omega^*_R \to Q^{n+1}\Omega^*_R \to 0 \qquad (12)$$

$$Q^{n+1}\Omega^*_R \xrightarrow{d} \Omega^{n+1}_R \to \Omega^{n+1}_R/B^{n+1}\Omega^*_R \to 0. \tag{13}$$

Note that the sequence (12) is exact on the left if and only if the Cartier map $\mathrm{Cart}\colon \Omega^n_{R^{(1)}} \to \mathrm{H}^n(\Omega^*_R)$ is injective, and that the sequence (13) is exact on the left if and only if the Cartier map $\mathrm{Cart}\colon \Omega^n_{R^{(1)}} \to \mathrm{H}^n(\Omega^*_R)$ is surjective (in which case the de Rham differential induces an isomorphism $Q^{n+1}\Omega^*_R \simeq B^{n+1}\Omega^*_R$).

Definition 9.5.4. Let R be an $\mathbf{F}_p$-algebra. We will say that R *has a universal Cartier isomorphism* if it satisfies the following pair of conditions, for every integer $n \geq 0$:

(a_n) The sequence

$$0 \to \Omega^n_{R^{(1)}} \xrightarrow{\mathrm{Cart}} \Omega^n_R/B^n\Omega^*_R \to Q^{n+1}\Omega^*_R \to 0$$

is exact. Moreover, $Q^{n+1}\Omega^*_R$ is a flat $R^{(1)}$-module.

(b_n) The sequence

$$0 \to Q^{n+1}\Omega^*_R \xrightarrow{d} \Omega^{n+1}_R \to \Omega^{n+1}_R/B^{n+1}\Omega^*_R \to 0$$

is exact. Moreover, the quotient $\Omega^{n+1}_R/B^{n+1}\Omega^*_R$ is a flat $R^{(1)}$-module.

Proposition 9.5.5. *Let k be a perfect ring of characteristic p and let R be a smooth k-algebra. Then R has a universal Cartier isomorphism.*

Proof. We will show that R satisfies conditions (a_n) and (b_n) of Definition 9.5.7 using descending induction on n. Note that both conditions are vacuous for $n \gg 0$, since Ω^*_R is concentrated in finitely many degrees. To carry out the inductive step, assume that (a_{n+1}) and (b_{n+1}) are satisfied. Theorem 3.3.6 implies that the Cartier map $\mathrm{Cart}\colon \Omega^n_{R^{(1)}} \to \mathrm{H}^n(\Omega^*_R)$ is an isomorphism, so that the sequences

$$0 \to \Omega^n_{R^{(1)}} \xrightarrow{\mathrm{Cart}} \Omega^n_R/B^n\Omega^*_R \to Q^{n+1}\Omega^*_R \to 0$$

$$0 \to Q^{n+1}\Omega^*_R \xrightarrow{d} \Omega^{n+1}_R \to \Omega^{n+1}_R/B^{n+1}\Omega^*_R \to 0$$

are exact. Assumption (a_{n+1}) guarantees that $\Omega^{n+1}_R/B^{n+1}\Omega^*_R$ can be realized as an extension of the flat $R^{(1)}$-module $Q^{n+2}\Omega^*_R$ by the flat $R^{(1)}$-module $\Omega^{n+1}_{R^{(1)}}$, and is therefore flat over $R^{(1)}$. This proves (b_n). Moreover, since Ω^{n+1}_R is also flat over R (hence also over $R^{(1)}$), it follows that $Q^{n+1}\Omega^*_R \simeq B^{n+1}\Omega^*_R$ is also flat over $R^{(1)}$, which proves (a_n). □

Corollary 9.5.6. *Let k be a field of characteristic p. Then the polynomial ring $k[t_1, \ldots, t_d]$ has a universal Cartier isomorphism.*

Proof. Writing k as a union of finitely generated subfields, we can reduce to the case where k is finitely generated over $\mathbf{F}_p$. In this case, $k[t_1, \ldots, t_d]$ is a localization of a smooth $\mathbf{F}_p$-algebra, so the desired result follows from Proposition 9.5.5. □

In good cases, the conditions of Definition 9.5.7 can be tested pointwise.

Definition 9.5.7. Let R be an $\mathbf{F}_p$-algebra and let k denote the residue field of R at a point $x \in \operatorname{Spec}(R)$. We will say that R *has a universal Cartier isomorphism at* x if it satisfies the following pair of conditions, for every integer $n \geq 0$:

$(a_{n,x})$ The sequence of vector spaces

$$0 \to \Omega^n_{R^{(1)}} \otimes_{R^{(1)}} k^{(1)} \xrightarrow{\text{Cart}} (\Omega^n_R / B^n \Omega^*_R) \otimes_{R^{(1)}} k^{(1)} \to Q^{n+1}\Omega^*_R \otimes_{R^{(1)}} k^{(1)} \to 0$$

is exact.

$(b_{n,x})$ The sequence of vector spaces

$$0 \to (Q^{n+1}\Omega^*_R) \otimes_{R^{(1)}} k^{(1)} \xrightarrow{d} \Omega^{n+1}_R \otimes_{R^{(1)}} k^{(1)} \to (\Omega^{n+1}_R / B^{n+1}\Omega^*_R) \otimes_{R^{(1)}} k^{(1)} \to 0$$

is exact.

Proposition 9.5.8. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then R has a universal Cartier isomorphism if and only if it has a universal Cartier isomorphism at x, for each $x \in \operatorname{Spec}(R)$.*

Proof. The "only if" direction is immediate. For the converse, suppose that R has a universal Cartier isomorphism at x, for each $x \in \operatorname{Spec}(R)$. We will show that R satisfies conditions (a_n) and (b_n) of Definition 9.5.7. The proof proceeds by induction on n. To prove (a_n), we note that the inductive hypothesis (together with Corollary 9.4.5) implies that the Cartier map $\text{Cart}\colon \Omega^n_{R^{(1)}} \to \Omega^n_R / B^n \Omega^n_R$ can be regarded as a morphism of flat $R^{(1)}$-modules. Consequently, assertion (a_n) follows from assertions $(a_{n,x})$ for $x \in \operatorname{Spec}(R)$ by virtue of Corollary 9.4.4. To prove (b_n), we observe that Ω^{n+1}_R is flat as an R-module (Corollary 9.4.5), hence also as an $R^{(1)}$-module (since the regularity of R guarantees that the Frobenius morphism $R^{(1)} \to R$ is flat [34]). Using (a_n), we see that the map $Q^{n+1}\Omega^*_R \to \Omega^{n+1}_R$ is a morphism of flat $R^{(1)}$-modules, so that assertion (b_n) follows from $(b_{n,x})$ for $x \in \operatorname{Spec}(R)$ (again by virtue of Corollary 9.4.4). □

Remark 9.5.9. Let R be a commutative $\mathbf{F}_p$-algebra and let x be a point of $\operatorname{Spec}(R)$, corresponding to a prime ideal $\mathfrak{p} \subseteq R$. Then R has a universal Cartier isomorphism at x if and only if the local ring $R_{\mathfrak{p}}$ has a universal Cartier isomorphism at x (where we abuse notation by identifying x with the closed point of $\operatorname{Spec}(R_{\mathfrak{p}})$).

Remark 9.5.10. Let R be a regular Noetherian local $\mathbf{F}_p$-algebra and let $\widehat{R}$ be its completion. Let x denote the closed point of $\operatorname{Spec}(\widehat{R})$, which we identify with its image in $\operatorname{Spec}(R)$. Using Corollary 9.4.6, we see that R has a universal Cartier isomorphism at x if and only if $\widehat{R}$ has a universal Cartier isomorphism at x.

Theorem 9.5.1 is a consequence of the following slightly more refined statement:

Theorem 9.5.11. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then R has a universal Cartier isomorphism.*

Proof. By virtue of Proposition 9.5.8, it will suffice to show that R has a universal Cartier isomorphism at x, for each point $x \in \operatorname{Spec}(R)$. Using Remark 9.5.9, we can reduce to the case where R is local and x is the closed point of $\operatorname{Spec}(R)$. By virtue of Remark 9.5.10, we can replace R by its completion $\widehat{R}$. Let k denote the residue field of R, so that the Cohen structure theorem supplies an isomorphism $R \simeq k[\![t_1,..,t_d]\!]$. Set $S = k[t_1,\ldots,t_d]$, and let $y \in \operatorname{Spec}(S)$ be the point corresponding to the maximal ideal $(t_1,\ldots,t_d) \subseteq S$. Using Remarks 9.5.9 and 9.5.10 again, we see that R has a universal Cartier isomorphism at x if and only if S has a universal Cartier isomorphism at y. Using Proposition 9.5.8, we are reduced to showing that S has a universal Cartier isomorphism, which follows from Corollary 9.5.6. □

9.6. **The de Rham Comparison for Regular Rings, Redux.** For any commutative $\mathbf{F}_p$-algebra R, the canonical map $R \to \mathcal{W}_1\Omega^0_R$ extends uniquely to a map of commutative differential graded algebras $\nu: \Omega^*_R \to \mathcal{W}_1\Omega^*_R$. In §4.3, we showed that this map is an isomorphism when R is a regular Noetherian ring (Theorem 4.3.1). Our argument proceeded in two steps: first, we showed that ν is an isomorphism when R is a smooth $\mathbf{F}_p$-algebra (Proposition 4.3.2); we then extended this result to regular Noetherian $\mathbf{F}_p$-algebras using Popescu's theorem. In this section, we give a different proof which avoids the use of Popescu's theorem (at the cost of requiring some of the "derived" technology introduced earlier in this section).

Theorem 9.6.1. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the comparison map $\nu: \Omega^*_R \to \mathcal{W}_1\Omega^*_R$ is an isomorphism of commutative differential graded algebras.*

Our proof will proceed in several steps. We begin by reformulating Theorem 9.5.1 as a statement about the *derived* de Rham complex. For any commutative $\mathbf{F}_p$-algebra R, Remark 9.2.4 supplies a comparison map $L\Omega_R \to \Omega^*_R$ (in the derived ∞-category $\mathcal{D}(\mathbf{Z})$), which is determined by the requirement that it depends functorially on R and reduces to the identity when R is a polynomial algebra over $\mathbf{F}_p$.

Proposition 9.6.2. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the comparison map $\rho_R: L\Omega_R \to \Omega^*_R$ is an isomorphism in $\mathcal{D}(\mathbf{Z})$.*

Proof. For each $n \geq 0$, Remark 9.2.4 supplies a comparison map

$$\rho_R^{\leq n}: \operatorname{Fil}_n^{\mathrm{conj}} L\Omega_R \to \tau^{\leq n}\Omega^*_R$$

in the ∞-category $\mathcal{D}(\mathbf{Z})$, where the domain of ρ^n_R is the nth stage of the conjugate filtration of Remark 9.2.7. Note that ρ_R can be identified with a filtered colimit of the maps $\rho_R^{\leq n}$. It will therefore suffice to show that each $\rho_R^{\leq n}$ is an isomorphism when R is regular and Noetherian. Proceeding by induction on n, we are reduced to showing that the induced map of filtration quotients

$$\rho_R^{=n} : (\bigwedge^n L_{R/\mathbf{F}_p})[-n] \to \mathrm{H}^n(\Omega_R^*)[-n]$$

is an isomorphism in $\mathcal{D}(\mathbf{Z})$; here $\bigwedge^n$ denotes the nonabelian derived functor of the nth exterior power. Our assumption that R is regular guarantees that $L_{R/\mathbf{F}_p}$ is flat (Proposition 9.4.1), so that we can identify $\bigwedge^n L_{R/\mathbf{F}_p}$ with Ω_R^n. Under this identification, the map $\rho_R^{=n}$ corresponds to the Cartier map of Proposition 3.3.4 (in degree n), which is an isomorphism by virtue of Theorem 9.5.1. □

Warning 9.6.3. Proposition 9.6.2 is not a formal consequence of the flatness of the cotangent complex $L_{R/\mathbf{F}_p}$. For example, let R be a nonreduced $\mathbf{F}_p$-algebra for which the cotangent complex $L_{R/\mathbf{F}_p}$ vanishes (Gabber has shown that such algebras exist; see [7]). In this case, one can use the conjugate filtration of Remark 9.2.7 to identify the comparison map $L\Omega_R \to \Omega_R^*$ with the Frobenius morphism $R^{(1)} \to R$, which is not an isomorphism when R is not reduced.

Note that if R is any simplicial commutative $\mathbf{F}_p$-algebra, then the identification

$$L\Omega_R \simeq LW\Omega_R \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$$

endows the cohomology ring $\mathrm{H}^*(L\Omega_R)$ with a Bockstein operator, which we will denote by β.

Corollary 9.6.4. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then there exists a unique isomorphism of differential graded algebras $\epsilon\colon (\Omega_R^*, d) \to (\mathrm{H}^*(L\Omega_R), d)$ for which the composite map*

$$\Omega_R^* \xrightarrow{\epsilon} \mathrm{H}^*(L\Omega_R) \xrightarrow{\rho_R} \mathrm{H}^*(\Omega_R^*)$$

is the Cartier morphism of Proposition 3.3.4.

Proof. Using Proposition 9.6.2, we see that there is a unique map of graded rings $\epsilon\colon \Omega_R^* \to \mathrm{H}^*(L\Omega_R)$ for which $\rho_R \circ \epsilon$ is the Cartier morphism. Moreover, Theorem 9.5.1 implies that ϵ is an isomorphism. To show that ϵ is compatible with differentials, it will suffice to show that $\epsilon(dx) = \beta\epsilon(x)$ for each $x \in R$. By functoriality, it suffices to prove this identity in the universal case $R = \mathbf{F}_p[x]$. In this case, the derived de Rham Witt complex is quasi-isomorphic to the completed de Rham complex $\widehat{\Omega}^*_{\mathbf{Z}_p[x]}$. Then $\epsilon(x)$ is the class of the cocycle x^p in the quotient $\widehat{\Omega}^*_{\mathbf{Z}_p[x]}/p\widehat{\Omega}^*_{\mathbf{Z}_p[x]}$, so that $\beta\epsilon(x)$ is represented by the cocycle $\frac{1}{p}d(x^p) = x^{p-1}dx$, and therefore agrees with $\epsilon(dx)$. □

Remark 9.6.5. For any simplicial commutative $\mathbf{F}_p$-algebra R, Construction 9.2.5 produces a canonical map

$$\alpha_R\colon LW\Omega_R \to (L\eta_p)(LW\Omega_R)$$

in the derived ∞-category $\mathcal{D}(\mathbf{Z})$. Extending scalars to $\mathbf{F}_p$, we obtain a map

$$\overline{\alpha}_R : L\Omega_R \to (\mathrm{H}^*(L\Omega_R), \beta)$$

in the derived ∞-category $\mathcal{D}(\mathbf{F}_p)$. In the case where R is a polynomial $\mathbf{F}_p$-algebra, the map $\epsilon : \Omega^*_R \to \mathrm{H}^*(L\Omega_R)$ of Corollary 9.6.4 supplies an *explicit* representative of $\overline{\alpha}_R$ at the level of commutative differential graded algebras. In other words, we have a commutative diagram in the ∞-category $\mathcal{D}(\mathbf{F}_p)$

$$\begin{tikzcd} L\Omega_R \arrow[rr, "\overline{\alpha}_R"] \arrow[dr, "\rho_R"'] & & \mathrm{H}^*(L\Omega_R) \\ & \Omega^*_R, \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

depending functorially on R. Using Remark 9.2.4, we see that this diagram commutes for *all* regular Noetherian $\mathbf{F}_p$-algebras R.

Proposition 9.6.6. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the map $\eta_R : LW\Omega_R \to \mathcal{W}\Omega^*_R$ of Theorem 9.3.1 is an isomorphism (in the ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p$).*

Proof. By virtue of Theorem 9.3.1, the map η_R exhibits $\mathcal{W}\Omega^*_R$ as a saturation of $LW\Omega_R$. It will therefore suffice to show that $LW\Omega_R$ is already saturated: that is, the map

$$\alpha_R : LW\Omega_R \to L\eta_p(LW\Omega_R)$$

of Construction 9.2.5 is an isomorphism in the ∞-category $\widehat{\mathcal{D}(\mathbf{Z})}_p$. Equivalently, it will suffice to show that the map

$$\overline{\alpha}_R : L\Omega_R \to (\mathrm{H}^*(L\Omega_R), \beta)$$

of Remark 9.6.5 is an isomorphism in $\mathcal{D}(\mathbf{F}_p)$. This follows from Remark 9.6.5, Proposition 9.6.2, and Corollary 9.6.4. $\square$

We are now ready to prove a weak version of Theorem 9.6.1:

Proposition 9.6.7. *Let R be a regular Noetherian $\mathbf{F}_p$-algebra. Then the comparison map $\nu : \Omega^*_R \to \mathcal{W}_1\Omega^*_R$ is a quasi-isomorphism.*

Proof. Let $\eta_R : LW\Omega_R \simeq \mathcal{W}\Omega^*_R$ be the isomorphism of Proposition 9.6.6 and let

$$\overline{\eta}_R : L\Omega_R = LW\Omega^*_R \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p \to \mathcal{W}\Omega^*_R \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p \simeq \mathcal{W}_1\Omega^*_R$$

be obtained from η_R by extending scalars along the map $\mathbf{Z}_p \to \mathbf{F}_p$. By virtue of Propositions 9.6.6 and 9.6.2, it will suffice to show that the diagram σ_R:

$$\begin{tikzcd} & L\Omega_R \arrow[dl, "\rho_R"'] \arrow[dr, "\overline{\eta}_R"] & \\ \Omega^*_R \arrow[rr, "\nu"] & & \mathcal{W}_1\Omega^*_R \end{tikzcd}$$

commutes in the derived category $D(\mathbf{F}_p)$. In fact, we make a stronger claim: it is possible to lift σ_R to a commutative diagram in the ∞-category $\mathcal{D}(\mathbf{F}_p)$ which depends functorially on R. By virtue of Remark 9.2.4, it suffices to produce this functorial lift when R is a polynomial algebra over $\mathbf{F}_p$, where the construction is evident. $\square$

Proof of Theorem 9.6.1. Let R be a regular Noetherian $\mathbf{F}_p$-algebra, and consider the diagram of graded algebras

$$\begin{tikzcd} \Omega^*_R \arrow[r, "\mathrm{Cart}"] \arrow[d, "\nu"] & \mathrm{H}^*(\Omega^*_R) \arrow[d, "\mathrm{H}^*(\nu)"] \\ \mathcal{W}_1\Omega^*_R \arrow[r, "F"] & \mathrm{H}^*(\mathcal{W}_1\Omega^*_R) \end{tikzcd}$$

where the top horizontal map is the Cartier morphism of Proposition 3.3.4 (which is an isomorphism by Theorem 9.5.1) and the bottom vertical map is induced by the Frobenius on $\mathcal{W}\Omega^*_R$ (and is therefore an isomorphism by Proposition 2.7.1 and Corollary 2.7.2). Proposition 9.6.7 guarantees that ν induces an isomorphism on cohomology. Consequently, to show that ν is an isomorphism, it will suffice to show that the diagram (9.6) is commutative. In degree 0, the commutativity is clear. Since Ω^*_R is generated as an algebra over Ω^0_R by elements of the form dx, it will suffice to show that

$$(\mathrm{H}^*(\nu)\circ\mathrm{Cart})(dx) = (F\circ\nu)(dx)$$

as elements of $\mathrm{H}^1(\mathcal{W}_1\Omega^*_R)$, for each $x\in R$. Let us abuse notation by identifying R with the ring $\mathcal{W}_1\Omega^0_R$ (see Theorem 6.5.3). Using Proposition 3.6.2, we can identify $\mathcal{W}\Omega^0_R$ with the ring of Witt vectors $W(R)$. For each element $x\in R$, let $[x]$ denote its Teichmüller representative in $\mathcal{W}\Omega^0_R$. Unwinding the definitions, we see that $(\mathrm{H}^*(\nu)\circ\mathrm{Cart})(dx)$ and $(F\circ\nu)(dx)$ are represented by the elements $[x]^{p-1}d[x]$ and $Fd[x]$ in $\mathcal{W}\Omega^1_R$. Since $\mathcal{W}\Omega^*_R$ is p-torsion-free, the desired result follows from the calculation

$$p(Fd[x]) = dF[x] = d[x]^p = p[x]^{p-1}d[x].$$

$\square$

10. The Crystalline Comparison for $A\Omega$

The main goal of this section is to provide a simpler account of the crystalline comparison isomorphism for the $A\Omega$-complexes from [11] using the theory developed in this paper. In [11], one can find two proofs of this result: one involving a comparison over Fontaine's A_{crys}, and the other involving relative de Rham-Witt complexes for mixed characteristic rings. In contrast, the proof presented here uses only the Hodge-Tate comparison and de Rham-Witt complexes for smooth algebras over perfect fields of characteristic p.

10.1. **Review of the Construction of $A\Omega$.** We begin by recalling some pertinent details of the construction of [11]. First, we introduce the basic rings involved.

Notation 10.1.1. Let C denote a complete nonarchimedean valued field of mixed characteristic p. We let $\mathcal{O}_C$ denote the valuation ring of C, $\mathfrak{m} \subseteq \mathcal{O}_C$ the maximal ideal of $\mathcal{O}_C$, $k = \mathcal{O}_C/\mathfrak{m}$ the residue field of $\mathcal{O}_C$, and $W = W(k)$ the ring of Witt vectors of k. We will assume that the field C is *perfectoid*: that is, the valuation on C is nondiscrete and the Frobenius map $\varphi\colon \mathcal{O}_C/p\mathcal{O}_C \to \mathcal{O}_C/p\mathcal{O}_C$ is surjective. Let $\mathcal{O}_C^\flat$ denote the tilt of $\mathcal{O}_C$, given by the inverse limit of the system

$$\cdots \xrightarrow{\varphi} \mathcal{O}_C/p\mathcal{O}_C \xrightarrow{\varphi} \mathcal{O}_C/p\mathcal{O}_C \xrightarrow{\varphi} \mathcal{O}_C/p\mathcal{O}_C.$$

Let A_{inf} denote the ring of Witt vectors $W(\mathcal{O}_C^\flat)$, and let $\varphi\colon A_{\mathrm{inf}} \to A_{\mathrm{inf}}$ denote the automorphism of A_{inf} induced by the Frobenius automorphism of $\mathcal{O}_C^\flat$. We let $\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_C$ denote the unique ring homomorphism for which the diagram

$$\begin{tikzcd} A_{\mathrm{inf}} \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_C \arrow[d] \\ \mathcal{O}_C^\flat \arrow[r] & \mathcal{O}_C/p\mathcal{O}_C \end{tikzcd}$$

commutes, and define $\tilde{\theta} \coloneqq \theta \circ \varphi^{-1}$. Note that the map $A_{\mathrm{inf}} \xrightarrow{\tilde{\theta}} \mathcal{O}_C \to k$ lifts uniquely to a ring homomorphism $A_{\mathrm{inf}} \to W$, which we will sometimes refer to as the *(crystalline) specialization map.*

Next, we name certain elements that play an important role (see [11, Example 3.16 and Proposition 3.17]).

Notation 10.1.2. In what follows, we will assume that the field C contains a system of primitive p^nth roots of unity $\epsilon_{p^n} \in \mu_{p^n}(C)$ satisfying $\epsilon_{p^n}^p = \epsilon_{p^{n-1}}$. The system $\{\epsilon_{p^n}\}_{n>1}$ then determines an element $\underline{\epsilon} \in \mathcal{O}_C^\flat$. Let $[\underline{\epsilon}] \in A_{\mathrm{inf}}$ denote the Teichmüller representative of $\underline{\epsilon}$ and set $\mu = [\underline{\epsilon}] - 1$, so that $\varphi^{-1}(\mu)$ divides μ. We will use the normalization of [11], so that

$$\xi = \frac{\mu}{\varphi^{-1}(\mu)} = \frac{[\underline{\epsilon}] - 1}{[\underline{\epsilon}^{1/p}] - 1}$$

is a generator of the ideal $\ker(\theta) \subseteq A_{\text{inf}}$, and

$$\tilde{\xi} = \varphi(\xi) = \frac{\varphi(\mu)}{\mu} = \frac{[\underline{\epsilon}^p] - 1}{[\underline{\epsilon}] - 1}$$

is a generator of the ideal $\ker(\tilde{\theta}) \subseteq A_{\text{inf}}$.

Finally, we fix the main geometric object of interest.

Notation 10.1.3. Let $\mathfrak{X}$ be a smooth formal $\mathcal{O}_C$-scheme. We follow the same conventions for formal schemes as in [11]: a smooth formal $\mathcal{O}_C$-scheme is one that locally has the form $\mathrm{Spf}(R)$ where R is a p-adically complete and p-torsion-free $\mathcal{O}_C$-algebra equipped with the p-adic topology and for which R/pR is smooth over $\mathcal{O}_C/p\mathcal{O}_C$. For such R, in this section we will write $\Omega^i_{R/\mathcal{O}_C}$ for the p-adically completed module of i-forms (i.e., we suppress the extra p-adic completion).

For every commutative ring Λ, we let $\mathcal{D}(\mathfrak{X}, \Lambda)$ denote the derived ∞-category of sheaves of Λ-modules on (the underlying topological space of) $\mathfrak{X}$, and we let $D(\mathfrak{X}, \Lambda)$ denote its homotopy category (that is, the usual derived category of sheaves of Λ-modules on $\mathfrak{X}$). Let $\mathfrak{X}_k$ denote the special fibre of $\mathfrak{X}$. As the topological spaces underlying $\mathfrak{X}$ and $\mathfrak{X}_k$ are identical, we can also view $D(\mathfrak{X}, \Lambda)$ as the derived category $D(\mathfrak{X}_k, \Lambda)$ of sheaves of Λ-modules on $\mathfrak{X}_k$.

The following summarizes the main construction from [11].

Construction 10.1.4 (The $A\Omega$-complex [11, §9])**.** We let $A\Omega_{\mathfrak{X}}$ denote the object of the derived category $D(\mathfrak{X}, A_{\text{inf}})$ given by the formula

$$A\Omega_{\mathfrak{X}} \coloneqq L\eta_\mu R\nu_* A_{\text{inf},X}$$

where $\nu: X_{proet} \to \mathfrak{X}$ is the nearby cycles map from the generic fibre X of $\mathfrak{X}$. This is a commutative algebra object of $D(\mathfrak{X}, A_{\text{inf}})$ which is equipped with a multiplicative isomorphism

$$\widetilde{\varphi}_{\mathfrak{X}}: A\Omega_{\mathfrak{X}} \xrightarrow{\simeq} \varphi_* L\eta_{\tilde{\xi}} A\Omega_{\mathfrak{X}} \tag{14}$$

that factors the commutative A_{inf}-algebra map

$$\varphi_{\mathfrak{X}}: A\Omega_{\mathfrak{X}} \to \varphi_* A\Omega_{\mathfrak{X}}$$

induced by $L\eta_\mu$ functoriality from the Frobenius automorphism φ_X of the sheaf $A_{\text{inf},X}$ on X_{proet}. We refer to $\widetilde{\varphi}_{\mathfrak{X}}$ informally as the *divided Frobenius* on $A\Omega_{\mathfrak{X}}$.

Finally, let us recall the Hodge-Tate comparison theorem, which provides us with our main tool for controlling the local structure of $A\Omega$.

Theorem 10.1.5 (Hodge-Tate comparison theorem)**.** *Let* $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ *be an affine open. Define the* E_∞*-algebra* $\widetilde{\Omega}_R := A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \in D(\mathcal{O}_C)$*. Then there is a natural map of* $E_\infty - \mathcal{O}_C$*-algebras* $R \to \widetilde{\Omega_R}$ *and the cohomology ring* $H^*(\widetilde{\Omega}_R)$ *is identified with the exterior algebra* $\bigwedge^*_R \Omega^1_{R/\mathcal{O}_C}$ *as a graded* R*-algebra.*

Remark 10.1.6. We are implicitly suppressing all Breuil-Kisin twists above.

Proof. This follows from [11, Theorem 9.2 (i)] (see [8, Proposition 7.5 and Remark 7.11] for a simpler argument). More precisely, these references give an identification $A\Omega_{\mathfrak{X}} \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \simeq \widetilde{\Omega}_{\mathfrak{X}}$ in the derived ∞-category of sheaves. The statement above follows by taking derived global sections over $\mathrm{Spf}(R)$ and using the vanishing of coherent sheaf cohomology on formal affines.

Alternately, we remark that in the sequel, we shall only use this result for small affine opens (see Definition 10.3.1), and here one can simply invoke [11, Theorem 9.4 (i)] with $r = 1$ and [11, Theorem 8.7]. □

10.2. **The First Formulation of the Main Comparison Theorem.** We follow the notation of §10.1. In particular, $\mathfrak{X}$ denotes a smooth formal $\mathcal{O}_C$-scheme. Our goal is to extract the de Rham-Witt complex of $\mathfrak{X}_k$ (viewed as a sheaf of strict Dieudonné algebras on $\mathfrak{X}_k$) from the pair $(A\Omega_{\mathfrak{X}}, \widetilde{\varphi}_{\mathfrak{X}})$ coming from Construction 10.1.4. A slightly imprecise version of this comparison can be formulated as follows, and gives a new proof of [11, Theorem 1.10 (i)].

Theorem 10.2.1. *There is a natural identification* $A\Omega_{\mathfrak{X}} \widehat{\otimes}_{A_{\mathrm{inf}}} W \simeq W\Omega^*_{\mathfrak{X}_k}$ *of commutative algebras in* $D(\mathfrak{X}, W)$ *that carries* $\varphi_{\mathfrak{X},W}$ *to* $\varphi_{\mathfrak{X}_k}$*. Moreover, this identification can be promoted to an isomorphism of presheaves of strict Dieudonné* W*-algebras (see Theorem 10.4.4 for a precise formulation).*

There are roughly two steps in the (formulation and) proof of this result. First, in §10.3, we explain how to represent $A\Omega_{\mathfrak{X}} \widehat{\otimes}^L_{A_{\mathrm{inf}}} W$ by a presheaf of strict Dieudonné W-algebras on a basis of the topology of $\mathfrak{X}$ (given by the small affine opens of Definition 10.3.1). Second, in §10.4, we compare the aforementioned presheaf with the one given by the de Rham-Witt complex using the universal property of the latter (Definition 4.1.1), and deduce Theorem 10.2.1 using the Hodge-Tate comparison theorem.

10.3. **Extracting a Presheaf of Strict Dieudonné Algebras from** $A\Omega_{\mathfrak{X}}$**.** We follow the notation of §10.1. In particular, $\mathfrak{X}$ denotes a smooth formal $\mathcal{O}_C$-scheme. In this section, we explain how to lift the object $A\Omega_{\mathfrak{X}} \widehat{\otimes}_{A_{\mathrm{inf}}} W \in D(\mathfrak{X}, W)$ naturally to a presheaf of strict Dieudonné W-algebras (Proposition 10.3.14). The strategy roughly is to construct a divided Frobenius isomorphism for $A\Omega_{\mathfrak{X}} \widehat{\otimes}_{A_{\mathrm{inf}}} W$ from the one for $A\Omega_{\mathfrak{X}}$ (Proposition 10.3.10), and then produce the desired rigidification by appealing to Corollary 7.4.9. As in the local study of [11], we restrict attention to the following collection of affine open subsets of $\mathfrak{X}$.

Definition 10.3.1 (Small affine opens)**.** We will say that an open subset $U \subseteq \mathfrak{X}$ is *small* if it is affine and admits an étale map to a torus, i.e., we can write U as a formal spectrum $\mathrm{Spf}(R)$, where R is a p-adically complete and p-torsion free $\mathcal{O}_C$-algebra for which there exists an étale $\mathcal{O}_C/p$-algebra map $(\mathcal{O}_C/p)[t_1^{\pm 1}, ..., t_d^{\pm 1}] \to R/pR$. In this case, we will simply write $A\Omega_R$ for the commutative algebra object

$$\mathrm{R}\Gamma(U, A\Omega_{\mathfrak{X}}|_U) = \mathrm{R}\Gamma(\mathrm{Spf}(R), A\Omega_{\mathrm{Spf}(R)})$$

of the derived category $D(A_{\mathrm{inf}})$.

Let us introduce the relevant category of presheaves.

Construction 10.3.2 (Presheaves on small affine opens)**.** Write $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ denote the poset of all small affine opens $U \subseteq \mathfrak{X}$, viewed as a category. For a commutative ring Λ, let $\mathcal{D}(\Lambda)$ denote the derived ∞-category of Λ, and let $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathcal{D}(\Lambda))$ for the ∞-category of $\mathcal{D}(\Lambda)$-valued presheaves on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$.

Note that each $\mathscr{F} \in \mathcal{D}(\mathfrak{X}, \Lambda)$ defines an object of $i_\Lambda(\mathscr{F}) \in \mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathcal{D}(\Lambda))$ by the formula

$$i_\Lambda(\mathscr{F})(U) = \mathrm{R}\Gamma(U, \mathscr{F}).$$

This construction determines a fully faithful embedding from the derived category $D(\mathfrak{X}, \Lambda)$ to the homotopy category of the ∞-category $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathcal{D}(\Lambda))$ (see Remark 10.3.5).

The symmetric monoidal structure on the ∞-category $\mathcal{D}(\Lambda)$ induces a symmetric monoidal structure on the functor ∞-category $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathcal{D}(\Lambda))$, given by pointwise (derived) tensor product over Λ. In particular, we have a natural identification

$$\mathrm{CAlg}(\mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathcal{D}(\Lambda))) \simeq \mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{D}(\Lambda)))$$

of ∞-categories of commutative algebra objects. Similar remarks apply to variants involving presheaves of more structured objects.

We say that an object $F \in \mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathcal{D}(\Lambda))$ is *p-complete* if $F(U)$ is p-complete, for each $U \in \mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$.

Remark 10.3.3. The ∞-category $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathcal{D}(\Lambda))$ can be identified with the derived ∞-category of the ordinary abelian category of presheaves of Λ-modules on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$. In particular, we can identify the homotopy category of $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}^{\mathrm{op}}, \mathcal{D}(\Lambda))$ with the derived category of $D(\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}, \Lambda)$ where we endow the poset $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ with the indiscrete Grothendieck topology (so that all presheaves are sheaves). Since $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ possesses a more natural Grothendieck topology coming from the Zariski topology of $\mathfrak{X}$, we prefer to not use this notation.

Remark 10.3.4. In analogy with Definition 10.3.1, let us say that an open subset $V \subseteq \mathfrak{X}_k$ is *small* if it is affine and admits an étale map to a torus. Using the fact that étale morphisms are finitely presented, it is easy to see that the identification of $\mathfrak{X}_k \simeq \mathfrak{X}$ of topological spaces preserves the notion of small affine opens, i.e., an open $U \subseteq \mathfrak{X}$ is small if and only if $U_k \subseteq \mathfrak{X}_k$ is so. Consequently, we can identify the poset $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ of small affine opens in $\mathfrak{X}$ with the corresponding poset $\mathcal{U}(\mathfrak{X}_k)_{\mathrm{sm}}$ for the special fibre $\mathfrak{X}_k$.

Remark 10.3.5 (Sheaves and presheaves)**.** The collection of small affine opens forms a basis for the topology of $\mathfrak{X}$. In particular, if one equips the category $\mathcal{U}(\mathfrak{X})$ with the standard Grothendieck topology (i.e., the one inherited from the Zariski topology on $\mathfrak{X}$), then the resulting category of sheaves is identified with the category of sheaves on $\mathfrak{X}$. In particular, the construction

$$i_\Lambda\colon \mathcal{D}(\mathfrak{X}, \Lambda) \to \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(\Lambda))$$

of Construction 10.3.2 determines a fully faithful embedding from the derived ∞-category of sheaves on $\mathfrak{X}$ to the derived ∞-category of presheaves on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$. This has a left adjoint

$$G_\Lambda\colon \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(\Lambda)) \to D(\mathfrak{X}, \Lambda)$$

given by sheafification for the Zariski topology.

For future reference, we also record a variant of this observation for p-complete objects. Let $\widehat{D}(\mathfrak{X}, W) \subseteq \mathcal{D}(\mathfrak{X}, W)$ denotes the full subcategory spanned by p-complete objects, and $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}}(W)) \subseteq \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(W))$ denote the full subcategory spanned by presheaves valued in the full subcategory $\widehat{\mathcal{D}}(W) \subseteq \mathcal{D}(W)$ of p-complete objects in $\mathcal{D}(W)$. Then the functor i_W discussed above restricts to a fully faithful functor

$$\widehat{i}_W\colon \widehat{\mathcal{D}}(\mathfrak{X}, W) \to \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}}(W))$$

which admits a left adjoint

$$\widehat{G}_W\colon \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}}(W)) \to \widehat{\mathcal{D}}(\mathfrak{X}, W)$$

given on objects by composing the sheafification functor G_W mentioned above with the derived p-completion functor $\mathcal{D}(\mathfrak{X}, W) \to \widehat{\mathcal{D}}(\mathfrak{X}, W)$. Unwinding definitions, we can write $\widehat{G}_W(K) = \varprojlim G_W(K \otimes^L_W W/p^n)$.

Let us write $A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \in \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(A_{\mathrm{inf}}))$ for the complex $A\Omega_{\mathfrak{X}}$ of Construction 10.1.4 regarded as a presheaf: that is, the image of $A\Omega_{\mathfrak{X}} \in \mathcal{D}(\mathfrak{X}, A_{\mathrm{inf}})$ under the functor $i_{A_{\mathrm{inf}}}$ of Construction 10.3.2. Beware that composition with the functor $L\eta_{\tilde{\xi}}$ does not preserve the essential image of the functor $i_{A_{\mathrm{inf}}}$: that is, it generally does not carry sheaves to sheaves. Nevertheless, we have the following:

Proposition 10.3.6. *The Frobenius map $\widetilde{\varphi}_{\mathfrak{X}}$ on $A\Omega_{\mathfrak{X}}$ lifts to an identification*

$$\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X}}\colon A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \simeq \varphi_* L\eta_{\tilde{\xi}} A\Omega^{\mathrm{sm}}_{\mathfrak{X}}$$

of commutative algebras in $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, D(A_{\mathrm{inf}}))$.

The proof below crucially uses the smallness restriction in the definition of $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$.

Proof. By definition, for each small affine open $U \subseteq \mathfrak{X}$, we have $A\Omega^{\mathrm{sm}}_{\mathfrak{X}}(U) := \mathrm{R}\Gamma(U, A\Omega_{\mathfrak{X}})$. By [11, Theorem 9.4], we have

$$A\Omega^{\mathrm{sm}}_{\mathfrak{X}}(U) \simeq L\eta_\mu \, \mathrm{R}\Gamma(U, A_{\mathrm{inf},U}).$$

In particular, the Frobenius map $\widetilde{\varphi}_{\mathfrak{X}}$ refines to give an identification

$$A\Omega^{\mathrm{sm}}_{\mathfrak{X}}(U) \simeq \varphi_* L\eta_{\tilde{\xi}} A\Omega^{\mathrm{sm}}_{\mathfrak{X}}(U)$$

for all small affine U, and hence we have an identification

$$\varphi^{\mathrm{sm}}_{\mathfrak{X}} \colon A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \simeq \varphi_* L\eta_{\tilde{\xi}} A\Omega^{\mathrm{sm}}_{\mathfrak{X}}$$

of commutative algebras in $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(A_{\mathrm{inf}}))$, as desired.. □

We can now introduce the crystalline specialization of $A\Omega$ with its divided Frobenius map.

Construction 10.3.7 (The crystalline specialization)**.** Let

$$A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} := A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \widehat{\otimes}^L_{A_{\mathrm{inf}}} W \in \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(W))$$

be the $\mathcal{D}(W)$-valued presheaf on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ defined by $A\Omega^{\mathrm{sm}}_{\mathfrak{X}}$ via p-adically completed base change along $A_{\mathrm{inf}} \to W$. Write $A\Omega_{R,W}$ for its value on a small affine open $\mathrm{Spf}(R) \subseteq \mathfrak{X}$, so $A\Omega_{R,W} := A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W$. The map $\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X}}$ induces an isomorphism

$$A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \simeq \left(\varphi_* L\eta_{\tilde{\xi}} A\Omega^{\mathrm{sm}}_{\mathfrak{X}}\right) \widehat{\otimes}^L_{A_{\mathrm{inf}}} W. \tag{15}$$

Since the crystalline specialization map $A_{\mathrm{inf}} \to W$ carries $\tilde{\xi}$ to p, the natural map $A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \to A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$ induces by functoriality a commutative A_{inf}-algebra map

$$\varphi_* L\eta_{\tilde{\xi}} A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \to \varphi_* L\eta_p A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}.$$

As the codomain of the preceding map is p-complete, we obtain a map

$$\left(\varphi_* L\eta_{\tilde{\xi}} A\Omega^{\mathrm{sm}}_{\mathfrak{X}}\right) \widehat{\otimes}^L_{A_{\mathrm{inf}}} W \to \varphi_* L\eta_p A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \tag{16}$$

Composing with the isomorphism in (15), we obtain a map

$$\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W} \colon A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \to \varphi_* L\eta_p A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$$

of commutative algebra objects of $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}}(W))$; we will refer to this map as the *divided Frobenius* on $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$.

To justify the "divided Frobenius" terminology, we prove the following:

Lemma 10.3.8. *The presheaf* $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \otimes^L_W W/p^n$ *takes coconnective values for all* $n \geq 1$. *Consequently, there is a natural map*

$$\varphi_* L\eta_p A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \to \varphi_* A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$$

of commutative algebras in $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}}(W))$.

Proof. Proceeding by induction on n, we can reduce to the case $n = 1$. We must show that for every small affine open $\mathrm{Spf}(R) \subseteq \mathfrak{X}$, the complex

$$(A\Omega_R \otimes^L_{A_{\mathrm{inf}}} W) \otimes^L_W W/p \simeq A\Omega_R \otimes^L_{A_{\mathrm{inf}}} k$$

is coconnective. By the Hodge-Tate comparison (Theorem 10.1.5), we have

$$A\Omega_R \otimes^L_{A_{\mathrm{inf}}, \widetilde{\theta}} \mathcal{O}_C \simeq \widetilde{\Omega}_R,$$

where $\widetilde{\Omega}_R$ is a coconnective R-complex with cohomology groups given by $\mathrm{H}^i(\widetilde{\Omega}_R) \simeq \Omega^i_{R/\mathcal{O}_C}$. The claim now follows by extending scalars along the map $\mathcal{O}_C \to k$ (and observing that each $\Omega^i_{R/\mathcal{O}_C}$ is a locally free R-module, hence flat over $\mathcal{O}_C$).

The second assertion is immediate from the first (note that there is a symmetric monoidal natural transformation $L\eta_p(K) \to K$ defined on the full subcategory of $\mathcal{D}(W)$ spanned by those objects K for which $k \otimes^L_W K \in \mathcal{D}(k)$ is coconnective). □

Composing the divided Frobenius $\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W}$ from Construction 10.3.7 with the map from Lemma 10.3.8 gives a map

$$\varphi^{\mathrm{sm}}_{\mathfrak{X},W} \colon A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} \to \varphi_* A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$$

of commutative algebras; we refer to this map as the *Frobenius* on $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$.

Remark 10.3.9. The object $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$ from Construction 10.3.7 belongs to the full subcategory $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}}(W)) \subseteq \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(W))$ of p-complete objects introduced in Remark 10.3.5. The completed sheafification functor $\widehat{G}_W$ of Remark 10.3.5 carries $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$ to the object $A\Omega_{\mathfrak{X},W} \coloneqq A\Omega_{\mathfrak{X}} \widehat{\otimes}^L_{A_{\mathrm{inf}}} W \in \widehat{\mathcal{D}}(\mathfrak{X}, W) \subseteq \mathcal{D}(\mathfrak{X}, W)$: this follows easily from the formula for $\widehat{G}_W$ given at the end of Remark 10.3.5.

Let us now show that the divided Frobenius $\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W}$ constructed above is an isomorphism; this relies on the Hodge-Tate comparison theorem.

Proposition 10.3.10. *The divided Frobenius map* $\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W}$ *of Construction 10.3.7 is an isomorphism of commutative algebra objects of* $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}}(W))$.

Proof. It is enough to show that the natural map

$$\left(\varphi_* L\eta_{\widetilde{\xi}} A\Omega^{\mathrm{sm}}_{\mathfrak{X}}\right) \widehat{\otimes}^L_{A_{\mathrm{inf}}} W \to \varphi_* L\eta_p A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$$

appearing in Construction 10.3.7 as (16) is a quasi-isomorphism. Unwinding definitions (and neglecting Frobenius twists), we must show that the natural map

$$\big(L\eta_{\tilde{\xi}} A\Omega^{\mathrm{sm}}_{\mathfrak{X}}\big)\widehat{\otimes}^L_{A_{\inf}} W \to L\eta_{\tilde{\xi}}(A\Omega^{\mathrm{sm}}_{\mathfrak{X}}\widehat{\otimes}^L_{A_{\inf}} W)$$

is a quasi-isomorphism, i.e., that $L\eta_{\tilde{\xi}}$ commutes with p-completed tensor product W; here we replace p with $\tilde{\xi}$ since they have the same image in W. Fix a small affine open $\mathrm{Spf}(R) \subseteq \mathfrak{X}$; we wish to show that the map

$$(L\eta_{\tilde{\xi}} A\Omega_R)\widehat{\otimes}^L_{A_{\inf}} W \to L\eta_{\tilde{\xi}}\big(A\Omega_R\widehat{\otimes}^L_{A_{\inf}} W\big)$$

is a quasi-isomorphism.

Observe that we can write W as p-adic completion of $\varinjlim_r A_{\inf}/\varphi^{-r}(\mu)$. Moreover, since the equation $\tilde{\xi} = p$ holds in each quotient ring $A_{\inf}/\varphi^{-r}(\mu)$, we can also realize W as the $\tilde{\xi}$-adic completion of $\varinjlim_r A_{\inf}/\varphi^{-r}(\mu)$. Let us first show that for $r \geq 1$, the natural map $A_{\inf} \to A_{\inf}/\varphi^{-r}(\mu)$ induces a quasi-isomorphism

$$(L\eta_{\tilde{\xi}} A\Omega_R) \otimes^L_{A_{\inf}} A_{\inf}/\varphi^{-r}(\mu) \simeq L\eta_{\tilde{\xi}}\big(A\Omega_R \otimes^L_{A_{\inf}} A_{\inf}/\varphi^{-r}(\mu)\big).$$

Using the criterion in [8, Lemma 5.16], it is enough to show that the $\mathcal{O}_C$-modules $\mathrm{H}^i(A\Omega_R \otimes^L_{A_{\inf},\tilde{\theta}} \mathcal{O}_C)$ have no $\varphi^{-r}(\mu)$-torsion for all i. Since the ring homomorphism $\tilde{\theta}$ carries $\varphi^{-r}(\mu)$ to the non-zero-divisor $\epsilon_{p^r} - 1 \in \mathcal{O}_C$, it enough to show that these cohomology groups are flat over $\mathcal{O}_C$. This is immediate from the Hodge-Tate comparison (Theorem 10.1.5), since each $\Omega^i_{R/\mathcal{O}_C}$ is a projective R-module of finite rank (and therefore $\mathcal{O}_C$-flat).

Since the functors $L\eta_{\tilde{\xi}}$ and $\otimes^L$ both preserve filtered colimits, we get a natural identification

$$(L\eta_{\tilde{\xi}} A\Omega_R) \otimes^L_{A_{\inf}} \varinjlim_r A_{\inf}/\varphi^{-r}(\mu) \simeq L\eta_{\tilde{\xi}}\big(A\Omega_R \otimes^L_{A_{\inf}} \varinjlim_r A_{\inf}/\varphi^{-r}(\mu)\big)$$

in the ∞-category $\mathcal{D}(A_{\inf})$. Proposition 10.3.10 now follows by applying derived $\tilde{\xi}$-adic completions to both sides, since W is the derived $\tilde{\xi}$-adic completion of $\varinjlim_r A_{\inf}/\varphi^{-r}(\mu)$ and the functor $L\eta_{\tilde{\xi}}(-)$ commutes with derived $\tilde{\xi}$-adic completions (see [11, Lemma 6.20]). □

We now use the divided Frobenius map $\widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W}$ and the equivalence of Corollary 7.4.9 to promote $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$ to a presheaf taking values in the ordinary category of strict Dieudonné algebras.

Construction 10.3.11. By Proposition 10.3.10, the pair $(A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}, \widetilde{\varphi}^{\mathrm{sm}}_{\mathfrak{X},W})$ provides a lift of $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$ along the forgetful functor

$$\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}(W)}^{\varphi_* L\eta_p}) \simeq \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}(W)})^{\varphi_* L\eta_p} \to \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(W)),$$

where we use the notation from Example 7.6.5 (applied to $R = W$ and $\sigma = \varphi$). Applying Example 7.6.5, we can identify this lift with a presheaf

$$A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W} \in \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathrm{Mod}_W(\mathbf{DC}_{\mathrm{str}}))$$

taking values in the ordinary category of W-module objects in the category $\mathbf{DC}_{\mathrm{str}}$ of strict Dieudonné complexes. As all our constructions are compatible with the commutative algebra structure on $A\Omega_{\mathfrak{X}}$, it follows that $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ is a commutative algebra object in $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathrm{Mod}_W(\mathbf{DC}_{\mathrm{str}}))$; see Example 7.6.7 for four equivalent descriptions of this category.

Definition 10.3.12. Let us regard the ring $W = W(k)$ of Witt vectors as a strict Dieudonné algebra (Example 3.1.8). A *strict Dieudonné W-algebra* is a strict Dieudonné algebra A^* equipped with a morphism of Dieudonné algebras $W \to A^*$. We let $\mathbf{DA}_{\mathrm{str}W/}$ denote the category of strict Dieudonné W-algebras, which we will view as a full subcategory of the category $\mathrm{CAlg}(\mathrm{Mod}_W(\mathbf{DC}))$ of commutative W-algebras in $\mathbf{DC}$.

Remark 10.3.13. By virtue of Remark 3.1.5 and Proposition 3.4.2, a commutative algebra object A^* of $\mathrm{Mod}_W(\mathbf{DC})$ belongs to $\mathbf{DA}_{\mathrm{str}W/}$ if and only if the underlying Dieudonné complex is strict, the groups A^n vanish for $n < 0$, and the Frobenius map satisfies the congruence $Fx \equiv x^p \pmod{VA^0}$ for each $x \in A^0$.

Proposition 10.3.14. *The presheaf $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ of Construction 10.3.11 takes values in the full subcategory $\mathbf{DA}_{\mathrm{str}W/} \subseteq \mathrm{CAlg}(\mathrm{Mod}_W(\mathbf{DC}))$ of strict Dieudonné W-algebras.*

To prove Proposition 10.3.14, we must show that for every small open $U \subseteq \mathfrak{X}$, the complex $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U)$ is a strict Dieudonné W-algebra: that is, it satisfies the requirements of Remark 10.3.13. Note that $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U)$ is strict by construction, and the groups $A\Omega^{\mathrm{sm},n}_{\mathfrak{X},W}(U)$ vanish for $n < 0$ by virtue of Lemma 10.3.8 (together with Proposition 2.7.1). We are therefore reduced to proving the following:

Proposition 10.3.15. *Let $U \subseteq \mathfrak{X}$ be a small open. Then the Frobenius endomorphism of the $\mathbf{F}_p$-algebra $\mathrm{H}^0(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U)/p)$ coincides with the map induced by the structure map $F\colon A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U) \to A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}(U)$.*

Proof. Write $U = \mathrm{Spf}(R)$. Unwinding the definitions, we must check that the Frobenius endomorphism of the $\mathbf{F}_p$-algebra $\mathrm{H}^0(A\Omega_R \otimes^L_{A_{\mathrm{inf}}} k)$ coincides with the one induced by the tensor product of the map $\varphi_R\colon A\Omega_R \to A\Omega_R$ and the Frobenius map $\varphi_k\colon k \to k$. Using the smallness of R, we can choose a map $R \to R_\infty$ as in [11, Definitions 8.5 and 8.6]. In particular, R_∞ is perfectoid, the map $R \to R_\infty$ is faithfully flat modulo p and a pro-étale Γ-torsor after inverting p (where $\Gamma = \mathbf{Z}_p(1)^{\oplus d}$), and $A\Omega_R \simeq L\eta_\mu R\Gamma(\Gamma, A_{\mathrm{inf}}(R_\infty))$ (by [11, Theorem 9.4 (iii)]). We therefore obtain a φ-equivariant homomorphism

$$\eta\colon A\Omega_R \to A_{\mathrm{inf}}(R_\infty).$$

Recall that if S is a perfectoid $\mathcal{O}_C$-algebra, then there is a Frobenius-equivariant isomorphism $A_{\inf}(S)\widehat{\otimes}^L_{A_{\inf}} W \simeq W(S_k)$ (see [11, Lemma 3.13]). Applying this to $S = R_\infty$, we obtain a φ-equivariant map

$$\mathrm{H}^0(\eta \otimes^L_{A_{\inf}} k)\colon \mathrm{H}^0(A\Omega_R \otimes^L_{A_{\inf}} k) \to R_{\infty,k}.$$

As φ acts as the Frobenius on $R_{\infty,k}$, it also acts as the Frobenius on any φ-stable subring. Consequently, to prove Proposition 10.3.15, it will suffice to show that the map $\mathrm{H}^0(\eta \otimes^L_{A_{\inf}} k)$ is injective. For this, we use the Hodge-Tate comparison isomorphism (Theorem 10.1.5), which carries $\mathrm{H}^0(\eta \otimes^L_{A_{\inf}} k)$ to map obtained by applying the functor H^0 to the map of chain complexes

$$\widetilde{\Omega}_R \otimes^L_{\mathcal{O}_C} k \to R_{\infty,k}.$$

We now observe that on 0th cohomology groups, this can be identified with the map $R_k \to R_{\infty,k}$ (using [11, Theorem 8.7] to identify the left hand side, and the construction of η to identify the map). Since $R \to R_\infty$ is faithfully flat modulo p, it is also faithfully flat (hence injective) after extending scalars to k. □

10.4. **Comparison with the de Rham-Witt Complex.** We follow the notation of §10.1. Let $\mathfrak{X}$ be a smooth formal $\mathcal{O}_C$-scheme, and let $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ be the presheaf of strict Dieudonné W-algebras given by Proposition 10.3.14. Our goal is to prove Theorem 10.2.1 by identifying $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ with the de Rham-Witt complex of the special fiber $\mathfrak{X}_k$.

Notation 10.4.1. Let $\mathcal{O}_{\mathfrak{X}_k}$ denote the structure sheaf of the k-scheme $\mathfrak{X}_k$ and let $\mathcal{O}^{\mathrm{sm}}_{\mathfrak{X}_k}$ denote its restriction to the category $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ of small open subsets of $\mathfrak{X}_k$. We let $W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}$ denote the presheaf of strict Dieudonné W-algebras on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ given by the formula

$$W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}(U) = W\Omega^*_{\mathcal{O}(U_k)}.$$

For each integer i, let $\Omega^{\mathrm{sm},i}_{\mathfrak{X}_k}$ denote the presheaf on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$ given by $U \mapsto \Omega^i_{\mathcal{O}(U_k)}$. Write

$$\eta_{W\Omega}\colon \mathcal{O}^{\mathrm{sm}}_{\mathfrak{X}_k} \to H^0(W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}/p)$$

for the natural map of commutative k-algebras induced by the Cartier isomorphism in degree zero.

Remark 10.4.2. Note that passage to the derived ∞-category (that is, composing with the forgetful functor $\mathbf{DA}_{\mathrm{str}W/} \to \widehat{\mathcal{D}}(W)$), carries the presheaf $W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}$ of Notation 10.4.1 to a commutative algebra object $W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \in \mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{\mathcal{D}}(W))$ equipped with a Frobenius endomorphism. Applying the completed sheafification functor $\widehat{G}_W$ of Remark 10.3.5 to this object yields the classical de Rham-Witt complex $W\Omega^*_{\mathfrak{X}_k} \in \widehat{\mathcal{D}}(\mathfrak{X}_k, W)$.

Our proof of Theorem 10.2.1 will use the following recognition criterion for the de Rham-Witt complex, which may be of independent interest:

Proposition 10.4.3. *Let A^* be a presheaf of strict Dieudonné W-algebras on $\mathcal{U}(\mathfrak{X}_k)_{\mathrm{sm}}$ equipped with a map*

$$\eta_A \colon \mathcal{O}^{\mathrm{sm}}_{\mathfrak{X}_k} \to \mathrm{H}^0(A^*/pA^*).$$

of commutative k-algebras. Then $\mathrm{H}^*(A/pA)$ has the structure of a presheaf of commutative differential graded algebras over k, with differential given by the Bockstein map associated to p, and η_A extends naturally to a map

$$\widetilde{\eta}_A \colon \Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \to \mathrm{H}^*(A/pA)$$

of presheaves of commutative differential graded algebras. If $\widetilde{\eta}_A$ is an isomorphism, then there is a unique isomorphism $W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \simeq A^*$ intertwining $\eta_{W\Omega}$ with η_A.

Proof. By the universal property of the de Rham-Witt complex (Definition 4.1.1 and Theorem 4.4.12), the map η_A lifts to uniquely to a map $\Psi \colon W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \to A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ which intertwines η_A with $\eta_{W\Omega}$. Now recall that the Cartier isomorphism gives an identification $\Omega^*_{\mathfrak{X}_k} \simeq \mathrm{H}^*(W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}/p)$ that intertwines the de Rham differential with the Bockstein differential. The map $\mathrm{H}^*(\Psi/p)$ intertwines the Bockstein differentials, and coincides with the map $\widetilde{\eta}_A$ in the proposition in degree 0 by construction under the Cartier isomorphism. By the universal property of the de Rham complex, it follows that $\mathrm{H}^*(\Psi/p) = \widetilde{\eta}_A$. In particular, Ψ/p is a quasi-isomorphism. Since the domain and codomain of Ψ are p-torsion-free and p-adically complete, it follows that Ψ is a quasi-isomorphism. Applying Theorem 7.3.4, we deduce that Ψ is an isomorphism. The uniqueness of Ψ is clear from the construction. □

Theorem 10.4.4. *There is a natural isomorphism $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W} \simeq W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}$ of presheaves of strict Dieudonné W-algebras on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$.*

Proof. We will show that the criterion of Proposition 10.4.3 applies to the presheaf $A^* := A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$. Recall that we have $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W} := A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \widehat{\otimes}^L_{A_{\mathrm{inf}}} W$ by definition (Construction 10.3.7). Moreover, $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ is a presheaf of commutative differential graded algebras over W which represents $A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$ as a commutative algebra object of the the derived category (Construction 10.3.11). In particular, $\mathrm{H}^0(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p)$ can be identified with $\mathrm{H}^0(A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \otimes^L_{A_{\mathrm{inf}}} k)$. By transitivity of tensor products, we obtain an isomorphism

$$A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \otimes^L_{A_{\mathrm{inf}}} k \simeq (A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C) \otimes^L_{\mathcal{O}_C} k.$$

We also have Hodge-Tate comparison isomorphism (Theorem 10.1.5)

$$\widetilde{\Omega}^{\mathrm{sm}}_{\mathfrak{X}} \simeq A\Omega^{\mathrm{sm}}_{\mathfrak{X}} \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C$$

of commutative algebras in $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \mathcal{D}(\mathcal{O}_C))$, where the left side is the presheaf which carries a small affine open $\mathrm{Spf}(R) \subseteq \mathfrak{X}$ to $\widetilde{\Omega}_R$. For such R, the R-module

$\mathrm{H}^i(\widetilde{\Omega}_R) \simeq \Omega^i_{R/\mathcal{O}_C}$ is locally free and hence flat over $\mathcal{O}_C$. It immediately follows that $\mathrm{H}^0(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p)$ is identified with $\mathcal{O}^{\mathrm{sm}}_{\mathfrak{X}_k}$ as a commutative k-algebra, which supplies the map η_A required in Proposition 10.4.3. Moreover, this analysis also shows that the cohomology ring $\mathrm{H}^*(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p)$ is an exterior algebra on

$$\mathrm{H}^1(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p) \simeq \mathrm{H}^1(\widetilde{\Omega}^{\mathrm{sm}}_{\mathfrak{X}}) \otimes_{\mathcal{O}_C} k \simeq \Omega^{\mathrm{sm},1}_{\mathfrak{X}_k},$$

where the target is the presheaf determined by sending a small affine open $\mathrm{Spf}(R)$ to $\Omega^1_{R_k}$. To apply Proposition 10.4.3, it remains to check that the Bockstein map

$$\mathrm{H}^0(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p) \xrightarrow{\beta_p} \mathrm{H}^1(A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}/p)$$

corresponds to the to the de Rham differential

$$\mathcal{O}^{\mathrm{sm}}_{\mathfrak{X}_k} \to \Omega^{\mathrm{sm},1}_{\mathfrak{X}_k}$$

under the preceding identifications. In fact, it is enough to check this compatibility after taking sections over a small affine open $\mathrm{Spf}(R)$. Over such an affine, we will deduce the desired compatibility from the de Rham comparison theorem for $A\Omega_R$. Because the crystalline specialization map $A_{\mathrm{inf}} \to W$ carries $\tilde{\xi}$ to p, we obtain a commutative diagram in the derived category $D(A_{\mathrm{inf}})$ comparing the Bockstein constructions for $\tilde{\xi}$ and p:

$$\begin{tikzcd}
A\Omega_R/\tilde{\xi} \simeq A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \arrow[r, "\tilde{\xi}"] \arrow[d] & A\Omega/\tilde{\xi}^2 \arrow[r] \arrow[d] & A\Omega_R/\tilde{\xi} \simeq A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \arrow[d] \\
A\Omega_R/\tilde{\xi} \otimes^L_{A_{\mathrm{inf}}} W \arrow[r, "\tilde{\xi}"] \arrow[d, "\simeq"] & A\Omega/\tilde{\xi}^2 \otimes^L_{A_{\mathrm{inf}}} W \arrow[r] \arrow[d, "\simeq"] & A\Omega_R/\tilde{\xi} \otimes^L_{A_{\mathrm{inf}}} W \arrow[d, "\simeq"] \\
(A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p \arrow[r, "p"] & (A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{Z}/p^2 \arrow[r] & (A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p.
\end{tikzcd}$$

Each row of this diagram is an exact triangle, and the second row is obtained from the first by extending scalars along the crystalline specialization $A_{\mathrm{inf}} \to W$. Comparing boundary maps on cohomology for the first and last row above gives a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^i(A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C) \arrow[r, "\beta_{\tilde{\xi}}"] \arrow[d] & \mathrm{H}^{i+1}(A\Omega_R \otimes^L_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C) \arrow[d] \\
\mathrm{H}^i((A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p) \arrow[r, "\beta_p"] & \mathrm{H}^{i+1}((A\Omega_R \widehat{\otimes}^L_{A_{\mathrm{inf}}} W) \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p).
\end{tikzcd}$$

Via the Hodge-Tate comparison (and ignoring Breuil-Kisin twists), this square identifies with

$$\begin{tikzcd} \Omega^i_{R/\mathcal{O}_C} \arrow[r, "\beta_{\tilde{\xi}}"] \arrow[d] & \Omega^{i+1}_{R/\mathcal{O}_C} \arrow[d] \\ \Omega^i_{R_k} \simeq \Omega^i_{R_k/k} \arrow[r, "\beta_p"] & \Omega^{i+1}_{R_k} \simeq \Omega^{i+1}_{R_k/k}. \end{tikzcd}$$

where the vertical maps are the natural ones. It will therefore suffice to show that the top horizontal map is given by the de Rham differential on $\Omega^*_{R/\mathcal{O}_C}$, which follows from the proof of the de Rham comparison theorem for $A\Omega$ (see [8, Proposition 7.9]). □

Proof of Theorem 10.2.1. Theorem 10.4.4 gives an identification

$$\Psi \colon W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \simeq A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$$

of presheaves of strict Dieudonné W-algebras on $\mathcal{U}(\mathfrak{X})_{\mathrm{sm}}$. This already proves the second statement of Theorem 10.2.1.

Let us now deduce the first statement. Passing to the derived category (i.e., composing with the forgetful functor $\mathbf{DA}_{\mathrm{str}/W} \to \widehat{D}(W)$), the isomorphism Ψ induces an isomorphism

$$\Theta \colon W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k} \simeq A\Omega^{\mathrm{sm}}_{\mathfrak{X},W}$$

of commutative algebra objects in $\mathrm{Fun}(\mathcal{U}(\mathfrak{X})^{\mathrm{op}}_{\mathrm{sm}}, \widehat{D}(W))$ that is compatible with the Frobenius maps. Applying the completed sheafification functor $\widehat{G}_W$ of Remark 10.3.5 to $A\Omega^{\mathrm{sm},*}_{\mathfrak{X},W}$ yields the complex $A\Omega_{\mathfrak{X},W} \in \widehat{D}(\mathfrak{X}, W)$ (Remark 10.3.9). Similarly, by Remark 10.4.2, applying $\widehat{G}_W$ to $W\Omega^{\mathrm{sm},*}_{\mathfrak{X}_k}$ gives the de Rham-Witt complex $W\Omega^*_{\mathfrak{X}_k} \in \widehat{D}(\mathfrak{X}, W)$. Combining these observations with the Frobenius equivariant isomorphism Θ, we obtain the first statement of Theorem 10.2.1. □

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