ULTRACATEGORIES

Contents

Introduction

A central aim of mathematical logic is to understand the relationship between the syntax of a logical theory T and its semantics. This relationship is particularly strong in the setting of first-order predicate logic: according to Gödel's celebrated *completeness theorem*, a first-order sentence φ is provable from the axioms of T (an *a priori* syntactic notion) if and only if it is true when interpreted in any model of T (an *a priori* semantic notion). In [9], Makkai proved a stronger form of Gödel's completeness theorem, which provides a complete recipe for reconstructing the syntax of a first-order theory T (up to an appropriate notion of equivalence) from its semantics. The goal of this paper is to give a new proof of Makkai's theorem and a reasonably self-contained exposition of the mathematics that surrounds it.

Before stating Makkai's result, let us recall an important classical precursor: the *Stone duality theorem* for Boolean algebras.

Definition 0.0.1. A *Boolean algebra* is a partially ordered set $(B, \leq)$ with the following properties:

- Every finite subset of B has a least upper bound. Equivalently, B contains a least element 0 and every pair of elements $x, y \in B$ have a least upper bound $x \vee y$.
- Every finite subset of B has a greatest lower bound. Equivalently, B has a largest element 1 and every pair of elements $x, y \in B$ have a greatest lower bound $x \wedge y$.
- Every element $x \in B$ has a *complement* $\overline{x}$, characterized by the identities

$$x \wedge \overline{x} = 0 \qquad x \vee \overline{x} = 1.$$

If B and B' are Boolean algebras, a *homomorphism of Boolean algebras* from B to B' is a function $\mu : B \to B'$ satisfying the identities

$$\mu(0) = 0 \qquad \mu(x \vee y) = \mu(x) \vee \mu(y)$$

$$\mu(1) = 1 \qquad \mu(x \wedge y) = \mu(x) \wedge \mu(y).$$

We let BAlg denote the category whose objects are Boolean algebras and whose morphisms are Boolean algebra homomorphisms.

For every Boolean algebra B, we let $\mathrm{Spec}(B) = \mathrm{Hom}_{\mathrm{BAlg}}(B, \{0,1\})$ denote the set of all Boolean algebra homomorphisms $\mu : B \to \{0,1\}$. Note that $\mathrm{Spec}(B)$ can be identified with a subset of the set $\prod_{x \in B}\{0,1\}$ of *all* functions from B to the two-element set $\{0,1\}$. Consequently, the product topology on $\prod_{x \in B}\{0,1\}$ induces a topology on $\mathrm{Spec}(B)$, which depends functorially on the Boolean algebra B.

Theorem 0.0.2 (Stone Duality Theorem). *The construction $B \mapsto \mathrm{Spec}(B)$ determines a fully faithful embedding $\mathrm{Spec} : \mathrm{BAlg}^{\mathrm{op}} \hookrightarrow \mathrm{Top}$ from (the opposite of) the category of Boolean algebras to the category of topological spaces. The essential image of this functor is the full subcategory $\mathrm{Stone} \subseteq \mathrm{Top}$ whose objects are Stone spaces: that is, topological spaces which are compact, Hausdorff, and totally disconnected.*

The Stone duality theorem can be understood as supplying an equivalence between syntax and semantics in the setting of *propositional* logic. Every Boolean algebra B can viewed as a theory of propositional logic, whose models are the points of the spectrum $\mathrm{Spec}(B)$. Theorem 0.0.2 implies that B can be recovered (up to isomorphism) from the set of models $\mathrm{Spec}(B)$, together with its topology. Makkai proved a generalization of the Stone daulity theorem in the setting of *coherent* logic, where the Boolean algebra B is replaced by a small pretopos $\mathcal{C}$ (see §A.4), and the set $\mathrm{Spec}(B)$ is replaced by the category of models $\mathrm{Mod}(\mathcal{C})$ (Definition A.4.5). Roughly speaking, it asserts that a small pretopos $\mathcal{C}$ can be recovered (up to equivalence) from its category of models $\mathrm{Mod}(\mathcal{C})$, together with some additional structure that plays the role of a "topology" on $\mathrm{Mod}(\mathcal{C})$. To motivate the precise statement, we need to review a bit of point-set topology.

Construction 0.0.3 (The Stone-Čech Compactification). Let S be a set and let $P(S)$ denote the collection of all subsets of S, which we regard as a Boolean algebra. We denote the spectrum $\mathrm{Spec}(P(S))$ by βS and refer to it as the *Stone-Čech compactification of* S. By definition, the points of βS can be identified with Boolean algebra homomorphism $\mu : P(S) \to \{0,1\}$, which we refer to as *ultrafilters on* S (see §1.1).

Every element $s \in S$ determines an ultrafilter δ_s on S, given by the formula

$$\delta_s : P(S) \to \{0,1\} \qquad \delta_s(I) = \begin{cases} 1 & \text{if } s \in I \\ 0 & \text{if } s \notin I. \end{cases}$$

The construction $s \mapsto \delta_s$ determines a map of sets $\delta : S \to \beta S$. One can show that the map δ exhibits βS as a "universal" compactification of S in the following precise sense (see Proposition 3.2.7):

- $(*)$ Let X be a compact Hausdorff space and let $f : S \to X$ be a function. Then there is a unique continuous function $\overline{f} : \beta S \to X$ satisfying $\overline{f} \circ \delta = f$.

Note that an ultrafilter μ on a set S can be viewed as a finitely additive $\{0,1\}$-valued measure defined on the collection of all subsets of S. In the situation of $(*)$, we will indicate the continuous extension $\overline{f}$ by the suggestive notation $\overline{f}(\mu) = \int_S f(s) d\mu$. Assertion $(*)$ can then be understood as articulating a special feature enjoyed by the underlying set of any compact Hausdorff space X: any map of sets $f : S \to X$ can be "integrated" with respect to an ultrafilter $\mu \in \beta S$ to produce a new point $\int_S f(s) d\mu$ of X. This integration procedure is determined by the topology on X: note that it is characterized by the normalization condition $\int_S f(s) d\delta_t = f(t)$ and the requirement that $\int_S f(s) d\mu$ depends continuously $\mu \in \beta S$. Conversely, if the map $f : S \to X$ has dense image, then the topology on X can be recovered from the map $\mu \mapsto \int_S f(s) d\mu$ (since any continuous surjection of compact Hausdorff spaces $\beta S \to X$ is a quotient map). We can therefore regard the construction $(f, \mu) \mapsto \int_S f(s) d\mu$ as a way of encoding the the topology on the set X.

The integration procedure above has an analogue in the setting of coherent logic. If $\mathcal{C}$ is a small pretopos and $\{M_s\}_{s \in S}$ is a collection of models of $\mathcal{C}$ indexed by a set S, then to each ultrafilter μ the *ultraproduct* of the models M_s indexed by μ (see Theorem 2.1.1). To emphasize the relationship with the preceding construction, we will denote this ultraproduct by $\int_S M_s d\mu$. For every fixed ultrafilter $\mu \in \beta S$, the construction $\{M_s\}_{s \in S} \mapsto \int_S M_s d\mu$ determines a functor

$$\int_S (\bullet) d\mu : \mathrm{Mod}(\mathcal{C})^S \to \mathrm{Mod}(\mathcal{C}).$$

These functors, together with certain natural transformations relating them, determine what we will call an *ultrastructure* on the category $\mathrm{Mod}(\mathcal{C})$ (see Definition 1.3.1). Using this structure, one can formulate an analogue of Theorem 0.0.2 (for a more precise formulation, see Theorem 2.3.1):

Theorem 0.0.4 (Makkai's Strong Conceptual Completeness Theorem)**.** *Let $\mathcal{C}$ be a small pretopos. Then $\mathcal{C}$ is equivalent to the category* $\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ *of* ultrafunctors *from the category of models* $\mathrm{Mod}(\mathcal{C})$ *to the category of sets: that is, functors intertwine the ultrastructures on the categories* $\mathrm{Mod}(\mathcal{C})$ *and* Set *(see Definition 1.4.2).*

In this paper, we give a proof of Theorem 0.0.4 which is very different from the proof which appears in [9]. Before describing our strategy, let us briefly comment on the notion of *ultrafunctor* which appears in the statement of Theorem 0.0.4. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, and let $F : \mathcal{M} \to \mathcal{N}$ be a functor. An *ultrastructure* on F is a family of isomorphisms $F(\int_S M_s d\mu) \simeq \int_S F(M_s) d\mu$, indexed by collections $\{M_s\}_{s \in S}$ of objects of $\mathcal{M}$ and ultrafilters μ on S, which are required to satisfy a handful of coherence conditions. More generally, we introduce notions of *left ultrastructure* and *right ultrastructure* on F, which are given respectively by families of morphisms

$$\sigma_\mu : F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu \qquad \gamma_\mu : \int_S F(M_s) d\mu \to F(\int_S M_s d\mu),$$

which are not required to be invertible (but are still required to satisfy some coherence conditions; see Definitions 1.4.1 and 8.1.1). We define an *ultrafunctor* (*left ultrafunctor*, *right ultrafunctor*) from $\mathcal{M}$ to $\mathcal{N}$ to be a functor $F : \mathcal{M} \to \mathcal{N}$ together with an ultrastructure (left ultrastructure, right ultrastructure) on F. The collection of ultrafunctors (left ultrafunctors, right ultrafunctors) from $\mathcal{M}$ to $\mathcal{N}$ can be organized into a category which we will denote by $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{N})$ ($\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$, $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathcal{N})$), so that we have inclusions

$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \supseteq \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{N}) \subseteq \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathcal{N}). \tag{1}$$

Remark 0.0.5. To appreciate the distinction between ultrafunctors, left ultrafunctors, and right ultrafunctors, it is instructive to examine the special case where $\mathcal{M} = \{*\}$ is a category having a single object and a single morphism, and $\mathcal{N} = \mathrm{Set}$ is the category of sets. In this case, specifying a functor $F : \mathcal{M} \to \mathcal{N}$ is equivalent to specifying the set $X = F(*)$. In this case:

- The functor F always admits a unique left ultrastructure. Moreover, the construction $F \mapsto X = F(*)$ induces an equivalence of categories $\mathrm{Fun}^{\mathrm{LUlt}}(\{*\}, \mathrm{Set}) \simeq \mathrm{Set}$ (here we can replace $\mathcal{N} = \mathrm{Set}$ with any other ultracategory; see Proposition 4.2.3).
- The unique left ultrastructure on the functor F is an ultrastructure if and only if the set X is finite. Consequently, the construction $F \mapsto X = F(*)$ induces an equivalence of categories $\mathrm{Fun}^{\mathrm{Ult}}(\{*\}, \mathrm{Set}) \simeq \mathrm{Fin}$, where $\mathrm{Fin} \subseteq \mathrm{Set}$ denotes the category of finite sets (this is a special case Theorem 0.0.4, applied to the pretopos $\mathcal{C} = \mathrm{Fin}$).
- There is a bijective correspondence between right ultrastructures on the functor F and compact Hausdorff topologies on the set X. More precisely, the construction $F \mapsto X = F(*)$ can be upgraded to an equivalence of categories $\mathrm{Fun}^{\mathrm{RUlt}}(\{*\}, \mathrm{Set}) \simeq \mathrm{Comp}$, where Comp denotes the category of compact Hausdorff spaces (Example 8.4.10). In particular, the functor F admits a unique right ultrastructure when the set X is finite, and otherwise admits many different right ultrastructures.

Specializing (1) to this case, we obtain the inclusions $\mathrm{Set} \supseteq \mathrm{Fin} \subseteq \mathrm{Comp}$.

The bulk of this paper is devoted to the proof of the following generalization of Theorem 0.0.4:

Theorem 0.0.6. *Let $\mathcal{C}$ be a small pretopos and let* $\mathrm{Shv}(\mathcal{C})$ *denote the associated coherent topos. Then* $\mathrm{Shv}(\mathcal{C})$ *is equivalent to the category of left ultrafunctors* $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$.

Let us now outline the contents of this paper. We begin in §1 by defining the notions of ultracategory (Definition 1.3.1) and (left) ultrafunctor (Definition 1.4.1) that we will use throughout this paper (beware that our definitions are somewhat different from those which appear [9]; see Warning 1.0.4). The simplest examples are given by categories $\mathcal{M}$ which admit small products and small filtered colimits: under these

assumptions, we can regard $\mathcal{M}$ as an ultracategory by associating to each family of objects $\{M_s\}_{s\in S}$ of $\mathcal{M}$ and each ultrafilter μ on S the *categorical ultraproduct* $\int_S M_s d\mu = \varinjlim_{\mu(S_0)} \prod_{s\in S_0} M_s$, where the colimit is taken over the collection of subsets $S_0 \subseteq S$ satisfying $\mu(S_0) = 1$ (partially ordered by reverse inclusion). However, many of the ultracategories we are interested in cannot be obtained directly in this way. For example, if $\mathcal{C}$ is a small pretopos, then the category of models $\mathrm{Mod}(\mathcal{C})$ need not admit products. Nevertheless, one can still define the ultraproduct of a collection of models $\{M_s\}_{s\in S}$ by the formula $\int_S M_s d\mu = \varinjlim_{\mu(S_0)} \prod_{s\in S_0} M_s$, provided that the product $\prod_{s\in S_0} M_s$ is formed in the larger category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \supseteq \mathrm{Mod}(\mathcal{C})$ (which admits all small limits and colimits). In §2, we recall the Łos ultraproduct theorem, which asserts (in this context) that $\int_S M_s d\mu$ is again a model of $\mathcal{C}$, and use it to endow $\mathrm{Mod}(\mathcal{C})$ with the structure of an ultracategory (Remark 2.1.2). We then apply this observation to give a more precise formulation of Theorem 0.0.6 (Theorem 2.2.2), and deduce some of its consequences:

- From the equivalence $\mathrm{Shv}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$, one can immediately deduce Deligne's completeness theorem (Theorem 2.2.10), which asserts that every coherent topos has "enough points." When restricted to Boolean pretopoi, this is essentially equivalent to the classical Gödel completeness theorem.
- In §2.3, we use some elementary compactness to show that the equivalence of categories $\mathrm{Shv}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ restricts to an equivalence of full subcategories $\mathcal{C} \simeq \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$, thereby obtaining a proof of Makkai's strong conceptual completeness theorem (see Theorem 2.3.1).
- From the strong conceptual completeness theorem, one can immediately deduce *Makkai duality*: the construction $\mathcal{C} \mapsto \mathrm{Mod}(\mathcal{C})$ determines a fully faithful embedding of 2-categories

  $$\{\text{Small pretopoi, pretopos functors}\}^{\mathrm{op}} \hookrightarrow \{\text{Ultracategories, ultrafunctors}\};$$

  see Corollary 2.3.3. Using Theorem 2.2.2, one can promote this to a fully faithful embedding

  $$\{\text{Coherent topoi, geometric morphisms}\} \hookrightarrow \{\text{Ultracategories, left ultrafunctors}\};$$

  see Remark 2.2.9.
- Let $\mathcal{E}$ be a small exact category and let $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ denote the category of *regular* functors from $\mathcal{E}$ to the category of sets: that is, functors which preserve finite limits and effective epimorphisms. In [10], Makkai proved that the essential image of the Barr embedding

  $$\mathcal{E} \hookrightarrow \mathrm{Fun}(\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}), \mathrm{Set})$$

  consists of those functors $F : \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}) \to \mathrm{Set}$ which preserve small products and small filtered colimits. In §2.4, we give a different (though arguably less elementary) proof, showing that it is an elementary consequence of the strong conceptual completeness theorem (Theorem 2.4.2).

Our approach to Theorem 2.2.2 is somewhat roundabout, and depends on having a good general understanding of ultracategories and (left) ultrafunctors. Heuristically, one can think of ultrastructure on a category $\mathcal{M}$ as playing the role of a (compact Hausdorff) topology on $\mathcal{M}$. In §3, we provide evidence for this heuristic by showing that if $\mathcal{M}$ is a category having only identity morphisms, then endowing $\mathcal{M}$ with an ultrastructure is equivalent to choosing a compact Hausdorff topology on the set $X = \mathrm{Ob}(\mathcal{M})$ of objects of $\mathcal{M}$ (Theorem 3.1.5). Moreover, we show that this equivalence identifies the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ with the category $\mathrm{Shv}(X)$ of set-valued sheaves on X (Theorem 3.4.4). This equivalence carries the category of ultrafunctors $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathrm{Set}) \subseteq \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ to the full subcategory $\mathrm{Loc}(X) \subseteq \mathrm{Shv}(X)$ consisting of sheaves $\mathscr{F}$ which are locally constant with finite stalks (or, equivalently, the category of covering spaces $\widetilde{X} \to X$ with finite fibers); see Theorem 3.4.11. We regard this as strong motivation for allowing left ultrafunctors (as opposed to only ultrafunctors) into the basic vocabulary of our theory: from the category $\mathrm{Shv}(X)$ we can completely recover the topology of the topological space X, but the category $\mathrm{Loc}(X)$ is a much weaker invariant (for example, if X is simply connected, then $\mathrm{Loc}(X)$ is equivalent to the category Fin of finite sets).

Remark 0.0.7. To every Boolean algebra B, one can associate a small pretopos $\mathcal{C}_B$ for which the category of models $\mathrm{Mod}(\mathcal{C}_B)$ is equivalent to the spectrum $\mathrm{Spec}(B)$, considered as a category having only identity

morphisms. Under this equivalence, the ultrastructure on $\mathrm{Mod}(\mathcal{C}_B)$ corresponds to the the topology on the spectrum $\mathrm{Spec}(B)$. From this observation, one can use Makkai duality to deduce *part* of the Stone duality theorem: namely, the assertion that the spectrum functor $\mathrm{Spec} : \mathrm{BAlg}^{\mathrm{op}} \to \mathrm{Top}$ is fully faithful. However, this observation does not really lead to an independent proof of Theorem 0.0.2, since we use Stone duality (implicitly and explicitly) in proving the main results of this paper.

In §4, we leverage the results of §3 to study ultracategories in general. Let $\mathcal{M}$ be any ultracategory, and let X be a compact Hausdorff space (regarded as an ultracategory having only identity morphisms). Motivated by the equivalence $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set}) \simeq \mathrm{Shv}(X)$ of Theorem 3.4.4), it will be useful to think of $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})$ as a category of "$\mathcal{M}$-valued sheaves on X." Fixing $\mathcal{M}$ and allowing X to vary, we obtain a functor $X \mapsto \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})^{\mathrm{op}}$ from (the opposite of) the category Comp of compact Hausdorff spaces to the 2-category of small categories. This construction can be encoded by fibration of categories $\mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}$, where $\mathrm{Comp}_{\mathcal{M}}$ is a category whose objects are pairs $(X, \mathcal{O}_X)$, where X is a compact Hausdorff space and $\mathcal{O}_X : X \to \mathcal{M}$ is a left ultrafunctor. In §4, we show that $\mathrm{Comp}_{\mathcal{M}}$ is a *stack* (with respect to the Grothendieck topology on Comp given by finite jointly surjective families; see Proposition 4.1.5). Moreover, the construction $\mathcal{M} \mapsto \mathrm{Comp}_{\mathcal{M}}$ determines a fully faithful embedding of 2-categories

$$\{\text{Ultracategories, left ultrafunctors}\} \hookrightarrow \{\text{Stacks of categories on Comp}\};$$

see Theorem 4.3.3. Consequently, it is possible to formulate the theory of ultracategories using the language of topological stacks. Beware, however, that the construction $\mathcal{M} \mapsto \mathrm{Comp}_{\mathcal{M}}$ is not essentially surjective. However, if we restrict our attention to *groupoids*, then the essential image has a very simply description: stack in groupoids on Comp has the form $\mathrm{Comp}_{\mathcal{M}}$ (for some small ultracategory $\mathcal{M}$) if and only if it is representable by a groupoid internal to the category of compact Hausdorff spaces (Theorem 4.4.7).

In §5, we specialize to the study of left ultrafunctors $F : \mathcal{M} \to \mathrm{Set}$ taking values in the category of sets. For every ultracategory $\mathcal{M}$, let $\mathrm{Stone}_{\mathcal{M}} \subseteq \mathrm{Comp}_{\mathcal{M}}$ denote the full subcategory spanned by those pairs $(X, \mathcal{O}_X)$, where X is a Stone space. Combining the results of §4 and §3, we construct a fully faithful embedding $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set}) \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ (Theorem 5.2.1). In the case where $\mathcal{M}$ admits small filtered colimits, the essential image of this embedding admits a simple description: it is the full subcategory $\mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ which preserve finite products and small filtered colimits (Theorem 5.3.3). It follows from this description that, under some mild set-theoretic assumptions, the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is a Grothendieck topos (Proposition 5.4.5).

The virtue of Theorem 5.2.1 is that it allows us describe left ultrafunctors $F : \mathcal{M} \to \mathrm{Set}$ (which are functors equipped with a large amount of additional structure) in terms of ordinary functors $\mathscr{F} : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ (which are *a priori* more amenable to study using standard category-theoretic tools). To make use of this in practice, we need to know something about the structure of the category $\mathrm{Stone}_{\mathcal{M}}$. In §6, we specialize to the situation where $\mathcal{M} = \mathrm{Mod}(\mathcal{C})$ is the category of models of a small pretopos $\mathcal{C}$. In this case, the category $\mathrm{Stone}_{\mathcal{M}}$ admits a concrete description which is independent of the theory of ultracategories: according to Theorem 6.3.14, there is a fully faithful embedding $\Gamma : \mathrm{Stone}_{\mathcal{M}} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ into the category $\mathrm{Pro}(\mathcal{C})$ of *pro-objects* of $\mathcal{C}$, whose essential image is the full subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ of *weakly projective* pro-objects of $\mathcal{C}$ (Definition 6.2.2).

In §7, we combine the preceding ideas to obtain a proof of Theorem 2.2.2. Our strategy is inspired by the work of Bhatt-Scholze on pro-étale sheaves in the setting of algebraic geometry (and earlier work of Scholze in the rigid-analytic setting). To any small pretopos $\mathcal{C}$, we can equip the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ with a Grothendieck topology and consider the category $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ of set-valued sheaves on $\mathrm{Pro}(\mathcal{C})$. This category of sheaves is generally very large (for example, it is not a Grothendieck topos), but contains a more manageable category $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C}))$ of *continuous sheaves* (Definition 7.1.4) which is equivalent to the category of sheaves on $\mathcal{C}$ itself (Corollary 7.1.5). The equivalence of Theorem 0.0.6 can then be realized as a composition

$$\mathrm{Shv}(\mathcal{C}) \simeq \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \xrightarrow{\circ\Gamma} \mathrm{Fun}_0(\mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})}^{\mathrm{op}}, \mathrm{Set}) \simeq \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}).$$

Here the essential point is to show that any functor $\mathrm{Stone}^{\mathrm{op}}_{\mathrm{Mod}(\mathcal{C})} \to \mathrm{Set}$ which preserves finite products and small filtered colimits extends (uniquely) to a continuous sheaf on the category $\mathrm{Pro}(\mathcal{C})$ (Proposition 7.2.5), which we prove by exploiting the relationship between ultraproducts and elementary embeddings in the category $\mathrm{Mod}(\mathcal{C})$.

By definition, an ultrastructure on a category $\mathcal{M}$ is given by a collection of ultraproduct functors $\int_S (\bullet) d\mu : \mathcal{M}^S \to \mathcal{M}$ and natural transformations relating them, which are required to satisfy some axioms (expressing the commutativity of various diagrams). This is a large amount of data which can be somewhat cumbersome to work with. We close this paper by explaining an alternative approach to the theory of ultracategories which is in some ways more efficient. In §8, we introduce the notion of an *ultracategory envelope* (Definition 8.2.2). By definition, an ultracategory envelope is a category $\mathcal{E}$ satisfying a few simple axioms (which do not refer to any additional structure on $\mathcal{E}$), which determine an ultrastructure on a certain full subcategory $\mathcal{E}^{\mathrm{cc}} \subseteq \mathcal{E}$. We show that the construction $\mathcal{E} \mapsto \mathcal{E}^{\mathrm{cc}}$ induces a bijection from the equivalence of ultracategory envelopes (considered as abstract categories) to equivalence classes of ultracategories (considered as categories with additional structure). In particular, every ultracategory $\mathcal{M}$ can be identified with $\mathrm{Env}(\mathcal{M})^{\mathrm{cc}}$ for an essentially unique ultracategory envelope $\mathrm{Env}(\mathcal{M})$, which we refer to as the *envelope of* $\mathcal{M}$. The category $\mathrm{Env}(\mathcal{M})$ admits a number of (equivalent) realizations, which we describe in §8.4.

Remark 0.0.8. The theory of ultracategory envelopes developed in §8 will play no role in our proof of Makkai's strong conceptual completeness theorem. As we will see, ultracategory envelopes are very well adapted to describing *right* ultrafunctors between ultracategories, while our approach is based on the classification of *left* ultrafunctors from $\mathrm{Mod}(\mathcal{C})$ to Set. However, one of the original motivations for the work described in this paper was to find a formulation (and proof) of Makkai's theorem which can be adapted easily to the setting of higher category theory. For this purpose, the formalism of ultracategory envelopes is much more convenient than the explicit approach of §1. We will return to this point in a future work.

For the convenience of the reader, we include some appendices which review the categorical background which is used in the body of this paper.

Warning 0.0.9. The definitions of ultracategory and ultrafunctor that we use in this paper are somewhat different from the definitions which appear in [9] (see Warning 1.0.4). For us, an object of the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ is a functor $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ together with a collection of isomorphisms

$$\sigma_\mu : F(\int_S M_s d\mu) \simeq \int_S F(M_s) d\mu$$

for which a relatively small number of diagrams are required to commute (see Definition 1.4.1). Makkai's definition is similar, but requires a much larger number of diagrams to commute. As a consequence, our category of ultrafunctors $\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ is *a priori* larger than the category of ultrafunctors introduced by Makkai. Consequently, our version Theorem 0.0.4 can be viewed as a slight strengthening of Makkai's original result: it shows that every ultrafunctor $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ in the sense of this paper is also an ultrafunctor in the more restrictive sense of Makkai (since it is given by evaluation at an object $C \in \mathcal{C}$).

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Conventions. We use the following notations:

- We write Set for the category of sets.
- We write Fin for the category of finite sets (regarded as a full subcategory of Set).
- We write Top for the category of topological spaces.
- We write Comp for the category of compact Hausdorff spaces (regarded as a full subcategory of Top).
- We write Stone for the category of Stone spaces (regarded as a full subcategory of Comp).

If X is a topological space, we let $\mathrm{Shv}(X)$ denote the category of set-valued sheaves on X. For each object $\mathscr{F} \in \mathrm{Shv}(X)$, we let $\mathscr{F}(U)$ denote its value on an open subset $U \subseteq X$ and $\mathscr{F}_x$ its stalk at a point $x \in X$

(given by the direct limit $\varinjlim_{x \in U} \mathscr{F}(U)$, taken over the collection of all open neighborhoods U of the point x). Every continuous map of topological spaces $f : X \to Y$ induces *pushforward* and *pullback* functors

$$f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y) \qquad f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X),$$

satisfying the formulae

$$(f_* \mathscr{F})(U) = \mathscr{F}(f^{-1}(U)) \qquad (f^* \mathscr{G})_x \simeq \mathscr{G}_{f(x)}$$

for $\mathscr{F} \in \mathrm{Shv}(X)$ and $\mathscr{G} \in \mathrm{Shv}(Y)$, respectively. Given a sheaf $\mathscr{G}$ on Y, we sometimes denote its pullback $f^* \mathscr{G}$ by $\mathscr{G}|_X$ (primarily in the case where X is a subspace of Y). We will need the following elementary fact:

Proposition 0.0.10 (Proper Base Change)**.** *Let $f : X \to Y$ be a continuous map of compact Hausdorff spaces and let $\mathscr{F}$ be a set-valued sheaf on X. Let $y \in Y$ be a point and let $X_y = f^{-1}\{y\}$ be the fiber of f over y. Then the canonical map*

$$(f_* \mathscr{F})_y \to (\mathscr{F}|_{X_y})(X_y)$$

is a bijection.

Concretely, Proposition 0.0.10 asserts that every global section of $\mathscr{F}|_{X_y}$ can be extended to a section of $\mathscr{F}$ over an open set of the form $f^{-1}(U)$, where U is an open neighborhood of the point $y \in Y$; moreover any two extensions coincide over $f^{-1}(V)$, for some smaller open neighborhood V of the point y.

Warning 0.0.11. Throughout this paper, we will generally ignore set-theoretic issues when working with categories that are not necessarily small. However, our notion of *ultrastructure* introduces set-theoretic issues of a new kind (which we will also disregard). By definition, an ultrastructure on a category $\mathcal{M}$ consists of a collection of functors

$$\int_S (\bullet) d\mu : \mathcal{M}^S \to \mathcal{M},$$

indexed by the collection of *all* sets S and *all* choices of ultrafilter μ on S. Consequently, this is a proper class of data even if we assume that the category $\mathcal{M}$ is small. One can address this (within the framework of Zermelo-Fraenkel set theory, say) as follows:

- (a) We assume throughout this paper that we have chosen a strongly inaccessible cardinal κ.
- (b) We say that a mathematical object (like a set or a topological space) is *small* if it has cardinality $< \kappa$. All mathematical objects other than categories are assumed to be small unless otherwise specified.
- (c) Whenever we speak of the ultraproduct $\int_S M_s d\mu$ of a family of objects $\{M_s\}_{s \in S}$, we assume that S is small.

Of course, this is just for convenience; none of the results of this paper depend on the existence of a strongly inaccessible cardinal in an essential way.

1. Ultracategories

Let $\{M_s\}_{s \in S}$ be a collection of nonempty sets. Every ultrafilter μ on S determines an equivalence relation $\simeq_\mu$ on the Cartesian product $\prod_{s \in S} M_s$, given by the formula

$$(\{x_s\}_{s \in S} \simeq_\mu \{y_s\}_{s \in S}) \Leftrightarrow \mu(\{s \in S : x_s = y_s\}) = 1.$$

We will refer to the quotient $(\prod_{s \in S} M_s)/\simeq_\mu$ as the *ultraproduct of $\{M_s\}_{s \in S}$ with respect to* μ and denote it by $\int_S M_s d\mu$. This quotient can be characterized by a universal mapping property: it is the direct limit

$$\varinjlim_{\mu(S_0)=1} (\prod_{s \in S_0} M_s),$$

taken over the collection of all subsets $S_0 \subseteq S$ satisfying $\mu(S_0) = 1$, partially ordered by reverse inclusion (see Example 1.2.6). This observation allows us to make sense of ultraproducts in a more general setting:

Construction 1.0.1 (Categorical Ultraproducts)**.** Let $\mathcal{M}$ be a category which admits small products and small filtered colimits. Suppose we are given a collection of objects $\{M_s\}_{s \in S}$ of $\mathcal{M}$, together with an ultrafilter μ on the set S. We let $\int_S M_s d\mu$ denote the direct limit $\varinjlim_{\mu(S_0)=1} (\prod_{s \in S_0} M_s)$, where the product and direct

limit are computed in the category $\mathcal{M}$. We will refer to $\int_S M_s d\mu$ as the *categorical ultraproduct of* $\{M_s\}_{s\in S}$ *with respect to* μ.

This section is devoted to the following:

Question 1.0.2. What are the essential properties of the ultraproduct construction $\{M_s\}_{s\in S} \mapsto \int_S M_s d\mu$?

In §1.2, we partially address Question 1.0.2 by observing that the categorical ultraproduct construction has the following features:

(*a*) If $\{M_s\}_{s\in S}$ is a collection of objects of $\mathcal{M}$ indexed by a set S and δ_{s_0} is the *principal* ultrafilter associated to an element $s_0 \in S$, then there is a canonical isomorphism $\epsilon_{S,s_0} : \int_S M_s \delta_{s_0} \simeq M_{s_0}$ (Example 1.2.7).

(*b*) If $\{N_t\}_{t\in T}$ is a collection of objects of $\mathcal{M}$ indexed by a set T, $\nu_\bullet = \{\nu_s\}_{s\in S}$ is a collection of ultrafilters on T indexed by a set S, and μ is an ultrafilter on S, then there is a canonical map

$$\Delta_{\mu,\nu_\bullet} : \int_T N_t d(\int_S \nu_s d\mu) \to \int_S (\int_T N_t d\nu_s) d\mu,$$

which we call the *categorical Fubini transformation*; here $\int_S \nu_s d\mu$ denotes the ultrafilter on T given by the formula $(\int_S \nu_s d\mu)(T_0) = \mu(\{s \in S : \nu_s(T_0) = 1\})$ (see Proposition 1.2.8).

In §1.3, we place these observations into an axiomatic framework by introducing the notion of an *ultracategory*. For any category $\mathcal{M}$, we define an *ultrastructure* on $\mathcal{M}$ to be a collection of functors

$$\int_S (\bullet) d\mu : \mathcal{M}^S \to \mathcal{M},$$

indexed by the collection of all sets S and all ultrafilters μ on S, together with natural transformations ϵ_{S,s_0} and $\Delta_{\mu,\nu_\bullet}$ as in (*a*) and (*b*) above (which are required to satisfy a few additional axioms). We define an *ultracategory* to be a category $\mathcal{M}$ together with an ultrastructure on $\mathcal{M}$ (Definition 1.3.1).

If $\mathcal{M}$ is a category which admits small products and small filtered colimits, then the categorical ultraproduct of Construction 1.0.1 determines an ultrastructure on $\mathcal{M}$, which we will refer to as the *categorical ultrastructure*. However, there are interesting examples of ultrastructures which do not arise in this way:

(1) Let $\mathcal{C}$ be a pretopos and let $\mathrm{Mod}(\mathcal{C})$ denote the category of models of $\mathcal{C}$, which we regard as a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$. It follows from the Łos ultraproduct theorem that the full subcategory $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ is closed under the formation of ultraproducts (see Theorem 2.1.1). Consequently, the categorical ultrastructure on $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ induces an ultrastructure on the category $\mathrm{Mod}(\mathcal{C})$, which usually cannot be obtained by applying Construction 1.0.1 directly to $\mathrm{Mod}(\mathcal{C})$ (because the category $\mathrm{Mod}(\mathcal{C})$ generally does not have products).

(2) Let X be a set, regarded as a category having only identity morphisms. In §3, we will show that there is a bijective correspondence between the set of ultrastructures on X and the collection of compact Hausdorff topologies on X (Theorem 3.1.5). These ultrastructures never arise from Construction 1.0.1, except in the trivial case where X has a single point.

Remark 1.0.3. Example (1) illustrates a general phenomenon. If $\mathcal{M}$ is an ultracategory containing a collection of objects $\{M_s\}_{s\in S}$ and μ is an ultrafilter on S, then the ultraproduct $\int_S M_s d\mu$ (given by the ultrastructure on $\mathcal{M}$) need not coincide with the categorical ultraproduct $\varinjlim_{\mu(S_0)=1} (\prod_{s\in S_0} M_s)$ of Construction 1.0.1 (in fact, the categorical ultraproduct might not even be defined, since $\mathcal{M}$ need not admit products or filtered colimits). However, we will show in §4.2 that one can always obtain $\int_S M_s d\mu$ by applying Construction 1.0.1 inside a larger category which contains $\mathcal{M}$ (Theorem 4.2.7). We will see in §8 that there is a canonical choice for this enlargement, which we will denote by $\mathrm{Env}(\mathcal{M})$ and refer to as the *ultracategory envelope* of $\mathcal{M}$.

For our purpose, the main virtue of axiomatizing the notion of ultracategory is that it allows us to precisely formulate what it means for a functor to "commute with ultraproducts." Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, and let $F : \mathcal{M} \to \mathcal{N}$ be a functor. We define an *ultrastructure* on F to be a collection of isomorphisms

$$\sigma_\mu : F(\int_S M_s) d\mu \xrightarrow{\sim} \int_S F(M_s) d\mu,$$

parametrized by all collections of objects $\{M_s\}_{s\in S}$ of $\mathcal{M}$ and all ultrafilters μ on S, satisfying a few natural constraints (see Definition 1.4.1). We define an *ultrafunctor* from $\mathcal{M}$ to $\mathcal{N}$ to be a pair $(F,\{\sigma_\mu\})$, where F is a functor from $\mathcal{M}$ to $\mathcal{N}$ and $\{\sigma_\mu\}$ is an ultrastructure on $\mathcal{M}$. The collection of all ultrafunctors from $\mathcal{M}$ to $\mathcal{N}$ forms a category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M},\mathcal{N})$, which we will study in §1.4. We also introduce a larger category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathcal{N})$ of *left ultrafunctors* from $\mathcal{M}$ to $\mathcal{N}$, which is defined in a similar way except that we do not require the morphisms σ_μ to be isomorphisms (there is also a dual notion of *right ultrafunctor*, which we will study in §8).

Warning 1.0.4. The theory of ultracategories developed in this paper is inspired by the work of Makkai ([9]). However, our definition of ultracategory differs from Makkai's definition in two respects:

- By our definition, an ultrastructure on a category $\mathcal{M}$ is completely determined by the ultraproduct functors
$$\int_S(\bullet)d\mu : \mathcal{M}^S \to \mathcal{M},$$
together with certain natural maps
$$\epsilon_{S,s_0} : \int_S M_s\delta_{s_0} \simeq M_{s_0} \qquad \Delta_{\mu,\nu_\bullet} : \int_T N_t d(\int_S \nu_s d\mu) \to \int_S(\int_T N_t d\nu_s)d\mu.$$
In [9], an ultrastructure consists of much more data (involving natural transformations between very complicated iterated ultraproducts).
- Our definition includes certain axioms that the maps ϵ_{S,s_0} and $\Delta_{\mu,\nu_\bullet}$ are required to satisfy. These axioms do not appear in [9].

1.1. Ultrafilters.

We begin with a brief review of the theory of ultrafilters, which will play an essential role throughout this paper.

Definition 1.1.1. Let S be a set and let $P(S)$ denote the Boolean algebra of all subsets of S. An *ultrafilter on* S is a Boolean algebra homomorphism
$$\mu : P(S) \to \{0,1\}.$$

Remark 1.1.2. Let S be a set and let $P(S)$ denote the collection of all subsets of S. To each ultrafilter μ on S, we can associate a subset $\mathcal{U}_\mu \subseteq P(S)$ given by
$$\mathcal{U}_\mu = \{S_0 \subseteq S : \mu(S_0) = 1\}.$$
The construction $\mu \mapsto \mathcal{U}_\mu$ determines a bijection from the set of ultrafilters on S to the collection of subsets $\mathcal{U} \subseteq P(S)$ with the following properties:

(a) The subset $\mathcal{U}$ is closed under finite intersections. That is, the set S belongs to $\mathcal{U}$, and for every pair $S_0, S_1 \in \mathcal{U}$, the intersection $S_0 \cap S_1$ also belongs to $\mathcal{U}$.
(b) For every subset $S_0 \subseteq S$, exactly one of the sets S_0 and $S \smallsetminus S_0$ belongs to $\mathcal{U}$.

Example 1.1.3 (Principal Ultrafilters)**.** Let S be a set. Then each element $s \in S$ determines an ultrafilter δ_s on S, given by the formula
$$\delta_s : P(S) \to \{0,1\} \qquad \delta_s(S_0) = \begin{cases} 1 & \text{if } s \in S_0 \\ 0 & \text{if } s \notin S_0. \end{cases}$$
We will refer to δ_s as the *principal ultrafilter associated to* s. We will say that an ultrafilter μ on S is *principal* if it has the form δ_s for some $s \in S$.

Construction 1.1.4 (Pushforward of Ultrafilters)**.** Let $f : S \to T$ be a map of sets and let μ be an ultrafilter on S. We define an ultrafilter $f_*\mu$ on T by the formula $(f_*\mu)(T_0) = \mu(f^{-1}(T_0))$. We will refer to $f_*\mu$ as the *pushforward of* μ *along* f.

Remark 1.1.5. Suppose that $f : S \to T$ is an injective map of sets. Then the pushforward map f_* is an injection
$$\{\text{Ultrafilters on } S\} \hookrightarrow \{\text{Ultrafilters on } T\},$$

whose image consists of those ultrafilters μ on T satisfying $\mu(f(S)) = 1$.

In particular, if $S = T_0$ is a subset of T and μ is an ultrafilter on T satisfying $\mu(T_0) = 1$, then μ restricts to an ultrafunctor $\mu_0 : \{\text{Subsets of } T_0\} \to \{0,1\}$.

We will need a generalization of Construction 1.1.4:

Construction 1.1.6 (Composition of Ultrafilters)**.** Let S and T be sets, and let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by the set S. For each ultrafilter μ on T, we let $\int_S \nu_s d\mu$ denote the ultrafilter on T given by the formula

$$(\int_S \nu_s d\mu)(T_0) = \mu(\{s \in S : \nu_s(T_0) = 1\}).$$

Example 1.1.7. Let $f : S \to T$ be a map of sets and let μ be an ultrafilter on S. Then the pushforward ultrafilter $f_*\mu$ of Construction 1.1.4 is given by the formula $f_*\mu = \int_S \delta_{f(s)} d\mu$. This follows from the calculation

$$\begin{aligned}(\int_S \delta_{f(s)} d\mu)(T_0) &= \mu(\{s \in S : \delta_{f(s)}(T_0) = 1\}) \\ &= \mu(\{s \in S : f(s) \in T_0\}) \\ &= \mu(f^{-1}T_0) \\ &= (f_*\mu)(T_0).\end{aligned}$$

Example 1.1.8. Let S and T be sets and let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by S. Let s_0 be an element of S and let δ_{s_0} denote the corresponding principal ultrafilter. Then the composite ultrafilter $\int_S \nu_s d\delta_{s_0}$ is equal to ν_{s_0}.

Remark 1.1.9 (Associativity)**.** Let T be a set, let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T, let $\{\mu_r\}_{r\in R}$ be a collection of ultrafilters on S, and let λ be an ultrafilter on R. Then we have an equality

$$\int_R (\int_S \nu_s d\mu_r) d\lambda = \int_S \nu_s d(\int_R \mu_r d\lambda).$$

Both sides coincide with the ultrafilter ρ on T given by the formula

$$\rho(T_0) = \lambda(\{r \in R : \mu_r(\{s \in S : \nu_s(T_0) = 1\}) = 1\}).$$

We will need the following existence result, which asserts that every *filter* on a set S can be extended to an ultrafilter.

Proposition 1.1.10. *Let S be a set and let $\mathcal{U}$ be a collection of subsets of S which is closed under finite intersections. If $\varnothing \notin \mathcal{U}$, then there exists an ultrafilter μ on S such that $\mu(S_0) = 1$ for each $S_0 \in \mathcal{U}$.*

Proof. Let Q be the collection of all subsets $\mathcal{V} \subseteq P(S)$ which are closed under finite intersections and satisfy $\varnothing \notin \mathcal{V}$. Let us regard Q as a partially ordered set with respect to inclusion. Applying Zorn's lemma, we deduce that $\mathcal{U} \in Q$ is contained in a maximal element $\mathcal{V} \in Q$. Define $\mu : P(S) \to \{0,1\}$ by the formula

$$\mu(S_0) = \begin{cases} 1 & \text{if } S_0 \in \mathcal{V} \\ 0 & \text{otherwise.} \end{cases}$$

We will complete the proof by showing that μ is an ultrafilter on S: that is, that the set $\mathcal{V}$ satisfies conditions (a) and (b) of Remark 1.1.2. Condition (a) is immediate. To prove (b), let S_0 be any subset of S; we must show that $\mathcal{V}$ contains either S_0 or the complement $S \smallsetminus S_0$ (it cannot contain both, since $\mathcal{V}$ is closed under finite intersections and does not contain $\varnothing$). Suppose otherwise, and set $\mathcal{V}_+ = \mathcal{V} \cup \{S_0 \cap I : I \in \mathcal{V}\}$. Then $\mathcal{V}_+$ is a subset of $P(S)$ which is closed under finite intersections and properly contains $\mathcal{V}$. Invoking the maximality of $\mathcal{V}$, we conclude that $\varnothing \in \mathcal{V}_+$: that is, we can choose a set $I \in \mathcal{V}$ such that $S_0 \cap I = \varnothing$. By the same reasoning, we can choose a set $J \in \mathcal{V}$ such that $(S \smallsetminus S_0) \cap J = \varnothing$. Since $\mathcal{V}$ is closed under finite intersections, we conclude that $\varnothing = I \cap J \in \mathcal{V}$, contradicting our assumption that $\mathcal{V}$ belongs to Q. □

1.2. **The Ultraproduct Construction.** In this section, we discuss the categorical ultraproduct of Construction 1.0.1 in more detail. With an eye toward future applications, we work in a slightly more general setting.

Notation 1.2.1. Let μ be an ultrafilter on a set S and set $\mathcal{U}_\mu = \{S_0 \subseteq S : \mu(S_0) = 1\}$ (Remark 1.1.2). We regard $\mathcal{U}_\mu$ as a partially ordered set with respect to inclusions. Note that since $\mathcal{U}_\mu$ is closed under finite intersections, the opposite partially ordered set $\mathcal{U}_\mu^{\mathrm{op}}$ is directed.

Construction 1.2.2 (Categorical Ultraproducts)**.** Let $\mathcal{M}^+$ be a category and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory. We will say that $\mathcal{M}$ *has ultraproducts in* $\mathcal{M}^+$ if the following conditions are satisfied:

- For every collection $\{M_s\}_{s\in S}$ of objects of $\mathcal{M}$ indexed by a set S, there exists a product $\prod_{s\in S} M_s$ in the category $\mathcal{M}^+$.
- For every collection $\{M_s\}_{s\in S}$ of objects of $\mathcal{M}$ indexed by a set S and and every ultrafilter μ on S, the diagram
$$(S_0 \in \mathcal{U}_\mu^{\mathrm{op}}) \mapsto (\prod_{s\in S_0} M_s)$$
admits a colimit (in the category $\mathcal{M}^+$) which belongs to the subcategory $\mathcal{M} \subseteq \mathcal{M}^+$. In this case, we denote this colimit by $\int_S M_s d\mu$ and refer to it as the *categorical ultraproduct of* $\{M_s\}_{s\in S}$ *indexed by* μ.

Notation 1.2.3. Let $\mathcal{M}^+$ be a category and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory which has ultraproducts in $\mathcal{M}^+$. Fix a set S and an ultrafilter μ on S. For every collection of objects $\{M_s\}_{s\in S}$ of $\mathcal{M}$, the categorical ultraproduct $\int_S M_s d\mu$ comes equipped with a family of maps
$$q_\mu^{S_0} : \prod_{s\in S_0} M_s \to \int_S M_s d\mu,$$
indexed by those subsets $S_0 \subseteq S$ satisfying $\mu(S_0) = 1$. In the special case $S_0 = S$, we will denote $q_\mu^{S_0}$ simply by $q_\mu : \prod_{s\in S} M_s \to \int_S M_s d\mu$.

Suppose we are given a collection $\{f_s : M_s \to N_s\}$ of morphisms in $\mathcal{M}$, indexed by a set S. We let $\int_S f_s d\mu$ denote the unique morphism from $\int_S M_s d\mu$ to $\int_S N_s d\mu$ in the category $\mathcal{M}$ with the property that, for every subset $S_0 \subseteq S$ satisfying $\mu(S_0) = 1$, the diagram
$$\begin{tikzcd} \prod_{s\in S_0} M_s \arrow[r, "\prod_{s\in S_0} f_s"] \arrow[d, "q_\mu^{S_0}"] & \prod_{s\in S_0} N_s \arrow[d, "q_\mu^{S_0}"] \\ \int_S M_s d\mu \arrow[r, "\int_S f_s d\mu"] & \int_S N_s d\mu \end{tikzcd}$$
commutes (in the category $\mathcal{M}^+$). The constructions
$$\{M_s\}_{s\in S} \mapsto \int_S M_s d\mu \qquad \{f_s\}_{s\in S} \mapsto \int_S f_s d\mu$$
determine a functor $\int_S (\bullet) d\mu : \mathcal{M}^S \to \mathcal{M}$, which we will refer to as the *categorical ultraproduct functor*. By construction, for each subset $S_0 \subseteq S$ satisfying $\mu(S_0) = 1$, the construction
$$\{M_s\}_{s\in S} \mapsto (q_\mu^{S_0} : \prod_{s\in S_0} M_s \to \int_S M_s d\mu)$$
is a natural transformation of functors.

Warning 1.2.4. Let $\mathcal{M}^+$ be a category and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a subcategory which has ultraproducts in $\mathcal{M}^+$. For any ultrafilter μ on a set S, the ultraproduct functor $\int_S (\bullet) d\mu : \mathcal{M}^S \to \mathcal{M}$ is given by the formula $\int_S M_s d\mu = \varinjlim_{\mu(S_0)=1} \prod_{s\in S_0} M_s$. Beware that the products appearing in this formula are formed in the category $\mathcal{M}^+$, and need not belong to $\mathcal{M}$. In particular, the ultraproduct functors $\int_S (\bullet) d\mu : \mathcal{M}^S \to \mathcal{M}$ are not necessarily intrinsic to $\mathcal{M}$: they depend on the structure of the larger category $\mathcal{M}^+$.

Example 1.2.5. Let $\mathcal{M}$ be a category which admits small products and small filtered colimits. Then $\mathcal{M}$ has ultraproducts in itself.

Example 1.2.6 (Ultraproducts of Sets)**.** Let $\{M_s\}_{s\in S}$ be a collection of sets indexed by a set S, let μ be an ultrafilter on S, and let $\int_S M_s d\mu$ be the categorical ultraproduct of Construction 1.2.2. Let $q_\mu : \prod_{s\in S} M_s \to \int_S M_s d\mu$ be the map of Notation 1.2.3, so that we can identify q_μ with a filtered direct limit of projection maps

$$\pi_{S_0} : \prod_{s\in S} M_s \to \prod_{s\in S_0} M_s.$$

If each of the sets M_s is nonempty, then each of the maps π_{S_0} is surjective and therefore q_μ is also surjective. In this case, we can identify $\int_S M_s d\mu$ with the quotient of the Cartesian product $\prod_{s\in S} M_s$ by an equivalence relation $\simeq_\mu$, given explicitly by

$$(\{x_s\}_{s\in S} \simeq_\mu \{y_s\}_{s\in S}) \Leftrightarrow (\mu(\{s \in S : x_s = y_s\}) = 1).$$

Beware that if one of the sets M_s is empty, then the map q_μ need not be surjective: in this case, the Cartesian product $\prod_{s\in S} M_s$ is empty, but the ultraproduct $\int_S M_s d\mu$ need not be.

Example 1.2.7 (Principal Ultrafilters)**.** Let $\mathcal{M}^+$ be a category and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory which has ultraproducts in $\mathcal{M}^+$. Let S be a set containing an element $s_0 \in S$, and let δ_{s_0} denote the principal ultrafilter associated to s_0. Then the partially ordered set $\mathcal{U}_{\delta_{s_0}} = \{S_0 \subseteq S : s_0 \in S_0\}$ has a least element, given by the singleton $\{s_0\}$. It follows that, for any collection of objects $\{M_s\}_{s\in S}$, we have a canonical isomorphism

$$\epsilon_{S,s_0} : \int_S M_s d\delta_{s_0} \simeq \prod_{s\in\{s_0\}} M_s = M_{s_0}.$$

The construction $\{M_s\}_{s\in S} \mapsto \epsilon_{S,s_0}$ determines a natural isomorphism $\epsilon_{S,s_0} : \int_S (\bullet) d\delta_{s_0} \xrightarrow{\sim} \mathrm{ev}_{s_0}$ of functors from $\mathcal{M}^S$ to $\mathcal{M}$; here $\mathrm{ev}_{s_0} : \mathcal{M}^S \to \mathcal{M}$ denotes the evaluation functor $\{M_s\}_{s\in S} \mapsto M_{s_0}$.

We now make an elementary observation concerning iterated ultraproducts.

Proposition 1.2.8. *Let $\mathcal{M}^+$ be a category and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory which has ultraproducts in $\mathcal{M}^+$. Let $\{M_t\}_{t\in T}$ be a collection of objects of $\mathcal{M}$ indexed by a set T, let $\nu_\bullet = \{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S. Let μ be an ultrafilter on S and let $\int_S \nu_s d\mu$ denote the composite ultrafunctor of Construction 1.1.6. Then there is a unique morphism*

$$\Delta_{\mu,\nu_\bullet} : \int_T M_t d(\int_S \nu_s d\mu) \to \int_S (\int_T M_t d\nu_s) d\mu$$

in the category $\mathcal{M}$ with the following property:

- $(*)$ *Let $S_0 \subseteq S$ and $T_0 \subseteq T$ be subsets such that $\mu(S_0) = 1$ and $\nu_s(T_0) = 1$ for each $s \in S_0$ (so that we also have $(\int_S \nu_s d\mu)(T_0) = 1$). Then the diagram*

$$\begin{tikzcd}
\prod_{t\in T_0} M_t \arrow[r, "\{q^{T_0}_{\nu_s}\}_{s\in S_0}"] \arrow[d, "q^{T_0}_{\int_S \nu_s d\mu}"] & \prod_{s\in S} (\int_T M_t d\nu_s) \arrow[d, "q^{S_0}_\mu"] \\
\int_T M_t d(\int_S \nu_s d\mu) \arrow[r, "\Delta_{\mu,\nu_\bullet}"] & \int_S (\int_T M_t d\nu_s) d\mu
\end{tikzcd}$$

commutes (in the category $\mathcal{M}^+$).

Proof. From the definition of the ultraproduct $\int_T M_t d(\int_S \nu_s d\mu)$ as a colimit, we see that there is a unique morphism

$$\Delta_{\mu,\nu_\bullet} : \int_T M_t d(\int_S \nu_s d\mu) \to \int_S (\int_T M_t d\nu_s) d\mu$$

for which the diagram of $(*)$ commutes in the special case where $S_0 = \{s \in S : \nu_s(T_0)\}$. It follows immediately from the definitions that the diagram commutes in general. ☐

Notation 1.2.9 (Categorical Fubini Transformation)**.** In the situation of Proposition 1.2.8, we will refer to the map $\Delta_{\mu,\nu_\bullet}$ as the *categorical Fubini transformation*. Note that is depends functorially on $\{M_t\}_{t\in T}$: that is, we can regard $\Delta_{\mu,\nu_\bullet}$ as a natural transformation of functors from $\mathcal{M}^T$ to $\mathcal{M}$, fitting into a diagram

$$\begin{tikzcd}
 & \mathcal{M}^S \arrow[dr, "\int_S(\bullet)d\mu"] \arrow[d, phantom, "\Uparrow \Delta_{\mu,\nu_\bullet}"] & \\
\mathcal{M}^T \arrow[ur, "\{\int_T(\bullet)d\nu_s\}_{s\in S}"] \arrow[rr, "\int_T(\bullet)d(\int_S \nu_s d\mu)"'] & {} & \mathcal{M}.
\end{tikzcd}$$

1.3. **Ultracategories.**

We now place the constructions of §1.2 into an axiomatic framework.

Definition 1.3.1. Let $\mathcal{M}$ be a category. An *ultrastructure* on $\mathcal{M}$ consists of the following data:

(1) For every set S and every ultrafilter μ on S, a functor

$$\int_S(\bullet)d\mu : \mathcal{M}^S \to \mathcal{M}$$

We will denote the value of this functor on an object $\{M_s\}_{s\in S} \in \mathcal{M}^S$ by $\int_S M_s d\mu$, and refer to it as the *ultraproduct of* $\{M_s\}_{s\in S}$ *with respect to* μ.

(2) For every family of objects $\{M_s\}_{s\in S}$ and every element $s_0 \in S$, an isomorphism

$$\epsilon_{S,s_0} : \int_S M_s d\delta_{s_0} \xrightarrow{\sim} M_{s_0};$$

here δ_{s_0} denotes the principal ultrafilter associated to s_0. We require that, for fixed S and s_0, these isomorphisms depend functorially on $\{M_s\}_{s\in S}$: that is, they determine a natural isomorphism $\epsilon_{S,s_0} : \int_S(\bullet)d\delta_{s_0} \xrightarrow{\sim} \mathrm{ev}_{s_0}$, where $\mathrm{ev}_{s_0} : \mathcal{M}^S \to \mathcal{M}$ denotes the evaluation function $\{M_s\}_{s\in S} \mapsto M_{s_0}$.

(3) For every family of objects $\{M_t\}_{t\in T}$ indexed by a set T, every family $\nu_\bullet = \{\nu_s\}_{s\in S}$ of ultrafilters on T indexed by a set S, and every ultrafilter μ on S, a morphism

$$\Delta_{\mu,\nu_\bullet} : \int_T M_t d(\int_S \nu_s d\mu) \to \int_S(\int_T M_t d\nu_s)d\mu$$

which we call the *Fubini transformation*.

For fixed S, T, μ, and $\nu_\bullet$, we require that these morphisms depend functorially on the family $\{M_t\}_{t\in T}$. That is, they determine a natural transformation of functors

$$\Delta_{\mu,\nu_\bullet} : \int_T(\bullet)d(\int_S \nu_s d\mu) \to \int_T(\int_S(\bullet)d\nu_s)d\mu$$

of functors from $\mathcal{M}^T$ to $\mathcal{M}$, fitting into a diagram

$$\begin{tikzcd}
 & \mathcal{M}^S \arrow[dr, "\int_S(\bullet)d\mu"] \arrow[d, phantom, "\Uparrow \Delta_{\mu,\nu_\bullet}"] & \\
\mathcal{M}^T \arrow[ur, "\{\int_T(\bullet)d\nu_s\}_{s\in S}"] \arrow[rr, "\int_T(\bullet)d(\int_S \nu_s d\mu)"'] & {} & \mathcal{M}.
\end{tikzcd}$$

These data are required to satisfy the following axioms:

(A) Let $\{M_t\}_{t\in T}$ be a collection of objects of $\mathcal{M}$ indexed by a set T, let $\nu_\bullet = \{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S, and let δ_{s_0} be the principal ultrafilter on S associated to an element $s_0 \in S$. Then the Fubini transformation

$$\Delta_{\delta_{s_0},\nu_\bullet} : \int_T M_t d(\int_S \nu_s d\delta_{s_0}) \to \int_S(\int_T M_t d\nu_s)d\delta_{s_0}$$

is the inverse of the isomorphism

$$\int_S (\int_T M_t d\nu_s) d\delta_{s_0} \xrightarrow{\epsilon_{S,s_0}} \int_T M_t d\nu_{s_0} = \int_T M_t d(\int_S \nu_s d\delta_{s_0}).$$

(B) Let $\{M_t\}_{t\in T}$ be a collection of objects of $\mathcal{M}$ indexed by a set T, let $f : S \to T$ be a monomorphism of sets, and let μ be an ultrafilter on a set S, so that the pushforward ultrafilter $f_*(\mu)$ is given by $\int_S \delta_{f(s)} d\mu$ (see Example 1.1.7). Then the Fubini transformation

$$\Delta_{\mu,\delta_{f(\bullet)}} : \int_T M_t d(f_*\mu) \to \int_S (\int_T M_t d\delta_{f(s)}) d\mu$$

is an isomorphism.

(C) Let $\{M_t\}_{t\in T}$ be a collection of objects of $\mathcal{M}$ indexed by a set T, let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S, let $\{\mu_r\}_{r\in R}$ be a collection of ultrafilters on S indexed by a set R, and let λ be an ultrafilter on R. Let ρ denote the ultrafilter on T given by $\rho = \int_R (\int_S \nu_s d\mu_r) d\lambda = \int_S \nu_s d(\int_R \mu_r d\lambda)$ Then the diagram of Fubini transformations

$$\begin{tikzcd}
\int_T M_t d\rho \arrow[r, "\Delta_{\lambda, \int_S \nu_s d\mu_\bullet}"] \arrow[d, "\Delta_{\int_R \mu_r d\lambda, \nu_\bullet}"] & \int_R (\int_T M_t d(\int_S \nu_s d\mu_r)) d\lambda \arrow[d, "\int_R \Delta_{\mu_r, \nu_\bullet} d\lambda"] \\
\int_S (\int_T M_t d\nu_s) d(\int_R \mu_r d\lambda) \arrow[r, "\Delta_{\lambda, \mu_\bullet}"] & \int_R (\int_S (\int_T M_t d\nu_s) d\mu_r) d\lambda
\end{tikzcd}$$

commutes in the category $\mathcal{M}$.

An *ultracategory* is a category $\mathcal{M}$ together with an ultrastructure on $\mathcal{M}$.

Warning 1.3.2. To avoid an unmanagable profusion of notation, we adopt the convention of using the same symbols $\int_S (\bullet) d\mu$, ϵ_{S,s_0}, and $\Delta_{\mu,\nu_\bullet}$ for the data appearing in Definition 1.3.1, for *all* ultracategories that we consider. This convention creates some danger of confusion: for example, if $\mathcal{M}$ and $\mathcal{N}$ are two ultracategories, then the symbol ϵ_{S,s_0} is used to denote both an isomorphism in the functor category $\mathrm{Fun}(\mathcal{M}^S, \mathcal{M})$ (which is supplied by the ultrastructure on $\mathcal{M}$) and an isomorphism in the functor category $\mathrm{Fun}(\mathcal{N}^S, \mathcal{N})$ (which is supplied by the ultrastructure on $\mathcal{N}$).

Notation 1.3.3. Let $\mathcal{M}$ be an ultracategory. Suppose we are given a collection of objects $\{M_t\}_{t\in T}$ of $\mathcal{M}$, a map of sets $f : S \to T$, and an ultrafilter μ on S. We let $\Delta_{\mu,f} : \int_T M_t d(f_*\mu) \to \int_S M_{f(s)} d\mu$ denote the composite map

$$\int_T M_t d(f_*\mu) = \int_T M_t d(\int_S \delta_{f(s)} d\mu) \xrightarrow{\Delta_{\mu,\delta_{f(\bullet)}}} \int_S (\int_T M_t d\delta_{f(s)}) d\mu \xrightarrow{\int_S \epsilon_{S,s_0} d\mu} \int_S M_{f(s)} d\mu.$$

We will refer to $\Delta_{\mu,f}$ as the *ultraproduct diagonal map*. Note that axiom (B) of Definition 1.3.1 is equivalent to the requirement that if f is an injective map of sets, then $\Delta_{\mu,f}$ is an isomorphism in $\mathcal{M}$.

Example 1.3.4 (Ultrapowers)**.** Let $\mathcal{M}$ be an ultracategory and let μ be an ultrafilter on a set S. For each object $M \in \mathcal{M}$, we let $M^\mu = \int_S M d\mu$ denote the object of $\mathcal{M}$ obtained by applying the ultraproduct functor $\int_S (\bullet) d\mu$ to the constant map $S \to \mathcal{M}$ taking the value M. We will refer to M^μ as the *ultrapower of M by μ*. Applying the construction of Notation 1.3.3 in the case where $T = \{t\}$ is a singleton and $f : S \to T$ is the constant map taking the value t, we obtain a map

$$M \simeq \int_T M d(f_*\mu) \to \int_S M d\mu = M^\mu.$$

We will denote this map by $\Delta_\mu : M \to M^\mu$ and refer to it as the *ultrapower diagonal*. Beware that Δ_μ is generally not an isomorphism.

We will frequently make use of the following elementary observation concerning Notation 1.3.3:

Proposition 1.3.5. *Let $\mathcal{M}$ be an ultracategory. Suppose we are given a family of objects $\{M_t\}_{t\in T}$ of $\mathcal{M}$ indexed by a set T, a pair of composable maps $R \xrightarrow{g} S \xrightarrow{f} T$, and an ultrafilter λ on R. Then the composite map*

$$\int_T M_t d(f\circ g)_*\lambda \xrightarrow{\Delta_{g_*\lambda,f}} \int_S M_{f(s)} d(g_*\lambda) \xrightarrow{\Delta_{\lambda,g}} \int_R M_{(f\circ g)(r)} d\lambda$$

coincides with the ultraproduct diagonal $\Delta_{\lambda,f\circ g}$.

Proof. Note that the morphisms $\Delta_{\lambda,g}\circ\Delta_{g_*(\lambda),f}$ and $\Delta_{\lambda,f\circ g}$ are given by clockwise and counterclockwise composition around the diagram

$$\begin{tikzcd}
\int_T M_t d(f\circ g)_*\lambda \arrow[r, "\Delta_{g_*(\lambda),\delta_{f(\bullet)}}"] \arrow[d, "\Delta_{\lambda,\delta_{(f\circ g)(\bullet)}}"] & \int_S(\int_T M_t d\delta_{f(s)})d(g_*\lambda) \arrow[r, "\int_S \epsilon_{T,f(s)}"] \arrow[d, "\Delta_{\lambda,\delta_{g(\bullet)}}"] & \int_S M_{f(s)} d(g_*\lambda) \arrow[d, "\Delta_{\lambda,\delta_{g(\bullet)}}"] \\
\int_R(\int_T M_t d\delta_{(f\circ g)(\bullet)})d\lambda \arrow[r, "\int_R \Delta_{\delta_{g(r)},\delta_{f(\bullet)}} d\lambda"] \arrow[dr, equal] & \int_R(\int_S(\int_T M_t d\delta_{f(s)})d\delta_{g(r)}d\lambda \arrow[r] \arrow[d, "\int_R \epsilon_{S,g(r)} d\lambda"] & \int_R(\int_S M_{f(s)} d\delta_{g(r)})d\lambda \arrow[d, "\int_R \epsilon_{S,g(r)} d\lambda"] \\
 & \int_R(\int_T M_t d\delta_{(f\circ g)(r)})d\lambda \arrow[r, "\int_R \epsilon_{T,\delta_{(f\circ g)(r)}}"] & \int_R M_{(f\circ g)(r)} d\lambda.
\end{tikzcd}$$

It will therefore suffice to show that this diagram commutes. For the squares on the right, this follows by functoriality; the upper left square commutes by axiom (C) of Definition 1.3.1, and the triangle in the lower left commutes by axiom (A). □

Proposition 1.3.5 has a counterpart for identity maps:

Corollary 1.3.6. *Let $\mathcal{M}$ be an ultracategory, let $\{M_s\}_{s\in S}$ be a collection of objects of $\mathcal{M}$ indexed by a set S, and let μ be an ultrafilter on S, which we identify with the pushforward of itself along the identity map $\mathrm{id}_S : S \to S$. Then the ultraproduct diagonal map*

$$\Delta_{\mu,\mathrm{id}_S} : \int_S M_s d\mu \to \int_S M_s d\mu$$

of Notation 1.3.3 is the identity map.

Proof. It follows from axiom (B) of Definition 1.3.1 that $\Delta_{\mu,\mathrm{id}_S}$ is an isomorphism. Consequently, to show that $\Delta_{\mu,\mathrm{id}_S}$ is the identity map, it will suffice to show that $\Delta_{\mu,\mathrm{id}_S}\circ\Delta_{\mu,\mathrm{id}_S} = \Delta_{\mu,\mathrm{id}_S}$, which is a special case of Proposition 1.3.5. □

We close this section by showing that ultraproducts satisfy the axiomatics of Definition 1.3.1:

Proposition 1.3.7. *Let $\mathcal{M}^+$ be a category and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory which has ultraproducts in $\mathcal{M}$. Then the functors $\int_S(\bullet)d\mu : \mathcal{M}^S \to \mathcal{M}$ of Construction 1.2.2 (together with the isomorphisms ϵ_{S,s_0} of Example 1.2.7 and the categorical Fubini transformations $\Delta_{\mu,\nu_\bullet}$ of Proposition 1.2.8) determine an ultrastructure on $\mathcal{M}$.*

Example 1.3.8 (The Categorical Ultrastructure)**.** Let $\mathcal{M}$ be a category which admits small products and small filtered colimits. Applying Proposition 1.3.7 in the case $\mathcal{M}^+ = \mathcal{M}$, we obtain an ultrastructure on the category $\mathcal{M}$, which we will refer to as the *categorical ultrastructure on* $\mathcal{M}$. We will show later that the categorical ultrastructure is "initial" among all possible ultrastructures on $\mathcal{M}$ (see Example 8.3.4).

Proof of Proposition 1.3.7. We must argue that $\mathcal{M}$ satisfies axioms (A), (B), and (C) of Definition 1.3.1. We consider each in turn.

(A) Let $\{M_t\}_{t\in T}$ be a collection of objects of $\mathcal{M}$, let $\nu_\bullet = \{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S, and let μ be an ultrafilter on S. Recall that the categorical Fubini transformation

$$\Delta_{\mu,\nu_\bullet} : \int_T M_t d(\int_S \nu_s d\mu) \to \int_S (\int_T M_t d\nu_s) d\mu$$

is characterized by the following requirement: if $S_0 \subseteq S$ and $T_0 \subseteq T$ are subsets with the property that $\mu(S_0) = 1$ and $\nu_s(T_0) = 1$ for each $s \in S_0$, then the diagram

$$\begin{tikzcd}
\prod_{t\in T_0} M_t \arrow[r, "\{q_{\nu_s}^{T_0}\}_{s\in S_0}"] \arrow[d, "q^{T_0}_{\int_S \nu_s d\mu}"] & \prod_{s\in S_0} \int_T M_t \nu_s \arrow[d, "q_\mu^{S_0}"] \\
\int_T M_t d(\int_S \nu_s d\mu) \arrow[r, "\Delta_{\mu,\nu_\bullet}"] & \int_S (\int_T M_t d\nu_s) d\mu
\end{tikzcd}$$

commutes. In the special case where $\mu = \delta_{s_0}$ is the principal ultrafilter associated to some element $s_0 \in S$, we can take $S_0 = \{s_0\}$ to obtain a commutative diagram

$$\begin{tikzcd}
\prod_{t\in T_0} M_t \arrow[r] \arrow[d] & \int_T M_t d\nu_{s_0} \arrow[d, "\epsilon^{-1}_{S,s_0}"] \\
\int_T M_t d(\int_S \nu_s d\mu) \arrow[r, "\Delta_{\delta_{s_0},\nu_\bullet}"] & \int_S (\int_T M_t d\nu_s) d\delta_{s_0}
\end{tikzcd}$$

for any subset $T_0 \subseteq T$ satisfying $\nu_{s_0}(T_0) = 1$. Passing to the direct limit over T_0, we obtain the desired identity.

(B) Let $\{M_t\}_{t\in T}$ be a collection of objects of $\mathcal{M}$ indexed by a set T, let $f : S \hookrightarrow T$ be a monomorphism of sets, and let μ be an ultrafilter on S. We wish to show that the ultraproduct diagonal

$$\Delta_{\mu,f} : \int_T M_t d(f_*\mu) \to \int_S M_{f(s)} d\mu$$

of Notation 1.3.3 is an isomorphism. This follows from the observation that the construction $S_0 \mapsto f(S_0)$ induces a cofinal map of partially ordered sets $\mathcal{U}_\mu^{\mathrm{op}} \to \mathcal{U}^{\mathrm{op}}_{f_*(\mu)}$.

(C) Let $\{M_t\}_{t\in T}$ be a collection of objects of $\mathcal{M}$ indexed by a set T, let $\{\nu_s\}_{s\in S}$ be a collection of ultrafilters on T indexed by a set S, let $\{\mu_r\}_{r\in R}$ be a collection of ultrafilters on S indexed by a set R, and let λ be an ultrafilter on R. Set $\rho = \int_R(\int_S \nu_s d\mu_r) d\lambda = \int_S \nu_s d(\int_R \mu_r d\lambda$, and let $T_0 \subseteq T$ be a subset with $\rho(T_0) = 1$. Set $S_0 = \{s \in S : \nu_s(T_0) = 1\}$ and $R_0 = \{r \in R : \mu_r(S_0) = 1\}$, so that $\lambda(R_0) = 1$. We then have a cubical diagram

$$\begin{tikzcd}
\prod_{t\in T_0} M_t \arrow[rr] \arrow[dr, "q_\rho^{T_0}"] \arrow[dd] & & \prod_{r\in R_0}\prod_{t\in T_0} M_t \arrow[dr] \arrow[dd] & \\
& \int_T M_t d\rho \arrow[rr, "\Delta_{\lambda,\int_S \nu_s d\mu_\bullet}"] \arrow[dd, "\Delta_{\int_R \mu_r d\lambda,\nu_\bullet}"] & & \int_R(\int_T M_t d(\int_S \nu_s d\mu_r))d\lambda \arrow[dd, "\int_R \Delta_{\mu_r,\nu_\bullet} d\lambda"] \\
\prod_{s\in S_0}\prod_{t\in T_0} M_t \arrow[rr] \arrow[dr] & & \prod_{r\in R_0}\prod_{s\in S_0}\prod_{t\in T_0} M_t \arrow[dr] & \\
& \int_S(\int_T M_t d\nu_s)d(\int_R \mu_r d\lambda) \arrow[rr, "\Delta_{\lambda,\mu_\bullet}"] & & \int_R(\int_S(\int_T M_t d\nu_s)d\mu_r)d\lambda
\end{tikzcd}$$

It then follows by a diagram chase that we have an identity

$$\Delta_{\lambda,\mu_\bullet} \circ \Delta_{\int_R \mu_r d\lambda,\nu_\bullet} \circ q_\rho^{T_0} = (\int_R \Delta_{\mu_r,\nu_\bullet} d\lambda) \circ \Delta_{\lambda,\int_S \nu_s d\mu_\bullet} \circ q_\rho^{T_0}$$

in the set $\mathrm{Hom}_{\mathcal{M}^+}(\prod_{t\in T_0} M_t, \int_R(\int_S(\int_T M_t d\nu_s)d\mu_r)d\lambda)$. Since the maps $\{q_\rho^{T_0} : \prod_{t\in T_0} M_t \to \int_T M_t d\rho\}$ exhibit the categorical ultraproduct $\int_T M_t d\rho$ as a direct limit of the diagram $\{\prod_{t\in T_0} M_t\}_{\rho(T_0)=1}$, it

follows that we must also have the identity

$$\Delta_{\lambda,\mu_\bullet} \circ \Delta_{\int_R \mu_r d\lambda, \nu_\bullet} = (\int_R \Delta_{\mu_r,\nu_\bullet} d\lambda) \circ \Delta_{\lambda, \int_S \nu_s d\mu_\bullet}$$

in $\mathrm{Hom}_{\mathcal{M}}(\int_T M_t d\rho, \int_R (\int_S (\int_T M_t d\nu_s) d\mu_r) d\lambda)$.

□

1.4. **Ultrafunctors.** We now introduce some terminology for articulating the relationship between different ultracategories.

Definition 1.4.1 (Ultrafunctors)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be categories with ultrastructure and let $F : \mathcal{M} \to \mathcal{N}$ be a functor. A *left ultrastructure on* F consists of the following data:

(∗) For every collection of objects $\{M_s\}_{s\in S}$ of the category $\mathcal{M}$ and every ultrafilter μ on S, a morphism $\sigma_\mu : F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu$ in the category $\mathcal{N}$.

These morphisms are required to satisfy the following conditions:

(0) For every collection of morphisms $\{f_s : M_s \to M'_s\}$ in the category $\mathcal{M}$ and every ultrafilter μ on S, the diagram

$$\begin{tikzcd} F(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] \arrow[d, "F(\int_S f_s d\mu)"] & \int_S F(M_s) d\mu \arrow[d, "\int_S F(f_s) d\mu"] \\ F(\int_S M'_s d\mu) \arrow[r, "\sigma_\mu"] & \int_S F(M'_s) d\mu \end{tikzcd}$$

commutes. In other words, we can regard σ_μ as a natural transformation

$$\sigma_\mu : F \circ \int_S (\bullet) d\mu \to \int_S (\bullet) d\mu \circ F^S$$

of functors from $\mathcal{M}^S$ to $\mathcal{N}$.

(1) For every collection $\{M_s\}_{s\in S}$ of objects of $\mathcal{M}$ indexed by a set S and every element $s_0 \in S$, the diagram

$$\begin{tikzcd} F(\int_S M_s d\delta_{s_0}) \arrow[rr, "\sigma_{\delta_{s_0}}"] \arrow[dr, "F(\epsilon_{S,s_0})"'] & & \int_S F(M_s) d\delta_{s_0} \arrow[dl, "\epsilon_{S,s_0}"] \\ & F(M_{s_0}) & \end{tikzcd}$$

commutes (in the category $\mathcal{N}$).

(2) For every collection $\{M_t\}_{t\in T}$ of objects of $\mathcal{M}$ indexed by a set T, every collection $\nu_\bullet = \{\nu_s\}_{s\in S}$ of ultrafilters on T indexed by a set S, and every ultrafilter μ on S, the diagram

$$\begin{tikzcd} F(\int_T M_t d(\int_S \nu_s d\mu)) \arrow[rr, "\sigma_{\int_S \nu_s d\mu}"] \arrow[d, "F(\Delta_{\mu,\nu_\bullet})"] & & \int_T F(M_t) d(\int_S \nu_s d\mu) \arrow[d, "\Delta_{\mu,\nu_\bullet}"] \\ F(\int_S (\int_T M_t d\nu_s) d\mu) \arrow[r, "\sigma_\mu"] & \int_S F(\int_T M_t d\nu_s) d\mu \arrow[r, "\int_S \sigma_{\nu_s} d\mu"] & \int_S (\int_T F(M_t) d\nu_s) d\mu \end{tikzcd}$$

commutes (in the category $\mathcal{N}$).

An *ultrastructure* on F is a left ultrastructure $\{\sigma_\mu\}$ for which each of the maps σ_μ is an isomorphism. A *left ultrafunctor* from $\mathcal{M}$ to $\mathcal{N}$ is a pair $(F, \{\sigma_\mu\})$, where F is a functor from M to N and $\{\sigma_\mu\}$ is a left ultrastructure on F. An *ultrafunctor* from $\mathcal{M}$ to $\mathcal{N}$ is a right ultrafunctor $(F, \{\sigma_\mu\})$ for which each σ_μ is an isomorphism.

Definition 1.4.2. Let $\mathcal{M}$ and $\mathcal{N}$ be categories with ultrastructure, let $F, F' : \mathcal{M} \to \mathcal{N}$ be functors from $\mathcal{M}$ to $\mathcal{N}$, and suppose that F and F' are equipped with left ultrastructures $\{\sigma_\mu\}$ and $\{\sigma'_\mu\}$, respectively. We

will say that a natural transformation $u : F \to F'$ is a *natural transformation of left ultrafunctors* if, for every collection of objects $\{M_s\}_{s \in S}$ of $\mathcal{M}$ and every ultrafilter μ on S, the diagram

$$\begin{tikzcd} F(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] \arrow[d, "u(\int_S M_s d\mu)"] & \int_S F(M_s) d\mu \arrow[d, "\int_S u(M_s) d\mu"] \\ F'(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] & \int_S F'(M_s) d\mu \end{tikzcd}$$

commutes (in the category $\mathcal{N}$).

We let $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$ denote the category whose objects are left ultrafunctors $(F, \{\sigma_\mu\})$ from $\mathcal{M}$ to $\mathcal{N}$ and whose morphisms are natural transformations of left ultrafunctors, and we let $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{N})$ denote the full subcategory of $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$ spanned by the ultrafunctors from $\mathcal{M}$ to $\mathcal{N}$.

Remark 1.4.3 (Colimits of Left Ultrafunctors)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. Suppose that we are given a diagram $\{F_\alpha\}$ in the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$ with the property that, for every object $M \in \mathcal{M}$, the diagram $\{F_\alpha(M)\}$ admits a colimit in $\mathcal{N}$. Then:

- The construction $(M \in \mathcal{M}) \mapsto \varinjlim_\alpha F_\alpha(M)$ determines a functor $F : \mathcal{M} \to \mathcal{N}$.
- There is a unique left ultrastructure on F for which each of the natural maps $\rho_\alpha : F_\alpha \to F$ is a natural transformation of left ultrafunctors.
- The maps ρ_α exhibit F as a colimit of the diagram $\{F_\alpha\}$ in the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$.

In particular, if the ultracategory $\mathcal{N}$ admits small colimits, then the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$ also admits small colimits, which are preserved by the forgetful functor $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}(\mathcal{M}, \mathcal{N})$.

Warning 1.4.4. The analogue of Remark 1.4.3 for ultrafunctors (as opposed to left ultrafunctors) is false. If $\{F_\alpha\}$ is a diagram in the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{N})$ of ultrafunctors which admits a pointwise colimit $F : \mathcal{M} \to \mathcal{N}$, then F inherits a left ultrastructure (by virtue of Remark 1.4.3) given by maps

$$\sigma_\mu : F(\int_S M_s d\mu) \simeq \varinjlim_\alpha F_\alpha(\int_S M_s d\mu) \simeq \varinjlim_\alpha \int_S F_\alpha(M_s) d\mu \to \int_S (\varinjlim_\alpha F_\alpha(M_s) d\mu) \simeq \int_S F(M_s) d\mu.$$

But these maps are generally not invertible, because the ultraproduct functors on $\mathcal{N}$ need not preserve colimits.

Construction 1.4.5 (Composition of Left Ultrafunctors)**.** Let $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}''$ be ultracategories. Let $(F, \{\sigma_\mu\})$ be a left ultrafunctor from $\mathcal{M}$ to $\mathcal{M}'$, and let $(F', \{\sigma'_\mu\})$ be a left ultrafunctor from $\mathcal{M}'$ to $\mathcal{M}''$. Then the composite functor $F' \circ F$ admits a left ultrastructure, which associates to each collection of objects $\{M_s\}_{s \in S}$ of $\mathcal{M}$ and each ultrafilter μ on S the composite map

$$(F' \circ F)(\int_S M_s d\mu) \xrightarrow{F'(\sigma_\mu)} F'(\int_S F(M_s) d\mu) \xrightarrow{\sigma'_\mu} \int_S (F' \circ F)(M_s) d\mu.$$

Note that if $\{\sigma_\mu\}$ and $\{\sigma'_\mu\}$ are ultrastructures, then this construction determines an ultrastructure on the functor $F' \circ F$. We therefore obtain composition laws

$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}', \mathcal{M}'') \times \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{M}') \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{M}'')$$

$$\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}', \mathcal{M}'') \times \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}') \to \mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{M}'').$$

Remark 1.4.6. We can use Construction 1.4.5 to construct (strict) 2-categories $\mathbf{Ult} \subset \mathbf{Ult}^{\mathrm{L}}$ as follows:

- The objects of $\mathbf{Ult}$ and $\mathbf{Ult}^{\mathrm{L}}$ are ultracategories.
- For every pair of ultracategories $\mathcal{M}$ and $\mathcal{N}$, the category of morphisms from $\mathcal{M}$ to $\mathcal{N}$ in $\mathbf{Ult}$ is given by $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathcal{N})$, and the category of morphisms from $\mathcal{M}$ to $\mathcal{N}$ in $\mathbf{Ult}^{\mathrm{L}}$ is given by $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$.
- The composition laws on $\mathbf{Ult}$ and $\mathbf{Ult}^{\mathrm{L}}$ are given by Construction 1.4.5.

More informally: $\mathbf{Ult}^{\mathrm{L}}$ is the 2-category in which objects are ultracategories, 1-morphisms are left ultrafunctors, and 2-morphisms are natural transformations of left ultrafunctors. The 2-category $\mathbf{Ult}$ is the (non-full) subcategory of $\mathbf{Ult}^{\mathrm{L}}$ whose morphisms are ultrafunctors.

Remark 1.4.7. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories and let $F : \mathcal{M} \to \mathcal{N}$ be an ultrafunctor. If F is an equivalence of categories having homotopy inverse $G : \mathcal{N} \to \mathcal{M}$, then G inherits the structure of an ultrafunctor and can be regarded as a homotopy inverse of F in the 2-category $\mathbf{Ult}$. Beware that the analogous statement is not necessarily true if we assume only that F is a left ultrafunctor (though in this case, G still inherits the structure of a *right* ultrafunctor; see Remark 8.1.4).

We give some examples of (left) ultrafunctors.

Proposition 1.4.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be categories which admit small products and small filtered colimits, and equip $\mathcal{M}$ and $\mathcal{N}$ with the category ultrastructures of Example 1.3.8. Let $F : \mathcal{M} \to \mathcal{N}$ is a functor which preserves small filtered colimits, then F can be regarded as a left ultrafunctor from $\mathcal{M}$ to $\mathcal{N}$. If F preserves small filtered colimits and small products, then it can be regarded as an ultrafunctor from $\mathcal{M}$ to $\mathcal{N}$.*

Proposition 1.4.8 is an immediate consequence of the following more general (and more precise) assertion:

Proposition 1.4.9. *Let $\mathcal{M}^+$ and $\mathcal{N}^+$ be categories and let $\mathcal{M} \subseteq \mathcal{M}^+$ and $\mathcal{N} \subseteq \mathcal{N}^+$ be full subcategories which admit ultraproducts in $\mathcal{M}^+$ and $\mathcal{N}^+$, respectively. Let $F^+ : \mathcal{M}^+ \to \mathcal{N}^+$ be a functor which carries $\mathcal{M}$ into $\mathcal{N}$ and satisfies the following additional condition:*

$(*)$ *For every collection of objects $\{M_s\}_{s\in S}$ of $\mathcal{M}$ and every ultrafilter μ on S, the maps*

$$F^+(q_\mu^{S_0}) : F^+(\prod_{s\in I} M_s) \to F^+(\int_S M_s d\mu)$$

exhibit $F^+(\int_S M_s d\mu)$ as a colimit of the diagram $\{F^+(\prod_{s\in I} M_s)\}_{\mu(I)=1}$ in the category $\mathcal{N}^+$.

Let $F = F^+|_{\mathcal{M}}$, which we regard as a functor from $\mathcal{M}$ to $\mathcal{N}$, and regard $\mathcal{M}$ and $\mathcal{N}$ as equipped with the ultrastructures give by Proposition 1.3.7. Then:

(a) *For every collection of objects $\{M_s\}_{s\in S}$ of $\mathcal{M}$ and every ultrafilter μ on S, there is a unique map $\sigma_\mu : F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu$ having the property that, for each subset $S_0 \subseteq S$ with $\mu(S_0) = 1$, the diagram*

$$\begin{tikzcd}
F^+(\prod_{s\in S_0} M_s) \arrow[r] \arrow[d, "F^+(q_\mu^{S_0})"] & \prod_{s\in S_0} F(M_s) \arrow[d, "q_\mu^{S_0}"] \\
F(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] & \int_S F(M_s) d\mu
\end{tikzcd}$$

commutes.

(b) *The morphisms $\{\sigma_\mu\}$ of (a) determine a left ultrastructure on the functor F.*

(c) *If F^+ preserves small products, then $\{\sigma_\mu\}$ is an ultrastructure on F (that is, each of the maps $\sigma_\mu : F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu$ is an isomorphism).*

Proof of Proposition 1.4.9. Assertion (a) follows immediately from $(*)$, and (c) is clear. To prove (b), we show that that the maps $\{\sigma_\mu\}$ satisfy condition (2) of Definition 1.4.1 (conditions (0) and (1) are immediate from the definitions). Fix a collection of objects $\{M_t\}_{t\in T}$ of $\mathcal{M}$ indexed by a set T, a collection of ultrafilters $\{\nu_s\}_{s\in S}$ on T indexed by a set S, and an ultrafilter μ on the set S. Set $\lambda = \int_S \nu_s d\mu$. We wish to show that

the diagram τ :

$$\begin{tikzcd}
\int_T F(M_t)d\lambda \arrow[r, "\Delta_{\mu,\nu_\bullet}"] & \int_S(\int_T F(M_t)d\nu_s)d\mu \\
 & \int_S F(\int_T M_t d\nu_s)d\mu \arrow[u, "\int_S \sigma_{\nu_s} d\mu"'] \\
F(\int_T M_t d\lambda) \arrow[uu, "\sigma_\lambda"'] \arrow[r, "F(\Delta_{\mu,\nu_\bullet})"] & F(\int_S(\int_T M_t d\nu_s)d\mu) \arrow[u, "\sigma_\mu"']
\end{tikzcd}$$

commutes (in the category $\mathcal{N}$). Let $u, v : F(\int_T M_t d\lambda) \rightrightarrows \int_S(\int_T F(M_t)d\nu_s)d\mu$ be the maps given by clockwise and counterclockwise composition around the diagram σ. To show that $u = v$, it will suffice (by virtue of $(*)$) to show that u and v agree after precomposition with $F(q_\lambda^{T_0}) : F(\prod_{t\in T_0} M_t) \to F(\int_T M_t d\lambda)$, for every subset $T_0 \subseteq T$ with $\lambda(T_0) = 1$. Set $S_0 = \{s \in S : \nu_s(T_0) = 1\}$, so that $\mu(S_0) = 1$. We are then reduced to verifying the commutativity of the diagram

$$\begin{tikzcd}
\int_T F(M_t) \arrow[rrr, "\Delta_{\mu,\nu_\bullet}"] & & & \int_S(\int_T F(M_t)d\nu_s)d\mu \\
 & & \prod_{s\in S_0}(\int_T F(M_t)d\nu_s) \arrow[ur, "q_\mu^{S_0}"] & \\
 & \prod_{t\in T_0} F(M_t) \arrow[ul, "q_\lambda^{T_0}"'] \arrow[ur, "\{q_{\nu_s}^{T_0}\}_{s\in S_0}"] & \prod_{s\in S_0} F(\int_T M_t d\nu_s) \arrow[u, "\prod_{s\in S_0} \sigma_{\nu_s}"'] \arrow[r, "q_\mu^{S_0}"] & \int_S F(\int_T M_t d\nu_s)d\mu \arrow[uu, "\int_S \sigma_{\nu_s} d\mu"'] \\
 & F^+(\prod_{t\in T_0} M_t) \arrow[u] \arrow[r, "F(\{q_{\nu_s}^{T_0}\}_{s\in S})"] \arrow[dl, "F(q_\lambda^{T_0})"'] & F^+(\prod_{s\in S_0}(\int_T M_t d\nu_s)) \arrow[u] \arrow[dr, "F(q_\mu^{S_0})"] & \\
F(\int_T M_t d\lambda) \arrow[uuuu, "\sigma_\lambda"'] \arrow[rrr, "F(\Delta_{\mu,\nu_\bullet})"] & & & F(\int_S(\int_T M_t d\nu_s)d\mu) \arrow[uu, "\sigma_\mu"']
\end{tikzcd}$$

in the category $\mathcal{N}^+$. Note that the inner region of the diagram commutes by the construction of the maps σ_{ν_s}, the upper region commutes by the construction of the Fubini transformation for the ultrastructure on $\mathcal{N}$, the lower region commutes by the construction of the Fubini transformation for the ultrastructure on $\mathcal{M}$, the region on the left commutes by the construction of σ_λ, the region on the upper right commutes by functoriality, and the region on the lower right commutes by the construction of σ_μ. □

Remark 1.4.10. Let $\mathcal{M}^+$ and $\mathcal{N}^+$ be categories which admit small products, let $\mathcal{M} \subseteq \mathcal{M}^+$ and $\mathcal{N} \subseteq \mathcal{N}^+$ be full subcategories which admit ultraproducts in $\mathcal{M}^+$ and $\mathcal{N}^+$, and let $\mathrm{Fun}'(\mathcal{M}^+, \mathcal{N}^+)$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{M}, \mathcal{N})$ spanned by those functors which carry $\mathcal{M}$ into $\mathcal{N}$ and satisfy condition $(*)$ of Proposition 1.4.9. Then the construction $F^+ \mapsto (F, \{\sigma_\mu\})$ of Proposition 1.4.9 determines a functor $\mathrm{Fun}'(\mathcal{M}^+, \mathcal{N}^+) \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$.

2. Ultracategories and Logic

Let $\mathcal{C}$ be a small pretopos. Recall that a *model* of $\mathcal{C}$ is a functor $M : \mathcal{C} \to \mathrm{Set}$ which preserves finite limits, finite coproducts, and effective epimorphisms (Definition A.4.5). We let $\mathrm{Mod}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ spanned by the models of $\mathcal{C}$. In §2.1, we recall the *Los ultraproduct theorem*, which (in this context) asserts that the category of models of $\mathcal{C}$ is closed under the formation of ultraproducts in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ (Theorem 2.1.1). In particular, the category $\mathrm{Mod}(\mathcal{C})$ inherits an ultrastructure (Remark 2.1.2), so we can consider (left) ultrafunctors $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$. In §2.2, we give a precise statement of the main result of this paper (Theorem 2.2.2), which establishes an equivalence between the category of left ultrafunctors

$\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ with the topos of sheaves $\mathrm{Shv}(\mathcal{C})$. In §2.3, we apply this result to deduce Makkai's strong conceptual completeness theorem, which supplies an equivalence of the category of ultrafunctors $\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ with the pretopos $\mathcal{C}$ itself (Theorem 2.3.1). In §2.4, we apply the strong conceptual completeness theorem to prove another result of Makkai, which characterizes the essential image of the Barr embedding of a small exact category $\mathcal{E}$ (Theorem 2.4.2).

2.1. **The Łos Ultraproduct Theorem.** We now recall the classical Łos ultraproduct theorem in a form which is convenient for our applications. Note that, for any category $\mathcal{C}$, the functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ admits small limits and colimits. In particular, for any collection of functors $\{M_s : \mathcal{C} \to \mathrm{Set}\}_{s \in S}$ and any ultrafilter μ on S, we can form the categorical ultraproduct $\int_S M_s d\mu$ (in the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$), which is described concretely by the formula

$$(\int_S M_s d\mu)(C) = \int_S (M_s(C)) d\mu = \varinjlim_{\mu(S_0)=1} \prod_{s \in S_0} M_s(C).$$

Theorem 2.1.1 (Łos Ultraproduct Theorem)**.** *Let $\mathcal{C}$ be a pretopos and let $\{M_s\}_{s \in S}$ be a collection of models of $\mathcal{C}$. For every ultrafilter μ on S, the ultraproduct $\int_S M_s d\mu$ (formed in the category* $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$*) is also a model of $\mathcal{C}$.*

Remark 2.1.2. Let $\mathcal{C}$ be a pretopos. Theorem 2.1.1 asserts that the category of models $\mathrm{Mod}(\mathcal{C})$ has ultraproducts in the larger category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$, in the sense of Construction 1.2.2. Applying Proposition 1.3.7, we obtain an ultrastructure on the category $\mathrm{Mod}(\mathcal{C})$.

Theorem 2.1.1 is an immediate consequence of the following feature of the category of sets:

Proposition 2.1.3. *Let S be a set and let μ be an ultrafilter on S. Then the ultraproduct functor $\int_S (\bullet) d\mu :$* $\mathrm{Set}^S \to \mathrm{Set}$ *of Construction 1.2.2 is a pretopos functor: that is, it preserves finite limits, finite coproducts, and effective epimorphisms.*

Proof. By construction, the ultraproduct functor $\int_S (\bullet) d\mu$ can be written as a filtered direct limit of functors of the form $\{M_s\}_{s \in S} \mapsto \prod_{s \in S_0} M_s$. Since each of these functors preserves finite limits, initial objects, and effective epimorphisms, it follows that $\int_S (\bullet) d\mu$ preserves finite limits, initial objects, and effective epimorphisms. To complete the proof, it will suffice to show that for every pair of objects $\{M_s\}_{s \in S}, \{N_s\}_{s \in S} \in \mathrm{Set}^S$, the canonical map

$$(\int_S M_s d\mu) \amalg (\int_S N_s d\mu) \to \int_S (M_s \amalg N_s) d\mu$$

is bijective. It follows from the left exactness of the ultraproduct functor that we can identify $\int_S M_s d\mu$ and $\int_S N_s d\mu$ with disjoint subsets of $\int_S (M_s \amalg N_s) d\mu$; we wish to show that every element $x \in \int_S (M_s \amalg N_s) d\mu$ belongs to one of these subsets. Without loss of generality, we may assume that x is represented by an element $\{x_s\}_{s \in S_0} \in \prod_{s \in S_0} (M_s \amalg N_s)$ for some subset $S_0 \subseteq S$ with $\mu(S_0) = 1$. Then we can write $S_0 = S_- \amalg S_+$, where $S_- = \{s \in S_0 : x_s \in M_s\}$ and $S_+ = \{s \in S_0 : x_s \in N_s\}$. We then have $\mu(S_-) + \mu(S_+) = \mu(S_0) = 1$. Without loss of generality, we may assume that $\mu(S_-) = 1$. In this case, x can also be represented by the tuple

$$\{x_s\}_{s \in S_-} \in \prod_{s \in S_-} M_s \subseteq \prod_{s \in S_-} (M_s \amalg N_s)$$

and therefore belongs to the image of the ultraproduct $\int_S M_s d\mu$. □

Proof of Theorem 2.1.1. Let $\{M_s\}_{s \in S}$ be a collection of models of $\mathcal{C}$ and let μ be an ultrafilter on the index set S. Then the ultraproduct $\int_S M_s d\mu$ (formed in the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$) can be identified with the composition

$$\mathcal{C} \xrightarrow{\{M_s\}_{s \in S}} \mathrm{Set}^S \xrightarrow{\int_S (\bullet) d\mu} \mathrm{Set}.$$

The first map is a pretoos functor by virtue of our assumption that each M_s is a model of $\mathcal{C}$, and the second map is a pretopos functor by virtue of Proposition 2.1.3. It follows that the composite map is also a pretopos functor. □

We close this section by noting another consequence of Proposition 2.1.3.

Corollary 2.1.4. *Let $\mathcal{M}$ be an ultracategory. Then the category of left ultrafunctors* $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ *admits finite limits, which are preserved by the forgetful functor* $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set}) \to \mathrm{Fun}(\mathcal{M}, \mathrm{Set})$*. Moreover, the full subcategory* $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M}, \mathrm{Set})$ *is closed under finite limits.*

Proof. Let $\{F_\alpha\}$ be a finite diagram in the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$, and define $F : \mathcal{M} \to \mathrm{Set}$ by the formula $F(M) = \varprojlim F_\alpha(M)$. For any collection of objects $\{M_s\}_{s \in S}$ of $\mathcal{M}$ and any ultrafilter μ on S, we have a diagram

$$\begin{tikzcd}
F(\int_S M_s d\mu) \arrow[r, dashed, "\sigma_\mu"] \arrow[d, "\sim"] & \int_S F(M_s) d\mu \arrow[d, "\sim"] \\
\varprojlim_\alpha F_\alpha(\int_S M_s d\mu) \arrow[r] & \varprojlim_\alpha \int_S F_\alpha(M_s) d\mu,
\end{tikzcd}$$

where the bottom horizontal map is supplied by the left ultrastructures on the functors F_α and the right vertical map is a bijection by virtue of Proposition 2.1.3. It follows that there is a unique map $\sigma_\mu : F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu$ which renders the diagram commutative. Moreover, if each F_α is an ultrafunctor, then the bottom horizontal map is bijective, so that σ_μ is an isomorphism. We leave it to the reader to verify that the collection of maps $\{\sigma_\mu\}$ is a left ultrastructure on F (hence an ultrastructure in the case where each F_α is an ultrafunctor), and that the projection maps $F \to F_\alpha$ exhibit F as a limit of the diagram $\{F_\alpha\}$ in the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$. □

2.2. **Statement of the Main Theorem.**

We are now ready to formulate the main result of this paper.

Construction 2.2.1 (The Evaluation Map)**.** Let $\mathcal{C}$ be a pretopos. For each object $C \in \mathcal{C}$, evaluation at C determines a functor $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \to \mathrm{Set}$ which preserves small limits and colimits. We let ev_C denote the restriction of this evaluation functor to the category $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$, given on objects by $\mathrm{ev}_C(M) = M(C)$. Invoking Proposition 1.4.9, we see that ev_C can be regarded as an ultrafunctor from $\mathrm{Mod}(\mathcal{C})$ to the category of sets, where $\mathrm{Mod}(\mathcal{C})$ is endowed with the ultrastructure of Remark 2.1.2. Moreover, the construction $C \mapsto \mathrm{ev}_C$ determines a functor

$$\mathrm{ev} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}),$$

which we will refer to as *the evaluation map* (see Remark 1.4.10).

Theorem 2.2.2. *Let $\mathcal{C}$ be a small pretopos and let*

$$\mathrm{ev} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}) \subseteq \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$$

denote the evaluation map of Construction 2.2.1. Then:

(1) *Let $T : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ be a left ultrafunctor, and define a functor $\mathscr{F}_T : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ by the formula $\mathscr{F}_T(C) = \mathrm{Hom}_{\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})}(\mathrm{ev}_C, T)$. Then $\mathscr{F}_T$ is a sheaf on $\mathcal{C}$ (with respect to the coherent topology of Definition B.5.3).*
(2) *The construction $T \mapsto \mathscr{F}_T$ induces an equivalence of categories $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}) \to \mathrm{Shv}(\mathcal{C})$, where $\mathrm{Shv}(\mathcal{C})$ is the topos of sheaves on $\mathcal{C}$ (with respect to the coherent topology).*

We will prove Theorem 2.2.2 in §7.

Corollary 2.2.3. *Let $\mathcal{C}$ be a small pretopos. Then the category of left ultrafunctors* $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ *is a (coherent) Grothendieck topos.*

Notation 2.2.4. If $\mathcal{X}$ and $\mathcal{Y}$ are Grothendieck topoi, we let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by those functors $f^* : \mathcal{X} \to \mathcal{Y}$ that preserve small colimits and finite limits (in other words, $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ denotes the category of *geometric morphisms* from $\mathcal{Y}$ to $\mathcal{X}$).

Example 2.2.5. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, and assume that the categories $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ and $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{N}, \mathrm{Set})$ are Grothendieck topoi. For every left ultrafunctor $F : \mathcal{M} \to \mathcal{N}$, precomposition with F induces a map

$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{N}, \mathrm{Set}) \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$$

which preserves small colimits (by Remark 1.4.3) and finite limits (by Corollary 2.1.4). We therefore obtain a map

$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathcal{N}) \to \mathrm{Fun}^*(\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{N},\mathrm{Set}),\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Set})).$$

Corollary 2.2.6. *Let $\mathcal{C}$ be a small pretopos and let $\mathcal{X}$ be a Grothendieck topos. Then composition with the evaluation map* $\mathrm{ev} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set}) \subseteq \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set})$ *induces a fully faithful embedding*

$$\mathrm{Fun}^*(\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set}),\mathcal{X}) \to \mathrm{Fun}(\mathcal{C},\mathcal{X}),$$

whose essential image is spanned by the pretopos functors from $\mathcal{C}$ to $\mathcal{X}$.

Proof. Combine Theorem 2.2.2 with Corollary C.3.6. □

Corollary 2.2.7. *Let $\mathcal{C}$ be a small pretopos, let $\mathcal{M}$ be an ultracategory, and assume that* $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Set})$ *is a Grothendieck topos. Then the comparison map*

$$\theta : \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Mod}(\mathcal{C})) \to \mathrm{Fun}^*(\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set}),\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Set}))$$

of Example 2.2.5 is an equivalence of categories.

Proof. Unwinding the definitions, we see that θ fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Mod}(\mathcal{C})) \arrow[r, "\theta"] \arrow[d] & \mathrm{Fun}^*(\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set}),\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Set})) \arrow[d, "\circ\mathrm{ev}"] \\
\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Fun}(\mathcal{C},\mathrm{Set})) \arrow[r] & \mathrm{Fun}(\mathcal{C},\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Set}))
\end{tikzcd}$$

where the bottom horizontal map is an equivalence of categories. It will therefore suffice to show that this diagram is a pullback square. Using Corollary 2.2.6, we are reduced to proving the following concrete statement:

($*$) Let $F : \mathcal{C} \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Set})$ be a functor. Then F is a pretopos functor if and only if, for each object $X \in \mathcal{M}$, the functor

$$F_X : \mathcal{C} \to \mathrm{Set} \qquad F_X(C) = F(C)(X)$$

is a pretopos functor (that is, a model of $\mathcal{C}$).

This is clear, since colimits and finite limits in the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Set})$ are computed pointwise (Remark 1.4.3 and Corollary 2.1.4). □

Example 2.2.8. In the situation of Corollary 2.2.7, suppose that $\mathcal{M} = \mathrm{Mod}(\mathcal{D})$ for some other small pretopos $\mathcal{D}$. Combining the identifications

$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set}) \simeq \mathrm{Shv}(\mathcal{C}) \qquad \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{D}),\mathrm{Set}) \simeq \mathrm{Shv}(\mathcal{D})$$

of Theorem 2.2.2 with Corollary 2.2.7, we obtain an equivalence of categories

$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Mod}(\mathcal{D})) \simeq \mathrm{Fun}^*(\mathrm{Shv}(\mathcal{D}),\mathrm{Shv}(\mathcal{C})).$$

Remark 2.2.9. Let $\mathbf{Ult}^{\mathrm{L}}$ denote the (strict) 2-category whose objects are ultracategory and whose morphisms are left ultrafunctors (Remark 1.4.6), and let $\mathbf{Ult}_0^{\mathrm{L}} \subseteq \mathbf{Ult}^{\mathrm{L}}$ denote the full subcategory spanned by those ultracategories of the form $\mathrm{Mod}(\mathcal{C})$, where $\mathcal{C}$ is a small pretopos. It follows from Corollary 2.2.7 that the construction $\mathcal{M} \mapsto \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M},\mathrm{Set})$ determines a fully faithful embedding from $\mathbf{Ult}_0^{\mathrm{L}}$ to the (strict) 2-category of topoi and geometric morphisms. By virtue of Theorem 2.2.2, the essential image of this embedding is the 2-category of coherent topoi (see Proposition C.6.4).

Theorem 2.2.2 immediately implies the following classical result of Deligne:

Theorem 2.2.10 (Deligne's Completeness Theorem)**.** *Let $\mathcal{X}$ be a coherent Grothendieck topos. Then $\mathcal{X}$ has enough points. In other words, if $f : X \to Y$ is a morphism in $\mathcal{X}$ with the property that $u^*(f)$ is bijective for every point $u^* \in \mathrm{Fun}^*(\mathcal{X},\mathrm{Set})$, then f is an isomorphism in $\mathcal{X}$.*

Proof. By virtue of Proposition C.6.4 and Theorem 2.2.2, we may assume without loss of generality that $\mathcal{X} = \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ for some small pretopos $\mathcal{C}$, so that f is a natural transformation between left ultrafunctors $X, Y : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$. If f is not an isomorphism, then there exists some model $M \in \mathrm{Mod}(\mathcal{C})$ for which the map $X(M) \to Y(M)$ is not bijective. Evaluation at M determines a functor

$$u^* : \mathcal{X} = \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}) \to \mathrm{Fun}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}) \to \mathrm{Set}$$

such that $u^*(f)$ is not bijective. Since small colimits and finite limits in $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ are computed pointwise (Remark 1.4.3 and Corollary 2.1.4), the functor u^* is a point of the topos $\mathcal{X}$. □

We note the following easy consequence of Deligne's theorem, which will be useful in the next section:

Corollary 2.2.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi and let $\lambda : \mathcal{D} \to \mathcal{C}$ be a functor. Assume that $\mathcal{C}$ is small and that for each model M of $\mathcal{C}$, the composite functor $M \circ \lambda \in \mathrm{Fun}(\mathcal{D}, \mathrm{Set})$ is a model of $\mathcal{D}$. Then λ is a pretopos functor.*

Proof. We must show that the functor λ preserves finite limits, finite coproducts, and effective epimorphisms. We give the proof for finite limits; the other properties follow by a similar argument. Suppose we are given a finite diagram $\{D_\alpha\}$ in the category $\mathcal{D}$ having limit $D = \varprojlim_\alpha D_\alpha$. We wish to show that the canonical map $u : \lambda(D) \to \varprojlim_\alpha \lambda(D_\alpha)$ is an isomorphism in the category $\mathcal{C}$. By virtue of Theorem 2.2.10 (and Corollary B.5.6), it will suffice to show that $M(u)$ is an isomorphism for each model M of $\mathcal{C}$. Since M preserves finite limits, we can identify $M(u)$ with the canonical map $(M \circ \lambda)(D) \to \varprojlim_\alpha (M \circ \lambda)(D_\alpha)$. This map is an isomorphism by virtue of our assumption that $M \circ \lambda$ is a model of $\mathcal{D}$. □

2.3. Application: Strong Conceptual Completeness.

We now turn to the original version of Makkai's strong conceptual completeness theorem.

Theorem 2.3.1 (Makkai)**.** *Let $\mathcal{C}$ be a small pretopos. Then the evaluation map $\mathrm{ev} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ of Construction 2.2.1 is an equivalence of categories.*

Remark 2.3.2. Theorem 2.3.1 is actually slightly stronger than the version proved by Makkai in [9], since our category of ultrafunctors $\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ is *a priori* larger than the one introduced by Makkai; see Warning 0.0.9.

Before giving the proof of Theorem 2.3.1, let us note some of its consequences. Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi and let $\lambda : \mathcal{D} \to \mathcal{D}$ be any functor. Then precomposition with λ induces a map $\lambda^* : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \to \mathrm{Fun}(\mathcal{D}, \mathrm{Set})$ which preserves small limits and colimits. Applying Remark 1.4.10, we see that λ^* induces an ultrafunctor from $\mathrm{Mod}(\mathcal{C})$ to $\mathrm{Fun}(\mathcal{D}, \mathrm{Set})$, which we will also denote by λ^*. Note that if λ is a pretopos functor, then we can regard λ^* as an ultrafunctor from $\mathrm{Mod}(\mathcal{C})$ to $\mathrm{Mod}(\mathcal{D})$.

Corollary 2.3.3 (Makkai Duality)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi and let $\mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{D}, \mathcal{C})$ denote the category of pretopos functors from $\mathcal{D}$ to $\mathcal{C}$. If $\mathcal{C}$ is small, then the construction $\lambda \mapsto \lambda^*$ induces an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{D}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Mod}(\mathcal{D})).$$

Proof. It follows from Theorem 2.3.1 that the construction $\lambda \mapsto \lambda^*$ induces an equivalence of categories

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C})), \mathrm{Set}) \simeq \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Fun}(\mathcal{D}, \mathrm{Set})).$$

Under this equivalence, the full subcategory $\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Mod}(\mathcal{D})) \subseteq \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Fun}(\mathcal{D}, \mathrm{Set}))$ can be corresponds to the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ spanned by those functors λ with the property that for each model M of $\mathcal{C}$, the composition $M \circ \lambda$ is a model of $\mathcal{D}$. By virtue of Corollary 2.2.11, this subcategory coincides with $\mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{D}, \mathcal{C})$. □

Remark 2.3.4. Let **Ult** denote the (strict) 2-category whose objects are ultracategories and whose morphisms are ultrafunctors (Remark 1.4.6) It follows from Corollary 2.3.3 that the construction $\mathcal{C} \mapsto \mathrm{Mod}(\mathcal{C})$

determines a fully faithful embedding of 2-categories $\{\text{Small pretopoi}\}^{\text{op}} \hookrightarrow \mathbf{Ult}$. This embedding fits into a commutative diagram

$$\begin{tikzcd} \{\text{Small pretopoi}\}^{\text{op}} \arrow[r, "\mathcal{C}\mapsto\mathrm{Mod}(\mathcal{C})"] \arrow[d, "\mathcal{C}\mapsto\mathrm{Shv}(\mathcal{C})"] & \mathbf{Ult} \arrow[d] \\ \{\text{Coherent topoi}\} \arrow[r] & \mathbf{Ult}^{\mathrm{L}}, \end{tikzcd}$$

where the bottom horizontal map is the fully faithful embedding of Remark 2.2.9 (with homotopy inverse given by $\mathcal{M} \mapsto \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$).

Corollary 2.3.5 (Makkai-Reyes Conceptual Completeness Theorem). *Let $\mathcal{C}$ and $\mathcal{D}$ be small pretopoi and let $\lambda : \mathcal{D} \to \mathcal{C}$ be a pretopos functor. If the induced map $\mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{D})$ is an equivalence of categories, then λ is an equivalence of categories.*

Proof. Combine Corollary 2.3.3 with Remark 1.4.7. □

We will deduce Theorem 2.3.1 from Theorem 2.2.2 together with the following observation:

Lemma 2.3.6. *Let $\mathcal{C}$ be a small pretopos and let $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ be a functor equipped with a left ultrastructure $\{\sigma_\mu\}$. Suppose that, for every collection of models $\{M_s\}_{s\in S}$ of $\mathcal{C}$ and every ultrafilter μ on S, the map $\sigma_\mu : F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu$ is surjective. Then F is a quasi-compact object of the topos $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$.*

Proof. Let $\{F_s\}_{s\in S}$ be the collection of all quasi-compact subobjects of F (in the Grothendieck topos $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$). We regard S as partially ordered by inclusion (so that $s \leq s'$ if and only if $F_s \subseteq F_{s'}$). Since the collection of quasi-compact subobjects of F is closed under finite unions, the partial ordering on S is directed. Applying Proposition 1.1.10, we can choose an ultrafilter μ on S such that, for every $t \in S$, we have $\mu(\{s \in S : s \geq t\}) = 1$.

Assume that F is not quasi-compact. Then, for each $s \in S$, we have $F_s \subsetneq F$. We can therefore choose a model M_s of $\mathcal{C}$ and an element $x_s \in F(M_s)$ which does not belong to $F_s(M_s)$. Let x denote the image of $\{x_s\}_{s\in S}$ in the ultraproduct $\int_S F(M_s) d\mu$. Using the surjectivity of the map $\sigma_\mu : F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu$, we conclude that there is an element $y \in F(\int_S M_s d\mu)$ satisfying $\sigma_\mu(y) = x$. Then y belongs to $F_t(\int_S M_s d\mu)$ for some $t \in S$. Using the commutativity of the diagram

$$\begin{tikzcd} F_t(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] \arrow[d] & \int_S F_t(M_s) d\mu \arrow[d] \\ F(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] & \int_S F(M_s) d\mu, \end{tikzcd}$$

we see that x can be lifted to an element $\widetilde{x}$ of the ultraproduct $\int_S F_t(M_s) d\mu$. Choose a subset $S_0 \subseteq S$ satisfying $\mu(S_0) = 1$ and a tuple $\{\widetilde{x}_s \in F_t(M_s)\}_{s\in S_0}$ representing $\widetilde{x}$. Shrinking S_0 if necessary, we may assume that $\widetilde{x}_s = x_s$ for each $s \in S_0$ and that $S_0 \subseteq \{s \in S : s \geq t\}$. Then, for any element $s \in S_0$, we conclude that

$$x_s = \widetilde{x}_s \in F_t(M_s) \subseteq F_s(M_s),$$

contradicting our choice of x_s. □

Proof of Theorem 2.3.1. Let $\mathcal{C}$ be a small pretopos; we wish to prove that the evaluation map $\mathrm{ev} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ is an equivalence of categories. Let $\theta : \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}) \to \mathrm{Shv}(\mathcal{C})$ be the equivalence of Theorem 2.2.2, so that the composition $\theta \circ \mathrm{ev} : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ is the Yoneda embedding. Applying Theorem C.6.5, we see that the evaluation map ev is a fully faithful embedding, whose essential image consists of those ultrafunctors $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ which are quasi-compact and quasi-separated when viewed as objects of the topos $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$. We will complete the proof by showing that every ultrafunctor F is quasi-compact and quasi-separated as an object of $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$. The quasi-compactness of F follows from Lemma 2.3.6. To prove that F is quasi-separated, we must show that for every pair of

quasi-compact objects $F_0, F_1 \in \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ equipped with maps $F_0 \to F \leftarrow F_1$, the fiber product $F_0 \times_F F_1$ is quasi-compact. It follows from Theorem 2.2.2 that the topos $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ is generated (under small colimits) by objects of the form ev_C, for $C \in \mathcal{C}$. We may therefore assume without loss of generality that F_0 and F_1 belong to the essential image of the evaluation functor, and therefore belong to the subcategory $\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}) \subseteq \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$. In this case, the fiber product $F_0 \times_F F_1$ is also an ultrafunctor (Corollary 2.1.4), and is therefore quasi-compact by Lemma 2.3.6. □

2.4. **Application: The Image of the Barr Embedding.** Let $\mathcal{E}$ be a regular category (Definition A.1.3). Recall that a functor $M : \mathcal{E} \to \mathrm{Set}$ is said to be *regular* if it preserves finite limits and carries effective epimorphisms in $\mathcal{E}$ to surjections in the category of sets. We let $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathrm{Set})$ spanned by the regular functors. For every object $E \in \mathcal{E}$, we let $\mathrm{ev}_E : \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}) \to \mathrm{Set}$ denote the functor given by evaluation at E, so that $\mathrm{ev}_E(M) = M(E)$. The construction $E \mapsto \mathrm{ev}_E$ is called the *Barr embedding*, due to the following result of [3]:

Theorem 2.4.1 (Barr)**.** *Let $\mathcal{E}$ be a small regular category. Then the construction $E \mapsto \mathrm{ev}_E$ induces a fully faithful embedding*

$$\mathcal{E} \hookrightarrow \mathrm{Fun}(\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}), \mathrm{Set}).$$

We refer the reader to §6.2 for a proof of Theorem 2.4.1 (which is essentially identical to Barr's original proof).

If $\mathcal{E}$ is an exact category, then the essential image of the Barr embedding admits a simple description, given by the following result of Makkai:

Theorem 2.4.2 (Makkai)**.** *Let $\mathcal{E}$ be a small exact category (Definition A.2.6). Then the functor* $\mathrm{ev} : \mathcal{E} \to \mathrm{Fun}(\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}), \mathrm{Set})$ *is a fully faithful embedding, whose essential image consists of those functors* $F : \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}) \to \mathrm{Set}$ *which preserve small products and small filtered colimits.*

In this section, we observe that Theorem 2.4.2 can be deduced from Makkai's strong conceptual completeness theorem (the reverse is also true: see Remark 6.0.2). The proof is based on the following general category-theoretic fact:

Proposition 2.4.3. *Let $\mathcal{E}$ be a small regular category. Then there exists a small pretopos $\mathcal{C}$ and a fully faithful regular functor $h : \mathcal{E} \to \mathcal{C}$ such that precomposition with h induces an equivalence of categories* $\mathrm{Mod}(\mathcal{C}) \subseteq \mathrm{Fun}^{\mathrm{reg}}(\mathcal{C}, \mathrm{Set}) \to \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$. *Moreover, if the category $\mathcal{E}$ is exact, then an object $C \in \mathcal{C}$ belongs to the essential image of f if and only if the evaluation functor*

$$\mathrm{ev}_C : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set} \qquad M \mapsto M(C)$$

commutes with finite products.

Remark 2.4.4. For any regular category $\mathcal{E}$, the full subcategory $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathcal{E}, \mathrm{Set})$ is closed under small products and small filtered colimits. In the situation of Proposition 2.4.3, the existence of an equivalence $\mathrm{Mod}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ guarantees that the category $\mathrm{Mod}(\mathcal{C})$ also admits small products. Moreover, if $C \in \mathcal{C}$ belongs to the essential image of h, then the evaluation functor $\mathrm{ev}_C : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ can be identified with the evaluation functor $\mathrm{ev}_E : \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}) \to \mathrm{Set}$ for some object $E \in \mathcal{E}$, and therefore commutes with small products (not just finite products).

Proof of Proposition 2.4.3. Let us regard the category $\mathcal{E}$ as equipped with the regular topology of Definition B.3.3, and let $\mathrm{Shv}(\mathcal{E})$ denote the associated category of sheaves. Let $h : \mathcal{E} \to \mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathrm{Set})$ be the Yoneda embedding. Since the regular topology on $\mathcal{E}$ is subcanonical (Corollary B.3.6), we can regard h as a functor from $\mathcal{E}$ to $\mathrm{Shv}(\mathcal{E})$. Note that the topos $\mathrm{Shv}(\mathcal{E})$ is coherent, and that the functor h takes values in the full subcategory $\mathrm{Shv}_{\mathrm{coh}}(\mathcal{E}) \subseteq \mathrm{Shv}(\mathcal{E})$ of coherent objects (Proposition C.6.3). Let $\mathcal{C} \subseteq \mathrm{Shv}_{\mathrm{coh}}(\mathcal{E})$ denote a small subcategory which is equivalent to $\mathrm{Shv}_{\mathrm{coh}}(\mathcal{E})$ and contains the essential image of h. Then $\mathcal{C}$ is a small pretopos (Corollary C.5.14), and we can regard h as a regular functor from $\mathcal{E}$ to $\mathcal{C}$. We claim that precomposition with h induces an equivalence of categories $\theta : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$. To prove this, we

observe that θ fits into a commutative diagram

$$\begin{tikzcd} & \mathrm{Fun}^*(\mathrm{Shv}(\mathcal{E}), \mathrm{Set}) \arrow[dl] \arrow[dr, "\circ h"] & \\ \mathrm{Mod}(\mathcal{C}) \arrow[rr, "\theta"] & & \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}), \end{tikzcd}$$

where the left and right vertical maps are equivalences by virtue of Corollary C.3.4 and Corollary C.3.6, respectively. It follows that the category $\mathrm{Mod}(\mathcal{C})$ admits small products (Remark 2.4.4).

We now prove the following:

($*$) Let $C \in \mathcal{C}$ be an object for which the evaluation functor $\mathrm{ev}_C : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ preserves finite products. Then there exists an object $E \in \mathcal{E}$ and an effective epimorphism $h_E \twoheadrightarrow C$ in the pretopos $\mathcal{C}$.

To prove ($*$), we first note that $\mathrm{Shv}(\mathcal{E})$ is generated by objects of the form h_E. We can therefore choose a collection of objects $\{E_i\}_{i \in I}$ of $\mathcal{E}$ and maps $\{u_i : h_{E_i} \to C\}_{i \in I}$ for which the induced map $\coprod_{i \in I} h_{E_i} \to C$ is an effective epimorphism in the topos $\mathrm{Shv}(\mathcal{E})$. Since C is a quasi-compact object of $\mathrm{Shv}(\mathcal{E})$, we may assume without loss of generality that the set I is finite. We claim that one of the maps u_i is an effective epimorphism. Assume otherwise. Then, for each $i \in I$, we can apply Deligne's completeness theorem (Theorem 2.2.10) to choose a model $M_i \in \mathrm{Mod}(\mathcal{C})$ for which the map of sets $M_i(u_i) : M_i(h_{E_i}) \to M_i(C)$ is not surjective. For each $i \in I$, choose an element $x_i \in M_i(C)$ which does not belong to the image of $M_i(u_i)$. Let M denote the product $\prod_{i \in I} M_i$, formed in the category $\mathrm{Mod}(\mathcal{C})$. Our assumption that the evaluation functor ev_C preserves finite products guarantees that the canonical map $M(C) \to \prod_{i \in I} M_i(C)$ is bijective. We can therefore choose an element $x \in M(C)$ having image x_i under each of the projection maps $M(C) \to M_i(C)$. Since the maps u_i induce a surjection $\coprod_{i \in I} M(h_{E_i}) \to M(C)$, there exists an index $j \in I$ such that x belongs to the image of the map $M(u_j) : M(h_{E_j}) \to M(C)$. Using the commutativity of the diagram

$$\begin{tikzcd} M(h_{E_j}) \arrow[r, "M(u_j)"] \arrow[d] & M(C) \arrow[d] \\ M_j(h_{E_j}) \arrow[r, "M_j(u_j)"] & M_j(C), \end{tikzcd}$$

we conclude that x_j belongs to the image of $M_j(u_j)$, contradicting our choice of x_j. This completes the proof of ($*$).

We next prove:

($*'$) Let E' be an object of $\mathcal{E}$ and let $C \in \mathcal{C}$ be a subobject of $h_{E'}$. If the evaluation functor ev_C commutes with finite products, then C belongs to the essential image of h.

To prove ($*'$), we note that ($*$) guarantees the existence of an object $E \in \mathcal{E}$ and an effective epimorphism $v : h_E \twoheadrightarrow C$. Then C can be identified with the image of the composite map $h_E \xrightarrow{v} C \hookrightarrow h_{E'}$. Since the functor h is fully faithful, we can assume that this map has the form h_u, for some map $u : E \to E'$ in $\mathcal{E}$. The regularity of the functor h then implies that $C = \mathrm{Im}(h_u)$ is isomorphic to $h_{\mathrm{Im}(u)}$.

Now assume that the category $\mathcal{E}$ is exact, and let $C \in \mathcal{C}$ be any object for which the evaluation functor $\mathrm{ev}_C : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ preserves finite products. Using ($*$), we can choose an object $E \in \mathcal{E}$ and an effective epimorphism $v : h_E \twoheadrightarrow C$ in $\mathcal{C}$. Let $D = h_E \times_C h_E \subseteq h_{E \times E}$ be the equivalence relation on h_E determined by v. Then the evaluation functor ev_D is given by the fiber product $\mathrm{ev}_{h_E} \times_{\mathrm{ev}_C} \mathrm{ev}_{h_E}$, and therefore preserves small products. It follows from ($*'$) that we can assume $D = h_R$ for some subobject $R \subseteq E \times E$ in the category $\mathcal{E}$. It is easy to see that R is an equivalence relation on E. Since $\mathcal{E}$ is exact, the equivalence relation R is effective: that is, there exists an effective epimorphism $E \twoheadrightarrow E/R$ in $\mathcal{E}$ such that $R = E \times_{E/R} E$ (as subobjects of $E \times E$). Because the functor h is regular, we can identify $h_{E/R}$ with the quotient of h_E by the equivalence relation $h_R = h_E \times_{\mathrm{ev}_C} h_E$: that is, with the object $C \in \mathcal{C}$. It follows that C belongs to the essential image of h, as desired. $\square$

Remark 2.4.5. Let $\mathcal{E}$ be a small regular category and let $h : \mathcal{E} \to \mathcal{C}$ be as in Proposition 2.4.3. Then the equivalence $\theta : \mathrm{Mod}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ is an equivalence of ultracategories, where we endow $\mathrm{Mod}(\mathcal{C})$ with the ultrastructure of Remark 2.1.2 and $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ with the categorical ultrastructure. To see this, we observe that θ can be written as the composition of the inclusion map $\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Fun}^{\mathrm{reg}}(\mathcal{C}, \mathrm{Set})$ (which has an evident ultrastructure, where we endow $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{C}, \mathrm{Set})$ with the categorical ultrastructure) with the restriction functor $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{C}, \mathrm{Set}) \xrightarrow{\circ h} \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ (which preserves small products and filtered colimits, and therefore inherits an ultrastructure from Proposition 1.4.9.

Proof of Theorem 2.4.2. Let $\mathcal{E}$ be a small exact category. According to Theorem 2.4.1, the construction $E \mapsto \mathrm{ev}_E$ induces a fully faithful embedding $\mathrm{ev} : \mathcal{E} \hookrightarrow \mathrm{Fun}(\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}), \mathrm{Set})$. It will therefore suffice to show that if $F : \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}) \to \mathrm{Set}$ is a functor which preserves small products and small filtered colimits, then there is a natural isomorphism $F \simeq \mathrm{ev}_E$ for some $E \in \mathcal{E}$ (the converse is clear, since small products and small filtered colimits in the category $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ are computed pointwise). Note that we can use Proposition 1.4.9 to endow F with the structure of an ultrafunctor (where $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ and Set are equipped with the categorical ultrastructures of Example 1.3.8).

Choose a small pretopos $\mathcal{C}$ and a regular functor $h : \mathcal{E} \to \mathcal{C}$ satisfying the requirements of Proposition 2.4.3. Then precomposition with h induces an equivalence of ultracategories $H : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ (Remark 2.4.5). Consequently, the composite functor $\mathrm{Mod}(\mathcal{C}) \xrightarrow{H} \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}) \xrightarrow{F} \mathrm{Set}$ admits an ultrastructure, and is therefore given by evaluation on some object $C \in \mathcal{C}$ (Theorem 2.3.1). It will therefore suffice to show that C belongs to the essential image of h. This follows from Proposition 2.4.3, since the functor $F \circ H$ commutes with finite products. $\square$

Remark 2.4.6. Let $h : \mathcal{E} \hookrightarrow \mathcal{C}$ be as in the proof of Theorem 2.4.2. Note that there is a asymmetry between the statements of Theorems 2.3.1 and 2.4.2. The first supplies an equivalence of $\mathcal{C}$ with the category $\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$, whose objects are functors $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ equipped with additional structure. The second supplies an equivalence of the smaller category $\mathcal{E}$ with a full subcategory of $\mathrm{Fun}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ spanned by functors F which satisfy certain *conditions*: namely, that F preserves small products and small filtered colimits. This apparent discrepancy can be resolved by observing that a functor $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ which preserves small products and small filtered colimits admits a *unique* ultrastructure (namely, the ultrastructure supplied by Proposition 1.4.9). This is a special case of a more general result about categorical ultrastructures, which we will prove in §8.3 (see Corollary 8.3.5).

3. Ultracategories and Topology

Recall that every set X can be regarded as a category, having the elements of X as objects and no non-identity morphisms. Of course, categories of this form are not very interesting. However, we will show in this section that they can nevertheless carry interesting ultrastructures. Our principal results can be summarized as follows:

(*a*) Let X be a set, regarded as a category having only identity morphisms. Then there is a canonical bijection
$$\{\text{Ultrastructures on } X\} \simeq \{\text{Compact Hausdorff topologies on } X\}.$$

(*b*) Let X be a compact Hausdorff space, and let $\mathscr{F}$ be a sheaf of sets on X. Then the construction $x \mapsto \mathscr{F}_x$ can be regarded as a left ultrafunctor from X (equipped with the ultrastructure determined by its topology) to the category Set (equipped with the categorical ultrastructure of Example 1.3.8). Moreover, this construction determines an equivalence of categories $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set}) \simeq \mathrm{Shv}(X)$.

(*c*) The equivalence of (*b*) restricts to an equivalence of categories $\mathrm{Fun}^{\mathrm{Ult}}(X, \mathrm{Set}) \simeq \mathrm{Loc}(X)$, where $\mathrm{Loc}(X) \subseteq \mathrm{Shv}(X)$ is the full subcategory spanned by those sheaves which are locally constant with finite fibers.

We begin in §3.1 by giving a precise formulation of (*a*) (Theorem 3.1.5). The proof is given in §3.3, using some standard facts about Stone-Čech compactifications which we review in §3.2. In §3.4, we give a precise formulation of (*b*) by associating to each left ultrafunctor $G : X \to \mathrm{Set}$ a certain sheaf of sets $\mathscr{F}_G$ on X,

and asserting that the construction $G \mapsto \mathscr{F}_G$ is an equivalence of categories (Theorem 3.4.4); assertion (c) is then an easy consequence (Theorem 3.4.11). The main step of the proof is to show that the stalk of the sheaf $\mathscr{F}_G$ at a point $x \in X$ can be identified with the value of the functor G at x (Proposition 3.4.6), which we prove in §3.5.

Remark 3.0.1. Let Comp denote the category of compact Hausdorff spaces and let $G : \mathrm{Comp} \to \mathrm{Set}$ denote the forgetful functor, which associates to each compact Hausdorff space X its underlying set. Theorem 3.1.5 is essentially a reformulation of the classical fact that the functor G is *monadic*: that is, it admits a left adjoint F (the Stone-Čech compactification functor of §3.2) and induces an equivalence of Comp with the category of algebras over the monad $G \circ F$ (see, for example, § VI.9 of [8]). Equipping a set X with the structure of an algebra for this monad is equivalent to specifying an ultrastructure on X, almost by definition. Our Theorem 3.4.4 admits a similar interpretation: it is equivalent to the monadicity of a functor $G^{+} : \mathrm{Comp}^{+} \to \mathrm{Set}^{+}$ which can be described as follows:

- The objects of the category Comp^{+} are pairs $(X, \mathscr{F})$, where X is a compact Hausdorff space and $\mathscr{F}$ is a sheaf of sets on X; here a morphism from $(X, \mathscr{F})$ to $(X', \mathscr{F}')$ is given by a continuous function $f : X \to X'$ together with a map of sheaves $f^* \mathscr{F}' \to \mathscr{F}$ on X.
- The objects of the category Set^{+} are pairs $(X, \mathscr{F})$, where X is a set with the discrete topology and $\mathscr{F}$ is a sheaf of sets on X, with morphisms defined in a similar way.
- The functor G^{+} carries a pair $(X, \mathscr{F})$ to $(X^{\mathrm{disc}}, \mathscr{F}|_{X^{\mathrm{disc}}})$, where X^{disc} denotes the underlying set of X endowed with the discrete topology, and $\mathscr{F}|_{X^{\mathrm{disc}}}$ denotes the pullback of $\mathscr{F}$ to X^{disc}.

More informally, the functor G^{+} is given on a pair $(X, \mathscr{F})$ by "forgetting" everything except for the underlying set of X and the collection of stalks $\{\mathscr{F}_x\}_{x \in X}$.

3.1. **Ultrasets.** We begin by observing that, when working with categories having only identity morphisms, our notion of ultracategory becomes dramatically simpler: since every morphism is an isomorphism and every diagram commutes, axioms (A), (B), and (C) of Definition 1.3.1 are automatically satisfied. We can therefore rephrase Definition 1.3.1 as follows:

Definition 3.1.1. Let X be a set. An *ultrastructure on* X consists of the following data:

(1) For every map of sets $f : S \to X$ and every ultrafilter μ on S, an element $\int_S f(x) d\mu \in X$.

This data is required to satisfy the following conditions:

(2) For every map of sets $f : S \to X$ and every element $s_0 \in S$, we have $\int_S f(s) d\delta_{s_0} = f(s_0)$.

(3) For every map of sets $f : T \to S$, every family $\nu_\bullet = \{\nu_s\}_{s \in S}$ of ultrafilters on T, and every ultrafilter μ on S, we have an identity
$$\int_T f(t) d(\int_S \nu_s d\mu) = \int_S (\int_T f(t) d\nu_s) d\mu.$$

An *ultraset* is a set X together with an ultrastructure on X.

Definition 3.1.2. Let X and Y be ultrasets. A *morphism of ultrasets* from X to Y is a function $g : X \to Y$ which satisfies the following condition: for every map of sets $f : S \to X$ and every ultrafilter μ on S, we have an identity
$$g(\int_S f(s) d\mu) = \int_S (g \circ f)(s) d\mu.$$
We let USet denote the category whose objects are ultrasets and whose morphisms are morphisms of ultrasets.

Remark 3.1.3. Let X and Y be ultrasets. Then we can regard X and Y as ultracategories having only identity morphisms. The category $\mathrm{Fun}^{\mathrm{Ult}}(X, Y)$ of ultrafunctors from X to Y has only identity morphisms, and its objects can be identified with morphisms of ultrasets from X to Y. In other words, we can identify the category USet of Definition 3.1.2 with a full subcategory of the 2-category **Ult** of ultracategories (Remark 1.4.6), whose objects are small ultracategories having only identity morphisms.

Proposition 3.1.4. *Let X be an ultraset. Then there is a unique topology on X for which a subset $K \subseteq X$ is closed if and only if it satisfies the following condition:*

($*$) *For every map of sets $f : S \to K$ and every ultrafilter μ on S, the point $\int_S f(s)d\mu$ belongs to K.*

Proof. It is clear that the collection of subsets of X satisfying condition ($*$) is stable under intersections. We will show that it is also stable under finite unions. Suppose that we are given a finite collection of subsets $\{K_i \subseteq X\}_{i\in I}$, where each K_i satisfies condition ($*$). Set $K = \bigcup_{i\in I} K_i$; we wish to show that K also satisfies ($*$). Let $f : S \to K$ be any map of sets and let μ be an ultrafilter on S. For each $i \in I$, set $S_i = f^{-1}(K_i)$. Since μ is an ultrafilter, there exists $i \in I$ such that $\mu(S_i) = 1$. In this case, μ restricts to an ultrafilter μ_i on the set S_i, and we have

$$\int_S f(S)d\mu = \int_{S_i} f(s)d\mu_i \in K_i \subseteq K$$

by virtue of our assumption that K_i satisfies ($*$). □

Note that if $g : X \to Y$ is a morphism of ultrasets, then it is automatically continuous if we equip X and Y with the topology of Proposition 3.1.4. We can therefore regard Proposition 3.1.4 as supplying a functor from the category of ultrasets USet to the category Top of topological spaces. We can now state our first result:

Theorem 3.1.5. *The construction of Proposition 3.1.4 determines a fully faithful functor* USet $\to$ Top*, whose essential image is the full subcategory* Comp $\subseteq$ Top *spanned by the compact Hausdorff spaces.*

We will prove Theorem 3.1.5 in §3.3.

Remark 3.1.6. Let $f : X \to Y$ be a morphism of ultrasets, and regard X and Y as equipped with the topology of Definition 3.1.2. Then f is a closed map: that is, for every closed subset $K \subseteq X$, the image $f(K) \subseteq Y$ is closed. To prove this, we observe that any function $g : S \to f(K)$ can be written as $f \circ \widetilde{g}$ for some function $\widetilde{g} : S \to K$. It follows that for every ultrafilter μ on S, we have

$$\int_S g(s)d\mu = \int_S f(\widetilde{g}(s))d\mu = f(\int_S \widetilde{g}(s)d\mu) \in f(K)$$

by virtue of our assumption that K is closed.

3.2. **Digression: The Stone-Čech Compactification.**

Let S be a set. We let βS denote the set of all ultrafilters on S. Then βS can be identified with the spectrum of a Boolean algebra (namely, the Boolean algebra $P(S)$ of all subsets of S), and therefore inherits the structure of a topological space; this topological space is called the *Stone-Čech compactification* of S. In this section, we recall the construction of the space βS and review its universal property (Proposition 3.2.7).

Notation 3.2.1. Let S be a set. For each subset $S_0 \subseteq S$, pushforward along the inclusion $S_0 \hookrightarrow S$ induces a monomorphism $\beta S_0 \hookrightarrow \beta S$. In what follows, we will often abuse notation by identifying βS_0 with its image under this monomorphism: by virtue of Remark 1.1.5, this image is given $\{\mu \in \beta S : \mu(S_0) = 1\} \subseteq \beta S$. With this convention, we have

$$\beta(S_0 \cap S_1) = (\beta S_0) \cap (\beta S_1) \quad \beta(S_0 \cup S_1) = (\beta S_0) \cup (\beta S_1) \qquad \beta(S \smallsetminus S_0) = (\beta S) \smallsetminus (\beta S_0).$$

Construction 3.2.2 (The Topology on βS)**.** Let S be a set. We will regard the collection of ultrafilters βS as a topological space by equipping it with the topology generated by sets of the form $\beta S_0 = \{\mu \in \beta S : \mu(S_0) = 1\}$ for $S_0 \subseteq S$. Since these sets are closed under the formation of finite intersections, they comprise a basis for the topology on βS.

Proposition 3.2.3. *Let S be a set. Then the Stone-Čech compactification βS is a Stone space (with respect to the topology of Construction 3.2.2). That is, βS is a compact Hausdorff space having a basis of closed and open sets.*

Proof. For every pair of distinct ultrafilters $\mu, \nu \in \beta S$, we can choose some subset $S_0 \subseteq S$ such that $\mu(S_0) \neq \nu(S_0)$. Then βS_0 and $\beta(S \smallsetminus S_0)$ are complementary open sets containing μ and ν. This immediately implies that βS is Hausdorff. Moreover, each of the basic open sets $\beta S_0 \subseteq \beta S$ is also closed, since it is the complement of the basic open set $\beta(S \smallsetminus S_0)$.

To complete the proof, it will suffice to show that the topological space βS is compact. To prove this, suppose we are given a covering of βS by a collection of basic open sets $\{\beta S_\alpha\}_{\alpha\in A}$. Let $\mathcal{U}$ denote the collection of all subsets of S which can be written as a finite intersection of sets of the form $S \smallsetminus S_\alpha$. By assumption, every ultrafilter μ on S satisfies $\mu(S_\alpha) = 1$ for some index α, so we have $\mu(J) = 0$ for $J = (S \smallsetminus S_\alpha) \in \mathcal{U}$. Invoking Proposition 1.1.10, we conclude that $\mathcal{U}$ contains the empty set. In other words, we can choose a finite subset $A_0 \subseteq A$ such that $\bigcap_{\alpha\in A_0}(S \smallsetminus S_\alpha) = \varnothing$, so that $\bigcup_{\alpha\in A_0} S_\alpha = S$. It follows that βS is covered by the finite collection of subsets $\{\beta S_\alpha\}_{\alpha\in A_0}$. □

Corollary 3.2.4. *Let S be a set. Then the construction $(S_0 \subseteq S) \mapsto (\beta S_0 \subseteq \beta S)$ induces a bijection*

$$\{\text{Subsets of } S\} \simeq \{\text{Closed and open subsets of } \beta S\}.$$

Proof. Note that for any subset $S_0 \subseteq S$ and any element $s \in S$, the principal ultrafilter δ_s is contained in $\beta S_0 = \{\mu \in \beta S : \mu(S_0) = 1\}$ if and only if $s \in I$. It follows that if S_0 and S_1 are distinct subsets of S, then $\beta S_0 \neq \beta S_1$ (as subsets of βS). We will complete the proof by showing that if $U \subseteq \beta S$ is closed and open, then $U = \beta S_0$ for some $S_0 \subseteq S$. The assumption that U is open guarantees that we can write $U = \bigcup \beta S_\alpha$ for some collection $\{S_\alpha \subseteq S\}_{\alpha\in A}$. The assumption that U is closed guarantees that U is compact (since βS is compact by Proposition 3.2.3), so we can assume without loss of generality that A is finite. Then $U = \beta S_0$ for $S_0 = \bigcup_{\alpha\in A} S_\alpha$. □

Corollary 3.2.5. *Let S be a set and let μ be an ultrafilter on S. Then μ is principal if and only if is an isolated point of the topological space βS.*

Proof. By virtue of Corollary 3.2.4, the point μ is isolated in βS if and only if there exists a subset $S_0 \subseteq S$ such that $\{\mu\} = \{\nu \in \beta S : \nu(S_0) = 1\}$. The set S_0 must be nonempty, so we can choose some point $s \in S_0$. Then the principal ultrafilter δ_s belongs to $\{\nu \in \beta S : \nu(S_0) = 1\} = \{\mu\}$, so that $\mu = \delta_s$. □

Remark 3.2.6. Let S be a set. Then the collection of principal ultrafilters $\{\delta_s\}_{s\in S}$ is dense in the Stone-Čech compactification βS. To see this, we observe that every nonempty open subset $U \subseteq \beta S$ contains a nonempty set of the form βS_0 for $S_0 \subseteq S$, and therefore contains the point δ_s for any element $s \in S_0$.

The Stone-Čech compactification βS can be characterized (up to homeomorphism) by the following universal mapping property:

Proposition 3.2.7. *Let S be a set, and let $\delta : S \to \beta S$ be the map which associates to each $s \in S$ the principal ultrafilter $\delta_s \in \beta S$. Then, for any compact Hausdorff space X, composition with δ induces a bijection*

$$\mathrm{Hom}_{\mathrm{Top}}(\beta S, X) \to \mathrm{Hom}_{\mathrm{Set}}(S, X).$$

Here $\mathrm{Hom}_{\mathrm{Top}}(\beta S, X)$ denotes the collection of continuous maps from βS to X.

Proof. Let $f : S \to X$ be any map of sets. We wish to show that there is a unique continuous map $\overline{f} : \beta S \to X$ satisfying $f = \overline{f} \circ \delta$. We will prove the existence of $\overline{f}$; uniqueness is immediate from Remark 3.2.6. For each subset $S_0 \subseteq S$, let $\overline{f(S_0)}$ denote the closure of the subset $f(S_0) \subseteq X$. We first prove the following:

($*$) For every ultrafilter μ on S, the intersection $\bigcap_{\mu(S_0)=1} \overline{f(S_0)}$ consists of a single point of X.

To prove ($*$), we first observe that for every finite collection of sets $S_1, \ldots, S_n \in\subseteq S$ satisfying $\mu(S_1) = \mu(S_2) = \cdots = \mu(S_n) = 1$, we have

$$\varnothing \neq f(S_1 \cap \cdots \cap S_n) \subseteq \overline{f(S_1)} \cap \cdots \cap \overline{f(S_n)}.$$

Consequently, the closed sets $\{\overline{f(S_0)}\}_{\mu(S_0)=1}$ have the finite intersection property. Since X is compact, it follows that the intersection $\bigcap_{\mu(S_0)=1} \overline{f(S_0)}$ is nonempty. Suppose we are given a pair of points $x, y \in \bigcap_{\mu(S_0)=1} \overline{f(I)}$. If $x \neq y$, then we can choose disjoint open sets $U, V \subseteq X$ satisfying $x \in U$ and $y \in V$. Then $f^{-1}(U) \cap f^{-1}(V) = \varnothing$, so either $\mu(f^{-1}(U)) = 0$ or $\mu(f^{-1}(V)) = 0$. Without loss of generality, we may assume that $\mu(f^{-1}(U)) = 0$. Then U is an open neighborhood of x which does not intersect $\overline{f(S - f^{-1}(U))}$, contradicting our assumption that x belongs to $\bigcap_{\mu(S_0)=1} \overline{f(S_0)}$. This completes the proof of ($*$).

For each ultrafilter μ on S, let $\int_S f(s)d\mu$ denote the unique point of the intersection $\bigcap_{\mu(S_0)=1} \overline{f(S_0)}$. Note that if $\mu = \delta_t$ is the principal ultrafilter associated to an element $t \in S$, then $\mu(\{t\}) = 1$, so we have $\int_S f(s)d\mu \in \overline{f(\{t\})} = \{f(t)\}$; that is, we have $\int_S f(s)d\delta_t = f(t)$. We will complete the proof by showing that the function

$$\overline{f} : \beta S \to X \qquad \overline{f}(\mu) = \int_S f(s)d\mu$$

is continuous. Let μ be an ultrafilter on S and let $U \subseteq X$ be an open neighborhood of the point $\int_S f(s)d\mu$; we wish to show that $\overline{f}^{-1}(U)$ contains an open neighborhood of μ. Choose an open neighborhood V of the point $\int_S f(s)d\mu$ satisfying $\overline{V} \subseteq U$, and set $S_0 = S \setminus f^{-1}(V)$. Then $f(S_0)$ is disjoint from V, so the point $\int_S f(s)d\mu$ is not contained in the closure $\overline{f(S_0)}$. It follows that $\mu(S_0) = 1$, so that we can regard βI as an open neighborhood of μ in βS. If ν belongs to this neighborhood, we have $\overline{f}(\nu) = \int_S f(s)d\nu \in \overline{f(S_0)} \subseteq \overline{V} \subseteq U$. □

Example 3.2.8 (Composition of Ultrafilters)**.** Let S and T be sets. Suppose we are given a map of sets $f : S \to \beta T$, which we can identify with a collection of ultrafilters $\{\nu_s\}_{s\in S}$ on T. Define $\overline{f} : \beta S \to \beta T$ by the formula $\overline{f}(\mu) = \int_S \nu_s d\mu$ (see Construction 1.1.6). Then $\overline{f}$ is the continuous extension of Proposition 3.2.7: that is the unique continuous function satisfying $\overline{f}(\delta_s) = f(s)$ for $s \in S$ (see Example 1.1.8). To verify the continuity of $\overline{f}$, it will suffice to show that for every closed and open subset $\beta T_0 \subseteq \beta T$, the inverse image $\overline{f}^{-1}(\beta T_0)$ is a closed and open subset of βS; this follows from the calculation

$$\begin{aligned}
\overline{f}^{-1}(\beta T_0) &= \{\mu \in \beta S : \overline{f}(\mu) \in \beta T_0\} \\
&= \{\mu \in \beta S : \overline{f}(\mu)(T_0) = 1\} \\
&= \{\mu \in \beta S : \mu(\{s \in S : \nu_s(T_0) = 1\}) = 1\} \\
&= \beta S_0,
\end{aligned}$$

where $S_0 = \{s \in S : \nu_s(T_0) = 1\}$.

3.3. **The Proof of Theorem 3.1.5.** We now turn to the proof of Theorem 3.1.5. We begin by using Proposition 3.2.7 to explicitly construct an ultrastructure on each compact Hausdorff space X.

Proposition 3.3.1. *Let X be a compact Hausdorff space. For every map of sets $f : S \to X$ and every ultrafilter μ on S, let $\int_S f(s)d\mu$ be defined as in the proof of Proposition 3.2.7. Then the construction $(f, \mu) \mapsto \int_S f(s)d\mu$ determines an ultrastructure on X (in the sense of Definition 3.1.1).*

Proof. The identity $\int_S f(s)d\delta_{s_0} = f(s_0)$ follows immediately from the definitions. We will verify condition (3) of Definition 3.1.1. Suppose we are given a map of sets $f : T \to X$ and a family of ultrafilters $\nu_\bullet = \{\nu_s\}_{s\in S}$ on T. Using Proposition 3.2.7, we see that there is a unique continuous map $\overline{f} : \beta T \to X$ satisfying $\overline{f}(\delta_t) = f(t)$ for each $t \in T$, given by the formula $\overline{f}(\nu) = \int_T f(t)d\nu$. Similarly, there is a unique continuous map $\overline{g} : \beta S \to \beta T$ satisfying $\overline{g}(\delta_s) = \nu_s$ for $s \in S$, given by the formula $\overline{g}(\mu) = \int_S \nu_s d\mu$ (Example 3.2.8). Since the composition $\overline{f} \circ \overline{g} : \beta S \to X$ is continuous, we have

$$\begin{aligned}
\int_T f(t)d(\int_S \nu_s d\mu) &= \overline{f}(\int_S \nu_s d\mu) \\
&= \overline{f}(\overline{g}(\mu)) \\
&= (\overline{f} \circ \overline{g})(\mu) \\
&= \int_S (\overline{f} \circ \overline{g})(\delta_s) d\mu \\
&= \int_S \overline{f}(\nu_s) d\mu \\
&= \int_S (\int_T f(t) d\nu_s) d\mu.
\end{aligned}$$

□

Example 3.3.2. Let T be a set. The Stone-Čech compactification βT is a compact Hausdorff space (Proposition 3.2.3), and therefore inherits an ultrastructure from Proposition 3.3.1. Concretely, this ultrastructure associates to each collection $\{\nu_s\}_{s\in S}$ of ultrafilters on T and each ultrafilter μ on S the composite ultrafilter $\int_{s\in S}\nu_s d\mu$ defined in Construction 1.1.6. This follows from the calculation of Example 3.2.8.

Remark 3.3.3. Let X be a compact Hausdorff space, and regard X equipped with the ultrastructure of Proposition 3.3.1. Then the topology of X agrees with the topology of Proposition 3.1.4. That is, a subset $K \subseteq X$ is closed if and only if, for every map $f : S \to K$ and every ultrafilter μ on S, the point $\int_S f(s)d\mu$ belongs to K. One direction is clear: if K is closed, then $\{\mu \in \beta S : \int_S f(s)d\mu \in K\}$ is a closed subset of βS which contains every principal ultrafilter, and therefore coincides with βS (Remark 3.2.6). Conversely, if K is closed in the sense of Proposition 3.1.4 and we choose $f : S \to K$ to be surjective, then the construction $(\mu \in \beta S) \mapsto (\int_S f(s)d\mu \in X)$ determines a continuous surjection $\overline{f} : \beta S \to X$ having image K, so that K is closed (since βS is compact and X is Hausdorff).

Remark 3.3.4. Let X be an ultraset, let $f : T \to X$ be a map of sets, and let $\overline{f} : \beta T \to X$ be the map given by $\overline{f}(\nu) = \int_T f(t)d\nu$. Then $\overline{f}$ is a morphism of ultrasets, in the sense of Definition 3.1.2 (where we regard βT as equipped with the ultrastructure of Example 3.3.2). This is precisely the content of axiom (3) of Definition 3.1.1 (for a more general version of this argument, see Proposition 4.2.8).

Proposition 3.3.5. *Let X be an ultraset, and regard X as equipped with the topology of Proposition 3.1.4. Then X is a compact Hausdorff space.*

Proof. Choose a surjection of sets $f : T \to X$, and define $\overline{f} : \beta T \to X$ via the formula $\overline{f}(\nu) = \int_T f(t)d\nu$. Then $\overline{f}$ is a morphism of ultrasets (Remark 3.3.4). Note that the topology on βT provided by Proposition 3.1.4 agrees with the topology of Proposition 3.2.3 (Remark 3.3.3). Applying Remark 3.1.6, we see that $\overline{f}$ is a closed map of topological spaces. Since it is surjective, it is a quotient map. Consequently, to show that X is a compact Hausdorff space, it will suffice to show that the fiber product $S = (\beta T) \times_X (\beta T)$ has closed image in the product $(\beta T) \times (\beta T)$.

Let us view the embedding $g : S \hookrightarrow (\beta T) \times (\beta T)$ as a collection of pairs $\{(\nu_s, \nu'_s)\}_{s\in S}$ of ultrafilters on T, indexed by S. Then g extends to a continuous map $\overline{g} : \beta S \to (\beta T) \times (\beta T)$, given by the formula

$$\overline{g}(\mu) = (\int_S \nu_s d\mu, \int_S \nu'_s d\mu)$$

(see Example 3.2.8). Since $\overline{g}$ is a continuous map between compact Hausdorff spaces, it has closed image. It will therefore suffice to show that g and $\overline{g}$ have the same image: that is, that $\overline{g}$ takes values in the fiber product $(\beta T) \times_X (\beta T) \subseteq (\beta T) \times (\beta T)$. This follows from the calculation

$$\begin{aligned} \overline{f}(\int_S \nu_s d\mu) &= \int_T f(t)d(\int_S \nu_s d\mu) \\ &= \int_S (\int_T f(t)d\nu_s)d\mu \\ &= \int_S (\int_T f(t)d\nu'_s)d\mu \\ &= \int_T f(t)d(\int_S \nu'_s d\mu) \\ &= \overline{f}(\int_S \nu'_s d\mu). \end{aligned}$$

□

Proof of Theorem 3.1.5. By virtue of Proposition 3.3.5, the functor USet → Top takes values in the full subcategory Comp ⊆ Top spanned by the compact Hausdorff spaces. Moreover, every compact Hausdorff space belongs to the image, by Proposition 3.3.1 (and Remark 3.3.3). It will therefore suffice to show that the functor is fully faithful. In other words, it will suffice to show that if X and Y are ultrasets, then every continuous function $f : X \to Y$ is a morphism of ultrasets (in the sense of Definition 3.1.2). Choose a map

of sets $g : S \to X$, and define functions

$$\overline{g}_X : \beta S \to X \qquad \overline{g}_Y : \beta S \to Y$$

by the formulae

$$\overline{g}_X(\mu) = \int_S g(s) d\mu \qquad \overline{g}_Y(\mu) = \int_S (f \circ g)(s) d\mu.$$

We wish to show that $\overline{g}_Y = f \circ \overline{g}_X$. Note that the functions $\overline{g}_Y$ and $\overline{g}_X$ are both morphisms of ultrasets (Remark 3.3.4), and are therefore continuous. It follows that both $\overline{g}_Y$ and $f \circ \overline{g}_X$ are continuous functions from βS to Y. Since they agree on the dense subset of βS consisting of principal ultrafilters (Remark 3.2.6) and the topology on Y is Hausdorff (Proposition 3.3.5), it follows that they coincide everywhere. □

3.4. **Sheaves as Left Ultrafunctors.** Let X be a topological space and let $\mathscr{F}$ be a sheaf of sets on X. For each point $x \in X$, let $\mathscr{F}_x$ denote the stalk of $\mathscr{F}$ at the point x, given by the direct limit $\varinjlim_{x \in U} \mathscr{F}(U)$ (where the colimit is taken over all open neighborhoods of the point x). We can then ask the following:

Question 3.4.1. Is it possible to reconstruct a sheaf $\mathscr{F}$ from the collection of stalks $\{\mathscr{F}_x\}_{x \in X}$, together with some additional data?

The goal of this section is to give an affirmative answer to Question 3.4.1 in the special case where X is a compact Hausdorff space. In this case, we can identify the topology on X with an ultrastructure (Theorem 3.1.5), which associates to each map $f : S \to X$ and each ultrafilter μ on S a point $x = \int_S f(s) d\mu$. Let ϕ_x be an element of of the stalk $\mathscr{F}_x$. Then we can write ϕ_x as the germ of a section $\phi_U \in \mathscr{F}(U)$, for some open neighborhood U of the point x. Our assumption $x = \int_S f(s) d\mu$ then guarantees that the set $S_U = \{s \in S : f(s) \in U\}$ satisfies $\mu(S_U) = 1$. For each $s \in S_U$, the section ϕ_U determines an element ϕ_s of the stalk $\mathscr{F}_{f(s)}$. Individually, these germs are not determined by ϕ_x: they depend also on the choice of the section ϕ_U. However, they are well-defined "almost everywhere," in the sense of the ultrafilter μ: that is, the image of $\{\phi_s\}_{s \in S_0}$ in the ultraproduct $\int_S \mathscr{F}_{f(s)}\, d\mu$ depends only on ϕ_x. This construction determines maps

$$\sigma_\mu : \mathscr{F}_{\int_S f(s) d\mu} \to \int_S \mathscr{F}_{f(s)}\, d\mu,$$

which supply a left ultrastructure on the functor $x \mapsto \mathscr{F}_x$. We will address Question 3.4.1 by showing that this left ultrastructure determines the sheaf $\mathscr{F}$: more precisely, the preceding construction induces an equivalence from the category $\mathrm{Shv}(X)$ of set-valued sheaves on X to the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set})$ (Theorem 3.4.4). To carry out the details, it will actually be more convenient to work with the inverse equivalence.

Construction 3.4.2. Let X be a compact Hausdorff space and let $G : X \to \mathrm{Set}$ be a left ultrafunctor, with left ultrastructure $\{\sigma_\mu\}$. For every open subset $U \subseteq X$, we let $\mathscr{F}_G(U)$ denote the subset of $\prod_{x \in U} G(x)$ consisting of those tuples $\phi_\bullet = \{\phi_x\}_{x \in U}$ satisfying the following condition:

- $(*)$ For every map of sets $f : S \to U \subseteq X$ and every ultrafilter μ on S satisfying $\int_S f(s) d\mu \in U$, we have an equality $q_\mu(\{\phi_{f(s)}\}_{s \in S}) = \sigma_\mu(\phi_{\int_S f(s) d\mu})$ in the ultraproduct $\int_S G(f(s)) d\mu$. That is, $\{\phi_{f(s)}\}_{s \in S}$ and $\phi_{\int_S f(s) d\mu}$ have the same image under the maps

$$\prod_{s \in S} G(f(s)) \xrightarrow{q_\mu} \int_S G(f(s)) d\mu \xleftarrow{\sigma_\mu} G(\int_S f(s) d\mu).$$

Note that if $\{\phi_x\}_{x \in U}$ satisfies condition $(*)$ and $V \subseteq U$ is an open subset of U, then the tuple $\{\phi_x\}_{x \in V}$ also satisfies condition $(*)$. It follows that the construction $U \mapsto \mathscr{F}_G(U)$ determines a presheaf (of sets) on the topological space X.

We begin with an elementary observation:

Lemma 3.4.3. *Let X be a compact Hausdorff space and let $G : X \to \mathrm{Set}$ be a left ultrafunctor. Then the construction $U \mapsto \mathscr{F}_G(U)$ determines a sheaf of sets on X.*

Proof. Let $\mathscr{G}$ denote the sheaf of sets on X given by the formula $\mathscr{G}(U) = \prod_{x \in U} G(x)$. By construction, the presheaf $\mathscr{F}_G$ is contained in $\mathscr{G}$. It will therefore suffice to show that if $\{U_\alpha\}$ is a collection of open subsets of X having union $U = \bigcup_\alpha U_\alpha$, and $\phi_\bullet \in \mathscr{G}(U)$ is a section having the property that its image in each $\mathscr{G}(U_\alpha)$ belongs to $\mathscr{F}_G(U_\alpha)$, then $\phi_\bullet$ belongs to $\mathscr{F}_G(U)$. Write $\phi_\bullet = \{\phi_x\}_{x \in U}$ for some collection of elements $\phi_x \in \mathscr{F}_x$. Suppose that $f : S \to U \subseteq X$ is a map of sets and that μ is an ultrafilter on S with the property that $\int_S f(s) d\mu$ belongs to U; we wish to prove the identity

$$q_\mu(\{\phi_{f(s)}\}_{s \in S}) = \sigma_\mu(\phi_{\int_S f(s) d\mu})$$

in the ultraproduct $\int_S G(f(s)) d\mu$. Choose an index α such that $\int_S f(s) d\mu$ belongs to the subset $U_\alpha \subseteq U$. Set $S_0 = \{s \in S : f(s) \in U_\alpha\}$. Then $\mu(S_0) = 1$, so we can replace S by S_0 and f by $f|_{S_0}$. In this case, the relevant identity follows from our assumption that $\{\phi_x\}_{x \in U_\alpha}$ belongs to $\mathscr{F}_G(U_\alpha)$. $\square$

We can now give a precise statement of our main result.

Theorem 3.4.4. *Let X be a compact Hausdorff space. Then the construction $G \mapsto \mathscr{F}_G$ induces an equivalence of categories* $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set}) \to \mathrm{Shv}(X)$.

Variant 3.4.5. Recall that a category $\mathcal{M}$ is said to be *compactly generated* if it admits small colimits and is generated under filtered colimits by a small collection of compact objects. Equivalently, $\mathcal{M}$ is compactly generated if there exists an equivalence $\mathcal{M} \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$, for some small category $\mathcal{C}$ which admits finite limits (which can then be identified with the opposite of the category of compact objects of $\mathcal{M}$). If $\mathcal{M}$ is compactly generated and X is a compact Hausdorff space, then Theorem 3.4.4 determines equivalence from the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})$ to the category $\mathrm{Shv}(X; \mathcal{M})$ of $\mathcal{M}$-valued sheaves on X, given by the composition

$$\begin{aligned}
\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M}) &\simeq \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})) \\
&\simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set})) \\
&\to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Shv}(X)) \\
&\simeq \mathrm{Shv}(X; \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})) \\
&\simeq \mathrm{Shv}(X; \mathcal{M}).
\end{aligned}$$

One can also describe this equivalence directly, using a variant of Construction 3.4.2.

Let X be a compact Hausdorff space containing a point y and let $G : X \to \mathrm{Set}$ be a left ultrafunctor. For every open subset $U \subseteq X$ containing y, the construction $\{\phi_x\}_{x \in U} \mapsto \phi_y$ determines a map $\mathscr{F}_G(U) \to G(y)$. These maps are compatible as U varies, and therefore induce a map of sets $\mathscr{F}_{G,y} \to G(y)$. The main ingredient in our proof of Theorem 3.4.4 is the following result, whose proof we defer to §3.5:

Proposition 3.4.6. *Let X be a compact Hausdorff space and let $G : X \to \mathrm{Set}$ be a left ultrafunctor. Then, for each point $y \in X$, the preceding construction induces a bijection $\mathscr{F}_{G,y} \to G(y)$.*

Corollary 3.4.7. *Let X be a compact Hausdorff space and let $\alpha : G \to H$ be a natural transformation of left ultrafunctors $G, H : X \to \mathrm{Set}$. Then α is an isomorphism if and only if, for each open subset $U \subseteq X$, the induced map $\mathscr{F}_G(U) \to \mathscr{F}_H(U)$ is an isomorphism.*

Corollary 3.4.8. *Let X be a compact Hausdorff space. Then the functor* $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set}) \to \mathrm{Shv}(X)$ *preserves small colimits.*

Proof. Since small colimits $\mathrm{Shv}(X)$ are computed stalkwise, it will suffice to show that for each point $x \in X$, the functor $G \mapsto \mathscr{F}_{G,x}$ commutes with small colimits. By virtue of Proposition 3.4.6, this is equivalent to the evaluation functor $G \mapsto G(x)$, which preserves small colimits by virtue of Remark 1.4.3. $\square$

Corollary 3.4.9. *Let X be a compact Hausdorff space and let $G : X \to \mathrm{Set}$ be a left ultrafunctor. Then every subsheaf of $\mathscr{F}_G$ has the form $\mathscr{F}_{G_0}$, for some uniquely determined subobject $G_0 \subseteq G$ in the category of left ultrafunctors* $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set})$.

Proof. Let $\mathscr{G}$ be a subsheaf of $\mathscr{F}_G$. For each $x \in X$, let $G_0(x)$ denote the inverse image of the stalk $\mathscr{G}_x \subseteq \mathscr{F}_{G,x}$ under the bijection of Proposition 3.4.6. By virtue of Proposition 3.4.6, it will suffice to show that the left ultrastructure on G restricts to a left ultrastructure on G_0. In other words, it will suffice to show that for every map of sets $f : S \to X$ and every ultrafilter μ on S, the map

$$\sigma_\mu : G(\int_S f(s) d\mu) \to \int_S G(f(s)) d\mu$$

given by the left ultrastructure on G carries each element $\eta \in G_0(\int_S f(s) d\mu)$ into the subset

$$\int_S G_0(f(s)) d\mu \subseteq \int_S G(f(s)) d\mu.$$

It follows from the definition of G_0 that there exists an open subset $U \subseteq X$ containing the point $\int_S f(s) d\mu$ and a tuple

$$\phi_\bullet = \{\phi_x\}_{x \in U} \in \mathscr{G}(U) \subseteq \mathscr{F}_G(U)$$

such that $\phi_{\int_S f(s) d\mu} = \eta$. Set $S_0 = \{s \in S : f(s) \in U\}$, so that $\mu(S_0) = 1$. The definition of $\mathscr{F}_G$ then gives $\sigma_\mu(\phi_{\int_S f(s) d\mu}) = q_\mu^{S_0}(\{\phi_{f(s)}\}_{s \in S_0})$. By construction, we have $\phi_x \in G_0(x)$ for each $x \in U$, so that $q_\mu^{S_0}(\{\phi_{f(s)}\}_{s \in S_0})$ belongs to the subset $\int_S G_0(f(s)) d\mu \subseteq \int_S G(f(s)) d\mu$. □

Corollary 3.4.10. *Let X be a compact Hausdorff space. Then the construction $G \mapsto \mathscr{F}_G$ induces a fully faithful functor* $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set}) \to \mathrm{Shv}(X)$.

Proof. Let $G, H : X \to \mathrm{Set}$ be left ultrafunctors, and let $\alpha : \mathscr{F}_G \to \mathscr{F}_H$ be a morphism of sheaves on X. Let $\Gamma(\alpha)$ denote the graph of α, considered as a subobject of the product $\mathscr{F}_G \times \mathscr{F}_H \simeq \mathscr{F}_{G \times H}$. By virtue of Corollary 3.4.9, we can write $\Gamma(\alpha) = \mathscr{F}_{\overline{H}}$ for some subobject $G' \subseteq G \times H$ in $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set})$. Since projection map $\pi : G' \to G$ induces an isomorphism of sheaves $\mathscr{F}_{G'} \to \mathscr{F}_G$, Corollary 3.4.7 shows that π is an isomorphism of left ultrafunctors. It follows that G' can be identified with the graph of a morphism $G \to H$ which is a lift of α. The uniqueness is immediate from Proposition 3.4.6. □

Proof of Theorem 3.4.4. Let X be a compact Hausdorff space. Then the construction $G \mapsto \mathscr{F}_G$ determines a fully faithful embedding from $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set})$ to $\mathrm{Shv}(X)$ (Corollary 3.4.10). By virtue of Corollary 3.4.8, the essential image of this embedding is closed under small colimits. Note that the category $\mathrm{Shv}(X)$ is generated under small colimits by subobjects of the final object, it will suffice to show that every subobject of the final object of $\mathrm{Shv}(X)$ has the form $\mathscr{F}_G$, for some left ultrafunctor $G : X \to \mathrm{Set}$. This is a special case of Corollary 3.4.9. □

We close this section by establishing a variant of Theorem 3.4.4:

Theorem 3.4.11. *Let X be a compact Hausdorff space. Then the construction $G \mapsto \mathscr{F}_G$ induces a fully faithful embedding* $\mathrm{Fun}^{\mathrm{Ult}}(X, \mathrm{Set}) \hookrightarrow \mathrm{Shv}(X)$. *The essential image of this embedding is spanned by those sheaves $\mathscr{F}$ on X which are locally constant with finite stalks.*

Proof. By virtue of Theorem 3.4.4, it will suffice to show that the following conditions on a left ultrafunctor $G : X \to \mathrm{Set}$ are equivalent:

- (a) The left ultrafunctor G is an ultrafunctor. That is, for each map $f : S \to X$ and each ultrafilter μ on S, the left ultrastructure map $\sigma_\mu : G(\int_S f(s) d\mu) \to \int_S G(f(s)) d\mu$ is a bijection.
- (b) The sheaf $\mathscr{F}_G$ is locally constant with finite stalks.

We first show that (a) implies (b). Assume that G is an ultrafunctor. Fix a point $y \in X$, a set S, and an ultrafilter μ on S. Then the ultrapower diagonal map $\Delta_\mu : G(y) \to G(y)^\mu$ in the category Set (see Example 1.3.4) can be identified with the image under G of the ultrapower diagonal map $y \to y^\mu = y$ in X (regarded as a category having only identity morphisms). It follows that Δ_μ is bijective: that is, for every map $f : S \to G(y)$, there exists a subset $S_0 \subseteq S$ such that $\mu(S_0) = 1$ and $f|_{S_0}$ is constant. In the special case where $S = G(y)$ and f is the identity map, this implies that every ultrafilter on S is principal. It follows that $G(y)$ must be finite, so that (by virtue of Proposition 3.4.6) the stalk $\mathscr{F}_{G,y}$ is finite.

We now show that the sheaf $\mathscr{F}_G$ is constant in some neighborhood of any point $y \in X$. Let n be the cardinality of the stalk $\mathscr{F}_{G,y}$. Choose an open neighborhood U of the point y and a finite collection of sections $\phi_{1,\bullet}, \ldots, \phi_{n,\bullet} \in \mathscr{F}_G(U)$ having distinct images in $\mathscr{F}_{G,y}$. Let us identify each $\phi_{i,\bullet}$ with a tuple $\{\phi_{i,x} \in G(x)\}_{x \in U}$ satisfying condition $(*)$ of Construction 3.4.2. Note that, since G is an ultrafunctor, condition $(*)$ has the following consequence:

$(*')$ Let $f : S \to U$ be a map of sets and let μ be an ultrafilter on S such that $\int_S f(s) d\mu$ belongs to μ. If $\phi_{i,f(s)} = \phi_{j,f(s)}$ for all $s \in S$, then $\phi_{i,\int_S f(s)d\mu} = \phi_{j,\int_S f(s)d\mu}$.

Let $V \subseteq U$ be an open neighborhood of y whose closure $\overline{V}$ is contained in U. For $1 \leq i < j \leq n$, let $K_{i,j} \subseteq \overline{V}$ denote the subset consisting of those points x for which $\phi_{i,x} = \phi_{j,x}$. It follows from $(*')$ that each of the sets $K_{i,j}$ is closed. By construction, the sets $K_{i,j}$ do not contain the point y.

Let $K' \subseteq \overline{V}$ be the set of all points x for which the set $G(x)$ contains some element $\psi_x \notin \{\phi_{1,x}, \ldots, \phi_{n,x}\}$. Note that, for every ultrafilter μ on the set K', the image $\{\psi_x\}_{x \in K'}$ under the map

$$\prod_{x \in K'} G(x) \to \int_{K'} G(x) d\mu \xrightarrow{\sigma_\mu^{-1}} G(\int_{K'} x d\mu)$$

does not belong to the set $\{\phi_{1,\int_{K'} x d\mu}, \ldots, \phi_{n,\int_{K'} x d\mu}\}$. It follows that K' is a closed subset of $\overline{V}$, which also does not contain the point y.

Let $W \subseteq V$ be an open neighborhood of y which is disjoint from the closed sets K' and $K_{i,j}$. For each point $x \in W$, the elements $\phi_{1,x}, \ldots, \phi_{n,x}$ are pairwise distinct and exhaust $G(x)$. It follows from Proposition 3.4.6 that the germs of the sections $\phi_1, \ldots, \phi_n$ are pairwise distinct and exhaust the stalk $\mathscr{F}_{G,x}$. Consequently, they determine an isomorphism of $\mathscr{F}_G|_W$ is isomorphic with the constant sheaf associated to the finite set $\{1, 2, \ldots, n\}$. This completes the proof that (a) implies (b).

Now suppose that $\mathscr{F}_G$ is locally constant; we wish to show that G is an ultrafunctor. Choose a map $f : S \to X$ and an ultrafilter μ on S, and set $x = \int_S f(s) d\mu$. We must show that the left ultrastructure map $\sigma_\mu : G(x) = G(\int_S f(s) d\mu) \to \int_S G(f(s)) d\mu$ is bijective. Choose an open neighborhood U of x such that $\mathscr{F}_G|_U$ is isomorphic to the constant sheaf $\underline{J}_U$, for some finite set J. Let V be an open neighborhood of x whose closure is contained in U, and set $S_0 = f^{-1}(V)$. Since x belongs to V, we must have $\mu(S_0) = 1$. We may therefore replace S by S_0 and μ by its restriction to S_0, and thereby reduce to the case where the function f takes values in V. Replacing X by the compact set $\overline{V} \subseteq X$, we can reduce to the case where $\mathscr{F}_G = \underline{J}_X$ is itself the constant sheaf associated to a finite set J. In this case, we have an isomorphism of sheaves $\mathscr{F}_G \simeq \mathscr{F}_{G'}$ where $G' : X \to \mathrm{Set}$ is the constant ultrafunctor taking the value J. It follows from Theorem 3.4.4 that that G is isomorphic to G' (as a left ultrafunctor), and is therefore also an ultrafunctor. $\square$

3.5. **The Proof of Proposition 3.4.6.** Throughout this section, we fix a compact Hausdorff space X and a left ultrafunctor $G : X \to \mathrm{Set}$. Our goal is to compute the stalks of the sheaf $\mathscr{F}_G$ of Construction 3.4.2. We begin by establishing a weak version of Proposition 3.4.6.

Lemma 3.5.1. *For each point $y \in X$, the map $\mathscr{F}_{G,y} = \varinjlim_{y \in U} \mathscr{F}_G(U) \to G(y)$ of Proposition 3.4.6 is injective.*

Proof. Let U be an open neighborhood of the point y and suppose we are given a pair of elements $\phi_\bullet = \{\phi_x\}_{x \in U}$ and $\psi_\bullet = \{\psi_x\}_{x \in U}$ of $\mathscr{F}_G(U)$ satisfying $\phi_y = \psi_y$ (as elements of the set $G(y)$). We wish to show that there exists an open subset $V \subseteq U$ containing the point y such that $\phi_x = \psi_x$ for each $x \in V$. Assume otherwise. Set $U_0 = \{x \in U : \phi_x \neq \psi_x\} \subseteq U$, and let $\mathcal{U}$ be the collection of all subsets of U having the form $U_0 \cap V$, where V is an open neighborhood of y. Then $\mathcal{U}$ is closed under finite intersections and does not contain the empty set. Applying Proposition 1.1.10, we deduce that there exists an ultrafilter μ on the set U such that $\mu(U_0 \cap V) = 1$, whenever V is an open neighborhood of y. We then have $\int_U x d\mu = y$. Invoking our assumption that $\phi_y = \psi_y$ and the definition of the sheaf $\mathscr{F}_G$, we deduce that $\phi_\bullet$ and $\psi_\bullet$ have the same image in the ultraproduct $\int_U G(x) d\mu$. It follows that $\mu(U_0) = 0$, contradicting the definition of U_0. $\square$

To complete the proof of Proposition 3.4.6, we must show that each element $\varphi_y \in G(y)$ can be extended to an element $\phi_\bullet \in \mathscr{F}_G(U)$ for some open neighborhood U of the point y. The proof will require some auxiliary constructions.

Notation 3.5.2. Choose a set T equipped with a bijection $f_0 : T \to X$. We regard f_0 as a continuous bijection of topological spaces, where T is endowed with the discrete topology. We will use the letter ν to denote a typical ultrafilter on T: that is, a point in the Stone-Čech compactification βT. Let $i : T \hookrightarrow \beta T$ be the canonical embedding, which assigns to each point $t \in T$ the corresponding principal ultrafilter $i(t) = \delta_t$. By virtue of Proposition 3.2.7, the map $f_0 : T \to X$ extends uniquely to a continuous map $f : \beta T \to X$, given concretely by the formula $f(\nu) = \int_T f_0(t) d\nu$.

Let $\mathscr{G}_0$ denote the sheaf of sets on the (discrete) topological space T whose stalks are given by $\mathscr{G}_{0t} = G(f_0(t))$, and let $\mathscr{G}$ denote the direct image $i_* \mathscr{G}_0$. Then $\mathscr{G}$ is a sheaf of sets on βT whose value on closed and open sets is given by the formula

$$\mathscr{G}(\beta T_0) = \prod_{t \in T_0} G(f_0(t)),$$

and whose stalk at a point $\nu \in \beta T$ is given by the formula $\mathscr{G}_\nu \simeq \int_T G(f_0(t)) d\nu$.

Choose a set S and a bijection $g_0 : S \to \beta T$, which we identify with a collection of ultrafilters $\{\nu_s\}_{s \in S}$ on the set T. We regard g_0 as a continuous bijection of topological spaces, where S is equipped with the discrete topology and βT with the topology of Construction 3.2.2. We will use the letter μ to denote a typical ultrafilter on S: that is, a point in the Stone-Čech compactification βS. Let $j : S \hookrightarrow \beta S$ be the canonical embedding, which assigns to each point $s \in S$ the corresponding principal ultrafilter $j(s) = \delta_s$. Let $\mathscr{H}_0 = g_0^* \mathscr{G}$ denote the sheaf of sets on S whose stalks are given by $\mathscr{H}_{0s} = \int_T G(f_0(t)) d\nu_s$. Let $\mathscr{H}$ denote the sheaf of sets on βS given by the direct image $j_* \mathscr{H}_0$. Then $\mathscr{H}$ is given on closed and open sets by the formula

$$\mathscr{H}(\beta S_0) = \prod_{s \in S_0} \int_T G(f_0(t)) d\nu_s,$$

and its stalk at a point $\mu \in \beta S$ is given by $\mathscr{H}_\mu \simeq \int_S (\int_T G(f_0(t)) d\nu_s) d\mu$

By virtue of Proposition 3.2.7, the bijection $g_0 : S \to \beta T$ admits a unique continuous extension $g : \beta S \to \beta T$, given concretely by the formula $g(\mu) = \int_S \nu_s d\mu$ (see Example 3.2.8). We have a canonical isomorphism $j^* g^* \mathscr{G} \simeq g_0^* \mathscr{G} = \mathscr{H}_0$ which induces a map of sheaves $u : g^* \mathscr{G} \to j_* \mathscr{H}_0 = \mathscr{H}$. Concretely, the map u is given on stalks by the map

$$(g^* \mathscr{G})_\mu = \mathscr{G}_{g(\mu)} = \int_T G(f_0(t)) d(\int_S \nu_s d\mu) \xrightarrow{\Delta_{\lambda, \mu_\bullet}} \int_S (\int_T G(f_0(t) d\nu_s) d\mu \simeq \mathscr{H}_\mu,$$

where $\Delta_{\lambda, \mu_\bullet}$ is the categorical Fubini transformation in the category of sets.

Let $h_0 : S \to T$ be the map of sets given by the composition

$$S \xrightarrow{g_0} \beta T \xrightarrow{f} X \xrightarrow{f_0^{-1}} T \xrightarrow{i} \beta T$$

For each $s \in S$, the right ultrastructure on the functor G determines a map

$$\mathscr{G}_{(i \circ h_0)(s)} \simeq \mathscr{G}_{0 h_0(s)} \simeq G((f \circ g_0)(s)) = G(\int_T f_0(t) d\nu_s) \xrightarrow{\sigma_{\nu_s}} \int_T G(f_0(t)) d\nu_s \simeq \mathscr{H}_{0s}$$

Let $h : \beta S \to \beta T$ denote the unique continuous extension of h_0, given concretely by $h(\mu) = h_{0*}(\mu)$. The preceding discussion then gives a map of sheaves $j^* h^* \mathscr{G} \to \mathscr{H}_0$ on the discrete space S, which we identify with a map $v : h^* \mathscr{G} \to j_* \mathscr{H}_0 = \mathscr{H}$ on the topological space βS. Concretely, the map v is given on stalks by

the composition

$$\begin{array}{ccc} (h^*\mathscr{G})_\mu & \simeq & \int_T G(f_0(t))d(h_{0*}\mu) \\ & \xrightarrow{\Delta_{\mu,h_0}} & \int_S G((f_0\circ h_0)(s))d\mu \\ & = & \int_S G(\int_T f_0(t)d\nu_s)d\mu \\ & \xrightarrow{\int_S \sigma_{\nu_s} d\mu} & \int_S(\int_T G(f_0(t))d\nu_s)d\mu. \end{array}$$

Note that we have a commutative diagram of topological spaces

$$\beta S \overset{g}{\underset{h}{\rightrightarrows}} \beta T \xrightarrow{f} X;$$

that is, we have $f\circ g = f' = f\circ h$ for some continuous map $f' : \beta S \to X$, given concretely by the formula $f'(\mu) = \int_S(\int_T f_0(t)d\nu_s)d\mu$. The maps u and v therefore determine maps $u', v' : f_*(\mathscr{G}) \to f'_*(\mathscr{H})$ in the category $\mathrm{Shv}(X)$.

Lemma 3.5.3. *The sheaf $\mathscr{F}_G$ of Construction 3.4.2 can be identified with the equalizer of the maps $u', v' : f_*(\mathscr{G}) \rightrightarrows f'_*(\mathscr{H})$.*

Proof. By construction, the direct image sheaf $f_*\mathscr{G} \simeq f_{0*}\mathscr{G}_0$ is given by

$$(f_*\mathscr{G})(U) \simeq \prod_{x\in U} G(x).$$

Let $\mathscr{E}$ denote the equalizer of u' and v', regarded as a subsheaf of $f_*\mathscr{G}$. Unwinding the definitions, we see that for each subset $U \subseteq X$, we can identify $\mathscr{E}(U)$ with the subset of $\prod_{x\in U} G(x)$ consisting of those tuples $\{\phi_x\}_{x\in U}$ which satisfy the following condition:

- $(*')$ For every ultrafilter ν on T such that $\int_T f_0(t)d\nu$ belongs to U, the elements $\{\phi_{f_0(t)}\}_{t\in f_0^{-1}(U)}$ and $\phi_{\int_T f_0(t)d\nu}$ have the same image under the maps

$$\prod_{t\in f_0^{-1}(U)} G(f_0(t)) \xrightarrow{q_\nu^{f_0^{-1}(U)}} \int_T G(f_0(t))d\nu \xleftarrow{\sigma_\nu} G(\int_T f_0(t)d\nu),$$

  where σ_ν is given by the left ultrastructure on G.

Note that condition $(*')$ follows immediately from condition $(*)$ of Construction 3.4.2 (applied to the subset $T_0 = f^{-1}(U) \subseteq T$, and the ultrafilter on T_0 given by the restriction of ν). That is, we can regard the sheaf $\mathscr{F}_G$ of Construction 3.4.2 as a subsheaf of $\mathscr{E}$. To complete the proof, we must show that $(*')$ implies $(*)$. To this end, suppose we are given an arbitrary map of sets $e : R \to U$ and an ultrafilter λ on R satisfying $\int_R e(r)d\lambda \in U$; we wish to show that $\{\phi_{e(r)}\}_{r\in R}$ and $\phi_{\int_R e(r)d\lambda}$ have the same image under the maps

$$\prod_{r\in R} G(e(r)) \xrightarrow{q_\lambda} \int_R G(e(r))d\lambda \xleftarrow{\sigma_\lambda} G(\int_R e(r)d\lambda).$$

Since f_0 is bijective, the map e factors uniquely as a composition $R \xrightarrow{e'} T_0 \xrightarrow{f_0} X$. The desired assertion now follows by applying $(*)$ to the ultrafilter $e'_*\lambda$ on $T_0 = f_0^{-1}(U)$. □

Let us now fix a point $y \in X$ and an element ϕ_y of the set $G(y)$. For each ultrafilter ν on T satisfying $f(\nu) = \int_T f_0(t)d\nu = y$, let ψ_ν denote the image of ϕ_y under the map

$$G(y) = G(\int_T f_0(t)d\nu) \xrightarrow{\sigma_\mu} \int_T G(f_0(t))d\nu = \mathscr{G}_\nu.$$

Lemma 3.5.4. *Let μ be a point of βS satisfying $f'(\mu) = y$. Let $u_\mu : \mathscr{G}_{g(\mu)} \to \mathscr{H}_\mu$ and $v_\mu : \mathscr{G}_{h(\mu)} \to \mathscr{H}_\mu$ be the maps induced by u and v, respectively. Then $u_\mu(\psi_{g(\mu)}) = v_\mu(\psi_{h(\mu)})$.*

Proof. This follows from the definition of u and v, together with the commutativity of the diagram

$$\begin{tikzcd}
G(y) \arrow[r, "\sigma_{h_{0*}\mu}"] \arrow[d, equal] \arrow[drr, "\sigma_\mu"'] & \int_T G(f_0(t))d(h_{0*}\mu) \arrow[r, "\Delta_{\mu,h_0}"] & \int_S G((f_0 \circ h_0)(s))d\mu \arrow[d, equal] \\
G(\int_T f_0(t)d(\int_S \nu_s d\mu)) \arrow[dr, "\sigma_{\int_S \nu_s d\mu}"] & & \int_S G(\int_T f_0(t)d\nu_s)d\mu \arrow[d, "\int_S \sigma_{\nu_s} d\mu"] \\
& \int_T G(f_0(t))d(\int_S \nu_s d\mu) \arrow[r, "\Delta_{\mu,\nu_\bullet}"] & \int_S(\int_T G(f_0(t))d\nu_s)d\mu.
\end{tikzcd}$$

□

Lemma 3.5.5. *There exists a global section ψ of the sheaf $\mathscr{G}|_{f^{-1}\{y\}}$ such that, for each point $\nu \in f^{-1}\{y\}$, the germ of ψ at ν is equal to ψ_ν.*

Proof. Let $\overline{y}$ denote the unique element of T satisfying $f_0(\overline{y}) = y$, and let $K \subseteq \beta S$ denote the inverse image of $\delta_{\overline{y}}$ under the map $h : \beta S \to \beta T$. Then $g : \beta S \to \beta T$ restricts to a map of topological spaces $g_K : K \to f^{-1}\{y\}$. Then u restricts to a map of sheaves $u_K : g_K^* \mathscr{G}|_{f^{-1}\{y\}} \to \mathscr{H}|_K$ on K, which we can identify with a map $\iota : \mathscr{G}|_{f^{-1}\{y\}} \to (u_{K*}\mathscr{H}|_K)$ on the fiber $f^{-1}\{y\}$. We first claim that ι is injective. To prove this, it suffices to observe that for each point $\nu \in f^{-1}\{y\}$, there exists a point $\mu \in K$ for which $u_K(\mu) = \nu$ and u induces a monomorphism of stalks $u_\mu : \mathscr{G}_\mu \to \mathscr{H}_\nu$. In fact, writing $\nu = \nu_s$ for some $s \in S$, we can take $\mu = \delta_s$ to be the principal ultrafilter at the point s, in which case the map u_μ is bijective.

For each point $\nu \in f^{-1}\{y\}$, let ψ'_ν denote the image of ψ_ν in the stalk $(u_{K*}\mathscr{H}|_K)_\nu$ (under the map ι). Since ι is a monomorphism, it will suffice to show that the germs $\{\psi'_\nu\}_{\nu \in f^{-1}\{y\}}$ determine a global section of the sheaf $u_{K*}\mathscr{H}|_K$. Applying Proposition 0.0.10 to the map $u_K : K \to f^{-1}\{y\}$, this can be reformulated as follows:

- $(\star)$ There exists a global section ξ of the sheaf $\mathscr{H}|_K$ having the property that, for every point $\mu \in K$, the germ ξ_μ of ξ at the point μ is coincides with the image of $\psi_{g(\mu)}$ under the map $\mathscr{G}_{g(\mu)} \to \mathscr{H}_\mu$ determined by u_K.

Let $\underline{G(y)}_K$ denote the constant sheaf on K with the value $G(y)$, so that v restricts to a map of sheaves

$$v_K : \underline{G(y)}_K \to \mathscr{H}|_K.$$

The map v_K carries the element $\phi_y \in G(y)$ to a global section ξ the sheaf $\mathscr{H}|_K$. It follows from Lemma 3.5.4 that ξ satisfies the requirement of $(\star)$. □

Proof of Proposition 3.4.6. Let $\mathscr{G}(f^{-1}\{y\})$ denote the set of global sections of the sheaf $\mathscr{G}|_{f^{-1}\{y\}}$, and define $\mathscr{H}(f'^{-1}\{y\})$ similarly. Since f and f' are proper maps, we can identify $\mathscr{G}(f^{-1}\{y\})$ and $\mathscr{H}(f'^{-1}\{y\})$ with the stalks $(f_*\mathscr{G})_y$ and $(f'_*\mathscr{H})_y$, respectively. Using Lemma 3.5.3, we obtain an equalizer diagram of sets

$$\mathscr{F}_{G,y} \to \mathscr{G}(f^{-1}\{y\}) \rightrightarrows \mathscr{H}(f'^{-1}\{y\}).$$

For each element $\phi_y \in G(y)$, the section $\psi \in \mathscr{G}(f^{-1}\{y\})$ of Lemma 3.5.5 belongs to this equalizer (by Lemma 3.5.4), and can therefore be identified with an element of $\mathscr{F}_{G,y}$. It follows immediately from the construction that the canonical map $\theta : \mathscr{F}_{G,y} \to G(y)$ carries this element to ϕ_y, which shows that θ is surjective; injectivity was established in Lemma 3.5.1. □

4. Ultracategories as Stacks

Let Comp denote the category of compact Hausdorff spaces. In §3, we showed that every compact Hausdorff space X can be regarded as an ultracategory (Proposition 3.3.1), and that this observation determines a fully faithful embedding from the ordinary category Comp to the 2-category $\mathbf{Ult}^{\mathrm{L}}$ of Remark 1.4.6 (see Theorem 3.1.5). In this section, we will use embedding as a tool to analyze the entire category $\mathbf{Ult}^{\mathrm{L}}$. To any ultracategory $\mathcal{M}$, we can associate a presheaf of categories on Comp, given by the construction

$X \mapsto \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})$. This presheaf can be encoded by a fibration of categories $\mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}$, where $\mathrm{Comp}_{\mathcal{M}}$ is a category whose objects are pairs $(X, \mathcal{O}_X)$, with X a compact Hausdorff space and $\mathcal{O}_X : X \to \mathcal{M}$ a left ultrafunctor (see Construction 4.1.1). The main results of this section can be summarized as follows:

- Let $\mathcal{M}$ be an ultracategory. In §4.1, we show that $\mathrm{Comp}_{\mathcal{M}}$ is a stack (not in groupoids) with respect to the coherent topology on the category of compact Hausdorff spaces (Proposition 4.1.5) In other words, the construction $X \mapsto \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})$ satisfies descent with respect to continuous surjections of compact Hausdorff spaces $X \twoheadrightarrow Y$.
- Every ultracategory $\mathcal{M}$ can be identified with a full subcategory of $\mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$: namely, the full subcategory consisting of those pairs $(X, \mathcal{O}_X)$, where the topological space X has a single point. In §4.2, we show that this full subcategory has ultraproducts in $\mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$ and that the ultrastructure on $\mathcal{M}$ can be recovered by applying the construction of Proposition 1.3.7 (Theorem 4.2.7). In particular, this proves that every ultrastructure on a category $\mathcal{M}$ can be realized by applying the categorical ultraproduct construction in a suitable enlargement $\mathcal{M}^{+}$ of $\mathcal{M}$ (we will return to this idea in §8).
- The construction $\mathcal{M} \mapsto \mathrm{Comp}_{\mathcal{M}}$ determines a functor from the 2-category of ultracategories $\mathbf{Ult}^{\mathrm{L}}$ (with morphisms given by left ultrafunctors) to the 2-category of stacks on Comp. In §4.3, we show that this functor is fully faithful (Theorem 4.3.3). Consequently, it is possible to reformulate theory of ultracategories entirely in the language of topological stacks. Our interest in this result is more pragmatic: it provides a strategy for analyzing the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$ between two ultracategories $\mathcal{M}$ and $\mathcal{N}$. In §5, we will exploit this strategy to obtain more precise information in the special case where $\mathcal{N} = \mathrm{Set}$ is the category of sets, which we will ultimately use in the proof of Theorem 2.2.2.
- Let us say that an ultracategory $\mathcal{M}$ is an *ultragroupoid* if the underlying category of $\mathcal{M}$ is a groupoid: that is, if every morphism in $\mathcal{M}$ is invertible. For every ultragroupoid $\mathcal{M}$, we can regard $\mathrm{Comp}_{\mathcal{M}}$ as a stack *in groupoids* on the category of compact Hausdorff spaces. In §4.4, we show that a stack in groupoids arises in this way (for some small ultragroupoid $\mathcal{M}$) if and only if it is *representable*: that is, if and only if it arises from groupoid object in the category of compact Hausdorff spaces itself (Theorem 4.4.7). For example, the structure of the classifying stack BG of a compact topological group G can be encoded by a suitable ultrastructure on its underlying category (the category having a single object with automorphism group G); see Example 4.4.10.

4.1. **The Category** $\mathrm{Comp}_{\mathcal{M}}$.

Let X be a compact Hausdorff space, which we regard as an ultracategory having only identity morphisms (see Proposition 3.3.1). If $\mathcal{M}$ is another ultracategory, we will use the symbol $\mathcal{O}_X$ to denote a left ultrafunctor from X to $\mathcal{M}$. We denote the value of this functor at a point $x \in X$ by $\mathcal{O}_{X,x}$. Heuristically, it is useful to think of $\mathcal{O}_X$ as a sheaf on X with values in $\mathcal{M}$, whose stalk at a point $x \in X$ is given by $\mathcal{O}_{X,x}$. In the case where $\mathcal{M}$ is the category of sets, this heuristic is made precise by Theorem 3.4.4 (and Proposition 3.4.6).

Construction 4.1.1 (The Category $\mathrm{Comp}_{\mathcal{M}}$)**.** Let Comp denote the category whose objects are compact Hausdorff spaces and whose morphisms are continuous functions. Let $\mathcal{M}$ be an ultracategory. We define a category $\mathrm{Comp}_{\mathcal{M}}$ as follows:

(1) The objects of $\mathrm{Comp}_{\mathcal{M}}$ are pairs $(X, \mathcal{O}_X)$, where X is a compact Hausdorff space and $\mathcal{O}_X : X \to \mathcal{M}$ is a left ultrafunctor.
(2) If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are objects of $\mathrm{Comp}_{\mathcal{M}}$, then a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair (f, α), where $f : X \to Y$ is a continuous function and $\alpha : \mathcal{O}_Y \circ f \to \mathcal{O}_X$ is a natural transformation of left ultrafunctors (here we view f as an ultrafunctor via Theorem 3.1.5).
(3) The composition of a pair of morphisms
$$(X, \mathcal{O}_X) \xrightarrow{(f,\alpha)} (Y, \mathcal{O}_Y) \xrightarrow{(g,\beta)} (Z, \mathcal{O}_Z)$$
in the category $\mathrm{Comp}_{\mathcal{M}}$ is given by the pair $(g \circ f, \gamma)$, where γ is the natural transformation of left ultrafunctors given by
$$\mathcal{O}_Z \circ g \circ f \xrightarrow{\beta} \mathcal{O}_Y \circ f \xrightarrow{\alpha} \mathcal{O}_X .$$

Note that the construction $(X, \mathcal{O}_X) \mapsto X$ determines a forgetful functor $\mathrm{Comp}_{\mathcal{M}} \to \mathcal{M}$.

Remark 4.1.2. Let $\mathcal{M}$ be an ultracategory, let X be a compact Hausdorff space, and let $\mathcal{O}_X : X \to \mathcal{M}$ be a left ultrafunctor. If $f : Y \to X$ is a continuous map of compact Hausdorff spaces, then we can regard f as an ultrafunctor, so that the composition

$$Y \xrightarrow{f} X \xrightarrow{\mathcal{O}_X} \mathcal{M}$$

has the structure of a left ultrafunctor from Y to $\mathcal{M}$. We will denote this left ultrafunctor by $\mathcal{O}_X|_Y$ (this notation will be used primarily, but not exclusively, in the case where Y is a closed subset of X and f is the inclusion map), so that we have a canonical map $(Y, \mathcal{O}_X|_Y) \to (X, \mathcal{O}_X)$ in the category $\mathrm{Comp}_{\mathcal{M}}$ (given by the pair $(f, \mathrm{id}_{\mathcal{O}_X|_Y})$.

If f is a closed immersion, then this map enjoys the following universal property: for every object $(Z, \mathcal{O}_Z)$ in $\mathrm{Comp}_{\mathcal{M}}$, the induced map

$$\mathrm{Hom}_{\mathrm{Comp}_{\mathcal{M}}}((Z, \mathcal{O}_Z), (Y, \mathcal{O}_X|_Y)) \hookrightarrow \mathrm{Hom}_{\mathrm{Comp}_{\mathcal{M}}}((Z, \mathcal{O}_Z), (X, \mathcal{O}_X))$$

whose image is the collection of morphisms $(f, \alpha) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ such that $f(Z) \subseteq Y$.

Remark 4.1.3. Let $\mathcal{M}$ be an ultracategory. Then the forgetful functor $\pi : \mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}$ is a fibration of categories. By definition, it is obtained by applying the Grothendieck construction to the functor

$$\mathrm{Comp}^{\mathrm{op}} \to \mathcal{C}\mathrm{at} \qquad X \mapsto \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})^{\mathrm{op}}.$$

A morphism $(f, \alpha) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ in $\mathrm{Comp}_{\mathcal{M}}$ is π-Cartesian if and only if $\alpha : \mathcal{O}_X|_Y \to \mathcal{O}_Y$ is an isomorphism of left ultrafunctors.

Example 4.1.4. Let Y be a compact Hausdorff space, regarded as an ultracategory having only identity morphisms. Then the category Comp_Y of Construction 4.1.1 is equivalent to the full subcategory of $\mathrm{Top}_{/Y}$ spanned by those continuous maps $f : X \to Y$, where X is a compact Hausdorff space; see Theorem 3.1.5.

The category Comp of compact Hausdorff spaces is a pretopos. Consequently, we can regard Comp as equipped with the coherent topology of Definition B.5.3. By definition, a collection of maps $\{f_i : X_i \to X\}_{i \in I}$ is a covering for this topology if and only if there exists some subset $I_0 \subseteq I$ for which the induced map $\coprod_{i \in I_0} X_i \to X$ is surjective. Our goal in this section is to prove the following:

Proposition 4.1.5. *Let $\mathcal{M}$ be an ultracategory. Then the fibration $\mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}$ is a stack with respect to the coherent topology on* Comp*. In other words, the construction $X \mapsto \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})$ satisfies (effective) descent for the coherent topology.*

We begin by showing that the fibration $\mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}$ is a stack for the extensive topology. Concretely, this reduces to the following assertion:

Lemma 4.1.6. *Let X be a compact Hausdorff space which is given as a disjoint union of finitely many closed (and open) subspaces $\{X_i \subseteq X\}_{i \in I}$. Let $\mathcal{M}$ be an ultracategory. Then the construction $\mathcal{O}_X \mapsto \{\mathcal{O}_X|_{X_i}\}_{i \in I}$ induces an equivalence of categories*

$$\theta : \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M}) \to \prod_{i \in I} \mathrm{Fun}^{\mathrm{LUlt}}(X_i, \mathcal{M}).$$

Remark 4.1.7. Lemma 4.1.6 can be generalized. Given any finite collection of ultracategories $\{\mathcal{N}_i\}_{i \in I}$, one can equip the disjoint union $\coprod_{i \in I} \mathcal{N}_i$ with the structure of an ultracategory. This ultracategory is then a coproduct of the collection $\{\mathcal{N}_i\}_{i \in I}$ in the 2-category $\mathbf{Ult}^{\mathrm{L}}$ of Remark 1.4.6): that is, for every ultracategory $\mathcal{M}$, we have an equivalence

$$\mathrm{Fun}^{\mathrm{LUlt}}(\coprod_{i \in I} \mathcal{N}_i, \mathcal{M}) \simeq \prod_{i \in I} \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{N}_i, \mathcal{M}).$$

Since we will not need this more general result, we leave the details to the reader.

Proof of Lemma 4.1.6. We first show that the functor θ is fully faithful. Suppose we are given a pair of left ultrafunctors $\mathcal{O}_X, \mathcal{O}'_X : X \to \mathcal{M}$, with left ultrastructures given by $\{\sigma_\mu\}$ and $\{\sigma'_\mu\}$, and a natural transformation of functors $\alpha : \mathcal{O}_X \to \mathcal{O}'_X$. For each $i \in I$, set $\mathcal{O}_{X_i} = \mathcal{O}_X|_{X_i}$, $\mathcal{O}'_{X_i} = \mathcal{O}'_X|_{X_i}$, and let $\alpha_i : \mathcal{O}_{X_i} \to \mathcal{O}'_{X_i}$ be the induced map. We must show that if each α_i is a natural transformation of left ultrafunctors, then α is also a natural transformation of left ultrafunctors. To prove this, let $f : S \to X$ be a map of sets and let μ be an ultrafilter on S. For each $i \in I$, let $S_i = f^{-1}(X_i)$. Since μ is an ultrafilter, there is a unique index $i \in I$ such that $\mu(S_i) = 1$. Let $u : S_i \hookrightarrow S$ denote the inclusion map, so that we can write $\mu = u_*(\mu_i)$ for some ultrafilter μ_i on S_i. Then μ restricts to an ultrafilter μ_i on the set S_i. We wish to prove the commutativity of the "back face" of the cubical diagram

$$\begin{tikzcd}
\mathcal{O}_{X,\int_S f(s)d\mu} \arrow[rr, "\sigma_\mu"] \arrow[dr, equal] \arrow[dd, "\alpha"] & & \int_S \mathcal{O}_{X,f(s)}\,d\mu \arrow[dr, "\Delta_{\mu_i,u}", "\sim"'] \arrow[dd, "\int_S \alpha d\mu" near end] & \\
& \mathcal{O}_{X_i,\int_{S_i} f(s)d\mu_i} \arrow[rr, "\sigma_\mu" near start] \arrow[dd, "\alpha_i" near start] & & \int_{S_i} \mathcal{O}_{X_i,f(s)}\,d\mu_i \arrow[dd, "\int_{S_i} \alpha d\mu_i"] \\
\mathcal{O}'_{X,\int_S f(s)d\mu} \arrow[rr, "\sigma'_\mu" near start] \arrow[dr, equal] & & \int_S \mathcal{O}'_{X,f(s)}\,d\mu \arrow[dr, "\Delta_{\mu_i,u}", "\sim"'] & \\
& \mathcal{O}'_{X_i,\int_{S_i} f(s)d\mu_i} \arrow[rr, "\sigma'_\mu"] & & \int_{S_i} \mathcal{O}'_{X_i,f(s)}\,d\mu_i,
\end{tikzcd}$$

where $\Delta_{\mu_i,u}$ denotes the ultraproduct diagonal map of Notation 1.3.3 (which is an isomorphism, since u is injective and $\mathcal{M}$ satisfies axiom (B) of Definition 1.3.1). We now observe that the left face commutes by construction, the right face by the functoriality of ultraproducts in $\mathcal{M}$, the top face because σ_μ is a left ultrastructure, the bottom face because σ'_μ is a left ultrastructure, and the front face because α_i is a natural transformation of left ultrafunctors.

We now prove that θ is essentially surjective. Suppose that we are given functors $\mathcal{O}_{X_i} : X_i \to \mathcal{M}$ equipped with left ultrastructures $\{\sigma^i_\mu\}$ for each $i \in I$. Let $\mathcal{O}_X : X \to \mathcal{M}$ be the amalgam of the functors $\mathcal{O}_{X_i}$. For every map of sets $f : S \to X$ and every ultrafilter μ on S, define S_i and μ_i as above. Then there is a unique map $\sigma_\mu : \mathcal{O}_{X,\int_S f(s)d\mu} \to \int_S \mathcal{O}_{X,f(s)}\,d\mu$ for which the diagram

$$\begin{tikzcd}
\mathcal{O}_{X,\int_S f(s)d\mu} \arrow[r, "\sigma_\mu"] \arrow[d, equal] & \int_S \mathcal{O}_{X,f(s)}\,d\mu \arrow[d, "\Delta_{\mu_i,u}", "\sim"'] \\
\mathcal{O}_{X_i,\int_{S_i} f(s)d\mu_i} \arrow[r, "\sigma^i_{\mu_i}"] & \int_{S_i} \mathcal{O}_{X_i,f(s)}\,d\mu_i
\end{tikzcd}$$

is commutative. Note that, if the function f factors through X_i, then $\Delta_{\mu_i,u}$ is the identity map (Corollary 1.3.6), so that $\sigma_\mu = \sigma^i_\mu$. We will complete the proof by showing that $\{\sigma_\mu\}$ is a left ultrastructure on $\mathcal{O}_X$: that is, it satisfies the axioms of Definition 1.4.1. Axiom (0) is immediate from the construction. To verify (1), suppose that $\mu = \delta_{s_0}$ is a principal ultrafilter, so that $f(s_0) \in X_i$ for some index $i \in I$. Then the ultrafilter μ_i appearing in the above construction is also the principal ultrafilter δ_{s_0} (on the set S_i). We then have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{X,f(s_0)} \arrow[r, "\sigma_\mu"] \arrow[d, equal] & \int_S \mathcal{O}_{X,f(s)}\,d\delta_{s_0} \arrow[r, "\epsilon_{S,s_0}"] \arrow[d, "\Delta_{\mu_i,u}"] & \mathcal{O}_{X,f(s_0)} \arrow[d, equal] \\
\mathcal{O}_{X_i,f(s_0)} \arrow[r, "\sigma^i_{\mu_i}"] & \int_{S_i} \mathcal{O}_{X_i,f(s)}\,d\delta_{s_0} \arrow[r, "\epsilon_{S_i,s_0}"] & \mathcal{O}_{X_i,f(s_0)}
\end{tikzcd}$$

where the bottom horizontal composition is the identity (since $\mathcal{O}_{X_i}$ is a left ultrafunctor), so the upper horizontal composition is the identity as well.

To verify axiom (2), suppose we are given a map $f : T \to X$, a family of ultrafilters $\{\nu_s\}_{s\in S}$ on T, and an ultrafilter μ on X. Then there is a unique index $i \in I$ such that $(\int_S \nu_s d\mu)(f^{-1}X_i) = 1$. Set $T_i = f^{-1}(X_i)$ and $S_i = \{s \in S : \int_T f(t)d\nu_s \in X_i\}$. Then μ restricts to an ultrafilter μ_i on S_i, and for each $s \in S_i$ the ultrafilter ν_s restricts to an ultrafilter ν_{si} on T_i. Let $u : S_i \hookrightarrow S$ and $v : T_i \hookrightarrow T$ be the inclusion maps. We wish to show the commutativity of the outer cycle in the diagram

$$\begin{tikzcd}
\mathcal{O}_{X,\int_T f(t)d(\int_S \nu_s d\mu} \arrow[rrr, equal] \arrow[dr, equal] \arrow[dddd, "\sigma_{\int_S \nu_s d\mu}"] & & & \mathcal{O}_{X,\int_S(\int_T f(t)d\nu_s)d\mu} \arrow[dl, equal] \arrow[dd, "\sigma_\mu"] \\
& \mathcal{O}_{X_i,\int_{T_i} f(t)d(\int_{S_i} \nu_{si}d\mu_i} \arrow[r, equal] \arrow[dd, "\sigma^i_{\int_{S_i}\nu_{si}d\mu_i}"] & \mathcal{O}_{X_i,\int_{S_i}(\int_{T_i} f(t)d\nu_{si})d\mu_i} \arrow[d, "\sigma^i_{\mu_i}"] & \\
& & \int_{S_i} \mathcal{O}_{X_i,\int_{T_i} f(t)d\nu_{si}}\, d\mu_i \arrow[d, "\int_{S_i}\sigma^i_{\nu_{si}}d\mu_i"] & \int_S \mathcal{O}_{X,\int_T f(t)d\nu_s}\, d\mu \arrow[l, "\sim"'] \arrow[dd, "\int_S \sigma_{\nu_s} d\mu"] \\
& \int_{T_i} \mathcal{O}_{X_i,f(t)}\, d(\int_{S_i} \nu_{si} d\mu_i) \arrow[r] & \int_{S_i}(\int_{T_i} \mathcal{O}_{X_i,f(t)}\, d\nu_{si})d\mu_i & \\
\int_T \mathcal{O}_{X,f(t)}\, d(\int_S \nu_s d\mu) \arrow[ur, "\Delta_{\int_{S_i}\nu_{si}d\mu_i,v}"] \arrow[rrr, "\Delta_{\mu,\nu_\bullet}"] & & & \int_S(\int_T \mathcal{O}_{X,f(t)}\, d\nu_s)d\mu. \arrow[ul, "\sim"']
\end{tikzcd}$$

This follows from a diagram chase; note that the inner cycle commutes by virtue of our assumption that $\{\sigma^i_\mu\}$ is a left ultrastructure on $\mathcal{O}_{X_i}$. □

Remark 4.1.8. Let $\mathcal{M}$ be an ultracategory. Then every finite collection of objects $\{(X_i, \mathcal{O}_{X_i})\}_{i\in I}$ admits a coproduct $(X, \mathcal{O}_X)$ in $\mathrm{Comp}_{\mathcal{M}}$, where $X = \coprod_{i\in I} X_i$ is the disjoint union of the underlying topological spaces X_i and $\mathcal{O}_X : X \to \mathcal{M}$ is the unique left ultrafunctor whose restriction to each subset $X_i \subseteq X$ coincides with $\mathcal{O}_{X_i}$ (Lemma 4.1.6).

Let us now make Proposition 4.1.5 more explicit.

Notation 4.1.9. Let $p : X \twoheadrightarrow \overline{X}$ be a continuous surjection of compact Hausdorff spaces and let $\mathcal{M}$ be an ultracategory. Let $\mathcal{O}_X : X \to \mathcal{M}$ be a left ultrafunctor (with left ultrastructure $\{\sigma_\mu\}$). A *descent datum* for $\mathcal{O}_X$ consists of a collection of isomorphisms $\phi_{x,y} : \mathcal{O}_{X,y} \to \mathcal{O}_{X,x}$, defined for all $(x, y) \in X \times_{\overline{X}} X$, with the following properties:

(i) The isomorphisms $\phi_{\bullet,\bullet}$ comprise a natural transformation of left ultrafunctors from $X \times_{\overline{X}} X$ to $\mathcal{M}$. In other words, for every pair of maps $f, f' : S \to X$ satisfying $p \circ f = p \circ f'$ and every ultrafilter μ on S, the diagram

$$\begin{tikzcd}
\mathcal{O}_{X,\int_S f'(s)d\mu} \arrow[r, "\sigma_\mu"] \arrow[d, "\phi_{\int_S f(s)d\mu,\int_S f'(s)d\mu}"] & \int_S \mathcal{O}_{X,f'(s)}\, d\mu \arrow[d, "\int_S \phi_{f(s),f'(s)}d\mu"] \\
\mathcal{O}_{X,\int_S f(s)d\mu} \arrow[r, "\sigma_\mu"] & \int_S \mathcal{O}_{X,f(s)}\, d\mu
\end{tikzcd}$$

commutes.

(ii) For every triple $(x, y, z) \in X \times_{\overline{X}} X \times_{\overline{X}} X$, we have the cocycle identity $\phi_{x,z} = \phi_{x,y} \circ \phi_{y,z}$.

If $\mathcal{O}_X, \mathcal{O}'_X : X \to \mathcal{M}$ are left ultrafunctors equipped with descent data $\{\phi_{x,y}\}$ and $\{\phi'_{x,y}\}$, then we will say that a natural transformation of left ultrafunctors $\alpha : \mathcal{O}_X \to \mathcal{O}'_X$ is *compatible* with $\{\phi_{x,y}\}$ and $\{\phi'_{x,y}\}$ if, for every pair $(x,y) \in X \times_{\overline{X}} X$, the diagram

$$\begin{tikzcd} \mathcal{O}_{X,y} \arrow[r, "\phi_{x,y}"] \arrow[d, "\alpha"] & \mathcal{O}_{X,x} \arrow[d, "\alpha"] \\ \mathcal{O}'_{X,y} \arrow[r, "\phi'_{x,y}"] & \mathcal{O}_{X,y} \end{tikzcd}$$

is commutative. We let $\mathrm{Fun}^{\mathrm{LUlt}}(X \twoheadrightarrow \overline{X}, \mathcal{M})$ denote the category whose objects are left ultrafunctors on X with descent data, and whose morphisms are natural transformations of left ultrafunctors that are compatible with descent data.

Example 4.1.10. Let $p : X \twoheadrightarrow \overline{X}$ be a continuous surjection of compact Hausdorff spaces. If $\mathcal{O}_{\overline{X}} : \overline{X} \to \mathcal{M}$ is a left ultrafunctor, then we can endow the composite left ultrafunctor $\mathcal{O}_X = \mathcal{O}_{\overline{X}} \circ p$ with a descent datum, where for each $(x,y) \in X \times_{\overline{X}} X$ we take the isomorphism $\phi_{x,y}$ to be the identity map from

$$\mathcal{O}_{X,x} = \mathcal{O}_{\overline{X},p(x)} = \mathcal{O}_{\overline{X},p(y)} = \mathcal{O}_{X,y}$$

to itself. This construction determines a functor $\mathrm{Fun}^{\mathrm{LUlt}}(\overline{X}, \mathcal{M}) \to \mathrm{Fun}^{\mathrm{LUlt}}(X \twoheadrightarrow \overline{X}, \mathcal{M})$.

Proposition 4.1.5 follows from Lemma 4.1.6 together with the following:

Proposition 4.1.11. *Let $p : X \twoheadrightarrow \overline{X}$ be a continuous surjection of compact Hausdorff spaces and let $\mathcal{M}$ be an ultracategory. Then the construction of Example 4.1.10 induces an equivalence of categories*

$$\theta : \mathrm{Fun}^{\mathrm{LUlt}}(\overline{X}, \mathcal{M}) \to \mathrm{Fun}^{\mathrm{LUlt}}(X \twoheadrightarrow \overline{X}, \mathcal{M}).$$

Proof. We first show that θ is fully faithful. Suppose that $\mathcal{O}_{\overline{X}}, \mathcal{O}_{\overline{X}'} : \overline{X} \to \mathcal{M}$ are functors equipped with left ultrastructures $\{\sigma_\mu\}$ and $\{\sigma'_\mu\}$, and let $\mathcal{O}_X, \mathcal{O}'_X$ denote their images under the functor θ. Let $\alpha : \mathcal{O}_X \to \mathcal{O}'_X$ be a natural transformation of left ultrafunctors which is compatible with descent data. For each point $\overline{x} \in \overline{X}$, we can choose a point $x \in X$ such that $p(x) = \overline{x}$. In this case, α determines a map

$$\overline{\alpha}(\overline{x}) : \mathcal{O}_{\overline{X},\overline{x}} = \mathcal{O}_{X,x} \xrightarrow{\alpha(x)} \mathcal{O}'_{X,x} = \mathcal{O}_{\overline{X},\overline{x}}.$$

The condition that $\overline{\alpha}$ is compatible with descent data guarantees that this map is independent of the choice of x, and therefore defines a natural transformation $\overline{\alpha} : \mathcal{O}_{\overline{X}} \to \mathcal{O}'_{\overline{X}}$. We wish to show that $\overline{\alpha}$ is a natural transformation of left ultrafunctors: that is, for every map of sets $\overline{f} : S \to \overline{X}$ and every ultrafilter μ on S, the diagram τ :

$$\begin{tikzcd} \mathcal{O}_{\overline{X},\int_S \overline{f}(s)d\mu} \arrow[r, "\sigma_\mu"] \arrow[d, "\overline{\alpha}"] & \int_S \mathcal{O}_{\overline{X},\overline{f}(s)}\, d\mu \arrow[d, "\int_S \overline{\alpha} d\mu"] \\ \mathcal{O}'_{\overline{X},\int_S \overline{f}(s)d\mu} \arrow[r, "\sigma'_\mu"] & \int_S \mathcal{O}'_{\overline{X},\overline{f}(s)}\, d\mu \end{tikzcd}$$

is commutative. Since p is surjective, we can write $\overline{f} = p \circ f$ for some map $f : S \to X$. We can then identify τ with the diagram

$$\begin{tikzcd} \mathcal{O}_{X,\int_S f(s)d\mu} \arrow[r, "\sigma_\mu"] \arrow[d, "\alpha"] & \int_S \mathcal{O}_{X,f(s)}\, d\mu \arrow[d, "\int_S \alpha d\mu"] \\ \mathcal{O}'_{X,\int_S f(s)d\mu} \arrow[r, "\sigma'_\mu"] & \int_S \mathcal{O}'_{X,f(s)}\, d\mu, \end{tikzcd}$$

which commutes because α is a natural transformation of left ultrafunctors.

We now prove that θ is essentially surjective. Let $\mathcal{O}_X : X \to \mathcal{M}$ be a left ultrafunctor equipped with a descent datum $\{\phi_{x,y}\}$. We wish to show that the pair $(\mathcal{O}_X, \{\phi_{x,y}\})$ belongs to the essential image of θ.

Note that, if we ignore the left ultrastructure on $\mathcal{O}_X$, then the descent datum $\{\phi_{x,y}\}$ allows us to choose an isomorphism $\mathcal{O}_X \simeq \mathcal{O}_{\overline{X}} \circ p$, for some functor $\mathcal{O}_{\overline{X}}$. We may therefore assume without loss of generality that $\mathcal{O}_X = \mathcal{O}_{\overline{X}} \circ p$ for some functor $\mathcal{O}_{\overline{X}} : \overline{X} \to \mathcal{M}$, and that each of the isomorphisms $\phi_{x,y}$ is the identity map; in this case, we wish to show that the left ultrastructure $\{\sigma_\mu\}$ on $\mathcal{O}_X$ arises from a left ultrastructure $\{\overline{\sigma}_\mu\}$ on the functor $\mathcal{O}_{\overline{X}}$. For any map of sets $\overline{f} : S \to \overline{X}$ and any ultrafilter μ on S, we can write $\overline{f} = p \circ f$ for some map $f : S \to X$; we then define $\overline{\sigma}_\mu$ to be the map

$$\mathcal{O}_{\overline{X}, \int_S \overline{f}(s) d\mu} = \mathcal{O}_{X, \int_S f(s) d\mu} \xrightarrow{\sigma_\mu} \int_S \mathcal{O}_{X, f(s)}\, d\mu = \int_S \mathcal{O}_{\overline{X}, \overline{f}(s)}\, d\mu.$$

Since the descent datum $\{\phi_{x,y} = \mathrm{id}\}$ satisfies condition (i) of Notation 4.1.9, this map does not depend on the choice of f. The fact that the maps $\{\overline{\sigma}_\mu\}$ satisfy axioms (0), (1), and (2) of Definition 1.4.1 follows immediately from the assumption that the same axioms hold for $\{\sigma_\mu\}$. □

4.2. **Free Objects of** $\mathrm{Comp}_{\mathcal{M}}$**.** In §1.3, we introduced the notion of an *ultrastructure* on a category $\mathcal{M}$ (Definition 1.3.1). Our definition of ultrastructure was intended to axiomatize the essential features of the categorical ultraproducts studied in §1.2: whenever $\mathcal{M}$ has ultraproducts in some larger category $\mathcal{M}^+$, the formation of ultraproducts determines an ultrastructure on $\mathcal{M}$ (Proposition 1.3.7). It is natural to ask the following:

Question 4.2.1. Let $\mathcal{M}$ be a category. Can every ultrastructure on $\mathcal{M}$ be obtained by embedding $\mathcal{M}$ into some larger category $\mathcal{M}^+$, such that $\mathcal{M}$ has ultraproducts in $\mathcal{M}^+$?

In this section, we give an affirmative answer to Question 4.2.1. To every ultracategory $\mathcal{M}$, we show that there is a *canonical* way to recover the ultrastructure via categorical ultraproducts in a larger category $\mathcal{M}^+$: namely, we can take $\mathcal{M}^+$ to be the opposite of the category $\mathrm{Comp}_{\mathcal{M}}$ of Construction 4.1.1 (Theorem 4.2.7).

Remark 4.2.2. For every ultracategory $\mathcal{M}$, there exists an embedding $\mathcal{M} \hookrightarrow \mathcal{M}^+$ satisfying the requirement of Question 4.2.1. However, the enlargement $\mathcal{M}^+$ is not uniquely determined. Moreover, the construction of this section is not the most efficient: to recover the ultrastructure on $\mathcal{M}$, we do not need to use the *entire* category $\mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$. We will return to this point in §8.

We begin by observing that every ultracategory $\mathcal{M}$ can be identified with the full subcategory of $\mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$ spanned by those pairs $(X, \mathcal{O}_X)$, where X consists of a single point. To see this, we need the following general fact about left ultrafunctors:

Proposition 4.2.3. *Let $X = \{x\}$ be a one-point space. Then, for every ultracategory $\mathcal{M}$, the evaluation functor*

$$\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M}) \to \mathcal{M} \qquad F \mapsto F(x)$$

is an equivalence of categories.

We postpone the proof of Proposition 4.2.3 for the moment.

Remark 4.2.4. Let $\mathcal{M}$ be an ultracategory. Proposition 4.2.3 implies that for every object $M \in \mathcal{M}$, there is a unique left ultrastructure $\{\sigma_\mu\}$ on the functor

$$F : X = \{x\} \to \mathcal{M} \qquad F(x) = M.$$

Concretely, this left ultrastructure associates to each set S and each ultrafilter μ on S a map

$$M = F(\int_S x d\mu) \xrightarrow{\sigma_\mu} \int_S F(x) d\mu = M^\mu,$$

which is given by the ultrapower diagonal of Example 1.3.4. Beware that this morphism is generally not invertible. In other words, there is generally no *ultrafunctor* $\{x\} \to \mathcal{M}$ taking the value M. For example, if $\mathcal{M} = \mathrm{Set}$ is the category of sets, then the left ultrafunctor F is an ultrafunctor if and only if M is finite.

Notation 4.2.5. Let $\mathcal{M}$ be an ultracategory. For each object $M \in \mathcal{M}$, we let $\underline{M}$ denote the object of $\mathrm{Comp}_{\mathcal{M}}$ given by the pair $(*, \mathcal{O})$, where $*$ denotes the one-point space and $\mathcal{O} : * \to \mathcal{M}$ is the unique left ultrafunctor taking the value M.

Remark 4.2.6. Let $\mathcal{M}$ be an ultracategory, let M be an object of $\mathcal{M}$, and let $(X, \mathcal{O}_X)$ be any object of $\mathrm{Comp}_{\mathcal{M}}$. Every morphism $\underline{M} \to (X, \mathcal{O}_X)$ in the category $\mathrm{Comp}_{\mathcal{M}}$ determines a point $x \in X$ together with a morphism $\mathcal{O}_{X,x} \to M$ in the category $\mathcal{M}$. It follows from Proposition 4.2.3 that this construction induces a bijection

$$\mathrm{Hom}_{\mathrm{Comp}_{\mathcal{M}}}(\underline{M}, (X, \mathcal{O}_X)) \to \coprod_{x \in X} \mathrm{Hom}_{\mathcal{M}}(\mathcal{O}_{X,x}, M).$$

In particular, the construction $M \mapsto \underline{M}$ determines a fully faithful embedding $\mathcal{M} \to \mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$.

We can now state the main result of this section.

Theorem 4.2.7. *Let $\mathcal{M}$ be an ultracategory and let $\mathcal{M}' \subseteq \mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$ denote the full subcategory spanned by those objects $(X, \mathcal{O}_X)$, where X consists of a single point. Then:*

- (a) *The category $\mathcal{M}'$ has ultraproducts in $\mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$, in the sense of Construction 1.2.2).*
- (b) *The construction $M \mapsto \underline{M}$ can be promoted to an equivalence of ultracategories $\mathcal{M} \to \mathcal{M}'$ (where $\mathcal{M}'$ is equipped with the ultrastructure of Proposition 1.3.7).*

The main ingredient in the proof of Theorem 4.2.7 is the fact that objects of the form $\underline{M}$ admit products in $\mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$ (or, equivalently, coproducts in the category $\mathrm{Comp}_{\mathcal{M}}$). We now construct these.

Proposition 4.2.8. *Let $\mathcal{M}$ be an ultracategory and let $\{M_t\}_{t \in T}$ be a collection of objects of $\mathcal{M}$ indexed by a set T. Let $\mathcal{O}_{\beta T} : \beta T \to \mathcal{M}$ be the functor given by $\mathcal{O}_{\beta T, \nu} = \int_T M_t d\nu$. Then $\mathcal{O}_{\beta T}$ admits a left ultrastructure $\{\sigma_\mu\}$, which assigns to each map of sets $S \to \beta T$ given by a family of ultrafilters $\nu_\bullet = \{\nu_s\}_{s \in S}$ and each ultrafilter μ on S the map*

$$\sigma_\mu : \mathcal{O}_{\beta T, \int_S \nu_s d\mu} = \int_T M_t d(\int_S \nu_s d\mu) \xrightarrow{\Delta_{\mu, \nu_\bullet}} \int_S (\int_T M_t d\nu_s) d\mu = \int_S \mathcal{O}_{\beta T, \nu_s} d\mu$$

determined by the Fubini transformation $\Delta_{\mu, \nu_\bullet}$.

Proof. We must show that the maps $\{\sigma_\mu\}$ satisfy conditions (0), (1), and (2) of Definition 1.4.1. Condition (0) is vacuous (since X has only identity morphisms). To verify condition (1), suppose we are given a collection of ultrafilters $\nu_\bullet = \{\nu_s\}_{s \in S}$ on the set T and an element $s_0 \in S$; we wish to show that the diagram

$$\begin{tikzcd}
\int_T M_t d(\int_S M_s d\delta_{s_0}) \arrow[rr, "\Delta_{\delta_{s_0}, \nu_\bullet}"] \arrow[dr, equal] & & \int_S (\int_T N_t d\nu_s) d\delta_{s_0} \arrow[dl, "\epsilon_{S, s_0}"] \\
& \int_T M_t d\nu_{s_0} &
\end{tikzcd}$$

commutes in the category $\mathcal{M}$, which is axiom (A) of Definition 1.3.1. To verify condition (2), we must show that for every collection $\nu_\bullet = \{\nu_s\}_{s \in S}$ of ultraflters on T, every collection $\mu_\bullet = \{\mu_r\}_{r \in R}$ of ultrafilters on S, and every ultrafilter λ on R, we have a commutative diagram

$$\begin{tikzcd}
\int_T M_t d(\int_S \nu_s d(\int_R \mu_r d\lambda)) \arrow[rr, "\Delta_{\int_R \mu_r d\lambda, \nu_\bullet}"] \arrow[d, equal] & & \int_S (\int_T M_t d\nu_s) d(\int_R \mu_r d\lambda) \arrow[d, "\Delta_{\lambda, \mu_\bullet}"] \\
\int_T M_t d(\int_R (\int_S \nu_s d\mu_r) d\lambda) \arrow[r, "\Delta_{\lambda, \int_S \nu_s d\mu_\bullet}"] & \int_R (\int_T M_t d(\int_S \nu_s d\mu_r)) d\lambda \arrow[r, "\int_R \Delta_{\mu_r, \nu_\bullet} d\lambda"] & \int_R (\int_S (\int_T N_t d\nu_s) d\mu_r) d\lambda,
\end{tikzcd}$$

which is axiom (C) of Definition 1.3.1. $\square$

In the situation of Proposition 4.2.8, the left ultrafunctor $\mathcal{O}_{\beta T}$ has a universal property.

Proposition 4.2.9. *Let $\mathcal{M}$ be an ultracategory, let $\{M_t\}_{t \in T}$ be a collection of objects of $\mathcal{M}$, and let $\mathcal{O}_{\beta T}$: $\beta T \to \mathcal{M}$ be the left ultrafunctor of Proposition 4.2.8. Let $\mathscr{F} : \beta T \to \mathcal{M}$ be any left ultrafunctor. For any natural transformation of left ultrafunctors $\alpha : \mathscr{F} \to \mathcal{O}_{\beta T}$ and any element $t \in T$, let α_t denote the composite map*

$$\mathscr{F}_{\delta_t} \xrightarrow{\alpha} \mathcal{O}_{\beta T}(\delta_t) = \int_T M_{t'} d\delta_t \xrightarrow{\epsilon_{T,t}} M_t.$$

Then the construction $\alpha \mapsto \{\alpha_t\}_{t\in T}$ induces a bijection

$$\theta : \mathrm{Hom}_{\mathrm{Fun}^{\mathrm{LUlt}}(\beta T, \mathcal{M})}(\mathscr{F}, \mathcal{O}_{\beta T}) \to \prod_{t \in T} \mathrm{Hom}_{\mathcal{M}}(\mathscr{F}_{\delta_t}, M_t).$$

Proof. Let $\{\sigma_\mu\}$ denote the left ultrastructure on $\mathcal{O}_{\beta T}$ constructed in Proposition 4.2.8, and let $\{\sigma'_\mu\}$ denote the left ultrastructure on $\mathscr{F}$. Let $\alpha : \mathscr{F} \to \mathcal{O}_{\beta T}$ be a natural transformation of left ultrafunctors. For each ultrafilter ν on T, we can write $\nu = \int_T \delta_t d\nu$, so that we have a commutative diagram

$$\begin{tikzcd}
\mathscr{F}_{\int_T \delta_t d\nu} \arrow[r, "\sigma'_\nu"] \arrow[d, "\alpha(\nu)"] & \int_T \mathscr{F}_{\delta_t} d\nu d\nu \arrow[d, "\int_T \alpha(\delta_t)"] \arrow[dr, "\int_T \alpha_t d\nu"] & \\
\mathcal{O}_{\beta T, \nu} \arrow[r, "\sigma_\nu"] & \int_T \mathcal{O}_{\beta T, \delta_t} d\nu \arrow[r, "\int_T \epsilon_{T,t} d\nu"] & \int_T N_t d\nu.
\end{tikzcd}$$

It follows from Corollary 1.3.6 that the composition of the lower horizontal maps is the identity from $\int_T N_t d\nu$ to itself. Consequently, $\alpha(\nu)$ is given by the composition

$$\mathscr{F}_\nu = \mathscr{F}_{\int_T \delta_t d\nu} \xrightarrow{\sigma'_\nu} \int_T \mathscr{F}_{\delta_t} d\nu \xrightarrow{\int_T \alpha_t d\nu} \int_T N_t d\nu = \mathcal{O}_{\beta T, \nu}$$

and is therefore completely determined by the maps $\{\alpha_t\}_{t\in T}$. This proves that θ is injective.

To show that θ is surjective, suppose we are given any collection of morphisms $\{f_t : \mathscr{F}_{\delta_t} \to M_t\}_{t\in T}$ in the category $\mathcal{M}$. Let $\alpha : \mathscr{F} \to \mathcal{O}_{\beta T}$ be the natural transformation which assigns to each ultrafilter ν on T the composite map

$$\mathscr{F}_\nu = \mathscr{F}_{\int_T \delta_t d\nu} \xrightarrow{\sigma'_\nu} \int_T \mathscr{F}_{\delta_t} d\nu \xrightarrow{\int_T f_t d\nu} \int_T N_t d\nu = \mathcal{O}_{\beta T, \nu}.$$

It follows from axiom (1) of Definition 1.4.1 that $\alpha_t = f_t$ for each $t \in T$. Consequently, to show that $\{f_t\}_{t\in T}$ belongs to the image of θ, it will suffice to verify that α is a natural transformation of left ultrafunctors. Suppose we are given a collection of ultrafilters $\{\nu_s\}_{s\in S}$ on T and an ultrafilter μ on S; we wish to show that the diagram

$$\begin{tikzcd}
\mathscr{F}_{\int_S \nu_s d\mu} \arrow[r, "\gamma'_\mu"] \arrow[d, equal] & \int_S \mathscr{F}_{\nu_s} d\mu \arrow[d, equal] \\
\mathscr{F}_{\int_T \delta_t d(\int_S \nu_s d\mu)} \arrow[d, "\gamma'_{\int_S \nu_s d\mu}"] & \int_S \mathscr{F}_{\int_T \delta_t d\nu_s} d\mu \arrow[d, "\int_S \gamma'_{\nu_s} d\mu"] \\
\int_T \mathscr{F}_{\delta_t} d(\int_S \nu_s d\mu) \arrow[r, "\Delta_{\mu,\nu_\bullet}"] \arrow[d, "\int_T f_t d(\int_S \nu_s d\mu)"] & \int_S (\int_T \mathscr{F}_{\delta_t} d\nu_s) d\mu \arrow[d, "\int_S (\int_T f_t d\nu_s) d\mu"] \\
\int_T N_t d(\int_S \nu_s d\mu) \arrow[r, "\Delta_{\mu,\nu_\bullet}"] & \int_S (\int_T N_t d\nu_s) d\mu)
\end{tikzcd}$$

commutes. The commutativity of the upper rectangle follows from our assumption that $\{\gamma'_\mu\}$ is a left ultrastructure, and the commutativity of the lower square from the naturality of the Fubini transformations in the ultracategory $\mathcal{M}$. $\square$

Proof of Proposition 4.2.3. Let $X = \{x\}$ be a single point, which we identify with the Stone-Čech compactification of itself. We first note that for every object $M \in \mathcal{M}$, Proposition 4.2.8 produces a left ultrafunctor $\mathcal{O} : X \to \mathcal{M}$ satisfying $\mathcal{O}(x) = \int_X M\delta_x$, so that there exists an isomorphism $\epsilon_{X,x} : \mathcal{O}(x) \simeq M$. This proves the essential surjectivity of the evaluation map $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M}) \to \mathcal{M}$. We will complete the proof by showing that it is fully faithful: that is, for every pair of left ultrafunctors $F, G : X \to \mathcal{M}$, the canonical map

$$\mathrm{Hom}_{\mathrm{Fun}^{\mathrm{LUlt}}(X,\mathcal{M})}(F,G) \to \mathrm{Hom}_{\mathcal{M}}(F(x), G(x))$$

is bijective. Set $M = G(x)$ and define $\mathcal{O}$ as above. It follows from Proposition 4.2.9 that the isomorphism $\epsilon^{-1}_{X,x} : G(x) \simeq \mathcal{O}(x)$ lifts uniquely to a natural transformation of left ultrafunctors $\alpha : G \to \mathcal{O}$, which is automatically an isomorphism. We may therefore assume without loss of generality that $G = \mathcal{O}$, in which case the desired result follows from Proposition 4.2.9. $\square$

Proposition 4.2.10. *Let $\mathcal{M}$ be an ultracategory, let $\{M_t\}_{t \in T}$ be a collection of objects of $\mathcal{M}$, and let $\mathcal{O}_{\beta T} : \beta T \to \mathcal{M}$ be the left ultrafunctor of Proposition 4.2.8. For each $t \in T$, let $e_t : \underline{M_t} \to (\beta T, \mathcal{O}_{\beta T})$ denote the morphism in* $\mathrm{Comp}_{\mathcal{M}}$ *corresponding to the point $\delta_t \in \beta T$ and the isomorphism*

$$\mathcal{O}_{\beta T, \delta_t} = \int_T M_t d\delta_t \xrightarrow{\epsilon_{T,t}} M_t.$$

Then the maps e_t exhibit $(\beta T, \mathcal{O}_{\beta T})$ as a coproduct of the family of objects $\{\underline{M}_t\}_{t \in T}$ in the category $\mathrm{Comp}_{\mathcal{M}}$.

Proof. Combine Proposition 4.2.9 with Proposition 3.2.7. $\square$

Proof of Theorem 4.2.7. Let $\{M_t\}_{t \in T}$ be a collection of objects of $\mathcal{M}$ and let $\mathcal{O}_{\beta T}$ be as in Proposition 4.2.8, so that Proposition 4.2.10 identifies $(\beta T, \mathcal{O}_{\beta T})$ with the product of $\{\underline{M_t}\}_{t \in T_0}$ in $\mathrm{Comp}^{\mathrm{op}}_{\mathcal{M}}$. For each ultrafilter ν on T, let $p_\nu : (\beta T, \mathcal{O}_{\beta T}) \to \underline{\int_T M_t d\nu}$ be the morphism of $\mathrm{Comp}^{\mathrm{op}}_{\mathcal{M}}$ which corresponds, under the identification of Remark 4.2.6, to the point $\nu \in \beta T$ and the identity map $\mathrm{id} : \mathcal{O}_{\beta T, \nu} \to \int_T M_t d\nu$. For each subset $T_0 \subseteq T$ satisfying $\nu(T_0) = 1$, we can identify βT_0 with a subset of βT containing ν, so that p_ν factors uniquely as a composition

$$(\beta T, \mathcal{O}_{\beta T}) \to (\beta T_0, \mathcal{O}_{\beta T}|_{\beta T_0}) \xrightarrow{p^{T_0}_\nu} \underline{\int_T M_t d\nu}$$

in the category $\mathrm{Comp}^{\mathrm{op}}_{\mathcal{M}}$). Here we can use Remark 4.1.8 to identify $(\beta T_0, \mathcal{O}_{\beta T}|_{\beta T_0})$ with the direct factor of $(\beta T, \mathcal{O}_{\beta T})$ given by the product $\prod_{t \in T_0} \underline{M_t}$. For any object $(X, \mathcal{O}_X)$ of $\mathrm{Comp}^{\mathrm{op}}_{\mathcal{M}}$, it follows from Remark 4.1.2 that composition with p_ν induces an injection

$$\mathrm{Hom}_{\mathrm{Comp}^{\mathrm{op}}_{\mathcal{M}}}(\underline{\int_T M_t d\nu}, (X, \mathcal{O}_X)) \to \mathrm{Hom}_{\mathrm{Comp}^{\mathrm{op}}_{\mathcal{M}}}((\beta T, \mathcal{O}_{\beta T}), (X, \mathcal{O}_X))$$

with image consisting of those maps which, at the level of the underlying topological spaces, induce the constant map $X \to \beta T$ taking the value $\{\nu\}$. Note that this is equivalent to the requirement that the map $X \to \beta T$ factors through the subset $\beta T_0 \subseteq \beta T$ whenever $\nu(T_0) = 1$, so that the maps $p^{T_0}_\nu$ exhibit $\underline{\int_T M_t d\nu}$ as a colimit of the diagram $\{(\beta T_0, \mathcal{O}_{\beta T}|_{\beta T_0})\}_{\nu(T_0)=1}$ in the category $\mathrm{Comp}^{\mathrm{op}}_{\mathcal{M}}$. This proves the existence of the ultraproduct

$$\int_T \underline{M_t} d\nu = \varinjlim_{\nu(T_0)=1} \prod_{t \in T_0} \underline{M_t},$$

and shows that there exists a unique isomorphism $\gamma_\nu : \int_T \underline{M_t} d\nu \simeq \underline{\int_T M_t d\nu}$ satisfying $\gamma_\nu \circ q_\nu = p_\nu$.

To complete the proof, it will suffice to show that the isomorphisms $\gamma_\nu : \int_T \underline{M_t} d\nu \simeq \underline{\int_T M_t d\nu}$ determine a (right) ultrastructure on the functor $M \mapsto \underline{M}$. We will verify requirement (2) of Definition 8.1.1, and leave the verification of (0) and (1) to the reader. Suppose we are given a collection $\{M_t\}_{t \in T}$ of objects of $\mathcal{M}$ indexed by a set T, a collection $\nu_\bullet = \{\nu_s\}_{s \in S}$ of ultrafilters on T indexed by a set S, and an ultrafilter μ on S. Let λ denote the ultrafilter on T given by $\int_S \nu_s d\mu$. We wish to show that the outer rectangle appearing

in the diagram

$$\begin{tikzcd}
\int_T \underline{M_t} d\lambda \arrow[rrr, "\gamma_\lambda"] \arrow[ddd, "\Delta_{\mu,\nu_\bullet}"] & & & \underline{\int_T M_t d\lambda} \arrow[ddd, "\underline{\Delta_{\mu,\nu_\bullet}}"] \\
 & \prod_{t\in T} \underline{M_t} \arrow[ul, "q_\lambda"'] \arrow[urr, "p_\lambda"] \arrow[d, "\{q_{\nu_s}\}_{s\in S}"] \arrow[dr, "\{p_{\nu_s}\}_{s\in S}"] & & \\
 & \prod_{s\in S} \int_T \underline{M_t} d\nu_s \arrow[r, "\prod_{s\in S} \gamma_{\nu_s}"] \arrow[dl, "q_\mu"'] & \prod_{s\in S} \underline{\int_T M_t d\nu_s} \arrow[d, "q_\mu"] \arrow[dr, "p_\mu"] & \\
\int_S (\int_T \underline{M_t} d\nu_s) d\mu \arrow[rr, "\int_S \gamma_{\nu_s} d\mu"] & & \int_S \underline{\int_T M_t d\nu_s} d\mu \arrow[r, "\gamma_\mu"] & \underline{\int_S (\int_T M_t d\nu_s) d\mu}
\end{tikzcd}$$

commutes in the category $\mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$. Since the map q_λ is an epimorphism, it will suffice to prove the commutativity of each inner cycle of the diagram. The triangles commute the construction of γ, the quadrilateral on the left commutes by the construction of the Fubini transformations in $\mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}}$, and the quadrilateral on the bottom commutes by functoriality. We are therefore reduced to proving the commutativity of the quadrilateral on the right. Let $\mathcal{O}_{\beta T} : \beta T \to \mathcal{M}$ be the left ultrafunctor obtained by applying Proposition 4.2.8 to the family of objects $\{M_t\}_{t\in T}$, and $\mathcal{O}_{\beta S}$ denote the left ultrafunctor $\beta S \to \mathcal{M}$ obtained by applying Proposition 4.2.8 to the family of objects $\{\int_T M_t d\nu_s\}_{s\in S}$. Unwinding the definition, we are reduced to proving the commutativity of the diagram

$$\begin{tikzcd}
\underline{\int_S (\int_T M_t d\nu_s) d\mu} \arrow[r, "\underline{\Delta_{\mu,\nu_\bullet}}"] \arrow[d, "(\mu{,}\mathrm{id})"] & \underline{\int_T M_t d\lambda} \arrow[d, "(\lambda{,}\mathrm{id})"] \\
(\beta S, \mathcal{O}_{\beta S}) \arrow[r, "(f{,}\alpha)"] & (\beta T, \mathcal{O}_{\beta T}),
\end{tikzcd}$$

in the category $\mathrm{Comp}_{\mathcal{M}}$, where the vertical maps are labelled using the classification of Remark 4.2.6, and (f,α) is determined by the requirements $f(\delta_s) = \nu_s$ for each $s \in S$ and α is given at the point δ_s by the composition

$$\mathcal{O}_{\beta T,\nu_s} = \int_T M_t d \int_S (\nu_{s'} d\delta_s) \xrightarrow{\Delta_{\delta_s,\nu_\bullet}} \int_S (\int_T M_t d\nu_{s'}) d\delta_s = \mathcal{O}_{\beta S,\delta_s}$$

(see Proposition 4.2.10). Unwinding the definitions, we wish to show that the map

$$\alpha(\mu) : \mathcal{O}_{\beta T,\lambda} = \int_T M_t d\lambda \to \int_S (\int_T M_t d\nu_s) d\mu = \mathcal{O}_{\beta S,\mu}$$

coincides with the Fubini transformation $\Delta_{\mu,\nu_\bullet}$. Writing $\mu = \int_S \delta_s d\mu$ and invoking our assumption that α is a natural transformation of left ultrafunctors, we deduce that $\alpha(\mu)$ fits into a commutative diagram

$$\begin{tikzcd}
\int_T M_t d\lambda \arrow[r, "\alpha(\mu)"] \arrow[d, "\Delta_{\mu,\nu_\bullet}"] & \int_S (\int_T M_t d\nu_s) d\mu \arrow[d, "\Delta_{\mu,\delta_\bullet}"] \\
\int_S (\int_T M_t d\nu_s) d\mu \arrow[r, "\int_S \Delta_{\mu,\delta_s} d\mu"] & \int_S (\int_S (\int_T M_t d\nu_s) d\delta_{s'}) d\mu.
\end{tikzcd}$$

Combining this observation with axiom (C) of Definition 1.3.1, we conclude that the maps

$$\alpha(\mu), \Delta_{\mu,\nu_\bullet} : \int_T M_t d\lambda \to \int_S (\int_T M_t d\nu_s) d\mu$$

agree after composing with the isomorphism

$$\Delta_{\mu,\delta_\bullet} : \int_S (\int_T M_t d\nu_s) d\mu \simeq \int_S (\int_S (\int_T M_t d\nu_s) d\delta_{s'}) d\mu,$$

and must therefore coincide. □

4.3. **Ultracategories as Topological Stacks.** Let $\mathcal{M}$ be an ultracategory. We proved in §4.1 that the category $\mathrm{Comp}_{\mathcal{M}}$ of Construction 4.1.1 can be regarded as a stack on the category of compact Hausdorff spaces: that is, the construction $X \mapsto \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})$ satisfies descent with respect to the coherent topology (Proposition 4.1.5). Our goal in this section is to show that the construction $\mathcal{M} \mapsto \mathrm{Comp}_{\mathcal{M}}$ is fully faithful (Theorem 4.3.3). To formulate this precisely, it will be convenient to introduce a bit of notation.

Definition 4.3.1. Let $\mathcal{C}$ be a category. Suppose we are given fibrations of categories $p : \mathcal{D} \to \mathcal{C}$ and $q : \mathcal{E} \to \mathcal{C}$. We define a category $\mathrm{Cart}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$ as follows:

- The objects of $\mathrm{Cart}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$ are functors $F : \mathcal{D} \to \mathcal{E}$ which satisfy $q \circ F = p$ and have the property that, for every p-Cartesian morphism u of $\mathcal{D}$, the image $F(u)$ is a q-Cartesian morphism of $\mathcal{E}$.
- If $F, G : \mathcal{C} \to \mathcal{D}$ are objects of $\mathrm{Cart}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$, then a morphism from F to G in $\mathrm{Cart}_{\mathcal{C}}(\mathcal{D}, \mathcal{E})$ is a natural transformation $\alpha : F \to G$ for which the induced natural map
$$p = q \circ F \xrightarrow{\alpha} q \circ G = p$$
is the identity transformation from p to itself.

Example 4.3.2. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. Then, for every left ultrafunctor $F : \mathcal{M} \to \mathcal{N}$, the construction $(X, \mathcal{O}_X) \mapsto (X, F \circ \mathcal{O}_X)$ determines a functor $\Phi_F : \mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}_{\mathcal{N}}$, which is an object of the category $\mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})$

Theorem 4.3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. Then the construction $F \mapsto \Phi_F$ induces an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \simeq \mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})^{\mathrm{op}}.$$

Remark 4.3.4. Let $\mathbf{Ult}^{\mathrm{L}}$ denote the strict 2-category whose objects are ultracategories and whose morphisms are left ultrafunctors (see Remark 1.4.6). Theorem 4.3.3 (together with Proposition 4.1.5) asserts that the construction $\mathcal{M} \mapsto \mathrm{Comp}_{\mathcal{M}}$ induces a fully faithful embedding of 2-categories

$$\mathbf{Ult}^{\mathrm{L}} \hookrightarrow \{\text{Stacks of categories on Comp}\}.$$

Remark 4.3.5. By virtue of Theorem 3.1.5, we can regard the category of Comp of compact Hausdorff spaces (with morphisms given by continuous maps) as a full subcategory of the 2-category $\mathbf{Ult}^{\mathrm{L}}$ of ultracategories (with morphisms given by left ultrafunctors). Theorem 4.3.3 asserts that the inclusion $\mathrm{Comp} \hookrightarrow \mathbf{Ult}^{\mathrm{L}}$ is *dense*: that is, the identity functor on $\mathbf{Ult}^{\mathrm{L}}$ is a left Kan extension of its restriction to Comp.

We now turn to the proof of Theorem 4.3.3. We begin by explicitly constructing a functor in the opposite direction. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, and let $\Phi : \mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}_{\mathcal{N}}$ be a morphism of stacks on Comp. Then the induced map of opposite categories $\Phi^{\mathrm{op}} : \mathrm{Comp}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Comp}_{\mathcal{N}}^{\mathrm{op}}$ satisfies condition $(*)$ of Proposition 1.4.9, and therefore (by virtue of Theorem 4.2.7) restricts to a left ultrafunctor $F : \mathcal{M} \to \mathcal{N}$ (characterized on objects by the formula $\Phi(\underline{M}) = \underline{F(M)}$). Let us denote this left ultrafunctor by $\Phi|_{\mathcal{M}}$. We first prove the following:

Lemma 4.3.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. Then the composition*

$$\mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})^{\mathrm{op}} \xrightarrow{\Phi \mapsto \Phi|_{\mathcal{M}}} \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \xrightarrow{F \mapsto \Phi_F} \mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})^{\mathrm{op}}$$

is naturally isomorphic to the identity functor.

Proof. Let $\Phi : \mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}_{\mathcal{N}}$ be an object of $\mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})$ and set $F = \Phi|_{\mathcal{M}}$, so that we have $\Phi(\underline{M}) = \underline{F(M)}$ for each object $M \in \mathcal{M}$. For each object $(X, \mathcal{O}_X)$ in the category $\mathrm{Comp}_{\mathcal{M}}$, we can

write $\Phi(X, \mathcal{O}_X) = (X, \Phi_X(\mathcal{O}_X))$ for some left ultrafunctor $\Phi_X(\mathcal{O}_X) : X \to \mathcal{N}$. Applying Φ to each of the canonical maps $\underline{\mathcal{O}_{X,x}} \to (X, \mathcal{O}_X)$, we obtain maps

$$\underline{F(\mathcal{O}_{X,x})} \to (X, \Phi_X(\mathcal{O}_X))$$

which are Cartesian with respect to the projection $\mathrm{Comp}_{\mathcal{N}} \to \mathrm{Comp}$ and therefore determine isomorphisms $\phi_x : \Phi_X(\mathcal{O}_X)_x \simeq F(\mathcal{O}_{X,x})$ in the ultracategory $\mathcal{N}$. We will show that the isomorphisms ϕ_x determine an isomorphism

$$\phi(X, \mathcal{O}_X) : \Phi(X, \mathcal{O}_X) = (X, \Phi_X(\mathcal{O}_X)) \simeq (X, F \circ \mathcal{O}_X) = \Phi_F(X, \mathcal{O}_X)$$

in the category $\mathrm{Comp}_{\mathcal{N}}$ (it is then easy to see that these isomorphisms depend functorially on $(X, \mathcal{O}_X)$ and induce the identity at the level of the underlying topological spaces, and therefore determine an isomorphism $\Phi \simeq \Phi_F$ in the category $\mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})$, depending functorially on Φ). To prove this, we must show that isomorphisms $\{\phi_x\}_{x \in X}$ comprise an isomorphism of left ultrafunctors from $\Phi_X(\mathcal{O}_X)$ to $F \circ \mathcal{O}_X$. Fix a map of sets $f : S \to X$ and an ultrafilter μ on S; we wish to show that the diagram

$$\begin{tikzcd}
\Phi_X(\mathcal{O}_X)_{\int_S f(s) d\mu} \arrow[r, "\phi_{\int_S f(s) d\mu}"] \arrow[d] & F(\mathcal{O}_{X, \int_S f(s) d\mu}) \arrow[d] \\
\int_S \Phi_X(\mathcal{O}_X)_{f(s)} \arrow[r, "\int_S \phi_{f(s)} d\mu"] & \int_S F(\mathcal{O}_{X, f(s)}\, d\mu)
\end{tikzcd}$$

commutes, where the vertical maps are given by the left ultrastructure on $\Phi_X(\mathcal{O}_X)$ and F, respectively. Using the universal property of Proposition 4.2.9, we can reduce to checking this after replacing $(X, \mathcal{O}_X)$ with $(\beta S, \mathcal{O}_{\beta S})$, where $\mathcal{O}_{\beta S} : \beta S \to \mathcal{M}$ is the left ultrafunctor obtained by applying Proposition 4.2.8 to the objects $\{\mathcal{O}_{X, f(s)}\}_{s \in S}$ of $\mathcal{M}$. In this case, the desired commutativity follows from the construction of the left ultrastructure on F. □

Proof of Theorem 4.3.3. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories; we wish to show that the construction $F \mapsto \Phi_F$ induces an equivalence of categories $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \simeq \mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})^{\mathrm{op}}$. Essential surjectivity follows from Lemma 4.3.6. It will therefore suffice to prove that the construction $F \mapsto \Phi_F$ is fully faithful. Suppose that we are given a pair of left ultrafunctors $F, G : \mathcal{M} \to \mathcal{N}$ and a natural transformation $\alpha : \Phi_G \to \Phi_F$. To every compact Hausdorff space X and every left ultrafunctor $\mathcal{O}_X : X \to \mathcal{M}$, α associates a natural transformation of left ultrafunctors $\alpha(X, \mathcal{O}_X) : F \circ \mathcal{O}_X \to G \circ \mathcal{O}_X$, which we can associate with a collection of maps $\{\alpha(X, \mathcal{O}_X)_x : F(\mathcal{O}_{X,x}) \to G(\mathcal{O}_{X,x})\}$ in the ultracategory $\mathcal{N}$. Taking $(X, \mathcal{O}_X) = \underline{M}$ for some object $M \in \mathcal{M}$, we obtain maps $\alpha_0(M) : F(M) \to G(M)$, depending functorially on M. We will show that α is the image of α_0 under the functor $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \simeq \mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})^{\mathrm{op}}$ (it is immediate that α_0 is uniquely determined by this requirement). For any object $(X, \mathcal{O}_X) \in \mathrm{Comp}_{\mathcal{M}}$ and any point $x \in X$, we have a commutative diagram

$$\begin{tikzcd}
\Phi_G(\underline{\mathcal{O}_{X,x}}) \arrow[r, "\alpha(\underline{\mathcal{O}_{X,x}})"] \arrow[d] & \Phi_F(\underline{\mathcal{O}_{X,x}}) \arrow[d] \\
\Phi_G(X, \mathcal{O}_X) \arrow[r, "\alpha(X, \mathcal{O}_X)"] & \Phi_F(X, \mathcal{O}_X)
\end{tikzcd}$$

in the category $\mathrm{Comp}_{\mathcal{N}}$, which shows that $\alpha(X, \mathcal{O}_X)_x$ can be identified with the map $\alpha_0(\mathcal{O}_{X,x}) : F(\mathcal{O}_{X,x}) \to G(\mathcal{O}_{X,x})$. Consequently, it will suffice to show that α_0 is a natural transformation of left ultrafunctors. Fix a collection of objects $\{M_s\}_{s \in S}$ in $\mathcal{M}$ and an ultrafilter μ on S; we wish to show that the diagram

$$\begin{tikzcd}
F(\int_S M_s d\mu) \arrow[r, "\alpha_0(\int_S M_s d\mu)"] \arrow[d] & G(\int_S M_s d\mu) \arrow[d] \\
\int_S F(M_s) d\mu \arrow[r, "\int_S \alpha_0(M_s) d\mu"] & \int_S G(M_s) d\mu
\end{tikzcd}$$

commutes, where the vertical maps are given by the left ultrastructures on the functors F and G. This follows from the fact that $\alpha(\beta S, \mathcal{O}_{\beta S})$ is a natural transformation of left ultrafunctors $\beta S \to \mathcal{N}$, where $\mathcal{O}_{\beta S} : \beta S \to \mathcal{M}$ is the left ultrafunctor obtained by applying Proposition 4.2.8 to the family $\{M_s\}_{s \in S}$. □

Remark 4.3.7. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. It follows from Theorem 4.3.3 and Lemma 4.3.6 that the construction $\Phi \mapsto \Phi_{\mathcal{M}}$ is also a left homotopy inverse to the equivalence of Theorem 4.3.3: that is, the composition

$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \xrightarrow{F \mapsto \Phi_F} \mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})^{\mathrm{op}} \xrightarrow{\Phi \mapsto \Phi|_{\mathcal{M}}} \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N})$$

is naturally isomorphic to the identity.

4.4. **Application: Classification of Ultragroupoids.** We now consider a particularly simple class of ultracategories.

Definition 4.4.1. An *ultragroupoid* is an ultracategory $\mathcal{M}$ for which the underlying category of $\mathcal{M}$ is a groupoid (that is, every morphism in $\mathcal{M}$ is an isomorphism).

Example 4.4.2. Every ultraset (in the sense of Definition 3.1.1) is an ultragroupoid.

According to Theorem 3.1.5, the category of ultrasets is equivalent to the category of compact Hausdorff spaces. Our goal in this section is to obtain an analogous description for the 2-category of *all* ultragroupoids (which we regard as a full subcategory of the 2-category **Ult** of Remark 1.4.6). First, we need some terminology.

Definition 4.4.3. A *topological groupoid* is a groupoid $\mathcal{C}$ which is equipped with topologies on the set $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ of objects and morphisms of $\mathcal{C}$, satisfying the following requirements:

- The function $s, t : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ taking a morphism $f : C \to D$ in $\mathcal{C}$ to its source $s(f) = C$ and its target $t(f) = D$ are both continuous.
- The function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ taking each object $C \in \mathcal{C}$ to the identity morphism id_C is continuous.
- The function $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ taking each morphism $f : C \to D$ to its inverse $f^{-1} : D \to C$ is continuous.
- The function $\mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ taking a pair of composable morphisms $C \xrightarrow{f} D \xrightarrow{g} E$ to its composition $g \circ f$ is continuous.

We will say that a topological groupoid $\mathcal{C}$ is *compact* if both $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are compact Hausdorff spaces.

Remark 4.4.4. Let $\mathcal{C}$ be a compact topological groupoid. Then we can regard $\mathcal{C}$ as an ultracategory as follows:

- For each set S and each ultrafilter μ on S, we let $\int_S (\bullet) d\mu : \mathcal{C}^S \to \mathcal{C}$ be the functor given on objects and morphisms by the maps
$$\int_S (\bullet) d\mu : \mathrm{Ob}(\mathcal{C})^S \to \mathrm{Ob}(\mathcal{C}) \qquad \int_S (\bullet) d\mu : \mathrm{Mor}(\mathcal{C})^S \to \mathrm{Mor}(\mathcal{C})$$
determined by the compact Hausdorff topologies on $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$, respectively.
- The isomorphisms $\epsilon_{S,s}$ and Fubini transformations $\Delta_{\mu,\nu_\bullet}$ are the identity maps.

Example 4.4.5. Let G be a compact topological group. Then G determines a compact topological groupoid (hence an ultragroupoid) BG, where we take the space of objects Ob(BG) to be a single point and the space of morphisms Mor(BG) to be the group G (with composition of morphisms given by the multiplication in G).

Notation 4.4.6. Let $\mathcal{C}$ be a compact topological groupoid. We let $\mathrm{Comp}_{\mathcal{C}}$ denote the category obtained by applying Construction 4.1.1 to $\mathcal{C}$, where we endow $\mathcal{C}$ with the ultrastructure of Remark 4.4.4. By definition, the objects of $\mathrm{Comp}_{\mathcal{C}}$ are pairs $(X, \mathcal{O}_X)$, where X is a compact Hausdorff space and $\mathcal{O}_X : X \to \mathcal{C}$ is an ultrafunctor (note that, since $\mathcal{C}$ is a groupoid, every left ultrafunctor $X \to \mathcal{C}$ is automatically an ultrafunctor).

We can now state our main result.

Theorem 4.4.7. *Let $\mathcal{G} \to \mathrm{Comp}$ be a stack in groupoids over the category Comp of compact Hausdorff spaces (where we endow Comp with the coherent topology). The following conditions are equivalent:*

(1) *The stack $\mathcal{G}$ is representable. That is, there exists a map of stacks $\mathcal{G}' \to \mathcal{G}$, locally surjective with respect to the coherent topology, such that both $\mathcal{G}'$ and the fiber product $\mathcal{G}' \times_{\mathcal{G}} \mathcal{G}'$ are representable by compact Hausdorff spaces.*
(2) *There exists a compact topological groupoid $\mathcal{C}$ and an equivalence of stacks $\mathcal{G} \simeq \mathrm{Comp}_{\mathcal{C}}$.*
(3) *There exists a small ultragroupoid $\mathcal{M}$ and an equivalence of stacks $\mathcal{G} \simeq \mathrm{Comp}_{\mathcal{M}}$.*

Before turning to the proof of Theorem 4.4.7, let us analyze the construction of Notation 4.4.6 in more detail. Let $\mathcal{C}$ be a compact topological groupoid. For every compact Hausdorff space X, let $\mathrm{Fun}^{\mathrm{cont}}(X, \mathcal{C})$ denote the groupoid whose objects are continuous maps $X \to \mathrm{Ob}(\mathcal{C})$ and whose morphisms are continuous maps $X \to \mathrm{Mor}(\mathcal{C})$. Using Theorem 3.1.5, we can identify $\mathrm{Fun}^{\mathrm{cont}}(X, \mathcal{C})$ with a full subcategory of the category of ultrafunctors $\mathrm{Fun}^{\mathrm{Ult}}(X, \mathcal{C})$: namely, the full subcategory spanned by those functors F equipped with an ultrastructure $\{\sigma_\mu\}$ where each σ_μ is an identity map.

Lemma 4.4.8. *Let S be a set and let βS be its Stone-Čech compactification. Then, for every compact topological groupoid $\mathcal{C}$, the preceding construction induces an equivalence of categories $\mathrm{Fun}^{\mathrm{cont}}(\beta S, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{Ult}}(\beta S, \mathcal{C})$.*

Proof. Let $\mathcal{O}_{\beta S} : \beta S \to \mathcal{C}$ be any ultrafunctor. For each $s \in S$, set $C_s = \mathcal{O}_{\beta S, \delta_s}$. Let $\mathcal{O}'_{\beta S} : \beta S \to \mathcal{C}$ be the left ultrafunctor obtained by applying Proposition 4.2.8 to the collection $\{C_s\}_{s \in S}$. The ultrastructure on $\mathcal{O}'_{\beta S}$ is then given by a collection of maps $\{\sigma'_\mu\}$ which are obtained from the Fubini transformations of $\mathcal{C}$, and are therefore identity maps. It follows that $\mathcal{O}'_{\beta S}$ belongs to the image of the map $\mathrm{Fun}^{\mathrm{cont}}(\beta S, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{Ult}}(\beta S, \mathcal{C})$. Moreover, the universal property of Proposition 4.2.9 supplies a map of left ultrafunctors $\mathcal{O}_{\beta S} \to \mathcal{O}'_{\beta S}$, which must be an isomorphism because $\mathcal{C}$ is a groupoid. □

Lemma 4.4.9. *Let $\mathcal{C}$ be a compact topological groupoid. Then the stack $\mathrm{Comp}_{\mathcal{C}}$ of Notation 4.4.6 can be obtained by stackifying the presheaf of groupoids $X \mapsto \mathrm{Fun}^{\mathrm{cont}}(X, \mathcal{C})$ (with respect to the coherent topology on Comp).*

Proof. We have a canonical map $\iota : \mathrm{Fun}^{\mathrm{cont}}(X, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{Ult}}(X, \mathcal{C}) = \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{C})$ which is fully faithful and depends functorially on X, whose target satisfies descent for the coherent topology (Proposition 4.1.5). It will therefore suffice to show that ι is locally surjective. That is, we must show that for any ultrafunctor $\mathcal{O}_X : X \to \mathcal{C}$, there exists a continuous surjection $X' \twoheadrightarrow X$ for which the composite map $X' \twoheadrightarrow X \xrightarrow{\mathcal{O}_X} \mathcal{C}$ is isomorphic (as an ultrafunctor) to some object of $\mathrm{Fun}^{\mathrm{cont}}(X', \mathcal{C})$. This follows from Lemma 4.4.8 (for example, we can take X' to be the Stone-Čech compactification of the underlying set of X). □

Example 4.4.10. Let G be a compact topological group and let BG be the topological groupoid of Example 4.4.5. We define a *principal G-bundle on X* to be a compact Hausdorff space $\mathcal{P}$ equipped with a continuous, free action of the group G and a homeomorphism $\mathcal{P}/G \simeq X$ (beware that this terminology is potentially misleading: we do not require that the quotient map $\mathcal{P} \to X$ admits sections locally on X). The collection of principal G-bundles on X forms a category which we will denote by $\mathrm{Bun}_G(X)$. We can identify $\mathrm{Fun}^{\mathrm{cont}}(X, \mathrm{BG})$ with the full subcategory of $\mathrm{Bun}_G(X)$ spanned by the trivial G-bundle $\mathcal{P} = X \times G$ (whose automorphism group is the group $\mathrm{Hom}_{\mathrm{Top}}(X, G)$ of all continuous maps from X into G). Then the construction $X \mapsto \mathrm{Bun}_G(X)$ can be identified with the stackification (for the coherent topology on the category Comp) of the subfunctor $X \mapsto \mathrm{Fun}^{\mathrm{cont}}(X, \mathrm{BG}) \subseteq \mathrm{Bun}_G(X)$: this follows from the observation that when $X = \beta S$ is the Stone-Čech compactification of a set S, then every principal G-bundle on X is trivial. Applying Lemma 4.4.9 in this case, we obtain an equivalence of categories $\mathrm{Fun}^{\mathrm{Ult}}(X, \mathrm{BG}) \simeq \mathrm{Bun}_G(X)$, depending functorially on X.

Proof of Theorem 4.4.7. By a standard argument, any representable stack $\mathcal{G}$ can be obtained as the stackification of the groupoid-valued presheaf $X \mapsto \mathrm{Fun}^{\mathrm{cont}}(X, \mathcal{C})$, where $\mathcal{C}$ is some groupoid object of the category

Comp (that is, a compact topological groupoid in the sense of Definition 4.4.3), and is therefore equivalent to $\mathrm{Comp}_{\mathcal{C}}$ by virtue of Lemma 4.4.9. This proves the implication (1) ⇒ (2), and the implication (2) ⇒ (3) is trivial. We will complete the proof by showing that (3) implies (1). Assume that $\mathcal{G} = \mathrm{Comp}_{\mathcal{M}}$ for some small ultragroupoid $\mathcal{M}$. Choose a compact Hausdorff space X and an essentially surjective left ultrafunctor $\mathcal{O}_X : X \to \mathcal{M}$ (this is possible by virtue of our assumption that $\mathcal{M}$ is small; for example, we can use the construction of Proposition 4.2.8). Note that $\mathcal{O}_X$ is automatically an ultrafunctor (since every morphism in $\mathcal{M}$ is invertible). The (2-categorical) fiber product $X \times_{\mathcal{M}} X$ can then be regarded as an ultracategory having only identity morphisms, which we can identify (by virtue of Theorem 3.1.5) with a compact Hausdorff space R. Then $\mathcal{O}_X$ induces a map of stacks $\theta : \mathrm{Comp}_X \to \mathrm{Comp}_{\mathcal{M}} = \mathcal{G}$, where the (2-categorical) fiber product $\mathrm{Comp}_X \times_{\mathrm{Comp}_{\mathcal{M}}} \mathrm{Comp}_X$ is represented by R. To complete the proof that $\mathcal{G}$ satisfies (1), it will suffice to show that the functor θ is locally essentially surjective (with respect to the coherent topology on Comp). Choose any object of $\mathcal{G}$, corresponding to a compact Hausdorff space Y equipped with a left ultrafunctor $\mathcal{O}_Y : Y \to \mathcal{M}$; we wish to show that $(Y, \mathcal{O}_Y)$ is locally in the essential image of θ. To prove this, we may assume without loss of generality that $Y = \beta S$ for some set S. For each element $s \in S$, choose a point $f(s) \in X$ and an isomorphism $\mathcal{O}_{Y,\delta_s} \simeq \mathcal{O}_{X,f(s)}$. Then f induces a continuous map $\overline{f} : Y = \beta S \to X$ (given by $\overline{f}(\mu) = \int_S f(s) d\mu$). Let $\mathcal{O}'_{\beta S} : \beta S \to \mathcal{M}$ be the left ultrafunctor obtained by applying Proposition 4.2.8 to the collection of objects $\{\mathcal{O}_{X,f(s)}\}$ of S, so that the universal property of Proposition 4.2.9 supplies natural transformations of left ultrafunctors

$$\mathcal{O}_Y \to \mathcal{O}'_{\beta S} \leftarrow \mathcal{O}_X|_Y.$$

Since $\mathcal{M}$ is a groupoid, these natural transformations must be invertible, which shows that $\mathcal{O}_Y \simeq \mathcal{O}_X|_Y$ belongs to the essential image of θ as desired. □

5. The Topos of Left Ultrafunctors

Our ultimate goal in this paper is to show that the category $\mathrm{Shv}(\mathcal{C})$ of sheaves on a small pretopos $\mathcal{C}$ can be identified with the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ (Theorem 2.2.2). One obstacle to proving this is that our definition of left ultrafunctor is somewhat unwieldy. To supply a left ultrastructure on a functor $F : \mathcal{M} \to \mathcal{N}$, one must supply a morphism $\sigma_\mu : F(\int_S M_s d\mu) \simeq \int_S F(M_s) d\mu$ for *every* collection of objects $\{M_s\}_{s \in S}$ of $\mathcal{M}$ and *every* ultrafilter μ on S. Our goal in this section is to show that (in various cases) the study of left ultrafunctors from $\mathcal{M}$ to $\mathcal{N}$ can be reduced to the study of ordinary functors between suitable enlargements of $\mathcal{M}$ and $\mathcal{N}$. Note that we have already proved one result of this type: according to Theorem 4.4.7, the datum of a left ultrafunctor $F : \mathcal{M} \to \mathcal{N}$ is equivalent to the datum of a morphism of stacks from $\mathrm{Comp}_{\mathcal{M}}$ to $\mathrm{Comp}_{\mathcal{N}}$. However, this result is not quite what we are after: it establishes an equivalence between functors $F : \mathcal{M} \to \mathcal{N}$ which respect a certain kind of structure (namely, the ultrastructures on $\mathcal{M}$ and $\mathcal{N}$) with functors $\Phi : \mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}_{\mathcal{N}}$ which respect a different kind of structure (namely, the fibrations $\mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp} \leftarrow \mathrm{Comp}_{\mathcal{N}}$). To find a remedy, it will be useful to restrict the class of compact Hausdorff spaces that we work with.

Notation 5.0.1 (The Category $\mathrm{Stone}_{\mathcal{M}}$)**.** Let Stone denote the category whose objects are Stone spaces (that is, topological spaces which are compact, Hausdorff, and totally disconnected) and whose morphisms are continuous functions. We will regard Stone as a full subcategory of the category Comp of compact Hausdorff spaces. For each ultracategory $\mathcal{M}$, we let $\mathrm{Stone}_{\mathcal{M}}$ denote the full subcategory of $\mathrm{Comp}_{\mathcal{M}}$ spanned by those pairs $(X, \mathcal{O}_X)$ where X is a Stone space. Note that if $F : \mathcal{M} \to \mathcal{N}$ is a left ultrafunctor, then the functor $\Phi_F : \mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp}_{\mathcal{N}}$ restricts to a functor $\mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}_{\mathcal{N}}$, which we will also denote by Φ_F.

The main results of this section can be summarized as follows:

- Let $\mathcal{M}$ and $\mathcal{N}$ be ultractegories. In §5.1, we show that the construction $F \mapsto \Phi_F$ induces a fully faithful embedding $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})^{\mathrm{op}}$, whose essential image can be explicitly identified (Theorem 5.1.4). This is essentially a restatement of Theorem 4.4.7, but is formulated in terms of the structure of $\mathrm{Stone}_{\mathcal{M}}$ and $\mathrm{Stone}_{\mathcal{N}}$ as abstract categories (rather than as categories fibered over over Stone).

- In §5.2, we specialize to the case where $\mathcal{N} = \mathrm{Set}$ is the category of sets. In this case, we show that every left ultrafunctor $F : \mathcal{M} \to \mathrm{Set}$ admits a canonical extension $F^{+} : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$, from which the left ultrastructure on F can be recovered using the construction of Proposition 1.4.9. This construction determines a fully faithful embedding θ from the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ to the category of ordinary functors $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ (Theorem 5.2.1).
- If $\mathcal{M}$ is an ultracategory which admits small filtered colimits, then the essential image of $\theta : \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set}) \hookrightarrow \mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ admits a particularly simple description. In §5.3, we show that a functor $\overline{F} : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ belongs to the essential image of θ if and only if it preserves small filtered colimits and finite coproducts (Theorem 5.3.3).
- The main step in our proof of Theorem 5.3.3 is to show that any functor $F : \mathcal{M} \to \mathrm{Set}$ which admits a left ultrastructure automatically preserves filtered colimits (Proposition 5.3.4). This result suggests that, under appropriate set-theoretic assumptions, it should be possible to recover a left ultrafunctor $F : \mathcal{M} \to \mathrm{Set}$ from its restriction to any sufficiently large full subcategory $\mathcal{M}_0 \subseteq \mathcal{M}$. In §5.4 we exploit this idea to show that if the ultrastructure on $\mathcal{M}$ is *accessible* (Definition 5.4.1), then the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is a Grothendieck topos (Proposition 5.4.5).

5.1. **Canonical Extensions of Left Ultrafunctors.**

We begin with a few general remarks.

Definition 5.1.1. Let $\mathcal{C}$ be a category which admits finite coproducts. We say that an object $C \in \mathcal{C}$ is *connected* if the functor $\mathrm{Hom}_{\mathcal{C}}(C, \bullet) : \mathcal{C} \to \mathrm{Set}$ preserves finite coproducts.

Example 5.1.2. Let $\mathcal{M}$ be an ultracategory and let $(X, \mathcal{O}_X)$ be an object of the category $\mathrm{Comp}_{\mathcal{M}}$ (Construction 4.1.1). Then $(X, \mathcal{O}_X)$ is connected as an object of $\mathrm{Comp}_{\mathcal{M}}$ (in the sense of Definition 5.1.1) if and only if X is connected as a topological space. This follows from the description of coproducts in $\mathrm{Comp}_{\mathcal{M}}$ supplied by Remark 4.1.8.

Variant 5.1.3. Let $\mathcal{M}$ be an ultracategory and let $(X, \mathcal{O}_X)$ be an object of $\mathrm{Stone}_{\mathcal{M}}$. The following conditions are equivalent:

(*a*) The pair $(X, \mathcal{O}_X)$ is a connected object of $\mathrm{Comp}_{\mathcal{M}}$.
(*b*) The pair $(X, \mathcal{O}_X)$ is a connected object of $\mathrm{Stone}_{\mathcal{M}}$.
(*c*) The topological space X has a single point (so that $(X, \mathcal{O}_X)$ is isomorphic to $\underline{M}$, for some object $M \in \mathcal{M}$).

We can now state the main result of this section:

Theorem 5.1.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. Then the construction $F \mapsto \Phi_F$ of Notation 5.0.1 induces a fully faithful embedding*
$$\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})^{\mathrm{op}},$$
whose essential image is spanned by those functors $\Phi : \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}_{\mathcal{N}}$ satisfying the following conditions:

(1) *The functor Φ preserves finite coproducts.*
(2) *The functor Φ carries connected objects of $\mathrm{Stone}_{\mathcal{M}}$ to connected objects of $\mathrm{Stone}_{\mathcal{N}}$.*
(3) *For each object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{M}}$ and each point $x \in X$, the maps $\{\Phi(\{x\}, \mathcal{O}_{X,x}) \to \Phi(U, \mathcal{O}_X|_U)\}$ exhibit $\Phi(\{x\}, \mathcal{O}_{X,x})$ as a limit of the diagram $\{\Phi(U, \mathcal{O}_X|_U)\}_{x \in U}$ in the category $\mathrm{Stone}_{\mathcal{N}}$. Here U ranges over all closed and open neighborhoods of the point x.*

Remark 5.1.5. Let $\mathcal{M}$ and $\mathcal{N}$ be left ultracategories, let $F : \mathcal{M} \to \mathcal{N}$ be a left ultrafunctor, and let $\Phi_F : \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}_{\mathcal{N}}$ be the functor of Notation 5.0.1. Let $\{M_s\}_{s \in S}$ be any collection of objects of $\mathcal{M}$, so that Φ induces a map
$$u : \coprod_{s \in S} \Phi(\underline{M_s}) \to \Phi(\coprod_{s \in S} \underline{M_s})$$
in the category $\mathrm{Stone}_{\mathcal{N}}$. Using the description of both coproducts supplied by Proposition 4.2.10, we can identify the underlying topological spaces of both sides with the Stone-Čech compactification βS. At each point $\mu \in \beta S$, u induces the map $\sigma_\mu : F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu$ given by the left ultrastructure on F. Consequently, the following assertions are equivalent:

(1) The left ultrafunctor F is an ultrafunctor: that is, each of the maps σ_μ is an isomorphism in $\mathcal{N}$.
(2) The functor Φ_F preserves coproducts of connected objects of $\mathrm{Stone}_{\mathcal{M}}$.

Remark 5.1.6. In the situation of Theorem 5.1.4, the inverse equivalence is easy to describe. Let $\Phi : \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}_{\mathcal{N}}$ be a functor which satisfies conditions (1), (2), and (3) of Theorem 5.1.4. Then the map of opposite categories $\Phi^{\mathrm{op}} : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Stone}_{\mathcal{N}}^{\mathrm{op}}$ satisfies the hypotheses of Proposition 1.4.9 and therefore restricts to a left ultrafunctor $\Phi|_{\mathcal{M}} : \mathcal{M} \to \mathcal{N}$, which is (up to canonical isomorphism) a preimage of Φ under the equivalence Theorem 5.1.4; see Remark 4.3.7.

Remark 5.1.7. Let $\Phi : \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}_{\mathcal{N}}$ be a functor which satisfies conditions (2) and (3) of Theorem 5.1.4, but not necessarily condition (1). Then the map of opposite categories $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Stone}_{\mathcal{N}}^{\mathrm{op}}$ still satisfies the requirements of Proposition 1.4.9, and therefore restricts to a left ultrafunctor $F = \Phi|_{\mathcal{M}} : \mathcal{M} \to \mathcal{N}$. However, if Φ does not satisfy condition (1), then we cannot conclude that Φ is isomorphic to the functor Φ_F.

The proof of Theorem 5.1.4 will require some preliminaries. We begin by treating the case where $\mathcal{N} = \{*\}$ is a category having a single object and a single morphism, so that $\mathrm{Stone}_{\mathcal{N}}$ can be identified with the category of Stone spaces.

Lemma 5.1.8. *Let $\mathcal{M}$ be an ultracategory, let $\Phi_0 : \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}$ denote the forgetful functor (given on objects by $\Phi_0(X, \mathcal{O}_X) = X$), and let $\Phi : \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}$ be any functor which satisfies conditions (1), (2), and (3) of Theorem 5.1.4. Then there is a unique natural transformation of functors $\alpha : \Phi \to \Phi_0$, and α is an isomorphism.*

Proof. For every Stone space X, let $\mathcal{U}_0(X)$ denote the Boolean algebra of closed and open subsets of X. Let $(X, \mathcal{O}_X)$ be an object of $\mathrm{Stone}_{\mathcal{M}}$. Since the functor Φ satisfies property (1), it carries summands of $(X, \mathcal{O}_X)$ (in the category $\mathrm{Stone}_{\mathcal{M}}$) to summands of $\Phi(X, \mathcal{O}_X)$ (in the category Stone). This induces a homomorphism of Boolean algebras $\mathcal{U}_0(X) \to \mathcal{U}_0(\Phi(X, \mathcal{O}_X))$. By virtue of Stone duality, this Boolean algebra homomorphism is given by taking inverse images along a continuous map $\alpha(X, \mathcal{O}_X) : \Phi(X, \mathcal{O}_X) \to X$. Here α is characterized by the requirement that, for each closed and open subset $U \subseteq X$, the inverse image $\alpha(X, \mathcal{O}_X)^{-1}(U) \subseteq \Phi(X, \mathcal{O}_X)$ is the image of the map $\Phi(U, \mathcal{O}_X|_U) \to \Phi(X, \mathcal{O}_X)$. From this, we see immediately that the construction $(X, \mathcal{O}_X) \mapsto \alpha(X, \mathcal{O}_X)$ is a natural transformation of functors (and that any other natural transformation of functors is equal to α).

We will complete the proof by showing that for each object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{M}}$, the map $\alpha(X, \mathcal{O}_X) : \Phi(X, \mathcal{O}_X) \to X$ is a homeomorphism. Since $\alpha(X, \mathcal{O}_X)$ is a continuous map between compact Hausdorff spaces, it will suffice to show that $\alpha(X, \mathcal{O}_X)$ is bijective. Fix a point $x \in X$; we wish to show that the inverse image $\alpha(X, \mathcal{O}_X)^{-1}\{x\}$ consists of a single point. Since Φ satisfies condition (3), this inverse image can be identified with the space $\Phi(\{x\}, \mathcal{O}_{X,x})$, so that the desired result follows from condition (2). □

Lemma 5.1.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, and let $\Phi : \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}_{\mathcal{N}}$ be a functor which fits into a commutative diagram*

$$\begin{tikzcd} \mathrm{Stone}_{\mathcal{M}} \arrow[rr, "\Phi"] \arrow[dr] & & \mathrm{Stone}_{\mathcal{N}} \arrow[dl] \\ & \mathrm{Stone}, & \end{tikzcd}$$

where the vertical maps are the forgetful functors. Then Φ satisfies conditions (1) and (3) of Theorem 5.1.4 if and only if it preserves the class of Cartesian morphisms with respect to the vertical fibrations.

Remark 5.1.10. In the situation of Lemma 5.1.9, the functor Φ automatically satisfies condition (2) of Theorem 5.1.4.

Proof of Lemma 5.1.9. For each object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{M}}$, let us write $\Phi(X, \mathcal{O}_X) = (X, \Phi_X(\mathcal{O}_X))$, where $\Phi_X : \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M}) \to \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{N})$ is the functor obtained from Φ^{op} by passing to the fiber over $X \in$ Stone. Then, for any map of Stone spaces $Y \to X$, we have a comparison map $\alpha_{Y/X} : \Phi_X(\mathcal{O}_X)|_Y \to$

$\Phi_Y(\mathcal{O}_X|_Y)$, and the functor Φ preserves Cartesian morphisms if and only if each of the maps $\alpha_{Y/X}$ is an isomorphism. Note that Φ satisfies condition (1) of Theorem 5.1.4 if and only if $\alpha_{Y/X}$ is an isomorphism whenever Y is a closed and open subset of X (see Remark 4.1.8). If this condition is satisfied, then Φ satisfies condition (3) of Theorem 5.1.4 if and only if $\alpha_{Y/X}$ is an isomorphism whenever Y consists of a single point. In this case, $\alpha_{Y/X}$ is an isomorphism in general: this follows by observing that, for each point $y \in Y$, the induced map $(\alpha_{Y/X})|_{\{y\}} : \Phi_X(\mathcal{O}_X)|_{\{y\}} \to \Phi_Y(\mathcal{O}_X|Y)|_{\{y\}}$ fits into a commutative diagram

$$\begin{tikzcd}
\Phi_X(\mathcal{O}_X)|_{\{y\}} \arrow[rr, "\alpha_{\{y\}/X}"] \arrow[dr, "\alpha_{Y/X}|_{\{y\}}"'] & & \Phi_{\{y\}}(\mathcal{O}_X|_{\{y\}}) \\
& \Phi_Y(\mathcal{O}_X|_Y)|_{\{y\}}. \arrow[ur, "\alpha_{\{y\}/Y}"'] &
\end{tikzcd}$$

□

For every compact Hausdorff space X, there exists a surjection of compact Hausdorff spaces $\pi : Y \twoheadrightarrow X$, where Y is a Stone space (for example, we can take Y to be the Stone-Čech compactification of the underlying set of X). It follows that the category of Stone spaces can be regarded as a *basis* for the coherent topology on the category Comp (Definition B.6.1). Combining this observation with Proposition 4.1.5, we obtain the following:

Lemma 5.1.11. *Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. Then the restriction functor*

$$\mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}}) \to \mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})$$

is an equivalence of categories.

Proof of Theorem 5.1.4. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, and consider the functors

$$\begin{aligned}
\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathcal{N}) &\xrightarrow{\theta} \mathrm{Cart}_{\mathrm{Comp}}(\mathrm{Comp}_{\mathcal{M}}, \mathrm{Comp}_{\mathcal{N}})^{\mathrm{op}} \\
&\xrightarrow{\theta'} \mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})^{\mathrm{op}} \\
&\xrightarrow{\theta''} \mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})^{\mathrm{op}}.
\end{aligned}$$

Here θ is given by the construction of Example 4.3.2, θ' is the restriction functor, and θ'' is the inclusion. We wish to show that the composition $\theta'' \circ \theta' \circ \theta$ is a fully faithful embedding, whose essential image is the full subcategory $\mathrm{Fun}'(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})^{\mathrm{op}}$ spanned by those functors which satisfy conditions (1), (2), and (3) of Theorem 5.1.4. Note that the functor θ is an equivalence of categories by virtue of Theorem 4.3.3, and the functor θ' is an equivalence of categories by virtue of Lemma 5.1.11. Moreover, Lemma 5.1.9 implies that θ'' fits into a (homotopy) fiber sequence

$$\mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})^{\mathrm{op}} \xrightarrow{\theta''} \mathrm{Fun}'(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathcal{N}})^{\mathrm{op}} \to \mathrm{Fun}'(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}).$$

From the description of $\mathrm{Fun}'(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone})$ given by Lemma 5.1.8, we conclude that θ'' is an equivalence of categories. □

5.2. **Set-Valued Left Ultrafunctors.** Let $\mathcal{M}$ be an ultracategory and let $F_0 : \mathcal{M} \to \mathrm{Set}$ be a functor. Our goal in this section is to show that every left ultrastructure on F_0 arises from a suitable extension $F : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ via the construction of Proposition 1.4.9. Moreover, there is a canonical choice for the extension F.

Theorem 5.2.1. *Let $\mathcal{M}$ be an ultracategory and let $\mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors F which satisfy the following pair of conditions:*

(a) *For every object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{M}}$ and every point $x \in X$, the canonical map*

$$\varinjlim_{x \in U} F(U, \mathcal{O}_X|_U) \to F(\{x\}, \mathcal{O}_{X,x})$$

is bijective. Here the colimit is taken over all closed and open neighborhoods U of the point x.

(*b*) *The functor F carries finite coproducts in* $\mathrm{Stone}_{\mathcal{M}}$ *to finite products in the category of sets.*

Then the construction of Proposition 1.4.9 induces an equivalence of categories

$$\mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set}).$$

Variant 5.2.2. In the statement of Theorem 5.2.1, we can replace the category Set with any compactly generated category (see Variant 3.4.5), endowed with the categorical ultrastructure of Example 1.3.8.

We will deduce Theorem 5.2.1 by combining Theorem 4.3.3 with the results of §3, which allow us to give a more concrete description of the category $\mathrm{Stone}_{\mathrm{Set}}$.

Remark 5.2.3. For any compact Hausdorff space X, Theorem 3.4.4 supplies an equivalence of categories $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set}) \simeq \mathrm{Shv}(X)$. Moreover, this equivalence depends functorially on X (by virtue of Proposition 3.4.6). It follows that the category $\mathrm{Stone}_{\mathrm{Set}}$ of Notation 5.0.1 is equivalent to another category $\mathrm{Stone}'_{\mathrm{Set}}$, which can be described concretely as follows:

- The objects of $\mathrm{Stone}'_{\mathrm{Set}}$ are pairs $(X, \mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X$ is a sheaf of sets on X.
- If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are objects of $\mathrm{Stone}'_{\mathrm{Set}}$, then a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ in $\mathrm{Stone}'_{\mathrm{Set}}$ is a pair (f, α), where $f : X \to Y$ is a continuous function and $\alpha : f^* \mathcal{O}_Y \to \mathcal{O}_X$ is a morphism of set-valued sheaves on X.

For the remainder of this section, we will abuse notation by identifying $\mathrm{Stone}_{\mathrm{Set}}$ with $\mathrm{Stone}'_{\mathrm{Set}}$: that is, we will think of the objects of $\mathrm{Stone}_{\mathrm{Set}}$ as given by pairs $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of sets, rather than a set-valued left ultrafunctor.

Construction 5.2.4. For each object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathrm{Set}}$, we let $\Gamma(X, \mathcal{O}_X)$ denote the set $\mathcal{O}_X(X)$ of global sections of the sheaf $\mathcal{O}_X$. This construction determines a functor $\Gamma : \mathrm{Stone}_{\mathrm{Set}}^{\mathrm{op}} \to \mathrm{Set}$ which preserves small products and small filtered colimits.

Remark 5.2.5. Let $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ be a filtered diagram in the category $\mathrm{Stone}_{\mathrm{Set}}$ having inverse limit $(X, \mathcal{O}_X)$, so that $X = \varprojlim X_\alpha$ is the inverse limit of the underlying topological spaces and $\mathcal{O}_X$ is given by the direct limit $\varprojlim (\mathcal{O}_{X_\alpha})|_X$. Then the canonical map

$$\varinjlim_{\alpha} \Gamma(X_\alpha, \mathcal{O}_{X_\alpha}) \to \Gamma(X, \mathcal{O}_X)$$

is a bijection.

Let $\mathcal{M}$ be an ultracategory and define $\mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathrm{Set}})$ as in Definition 4.3.1. Note that, for any functor $\Phi : \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}_{\mathrm{Set}}$ belonging to $\mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathrm{Set}})$, the composition $\Gamma \circ \Phi^{\mathrm{op}} : \mathrm{Stone}_{\mathrm{Set}}^{\mathrm{op}} \to \mathrm{Set}$ satisfies conditions (*a*) and (*b*) of Theorem 5.2.1. Moreover, since the functor Γ commutes with small products and small filtered colimits, the functors Φ^{op} and $\Gamma \circ \Phi^{\mathrm{op}}$ restrict to the same set-valued left ultrafunctor on $\mathcal{M}$. Combining this observation with Lemma 5.1.11 and Theorem 4.3.3, we deduce that the composition

$$\mathrm{Cart}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathrm{Set}})^{\mathrm{op}} \xrightarrow{\Gamma\circ} \mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set}) \xrightarrow{\Phi \mapsto \Phi|_{\mathcal{M}}} \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$$

is an equivalence of categories. We can therefore reformulate Theorem 5.2.1 as follows:

Proposition 5.2.6. *Let $\mathcal{M}$ be an ultracategory. Then composition with the functor* $\Gamma : \mathrm{Stone}_{\mathrm{Set}}^{\mathrm{op}} \to \mathrm{Set}$ *of Construction 5.2.4 induces an equivalence of categories*

$$\Theta : \mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathrm{Set}})^{\mathrm{op}} \to \mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set}).$$

Proof. Let $F : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ be any functor and let $(X, \mathcal{O}_X)$ be an object of $\mathrm{Stone}_{\mathcal{M}}$. If F satisfies condition (*b*) of Theorem 5.2.1, then the construction $(U \subseteq X) \mapsto F(U, \mathcal{O}_X|_U)$ carries disjoint unions of closed and open subsets of X to products in the category of sets, and therefore admits an essentially unique extension to a sheaf of sets on X which we will denote by $F(\mathcal{O}_X)$. In this case, we can regard the construction $(X, \mathcal{O}_X) \mapsto (X, F(\mathcal{O}_X))$ as a functor from $\mathrm{Stone}_{\mathcal{M}}$ to $\mathrm{Stone}_{\mathrm{Set}}$. This functor belongs to $\mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathrm{Set}})$ if

and only if, for each object $(X, \mathcal{O}_X)$ in $\mathrm{Stone}_{\mathcal{M}}$ and each map of Stone spaces $f : Y \to X$, the canonical map $F(\mathcal{O}_X)|_Y \to F(\mathcal{O}_X|_Y)$ is an isomorphism of set-valued sheaves on Y. As in the proof of Lemma 5.1.9, it suffices to verify this condition in the case where Y is a point, in which case it translates to the condition that the functor F satisfies condition (a) of Theorem 5.2.1. We therefore obtain a functor

$$\Lambda : \mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathrm{Set}})^{\mathrm{op}},$$

given on objects by the formula $\Lambda(F)(X, \mathcal{O}_X) = (X, F(\mathcal{O}_X))$. Note that the composition $\Theta \circ \Lambda$ is the identity functor from $\mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ to itself. We will complete the proof by observing that there is a canonical isomorphism $\alpha : \mathrm{id} \to \Lambda \circ \Theta$ of functors from the category $\mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathrm{Set}})$ to itself, which assigns to each functor $\Phi \in \mathrm{Cart}_{\mathrm{Stone}}(\mathrm{Stone}_{\mathcal{M}}, \mathrm{Stone}_{\mathrm{Set}})$ the natural transformation

$$\alpha_\Phi : \Phi \to \Lambda(\Theta(\Phi))$$

given on each object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{M}}$ by the map of sheaves given on closed and open sets by the bijection

$$((U \subseteq X) \mapsto \Phi_X(\mathcal{O}_X)(U)) \xrightarrow{\sim} ((U \subseteq X) \mapsto \Phi_U(\mathcal{O}_X|_U)(U)).$$

$\square$

Remark 5.2.7. Let $\mathcal{M}$ be an ultracategory and let $F : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ be a functor satisfying conditions (a) and (b) of Theorem 5.2.1, so that we can regard $F_0 = F|_{\mathcal{M}}$ as a left ultrafunctor from $\mathcal{M}$ to Set. The following conditions are equivalent:

(1) The left ultrafunctor $F_0 : \mathcal{M} \to \mathrm{Set}$ is an ultrafunctor.
(2) For every collection of objects $\{M_s\}_{s \in S}$ indexed by a set S, the canonical map $F(\coprod_{s \in S} \underline{M_s}) \to \prod_{s \in S} F(\underline{M_s})$ is bijective.

The implication (1) ⇒ (2) follows from Proposition 1.4.9. For the converse, we observe that (by the proof of Proposition 5.2.6), the functor F can be recovered from F_0 by the construction

$$(X, \mathcal{O}_X) \mapsto \Gamma(X, F_0(\mathcal{O}_X)).$$

Suppose that F_0 is an ultrafunctor. If $(X, \mathcal{O}_X) = \coprod_{s \in S} \underline{M_s}$ is the object of $\mathrm{Stone}_{\mathcal{M}}$ obtained by applying Proposition 4.2.8 to the collection $\{M_s\}_{s \in S}$ of objects of $\mathcal{M}$, then $(X, F_0(\mathcal{O}_X))$ is the object of $\mathrm{Stone}_{\mathrm{Set}}$ obtained by applying Proposition 4.2.8 to the collection of sets $\{F_0(M_s)\}_{s \in S}$, so that $\Gamma(X, F_0(\mathcal{O}_X)) \simeq \prod_{s \in S} F_0(M_s)$.

5.3. **Left Ultrafunctors and Filtered Colimits.** If $\mathcal{M}$ is an ultracategory which admits small filtered colimits, then the equivalence of Theorem 5.2.1 can be formulated more simply. First, we need an elementary observation.

Proposition 5.3.1. *Let $\mathcal{M}$ be an ultracategory which admits small filtered colimits. Then:*

(a) The category $\mathrm{Comp}_{\mathcal{M}}$ admits small filtered limits.
(b) The full subcategory $\mathrm{Stone}_{\mathcal{M}} \subseteq \mathrm{Comp}_{\mathcal{M}}$ is closed under small filtered limits..

In particular, the category $\mathrm{Stone}_{\mathcal{M}}$ admits small filtered limits.

Proof. Let $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ be a small filtered diagram in the category $\mathrm{Comp}_{\mathcal{M}}$. Then the underlying diagram of topological spaces $\{X_\alpha\}$ admits an inverse limit $X = \varprojlim X_\alpha$, which is also a compact Hausdorff space. For each index α, let $\pi_\alpha : X \to X_\alpha$ be the projection map. Then $\{\pi_\alpha^* \mathcal{O}_{X_\alpha}\}$ is a filtered diagram in the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathcal{M})$, and therefore admits a colimit $\mathcal{O}_X = \varinjlim(\mathcal{O}_{X_\alpha} \circ \pi_\alpha)$ (see Remark 1.4.3). It is straightforward to verify that $(X, \mathcal{O}_X)$ can be regarded as an inverse limit of the diagram $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ in the category $\mathrm{Comp}_{\mathcal{M}}$. Moreover, if each X_α is a Stone space, then X is also a Stone space. $\square$

Variant 5.3.2. In the situation of Proposition 5.3.1, suppose that we assume only that $\mathcal{M}$ admits small κ-filtered colimits, for some regular cardinal κ. In this case, the categories $\mathrm{Comp}_{\mathcal{M}}$ and $\mathrm{Stone}_{\mathcal{M}}$ also admit small κ-filtered limits. Moreover, these limits are preserved by the forgetful functors

$$\mathrm{Comp}_{\mathcal{M}} \to \mathrm{Comp} \qquad \mathrm{Stone}_{\mathcal{M}} \to \mathrm{Stone}.$$

We can now state our main result:

Theorem 5.3.3. *Let $\mathcal{M}$ be an ultracategory which admits small filtered colimits, and let* $\mathrm{Fun}'(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ *denote the full subcategory of* $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ *spanned by those functors which preserve small filtered colimits and finite products. Then the construction of Proposition 1.4.9 induces an equivalence of categories* $\mathrm{Fun}'(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$.

We will deduce Theorem 5.3.3 by combining Theorem 5.2.1 with the following general result, which may be of independent interest:

Proposition 5.3.4. *Let $\mathcal{M}$ be an ultracategory and let $F : \mathcal{M} \to \mathrm{Set}$ be a functor. If there exists a left ultrastructure on F, then F preserves all small filtered colimit which exist in $\mathcal{M}$.*

Proof of Theorem 5.3.3. By virtue of Theorem 5.2.1, we must show that the category $\mathrm{Fun}'(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ appearing in the statement of Theorem 5.3.3 coincides with the category $\mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ in the statement of Theorem 5.2.1. In other words, we must show that if $F : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ is a functor which satisfies conditions (a) and (b) of Theorem 5.2.1, then F commutes with small filtered colimits.

For each object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{M}}$, let $F(\mathcal{O}_X)$ denote the sheaf of sets on X given on closed and open subsets $U \subseteq X$ by the formula $F(\mathcal{O}_X)(U) = F(U, \mathcal{O}_X|_U)$ (see Corollary B.6.5).

Let $\{(X_\alpha, \mathcal{O}_{X_\alpha})\}$ be a filtered diagram in the category $\mathrm{Stone}_{\mathcal{M}}$, having an inverse limit $(X, \mathcal{O}_X)$. For each index α, let $\pi_\alpha : X \to X_\alpha$ denote the projection map, and let $\mathcal{O}_X = \varinjlim_\alpha (\mathcal{O}_{X_\alpha} \circ \pi_\alpha)$ be as in the proof of Proposition 5.3.1. We wish to show that the canonical map

$$\theta : \varinjlim_{\alpha} F(X_\alpha, \mathcal{O}_{X_\alpha}) \to F(X, \mathcal{O}_X)$$

is a bijection. Equivalently, we wish to show that the composite map

$$\varinjlim_{\alpha} F(\mathcal{O}_{X_\alpha})(X_\alpha) \xrightarrow{\theta'} (\varinjlim_{\alpha} \pi_\alpha^* F(\mathcal{O}_{X_\alpha}))(X) \xrightarrow{\theta''} (F(\mathcal{O}_X))(X).$$

Here θ' is is bijective by virtue of Remark 5.2.5. To show that θ'' is bijective, we will prove the stronger assertion that the comparison map $u : \varinjlim_\alpha \pi_\alpha^* F(\mathcal{O}_{X_\alpha}) \to F(\mathcal{O}_X)$ is an isomorphism in the category of sheaves on X. Let u_x denote the map of sets obtained from u by passing to stalks at some point $x \in X$. Setting $x_\alpha = \pi_\alpha(x)$, we see that u_x can be identified with the upper horizontal map in a commutative diagram

$$\begin{tikzcd}
\varinjlim_\alpha F(\mathcal{O}_{X_\alpha})_{x_\alpha} \arrow[r, "u_x"] \arrow[d] & F(\mathcal{O}_X)_x \arrow[d] \\
\varinjlim_\alpha F(\{x_\alpha\}, \mathcal{O}_{X, x_\alpha}) \arrow[r, "\theta'"] & F(\{x\}, \mathcal{O}_{X,x}).
\end{tikzcd}$$

Since F satisfies condition (a) of Theorem 5.2.1, the vertical maps in this diagram are bijective (see Construction 7.5.5). We are therefore reduced to showing that the lower horizontal map is bijective, which follows from the fact that the restriction $F|_{\mathcal{M}}$ admits a left ultrastructure and therefore preserves filtered colimits (Proposition 5.3.4). □

The proof of Proposition 5.3.4 will require some preliminaries.

Notation 5.3.5. Let S be a partially ordered set. For each $s \in S$, set $S_{\geq s} = \{t \in S : t \geq s\}$. We will say that an ultrafilter μ on S is *cofinal* if $\mu(S_{\geq s}) = 1$ for each $s \in S$. In this case, μ restricts to an ultrafilter on each of the subsets $S_{\geq s}$, which we will denote by $\mu_{\geq s}$. Note that there exists a cofinal ultrafilter on S if and only if S is *directed*: that is, every finite subset of S has an upper bound (this is a consequence of Proposition 1.1.10).

Construction 5.3.6. Let $\mathcal{M}$ be an ultracategory and suppose we are given a diagram $(\{M_s\}, \{\varphi_{s,t} : M_s \to M_t\})$ in $\mathcal{M}$, indexed by a partially ordered set S. Let μ be a cofinal ultrafilter on S. For each $s_0 \in S$, let $w_s : M_s \to \int_S M_t d\mu$ denote the composite map

$$M_s \xrightarrow{\Delta_{\mu_{\geq s}}} \int_{S_{\geq s}} M_s d\mu_{\geq s} \xrightarrow{\int_{S_{\geq s}} \varphi_{s,t} d\mu_{\geq s}} \int_{S_{\geq s}} M_t d\mu_{\geq s} \xrightarrow{\Delta_{\mu_{\geq s}, \iota}} \int_S M_t d\mu,$$

where $\Delta_{\mu_{\geq s}}$ is the ultrapower diagonal map of Example 1.3.4 and $\Delta_{\mu_{\geq s},\iota}$ is the isomorphism induced by the inclusion $\iota : S_{\geq s} \hookrightarrow S$ (Notation 1.3.3). Note that the maps w_s satisfy $w_s = w_t \circ \varphi_{s,t}$ for $s \leq t$. Consequently, if the diagram $(\{M_s\}, \{\varphi_{s,t} : M_s \to M_t\})$ admits a colimit in $\mathcal{M}$, we obtain a canonical map

$$w : \varinjlim_{s\in S} M_s \to \int_S M_t d\mu.$$

Remark 5.3.7. In the situation of Construction 5.3.6, let $\{\psi_t : M_t \to M\}$ be a collection of morphisms in $\mathcal{M}$ which exhibit M as a colimit of the diagram $(\{M_s\}, \{\varphi_{s,t} : M_s \to M_t\})$. Then the composite map

$$M \xrightarrow{w} \int_S M_t d\mu \xrightarrow{\int_S \psi_t d\mu} \int_S M d\mu = M^\mu$$

coincides with the ultrapower diagonal Δ_μ of Example 1.3.4.

Remark 5.3.8. In the situation of Construction 5.3.6, suppose that $\mathcal{M} = \text{Set}$ is the category of sets (equipped with the categorical ultrastructure of Example 1.3.8). In this case, the map $w : M \to \int_S M_t d\mu$ is injective. To prove this, suppose we are given a pair of elements $x, y \in M_s$ for some $s \in S$ satisfying $w_s(x) = w_s(y)$. Unwinding the definitions, we see that the set $S' = \{t \in S_{\geq s} : \varphi_{s,t}(x) = \varphi_{s,t}(y)\}$ satisfies $\mu(S') = 1$. In particular, S' is nonempty, so that x and y have the same image in M_t for some $t \geq s$.

Proof of Proposition 5.3.4. Let $\mathcal{M}$ be an ultracategory and let $F : \mathcal{M} \to \text{Set}$ be a functor which admits a left ultrastructure $\{\sigma_\mu\}$. We wish to show that, for every small filtered category $\mathcal{I}$ and every diagram $U : \mathcal{I} \to \mathcal{M}$ which admits a colimit, the canonical map $\theta : \varinjlim(F \circ U) \to F(\varinjlim(U))$ is a bijection. Then there exists a directed partially ordered set S and a cofinal functor $S \to \mathcal{I}$ (see, for example, [13, Tag 0032]). Replacing $\mathcal{I}$ by S, we may assume that $\mathcal{I} = S$ is a directed partially ordered set, so that the diagram U is given by a collection of objects $\{M_s\}_{s\in S}$ and transition morphisms $\{\varphi_{s,t} : M_s \to M_t\}_{s\leq t}$. Let $\{\psi_s : M_s \to M\}_{s\in S}$ be a collection of morphisms in $\mathcal{M}$ which exhibit M as a colimit of the functor U. Choose a cofinal ultrafilter μ on S. Applying Construction 5.3.6 in the ultracategories $\mathcal{M}$ and Set, we obtain maps

$$w : M = \varinjlim M_s \to \int_S M_t d\mu \qquad w' : \varinjlim F(M_s) \to \int_S F(M_t) d\mu.$$

These maps fit into a commutative diagram

$$\begin{tikzcd}
\varinjlim F(M_s) \arrow[r, "w'"] \arrow[d, "\theta"] & \int_S F(M_t)d\mu \arrow[r, "\int_S F(\psi_t)d\mu"] & F(M)^\mu \\
F(M) \arrow[r, "F(w)"] & F(\int_S M_t d\mu) \arrow[u, "\sigma_\mu"] \arrow[r, "F(\int_S \psi_t d\mu)"] & F(M^\mu). \arrow[u, "\sigma_\mu"]
\end{tikzcd}$$

Since the map w' is injective (Remark 5.3.8), we immediately deduce that θ is injective. To prove surjectivity, suppose we are given an element $x \in F(M)$. Then $(\sigma_\mu \circ F(w))(x)$ is an element y of the ultraproduct $\int_S F(M_t) d\mu$ which we can represent by a collection of elements $\{y_t \in F(M_t)\}_{t\in S_0}$ for some subset $S_0 \subseteq S$ satisfying $\mu(S_0) = 1$. Note that the composition $(\int_S \psi_t d\mu) \circ w$ coincides with the ultrapower diagonal $\Delta_\mu : M \to M^\mu$ (Remark 5.3.7). It follows that the composition

$$F(M) \xrightarrow{F(w)} F(\int_S M_t d\mu) \xrightarrow{F(\int_S \psi_t d\mu)} F(M^\mu) \xrightarrow{\sigma_\mu} F(M)^\mu$$

agrees with the ultrapower diagonal $\Delta_\mu : F(M) \to F(M)^\mu$ in the category of sets. We therefore have an identity $(\int_S F(\psi_t) d\mu)(y) = \Delta_\mu(x)$ in the ultrapower $F(M)^\mu$, which translates concretely to the statement that the set $S_1 = \{t \in S_0 : F(\psi_t)(y_t) = x\}$ satisfies $\mu(S_1) = 1$. In particular, the set S_1 is nonempty. Choosing $t \in S_1$, we conclude that x belongs to the image of the map $F(\psi_t) : F(M_t) \to F(M)$ and therefore also to the image of θ. $\square$

5.4. **Accessible Ultrastructures.** Let $\mathcal{M}$ be an ultracategory and let $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ denote the category of set-valued left ultrafunctors on $\mathcal{M}$. In §2 and §3, we have seen two examples in which the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is a Grothendieck topos:

- If $\mathcal{M} = \mathrm{Mod}(\mathcal{C})$ is the category of models of a small pretopos $\mathcal{C}$ (endowed with the ultrastructure of Remark 2.1.2), then the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is equivalent to the topos $\mathrm{Shv}(\mathcal{C})$ of sheaves on $\mathcal{C}$ (Theorem 2.2.2).
- If $\mathcal{M} = X$ is a compact Hausdorff space (regarded as an ultracategory having only identity morphisms), then the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is equivalent to the topos $\mathrm{Shv}(X)$ of sheaves on X (Theorem 3.4.4).

Our goal in this section is to show that this is a rather general phenomenon. Of course, it is not completely general: to guarantee that the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is a reasonable mathematical object, we will need to assume that the ultracategory $\mathcal{M}$ is "not too big". To formulate this precisely, it is convenient to use the language of *accessible categories* (see [1]).

Definition 5.4.1. Let $\mathcal{M}$ be a category. We will say that an ultrastructure on $\mathcal{M}$ is *accessible* if the following conditions are satisfied:

- For some regular cardinal κ, the category $\mathcal{M}$ admits small κ-filtered colimits.
- The category $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$ of Construction 4.1.1 is accessible.

Remark 5.4.2. Let $\mathcal{M}$ be an ultracategory which admits small κ-filtered colimits, for some regular cardinal κ. Then the forgetful functor $F : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Stone}^{\mathrm{op}}$ preserves small κ-filtered colimits (Variant 5.3.2). If the ultrastructure on $\mathcal{M}$ is accessible, then F is an accessible functor between accessible categories. It follows that the category $\mathcal{M}$ is also accessible (since it can be realized as the 2-categorical fiber product $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \times_{\mathrm{Stone}} \{*\}$).

Example 5.4.3. Let $\mathcal{C}$ be a small pretopos. Then the ultrastructure on $\mathrm{Mod}(\mathcal{C})$ is accessible; this follows from the description of the category $\mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})}$ supplied by Theorem 6.3.14.

Example 5.4.4. Let X be a compact Hausdorff space, regarded as a category having only identity morphisms. Then the ultrastructure of Proposition 3.3.1 is accessible. This follows from the description of $\mathrm{Stone}_X \subseteq \mathrm{Comp}_X$ supplied by Example 4.1.4.

We can now state the main result of this section:

Proposition 5.4.5. *Let $\mathcal{M}$ be a category equipped with an accessible ultrastructure. Then the category of left ultrafunctors* $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ *is a Grothendieck topos.*

The main content of Proposition 5.4.5 is contained in the following result, which does not require any set-theoretic assumptions on $\mathcal{M}$:

Proposition 5.4.6. *Let $\mathcal{M}$ be an ultracategory. Then the category of left ultrafunctors* $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ *is a pretopos which admits small colimits. Moreover, for every morphism $\alpha : F \to G$ in* $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$*, the pullback functor* $\alpha^* : \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})_{/G} \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})_{/F}$ *preserves small colimits.*

Proof. The category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ admits small colimits by Remark 1.4.3 and finite limits by Corollary 2.1.4. For every natural transformation of left ultrafunctors $\alpha : F \to G$ and every small diagram $\{G_\alpha\}$ in $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})_{/G}$, the canonical map

$$\varinjlim_{\alpha}(G_\alpha \times_G F) \to (\varinjlim_{\alpha} G_\alpha) \times_G F$$

is an isomorphism of left ultrafunctors because it induces a bijection after evaluating on any object $M \in \mathcal{M}$ (since colimits and fiber products in $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ are computed pointwise, and the formation of colimits in Set is compatible with pullback). In particular, the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is extensive. To complete the proof that it is a pretopos, it will suffice to show that for every left ultrafunctor $F \in \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ and every equivalence relation $R \subseteq F \times F$ in $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$, the canonical map

$$R \to F \times_{F/R} F$$

is an isomorphism of left ultrafunctors, where $F/R = \mathrm{Coeq}(R \rightrightarrows F)$ denotes the quotient of F by the equivalence relation R. This also follows from the corresponding assertion in the category of sets, since the coequalizer defining F/R and the fiber product $F \times_{F/R} F$ are computed pointwise. □

Proof of Proposition 5.4.5. Let $\mathcal{M}$ be a category equipped with an accessible ultrastructure; we wish to show that the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is a Grothendieck topos. By virtue of Proposition 5.4.6 and Remark C.1.9, it will suffice to show that the category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is accessible. Using Theorem 5.2.1, we can identify $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ with the full subcategory $\mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors $F : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ satisfying conditions (a) and (b) of Theorem 5.2.1.

Choose a regular cardinal κ such that $\mathcal{M}$ admits small κ-filtered colimits. Enlarging κ if necessary, we may further assume that the category $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$ is κ-accessible: that is, it is equivalent to $\mathrm{Ind}_{\kappa}(\mathcal{E})$, where $\mathcal{E}$ is a small category. Let us abuse notation by identifying $\mathcal{E}$ with its image in $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$. Then $\mathcal{E}$ can be identified with the full subcategory of $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$ spanned by the κ-compact objects. iConsequently, the full subcategory $\mathcal{E} \subseteq \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$ is closed under finite products, and also under the formation of direct factors. Note that, for every functor $F : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ satisfying (a) and (b) of Theorem 5.2.1, the restriction $F|_{\mathcal{M}} : \mathcal{M} \to \mathrm{Set}$ admits a left ultrastructure (Proposition 1.4.9), and therefore preserves small κ-filtered colimits (Proposition 5.3.4). Arguing as in the proof of Theorem 5.3.3, we see that the functor F itself preserves small κ-filtered colimits.

Let $\mathrm{Fun}_{\kappa}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors which preserve small κ-filtered colimits, so that the restriction functor

$$\mathrm{Fun}_{\kappa}(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}(\mathcal{E}, \mathrm{Set})$$

is an equivalence of categories. Let $\mathrm{Fun}_0(\mathcal{E}, \mathrm{Set})$ denote the essential image of $\mathrm{Fun}_0(\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}, \mathrm{Set})$ under this equivalence, so that $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$ is equivalent to $\mathrm{Fun}_0(\mathcal{E}, \mathrm{Set})$. We will complete the proof by showing that $\mathrm{Fun}_0(\mathcal{E}, \mathrm{Set})$ is an accessible subcategory of the presheaf category $\mathrm{Fun}(\mathcal{E}, \mathrm{Set})$. To prove this, it suffices to observe that a functor $F_0 : \mathcal{E} \to \mathrm{Set}$ belongs to $\mathrm{Fun}_0(\mathcal{E}, \mathrm{Set})$ if and only if satisfies the following analogues of (a) and (b):

- (a') Let $F : \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}} \to \mathrm{Set}$ be a left Kan extension of F_0. Then, for each object $(X, \mathcal{O}_X)$ in $\mathcal{E}$, the canonical map
$$\varinjlim_{x \in U} F(U, \mathcal{O}_X|_U) \to F(\{x\}, \mathcal{O}_{X,x})$$
is bijective.
- (b') For every finite collection of objects $(X_i, \mathcal{O}_{X_i})$ of $\mathcal{E}$, the canonical map
$$F(\coprod X_i, \mathcal{O}_{\amalg X_i}) \to \prod F_0(X_i, \mathcal{O}_{X_i})$$
is bijective.

This characterization exhibits $\mathrm{Fun}_0(\mathcal{E}, \mathrm{Set})$ as an intersection of a bounded number of accessible subcategories of $\mathrm{Fun}(\mathcal{E}, \mathrm{Set})$, so that $\mathrm{Fun}_0(\mathcal{E}, \mathrm{Set})$ is itself accessible. □

Let $\mathcal{M}$ be a category equipped with an accessible ultrastructure. For each object $M \in \mathcal{M}$, evaluation on M induces a functor

$$\mathrm{ev}_M : \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set}) \to \mathrm{Set}$$

which preserves small colimits and finite limits (Remark 1.4.3) and finite limits (Corollary 2.1.4). We can therefore regard ev_M as a *point* of the Grothendieck topos $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set})$. Moreover, the construction $M \mapsto \mathrm{ev}_M$ determines a functor

$$\mathrm{ev} : \mathcal{M} \to \mathrm{Fun}^*(\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set}), \mathrm{Set}).$$

This functor is an equivalence in the following cases:

- If $\mathcal{M}$ is the ultracategory $\mathrm{Mod}(\mathcal{C})$ of models of a small pretopos $\mathcal{C}$, then the functor ev is an equivalence of categories; it is homotopy inverse to the equivalence of Corollary 2.2.6.

- If $\mathcal{M} = X$ is a compact Hausdorff space (regarded as an ultracategory having only identity morphisms), then the functor ev is an equivalence of categories (every point of the sheaf topos $\mathrm{Shv}(X)$ is determined by a point of the topological space X).

However, the functor ev is not an equivalence in general.

Counterexample 5.4.7. Let G be a compact topological group and let BG be the ultracategory of Example 4.4.5. Let e denote the unique object of the category BG, so that we can identify G with the automorphism group $\mathrm{Aut}_{\mathrm{BG}}(e)$. Let us abuse notation by identifying e with a left ultrafunctor from the one-object category $*$ to BG, so that we have a homotopy pullback square

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\pi"'] & * \arrow[d, "e"] \\ * \arrow[r, "e"'] \arrow[ur, Rightarrow, "\alpha"] & \mathrm{BG} \end{tikzcd}$$

in the 2-category **Ult** of Remark 1.4.6.

Let $F : \mathrm{BG} \to \mathrm{Set}$ be a left ultrafunctor. Then α determines an automorphism φ of the composite left ultrafunctor

$$G \xrightarrow{\pi} * \xrightarrow{e} \mathrm{BG} \xrightarrow{F} \mathrm{Set},$$

which we can identify (under the equivalence of Theorem 3.4.4) with the constant sheaf $\underline{F(e)}_G$. At each point $g \in G$, the automorphism φ determines an automorphism of the stalk $\underline{F(e)}_{G,g} \simeq F(e)$, which is simply given by the action of $g \in G = \mathrm{Aut}_{\mathrm{BG}}(e)$ on $F(e)$ by the functoriality of F. The fact that φ is automorphism of sheaves guarantees that this automorphism is constant on the identity component $G^\circ \subseteq G$. Consequently, the action of G° on the object $e \in \mathrm{BG}$ induces the trivial action of G° on the image $\mathrm{ev}(e) \in \mathrm{Fun}^*(\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{BG}, \mathrm{Set}), \mathrm{Set})$. In particular, if G° is nontrivial, then the evaluation functor

$$\mathrm{ev} : \mathrm{BG} \to \mathrm{Fun}^*(\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{BG}, \mathrm{Set}), \mathrm{Set})$$

cannot be an equivalence of categories.

With a bit more effort, one can show that the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{BG}, \mathrm{Set})$ is equivalent to the Grothendieck topos of sets equipped with a continuous action of the profinite group $\pi_0(G) = G/G^\circ$. In particular, we can identify ev with the canonical map $\mathrm{BG} \to \mathrm{B}(\pi_0(G))$.

6. The Category $\mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})}$

Let $\mathcal{C}$ be a small pretopos and let $\mathrm{Mod}(\mathcal{C})$ denote the category of models of $\mathcal{C}$, which we endow with the ultrastructure of Remark 2.1.2. The ultimate goal of this paper is to prove Theorem 2.2.2, which asserts that the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ is equivalent to the topos $\mathrm{Shv}(\mathcal{C})$ of sheaves on $\mathcal{C}$. To prove this, we will take advantage of the characterization of (set-valued) left ultrafunctors established in §5. According to Theorem 5.3.3, the datum of a left ultrafunctor $\mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ is equivalent to the datum of a functor $F : \mathrm{Stone}^{\mathrm{op}}_{\mathrm{Mod}(\mathcal{C})} \to \mathrm{Set}$ which preserves finite products and small filtered colimits. In some sense, this is progress: by trading the category of models $\mathrm{Mod}(\mathcal{C})$ for the larger category $\mathrm{Stone}^{\mathrm{op}}_{\mathrm{Mod}(\mathcal{C})}$, we can reduce Theorem 2.2.2 to a statement about ordinary categories and ordinary functors, rather than ultracategories and (left) ultrafunctors. The caveat is that the ultrastructure on $\mathrm{Mod}(\mathcal{C})$ is embedded into the *definition* of the category $\mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})}$: recall that the objects of $\mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})}$ are given by pairs $(X, \mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X$ is a left ultrafunctor from X to $\mathrm{Mod}(\mathcal{C})$. Our goal in this section is to show that the category $\mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})}$ has an alternative description, which is completely independent of the theory of ultracategories:

Theorem 6.0.1. *Let $\mathcal{C}$ be a small pretopos. Then there is a canonical equivalence of categories $\mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})} \simeq \mathrm{Fun}^{\mathrm{reg}}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$. Here $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{C}, \mathrm{Set})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ spanned by those functors $P : \mathcal{C} \to \mathrm{Set}$ which are* regular*: that is, which preserve finite limits and effective epimorphisms.*

Remark 6.0.2. Let $\mathcal{C}$ be a small pretopos. Combining Theorem 6.0.1 with Theorem 5.3.3, we obtain a fully faithful embedding

$$\mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set}) \hookrightarrow \mathrm{Fun}(\mathrm{Fun}^{\mathrm{reg}}(\mathcal{C},\mathrm{Set}),\mathrm{Set}).$$

Moreover, the essential image of this embedding contains all functors $F : \mathrm{Fun}^{\mathrm{reg}}(\mathcal{C},\mathrm{Set}) \to \mathrm{Set}$ which preserve small products and small filtered colimits (see Remark 5.2.7). From the construction, it will follow immediately that the composition

$$\mathcal{C} \xrightarrow{\mathrm{ev}} \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set}) \hookrightarrow \mathrm{Fun}(\mathrm{Fun}^{\mathrm{reg}}(\mathcal{C},\mathrm{Set}),\mathrm{Set})$$

is the Barr embedding of $\mathcal{C}$ (see §2.4); here ev denotes the evaluation map of Construction 2.2.1. Consequently, from Makkai's description of the image the Barr embedding (Theorem 2.4.2) we can deduce Makkai's strong conceptual completeness theorem (Theorem 2.3.1), which asserts that the evaluation map $\mathrm{ev} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set})$ is an equivalence of categories (recall that we proved the reverse implication in §2.4). We will not make use of this observation: instead, we will apply Theorem 6.0.1 in §7 to prove a stronger version of Makkai's conceptual completeness theorem (Theorem 2.2.2), which describes the larger category $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}),\mathrm{Set})$ of *left* ultrafunctors from $\mathrm{Mod}(\mathcal{C})$ to Set.

Let us now outline our approach to Theorem 6.0.1. In §6.3, we introduce a category $\mathrm{Stone}_{\mathcal{C}}$ whose objects are pairs $(X,\mathcal{O}_X)$, where X is a Stone space and $\mathcal{O}_X$ is a pretopos functor from $\mathcal{C}$ to the topos $\mathrm{Shv}(X)$ (Definition 6.3.8). Using the results of §3, we construct an equivalence of categories $\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})}$ (see Warning 6.3.9). We then construct a comparison functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Fun}^{\mathrm{reg}}(\mathcal{C},\mathrm{Set})^{\mathrm{op}}$, given concretely by the formula $\Gamma(X,\mathcal{O}_X)(C) = \mathcal{O}_X^C(X)$ (where $\mathcal{O}_X^C$ denotes the image of an object $C \in \mathcal{C}$ under the pretopos functor $\mathcal{O}_X$). We deduce Theorem 6.0.1 from the more precise assertion that the functor Γ is an equivalence of categories, which we prove in §6.4.

Note that the category $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{C},\mathrm{Set})$ of regular functors from $\mathcal{C}$ to Set is contained in the larger category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C},\mathrm{Set})$ of functors $P : \mathcal{C} \to \mathrm{Set}$ which preserve finite limits. Throughout this section, it will be convenient to phrase our results in terms of the *opposite category* $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C},\mathrm{Set})^{\mathrm{op}}$, which we denote by $\mathrm{Pro}(\mathcal{C})$ and refer to as the *category of pro-objects of* $\mathcal{C}$. In §6.1, we recall some standard facts about the category $\mathrm{Pro}(\mathcal{C})$ and study properties that it inherits from $\mathcal{C}$. In particular, we show that if $\mathcal{C}$ is a small pretopos, then the category $\mathrm{Pro}(\mathcal{C})$ is regular and extensive (Corollary 6.1.20), which we will exploit in §7 to study *sheaves* on $\mathrm{Pro}(\mathcal{C})$. The category $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{C},\mathrm{Set})^{\mathrm{op}}$ can then be identified with the full subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ of *weakly projective* pro-objects of $\mathcal{C}$ (Definition 6.2.2). In §6.2, we recall the proof (due to Barr) that the Barr embedding of a small regular category is fully faithful (Theorem 2.4.1). This is a consequence of the stronger assertion that the category $\mathrm{Pro}(\mathcal{C})$ has "enough" weakly projective objects (Proposition 6.2.12), which will be needed in §7.

6.1. **Pro-Objects.** We begin with a short review of the theory of pro-objects of a small category $\mathcal{C}$. To simplify the discussion, we will confine our attention to the case where the category $\mathcal{C}$ admits finite limits.

Definition 6.1.1. Let $\mathcal{C}$ be a small category which admits finite limits. A *pro-object of* $\mathcal{C}$ is a functor $P : \mathcal{C} \to \mathrm{Set}$ which preserves finite limits. If P and Q are two such functors, then a *morphism of pro-objects* from P to Q is a natural transformation of functors $\alpha : Q \to P$. We let $\mathrm{Pro}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C},\mathrm{Set})^{\mathrm{op}}$ denote the category whose objects are pro-objects of $\mathcal{C}$.

Example 6.1.2. Let $\mathcal{C}$ be a small category which admits finite limits. Then every object $C \in \mathcal{C}$ determines a pro-object of $\mathcal{C}$, given the functor $\mathrm{Hom}_{\mathcal{C}}(C,\bullet)$ corepresented by C. By virtue of Yoneda's lemma, the construction $C \mapsto \mathrm{Hom}_{\mathcal{C}}(C,\bullet)$ determines a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$. We will generally abuse notation by identifying $\mathcal{C}$ with its essential image under this embedding: that is, we will not distinguish between an object $C \in \mathcal{C}$ and the corresponding pro-object $\mathrm{Hom}_{\mathcal{C}}(C,\bullet)$. Moreover, if $X : \mathcal{C} \to \mathrm{Set}$ is any pro-object of $\mathcal{C}$, we will identify $X(C)$ with the set $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X,C)$ of morphisms from X to C in the category $\mathrm{Pro}(\mathcal{C})$.

Example 6.1.3 (Profinite Sets)**.** Let X be a topological space and let Fin denote the category of finite sets. We let $\Gamma(X) : \mathrm{Fin} \to \mathrm{Set}$ denote the functor given by the formula $\Gamma(X)(S) = \mathrm{Hom}_{\mathrm{Top}}(X,S)$: that is, it

carries a finite set S to the collection of locally constant S-valued functions on X. Since the inclusion functor $\mathrm{Fin} \hookrightarrow \mathrm{Top}$ preserves finite limits, the functor $\Gamma(X)$ preserves finite limits, and can therefore be regarded as an object of the category $\mathrm{Pro}(\mathrm{Fin})$ of profinite sets. The construction $X \mapsto \Gamma(X)$ induces an equivalence of categories $\mathrm{Stone} \simeq \mathrm{Pro}(\mathrm{Fin})$.

Remark 6.1.4. Let $\mathcal{C}$ be a small category which admits finite limits. Since the formation of finite limits commutes with filtered colimits in the category Set, the full subcategory $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ is closed under (small) filtered colimits. Passing to opposite categories, we conclude:

- The category $\mathrm{Pro}(\mathcal{C})$ admits small filtered limits.
- For every object $C \in \mathcal{C}$, the evaluation functor $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\bullet, C)$ carries small filtered limits in $\mathrm{Pro}(\mathcal{C})$ to filtered colimits of sets.

Remark 6.1.5. Let $\mathcal{C}$ be a small category which admits finite limits, and suppose we are given a small filtered diagram $\{C_\alpha\}$ in $\mathcal{C}$. Then, in the category $\mathrm{Pro}(\mathcal{C})$, this diagram admits an inverse limit $\varprojlim C_\alpha$, given concretely by the functor $D \mapsto \varinjlim \mathrm{Hom}_{\mathcal{C}}(C_\alpha, D)$. Beware that this is generally different from the inverse limit of the diagram $\{C_\alpha\}$ in $\mathcal{C}$ (if such a limit exists).

Remark 6.1.6. Let $\mathcal{C}$ be a small category which admits finite limits. One can show that every object of $\mathrm{Pro}(\mathcal{C})$ arises as the inverse limit of a filtered diagram in $\mathcal{C}$. Consequently, the category $\mathrm{Pro}(\mathcal{C})$ can be described more informally as follows:

- The objects of $\mathrm{Pro}(\mathcal{C})$ are "formal" inverse limits $\varprojlim C_\alpha$, where $\{C_\alpha\}$ is a small filtered diagram in $\mathcal{C}$.
- Given small filtered diagrams $\{C_\alpha\}$ and $\{D_\beta\}$, we have canonical bijections

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\varprojlim_\alpha C_\alpha, \varprojlim_\beta D_\beta) &\simeq \varprojlim_\beta \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\varprojlim_\alpha C_\alpha, D_\beta) \\ &\simeq \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}_{\mathcal{C}}(C_\alpha, D_\beta). \end{aligned}$$

Remark 6.1.7. Let $\mathcal{C}$ be a small category which admits finite limits. Then the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ can be characterized (up to equivalence) by a universal mapping property. Let $\mathcal{D}$ be any category which admits small filtered limits. Then precomposition with the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ induces an equivalence of categories $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve small filtered limits. More informally: the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ is obtained from $\mathcal{C}$ by "freely" adjoining small filtered limits. See Proposition 8.7.3 of [6], Expose 1.

Remark 6.1.8 (Functoriality)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be small categories which admit finite limits and let $g : \mathcal{C} \to \mathcal{D}$ be a left exact functor. Then precomposition with g induces a functor $F : \mathrm{Pro}(\mathcal{D}) \to \mathrm{Pro}(\mathcal{C})$. This functor F admits a right adjoint $G : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$. To prove this, we must show that for every pro-object $X \in \mathrm{Pro}(\mathcal{C})$, the functor $(Y \in \mathrm{Pro}(\mathcal{D})) \mapsto \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(F(Y), X)$ is representable by an object of $\mathrm{Pro}(\mathcal{D})$. Writing X as a filtered limit of objects of $\mathcal{C}$ (Remark 6.1.6), we can assume that X belongs to $\mathcal{C}$. In this case, the desired result follows from the bijections $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(F(Y), X) \simeq (Y \circ g)(X) \simeq \mathrm{Hom}_{\mathrm{Pro}(\mathcal{D})}(Y, g(X))$.

It follows from the above argument that the restriction $G|_{\mathcal{C}}$ is given by the composition of $g : \mathcal{C} \to \mathcal{D}$ with the fully faithful embedding $\mathcal{D} \hookrightarrow \mathrm{Pro}(\mathcal{D})$. Moreover, the functor G preserves filtered limits (since it is a right adjoint). Using the universal property of Remark 6.1.7, we see that the functor G is characterized (up to isomorphism) by these requirements.

Remark 6.1.9. In the situation of Remark 6.1.8, suppose that we drop the assumption that the functor $g : \mathcal{C} \to \mathcal{D}$ is left exact. In this case, the universal property of Remark 6.1.7 guarantees that g admits an essentially unique extension $G : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$ which commutes with filtered limits. Concretely, the functor G can be constructed as a right Kan extension of the composition $\mathcal{C} \xrightarrow{g} \mathcal{D} \hookrightarrow \mathrm{Pro}(\mathcal{D})$. However, it generally cannot be characterized as in Remark 6.1.8: if the functor g does not preserve finite limits, then neither does G, so G cannot admit a left adjoint.

Proposition 6.1.10. *Let $\mathcal{C}$ be a small category which admits finite limits, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under finite limits, and let $G : \mathrm{Pro}(\mathcal{C}_0) \to \mathrm{Pro}(\mathcal{C})$ be the essentially unique extension of the inclusion map $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ which commutes with filtered limits. Then G is fully faithful, and the essential image of G consists of those pro-objects $X \in \mathrm{Pro}(\mathcal{C})$ which satisfy the following condition:*

- $(*)$ *For $C \in \mathcal{C}$, every morphism $f : X \to C$ admits a factorization $X \to C_0 \to C$ where C_0 belongs to the subcategory $\mathcal{C}_0$.*

Proof. Let $F : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}_0)$ be the left adjoint to G, given concretely by the restriction map

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0, \mathrm{Set})^{\mathrm{op}}.$$

To prove (1), we must show that for each object $Y \in \mathrm{Pro}(\mathcal{C}_0)$, the counit map $(F \circ G)(Y) \to Y$ is an isomorphism. Since both F and G commute with filtered limits, we may assume without loss of generality that Y belongs to $\mathcal{C}_0$, which case the result is obvious. We now prove (2). By definition, an object $X \in \mathrm{Pro}(\mathcal{C})$ belongs to the essential image of G if and only if it can be written as a formal inverse limit "$\varprojlim C_\alpha$", where each C_α belongs to $\mathcal{C}_0$. If X satisfies this condition, then every morphism $X \to C$ factors through one of the projection maps C_α, so that X satisfies condition $(*)$.

We now prove the converse. Assume that X satisfies $(*)$. Set $Y = (G \circ F)(X)$ and let $u : X \to Y$ be the unit map; we wish to show that u is an isomorphism. To prove this, we show that composition with u induces a bijection $\theta_C : \mathrm{Hom}_{\mathcal{C}}(Y, C) \to \mathrm{Hom}_{\mathcal{C}}(X, C)$ for each object $C \in \mathcal{C}$. By construction, the map θ_C is bijective when C belongs to $\mathcal{C}_0$. From our assumption that X satisfies $(*)$, we immediately deduce that θ_C is surjective for any object $C \in \mathcal{C}$. To prove injectivity, suppose we are given a pair of morphisms $f, f' : Y \to C$ satisfying $f \circ u = f' \circ u$. Since Y satisfies condition $(*)$, the induced map $(f, f') : Y \to C \times C$ factors as a composition $Y \xrightarrow{g} C_0 \xrightarrow{e} C \times C$ for some $C_0 \in \mathcal{C}$. Using our assumption that X satisfies $(*)$, we deduce that the map $(g \circ u, f \circ u) : X \to C_0 \times_{C \times C} C$ factors as a composition $X \xrightarrow{h} C_1 \xrightarrow{e'} C_0 \times_{C \times C} C$ for some $C_1 \in \mathcal{C}$. Using the bijectivity of θ_{C_1}, we can write $h = h' \circ u$ for some map $h' : Y \to C_1$. The composition $Y \xrightarrow{h'} C' \xrightarrow{e'} C_0 \times_{C \times C} C$ then gives a pair of maps $g' : Y \to C_0$ and $f'' : Y \to C$ satisfying $g' \circ u = g \circ u$ and $e \circ g' = (f'', f'')$. Using the injectivity of θ_{C_0}, we conclude that $g' = g$, so that $(f'', f'') = e \circ g' = e \circ g = (f, f')$ and therefore $f = f'$. $\square$

Lemma 6.1.11. *Let $\mathcal{C}$ be a small category which admits finite limits. Then the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves finite limits and all colimits which exist in $\mathcal{C}$.*

Proof. The first assertion follows from the observation that for every pro-object $X \in \mathrm{Pro}(\mathcal{C})$, the functor

$$(C \in \mathcal{C}) \mapsto \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, C)$$

can be identified with X, and is therefore left exact. The second assertion follows from the fact that the Yoneda embedding

$$\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \qquad C \mapsto \mathrm{Hom}_{\mathcal{C}}(C, \bullet)$$

carries colimits in the category $\mathcal{C}$ to limits in the functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$. $\square$

Lemma 6.1.12. *Let $\mathcal{C}$ be a small category which admits finite limits and let I be a finite partially ordered set (regarded as a category). Then the inclusion map $\mathrm{Fun}(I, \mathcal{C}) \hookrightarrow \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$ admits an (essentially unique) extension to an equivalence of categories $\mathrm{Pro}(\mathrm{Fun}(I, \mathcal{C})) \simeq \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$.*

Proof. See Proposition 3.3 of the appendix to [2]. $\square$

Example 6.1.13. Applying Lemma 6.1.12 in the case $I = \{0 < 1\}$, we see that every morphism $f : C \to D$ in $\mathrm{Pro}(\mathcal{C})$ can be obtained as the limit of a filtered diagram of morphisms $\{f_\alpha : C_\alpha \to D_\alpha\}$ between objects of $\mathcal{C}$.

Lemma 6.1.14. *Let $\mathcal{C}$ be a small category which admits finite limits. Then the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ admits small limits. In particular, $\mathrm{Pro}(\mathcal{C})$ also admits finite limits.*

Proof. By virtue of Remark 6.1.4; it will suffice to show that the category $\mathrm{Pro}(\mathcal{C})$ has a final object and admits pullbacks. We argue more generally that, for any diagram

$$I \to \mathrm{Pro}(\mathcal{C}) \qquad i \mapsto X_i$$

indexed by a finite partially ordered set I, there exists a limit $\varprojlim_{i \in I} X_i$ in the category $\mathrm{Pro}(\mathcal{C})$. Using Lemma 6.1.12 (together with the "Fubini theorem" for inverse limits), we can reduce to the case where each X_i is an object of $\mathcal{C}$. In this case, the desired limit exists (and belongs to the essential image of the embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$) by virtue of Lemma 6.1.11. □

Proposition 6.1.15. *Let $\mathcal{C}$ be a small regular category (Definition A.1.3). Then:*

(1) *The category $\mathrm{Pro}(\mathcal{C})$ is regular.*
(2) *A morphism $f : X \to Y$ in $\mathrm{Pro}(\mathcal{C})$ is a monomorphism if and only if it can be written as a small filtered limit of monomorphisms in $\mathcal{C}$.*
(3) *A morphism $f : X \to Y$ in $\mathrm{Pro}(\mathcal{C})$ is an effective epimorphism if and only if it can be written as a small filtered limit of effective epimorphisms in $\mathcal{C}$.*

Proof. To prove (1), we must show that $\mathrm{Pro}(\mathcal{C})$ satisfies axioms $(R1)$, $(R2)$, and $(R3)$ of Definition A.1.3. Assertion $(R1)$ follows from Lemma 6.1.14. To prove $(R2)$, fix any morphism $f : X \to Z$ in $\mathrm{Pro}(\mathcal{C})$. Invoking Lemma 6.1.12, we can write f as the limit of a diagram $\{f_i : X_i \to Z_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$, where $\mathcal{I}$ is a small filtered category and each f_i is a morphism in $\mathcal{C}$. Since $\mathcal{C}$ is regular, each of the morphism f_i admits a factorization

$$X_i \xrightarrow{g_i} Y_i \xrightarrow{h_i} Z_i,$$

where g_i is an effective epimorphism in $\mathcal{C}$ and h_i is a monomorphism in $\mathcal{C}$. It follows from Proposition A.1.4 that we can regard the construction $i \mapsto Y_i$ as a functor $\mathcal{I}^{\mathrm{op}} \to \mathcal{C}$, and the construction $i \mapsto g_i, h_i$ as natural transformations of functors. Passing to the inverse limit, we deduce that f factors as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$ where Y denotes the inverse limit $\varprojlim_{i \in \mathcal{I}^{\mathrm{op}}} Y_i$, formed in the category $\mathrm{Pro}(\mathcal{C})$. Note that the diagonal map $\delta : Y \to Y \times_Z Y$ can be identified with an inverse limit of diagonal maps $\delta_i : Y_i \to Y_i \times_{Z_i} Y_i$. Each h_i is a monomorphism in $\mathcal{C}$ and therefore also in $\mathrm{Pro}(\mathcal{C})$ (Lemma 6.1.11), so each δ_i is an isomorphism and therefore δ is also an isomorphism. It follows that h is a monomorphism in $\mathrm{Pro}(\mathcal{C})$. We will will show that g is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. For this, we wish to show that for each object $C \in \mathrm{Pro}(\mathcal{C})$, the diagram of sets

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(Y, C) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, C) \rightrightarrows \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X \times_Y X, C)$$

is an equalizer. Using Remark 6.1.6, we can reduce to the case where C belongs to $\mathcal{C}$. In this case, we can realize the preceding diagram as a filtered colimit of diagrams

$$\mathrm{Hom}_{\mathcal{C}}(Y_i, C) \to \mathrm{Hom}_{\mathcal{C}}(X_i, C) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(X_i \times_{Y_i} X_i, C).$$

We conclude by observing that each of these diagrams is an equalizer (since g_α is an effective epimorphism in $\mathcal{C}$), and the collection of equalizer diagrams in Set is closed under filtered colimits. This completes the verification of axiom $(R2)$.

We now prove (3) (the proof of (2) is similar). If f is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$, then the morphism $h : Y \to Z$ is an isomorphism (Remark A.1.7). It follows that f can be identified with the limit of the diagram $\{g_i : X_i \to Y_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ of effective epimorphisms in $\mathcal{C}$. Conversely, suppose that the diagram $\{f_i : X_i \to Z_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ can be chosen so that each f_i is an effective epimorphism. Then each of the morphisms $h_i : Y_i \to Z_i$ will be an isomorphism in $\mathcal{C}$, so that the map $h : Y \to Z$ is an isomorphism in $\mathrm{Pro}(\mathcal{C})$ and $f = h \circ g$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$.

We now complete the proof by showing that the category $\mathrm{Pro}(\mathcal{C})$ satisfies axiom $(R3)$ of Definition A.1.3. Fix a pullback diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Z' \arrow[r, "u"] & Z
\end{tikzcd}$$

in $\mathrm{Pro}(\mathcal{C})$, where f is an effective epimorphism; we wish to show that f' is also an effective epimorphism. Using Lemma 6.1.12, we can assume that f and u are given as the limits of diagrams $\{f_i : X_i \to Z_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ and $\{u_i : Z'_i \to Z_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$, where $\mathcal{I}$ is a small filtered category. Defining g_i and h_i as above, we obtain diagrams

$$\begin{tikzcd}
X_i \times_{Z_i} Z'_i \arrow[r] \arrow[d] & X_i \arrow[d, "g_i"] \\
Y_i \times_{Z_i} Z'_i \arrow[r] \arrow[d] & Y_i \arrow[d, "h_i"] \\
Z'_i \arrow[r, "u_i"] & Z_i,
\end{tikzcd}$$

where each square is a pullback, depending functorially on $i \in \mathcal{I}^{\mathrm{op}}$. Passing to the inverse limit, we obtain a commutative diagram of pro-objects

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "g'"] & X \arrow[d, "g"] \\
\varprojlim(Y_i \times_{Z_i} Z'_i) \arrow[r] \arrow[d, "h'"] & Y \arrow[d, "h"] \\
Z' \arrow[r, "u"] & Z,
\end{tikzcd}$$

in which both squares are pullbacks in $\mathrm{Pro}(\mathcal{C})$. Since f is an effective epimorphism, the map h is an isomorphism, so h' is also an isomorphism. We are therefore reduced to showing that g' is an effective epimorphism. This follows from (3), since it an inverse limit of effective epimorphisms $X_i \times_{Z_i} Z'_i \to Y_i \times_{Z_i} Z'_i$ in the regular category $\mathcal{C}$. ☐

Remark 6.1.16. Let $\mathcal{C}$ be a small regular category. Then the collection of effective epimorphisms in $\mathrm{Pro}(\mathcal{C})$ is closed under small filtered limits.

Remark 6.1.17. Let $\mathcal{C}$ be a small regular category. Then the collection of effective epimorphisms in $\mathrm{Pro}(\mathcal{C})$ is closed under the formation of products. To prove this, we can use Remark 6.1.16 to reduce to the case of finite products, in which case the desired result follows from the regularity of $\mathrm{Pro}(\mathcal{C})$ (Corollary A.1.10).

Lemma 6.1.18. *Let $\mathcal{C}$ be a small category which admits finite limits. Then the category $\mathrm{Pro}(\mathcal{C})$ admits small colimits. Moreover, the formation of finite colimits in $\mathrm{Pro}(\mathcal{C})$ commutes with filtered limits.*

Proof. The first assertion follows from the observation that the full subcategory $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ is closed under inverse limits, and the second from the observation that it is also closed under filtered colimits (together with the observation that filtered colimits commute with finite limits in the category Set). ☐

Proposition 6.1.19. *Let $\mathcal{C}$ be a small category which admits finite limits. If $\mathcal{C}$ is extensive (Definition A.3.2), then the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ is also extensive.*

Proof. The existence of coproducts in $\mathrm{Pro}(\mathcal{C})$ follows from Lemma 6.1.18. We next show that coproducts in $\mathrm{Pro}(\mathcal{C})$ are disjoint. Let X and Y be objects of $\mathrm{Pro}(\mathcal{C})$. Using Lemma 6.1.12 and Remark 6.1.6, we can write X and Y as the limits of diagrams $\{X_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ and $\{Y_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ in the category $\mathcal{C}$, where $\mathcal{I}$ is a small filtered category. For each $i \in I$, let $X_i \amalg Y_i$ denote a coproduct of X_i with Y_i in the category $\mathcal{C}$. Using Lemmas 6.1.18 and 6.1.11, we see that the limit of the diagram $\{X_i \amalg Y_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ can be identified with the coproduct of X and Y in the category $\mathrm{Pro}(\mathcal{C})$. Under this identification, the tautological maps $X \to X \amalg Y \leftarrow Y$ can be realized as limits of the maps $X_i \hookrightarrow X_i \amalg Y_i \hookleftarrow Y_i$, which are monomorphisms in $\mathcal{C}$ (hence also in $\mathrm{Pro}(\mathcal{C})$) by virtue of our assumption that coproducts in $\mathcal{C}$ are disjoint. Moreover, the fiber product $X \times_{(X \amalg Y)} Y$ can be identified with the limit of the diagram $\{X_i \times_{X_i \amalg Y_i} Y_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ which carries each object $i \in \mathcal{I}^{\mathrm{op}}$ to an initial object of $\mathcal{C}$.

Since the inclusion $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves initial objects (Lemma 6.1.11), it follows that $X \times_{(X \amalg Y)} Y$ is an initial object of $\mathrm{Pro}(\mathcal{C})$.

To complete the proof that $\mathrm{Pro}(\mathcal{C})$ is extensive, we must show that the formation of finite coproducts in $\mathrm{Pro}(\mathcal{C})$ is compatible with pullback. Suppose we are given a morphism $f : X \to Y$ in $\mathrm{Pro}(\mathcal{C})$ and a collection of morphisms $u_j : Z_j \to Y$ indexed by some finite set J. We wish to show that the canonical map

$$\theta : \coprod_{j \in J} (X \times_Y Z_j) \to X \times_Y (\coprod_{j \in J} Z_j)$$

is an isomorphism in $\mathrm{Pro}(\mathcal{C})$. Using Lemma 6.1.12 and Remark 6.1.6, we can assume that the morphisms f and u_j are realized as the limits of diagrams $\{f_i : X_i \to Y_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ and $\{u_{i,j} : Z_{i,j} \to Y_i\}_{i \in \mathcal{I}^{\mathrm{op}}}$ for some small filtered category $\mathcal{I}$, where each of the pro-objects X_i, Y_i, and $Z_{i,j}$ belongs to $\mathcal{C}$. Invoking Lemma 6.1.18, we can write θ as a limit of maps

$$\theta_i : \coprod_{j \in J} (X_i \times_{Y_i} Z_{i,j}) \to X_i \times_{Y_i} (\coprod_{j \in J} Z_{i,j}),$$

where the coproduct and fiber product can be formed either in $\mathrm{Pro}(\mathcal{C})$ or in the smaller category $\mathcal{C}$ (Lemma 6.1.11). Since the formation of finite coproducts in $\mathcal{C}$ commutes with pullback, each of the maps θ_i is an isomorphism in $\mathcal{C}$, so that θ is an isomorphism in $\mathrm{Pro}(\mathcal{C})$. □

Corollary 6.1.20. *Let $\mathcal{C}$ be a small regular extensive category. Then* $\mathrm{Pro}(\mathcal{C})$ *is also regular and extensive.*

Proof. Combine Propositions 6.1.15 and 6.1.19. □

Warning 6.1.21. It is not true that if $\mathcal{C}$ is a small pretopos, then the category $\mathrm{Pro}(\mathcal{C})$ is also a pretopos. For example, let $\mathcal{C}$ be the category of finite sets. Then the category $\mathrm{Pro}(\mathcal{C})$ of profinite sets can be identified with the category Stone of Stone spaces (Example 6.1.3). Let $C \in \mathrm{Stone}$ be the Cantor set, which we identify with the collection of infinite sequences $(n_1, n_2, n_3, \ldots)$ where $n_i \in \{0, 1\}$. The construction

$$(n_1, n_2, n_3, \ldots) \mapsto \sum \frac{n_i}{2^i}$$

defines a continuous surjection $C \to [0,1]$, and the fiber product $R = C \times_{[0,1]} C$ can be regarded as an equivalence relation on C in the category of Stone spaces. However, this equivalence relation is *not* effective: given any Stone space X, a continuous map $C \to X$ which equalizes the two projection maps $R \rightrightarrows C$ must factor through a continuous map $[0,1] \to X$. Such a map is automatically constant (since X is totally disconnected), so that $C \times_X C = C \times C$ is strictly larger than the subset $R \subseteq C \times C$.

We close this section by giving an alternate characterization of the the class effective epimorphisms in $\mathrm{Pro}(\mathcal{C})$, in the case where $\mathcal{C}$ is both regular and extensive.

Notation 6.1.22. Let $\mathcal{C}$ be a small regular extensive category. Then, for every object $C \in \mathcal{C}$, the collection of subobjects of C forms a distributive lattice $\mathrm{Sub}(C)$ (Example A.5.2). We let $\widehat{C}$ denote the spectrum $\mathrm{Spec}(\mathrm{Sub}(C))$, in the sense of Definition A.5.3, which we regard as a topological space. The construction $C \mapsto \widehat{C}$ determines a functor from $\mathcal{C}$ to the category Top of topological spaces. Using Remark 6.1.7, we see that this functor admits an essentially unique extension to a functor

$$\mathrm{Pro}(\mathcal{C}) \to \mathrm{Top} \qquad P \mapsto \widehat{P}$$

which commutes with filtered inverse limits.

Warning 6.1.23. The notation 6.1.22 is potentially misleading. Let $\mathcal{C}$ be a small regular extensive category. Then $\mathrm{Pro}(\mathcal{C})$ is also a regular extensive category. Consequently, for every pro-object $P \in \mathrm{Pro}(\mathcal{C})$, the partially ordered set $\mathrm{Sub}(P)$ of subobjects of P is a distributive lattice. However, the topological space $\widehat{P}$ of Notation 6.1.22 is *not* the spectrum $\mathrm{Spec}(\mathrm{Sub}(P))$. Writing P as the limit of a filtered diagram $\{C_\alpha\}$ of objects of $\mathcal{C}$, Remark A.5.6, we have

$$\widehat{P} \simeq \varprojlim \widehat{C}_\alpha \simeq \varprojlim \mathrm{Spec}(\mathrm{Sub}(C_\alpha)) \simeq \mathrm{Spec}(\varinjlim \mathrm{Sub}(C_\alpha)) = \mathrm{Spec}(\mathrm{Sub}^0(P)),$$

where $\mathrm{Sub}^0(P)$ denotes the sublattice of $\mathrm{Sub}(P)$ consisting of those subobjects $P_0 \subseteq P$ for which there exists a pullback square

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d] & P \arrow[d] \\ C_0 \arrow[r, "j"] & C, \end{tikzcd}$$

where $j : C_0 \hookrightarrow C$ is a monomorphism in the category $\mathcal{C}$.

Proposition 6.1.24. *Let $\mathcal{C}$ be a small regular extensive category and let $f : P \to Q$ be a morphism in $\mathrm{Pro}(\mathcal{C})$. The following conditions are equivalent:*

(a) *The morphism f is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$.*
(b) *The map of topological spaces $\widehat{f} : \widehat{P} \to \widehat{Q}$ is surjective.*

Proof. Suppose first that f is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. Then f can be realized as a filtered inverse limit of effective epimorphisms $f_\alpha : P_\alpha \twoheadrightarrow Q_\alpha$ in the category $\mathcal{C}$ (Proposition 6.1.15). It follows that each of the inverse limit maps $f_\alpha^{-1} : \mathrm{Sub}(Q_\alpha) \to \mathrm{Sub}(P_\alpha)$ is a monomorphism of distributive lattices. Passing to the colimit over α, we obtain a monomorphism of distributive lattices $\mathrm{Sub}^0(Q) \to \mathrm{Sub}^0(P)$ (where $\mathrm{Sub}^0(P)$ and $\mathrm{Sub}^0(Q)$ are defined as in Warning 6.1.23), hence a surjection of topological spaces

$$\widehat{P} \simeq \mathrm{Spec}(\mathrm{Sub}^0(P)) \to \mathrm{Spec}(\mathrm{Sub}^0(Q)) \to \widehat{Q}$$

(see Proposition A.5.7).

Now suppose that (b) is satisfied; we wish to show that f is an effective epimorphism. Using Proposition 6.1.15, we can factor f as a composition $P \overset{f'}{\twoheadrightarrow} P' \overset{f''}{\hookrightarrow} Q$, where f' is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$ and f'' is a monomorphism in $\mathrm{Pro}(\mathcal{C})$. Since the surjection $\widehat{f} : \widehat{P} \to \widehat{Q}$ factors through $\widehat{f''} : \widehat{P'} \to \widehat{Q}$, the map $\widehat{f''}$ is also surjective. We may therefore replace f by f'' and thereby reduce to the case where f is a monomorphism in $\mathrm{Pro}(\mathcal{C})$. Using Proposition 6.1.15, we can write f as the limit of a diagram $\{f_\alpha : P_\alpha \hookrightarrow Q_\alpha\}$. In this case, we can identify P with the intersection (in the partially ordered set $\mathrm{Sub}(Q)$) of the subobjects $S_\alpha = P_\alpha \times_{Q_\alpha} Q$. Each of these subobjects belongs to the lattice $\mathrm{Sub}^0(Q)$ of Warning 6.1.23), and has the property that $P \cap S_\alpha = P$. It follows from assumption (b) and Proposition A.5.7 that the intersection map $S \mapsto S \cap P$ induces a monomorphism of distributive lattices $\mathrm{Sub}^0(Q) \to \mathrm{Sub}^0(P)$, so that each S_α must coincide with Q and therefore $P = Q$ (as subobjects of Q). □

6.2. Weak Projectives and the Barr Embedding.

Let $\mathcal{C}$ be a regular category. Recall that an object $P \in \mathcal{C}$ is said to be *projective* if every effective epimorphism $\overline{P} \twoheadrightarrow P$ in the category $\mathcal{C}$ admits a section $s : P \to \overline{P}$.

Proposition 6.2.1. *Let $\mathcal{C}$ be a regular category and let P be an object of $\mathcal{C}$. The following conditions are equivalent:*

(a) *The object P is projective.*
(b) *For every effective epimorphism $f : C \twoheadrightarrow D$ in $\mathcal{C}$, postcomposition with f induces a surjection $\mathrm{Hom}_{\mathcal{C}}(P, C) \to \mathrm{Hom}_{\mathcal{C}}(P, D)$.*

Proof. Assume first that P is projective, and let $f : C \twoheadrightarrow D$ be an effective epimorphism. For any morphism $g : P \to D$, we can form a pullback square

$$\begin{tikzcd} \overline{P} \arrow[r, "g'"] \arrow[d, "f'"] & C \arrow[d, "f"] \\ P \arrow[r, "g"] & D. \end{tikzcd}$$

Using axiom $(R3)$ of Definition A.1.3, we conclude that f' is also an effective epimorphism. Our assumption that P is projective guarantees that f' admits a section $s : P \to \overline{P}$. Then the composition $g' \circ s \in \mathrm{Hom}_{\mathcal{C}}(P, C)$ is a preimage of g under the map $\mathrm{Hom}_{\mathcal{C}}(P, C) \to \mathrm{Hom}_{\mathcal{C}}(P, D)$. This shows that $(a) \Rightarrow (b)$.

Conversely, suppose that (b) is satisfied; we wish to show that P is projective. Choose an effective epimorphism $f : \overline{P} \to P$. It follows from (b) that postcomposition with f induces a surjection $\mathrm{Hom}_{\mathcal{C}}(P, \overline{P}) \to \mathrm{Hom}_{\mathcal{C}}(P, P)$ is surjective. In particular, there exists a map $s : P \to \overline{P}$ such that $f \circ s = \mathrm{id}_P$. □

Let $\mathcal{C}$ be a small regular category. Then the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ is also regular (Proposition 6.1.15). Beware that a projective object P of the category $\mathcal{C}$ need not be projective when regarded as an object of $\mathrm{Pro}(\mathcal{C})$. In general, projective objects of $\mathrm{Pro}(\mathcal{C})$ might be in short supply. We therefore introduce the following variant notion:

Definition 6.2.2. Let $\mathcal{C}$ be a small regular category and let X be a pro-object of $\mathcal{C}$. We will say that X is *weakly projective* if, for every effective epimorphism $f : C \twoheadrightarrow D$ in the category $\mathcal{C}$, composition with f induces a surjection $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, C) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X, D)$. We let $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Pro}(\mathcal{C})$ spanned by the weakly projective pro-objects of $\mathcal{C}$.

Remark 6.2.3. Let $\mathcal{C}$ be a regular category. A pro-object $X \in \mathrm{Pro}(\mathcal{C})$ is weakly projective if and only if it is regular when regarded as a functor from $\mathcal{C}$ to the category of sets: that is, the functor $X : \mathcal{C} \to \mathrm{Set}$ preserves finite limits and effective epimorphisms.

Example 6.2.4. Let $\mathcal{C}$ be a small regular category and let P be an object of $\mathcal{C}$. Then P is projective as an object of $\mathcal{C}$ if and only if it is weakly projective as an object of $\mathrm{Pro}(\mathcal{C})$ (this is the content of Proposition 6.2.1).

Example 6.2.5. Let $\mathcal{C}$ be a small regular category. Then every projective object of $\mathrm{Pro}(\mathcal{C})$ is weakly projective.

Example 6.2.6. Let Fin denote the category of finite sets. Then Fin is a regular category in which every object is projective: that is, every effective epimorphism in Fin admits a section. It follows that every pro-object of Fin is weakly projective. However, not every object of Pro(Fin) is projective. For example, the infinite product $\prod_{n \geq 0} \{0, 1\}$ is a non-projective object of Pro(Fin). Under the equivalence $\mathrm{Pro}(\mathrm{Fin}) \simeq \mathrm{Stone}$ of Example 6.1.3, the projective objects of Pro(Fin) correspond to Stone spaces X which are *extremally disconnected*: that is, those which can be realized as a retract of the Stone-Čech compactification βS, for some set S.

Remark 6.2.7. Let $\mathcal{C}$ be a small regular category. Then the collection of weakly projective pro-objects is closed under the formation of filtered limits in $\mathrm{Pro}(\mathcal{C})$. This follows from Remark 6.2.3, since the collection of surjective morphisms in Set is closed under the formation of filtered colimits.

Example 6.2.8. Let X be a quasi-compact, quasi-separated scheme and let $\mathcal{C}$ be the pretopos of constructible étale sheaves (of sets) on X. Then the final object $\mathbf{1} \in \mathcal{C}$ is projective as an object of $\mathrm{Pro}(\mathcal{C})$ if and only if X is affine and w-contractible in the sense of Bhatt-Scholze (see [4]).

Remark 6.2.9. Let $\mathcal{C}$ be a small regular category. The category $\mathrm{Pro}(\mathcal{C})$ is sensitive to the precise structure of $\mathcal{C}$: one can recover $\mathcal{C}$ (up to equivalence) as the full subcategory spanned by the cocompact objects of $\mathrm{Pro}(\mathcal{C})$. However, the subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ is a coarser invariant. If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a full subcategory which is closed under finite limits having the property that every object $C \in \mathcal{C}$ admits an effective epimorphism $C_0 \twoheadrightarrow C$, with $C_0 \in \mathcal{C}_0$, then every weakly projective pro-object of $\mathcal{C}$ belongs to the essential image of the embedding $\mathrm{Pro}(\mathcal{C}_0) \hookrightarrow \mathrm{Pro}(\mathcal{C})$ (see Proposition 6.1.10). If $\mathcal{C}_0$ is closed under the formation of images in $\mathcal{C}$, then it is also a small regular category and the embedding $\mathrm{Pro}(\mathcal{C}_0) \hookrightarrow \mathrm{Pro}(\mathcal{C})$ induces an equivalence $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}_0) \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$.

Remark 6.2.10. Let $\mathcal{C}$ be a small regular category. Then the collection of regular functors from $\mathcal{C}$ to the category of sets is closed under the formation of products in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ (this follows from the observation that the collection of surjections in Set is closed under products, by the axiom of choice). It follows that the collection of weakly projective pro-objects of $\mathcal{C}$ is closed under the formation of coproducts in $\mathrm{Pro}(\mathcal{C})$.

Remark 6.2.10 admits the following converse:

Proposition 6.2.11. *Let $\mathcal{C}$ be a small regular category and let $P, Q \in \operatorname{Pro}(\mathcal{C})$ be pro-objects of $\mathcal{C}$. If the coproduct $P \amalg Q$ is weakly projective, then P and Q are weakly projective.*

Proof. We will show that P is weakly projective. Let $q : C \twoheadrightarrow D$ be an effective epimorphism in $\mathcal{C}$ and let $f : P \to D$ be a morphism of pro-objects; we wish to show that there exists a morphism $\overline{f} : P \to C$ such that $f = q \circ \overline{f}$. Let $\mathbf{1}$ be a final object of $\mathcal{C}$ and let $g : Q \to \mathbf{1}$ be the unique map. Since $P \amalg Q$ is weakly projective and the map $(q \amalg \mathrm{id}) : C \amalg \mathbf{1} \to D \amalg \mathbf{1}$ is an effective epimorphism, the map $(f \amalg g) : P \amalg Q \to D \amalg \mathbf{1}$ factors as a composition

$$P \amalg Q \xrightarrow{h} C \amalg \mathbf{1} \xrightarrow{q \amalg \mathrm{id}} D \amalg \mathbf{1}.$$

Applying Proposition A.3.7, we obtain a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\overline{f}"] \arrow[d] & C \arrow[r, "q"] \arrow[d] & D \arrow[d] \\
P \amalg Q \arrow[r, "h"] & C \amalg \mathbf{1} \arrow[r, "q \amalg \mathrm{id}"] & D \amalg \mathbf{1}
\end{tikzcd}$$

where both squares are pullbacks and the upper vertical composition coincides with f. □

Let $\mathcal{C}$ be a small regular category and let X be a pro-object of $\mathcal{C}$. One can apply Quillen's "small object argument" to construct an effective epimorphism of pro-objects $\overline{X} \twoheadrightarrow X$, where $\overline{X}$ is weakly projective. For later use, we record the following more precise statement:

Proposition 6.2.12. *Let $\mathcal{C}$ be a small regular category. Then there exists a functor $\lambda : \operatorname{Pro}(\mathcal{C}) \to \operatorname{Pro}(\mathcal{C})$ and a natural transformation $\rho : \lambda \to \mathrm{id}_{\operatorname{Pro}(\mathcal{C})}$ with the following properties:*

(1) *For each object $X \in \operatorname{Pro}(\mathcal{C})$, the object $\lambda(X) \in \operatorname{Pro}(\mathcal{C})$ is weakly projective.*
(2) *For each object $X \in \operatorname{Pro}(\mathcal{C})$, the map $\rho(X) : \lambda(X) \to X$ is an effective epimorphism in* $\operatorname{Pro}(\mathcal{C})$.
(3) *The functor λ preserves small filtered limits.*

Proof. Let $\{f_i : C_i \to D_i\}_{i \in I}$ be a set of representatives for all isomorphism classes of effective epimorphisms in $\mathcal{C}$. For each object $X \in \operatorname{Pro}(\mathcal{C})$, set

$$C(X) = \prod_{i \in I} \Big(\prod_{\eta : X \to D_i} C_i \Big) \qquad D(X) = \prod_{i \in I} \Big(\prod_{\eta : X \to D_i} D_i \Big),$$

where in both cases the inner product is indexed by the set $\operatorname{Hom}_{\operatorname{Pro}(\mathcal{C})}(X, D_i)$ of all maps from X to D_i in the category $\operatorname{Pro}(\mathcal{C})$. By construction, we have a canonical map $X \to D(X)$; let $\lambda_1(X)$ denote the fiber product $C(X) \times_{D(X)} X$. It follows from Remark 6.1.17 that the natural map $C(X) \to D(X)$ is an effective epimorphism in $\operatorname{Pro}(\mathcal{C})$. Since the category $\operatorname{Pro}(\mathcal{C})$ is regular, the projection map $\lambda_1(X) \to X$ is also an effective epimorphism in $\operatorname{Pro}(\mathcal{C})$.

For $n > 1$, we define $\lambda_n(X)$ by the formula $\lambda_n(X) = \lambda_1(\lambda_{n-1}(X))$, so that we have an inverse system

$$\cdots \twoheadrightarrow \lambda_3(X) \twoheadrightarrow \lambda_2(X) \twoheadrightarrow \lambda_1(X) \twoheadrightarrow X$$

of effective epimorphisms in $\mathcal{C}$. Set $\lambda(X) = \varprojlim \lambda_n(X)$. We have an evident projection map $\rho(X) : \lambda(X) \to X$, depending functorially on X. We claim that the functor $X \mapsto \lambda(X)$ and the natural transformation $X \mapsto \rho(X)$ satisfy the requirements of Proposition 6.2.12:

(1) For each pro-object $X \in \operatorname{Pro}(\mathcal{C})$, the pro-object $\lambda(X)$ is weakly projective. Choose an index $i \in I$ and a morphism $g : \lambda(X) \to D_i$; we wish to show that g factors through the effective epimorphism $f_i : C_i \to D_i$. Using Remark 6.1.4, we see that g factors as a composition $\lambda(X) \to \lambda_n(X) \xrightarrow{g_n} D_i$ for some $n \gg 0$. It now suffices to observe that, by construction, the composite map

$$\lambda_{n+1}(X) = \lambda_1(\lambda_n(X)) \to \lambda_n(X) \xrightarrow{g_n} D_i$$

factors through f_i.

(2) For each pro-object $X \in \mathrm{Pro}(\mathcal{C})$, the projection map $\rho(X) : \lambda(X) \to X$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. This follows from Remark 6.1.16 (since each of the transition maps $\lambda_{n+1}(X) \twoheadrightarrow \lambda_n(X)$ is an effective epimorphism).

(3) To show that the functor λ commutes with small filtered limits, it will suffice to show that each $\lambda_n : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C})$ commutes with small filtered limits. Writing λ_n as an n-fold iterate of the functor λ_1, we can reduce to the case $n = 1$. By construction, we have a pullback diagram of functors

$$\begin{tikzcd} (X \mapsto \lambda_1(X)) \arrow[r] \arrow[d] & (X \mapsto C(X)) \arrow[d] \\ (X \mapsto X) \arrow[r] & (X \mapsto D(X)). \end{tikzcd}$$

It will therefore suffice to establish that the functors $X \mapsto C(X)$ and $X \mapsto D(X)$ commute with filtered limits limits, which follows easily from Remark 6.1.4.

□

Proposition 6.2.12 is essentially contained in [3], where it used to prove the following:

Theorem 2.4.1 (Barr)**.** *Let $\mathcal{E}$ be a small regular category. Then the construction $E \mapsto \mathrm{ev}_E$ induces a fully faithful embedding*

$$\mathcal{E} \hookrightarrow \mathrm{Fun}(\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}), \mathrm{Set}).$$

Proof. Since the category $\mathcal{E}$ is regular, the category of pro-objects $\mathrm{Pro}(\mathcal{E})$ is also regular (Proposition 6.1.15). Let us regard $\mathrm{Pro}(\mathcal{E})$ as equipped with the regular topology of Definition B.3.3. Since the regular topology is subcanonical (Corollary B.3.6), the Yoneda embedding $\mathrm{Pro}(\mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathrm{Pro}(\mathcal{E})^{\mathrm{op}}, \mathrm{Set})$ factors through the category $\mathrm{Shv}(\mathrm{Pro}(\mathcal{E})) \subseteq \mathrm{Fun}(\mathrm{Pro}(\mathcal{E})^{\mathrm{op}}, \mathrm{Set})$. Proposition 6.2.12 implies that the subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{E}) \subseteq \mathrm{Pro}(\mathcal{E})$ is a basis for the regular topology on $\mathcal{E}$. In particular, $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{E})$ inherits a Grothendieck topology and the restriction functor $\mathrm{Shv}(\mathrm{Pro}(\mathcal{E})) \to \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{E}))$ is an equivalence of categories (Propositions B.6.3 and B.6.4). We now observe that the evaluation map $\mathrm{ev} : \mathcal{E} \to \mathrm{Fun}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{E}), \mathrm{Set})$ factors as a composition

$$\mathcal{E} \subseteq \mathrm{Pro}(\mathcal{E}) \hookrightarrow \mathrm{Shv}(\mathrm{Pro}(\mathcal{E})) \simeq \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{E})) \subseteq \mathrm{Fun}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{E})^{\mathrm{op}}, \mathrm{Set}) = \mathrm{Fun}(\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}), \mathrm{Set}).$$

□

Remark 6.2.13. Let X be a quasi-compact, quasi-separated scheme. By Theorem 1.5 of [4], one can choose a faithfully flat pro-étale map $U \to X$, where U is a w-contractible affine scheme. Using the terminology of this section, this result asserts the existence of an effective epimorphism $P \twoheadrightarrow \mathbf{1}$ in $\mathrm{Pro}(\mathcal{C})$, where $\mathcal{C}$ denotes the pretopos of constructible étale sheaves on X and P is a *projective* object of $\mathrm{Pro}(\mathcal{C})$. A relative version of the same argument shows that every object $X \in \mathrm{Pro}(\mathcal{C})$ admits an effective epimorphism $P \twoheadrightarrow X$, where P is projective. However, it seems unlikely that the same result can be extended to an arbitrary pretopos.

6.3. **Classification of Weak Projectives.** Let $\mathcal{C}$ be a small pretopos. Recall that a *model* of $\mathcal{C}$ is a functor $M : \mathcal{C} \to \mathrm{Set}$ which preserves finite limits, finite coproducts, and effective epimorphisms. In particular, every model of $\mathcal{C}$ can be regarded as a weakly projective pro-object of $\mathcal{C}$. Our goal in this section is to formulate a partial converse: every weakly projective pro-object can be viewed as a "continuous family" of models of $\mathcal{C}$, parametrized by a Stone space (Theorem 6.3.14). To make this precise, we need to introduce some terminology.

Definition 6.3.1. Let $\mathcal{C}$ be a pretopos and let X be a topological space. Let $\mathrm{Shv}(X)$ denote the category of set-valued sheaves on X (which is a Grothendieck topos, and therefore also a pretopos). An *X-model of $\mathcal{C}$* is a pretopos functor $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X)$. Given such a functor $\mathcal{O}_X$, we will denote the value of $\mathcal{O}_X$ on an object $C \in \mathcal{C}$ by $\mathcal{O}_X^C$. This is a sheaf of sets on X, whose value on an open set $U \subseteq X$ we denote by $\mathcal{O}_X^C(U)$.

Example 6.3.2. If the topological space X consists of a single point, we can identify $\mathrm{Shv}(X)$ with the category of sets. In this case, we can identify X-models of a coherent category $\mathcal{C}$ with models of $\mathcal{C}$.

Example 6.3.3. Let X be a compact Hausdorff space. In this case, we can use Theorem 3.4.4 to identify the topos $\mathrm{Shv}(X)$ with the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set})$. For every pretopos $\mathcal{C}$, we have a canonical equivalence of categories

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Shv}(X)) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set})) \simeq \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Fun}(\mathcal{C}, \mathrm{Set})).$$

Since colimits and finite limits in $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Set})$ are computed pointwise (Remarks 1.4.3 and Corollary 2.1.4), this restricts to an equivalence of categories $\mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{C}, \mathrm{Shv}(X)) \simeq \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{C}, \mathrm{Set})) = \mathrm{Fun}^{\mathrm{LUlt}}(X, \mathrm{Mod}(\mathcal{C}))$, where we endow the category of models $\mathrm{Mod}(\mathcal{C})$ with the ultrastructure of Remark 2.1.2. In other words, we can identify X-models of $\mathcal{C}$ (in the sense of Definition 6.3.1) with left ultrafunctors $X \to \mathrm{Mod}(\mathcal{C})$ (in the sense of Definition 1.4.1).

Construction 6.3.4. Let $f : X \to Y$ be a continuous function between topological spaces, and let $f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ denote the functor given by pullback along f. Then f^* is a pretopos functor. In particular, for any pretopos $\mathcal{C}$, postcomposition with f^* induces a functor $\mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{C}, \mathrm{Shv}(Y)) \to \mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{C}, \mathrm{Shv}(X))$. If $\mathcal{O}_Y$ is a Y-model of $\mathcal{C}$, we will denote its image under this functor by $f^* \mathcal{O}_Y$ or by $\mathcal{O}_Y|_X$ (we will use the latter notation primarily in the case where X is given as a subset of Y).

Example 6.3.5. Let $\mathcal{C}$ be a pretopos, let X be a topological space, and let $\mathcal{O}_X$ be an X-model of $\mathcal{C}$. For each point $x \in X$, pullback along the inclusion map $\{x\} \hookrightarrow X$ determines an $\{x\}$-model of $\mathcal{C}$, which we can identify with an object of $\mathrm{Mod}(\mathcal{C})$ (Example 6.3.2). We will denote this model by $\mathcal{O}_{X,x}$ and refer to it as the *stalk of $\mathcal{O}_X$ at x*. Concretely, the stalk $\mathcal{O}_{X,x}$ is a functor $\mathcal{C} \to \mathrm{Set}$ given by the construction $C \mapsto \varinjlim_{x \in U} \mathcal{O}_X^C(U)$, where the colimit is taken over the collection of all open neighborhoods U of the point x.

Remark 6.3.6. Let $\mathcal{C}$ be a pretopos and let X be a topological space. One can think of an X-model $\mathcal{O}_X$ of $\mathcal{C}$ as given by a collection of models $\{\mathcal{O}_{X,x}\}_{x \in X}$ depending "continuously" on the point $x \in X$. Beware that the category $\mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{C}, \mathrm{Shv}(X))$ is generally *not* equivalent to the category of $\mathrm{Mod}(\mathcal{C})$-valued sheaves on X (though they are equivalent in certain cases; see Example 6.3.10). Given an X-model $\mathcal{O}_X$ of $\mathcal{C}$, there is generally no way to construct a model of $\mathcal{C}$ by "evaluating" on an open set $U \subseteq X$: the construction $(C \in \mathcal{C}) \mapsto (\mathcal{O}_X^C(U) \in \mathrm{Set})$ is usually not a pretopos functor (see Proposition 6.3.12).

Remark 6.3.7. Let $\mathcal{C}$ be a pretopos, let X be a topological space, and let $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X)$ be any functor. The following conditions are equivalent:

(a) The functor $\mathcal{O}_X$ is an X-model of $\mathcal{C}$.

(b) For every point $x \in X$, the functor $(C \in \mathcal{C}) \mapsto ((\mathcal{O}_X^C)_x \in \mathrm{Set})$ is a model of $\mathcal{C}$.

The implication $(a) \Rightarrow (b)$ is Example 6.3.5, and the converse follows from the observation that the stalk functors $\mathrm{Shv}(X) \to \mathrm{Set}$ detect isomorphisms.

Definition 6.3.8. Let $\mathcal{C}$ be a pretopos. We define a category $\mathrm{Top}_{\mathcal{C}}$ as follows:

- The objects of $\mathrm{Top}_{\mathcal{C}}$ are pairs $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X)$ is an X-model of $\mathcal{C}$ (Definition 6.3.1).
- A morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ in the category $\mathrm{Top}_{\mathcal{C}}$ consists of a pair (f, α), where $f : X \to Y$ is a continuous function and $\alpha : f^* \mathcal{O}_Y \to \mathcal{O}_X$ is a natural transformation of functors from $\mathcal{C}$ to $\mathrm{Shv}(X)$.
- The composition of a pair of morphisms $(X, \mathcal{O}_X) \xrightarrow{(f,\alpha)} (Y, \mathcal{O}_Y) \xrightarrow{(g,\beta)} (Z, \mathcal{O}_Z)$ is given by the pair $(g \circ f, \alpha \circ f^*(\beta))$.

We let $\mathrm{Comp}_{\mathcal{C}} \subseteq \mathrm{Top}_{\mathcal{C}}$ denote the full subcategory spanned by those pairs $(X, \mathcal{O}_X)$ where X is a compact Hausdorff space, and $\mathrm{Stone}_{\mathcal{C}} \subseteq \mathrm{Comp}_{\mathcal{C}}$ the full subcategory spanned by those pairs $(X, \mathcal{O}_X)$ where X is a Stone space.

Warning 6.3.9. We have now attached two different meanings to the notation $\mathrm{Comp}_{\mathcal{C}}$ (and $\mathrm{Stone}_{\mathcal{C}}$):

- In the case where $\mathcal{C}$ is a pretopos, the category $\mathrm{Comp}_{\mathcal{C}}$ of Definition 6.3.8 consists of pairs $(X, \mathcal{O}_X)$, where X is a compact Hausdorff space and $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X)$ is a pretopos functor.

- In the case where $\mathcal{C}$ is an ultracategory, the category $\mathrm{Comp}_{\mathcal{C}}$ of Construction 4.1.1 consists of pairs $(X, \mathcal{O}_X)$, where X is a compact Hausdorff space and $\mathcal{O}_X : X \to \mathcal{C}$ is a left ultrafunctor.

In the case where $\mathcal{C}$ is *both* a pretopos and an ultracategory (for example, if $\mathcal{C} = \mathrm{Set}$ is the category of sets), these definitions are not compatible. To avoid confusion, we will use a subscript $\mathcal{C}$ to indicate that we are considering the case of a pretopos, and a subscript $\mathcal{M}$ to indicate that we are considering the case of an ultracategory (there is little danger of confusion in any case, since we will generally employ Definition 6.3.8 in the case where $\mathcal{C}$ is a *small* pretopos having no obvious ultrastructure).

However, these definitions are compatible in a different sense. For any pretopos $\mathcal{C}$, we can equip the category of models $\mathrm{Mod}(\mathcal{C})$ with the ultrastructure of Remark 2.1.2. Then, for any compact Hausdorff space X, we can identify pretopos functors $\mathcal{C} \to \mathrm{Shv}(X)$ with left ultrafunctors $X \to \mathrm{Mod}(\mathcal{C})$ (Example 6.3.3). This identification depends functorially on X, and therefore gives rise to an equivalence from the category $\mathrm{Comp}_{\mathcal{C}}$ (introduced in Definition 6.3.8) to the category $\mathrm{Comp}_{\mathrm{Mod}(\mathcal{C})}$ (introduced in Construction 4.1.1). This equivalence is the identity at the level of topological spaces, and therefore restricts to an equivalence $\mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Stone}_{\mathrm{Mod}(\mathcal{C})}$.

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Example 6.3.10 (Ringed Spaces)**.** Let CRing denote the category of commutative rings and let $\mathcal{X} \subseteq \mathrm{Fun}(\mathrm{CRing}, \mathrm{Set})$ be the full subcategory spanned by those functors $F : \mathrm{CRing} \to \mathrm{Set}$ that commute with filtered colimits. Then $\mathcal{X}$ is a coherent Grothendieck topos, called the *classifying topos of commutative rings.* Let $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ be the pretopos of coherent objects of $\mathcal{X}$. For any pretopos $\mathcal{C}$, we can identify pretopos functors $\mathcal{X}_{\mathrm{coh}} \to \mathcal{C}$ with commutative ring objects of $\mathcal{C}$. In particular, we can identify models of $\mathcal{X}_{\mathrm{coh}}$ with commutative rings, and X-models of $\mathcal{X}_{\mathrm{coh}}$ with sheaves of commutative rings on X (for any topological space X). This identification induces an equivalence of $\mathrm{Top}_{\mathcal{X}_{\mathrm{coh}}}$ with the category of ringed spaces.

Construction 6.3.11 (The Global Sections Functor)**.** Let $\mathcal{C}$ be a small pretopos, let X be a topological space, and let $\mathcal{O}_X : \mathcal{C} \to \mathrm{Shv}(X)$ be an X-model of $\mathcal{C}$. We let $\Gamma(X, \mathcal{O}_X) : \mathcal{C} \to \mathrm{Set}$ denote the functor given on objects by $\Gamma(X, \mathcal{O}_X)(C) = \mathcal{O}_X^C(X)$.

Recall that a topological space X is said to be *zero-dimensional* if every open covering of X can be refined to a covering of X by *disjoint* open sets.

Proposition 6.3.12. *Let $\mathcal{C}$ be a pretopos, let $(X, \mathcal{O}_X)$ be an object of $\mathrm{Top}_{\mathcal{C}}$, and let $\Gamma(X, \mathcal{O}_X) : \mathcal{C} \to \mathrm{Set}$ be the functor of Construction 6.3.11. Then:*

1. *The functor $\Gamma(X, \mathcal{O}_X)$ preserves finite limits: that is, it can be regarded as a pro-object of $\mathcal{C}$.*
2. *If X is zero-dimensional, then the pro-object $\Gamma(X, \mathcal{O}_X)$ is weakly projective.*
3. *The functor $\Gamma(X, \mathcal{O}_X)$ preserves finite coproducts if and only if X is connected.*

Proof. Assertion (1) follows from the observation that the evaluation functor

$$\mathrm{Shv}(X) \to \mathrm{Set} \qquad \mathscr{F} \mapsto \mathscr{F}(U)$$

is left exact. To prove (2), suppose that X is zero-dimensional. We wish to show that the functor $C \mapsto \mathcal{O}_X^C(X)$ carries effective epimorphisms in $\mathcal{C}$ to surjections of sets. Let $f : C \twoheadrightarrow D$ be an effective epimorphism in $\mathcal{C}$. Then the induced map $\mathcal{O}_X^C \twoheadrightarrow \mathcal{O}_X^D$ is an effective epimorphism in the sheaf category $\mathrm{Shv}(X)$: that is, it is surjective on stalks. In particular, if $s \in \mathcal{O}_X^D(X)$ is a global section of the sheaf $\mathcal{O}_X^D$, then we can choose a covering $\{U_\alpha\}$ of X such that each restriction $s|_{U_\alpha} \in \mathcal{O}_X^D(U_\alpha)$ can be lifted to a section $\overline{s}_\alpha \in \mathcal{O}_X^C(U_\alpha)$. Using the assumption that X is zero-dimensional, we can assume that the open sets U_α are disjoint. It follows that there is a unique section $\overline{s} \in \mathcal{O}_X^C(X)$ satisfying $\overline{s}|_{U_\alpha} = \overline{s}_\alpha$ for each index α. This section is a preimage of s under the map $\mathcal{O}_X^C(X) \to \mathcal{O}_X^D(X)$ determined by f.

We now prove (3). Let $\mathbf{1}$ denote a final object of $\mathcal{C}$. For every finite set S, let $\underline{S}$ denote the constant sheaf on X with value S. Since the functor $\mathcal{O}_X$ preserves final objects and finite coproducts, it carries the coproduct $\coprod_{s \in S} \mathbf{1}$ to the sheaf $\underline{S}$. Consequently, the functor $\Gamma(X, \mathcal{O}_X)$ carries the coproduct $\coprod_{s \in S} \mathbf{1}$ to the set $\mathrm{Hom}_{\mathrm{Top}}(X, S)$ of locally-constant S-valued functions on X. Using the criterion of Proposition A.3.10,

we see that $\Gamma(X, \mathcal{O}_X)$ preserves finite coproducts if and only if the canonical map $S \to \mathrm{Hom}_{\mathrm{Top}}(X, S)$ is bijective when S is either empty or has two elements; that is, if and only if X is connected. □

Construction 6.3.13. Let $\mathcal{C}$ be a pretopos. The construction $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$ determines a functor from the category $\mathrm{Top}_{\mathcal{C}}$ of Definition 6.3.8 to the category of pro-objects $\mathrm{Pro}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$. Since every Stone space X is zero-dimensional, Proposition 6.3.12 shows that this construction restricts to a functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$.

We can now state the main result of this section:

Theorem 6.3.14. *Let $\mathcal{C}$ be a small pretopos. Then the functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ of Construction 6.3.13 is an equivalence of categories.*

We will give a detailed proof of Theorem 6.3.14 in §6.4. Let us give an informal sketch of the main idea. Suppose that $P : \mathcal{C} \to \mathrm{Set}$ is a weakly projective pro-object: that is, a functor which preserves finite limits and effective epimorphisms. We would like to show that there is an essentially unique object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$ with an isomorphism $P \simeq \Gamma(X, \mathcal{O}_X)$. We observe that the object $(X, \mathcal{O}_X)$ must have the following features:

- (a) Since $\mathcal{O}_X$ preserves final objects and finite coproducts, we can identify $\mathcal{O}_X^{\mathbf{1} \amalg \mathbf{1}}$ with the constant sheaf on X associated to the two-element set $\{0, 1\}$. Consequently, the set $P(\mathbf{1} \amalg \mathbf{1})$ can be identified with the Boolean algebra of closed and open subsets of X, from which we can recover the topology of X (by Stone duality).
- (b) Since X is a Stone space, each of the sheaves $\mathcal{O}_X^C$ is determined by its values on closed and open subsets $U \subseteq X$. These can be recovered from the functor P by the formula
$$\mathcal{O}_X^C(U) \simeq \mathcal{O}_X^{C \amalg \mathbf{1}}(X) \times_{\mathcal{O}_X^{\mathbf{1} \amalg \mathbf{1}}(X)} \{U\} \simeq P(C \amalg \mathbf{1}) \times_{P(\mathbf{1} \amalg \mathbf{1})} \{U\}.$$

This analysis suggests a recipe for reconstructing the pair $(X, \mathcal{O}_X)$ from the functor P; the essential content of Theorem 6.3.14 is that this recipe works if (and only if) the functor P preserves finite limits and effective epimorphisms.

Remark 6.3.15. Theorem 6.3.14 immediately implies (a version of) Deligne's completeness theorem. Let $\mathcal{C}$ be a small pretopos, and let $u : C \to C'$ be a morphism in $\mathcal{C}$ with the property that, for every model M of $\mathcal{C}$, the map $M(u) : M(C) \to M(C')$ is bijective. It follows that, for each object $(X, \mathcal{O}_X)$ in $\mathrm{Top}_{\mathcal{C}}$, the induced map $\mathcal{O}_X^C \to \mathcal{O}_X^{C'}$ is an isomorphism of sheaves on X, and therefore induces a bijection $\Gamma(X, \mathcal{O}_X)(C) \to \Gamma(X, \mathcal{O}_X)(C')$. Theorem 6.3.14 then guarantees that composition with u induces a bijection $\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P, C) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P, C')$ for every weakly projective pro-object of $\mathrm{Pro}(\mathcal{C})$, so that u is an isomorphism by virtue of Proposition 6.2.12. See Theorem 2.2.10 for a (slightly) stronger version.

6.4. **The Proof of Theorem 6.3.14.**

Let $\mathcal{C}$ be a small pretopos. Our goal in this section is to prove Theorem 6.3.14 by showing that the global sections functor
$$\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$$
is an equivalence of categories. The proof will proceed in several steps. We first argue that the functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ is conservative.

Lemma 6.4.1. *Let X be a Stone space and let $f : \mathscr{F} \to \mathscr{G}$ be a morphism in $\mathrm{Shv}(X)$. Suppose that f induces a bijection $\theta : (\mathscr{F} \amalg \mathbf{1})(X) \to (\mathscr{G} \amalg \mathbf{1})(X)$ (where $\mathbf{1}$ denotes the final object of $\mathrm{Shv}(X)$). Then f is an isomorphism.*

Proof. Unwinding the definitions, we see that θ is the map
$$\coprod_U \mathscr{F}(U) \to \coprod_U \mathscr{G}(U)$$
induced by f, where both coproducts are indexed by the collection of all closed and open subsets of X. Consequently, if θ is a bijection, then f induces a bijection $\mathscr{F}(U) \to \mathscr{G}(U)$ whenever $U \subseteq X$ is closed and open. Since X is a Stone space, the closed and open subsets of X form a basis for the topology of X. It follows that f is an isomorphism of sheaves. □

Lemma 6.4.2. *Let $\mathcal{C}$ be a small pretopos and let $(f,\alpha):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a morphism in the category* $\mathrm{Stone}_{\mathcal{C}}$. *If the induced map $\theta:\Gamma(X,\mathcal{O}_X)\to\Gamma(Y,\mathcal{O}_Y)$ is an isomorphism in* $\mathrm{Pro}(\mathcal{C})$, *then (f,α) is an isomorphism in* $\mathrm{Stone}_{\mathcal{C}}$.

Proof. Let $\mathbf{1}$ denote a final object of $\mathcal{C}$. For each finite set S, let $S_{\mathcal{C}}=\coprod_{s\in S}\mathbf{1}$ denote the corresponding object of $\mathcal{C}$. Then we can identify $\mathcal{O}_X^{S_{\mathcal{C}}}$ and $\mathcal{O}_Y^{S_{\mathcal{C}}}$ with the constant sheaves $\underline{S}_X$ and $\underline{S}_Y$ on X and Y, respectively. If θ is an isomorphism of pro-objects, then composition with θ induces a bijection

$$\Gamma(Y,\underline{S}_Y)=\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Gamma(Y,\mathcal{O}_Y),S_{\mathcal{C}})\to\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Gamma(X,\mathcal{O}_X),S_{\mathcal{C}})=\Gamma(X,\underline{S}_X).$$

Equivalently, composition with f induces a bijection $\mathrm{Hom}_{\mathrm{Top}}(Y,S)\to\mathrm{Hom}_{\mathrm{Top}}(X,S)$ for every finite set S. Since X and Y are Stone spaces, it follows that f is a homeomorphism (Example 6.1.3). To complete the proof, it will suffice to show that α induces an isomorphism $f^*\mathcal{O}_Y^C\to\mathcal{O}_X^C$ in $\mathrm{Shv}(X)$, for each object $C\in\mathcal{C}$. Using the criterion of Lemma 6.4.1, this follows from the bijectivity of the map the bijectivity of the map

$$\mathcal{O}_Y^{C\amalg\mathbf{1}}(Y)\simeq\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Gamma(Y,\mathcal{O}_Y),C\amalg\mathbf{1})\to\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Gamma(X,\mathcal{O}_X),C\amalg\mathbf{1})\simeq\mathcal{O}_X^{C\amalg\mathbf{1}}(X).$$

□

Our next goal is to explain how to reconstruct the underlying topological space of an object $(X,\mathcal{O}_X)\in\mathrm{Stone}_{\mathcal{C}}$ directly from the pro-object $\Gamma(X,\mathcal{O}_X)$.

Construction 6.4.3. Let $\mathcal{C}$ be a small pretopos and let $\mathbf{1}$ be a final object of $\mathcal{C}$. Let Fin denote the category of finite sets, and let $g:\mathrm{Fin}\to\mathcal{C}$ be the functor given on objects by $g(S)=\coprod_{s\in S}\mathbf{1}$. Then g is left exact, and therefore induces a pair of adjoint functors

$$\mathrm{Pro}(\mathcal{C})\underset{G}{\overset{F}{\rightleftarrows}}\mathrm{Pro}(\mathrm{Fin})$$

where F is given by precomposition with g and $G|_{\mathrm{Fin}}\simeq g$ (see Remark 6.1.8). Let $\Phi:\mathrm{Stone}\to\mathrm{Pro}(\mathcal{C})$ denote the composition of G with the equivalence of categories $\mathrm{Stone}\simeq\mathrm{Pro}(\mathrm{Fin})$ of Example 6.1.3. Then Φ has a left adjoint $\mathrm{Pro}(\mathcal{C})\to\mathrm{Stone}$ (obtained by composing F with the inverse equivalence), which we will denote by $(P\in\mathrm{Pro}(\mathcal{C}))\mapsto(|P|\in\mathrm{Stone})$. These functors can be described more concretely as follows:

- If X is a Stone space given as the limit of a filtered diagram of finite sets $\{S_\alpha\}$, then $\Phi(X)\in\mathrm{Pro}(\mathcal{C})$ is given by the limit of the diagram $\{\coprod_{s\in S_\alpha}\mathbf{1}\}$, where $\mathbf{1}$ denotes a final object of $\mathcal{C}$.
- If P is a pro-object of $\mathcal{C}$, then the Stone space $|P|$ is characterized (up to homeomorphism) by the requirement that for any finite set S, we have a bijection

$$\mathrm{Hom}_{\mathrm{Top}}(|P|,S)\simeq\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P,\coprod_{s\in S}\mathbf{1})=P(\coprod_{s\in S}\mathbf{1}).$$

Example 6.4.4. Let $\mathcal{C}$ be a small pretopos, let $(X,\mathcal{O}_X)$ be an object of $\mathrm{Stone}_{\mathcal{C}}$, and let $\Gamma(X,\mathcal{O}_X)$ be the corresponding pro-object. Then we have a canonical homeomorphism $|\Gamma(X,\mathcal{O}_X)|\simeq X$.

Lemma 6.4.5. *Let $\mathcal{C}$ be a small pretopos and let P be a pro-object of $\mathcal{C}$, which we view as a left-exact functor $P:\mathcal{C}\to\mathrm{Set}$. Then P preserves finite coproducts if and only if the Stone space $|P|$ is a point.*

Proof. This is a restatement of Proposition A.3.10. □

Lemma 6.4.6. *Let $f:X\to Y$ be a continuous function between Stone spaces, let $\mathcal{C}$ be a small pretopos, and suppose we are given a pullback diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ \Phi(X) \arrow[r, "\Phi(f)"] & \Phi(Y) \end{tikzcd}$$

in the category $\mathrm{Pro}(\mathcal{C})$. *If Q is weakly projective, then P is also weakly projective.*

Proof. By virtue of Lemma 6.1.12 (and Example 6.1.3), we can realize f as the limit of a filtered diagram $\{f_\alpha : X_\alpha \to Y_\alpha\}$, where each f_α is a map between finite sets. Then P is a limit of the filtered diagram of pro-objects $\{Q \times_{\Phi(Y_\alpha)} \Phi(X_\alpha)\}$. Since the collection of weakly projective objects of $\mathrm{Pro}(\mathcal{C})$ is closed under filtered limits (Remark 6.2.7), it will suffice to show that each of the fiber products $Q \times_{\Phi(Y_\alpha)} \Phi(X_\alpha)$ is weakly projective. In other words, we may assume without loss of generality that X and Y are finite sets (with the discrete topology).

Let us identify $\Phi(X)$ with the coproduct $\coprod_{x \in X} \mathbf{1}$, where $\mathbf{1}$ is a final object of $\mathcal{C}$ (moreover, this coproduct can be taken either in the category $\mathcal{C}$ or $\mathrm{Pro}(\mathcal{C})$, by virtue of Lemma 6.1.11). Since $\mathrm{Pro}(\mathcal{C})$ is extensive (Proposition 6.1.19), it follows that P decomposes as a coproduct $\coprod_{x \in X} \Phi(\{x\}) \times_{\Phi(Y)} Q$. Because the collection of weakly projective objects of $\mathrm{Pro}(\mathcal{C})$ is closed under coproducts (Remark 6.2.10), it will suffice to show that each summand $\Phi(\{x\}) \times_{\Phi(Y)} Q$ is weakly projective. In other words, we may assume without loss of generality that the set X consists of a single element x.

For each $y \in Y$, let Q_y denote the fiber product $\Phi(\{y\}) \times_{\Phi(Y)} Q$. Since $\mathrm{Pro}(\mathcal{C})$ is extensive (Proposition 6.1.19), we can identify Q with the coproduct $\coprod_{y \in Y} Q_y$. Applying Proposition 6.2.11, we deduce that each Q_y is weakly projective. In particular, the object $P = Q_{f(x)}$ is weakly projective, as desired. □

Lemma 6.4.7. *Let $f : X \to Y$ be a continuous function between Stone spaces, let $\mathcal{C}$ be a small pretopos, and suppose we are given a pullback diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ \Phi(X) \arrow[r, "\Phi(f)"] & \Phi(Y) \end{tikzcd}$$

in $\mathrm{Pro}(\mathcal{C})$. *Then the associated diagram of Stone spaces σ:*

$$\begin{tikzcd} {|P|} \arrow[r] \arrow[d] & {|Q|} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is also a pullback square.

Proof. Arguing as in the proof of Lemma 6.4.6 (and using the fact that the functor $|\,| : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Stone}$ commutes with filtered limits), we can reduce to the case where X and Y are finite sets with the discrete topology. For each $y \in Y$, set $Q_y = \Phi(\{y\}) \times_{\Phi(Y)} Q$. Since $\mathrm{Pro}(\mathcal{C})$ is extensive, the canonical maps

$$\coprod_{x \in X} Q_{f(x)} \to P \qquad \coprod_{y \in Y} Q_y \to Q$$

are isomorphisms. Because the functor $|\,| : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Stone}$ preserves finite coproducts (in fact, all colimits), we can identify σ with the diagram of Stone spaces

$$\begin{tikzcd} \coprod_{x \in X} |Q_{f(x)}| \arrow[r] \arrow[d] & \coprod_{y \in Y} |Q_y| \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd}$$

which is evidently a pullback square. □

Lemma 6.4.8. *Let $\mathcal{C}$ be a small pretopos, let P be a weakly projective pro-object of $\mathcal{C}$, and let $u : P \to \Phi(|P|)$ be the unit map for the adjunction of Construction 6.4.3. Let x be a point of the Stone space $|P|$, and form*

a pullback diagram

$$\begin{tikzcd} P_x \arrow[r] \arrow[d] & P \arrow[d] \\ \Phi(\{x\}) \arrow[r] & \Phi(|P|). \end{tikzcd}$$

Then P_x is a model of $\mathcal{C}$ (when viewed as a left exact functor from $\mathcal{C}$ to Set*).*

Proof. It follows from Lemma 6.4.6 that the functor $P_x : \mathcal{C} \to \mathrm{Set}$ preserves finite limits and effective epimorphisms. It will therefore suffice to show that P_x preserves finite coproducts. Using the criterion of Lemma 6.4.5, we are reduced to showing that the Stone space $|P_x|$ consists of a single point. Equivalently, we must show that the diagram of Stone spaces

$$\begin{tikzcd} {|P_x|} \arrow[r] \arrow[d] & {|P|} \arrow[d, "\mathrm{id}"] \\ \{x\} \arrow[r] & {|P|} \end{tikzcd}$$

is a pullback square, which is a special case of Lemma 6.4.7. □

Lemma 6.4.9. *Let $\mathcal{C}$ be a small pretopos. Then the functor $\Phi : \mathrm{Stone} \to \mathrm{Pro}(\mathcal{C})$ preserves finite coproducts.*

Proof. Let $\{X_i\}_{i \in I}$ be a finite collection of Stone spaces; we wish to show that the canonical map $\theta : \coprod_{i \in I} \Phi(X_i) \to \Phi(\coprod_{i \in I} X_i)$ is an isomorphism in $\mathrm{Pro}(\mathcal{C})$. By virtue of Lemma 6.1.12 and Example 6.1.3, we can assume that each X_i is written as the limit of a diagram $\{X_{i,j}\}_{j \in \mathcal{J}^{\mathrm{op}}}$, where $\mathcal{J}$ is a filtered category which is independent of i. Since the formation of finite coproducts in $\mathrm{Stone} \simeq \mathrm{Pro}(\mathrm{Fin})$ and $\mathrm{Pro}(\mathcal{C})$ commutes with filtered limits (Lemma 6.1.18), we can identify θ with the limit of a diagram of morphisms $\{\theta_j : \coprod_{i \in I} \Phi(X_{i,j}) \to \Phi(\coprod_{i \in I} X_{i,j})\}_{j \in \mathcal{J}^{\mathrm{op}}}$. It will therefore suffice to show that each θ_j is an isomorphism. In other words, we are reduced to showing that the restriction $\Phi|_{\mathrm{Fin}}$ preserves finite coproducts, where we identify the category Fin of finite sets with a full subcategory of Stone. This follows immediately from the construction of Φ. □

Lemma 6.4.10. *Let $\mathcal{C}$ be a small pretopos, let P be a weakly projective pro-object of $\mathcal{C}$, and let $|P|$ be the Stone space of Construction 6.4.3. Let $\mathcal{U}_0(|P|)$ denote the collection of all closed and open subsets of $|P|$, which we regard as a partially ordered set with respect to inclusion. Then:*

(1) *For each object $C \in \mathcal{C}$, the construction*

$$(U \in \mathcal{U}_0(|P|)^{\mathrm{op}}) \mapsto \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Phi(U) \times_{\Phi(|P|)} P, C)$$

extends (in an essentially unique way) to a sheaf of sets $\mathcal{O}^C_{|P|}$ on the Stone space $|P|$.

(2) *The functor*

$$\mathcal{O}_{|P|} : \mathcal{C} \to \mathrm{Shv}(|P|) \qquad C \mapsto \mathcal{O}^C_{|P|}$$

is coherent: that is, it is a $|P|$-model of $\mathcal{C}$, in the sense of Definition 6.3.1.

Proof. To prove (1), it will suffice to show that if $U \subseteq |P|$ is given as a finite union of pairwise disjoint closed and open sets $\{U_i\}_{i \in I}$, then the canonical map

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Phi(U) \times_{\Phi(|P|)} P, C) \to \prod_{i \in I} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Phi(U_i) \times_{\Phi(|P|)} P, C).$$

In fact, we claim that $\Phi(U) \times_{\Phi(|P|)} P$ is a coproduct of the objects $\{\Phi(U_i) \times_{\Phi(|P|)} P\}_{i \in I}$ in the category $\mathrm{Pro}(\mathcal{C})$. Since $\mathrm{Pro}(\mathcal{C})$ is extensive (Proposition 6.1.19), we are reduced to showing that $\Phi(U)$ is a coproduct of the objects $\{\Phi(U_i)\}_{i \in I}$. This is clear, since U is a coproduct of the objects U_i in the category of Stone spaces, and the functor Φ preserves coproducts (Lemma 6.4.9).

For each object $C \in \mathcal{C}$ and each point $x \in |P|$, let $\mathcal{O}^C_{|P|,x}$ denote the stalk of the sheaf $\mathcal{O}^C_{|P|}$ at the point x. To prove (2), it will suffice (by virtue of Remark 6.3.7) to show that the functor

$$\mathcal{O}_{|P|,x} : \mathcal{C} \to \mathrm{Set} \qquad C \mapsto \mathcal{O}^C_{|P|,x}$$

is a model of $\mathcal{C}$ for each point $x \in |P|$. This is a restatement of Lemma 6.4.8. □

In the situation of Lemma 6.4.10, we can regard the pair $(|P|, \mathcal{O}_{|P|})$ as an object of the category $\mathrm{Stone}_{\mathcal{C}}$. By definition, the functor $\Gamma(|P|, \mathcal{O}_{|P|})$ is given by the construction

$$(C \in \mathcal{C}) \mapsto \mathcal{O}^C_{|P|}(|P|) = \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Phi(|P|) \times_{\Phi(|P|)} P, C) = \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(P, C).$$

In other words, we have a canonical isomorphism of pro-objects $v : \Gamma(|P|, \mathcal{O}_{|P|}) \simeq P$.

Lemma 6.4.11. *Let $\mathcal{C}$ be a small regular extensive category, let P be a weakly projective pro-object of $\mathcal{C}$, and let $v : \Gamma(|P|, \mathcal{O}_{|P|}) \simeq P$ be the isomorphism described above. Then, for any object $(X, \mathcal{O}_X)$ in* $\mathrm{Stone}_{\mathcal{C}}$*, composition with v induces a bijection*

$$\mathrm{Hom}_{\mathrm{Stone}(\mathcal{C})}((X, \mathcal{O}_X), (|P|, \mathcal{O}_{|P|})) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Gamma(X, \mathcal{O}_X), P).$$

Proof. Fix a continuous map $f : X \to |P|$, and let $\mathrm{Hom}^f_{\mathrm{Stone}(\mathcal{C})}((X, \mathcal{O}_X), (|P|, \mathcal{O}_{|P|}))$ denote the summand of $\mathrm{Hom}_{\mathrm{Stone}(\mathcal{C})}((X, \mathcal{O}_X), (|P|, \mathcal{O}_{|P|}))$ consisting of those maps $(X, \mathcal{O}_X) \to (|P|, \mathcal{O}_{|P|})$ for which the underlying continuous map $X \to |P|$ coincides with f. For each object $C \in \mathcal{C}$, let $f_* \mathcal{O}^C_X$ denote the direct image of the sheaf $\mathcal{O}^C_X$ under the function f. The construction $C \mapsto f_* \mathcal{O}^C_X$ determines a left exact functor $\mathcal{C} \to \mathrm{Shv}(|P|)$, which we will denote by $f_* \mathcal{O}_X$ (beware that this need not be a $|P|$-model of $\mathcal{C}$). Let $\mathcal{U}$ denote the partially ordered set of closed and open subsets of the Stone space $|P|$. Since $\mathcal{U}$ is a basis for the topology of $|P|$, we can identify $\mathrm{Shv}(|P|)$ with a full subcategory of $\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{Set})$. Under this identification, $f_* \mathcal{O}_X$ and $\mathcal{O}_{|P|}$ can be viewed as functors from $\mathcal{U}^{\mathrm{op}}$ to the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$, or equivalently as functors from $\mathcal{U}$ to the category $\mathrm{Pro}(\mathcal{C})$. Unwinding the definitions, we have a canonical bijection

$$\mathrm{Hom}^f_{\mathrm{Stone}(\mathcal{C})}((X, \mathcal{O}_X), (|P|, \mathcal{O}_{|P|})) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}, \mathrm{Pro}(\mathcal{C}))}(f_* \mathcal{O}_X, \mathcal{O}_{|P|}).$$

For every pro-object Q of $\mathcal{C}$, let $\underline{Q}$ denote the constant functor $\mathcal{U}_0(X) \to \mathrm{Pro}(\mathcal{C})$ taking the value Q. Let $T : \mathcal{U} \to \mathrm{Stone}$ be the forgetful functor. By definition, we have a pullback diagram

$$\begin{tikzcd} \mathcal{O}_{|P|} \arrow[r] \arrow[d] & \underline{P} \arrow[d] \\ \Phi \circ T \arrow[r] & \underline{\Phi(|P|)} \end{tikzcd}$$

in the category $\mathrm{Fun}(\mathcal{U}, \mathrm{Pro}(\mathcal{C}))$. We therefore obtain a pullback diagram of sets

$$\begin{tikzcd} \mathrm{Hom}^f_{\mathrm{Stone}(\mathcal{C})}((X, \mathcal{O}_X), (|P|, \mathcal{O}_{|P|})) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}, \mathrm{Pro}(\mathcal{C}))}(f_* \mathcal{O}_X, \underline{P}) \arrow[d] \\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}, \mathrm{Pro}(\mathcal{C}))}(f_* \mathcal{O}_X, \Phi \circ T) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}, \mathrm{Pro}(\mathcal{C})}(f_* \mathcal{O}_X, \underline{\Phi(|P|)}). \end{tikzcd}$$

Using the adjunction of Construction 6.4.3 (and the observation that a functor of the form $\underline{Q}$ is a right Kan extension of its restriction to the full subcategory $\{X\} \subseteq \mathcal{U}$), we can rewrite this diagram as

$$\begin{tikzcd} \mathrm{Hom}^f_{\mathrm{Stone}(\mathcal{C})}((X, \mathcal{O}_X), (|P|, \mathcal{O}_{|P|})) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Gamma(X, \mathcal{O}_X), P) \arrow[d] \\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}, \mathrm{Stone})}(|f_* \mathcal{O}_X|, T) \arrow[r] & \mathrm{Hom}_{\mathrm{Stone}}(|\Gamma(X, \mathcal{O}_X)|, |P|). \end{tikzcd}$$

Here the right vertical map is independent of the function $f : X \to |P|$. Passing to a coproduct over all choices of f, we obtain a pullback square

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Stone}(\mathcal{C})}((X, \mathcal{O}_X), (|P|, \mathcal{O}_{|P|})) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\Gamma(X, \mathcal{O}_X), P) \arrow[d] \\ \coprod_{f : X \to |P|} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U}, \mathrm{Stone})}(|f_* \mathcal{O}_X|, T) \arrow[r, "\theta"] & \mathrm{Hom}_{\mathrm{Stone}}(|\Gamma(X, \mathcal{O}_X)|, |P|), \end{tikzcd}$$

where the top horizontal map is induced by composition with v. To prove that this map is bijective, it will suffice to show that θ is bijective. For each continuous function $f : X \to |P|$, let us regard the construction $U \mapsto f^{-1}(U)$ as a functor from the partially ordered set U to the category of Stone spaces Stone. Using Example 6.4.4, we can identify θ with the canonical map

$$\coprod_{f:X\to|P|} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{U},\mathrm{Stone})}(f^{-1}, T) \to \mathrm{Hom}_{\mathrm{Stone}}(X, |P|).$$

Note that, given a continuous function $f : X \to |P|$, we can identify objects of $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{U},\mathrm{Stone})}(f^{-1}, T)$ with continuous maps $g : X = f^{-1}(|P|) \to T(|P|) = P$ having the property that, for each closed and open subset $U \subseteq |P|$, the map g carries $f^{-1}(U)$ into U. This condition is satisfied only when $g = f$, so the set $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{U},\mathrm{Stone})}(f^{-1}, T)$ has a single element having image $f \in \mathrm{Hom}_{\mathrm{Stone}}(X, |P|)$. Taking the disjoint union over all possible values for the function f, we conclude that θ is a bijection. □

Proof of Theorem 6.3.14. Let $\mathcal{C}$ be a small pretopos category. We wish to show that the functor

$$\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$$

is an equivalence fo categories. By virtue of Lemma 6.4.11, the functor Γ admits a right adjoint G, given on objects by the formula $G(P) = (|P|, \mathcal{O}_{|P|})$. Moreover, for each object $P \in \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$, the counit map $v : (\Gamma \circ G)(P) = \Gamma(|P|, \mathcal{O}_{|P|}) \to P$ is an isomorphism. It follows that the functor G is fully faithful. Consequently, to show that Γ is an equivalence of categories, it will suffice to show that it is conservative, which follows from Lemma 6.4.2. □

7. The Main Theorem

Let $\mathcal{C}$ be a small pretopos. The goal of this section is to prove Theorem 2.2.2, which asserts that the category of left ultrafunctors $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ is equivalent to the category of sheaves $\mathrm{Shv}(\mathcal{C})$. Our strategy will be to introduce a third category $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C}))$ which is equipped with forgetful functors

$$\mathrm{Shv}(\mathcal{C}) \xleftarrow{\phi} \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \xrightarrow{\psi} \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}),$$

and to show that both of these functors as equivalences.

We begin by constructing the equivalence $\phi : \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathcal{C})$. Recall that if $\mathcal{C}$ is a small pretopos, then the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ is regular and extensive, and can therefore be equipped with the coherent topology of Definition B.5.3. We let $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ denote the category of sheaves for the coherent topology, and $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \subseteq \mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ the full subcategory spanned by those sheaves $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ which commute with filtered colimits (Definition 7.1.4). In §7.1 we observe that precomposition with the inclusion $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ induces an equivalence of categories $\phi : \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathcal{C})$ (Corollary 7.1.5).

Recall that every model M of $\mathcal{C}$ can be regarded as a pro-object of $\mathcal{C}$; more precisely, we can regard the category of models $\mathrm{Mod}(\mathcal{C})$ as a full subcategory of $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}} = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$. It follows from the Łos ultraproduct theorem that the category $\mathrm{Mod}(\mathcal{C})$ is closed under the formation of categorical ultraproducts in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$. Consequently, if $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ is a functor which commutes with filtered colimits, then the restriction $\mathscr{F}|_{\mathrm{Mod}(\mathcal{C})} : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ inherits the structure of a left ultrafunctor (Proposition 1.4.9). This construction determines a forgetful functor $\psi : \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$, and we would like to show that ψ is an equivalence of categories.

Let $\mathrm{Stone}_{\mathcal{C}}$ denote the category introduced in Definition 6.3.8. Then Theorem 6.3.14 supplies a fully faithful embedding $\Gamma : \mathrm{Stone}_{\mathcal{C}} \hookrightarrow \mathrm{Pro}(\mathcal{C})$, whose essential image is the full subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ of weakly projective pro-objects of $\mathcal{C}$. In §7.2, we use the functor Γ to transport the coherent topology on $\mathrm{Pro}(\mathcal{C})$ to a topology on $\mathrm{Stone}_{\mathcal{C}}$, which we will refer to as the *elementary topology* (Definition 7.2.1). It follows formally that precomposition with Γ induces an equivalence of categories $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Stone}_{\mathcal{C}})$, where $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Stone}_{\mathcal{C}})$ denotes the category of functors $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ which are sheaves with respect to the elementary topology and commute with filtered colimits (Corollary 7.2.4).

Every functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ which is a sheaf for the elementary topology commutes with finite products. In §7.5, we prove that that the converse holds if $\mathscr{F}$ commutes with filtered colimits (Proposition 7.2.5). In other words, we can identify $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Stone}_{\mathcal{C}})$ with the full subcategory of $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors which preserve finite products and small filtered colimits. Using this identification, we see that the functor ψ factors as a composition

$$\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Stone}_{\mathcal{C}}) \xrightarrow{\psi'} \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}),$$

where ψ' is an equivalence by virtue of Theorem 5.3.3. In §7.3, we use this strategy to complete the proof of Theorem 2.2.2.

To prove Proposition 7.2.5, we will need to be able to recognize when a functor $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf for the elementary topology. For this, it is convenient to give an alternative definition of the elementary topology, which does not make reference to the embedding $\Gamma : \mathrm{Stone}_{\mathcal{C}} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ of Theorem 6.3.14. In §7.4, we show that a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in $\mathrm{Stone}_{\mathcal{C}}$ is a covering for the elementary topology if and only if, for each point $y \in Y$, it is possible to choose a point $x \in X$ for which $f(x) = y$ and the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an *elementary embedding* of models of $\mathcal{C}$ (Theorem 7.4.5). In §7.5, we prove Proposition 7.2.5 by combining this characterization of elementary coverings with a standard argument relating ultraproducts and elementary embeddings (Lemma 7.5.4).

7.1. Sheaves on $\mathrm{Pro}(\mathcal{C})$.

Let $\mathcal{C}$ be a small pretopos. Then the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ is regular and extensive (Corollary 6.1.20). We will regard $\mathrm{Pro}(\mathcal{C})$ as equipped with the *coherent topology* of Construction B.5.3. We let $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ denote the category of set-valued sheaves with respect to the coherent topology on $\mathrm{Pro}(\mathcal{C})$.

Remark 7.1.1. Let $\mathcal{C}$ be a small pretopos. By virtue of Proposition B.5.5, we can identify $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ with the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathcal{C})^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors $\mathscr{F}$ satisfying the following pair of conditions:

- (i) For every finite collection of objects $\{P_i\}_{i \in I}$ of $\mathrm{Pro}(\mathcal{C})$, the canonical map $\mathscr{F}(\mathrm{Pro}(\mathcal{C})) \to \prod_{i \in I} \mathscr{F}(C_i)$ is bijective.
- (ii) For every effective epimorphism $P \twoheadrightarrow Q$ in $\mathrm{Pro}(\mathcal{C})$, the diagram of sets
$$\mathscr{F}(P) \to \mathscr{F}(Q) \rightrightarrows \mathscr{F}(Q \times_P Q)$$
is an equalizer.

Warning 7.1.2. Let $\mathcal{C}$ be a small pretopos. Then the category of pro-objects $\mathrm{Pro}(\mathcal{C})$ need not be small, or even essentially small (in fact, $\mathrm{Pro}(\mathcal{C})$ is essentially small if and only if the pretopos $\mathcal{C}$ is trivial). Consequently, one should exercise some care when working with sheaves on $\mathrm{Pro}(\mathcal{C})$:

- If $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ is a presheaf of (small) sets on $\mathrm{Pro}(\mathcal{C})$, then one cannot generally sheafify $\mathscr{F}$ to produce a sheaf of (small) sets; the standard sheafification process involves passage to a direct limit over all possible coverings, which need not exist in the category of (small) sets.
- The category $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ is not a Grothendieck topos (except in the case where $\mathcal{C}$ is trivial).

We now compare the category $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ with the topos $\mathrm{Shv}(\mathcal{C})$.

Proposition 7.1.3. *Let $\mathcal{C}$ be a small pretopos and let $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ be a functor. Then:*

1. *If $\mathscr{F}$ is a sheaf with respect to the coherent topology on $\mathrm{Pro}(\mathcal{C})$, then the restriction $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf with respect to the coherent topology on $\mathcal{C}$.*
2. *If $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf with respect to the coherent topology on $\mathcal{C}$ and the functor $\mathscr{F}$ commutes with filtered colimits, then $\mathscr{F}$ is a sheaf with respect to the coherent topology on $\mathrm{Pro}(\mathcal{C})$.*

Proof. Assertion (1) follows immediately from the characterization of sheaves given in Proposition B.5.5. We will prove (2). Assume that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf and that the functor $\mathscr{F}$ carries filtered limits in $\mathrm{Pro}(\mathcal{C})$ to filtered colimits of sets. We will show that $\mathscr{F}$ satisfies conditions (i) and (ii) of Remark 7.1.1:

(i) Suppose we are given a finite collection of objects $\{P_i\}_{i\in I}$ in $\mathrm{Pro}(\mathcal{C})$ having coproduct P. We wish to show that the canonical map

$$\theta : \mathscr{F}(P) \to \prod_{i\in I} \mathscr{F}(P_i)$$

is bijective. Using Lemma 6.1.12, we can write each P_i as the limit of a diagram $\{P_{i,\alpha}\}_{\alpha\in\mathcal{A}^{\mathrm{op}}}$ indexed by the same filtered category $\mathcal{A}$. Using Lemmas 6.1.18, we can identify P with the limit of the coproduct diagram $\{\coprod_{i\in I} P_{i,\alpha}\}_{\alpha\in\mathcal{A}^{\mathrm{op}}}$, where each $P_{i,\alpha}$ belongs to $\mathcal{C}$. Consequently, our assumption on $\mathscr{F}$ allows us to identify θ with a filtered colimit of comparison maps

$$\theta_\alpha : \mathscr{F}(\coprod_{i\in I} P_{i,\alpha}) \to \prod_{i\in I} \mathscr{F}(P_{i,\alpha}),$$

each of which is bijective by virtue of our assumption that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf with respect to the coherent topology on $\mathcal{C}$ (Proposition B.5.5).

(ii) Let $f : P \twoheadrightarrow Q$ be an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$. We wish to show that the induced map

$$\rho : \mathscr{F}(P) \to \mathrm{Eq}(\mathscr{F}(Q) \rightrightarrows \mathscr{F}(Q \times_P Q))$$

is a bijection. Using Proposition 6.1.15, we can write f as the limit of a filtered diagram $\{f_\alpha : P_\alpha \twoheadrightarrow Q_\alpha\}$ of effective epimorphisms in the category $\mathcal{C}$. Invoking our assumption on $\mathscr{F}$, we can write ρ as a filtered colimit of maps

$$\rho_\alpha : \mathscr{F}(P_\alpha) \to \mathrm{Eq}(\mathscr{F}(Q_\alpha) \rightrightarrows \mathscr{F}(Q_\alpha \times_{P_\alpha} Q_\alpha)),$$

each of which is bijective by virtue of our assumption that $\mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ is a sheaf with respect to the coherent topology on $\mathcal{C}$ (Proposition B.5.5).

□

Definition 7.1.4. Let $\mathcal{C}$ be a small pretopos. We will say that a sheaf $\mathscr{F} \in \mathrm{Pro}(\mathcal{C})$ is *continuous* if, for every object $X \in \mathrm{Pro}(\mathcal{C})$ given as the limit of a filtered diagram $\{X_\alpha\}$ in $\mathcal{C}$, the canonical map

$$\varinjlim_{\alpha} \mathscr{F}(X_\alpha) \to \mathscr{F}(X)$$

is a bijection. Let $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C}))$ denote the full subcategory of $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ spanned by the continuous sheaves on $\mathrm{Pro}(\mathcal{C})$.

Corollary 7.1.5. *Let $\mathcal{C}$ be a small pretopos. Then the restriction functor $\mathscr{F} \mapsto \mathscr{F}|_{\mathcal{C}^{\mathrm{op}}}$ induces an equivalence of categories* $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathcal{C})$.

Proof. Combine Proposition 7.1.3 with Remark 6.1.7. □

We close this section by describing the relationship of our constructions with the theory of pro-étale sheaves in algebraic geometry introduced by Bhatt and Scholze in [4]. The following discussion will play no further role in this paper and can safely be omitted by the reader. We begin with a general observation.

Remark 7.1.6. Let $\mathcal{C}$ be a small pretopos, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under finite limits with the following additional property:

$(*)$ For every object $C \in \mathcal{C}$, there exists an effective epimorphism $C_0 \twoheadrightarrow C$ with $C_0 \in \mathcal{C}_0$.

Then we can identify the category of pro-objects $\mathrm{Pro}(\mathcal{C}_0)$ with a full subcategory of $\mathrm{Pro}(\mathcal{C})$ (Proposition 6.1.10). Condition $(*)$ guarantees that $\mathrm{Pro}(\mathcal{C}_0)$ contains all weakly projective pro-objects of $\mathcal{C}$ (Remark 6.2.9), and therefore forms a basis with respect to the coherent topology on $\mathrm{Pro}(\mathcal{C})$. Applying Proposition B.6.3, we see that $\mathrm{Pro}(\mathcal{C}_0)$ inherits a Grothendieck topology for which the restriction functor $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}(\mathcal{C}_0))$ is an equivalence of categories.

In the special case where $\mathcal{C}_0$ is closed under the formation of coproducts and images in $\mathcal{C}$, we can characterize the induced topology on $\mathrm{Pro}(\mathcal{C}_0)$ more intrinsically: it is the coherent topology on $\mathrm{Pro}(\mathcal{C}_0)$ (which is a regular extensive category by virtue of Corollary 6.1.20).

Example 7.1.7 (The Pro-Étale Topology of a Scheme)**.** Let X be a scheme. Following Bhatt-Scholze ([4]), we say that a morphism of schemes $f : U \to X$ is *weakly étale* if both f and the diagonal map $\delta : U \to U \times_X U$ are flat. Let Sch_X denote the category of all X-schemes and let $\mathrm{Sch}_X^{\text{proét}}$ denote the full subcategory spanned by those X-schemes U for which the structure morphism $f : U \to X$ is weakly étale. The category $\mathrm{Sch}_X^{\text{proét}}$ admits a Grothendieck topology, where a collection of maps $\{f_i : V_i \to U\}_{i \in I}$ is a covering if and only if, for every quasi-compact open subset $U_0 \subseteq U$, there exists a finite subset $I_0 \subseteq I$ for which the maps $\{V_i \times_U U_0 \to U_0\}_{i \in I_0}$ are mutually surjective. We refer to this topology as the *pro-étale topology on* $\mathrm{Sch}_X^{\text{proét}}$.

For simplicity, let us assume that $X = \mathrm{Spec}(R)$ is affine. Let $\mathcal{C}_0$ be the category of X-schemes of the form $\mathrm{Spec}(A)$, where A is an étale R-algebra. Then the inclusion $\mathcal{C}_0 \hookrightarrow \mathrm{Sch}_X^{\text{proét}}$ extends to a fully faithful embedding $\mathrm{Pro}(\mathcal{C}_0) \hookrightarrow \mathrm{Sch}_X^{\text{proét}}$ whose essential image consists of those X-schemes of the form $\mathrm{Spec}(A)$, where A is an *ind-étale* R-algebra (that is, an R-algebra which can be written as a filtered colimit of étale R-algebras). Under this embedding, the functor $P \mapsto \widehat{P}$ of Notation 6.1.22 corresponds to the forgetful functor from R-schemes to topological spaces. It follows from Proposition 6.1.24 that a collection of morphisms $\{U_i \to U\}_{i \in I}$ is a covering with respect to the coherent topology on $\mathrm{Pro}(\mathcal{C}_0)$ if and only if it is a covering with respect to the pro-étale topology of [4]. Moreover, the image of the inclusion $\mathrm{Pro}(\mathcal{C}_0) \hookrightarrow \mathrm{Sch}_X^{\text{proét}}$ forms a basis for the pro-étale topology (see Lemma 4.2.4 of [4]), so Proposition B.6.3 supplies an equivalence of categories $\alpha : \mathrm{Shv}(\mathrm{Sch}_X^{\text{proét}}) \to \mathrm{Shv}(\mathrm{Pro}(\mathcal{C}_0))$.

The category $\mathcal{C}_0$ is usually not a pretopos. However, it can be identified with a full subcategory of the pretopos $\mathcal{C}$ of constructible set-valued sheaves on X. Moreover, the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ satisfies condition $(*)$ of Remark 7.1.6, and therefore induces an equivalence of categories $\beta : \mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}(\mathcal{C}_0))$. Combined with the preceding analysis, we obtain an equivalence

$$\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \simeq \mathrm{Shv}(\mathrm{Pro}(\mathcal{C}_0)) \simeq \mathrm{Shv}(\mathrm{Sch}_X^{\text{proét}})$$

between the category $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ studied in this section with the category $\mathrm{Shv}(\mathrm{Sch}_X^{\text{proét}})$ of pro-étale sheaves studied in [4].

The preceding discussion can be extended to the case where X is any quasi-compact, quasi-separated scheme; (with some minor modifications, since the category $\mathcal{C}_0$ need not admit finite limits).

7.2. Sheaves on $\mathrm{Stone}_{\mathcal{C}}$.

Using the results of §6 to "restrict" the coherent topology on $\mathrm{Pro}(\mathcal{C})$ to a topology on the category $\mathrm{Stone}_{\mathcal{C}}$ of Definition 6.3.8, which we will refer to as the *elementary topology* (for reasons which will become clear in §7.4).

Definition 7.2.1. Let $\mathcal{C}$ be a small pretopos. We will say that a collection of morphisms

$$\{f_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)\}_{i \in I}$$

in the category $\mathrm{Stone}_{\mathcal{C}}$ is an *elementary covering* if the collection of maps

$$\{\Gamma(f_i) : \Gamma(X_i, \mathcal{O}_{X_i}) \to \Gamma(X, \mathcal{O}_X)\}_{i \in I}$$

is a covering with respect to the coherent topology on $\mathrm{Pro}(\mathcal{C})$: that is, if there exists a finite subset $I_0 \subseteq I$ for which the induced map

$$\coprod_{i \in I_0} \Gamma(X_i, \mathcal{O}_{X_i}) \to \Gamma(X, \mathcal{O}_X)$$

is an effective epimorphism of pro-objects.

Proposition 7.2.2. *Let $\mathcal{C}$ be a small pretopos. Then:*

(1) *The collection of elementary coverings determines a Grothendieck topology on the category* $\mathrm{Stone}_{\mathcal{C}}$*, which we will refer to as the elementary topology.*
(2) *Precomposition with the global sections functor* $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C})$ *induces an equivalence of categories* $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}})$*, where we endow* $\mathrm{Pro}(\mathcal{C})$ *with the coherent topology and* $\mathrm{Stone}_{\mathcal{C}}$ *with the elementary topology.*

Proof. It follows from Proposition 6.2.12 that the full subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}) \subseteq \mathrm{Pro}(\mathcal{C})$ forms a basis for the coherent topology on $\mathrm{Pro}(\mathcal{C})$. It follows from Proposition B.6.3 that the subcategory $\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ inherits a Grothendieck topology for which the restriction functor $\mathrm{Shv}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C}))$ is an equivalence of categories. Proposition 7.2.2 follows by transporting this topology along the equivalence of categories $\Gamma : \mathrm{Stone}_{\mathcal{C}} \simeq \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ supplied by Theorem 6.3.14. □

Definition 7.2.3. Let $\mathcal{C}$ be a small pretopos and let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ be a sheaf for the elementary topology. We will say that $\mathscr{F}$ is *continuous* if it commutes with small filtered colimits. We let $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Stone}_{\mathcal{C}}, \mathrm{Set})$ denote the full subcategory of $\mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}})$ spanned by the continuous sheaves.

Corollary 7.2.4. *Let $\mathcal{C}$ be a small pretopos. Then composition with the global sections functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ induces an equivalence of categories $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Stone}_{\mathcal{C}})$.*

Proof. By virtue of Proposition 7.2.2, it will suffice to show that a sheaf $\mathscr{F} \in \mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))$ is continuous if and only if the composition $\mathscr{F} \circ \Gamma \in \mathrm{Shv}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}})$ is continuous. The "only if" direction is clear (since the global sections functor $\Gamma : \mathrm{Stone}_{\mathcal{C}} \to \mathrm{Pro}(\mathcal{C})$ commutes with filtered limits). To prove the converse, let $\lambda : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ be the functor of Proposition 6.2.12, so that every object $P \in \mathrm{Pro}(\mathcal{C})$ is equipped with an epimorphism $\rho_P : \lambda(P) \twoheadrightarrow P$. Let $\mu : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})$ be the functor given on objects by the formula $\mu(P) = \lambda(\lambda(P) \times_P \lambda(P))$. Our assumption that $\mathscr{F}$ is a sheaf guarantees that the coequalizer diagram $\mu(X) \rightrightarrows \lambda(X) \to X$ in $\mathrm{Pro}(\mathcal{C})$ induces an isomorphism

$$\mathscr{F}(P) \simeq \mathrm{Eq}(\mathscr{F}(\lambda(P)) \rightrightarrows \mathscr{F}(\mu(P))),$$

depending functorially on P. By construction, the functors λ and μ commute with filtered inverse limits. Consequently, if $\mathscr{F}|_{\mathrm{Pro}^{\mathrm{wp}}(\mathcal{C})^{\mathrm{op}}}$ commutes with filtered colimits, then the functor

$$P \mapsto \mathscr{F}(P) \simeq \mathrm{Eq}(\mathscr{F}(\lambda(P)) \rightrightarrows \mathscr{F}(\mu(P)))$$

also commutes with filtered colimits. Corollary 7.2.4 now follows from Theorem 6.3.14. □

Our proof of Theorem 2.2.2 will make use of the following characterization of continuous sheaves on $\mathrm{Stone}_{\mathcal{C}}$, which we prove in §7.5:

Proposition 7.2.5. *Let $\mathcal{C}$ be a small pretopos and let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ be a functor which commutes with filtered colimits. The following conditions are equivalent:*

(*a*) *The functor $\mathscr{F}$ is a sheaf with respect to the elementary topology.*
(*b*) *The functor $\mathscr{F}$ carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products in Set.*

7.3. **The Proof of Theorem 2.2.2.**

We are now almost ready to prove the main result of this paper.

Proposition 7.3.1. *Let $\mathcal{C}$ be a small pretopos. Then the construction of Proposition 1.4.9 (applied to the inclusion $\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$) induces an equivalence of categories*

$$\psi : \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}).$$

Proof. Note that the inclusion $\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$ is isomorphic to the composite functor

$$\mathrm{Mod}(\mathcal{C}) \xrightarrow{M \mapsto \underline{M}} \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \xrightarrow{\Gamma} \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}.$$

Consequently, the functor ψ factors (up to isomorphism) as a composition

$$\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \xrightarrow{\mathscr{F} \mapsto \mathscr{F} \circ \Gamma} \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Stone}_{\mathcal{C}}) \xrightarrow{\psi'} \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}),$$

where ψ' obtained by applying the construction of Proposition 1.4.9 to the fully faithful embedding $\mathrm{Mod}(\mathcal{C}) \hookrightarrow \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}$. The first map is an equivalence by virtue of Corollary 7.2.4. According to Proposition 7.2.5, $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Stone}_{\mathcal{C}})$ is the full subcategory of $\mathrm{Fun}(\mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors which preserve finite products and small filtered colimits. Consequently, the functor ψ' is an equivalence by virtue of Theorem 5.2.1 (and Warning 6.3.9). □

Proof of Theorem 2.2.2. Let $\mathcal{C}$ be a small pretopos. Then the evaluation map $\mathrm{ev} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set})$ of Construction 2.2.1 fits into a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{ev}"] \arrow[d, "h"] & \mathrm{Fun}^{\mathrm{Ult}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}) \arrow[d, hook] \\ \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \arrow[r, "\psi", "\sim"'] & \mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathrm{Set}), \end{tikzcd}$$

where ψ is the equivalence of Proposition 7.3.1 and $h : \mathcal{C} \to \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C}))$ associates to each object $C \in \mathcal{C}$ the functor $h_C : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set}) \to \mathrm{Set}$ given by evaluation at C (which is a sheaf, since it is representable by the image of C in the category $\mathrm{Pro}(\mathcal{C})$). We can therefore reformulate Theorem 2.2.2 as follows:

(1) For each object $\mathscr{F} \in \mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C}))$, the construction

$$(C \in \mathcal{C}) \mapsto \mathrm{Hom}_{\mathrm{Shv}(\mathrm{Pro}(\mathcal{C}))}(h_C, \widehat{\mathscr{F}})$$

determines a functor $\mathscr{F}_0 : \mathcal{C} \to \mathrm{Set}$ which is a sheaf for the coherent topology on $\mathcal{C}$.

(2) The construction $\mathscr{F} \mapsto \mathscr{F}_0$ induces an equivalence of categories $\mathrm{Shv}^{\mathrm{cont}}(\mathrm{Pro}(\mathcal{C})) \to \mathrm{Shv}(\mathcal{C})$.

This is precisely the content of Corollary 7.1.5. □

7.4. **Effective Epimorphisms and Elementary Embeddings.** Let $\mathcal{C}$ be a small pretopos. Our goal in this section is to give an alternative description of the elementary topology on the category $\mathrm{Stone}_{\mathcal{C}}$, which does not make reference to the embedding $\Gamma : \mathrm{Stone}_{\mathcal{C}} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ of Theorem 6.3.14. First, we need to review some terminology.

Definition 7.4.1. Let $\mathcal{C}$ be a pretopos and let $f : M \to N$ be a morphism in the category of models $\mathrm{Mod}(\mathcal{C})$. We will say that f is an *elementary embedding* if, for every object $C \in \mathcal{C}$, the induced map $M(f) : M(C) \to N(C)$ is a monomorphism of sets.

Example 7.4.2. Let $\mathcal{C}$ be a pretopos, let M be a model of $\mathcal{C}$, and let μ be an ultrafilter on a set S. Then the ultrapower diagonal $\Delta_\mu : M \to M^\mu$ of Example 1.3.4 is an elementary embedding. This follows from the observation that, for every set X, the ultrapower diagonal $X \to X^\mu$ is a monomorphism of sets (Lemma 7.5.2).

Proposition 7.4.3. *Let $\mathcal{C}$ be a small pretopos and let $f : M \to N$ be a morphism in* $\mathrm{Mod}(\mathcal{C})$*. The following conditions are equivalent:*

(1) *The morphism f is an elementary embedding.*

(2) *For every object $C \in \mathcal{C}$ and every subobject $C_0 \subseteq C$, the diagram*

$$\begin{tikzcd} M(C_0) \arrow[r, "f"] \arrow[d] & N(C_0) \arrow[d] \\ M(C) \arrow[r, "f"] & N(C) \end{tikzcd}$$

is a pullback square (in the category of sets).

Proof. We first show that (2) implies (1). Let C be an object of $\mathcal{C}$. Then the diagonal map $\delta : C \to C \times C$ exhibits C as a subobject of $C \times C$. If condition (2) is satisfied, then the upper square in the diagram

$$\begin{tikzcd} M(C) \arrow[r] \arrow[d] & N(C) \arrow[d] \\ M(C \times C) \arrow[r] \arrow[d] & N(C \times C) \arrow[d] \\ M(C) \times M(C) \arrow[r] & N(C) \times N(C) \end{tikzcd}$$

is a pullback. Since M and N preserve finite products, the lower vertical maps are isomorphisms. It follows that the outer rectangle is also a pullback square, so that the map $M(C) \to N(C)$ is injective.

We now show that (1) implies (2). Assume that f is an elementary embedding. Let C be an object of $\mathcal{C}$ and let $C_0 \subseteq C$ a subobject. We wish to show that the diagram σ :

$$\begin{tikzcd} M(C_0) \arrow[r] \arrow[d] & N(C_0) \arrow[d] \\ M(C) \arrow[r] & N(C) \end{tikzcd}$$

is a pullback square. Let $\mathbf{1}$ denote the final object of $\mathcal{C}$. Replacing C by $C \amalg \mathbf{1}$ and C_0 by $C_0 \amalg \mathbf{1}$ (and using the fact that M and N preserve finite coproducts), we can reduce to the case where the projection map $C_0 \to \mathbf{1}$ is an effective epimorphism.

Let R denote the subobject of $C \times C$ given by the union of C (embedded diagonally in $C \times C$) with the product $C_0 \times C_0$. Since $\mathcal{C}$ is a pretopos, the equivalence relation R is effective. Let $D = C/R$ denote the coequalizer of the diagram $R \rightrightarrows C$, and let $D_0 \subseteq D$ denote the image of the composite map $C_0 \hookrightarrow C \twoheadrightarrow D$. Note that effective epimorphisms $C_0 \twoheadrightarrow D_0$ and $C_0 \twoheadrightarrow \mathbf{1}$ determine the same equivalence relation on C_0 and are therefore equivalent: that is, D_0 is a final object of $\mathcal{C}$.

Since the formation of images in $\mathcal{C}$ is compatible with pullback, we can identify the fiber product $D_0 \times_D C$ with the image of the map $q : C_0 \times_D C \to C$ given by projection onto the second factor. Using the distributivity of the lattice $\mathrm{Sub}(C \times C)$, we compute

$$\begin{aligned} C_0 \times_D C &= (C_0 \times C) \cap (C \times_D C) \\ &= (C_0 \times C) \cap R \\ &= (C_0 \times C) \cap (C \cup (C_0 \times C_0)) \\ &= ((C_0 \times C) \cap C) \cup ((C_0 \times C) \cap (C_0 \times C_0)) \\ &= C_0 \cup (C_0 \times C_0) \\ &= C_0 \times C_0. \end{aligned}$$

It follows that the image of q is contained in D_0, so that the diagram

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & D_0 \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a pullback square.

Using the left exactness of the functor N, we deduce that σ can be extended to a commutative diagram

$$\begin{tikzcd} M(C_0) \arrow[r] \arrow[d] & N(C_0) \arrow[r] \arrow[d] & N(D_0) \arrow[d] \\ M(C) \arrow[r] & N(C) \arrow[r] & N(D) \end{tikzcd}$$

where the right square is a pullback. Consequently, to show that σ is a pullback, it will suffice to show that the outer rectangle is a pullback. This outer rectangle fits into another commutative diagram

$$\begin{tikzcd} M(C_0) \arrow[r] \arrow[d] & M(D_0) \arrow[r] \arrow[d] & N(D_0) \arrow[d] \\ M(C) \arrow[r] & M(D) \arrow[r] & N(D) \end{tikzcd}$$

where the left square is a pullback (by virtue of the left exactness of M). It will therefore suffice to show that the canonical map

$$\theta : M(D_0) \to N(D_0) \times_{M(D)} N(D)$$

is a bijection. Since D_0 is a final object of $\mathcal{C}$ and the functor M is left exact, the set $M(D_0)$ is a singleton. It will therefore suffice to show that the fiber product $N(D_0) \times_{M(D)} N(D)$ has at most one element. This is clear, since the set $N(D_0)$ is also a singleton and the map $M(D) \to N(D)$ is injective (by virtue of our assumption that f is an elementary embedding). □

In the setting of classical first-order logic, the requirement of Definition 7.4.1 is vacuous. Recall that a pretopos $\mathcal{C}$ is said to be *Boolean* if, for every object $X \in \mathcal{C}$, the partially ordered set $\mathrm{Sub}(X)$ is a Boolean algebra (in other words, every subobject of X is a summand of X).

Proposition 7.4.4. *Let $\mathcal{C}$ be a Boolean pretopos. Then every morphism $f : M \to N$ in $\mathrm{Mod}(\mathcal{C})$ is an elementary embedding.*

Proof. We will show that f satisfies criterion (2) of Proposition 7.4.3. Let C be an object of $\mathcal{C}$ and let $C_0 \subseteq C$ be a subobject; we wish to show that the diagram σ :

$$\begin{tikzcd} M(C_0) \arrow[r, "f"] \arrow[d] & N(C_0) \arrow[d] \\ M(C) \arrow[r, "f"] & N(C) \end{tikzcd}$$

is a pullback. Since $\mathcal{C}$ is Boolean, the partially ordered set $\mathrm{Sub}(C)$ is a Boolean algebra. We can therefore choose another subobject $C_1 \in \mathrm{Sub}(C)$ which is complementary to C_0, so that $C_0 \cup C_1 = C$ and $C_0 \cap C_1 = \varnothing$. It follows that inclusions $C_0 \hookrightarrow C \hookleftarrow C_1$ exhibit C as a coproduct of C_0 with C_1. Since the functors M and N preserve finite coproducts, we can identify σ with the diagram

$$\begin{tikzcd} M(C_0) \arrow[r] \arrow[d] & N(C_0) \arrow[d] \\ M(C_0) \amalg M(C_1) \arrow[r] & N(C_0) \amalg N(C_1), \end{tikzcd}$$

so that the desired result follows from Proposition A.3.7 (or by direct inspection). □

The relevance of Definition 7.4.1 for us is the following:

Theorem 7.4.5. *Let $\mathcal{C}$ be a small pretopos and let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism in $\mathrm{Stone}_{\mathcal{C}}$. The following conditions are equivalent:*

(1) *The induced map $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$.*

(2) *For each point $y \in Y$, there exists a point $x \in X$ such that $f(x) = y$ and the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an elementary embedding in* $\mathrm{Mod}(\mathcal{C})$.

Corollary 7.4.6. *Let $\mathcal{C}$ be a small pretopos and let $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism in* $\mathrm{Stone}_{\mathcal{C}}$*. If the induced map $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ is an effective epimorphism in* $\mathrm{Pro}(\mathcal{C})$*, then the underlying map of topological spaces $X \to Y$ is surjective. The converse holds if $\mathcal{C}$ is Boolean.*

Proof. Combine Theorem 7.4.5 with Proposition 7.4.4. □

Corollary 7.4.7. *Let $\mathcal{C}$ be a small pretopos and let $\{f_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)\}_{i \in I}$ be a collection of morphisms in* $\mathrm{Stone}_{\mathcal{C}}$*. Then the morphisms f_i are an elementary covering (in the sense of Definition 7.2.1) if and only if there exists a finite subset $I_0 \subseteq I$ satisfying the following condition:*

($*$) *For every point $x \in X$, there exists an index $i \in I_0$ and a point $y \in X_i$ such that $f_i(y) = x$ and the map $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,y}$ is an elementary embedding in* $\mathrm{Mod}(\mathcal{C})$.

Corollary 7.4.8. *[Amalgamation] Let $\mathcal{C}$ be a small pretopos, let $f : M \to N$ be an arbitrary morphism in* $\mathrm{Mod}(\mathcal{C})$*, and let $g : M \to M'$ be an elementary embedding in* $\mathrm{Mod}(\mathcal{C})$*. Then there exists a commutative diagram*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & M' \arrow[d] \\ N \arrow[r, "g'"] & N' \end{tikzcd}$$

in $\mathrm{Mod}(\mathcal{C})$*, where g' is also an elementary embedding.*

Proof. By virtue of Corollary 7.4.7, the elementary embedding g determines an elementary covering $\underline{g} : \underline{M}' \to \underline{M}$ in the category $\mathrm{Stone}_{\mathcal{C}}$. Since the elementary coverings give rise to a Grothendieck topology on $\mathrm{Stone}_{\mathcal{C}}$, we can find a collection of commutative diagrams

$$\begin{tikzcd} (X_i, \mathcal{O}_{X_i}) \arrow[r] \arrow[d] & \underline{M}' \arrow[d, "\underline{g}"] \\ \underline{N} \arrow[r, "\underline{f}"] & \underline{M} \end{tikzcd}$$

in $\mathrm{Stone}_{\mathcal{C}}$ where the left vertical maps form an elementary covering. Using Corollary 7.4.7, we conclude that there is an index i and a point $x \in X_i$ for which the left vertical map induces an elementary embedding $g' : N \to \mathcal{O}_{X_i,x}$, in which case we obtain a commutative diagram of models

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"] & M' \arrow[d] \\ N \arrow[r, "g'"] & \mathcal{O}_{X_i,x} \end{tikzcd}$$

with the desired property. □

Proof of Theorem 7.4.5. We proceed in several steps. Let $f : P \to Q$ be an arbitrary morphism in $\mathrm{Pro}(\mathcal{C})$. Consider the following assertion:

(i) The map $f : P \to Q$ is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$.

We claim that (i) is equivalent to the following:

(ii) For every monomorphism $u : U \hookrightarrow V$ in $\mathrm{Pro}(\mathcal{C})$ and every commutative square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"] & U \arrow[d, "u"] \\ Q \arrow[r] \arrow[ur, dashed] & V, \end{tikzcd}$$

there exists a dotted arrow rendering both triangles commutative.

The implication $(i) \Rightarrow (ii)$ is clear. Conversely, if (ii) is satisfied for the inclusion $\mathrm{Im}(u) \hookrightarrow Q$, then assertion (i) follows. Note that, in the situation of (ii), the dotted arrow is automatically unique (by virtue of our assumption that the morphism $U \to V$ is a monomorphism). According to Proposition 6.1.15, every monomorphism in $\mathrm{Pro}(\mathcal{C})$ can be realized as a filtered limit of monomorphisms in $\mathcal{C}$. Consequently, (ii) is equivalent to the following *a priori* weaker condition:

(iii) For every monomorphism $u : C_0 \hookrightarrow C$ in the category $\mathcal{C}$ and every square diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C_0 \arrow[d, "u"] \\ Q \arrow[r] \arrow[ur, dashed] & C, \end{tikzcd}$$

there exists a dotted arrow rendering both triangles commutative.

Let us now suppose that $P = \Gamma(X, \mathcal{O}_X)$ and $Q = \Gamma(Y, \mathcal{O}_Y)$ for some objects $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y) \in \mathrm{Stone}_{\mathcal{C}}$. Unwinding the definitions, we can rephrase (iii) as follows:

(iv) Let $C \in \mathcal{C}$ be an object and let $s_Y \in \Gamma(Y, \mathcal{O}_Y)(C) = \mathcal{O}_Y^C(Y)$ be a section having image $s_X \in \mathcal{O}_X^C(X)$. Suppose that C_0 is a subobject of C and that s_X lifts to a section of the subsheaf $\mathcal{O}_X^{C_0} \subseteq \mathcal{O}_X^C$. Then s_Y lifts to a section of the subsheaf $\mathcal{O}_Y^{C_0} \subseteq \mathcal{O}_Y^C$.

We now restate (v) in contrapositive form:

(v) Let $C \in \mathcal{C}$ be an object and let $C_0 \in \mathrm{Sub}(C)$ be a subobject. Suppose we are given a global section $s_Y \in \mathcal{O}_Y^C(Y)$ having image $s_X \in \mathcal{O}_X^C(X)$. If there exists a point $y \in Y$ such that the stalk $s_{Y,y}$ does not belong to $\mathcal{O}_{Y,y}^{C_0}$, then there exists a point $x \in X$ such that $s_{X,x}$ does not belong to $\mathcal{O}_{X,x}^{C_0}$.

Note that assertion (v) follows immediately from (2), together with the criterion of Proposition 7.4.3 (if (2) is satisfied, then we can choose a point $x \in X$ for which $f(x) = y$ and the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an elementary embedding). We will complete the proof by showing that assertion (iv) implies (2).

Assume that (iv) is satisfied, and fix a point $y \in Y$. We wish to show that there exists a point $x \in X$ such that $f(x) = y$ and the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is an elementary embedding. Suppose otherwise. Then, for each point $x \in f^{-1}(y)$, the induced map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is not an elementary embedding. It follows that we can choose an object $C(x) \in \mathcal{C}$, a subobject $C_0(x) \in \mathrm{Sub}(C(x))$, and an element of $\eta_x \in \mathcal{O}_{Y,y}^{C(x)} \smallsetminus \mathcal{O}_{Y,y}^{C_0(x)}$ for which the image of η_x in $\mathcal{O}_{X,x}^{C(x)}$ belongs to $\mathcal{O}_{X,x}^{C_0(x)}$. Let U_x be an open neighborhood of x in the fiber $f^{-1}\{y\}$ for which the image of η_x in $\mathcal{O}_{X,x'}^{C(x)}$ belongs to $\mathcal{O}_{Y,y}^{C_0(x)}$, for each $x' \in U_x$. Since the fiber $f^{-1}\{y\}$ is compact, we can choose finitely many points $x_1, \ldots, x_n \in f^{-1}\{y\}$ for which the open sets $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover the fiber $f^{-1}\{y\}$. Set $C = C(x_1) \times \cdots \times C(x_n)$, and let $C_0 \subseteq C$ be the union of the subobjects $C_0(x_i) \times \prod_{j \neq i} C(x_j)$. Then we can identify $\{\eta_{x_i}\}_{1 \leq i \leq n}$ with a point $\eta \in \mathcal{O}_{Y,y}^C$. By construction, η does not belong to $\mathcal{O}_{Y,y}^{C_0}$, but the image of η in $\mathcal{O}_{X,x}^C$ belongs to $\mathcal{O}_{X,x}^{C_0}$ for each $x \in f^{-1}\{y\}$.

Choose a lift of η to a point $s_V \in \mathcal{O}_Y^C(V)$, for some open neighborhood V of Y. Let $s_{f^{-1}(V)}$ denote the image of V in $\mathcal{O}_X^C(f^{-1}(V))$. Then there is a largest open subset $W \subseteq f^{-1}(V)$ for which the restriction $s_{f^{-1}(V)}|_W$ is a section of the subsheaf $\mathcal{O}_X^{C_0} \subseteq \mathcal{O}_X^C$. By construction, the open set W contains the fiber $f^{-1}\{y\}$. Since f is a proper map, we can choose a smaller open set $V' \subseteq V$ such that $y \in V'$ and $f^{-1}(V') \subseteq W$. Replacing V by V', we can assume that $s_{f^{-1}(V)}$ belongs to $\mathcal{O}_X^{C_0}(f^{-1}(V))$.

Shrinking V further if necessary, we can arrange that V is both open and closed. In this case, we can extend s_V to a global section s_Y of the sheaf $\mathcal{O}_Y^{C \amalg \mathbf{1}} \simeq \mathcal{O}_Y^C \amalg \underline{\mathbf{1}}$ (which is equal to s_V on the open set V, and carries the complement of V to the second summand of $\mathcal{O}_Y^{C \amalg \mathbf{1}}$). Replacing C by the coproduct $C \amalg \mathbf{1}$ and C_0 by the coproduct $C_0 \amalg \mathbf{1}$, we can assume that $V = Y$: that is, that s_V is a global section of $\mathcal{O}_Y^C$. It then follows from (iv) that s_V is also a global section of the subsheaf $\mathcal{O}_Y^{C_0} \subseteq \mathcal{O}_Y^C$, contradicting our choice of η. $\square$

7.5. **The Proof of Proposition 7.2.5.** Let $\mathcal{C}$ be a small pretopos, and let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. If $\mathscr{F}$ is a sheaf for the elementary topology, then it carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products in Set. Our goal in this section is to prove Proposition 7.2.5, which asserts that the converse holds provided that $\mathscr{F}$ commutes with filtered colimits. To prove this, we must show that $\mathscr{F}$ satisfies descent for every elementary covering $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. We begin by treating the case where the underlying topological spaces X and Y consist of a single point. By virtue of Corollary 7.4.7, this reduces a concrete statement about the behavior of the functor $F = \mathscr{F}|_{\mathrm{Mod}(\mathcal{C})}$ with respect to elementary embeddings:

Proposition 7.5.1. *Let $\mathcal{C}$ be a small pretopos, let $f : M \to N$ be an elementary embedding in* $\mathrm{Mod}(\mathcal{C})$*, and let $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ be a left ultrafunctor. Then the map of sets $F(f) : F(M) \to F(N)$ is injective. Moreover, the image of $F(f)$ consists of those elements $x \in F(N)$ which satisfy the following condition:*

- $(*)$ *For every pair of morphisms $u, v : N \to P$ in the category* $\mathrm{Mod}(\mathcal{C})$ *satisfying $u \circ f = v \circ f$, we $F(u)(x) = F(v)(x)$ in the set $F(P)$.*

The proof of Proposition will require some preliminaries. We first make some elementary observations concerning ultraproducts in the category of sets:

Lemma 7.5.2. *Let X and S be sets, let μ be an ultrafilter on S, and let $\Delta_\mu : X \to X^\mu$ be the ultrapower diagonal of Example 1.3.4. Then Δ_μ is injective.*

Proof. Without loss of generality we may assume that X is nonempty, so that we can identify X^μ with the set of equivalence classes of functions $f : S \to X$ with respect to an equivalence relation $\sim_\mu$, where $f \sim_\mu g$ if and only if $\mu(\{s \in S : f(s) = g(s)\})) = 1$. If f is the constant function with value $x \in X$ and g is a constant function with value $y \in S$ (representing $\Delta_\mu(x)$ and $\Delta_\mu(y)$, respectively) then we have

$$(f \sim_\mu g) \Leftrightarrow (\mu(\{s \in S : x = y\}) = 1) \Leftrightarrow (x = y).$$

□

Lemma 7.5.3. *Let $u : X \hookrightarrow Y$ be an injective map of sets and let μ be an ultrafilter on a set S. Then the diagram*

$$\begin{tikzcd} X \arrow[r, "\Delta_\mu"] \arrow[d, "u"] & X^\mu \arrow[d, "\int_S u d\mu"] \\ Y \arrow[r, "\Delta_\mu"] & Y^\mu \end{tikzcd}$$

is a pullback square (in the category of sets).

Proof. Let $y \in Y$ be an element with the property that $\Delta_\mu(y) = (\int_S u d\mu)(x)$, for some $x \in X^\mu$; we wish to show that y belongs to the image of u. Then X^μ is nonempty, so X is nonempty; we may therefore assume without loss of generality that x is represented by a tuple $\{x_s\}_{s \in S} \in X^S$. Set $I = \{s \in S : u(x_s) = y\}$. The equality $\Delta_\mu(y) = (\int_S u d\mu)(x)$ guarantees that $\mu(I) = 1$. In particular, the set I is nonempty. Choose an element $s \in I$. Then $y = u(x_s)$ belongs to the image of u. □

Lemma 7.5.4. *Let $\mathcal{C}$ be a small pretopos and let $f : M \to N$ be an elementary embedding in* $\mathrm{Mod}(\mathcal{C})$*. Then there exists a set S, an ultrafilter μ on S, and a morphism $g : N \to M^\mu$ for which the composite map*

$$M \xrightarrow{f} N \xrightarrow{g} M^\mu$$

coincides with the ultrapower diagonal $\delta_M : M \hookrightarrow M^\mu$ of Example 1.3.4.

Proof. To avoid confusion, let us use the notation T_M to denote the image of a model $M \in \mathrm{Mod}(\mathcal{C})$ under the inclusion $\mathrm{Mod}(\mathcal{C})^{\mathrm{op}} \hookrightarrow \mathrm{Pro}(\mathcal{C})$. The elementary embedding $f : M \to N$ can then be identified with a map of pro-objects $T_N \twoheadrightarrow T_M$, which we will denote by T_f. Our assumption that f is elementary guarantees that T_f is an effective epimorphism in $\mathrm{Pro}(\mathcal{C})$ (Theorem 7.4.5), and can therefore be realized as the limit of a filtered diagram $\{f_s : C_s \twoheadrightarrow D_s\}$ of effective epimorphisms in $\mathcal{C}$. Without loss of generality, we can assume that this limit is indexed by (the opposite of) a directed partially ordered set $(S, \leq)$.

For each $s \in S$, set $P_s = C_s \times_{D_s} T_M$ (where the fiber product is formed in $\mathrm{Pro}(\mathcal{C})$. Since M is a model, the pro-object T_M is weakly projective so that each of the maps $T_M \to D_s$ factors through f_s. A choice of factorization then determines a map $u_s : T_M \to P_s$ which is a section of of the projection map $P_s \to T_M$ (note that we do not require, and usually cannot arrange, that the morphisms u_s depend functorially on s). To avoid confusion, let us write N_s for the image of P_s in the opposite category $\mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \subseteq \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$, so that each u_s can be viewed as a natural transformation of functors $N_s \to M$. Moreover, we can identify N with the colimit $\varinjlim_{s \in S} N_s$ (as an object of the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$).

Since the partial ordering on S is directed, we can choose a cofinal ultrafilter μ on S. We now define $g : N \to M^\mu$ to be the composition

$$N \xrightarrow{w} \int_S N_s d\mu \xrightarrow{\int_S u_s d\mu} \int_S M d\mu = M^\mu,$$

where w is the morphism defined by applying Construction 5.3.6 (in the ultracategory $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$). The identity $g \circ f = \Delta_\mu$ follows from Remark 5.3.7. □

Proof of Proposition 7.5.1. Let $\mathcal{C}$ be a small pretopos, let $f : M \to N$ be an elementary embedding in $\mathrm{Mod}(\mathcal{C})$, and let $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ be a left ultrafunctor. Using Lemma 7.5.4, we can choose a set S, an ultrafilter μ on S, and a morphism $g : N \to M^\mu$ such that $g \circ f$ is the ultrapower embedding $\Delta_\mu(M) : M \to M^\mu$. Let $\{\sigma_\mu\}$ be a left ultrastructure on F, so that the composite map

$$F(M) \xrightarrow{F(\Delta_\mu(M))} F(M^\mu) \xrightarrow{\sigma_\mu} F(M)^\mu$$

is the ultrapower diagonal of the set $F(M)$, and therefore injective (Lemma 7.5.2). It follows that the composition $F(\Delta_\mu(M)) = F(g) \circ F(f)$ is also injective, so that $F(f) : F(M) \to F(N)$ is injective. We will complete the proof by showing that if $x \in F(N)$ is an element satisfying condition $(*)$ of Proposition 7.5.1, then x belongs to the image of $F(f)$. We have a commutative diagram

$$\begin{tikzcd}
F(M) \arrow[r, "F(f)"] \arrow[d, "F(\Delta_\mu(M))"] & F(N) \arrow[d, "F(\Delta_\mu(N))"] \\
F(M^\mu) \arrow[r, "F(f^\mu)"] \arrow[d, "\sigma_\mu"] & F(N^\mu) \arrow[d, "\sigma_\mu"] \\
F(M)^\mu \arrow[r, "F(f)^\mu"] & F(N)^\mu,
\end{tikzcd}$$

where the outer rectangle is a pullback square by Lemma 7.5.3. Consequently, to show that x belongs to the image of $F(f)$, it will suffice to show that $F(\Delta_\mu(N))$ belongs to the image of the map $F(f^\mu) : F(M^\mu) \to F(N^\mu)$. In fact, we claim that $F(\Delta_\mu(N))(x) = F(f^\mu)(F(w)(x))$. To prove this, it will suffice (by virtue of our assumption that x satisfies $(*)$) to show that the maps $\Delta_\mu(N), (f^\mu \circ w) : N \to N^\mu$ have the same restriction to M. This follows from Remark 5.3.7 together with the commutativity of the diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\Delta_\mu(M)"] & N \arrow[d, "\Delta_\mu(N)"] \\
M^\mu \arrow[r, "f^\mu"] & N^\mu.
\end{tikzcd}$$

□

To deduce Proposition 7.2.5 from Proposition 7.5.1, we will need a bit of notation.

Construction 7.5.5. Let $\mathcal{C}$ be a small pretopos and let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. Assume that $\mathscr{F}$ carries finite coproducts in $\mathrm{Stone}_{\mathcal{C}}$ to products in the category of sets. Fix an object $(X, \mathcal{O}_X)$ in the category $\mathrm{Stone}_{\mathcal{C}}$, and let $\mathcal{U}_0(X)$ denote the collection of all closed and open subsets of X. We define a functor

$$\mathscr{F}(\mathcal{O}_X) : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathrm{Set}$$

by the formula $\mathscr{F}(\mathcal{O}_X)(U) = \mathscr{F}(U, \mathcal{O}_X|_U)$. It follows from our assumption on $\mathscr{F}$ that the functor $\mathscr{F}(\mathcal{O}_X)$ carries disjoint unions in $\mathcal{U}_0(X)$ to products in Set, and therefore extends uniquely to a set-valued sheaf on X (Corollary B.6.5) which we will also denote by $\mathscr{F}(\mathcal{O}_X)$. The stalks of this sheaf are given by the formula

$$\mathscr{F}(\mathcal{O}_X)_x \simeq \varinjlim_U \mathscr{F}(\mathcal{O}_X)(U) = \varinjlim_U \mathscr{F}(U, \mathcal{O}_X|_U),$$

where the colimit is taken over the collection of all closed and open neighborhoods $U \subseteq X$ of the point x. Consequently, there is a natural map $\mathscr{F}(\mathcal{O}_X)_x \to \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$, which is an isomorphism if $\mathscr{F}$ commutes with filtered colimits.

Remark 7.5.6. Let $\mathcal{C}$ be a small pretopos and let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ be a functor which preserves finite products and small filtered colimits. Then, for any object $(X, \mathcal{O}_X) \in \mathrm{Stone}_{\mathcal{C}}$, the canonical map

$$\rho : \mathscr{F}(X, \mathcal{O}_X) \to \prod_{x \in X} \mathscr{F}(\{x\}, \mathcal{O}_{X,x})$$

is injective. This follows from Construction 7.5.5, since a global section s of the sheaf $\mathscr{F}(\mathcal{O}_X)$ is determined by the collection of stalks $\{s_x \in \mathscr{F}(\mathcal{O}_X)_x \simeq \mathscr{F}(\{x\}, \mathcal{O}_{X,x})\}$.

Proof of Proposition 7.2.5. Let $\mathcal{C}$ be a small pretopos and let $\mathscr{F} : \mathrm{Stone}_{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Set}$ be a functor which preserves finite products and small filtered colimits. We wish to show that $\mathscr{F}$ is a sheaf with respect to the elementary topology. Define $F : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Set}$ by the formula $F(M) = \mathscr{F}(\underline{M})$. Note that F admits a left ultrastructure, and therefore satisfies the conclusions of Proposition 7.5.1.

We first argue that $\mathscr{F}$ is a separated presheaf. Suppose we are given a collection of morphisms

$$\{(X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)\}_{i \in I}$$

which comprise a covering with respect to the elementary topology; we wish to show that the induced map

$$\mathscr{F}(X, \mathcal{O}_X) \to \prod_{i \in I} \mathscr{F}(X_i, \mathcal{O}_{X_i})$$

is injective. This map fits into a commutative diagram

$$\begin{tikzcd}
\mathscr{F}(X, \mathcal{O}_X) \arrow[r] \arrow[d] & \prod_{i \in I} \mathscr{F}(X_i, \mathcal{O}_{X_i}) \arrow[d] \\
\prod_{x \in X} \mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \arrow[r] & \prod_{i \in I} \prod_{\overline{x} \in X_i} \mathscr{F}(\{\overline{x}\}, \mathcal{O}_{X_i, \overline{x}}),
\end{tikzcd}$$

where the vertical maps are injective by virtue of Remark 7.5.6. It will therefore suffice to show that the lower horizontal map is injective. This is clear: for each point $x \in X$, we can choose an index $i \in I$ and a point $\overline{x} \in X_i$ lying over x for which the map of models $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i, \overline{x}}$ is an elementary embedding, so that Proposition 7.5.1 guarantees the injectivity of the induced map $\mathscr{F}(\{x\}, \mathcal{O}_{X,x}) \to \mathscr{F}(\{\overline{x}\}, \mathcal{O}_{X_i, \overline{x}})$.

To complete the proof that $\mathscr{F}$ is a sheaf, suppose that we are given a collection of elements $s_i \in \mathscr{F}(X_i, \mathcal{O}_{X_i})$ satisfying the following compatibility condition:

$(*)$ For every commutative diagram

$$\begin{tikzcd}
(W, \mathcal{O}_W) \arrow[r] \arrow[d] & (X_i, \mathcal{O}_{X_i}) \arrow[d] \\
(X_j, \mathcal{O}_{X_j}) \arrow[r] & (X, \mathcal{O}_X)
\end{tikzcd}$$

in $\mathrm{Stone}_{\mathcal{C}}$, the elements s_i and s_j have the same image in $\mathscr{F}(W, \mathcal{O}_W)$.

We wish to show that there exists an element $s \in \mathscr{F}(X, \mathcal{O}_X)$ having image s_i in each $\mathscr{F}(X_i, \mathcal{O}_{X_i})$ (the uniqueness of s is automatic by the preceding argument).

Without loss of generality, we may assume that the set I is finite. For each point $y \in X_i$, we let $s_{i,y}$ denote the image of s_i under the map $\mathscr{F}(X_i, \mathcal{O}_{X_i}) \to \mathscr{F}(\{y\}, \mathcal{O}_{X_i,y}) \simeq \mathscr{F}_0(\mathcal{O}_{X_i,y})$. Fix a point $x \in X$. Choose an

index $i \in I$ and a point $y \in X_i$ lying over x for which the map of models $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,y}$ is an elementary embedding. Applying $(*)$ in the case where W is a single point, we deduce the following:

$(*')$ For each index $j \in I$, each point $z \in X_j$ lying over x, and each commutative diagram

$$\begin{tikzcd} M & \mathcal{O}_{X_i,y} \arrow[l] \\ \mathcal{O}_{X_j,z} \arrow[u] & \mathcal{O}_{X,x} \arrow[l] \arrow[u] \end{tikzcd}$$

in $\mathrm{Mod}(\mathcal{C})$, the elements $s_{i,y}$ and $s_{j,z}$ have the same image in $F(M)$.

Applying $(*')$ in the case $i = j$ and $z = y$ and invoking (c), we conclude that $s_{i,y}$ has a unique preimage s_x under the map $F(\mathcal{O}_{X,x}) \to F(\mathcal{O}_{X_i,y})$. We claim that the element s_x satisfies the following additional condition:

$(*'')$ For each index $j \in I$ and each point $j \in X_j$ lying over x, the canonical map $\rho : F(\mathcal{O}_{X,x}) \to F(\mathcal{O}_{X_j,z})$ carries s_x to $s_{j,z}$.

To prove $(*'')$, we note that Corollary 7.4.8 guarantees the existence of a commutative diagram

$$\begin{tikzcd} M & \mathcal{O}_{X_i,y} \arrow[l] \\ \mathcal{O}_{X_j,z} \arrow[u, "u"] & \mathcal{O}_{X,x} \arrow[l] \arrow[u] \end{tikzcd}$$

in the category $\mathrm{Mod}(\mathcal{C})$, where the map u is an elementary embedding. It then follows from $(*')$ that $\rho(s_x)$ and $s_{j,z}$ have the same image under the map $F(u) : F(\mathcal{O}_{X_j,z}) \to F(M)$. Since u is an elementary embedding, Proposition 7.5.1 guarantees that $F(u)$ is injective, so we must have $\rho(s_x) = s_{j,z}$.

Let $\mathscr{F}(\mathcal{O}_X)$ be the sheaf of sets on X given by Construction 7.5.5, so that we can identify each of the sets $F(\mathcal{O}_{X,x})$ with the stalk of $\mathscr{F}(\mathcal{O}_X)$ at the point x. It follows that we can lift s_x to an element $s_{U(x)} \in \mathscr{F}(\mathcal{O}_X)(U(x)) = \mathscr{F}(U(x), \mathcal{O}_X|_{U(x)})$ for some closed and open neighborhood $U(x)$ of the point x. For each $j \in I$, let $U_j(x)$ denote the inverse image of $U(x)$ in X_j and let s'_j denote the image of $s_{U(x)}$ under the canonical map $\mathscr{F}(U(x), \mathcal{O}_X|_{U(x)}) \to \mathscr{F}(U_j(x), \mathcal{O}_{X_j}|_{U_j(x)})$. Then we can identify s'_j with a section of the sheaf $\mathscr{F}(\mathcal{O}_{X_j})$ over the open set $U_j(x)$, and s_j with a global section of the sheaf $\mathscr{F}(\mathcal{O}_{X_j})$. Let $V_j(x) \subseteq U_j(x)$ denote the open subset of $U_j(x)$ consisting of those points z for which s_j and s'_j have the same image in the stalk $\mathscr{F}(\mathcal{O}_{X_j})_z \simeq F(\mathcal{O}_{X_j,z})$. It follows from $(*'')$ that the set $V_j(x)$ contains the preimage $X_j \times_X \{x\}$ of the point x. Because each of the maps $X_j \to X$ is proper (since both X_j and X are compact and Hausdorff), we can choose an open subset $U'(x) \subseteq U(x)$ containing x which satisfies $X_j \times_X U'(x) \subseteq V_j(x)$ for each $j \in I$. Since X is a Stone space, we may further assume that $U'(x)$ is closed. Replacing $U(x)$ by $U'(x)$, we may assume that $U(x)$ has been chosen so that $s_{U(x)}$ and s_j have the same image in $\mathscr{F}(U_j(x), \mathcal{O}_{X_j}|_{U_j(x)})$ for each $j \in J$.

Because X is compact, the open covering $\{U(x)\}_{x \in X}$ admits a finite subcover $U(x_1), U(x_2), \ldots, U(x_n)$. Since X is a Stone space, we can further assume (by shrinking the open sets $U(x_m)$ if necessary) that the sets $U(x_1), U(x_2), \ldots, U(x_n)$ are disjoint. Applying assumption (a'), we deduce that there is a unique element $s \in \mathscr{F}(X, \mathcal{O}_X)$ having image $s_{U(x_m)}$ in each $\mathscr{F}(U(x_m), \mathcal{O}_X|_{U(x_m)})$. It follows immediately from the construction that each of the maps $\mathscr{F}(X, \mathcal{O}_X) \to \mathscr{F}(X_j, \mathcal{O}_{X_j})$ carries s to s_j. □

8. The Envelope of an Ultracategory

In §5, we proved that every ultracategory $\mathcal{M}$ can be obtained from the construction of Proposition 1.3.7. That is, one can obtain any ultrastructure on a category $\mathcal{M}$ by embedding it into a larger category $\mathcal{M}^+$ in such a way that $\mathcal{M}$ has ultraproducts in $\mathcal{M}^+$. However, the category $\mathcal{M}^+$ is not uniquely determined. Moreover, the specific construction that we studied in §5 is not the most economical: the category $\mathrm{Stone}^{\mathrm{op}}_{\mathcal{M}}$ of Construction 4.1.1 contains many objects which do not arise as products of objects of $\mathcal{M}$, and are therefore not needed in the construction of ultraproducts in $\mathcal{M}$.

In this section, we show that for every ultracategory $\mathcal{M}$, there is an "optimal" choice for an embedding $\mathcal{M} \hookrightarrow \mathrm{Env}(\mathcal{M})$ which induces the ultrastructure on $\mathcal{M}$. We will refer to the category $\mathrm{Env}(\mathcal{M})$ as the *envelope of* $\mathcal{M}$. It is characterized (up to equivalence) by the following features:

- (a) The full subcategory $\mathcal{M}$ has ultraproducts in $\mathrm{Env}(\mathcal{M})$ (giving rise to the ultrastructure on $\mathcal{M}$ via the construction of Proposition 1.3.7).
- (b) Every object of $\mathrm{Env}(\mathcal{M})$ can be written as a product of objects belonging to $\mathcal{M}$.
- (c) Every object of $\mathcal{M}$ is *coconnected* when viewed as an object of $\mathrm{Env}(\mathcal{M})$ (see Definition 8.2.1).

The embedding $\mathcal{M} \hookrightarrow \mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$ of Remark 4.2.6 has properties (a) and (c), but not property (b). However, this is easily remedied: we can simply take $\mathrm{Env}(\mathcal{M})$ to be the full subcategory of of $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$ generated by $\mathcal{M}$ under products. This category admits several other concrete descriptions, which we outline in §8.4.

The data of an ultracategory $\mathcal{M}$ and its envelope $\mathrm{Env}(\mathcal{M})$ are equivalent: either can be reconstructed from the other. Consequently, it is possible to entirely dispense with ultracategories and work only with their envelopes. The advantage of this approach is that one does not need to keep track of any additional information: the ultrastructure on $\mathcal{M}$ is completely determined by the structure of $\mathrm{Env}(\mathcal{M})$ *as an abstract category*. Moreover, there is a relatively simple characterization of those categories which arise in this way. In §8.2, we introduce the notion of an *ultracategory envelope* (Definition 8.2.2). By definition, an ultracategory envelope is a category $\mathcal{E}$ satisfying a few simple axioms. These axioms guarantee that $\mathcal{E}$ induces an ultrastructure on a certain full subcategory $\mathcal{E}^{\mathrm{cc}} \subseteq \mathcal{E}$, and that $\mathcal{E}$ can be recovered (up to equivalence) as the envelope of $\mathcal{E}^{\mathrm{cc}}$. To prove this, we show in §8.6 that the relationship between $\mathcal{E}$ and $\mathcal{E}^{\mathrm{cc}}$ is governed by a universal mapping property (Theorem 8.2.6). This property is most conveniently formulated using the language of *right ultrafunctors*, which we introduce in §8.1.

8.1. **Right Ultrafunctors.**

We now consider a variant of Definition 1.4.1:

Definition 8.1.1 (Right Ultrafunctors)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be categories with ultrastructure and let $F : \mathcal{M} \to \mathcal{N}$ be a functor. A *right ultrastructure on* F consists of the following data:

- $(*)$ For every collection of objects $\{M_s\}_{s\in S}$ of the category $\mathcal{M}$ and every ultrafilter μ on S, a morphism $\gamma_\mu : \int_S F(M_s)d\mu \to F(\int_S M_s d\mu)$ in the category $\mathcal{N}$.

These morphisms are required to satisfy the following conditions:

- (0) For every collection of morphisms $\{f_s : M_s \to M'_s\}$ in the category $\mathcal{M}$ and every ultrafilter μ on S, the diagram

$$\begin{tikzcd} \int_S F(M_s)d\mu \arrow[r, "\gamma_\mu"] \arrow[d, "\int_S F(f_s)d\mu"] & F(\int_S M_s dd\mu) \arrow[d, "F(\int_S f_s d\mu)"] \\ \int_S F(M'_s)d\mu \arrow[r, "\gamma_\mu"] & F(\int_S M'_s d\mu) \end{tikzcd}$$

  commutes. In other words, we can regard γ_μ as a natural transformation

$$\gamma_\mu : (\int_S (\bullet) d\mu) \circ F^S \to F \circ \int_S (\bullet) d\mu$$

  of functors from $\mathcal{M}^S$ to $\mathcal{N}$.

- (1) For every collection $\{M_s\}_{s\in S}$ of objects of $\mathcal{M}$ indexed by a set S and every element $s_0 \in S$, the diagram

$$\begin{tikzcd} \int_S F(M_s)d\delta_{s_0} \arrow[rr, "\gamma_{\delta_{s_0}}"] \arrow[dr, "F(\epsilon_{S,s_0})"'] & & F(\int_S M_s d\delta_{s_0}) \arrow[dl, "\epsilon_{S,s_0}"] \\ & F(M_{s_0}) & \end{tikzcd}$$

  commutes (in the category $\mathcal{N}$).

(2) For every collection $\{M_t\}_{t\in T}$ of objects of $\mathcal{M}$ indexed by a set T, every collection $\nu_\bullet = \{\nu_s\}_{s\in S}$ of ultrafilters on T indexed by a set S, and every ultrafilter μ on S, the diagram

$$\begin{tikzcd} \int_T F(M_t)d(\int_S \nu_s d\mu) \arrow[rr, "\gamma_{\int_S \nu_s d\mu}"] \arrow[d, "\Delta_{\mu,\nu_\bullet}"] & & F(\int_T M_t d(\int_S \nu_s d\mu)) \arrow[d, "F(\Delta_{\mu,\nu_\bullet})"] \\ \int_S(\int_T F(M_t)d\nu_s)d\mu \arrow[r, "\int_S \gamma_{\nu_s} d\mu"] & \int_S F(\int_T M_t d\nu_s)d\mu \arrow[r, "\gamma_\mu"] & F(\int_S(\int_T M_t d\nu_s)d\mu) \end{tikzcd}$$

commutes (in the category $\mathcal{N}$).

A *right ultrafunctor* from $\mathcal{M}$ to $\mathcal{N}$ is a pair $(F,\{\gamma_\mu\})$, where F is a functor from M to N and $\{\gamma_\mu\}$ is a right ultrastructure on F.

Example 8.1.2. Let $\mathcal{M} = \{*\}$ denote the category having a single object and a single morphism (so that $\mathcal{M}$ admits a unique ultrastructure), and let $\mathcal{N}$ denote the category of sets (endowed with the categorical ultrastructure). Then the datum of a functor $F : \mathcal{M} \to \mathcal{N}$ is equivalent to the datum of the set $X = F(*)$. A right ultrastructure $\{\gamma_\mu\}$ on F associates to each ultrafilter μ on a set S a map

$$X^S \xrightarrow{q_\mu} \int_S F(*)d\mu \xrightarrow{\gamma_\mu} F(\int_S *d\mu) = X,$$

which we will denote by $f \mapsto \int_S f(s)d\mu$. Using axioms (1) and (2) of Definition 8.1.1, we see that the construction $(f,\mu) \mapsto \int_S f(s)d\mu$ endows X with the structure of an ultraset (Definition 3.1.1), or equivalently with the structure of a compact Hausdorff space (Theorem 3.1.5).

The rest of this section is devoted to some general remarks about the theory of right ultrafunctors. For the most part, these can be regarded as counterparts to observations that we made in §1.4 concerning left ultrafunctors. Beware that there is no formal mechanism for reducing questions about right ultrafunctors to questions about left ultrafunctors, because the theory of ultracategories is not "self-dual" (an ultrastructure on a category $\mathcal{M}$ does not induce an ultrastructure on the opposite category $\mathcal{M}^{\mathrm{op}}$, because the Fubini transformations for the ultrastructure on $\mathcal{M}$ need not be invertible).

Remark 8.1.3. Let $F : \mathcal{M} \to \mathcal{N}$ be a functor between ultracategories and let $\{\sigma_\mu\}$ be an ultrastructure on F (in the sense of Definition 1.4.1). Then the collection of inverse maps $\{\sigma_\mu^{-1}\}$ is a left ultrastructure on F. Conversely, if $\{\gamma_\mu\}$ is a right ultrastructure on F for which each of the maps $\gamma_\mu : \int_S F(M_s)d\mu \to F(\int_S M_s d\mu)$ is an isomorphism, then the collection of inverse maps $\{\gamma_\mu^{-1}\}$ is an ultrastructure on F.

Remark 8.1.4 (Adjoint Functors)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories and suppose we are given a pair of adjoint functors

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "F"] & \mathcal{N}, \arrow[l, shift left, "G"] \end{tikzcd}$$

with unit map $u : \mathrm{id}_\mathcal{M} \to G\circ F$ and counit $v : F\circ G \to \mathrm{id}_\mathcal{N}$. Then:

- Every right ultrastructure $\{\gamma_\mu\}$ on G determines a left ultrastructure on F, which assigns to each collection $\{M_s\}_{s\in S}$ and each ultrafilter μ on S the composite map

$$\begin{aligned} F(\int_S M_s d\mu) &\xrightarrow{F(\int_S u(M_s)d\mu)} F\int_S (G\circ F)(M_s)d\mu \\ &\xrightarrow{F(\gamma_\mu)} (F\circ G)(\int_S F(M_s)d\mu) \\ &\xrightarrow{v(\int_S F(M_s)d\mu)} \int_S F(M_s)d\mu. \end{aligned}$$

- Every left ultrastructure $\{\sigma_\mu\}$ on F determines a right ultrastructure on G, which assigns to each collection of objects $\{N_t\}_{t\in T}$ and each ultrafilter ν on T the composite map

$$\begin{aligned}\int_T G(N_t)d\nu &\xrightarrow{u(\int_T G(N_t)d\nu)} (G\circ F)(\int_T G(N_t)d\nu)\\ &\xrightarrow{G(\sigma_\nu)} G(\int_T (F\circ G)(N_t)d\nu)\\ &\xrightarrow{G(\int_T v(N_s)d\nu)} G(\int_S N_s d\nu).\end{aligned}$$

These constructions determine mutually inverse bijections

$$\{\text{Left ultrastructures on } F\} \simeq \{\text{Right ultrastructures on } G\}.$$

Definition 8.1.5. Let $\mathcal{M}$ and $\mathcal{N}$ be categories with ultrastructure, let $F, F' : \mathcal{M} \to \mathcal{N}$ be functors from $\mathcal{M}$ to $\mathcal{N}$, and suppose that F and F' are equipped with right ultrastructures $\{\gamma_\mu\}$ and $\{\gamma'_\mu\}$, respectively. We will say that a natural transformation $u : F \to F'$ is a *natural transformation of right ultrafunctors* if, for every collection of objects $\{M_s\}_{s\in S}$ of $\mathcal{M}$ and every ultrafilter μ on S, the diagram

$$\begin{tikzcd}
\int_S F(M_s)d\mu \arrow[r, "\gamma_\mu"] \arrow[d, "\int_S u(M_s)d\mu"] & F(\int_S M_s d\mu) \arrow[d, "u(\int_S M_s d\mu)"] \\
\int_S F'(M_s)d\mu \arrow[r, "\gamma'_\mu"] & F'(\int_S M_s d\mu)
\end{tikzcd}$$

commutes (in the category $\mathcal{N}$).

We let $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{N})$ denote the category whose objects are right ultrafunctors $(F,\{\gamma_\mu\})$ from $\mathcal{M}$ to $\mathcal{N}$ and whose morphisms are natural transformations of right ultrafunctors.

Remark 8.1.6. Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. Then the construction $(F,\{\sigma_\mu\}) \mapsto (F,\{\sigma_\mu^{-1}\})$ determines a fully faithful embedding $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M},\mathcal{N}) \to \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{N})$, whose essential image consists of those right ultrafunctors $(F,\{\gamma_\mu\})$ for which each of the maps γ_μ is invertible (see Remark 8.1.3).

Remark 8.1.7 (Limits of Right Ultrafunctors)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories. Suppose that we are given a diagram $\{F_\alpha\}$ in the category $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{N})$ with the property that, for every object $M \in \mathcal{M}$, the diagram $\{F_\alpha(M)\}$ admits a limit in $\mathcal{N}$. Then:

- The construction $(M \in \mathcal{M}) \mapsto \varprojlim_\alpha F_\alpha(M)$ determines a functor $F : \mathcal{M} \to \mathcal{N}$.
- There is a unique right ultrastructure on F for which each of the natural maps $\lambda_\alpha : F \to F_\alpha$ is a natural transformation of left ultrafunctors.
- The maps λ_α exhibit F as a limit of the diagram $\{F_\alpha\}$ in $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{N})$.

In particular, if the ultracategory $\mathcal{N}$ admits small limits, then the category $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{N})$ also admits small limits, which are preserved by the forgetful functor $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{N}) \to \mathrm{Fun}(\mathcal{M},\mathcal{N})$.

Construction 8.1.8 (Composition of Right Ultrafunctors)**.** Let $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}''$ be ultracategories. Let $(F,\{\gamma_\mu\})$ be a right ultrafunctor from $\mathcal{M}$ to $\mathcal{M}'$, and let $(F',\{\gamma'_\mu\})$ be a right ultrafunctor from $\mathcal{M}'$ to $\mathcal{M}''$. Then the composite functor $F' \circ F$ admits a right ultrastructure, which associates to each collection of objects $\{M_s\}_{s\in S}$ of $\mathcal{M}$ and each ultrafilter μ on S the composite map

$$\int_S (F'\circ F)(M_s)d\mu \xrightarrow{\gamma'_\mu} F'(\int_S F(M_s)d\mu) \xrightarrow{F'(\gamma_\mu)} (F'\circ F)(\int_S M_s d\mu).$$

This construction determines a composition law

$$\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}',\mathcal{M}'') \to \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{M}') \to \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{M}'').$$

Remark 8.1.9. We can use Construction 8.1.8 to construct a (strict) 2-category $\mathbf{Ult}^{\mathrm{R}}$ as follows:

- The objects of $\mathbf{Ult}^{\mathrm{R}}$ are ultracategories.

- For every pair of objects $\mathcal{M}, \mathcal{N} \in \mathbf{Ult}^{\mathrm{R}}$, the category of morphisms from $\mathcal{M}$ to $\mathcal{N}$ is given by $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathcal{N})$.
- The composition law on $\mathbf{Ult}^{\mathrm{R}}$ is given by Construction 8.1.8.

More informally: $\mathbf{Ult}^{\mathrm{R}}$ is the category whose objects are ultracategories, whose morphisms are right ultrafunctors, and whose 2-morphisms are natural transformations of right ultrafunctors.

We close this section by establishing a counterpart of Proposition 1.4.9:

Proposition 8.1.10. *Let $\mathcal{M}^+$ and $\mathcal{N}^+$ be categories which admit small products and let $F^+ : \mathcal{M}^+ \to \mathcal{N}^+$ be a functor which preserves small products. Suppose that $\mathcal{M} \subseteq \mathcal{M}^+$ is a full subcategory which has ultraproducts in $\mathcal{M}^+$, that $\mathcal{N} \subseteq \mathcal{N}^+$ is a full subcategory which has ultraproducts in $\mathcal{N}^+$, and that F^+ carries objects of $\mathcal{M}$ to objects of $\mathcal{N}$, so that we can regard $F = F^+|_{\mathcal{M}}$ as a functor from $\mathcal{M}$ to $\mathcal{N}$. Then:*

(i) For every collection of objects $\{M_s\}_{s \in S}$ of $\mathcal{M}$ and every ultrafilter μ on S, there is a unique map $\gamma_\mu : \int_S F(M_s) d\mu \to F(\int_S F(M_s) d\mu)$ having the property that, for each subset $S_0 \subseteq S$ with $\mu(S_0) = 1$, the diagram

$$\begin{tikzcd}
\prod_{s \in S_0} F(M_s) \arrow[d, "q_\mu^{S_0}"] & F^+(\prod_{s \in I} M_s) \arrow[l, "\sim"'] \arrow[d, "F^+(q_\mu^{S_0})"] \\
\int_S F(M_s) d\mu \arrow[r, "\gamma_\mu"] & F(\int_S M_s d\mu)
\end{tikzcd}$$

commutes (in the category $\mathcal{N}^+$).
(ii) The morphisms $\{\gamma_\mu\}$ of (i) determine a right ultrastructure on the functor F.

Proof. Assertion (i) follows from the fact that the maps $\{q_\mu^{S_0} : \prod_{s \in S_0} F(M_s) \to \int_S F(M_s) d\mu\}$ exhibit $\int_S F(M_s) d\mu$ as a colimit of the diagram $\{\prod_{s \in S_0} F(M_s)\}_{\mu(S_0)=1}$ (together with our assumption that F^+ commutes with products). To prove (ii), we argue that the morphisms $\{\gamma_\mu\}$ satisfy condition (2) of Definition 8.1.1 (conditions (0) and (1) are immediate from the construction). Fix a collection of objects $\{M_t\}_{t \in T}$ of $\mathcal{M}$ indexed by a set T, a collection of ultrafilters $\{\nu_s\}_{s \in S}$ on T indexed by a set S, and an ultrafilter μ on the set S. Set $\lambda = \int_S \nu_s d\mu$. We wish to show that the diagram σ :

$$\begin{tikzcd}
\int_T F(M_t) d(\lambda) \arrow[r, "\Delta_{\mu,\nu_\bullet}"] \arrow[dd, "\gamma_\lambda"] & \int_S(\int_T F(M_t) d\nu_s) d\mu \arrow[d, "\int_S \gamma_{\nu_s} d\mu"] \\
 & \int_S F(\int_T M_t d\nu_s) d\mu \arrow[d, "\gamma_\mu"] \\
F(\int_T M_t d\lambda) \arrow[r, "F(\Delta_{\mu,\nu_\bullet})"] & F(\int_S(\int_T M_t d\nu_s) d\mu)
\end{tikzcd}$$

commutes (in the category $\mathcal{N}$). Let $u, v : \int_T F(M_t) d(\lambda) \rightrightarrows F(\int_S(\int_T M_t d\nu_s) d\mu)$ be the maps given by clockwise and counterclockwise composition around the diagram σ. To show that $u = v$, it will suffice to show that $u \circ q_\lambda^{T_0} = v \circ q_\lambda^{T_0}$ for every subset $T_0 \subseteq T$ satisfying $\lambda(T_0) = 1$. Set $S_0 = \{s \in S : \nu_s(T_0) = 1\}$, so that

$\mu(S_0) = 1$. Consider the diagram

$$\begin{tikzcd}
\int_T F(M_t) \arrow[rrr, "\Delta_{\mu,\nu_\bullet}"] \arrow[dddd, "\gamma_\lambda"'] & & & \int_S(\int_T F(M_t) d\nu_s) d\mu \arrow[dd, "\int_S \gamma_{\nu_s} d\mu"] \\
 & & \prod_{s \in S_0}(\int_T F(M_t) d\nu_s) \arrow[ur, "q_\mu^{S_0}"] \arrow[d, "\prod_{s \in S_0} \gamma_{\nu_s}"] & \\
 & \prod_{t \in T_0} F(M_t) \arrow[uul, "q_\lambda^{T_0}"'] \arrow[ur, "\{q_{\nu_s}^{T_0}\}_{s \in S_0}"] & \prod_{s \in S_0} F(\int_T M_t d\nu_s) \arrow[r, "q_\mu^{S_0}"] & \int_S F(\int_T M_t d\nu_s) d\mu \arrow[dd, "\gamma_\mu"] \\
 & F^+(\prod_{t \in T_0} M_t) \arrow[u, "\sim"] \arrow[r, "F(\{q_{\nu_s}^{T_0}\}_{s \in S})"] \arrow[dl, "F(q_\lambda^{T_0})"'] & F^+(\prod_{s \in S_0}(\int_T M_t d\nu_s)) \arrow[u, "\sim"] \arrow[dr, "F(q_\mu^{S_0})"] & \\
F(\int_T M_t d\lambda) \arrow[rrr, "F(\Delta_{\mu,\nu_\bullet})"] & & & F(\int_S(\int_T M_t d\nu_s) d\mu)
\end{tikzcd}$$

in the category $\mathcal{N}^+$. Note that the inner region of the diagram commutes by the construction of the maps γ_{ν_s}, the upper region commutes by the construction of the Fubini transformation for the ultrastructure on $\mathcal{N}$, the lower region commutes by the construction of the Fubini transformation for the ultrastructure on $\mathcal{M}$, the region on the left commutes by the construction of γ_λ, the region on the upper right commutes by functoriality, and the region on the lower right commutes by the construction of γ_μ. It follows by a diagram chase that $u \circ q_\lambda^{T_0} = v \circ q_\lambda^{T_0}$, as desired. □

Remark 8.1.11. In the situation of Proposition 8.1.10, the maps $\{\gamma_\mu\}$ are invertible if and only if the functor F^+ satisfies condition $(*)$ of Proposition 1.4.9. In this case, the right ultrastructure on F given by Proposition 8.1.10 is given by the image, under the identification of Remark 8.1.3, of of the the ultrastructure on F supplied by Proposition 1.4.9.

8.2. **Ultracategory Envelopes.** In this section, we introduce the notion of an *ultracategory envelope* (Definition 8.2.2), and show that it is equivalent to the notion of a ultracategory introduced in §1. First, we need some terminology.

Definition 8.2.1. If $\mathcal{E}$ is a category which admits finite products, then we say that an object $X \in \mathcal{E}$ is *coconnected* if it is connected when viewed as an object of the opposite category $\mathcal{E}^{\mathrm{op}}$. In other words, we say that X is coconnected if the functor $\mathrm{Hom}_{\mathcal{E}}(\bullet, X)$ carries finite products in the category $\mathcal{E}$ to disjoint unions in the category of sets. We let $\mathcal{E}^{\mathrm{cc}}$ denote the full subcategory of $\mathcal{E}$ spanned by the coconnected objects.

Definition 8.2.2. An *ultracategory envelope* is a category $\mathcal{E}$ which satisfies the following axioms:

- $(E1)$ The category $\mathcal{E}$ admits small products.
- $(E2)$ Every object $X \in \mathcal{E}$ can be written as a (small) product $\prod_{s \in S} X_s$, where each factor X_s is a coconnected object of $\mathcal{E}$.
- $(E3)$ The full subcategory $\mathcal{E}^{\mathrm{cc}} \subseteq \mathcal{E}$ of coconnected objects has ultraproducts in $\mathcal{E}$. In other words, for every collection $\{X_s\}_{s \in S}$ of coconnected objects of $\mathcal{E}$ and every ultrafilter μ on S, the direct limit

$$\int_S X_s d\mu = \varinjlim_{\mu(S_0)=1} \prod_{s \in S_0} X_s$$

exists and is a coconnected object of $\mathcal{E}$.

Remark 8.2.3. Let $\mathcal{E}$ be an ultracategory envelope and let $\mathcal{E}^{\mathrm{cc}} \subseteq \mathcal{E}$ be the full subcategory spanned by the coconnected objects. In what follows, we will always regard $\mathcal{E}^{\mathrm{cc}}$ as an ultracategory by equipping it with the ultrastructure supplied by Proposition 1.3.7.

Remark 8.2.4. Let $\mathcal{M}^+$ be a category and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a subcategory which has ultraproducts in $\mathcal{M}^+$. Let $\mathcal{E} \subseteq \mathcal{M}^+$ be the full subcategory spanned by products of objects that belong to $\mathcal{M}$. Then $\mathcal{E}$ automatically satisfies conditions $(E1)$ and $(E3)$ of Definition 8.2.2. If every object of $\mathcal{M}$ is coconnected as an object of $\mathcal{E}$, then $\mathcal{E}$ also satisfies $(E2)$ and is therefore an ultracategory envelope. Moreover, in this case, we have $\mathcal{E}^{\mathrm{cc}} \simeq \mathcal{M}$.

In §8.4, we will show that every ultracategory $\mathcal{M}$ arises (up to equivalence) from the construction of Remark 8.2.3:

Theorem 8.2.5. *Let $\mathcal{M}$ be an ultracategory. Then there exists an ultracategory envelope* $\mathrm{Env}(\mathcal{M})$ *and an equivalence of ultracategories* $\mathcal{M} \simeq \mathrm{Env}(\mathcal{M})^{\mathrm{cc}}$.

In the situation of Theorem 8.2.5, we will refer to the category $\mathrm{Env}(\mathcal{M})$ as the *envelope of* $\mathcal{M}$. It is determined (up to equivalence) by the ultracategory $\mathcal{M}$. This is a consequence of the following universal property, which we will establish in §8.6:

Theorem 8.2.6. *Let $\mathcal{E}$ be an ultracategory envelope, let $\mathcal{M}^+$ be a category which admits small products, and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory which has ultraproducts in $\mathcal{M}^+$. Let* $\mathrm{Fun}'(\mathcal{E}, \mathcal{M}^+)$ *denote the full subcategory of* $\mathrm{Fun}(\mathcal{E}, \mathcal{M}^+)$ *spanned by those functors F which preserve small products and carry coconnected objects of $\mathcal{E}$ to objects of $\mathcal{M}$. Then the construction of Proposition 8.1.10 induces an equivalence of categories* $\mathrm{Fun}'(\mathcal{E}, \mathcal{M}^+) \to \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{E}^{\mathrm{cc}}, \mathcal{M})$.

It follows from Theorems 8.2.5 and 8.2.6 that the notions of ultracategory and ultracategory envelope are interchangeable.

Notation 8.2.7. Let $\mathcal{E}$ and $\mathcal{E}'$ be ultracategory envelopes. We say that a functor $F : \mathcal{E} \to \mathcal{E}'$ is a *functor of ultracategory envelopes* if F preserves small products and carries coconnected objects of $\mathcal{E}$ to coconnected objects of $\mathcal{E}'$. We let $\mathrm{Fun}^{\mathrm{Env}}(\mathcal{E}, \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{E}')$ spanned by the functors of ultracategory envelopes. We let $\mathcal{C}\mathrm{at}^{\mathrm{Env}}$ denote the (strict) 2-category whose objects are ultracategory envelopes, where the category of morphisms from $\mathcal{E}$ to $\mathcal{E}'$ in $\mathcal{C}\mathrm{at}^{\mathrm{Env}}$ is given by $\mathrm{Fun}^{\mathrm{Env}}(\mathcal{E}, \mathcal{E}')$. Note that we can regard $\mathcal{C}\mathrm{at}^{\mathrm{Env}}$ as a (non-full) subcategory of the 2-category of categories.

Corollary 8.2.8. *The construction $\mathcal{E} \mapsto \mathcal{E}^{\mathrm{cc}}$ induces an equivalence of 2-categories $\mathcal{C}\mathrm{at}^{\mathrm{Env}} \to \mathbf{Ult}^{\mathrm{R}}$; here $\mathcal{C}\mathrm{at}^{\mathrm{Env}}$ is the 2-category of ultracategory envelopes (Notation 8.2.7) and $\mathbf{Ult}^{\mathrm{R}}$ is the 2-category of Remark 8.1.9.*

Proof. Essential surjectivity follows from Theorem 8.2.5. It will therefore suffice to show that for every pair of ultracategory envelopes $\mathcal{E}$ and $\mathcal{E}'$, the construction of Proposition 8.1.10 induces an equivalence of categories $\mathrm{Fun}^{\mathrm{Env}}(\mathcal{E}, \mathcal{E}') \to \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{E}^{\mathrm{cc}}, \mathcal{E}'^{\mathrm{cc}})$, which is a special case of Theorem 8.2.6. □

8.3. **Application: Classification of Right Ultrafunctors.** Let $\mathcal{C}$ be a small exact category and let $\mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set})$ denote the category of regular functors from $\mathcal{E}$ to the category of sets. In §2.4, we noted that a functor $F : \mathrm{Fun}^{\mathrm{reg}}(\mathcal{E}, \mathrm{Set}) \to \mathrm{Set}$ which preserves small products and small filtered colimits admits a unique ultrastructure (Remark 2.4.6). In this section, we prove a more general form of this result (Corollary 8.3.5), which we deduce from a more general statement in the setting of *right* ultrastructures. Our starting point is the following result, which is obtained by applying Theorem 8.2.6 in the special case $\mathcal{M} = \mathcal{M}^+$:

Proposition 8.3.1. *Let $\mathcal{E}$ be an ultracategory envelope and let $\mathcal{M}$ be a category which admits small products and filtered colimits. Then the construction of Proposition 8.1.10 induces an equivalence of categories*

$$\mathrm{Fun}^{\Pi}(\mathcal{E}, \mathcal{M}) \to \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{E}^{\mathrm{cc}}, \mathcal{M}).$$

Here $\mathrm{Fun}^{\Pi}(\mathcal{E}, \mathcal{M})$ *denotes the full subcategory of* $\mathrm{Fun}(\mathcal{E}, \mathcal{M})$ *spanned by those functors which preserve small products, and we regard $\mathcal{M}$ as endowed with the categorical ultrastructure of Example 1.3.8.*

Corollary 8.3.2. *Let $\mathcal{M}$ be an ultracategory which admits small products. Let $\mathcal{N}$ be a category which admits small products and small filtered colimits, which we endow with the categorical ultrastructure of Example 1.3.8. Then the forgetful functor* $\theta : \mathrm{Fun}^{\mathrm{RUlt},\Pi}(\mathcal{M}, \mathcal{N}) \to \mathrm{Fun}^{\Pi}(\mathcal{M}, \mathcal{N})$ *is an equivalence of categories. Here*

$\mathrm{Fun}^{\Pi}(\mathcal{M},\mathcal{N})$ *denotes the full subcategory of* $\mathrm{Fun}(\mathcal{M},\mathcal{N})$ *spanned by those functors which preserve small products, and* $\mathrm{Fun}^{\mathrm{RUlt},\Pi}(\mathcal{M},\mathcal{N}) \subseteq \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M},\mathcal{N})$ *is defined similarly.*

Remark 8.3.3. We can state Corollary 8.3.2 more informally as follows: if $\mathcal{M}$ and $\mathcal{N}$ are ultracategories which admit small products, and the ultrastructure on $\mathcal{N}$ is categorical, then any functor $F : \mathcal{M} \to \mathcal{N}$ which preserves small products admits a unique right ultrastructure.

Proof of Corollary 8.3.2. By virtue of Theorem 8.2.5, we may assume without loss of generality that $\mathcal{M} = \mathcal{E}^{\mathrm{cc}}$ for some ultracategory envelope $\mathcal{E}$. In this case, we can use Proposition 8.3.1 to identify θ with the restriction functor $\mathrm{Fun}'(\mathcal{E},\mathcal{N}) \to \mathrm{Fun}^{\Pi}(\mathcal{E}^{\mathrm{cc}},\mathcal{N})$, where $\mathrm{Fun}'(\mathcal{E},\mathcal{N})$ is the full subcategory of $\mathrm{Fun}(\mathcal{E},\mathcal{N})$ spanned by those functors F for which both F and $F|_{\mathcal{E}^{\mathrm{cc}}}$ preserve small products. To show that θ is an equivalence of categories, it will suffice to prove the following:

- (i) Every functor $F_0 : \mathcal{E}^{\mathrm{cc}} \to \mathcal{N}$ admits a right Kan extension $F : \mathcal{E} \to \mathcal{N}$.
- (ii) Let $F : \mathcal{E} \to \mathcal{N}$ be a functor for which the restriction $F_0 = F|_{\mathcal{E}^{\mathrm{cc}}}$ preserves small products. Then F is a right Kan extension of F_0 if and only if F preserves small products.

We first prove (i). Assume that $F_0 : \mathcal{E}^{\mathrm{cc}} \to \mathcal{N}$ is a functor which preserves small products, and let X be an object of $\mathcal{E}$. Then we can factor X as a product $\prod^{\mathcal{E}}_{s\in S} X_s$, where each X_s is coconnected and the superscript indicates that the product is formed in the category $\mathcal{E}$. Since $\mathcal{E}^{\mathrm{cc}}$ admits small products, the collection of objects $\{X_s\}_{s\in S}$ also admits a product in the subcategory $\mathcal{E}^{\mathrm{cc}}$, which we will denote by $\prod^{\mathcal{E}^{\mathrm{cc}}}_{s\in S} X_s$. We then have a canonical map $u : \prod^{\mathcal{E}^{\mathrm{cc}}}_{s\in S} X_s \to \prod^{\mathcal{E}}_{s\in S} X_s$, and composition with u induces a bijection

$$\mathrm{Hom}_{\mathcal{E}^{\mathrm{cc}}}(Y, \prod^{\mathcal{E}^{\mathrm{cc}}}_{s\in S} X_s) \to \mathrm{Hom}_{\mathcal{E}}(Y, \prod^{\mathcal{E}}_{s\in S} X_s)$$

for every coconnected object $Y \in \mathcal{E}^{\mathrm{cc}}$. It follows that $\prod^{\mathcal{E}^{\mathrm{cc}}}_{s\in S} X_s$ is a final object of the category $\mathcal{E}^{\mathrm{cc}} \times_{\mathcal{E}} \mathcal{E}_{/X}$, so that the inverse limit

$$\varprojlim_{Y \in \mathcal{E}^{\mathrm{cc}} \times_{\mathcal{E}} \mathcal{E}_{/X}} F_0(Y)$$

exists and is equivalent to $F_0(\prod^{\mathcal{E}^{\mathrm{cc}}}_{s\in S} X_s)$. This proves (i), and the following version of (ii):

- (ii') A functor $F : \mathcal{E} \to \mathcal{N}$ is a right Kan extension of $F_0 = F|_{\mathcal{E}^{\mathrm{cc}}}$ if and only if, for every collection of objects $\{X_s\}_{s\in S}$ of $\mathcal{E}^{\mathrm{cc}}$, the canonical map

$$F_0(\prod^{\mathcal{E}^{\mathrm{cc}}}_{s\in S} X_s) \to F(\prod^{\mathcal{E}}_{s\in S} X_s)$$

  is an isomorphism.

We conclude by observing that if F_0 preserves small products, then the criterion of (ii') is equivalent to the requirement that F also preserves small products. □

Example 8.3.4. Let $\mathcal{M}$ be an ultracategory. Assume that the underlying category of $\mathcal{M}$ admits small products and filtered colimits. Let $\mathcal{N} = \mathcal{M}$ denote the same category, but equipped with the categorical ultrastructure of Example 1.3.8. It follows from Corollary 8.3.2 that there is a unique right ultrastructure on the identity functor $\mathrm{id} : \mathcal{M} \simeq \mathcal{N}$. For every collection of objects $\{M_s\}_{s\in S}$ and every ultrafilter μ on S, this ultrastructure determines a canonical map

$$\gamma_\mu : \varinjlim_{\mu(S_0)=1} \prod_{s\in S_0} M_s \to \int_S M_s d\mu,$$

where the left hand side is the categorical ultraproduct of Construction 1.2.2 and the right hand side is supplied by the ultrastructure on $\mathcal{M}$. This map can be described concretely: for example, the composition $(\gamma_\mu \circ q_\mu) : \prod_{s\in S} M_s \to \int_S M_s d\mu$ is given by the composition

$$(\prod_{s\in S} M_s) \xrightarrow{\Delta_\mu} (\prod_{s\in S} M_s)^\mu = \int_S (\prod_{s\in S} M_s) d\mu \to \int_S M_s d\mu,$$

where Δ_μ is the ultrapower diagonal of Example 1.3.4.

Corollary 8.3.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be categories which admit small products and small filtered colimits, and egard $\mathcal{M}$ and $\mathcal{N}$ as equipped with the categorical ultrastructures of Example 1.3.8. Let* $\mathrm{Fun}^{\mathrm{Ult},\Pi}(\mathcal{M},\mathcal{N})$ *denote the full subcategory of* $\mathrm{Fun}^{\mathrm{Ult}}(\mathcal{M},\mathcal{N})$ *spanned by those functors which preserve small products. Then the forgetful functor* $\mathrm{Fun}^{\mathrm{Ult},\Pi}(\mathcal{M},\mathcal{N}) \to \mathrm{Fun}(\mathcal{M},\mathcal{N})$ *is a fully faithful embedding, whose essential image is spanned by those ultrafunctors $F : \mathcal{M} \to \mathcal{N}$ which preserve small products and small filtered colimits.*

Remark 8.3.6. Corollary 8.3.5 implies in particular that if $F : \mathcal{M} \to \mathcal{N}$ is a functor which preserves small products and small filtered colimits, then it admits a *unique* ultrastructure (namely, the ultrastructure given by Proposition 1.4.9).

Proof of Corollary 8.3.5. It follows from Corollary 8.3.2 that the forgetful functor $\theta : \mathrm{Fun}^{\mathrm{Ult},\Pi}(\mathcal{M},\mathcal{N}) \to \mathrm{Fun}(\mathcal{M},\mathcal{N})$ is fully faithful. Any functor $F : \mathcal{M} \to \mathcal{N}$ belonging to the essential image of θ must preserve small products (by definition) and small filtered colimits (by Proposition 5.3.4). Conversely, if $F : \mathcal{M} \to \mathcal{N}$ preserves small products and small filtered colimits, then it admits an ultrastructure by virtue of Proposition 1.4.9, and therefore belongs to the essential image of θ. □

8.4. **Construction of the Envelope.** Let $\mathcal{M}$ be an ultracategory. Our goal in this section is to prove Theorem 8.2.5, which asserts the existence of an ultracategory envelope $\mathrm{Env}(\mathcal{M})$ and an equivalence of ultracategories $\mathcal{M} \simeq \mathrm{Env}(\mathcal{M})^{\mathrm{cc}}$. We give a quick proof based on the constructions of §5. However, we also outline two other constructions of the category $\mathrm{Env}(\mathcal{M})$, which are independent of the ideas developed in §5 (see Remark 8.4.5 and Proposition 8.4.7).

Definition 8.4.1. Let $\mathcal{M}$ be an ultracategory. We will say that an object $(X, \mathcal{O}_X)$ of $\mathrm{Stone}_{\mathcal{M}}$ is *free* if it can be written as a small coproduct of objects of the form $\underline{M}$, where M belongs to $\mathcal{M}$. Equivalently, an object $(X, \mathcal{O}_X)$ of $\mathrm{Stone}_{\mathcal{M}}$ is free if it is isomorphic to $(\beta T, \mathcal{O}_{\beta T})$, where $\mathcal{O}_{\beta T}$ is the ultrafunctor associated by Proposition 4.2.8 to a collection of objects $\{M_t\}_{t \in T}$.

We let $\mathrm{Env}(\mathcal{M})$ denote the full subcategory of $\mathrm{Stone}^{\mathrm{op}}_{\mathcal{M}}$ spanned by the free objects $(X, \mathcal{O}_X)$. We will refer to $\mathrm{Env}(\mathcal{M})$ as the *envelope of* $\mathcal{M}$.

Example 8.4.2. Let Y be a compact Hausdorff space, regarded as an ultracategory having only identity morphisms. Then the envelope $\mathrm{Env}(Y)$ can be identified with the opposite of the full subcategory of $\mathrm{Top}_{/Y}$ spanned by those continuous maps $f : X \to Y$, where X is a topological space of the form βS, for some set S; see Example 4.1.4.

Example 8.4.3. Let $\mathcal{C}$ be a small pretopos and regard the category of models $\mathrm{Mod}(\mathcal{C})$ as endowed with the ultrastructure of Remark 2.1.2. Then the envelope $\mathrm{Env}(\mathrm{Mod}(\mathcal{C}))$ can be identified with the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ which contains $\mathrm{Mod}(\mathcal{C})$ and is closed under small products. This follows from Theorem 6.3.14. However, it can also be proved directly, by showing that the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ generated by $\mathrm{Mod}(\mathcal{C})$ under products satisfies the axioms of Definition 8.2.2. The essential observation is that every model of M of $\mathcal{C}$ is coconnected when viewed as an object of $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set}) = \mathrm{Pro}(\mathcal{C})^{\mathrm{op}}$ (beware that M is usually *not* coconnected as an object of the larger category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$).

Theorem 8.2.5 is a consequence of the following more precise assertion:

Theorem 8.4.4. *Let $\mathcal{M}$ be an ultracategory. Then the category* $\mathrm{Env}(\mathcal{M})$ *of Definition 8.4.1 is an ultracategory envelope. Moreover, the construction $M \mapsto \underline{M}$ induces an equivalence of ultracategories* $\mathcal{M} \to \mathrm{Env}(\mathcal{M})^{\mathrm{cc}}$.

Proof. By virtue of Theorem 4.2.7 and Remark 8.2.4, it will suffice to show that for each object $M \in \mathcal{M}$, the object $\underline{M} \in \mathrm{Env}(\mathcal{M})$ is coconnected. This is a consequence of Example 5.1.2. □

Remark 8.4.5. Let $\mathcal{M}$ be an ultracategory. One we have granted the existence of an envelope $\mathrm{Env}(\mathcal{M})$, it is not difficult to work out the structure of $\mathrm{Env}(\mathcal{M})$ directly from the definitions. Note that every object of $\mathrm{Env}(\mathcal{M})$ must factor as a product of objects $\{M_s\}_{s \in S}$ belonging to $\mathcal{M}$ (moreover, this factorization is essentially unique: see Proposition 8.5.5). Moreover, giving a map from a product $\prod_{t \in T} N_t$ to a product

$\prod_{s\in S} M_s$ is equivalent to giving a family of maps $\{u_s : \prod_{t\in T} N_t \to M_s\}_{s\in S}$. Each of the maps u_s then factors uniquely as a composition

$$\prod_{t\in T} N_t \xrightarrow{q_{\nu_s}} \int_T N_t d\nu_s \xrightarrow{g_s} M_s$$

for some morphism g_s in the category $\mathcal{M}$ (see Lemma 8.6.2). Moreover, the composition law on morphisms in $\mathrm{Env}(\mathcal{M})$ is determined by the ultrastructure on the category $\mathcal{M} \simeq \mathrm{Env}(\mathcal{M})^{\mathrm{cc}}$. This analysis supplies an equivalence of $\mathrm{Env}(\mathcal{M})$ with a category $\mathrm{Env}'(\mathcal{M})$, which can be described explicitly as follows:

- An object of $\mathrm{Env}'(\mathcal{M})$ is a set S together with a collection $\{M_s\}_{s\in S}$ of objects of $\mathcal{M}$ which is indexed by S. We will denote such an object simply by $\{M_s\}_{s\in S}$.
- Let $\{M_s\}_{s\in S}$ and $\{N_t\}_{t\in T}$ be objects of $\mathrm{Env}'(\mathcal{M})$. A *morphism from* $\{N_t\}_{t\in T}$ *to* $\{M_s\}_{s\in S}$ is a collection of pairs $\{(\nu_s, g_s)\}_{s\in S}$, indexed by S, where each ν_s is an ultrafilter on T and each g_s is a morphism from $\int_T N_t d\nu_s$ to M_s in the category $\mathcal{M}$.
- Let $g : \{N_t\}_{t\in T} \to \{M_s\}_{s\in S}$ be a morphism in $\mathrm{Env}'(\mathcal{M})$ given by $\{(\nu_s, g_s)\}_{s\in S}$, and let $f : \{M_s\}_{s\in S} \to \{L_r\}_{r\in R}$ be a morphism in $\mathrm{Env}'(\mathcal{M})$ given by $\{(\mu_r, f_r)\}_{r\in R}$. Then the composition $(f\circ g) : \{N_t\}_{t\in T} \to \{L_r\}_{r\in R}$ is defined to be $\{(\lambda_r, h_r)\}_{r\in R}$, where each λ_r denotes the ultrafilter on T given by $\int_S \nu_s d\mu_r$, and each $h_r : \int_T N_t d\lambda_r \to L_r$ is the morphism given by the composition

$$\int_T N_t d\lambda_r = \int_T N_t d(\int_S \nu_s d\mu_r) \xrightarrow{\Delta_{\mu_r,\nu_\bullet}} \int_S (\int_T N_t d\nu_s) d\mu_r \xrightarrow{\int_S g_s d\mu_r} \int_S M_s d\mu_r \xrightarrow{f_r} L_r.$$

It is not difficult (albeit somewhat tedious) to prove Theorem 8.2.5 directly by showing that the category $\mathrm{Env}'(\mathcal{M})$ is an ultracategory envelope, and that the construction $M \mapsto \{M\}$ defines an equivalence of ultracategories from $\mathcal{M}$ to the full subcategory of coconnected objects of $\mathrm{Env}'(\mathcal{M})$.

Remark 8.4.6. Let us define a *quasi-ultracategory* to be a category $\mathcal{M}$ equipped with ultraproduct functors

$$\mathcal{M}^S \to \mathcal{M} \qquad \{M_s\}_{s\in S} \mapsto \int_S M_s d\mu,$$

together with natural transformations

$$\epsilon_{S,s_0} : \int_S M_s d\delta_{s_0} \simeq M_{s_0} \qquad \Delta_{\mu,\nu_\bullet} : \int_T N_t d(\int_S \nu_s d\mu) \to \int_S (\int_T N_t d\nu_s) d\mu,$$

which are not required to satisfy any further conditions (that is, we omit axioms (A), (B), and (C) of Definition 1.3.1).

Let us say that a quasi-ultracategory $\mathcal{M}$ *admits an envelope* if it is equivalent (as a quasi-ultracategory) to the category of coconnected objects $\mathcal{E}^{\mathrm{cc}}$ of some ultracategory envelope $\mathcal{E}$. Taken together, Theorem 8.2.5 and Proposition 1.3.7 assert that a quasi-ultracategory $\mathcal{M}$ admits an envelope if and only if it is an ultracategory: that is, if and only if it satisfies axioms (A), (B), and (C). Consequently, any construction of the envelope of an ultracategory $\mathcal{M}$ must make essential use of these axioms at some point. Here it is instructive to contrast the approaches of Definition 8.4.1 and Remark 8.4.5:

- In the construction of Remark 8.4.5, axioms (A) and (C) are needed immediately to show that the category $\mathrm{Env}'(\mathcal{M})$ is well-defined. In fact, axiom (C) is precisely equivalent to the associativity of the composition law on $\mathrm{Env}'(\mathcal{M})$, and axiom (A) is equivalent to the assertion that, for every object $\{M_s\}_{s\in S}$ of $\mathrm{Env}'(\mathcal{M})$, the morphism $\{(\delta_s, \epsilon_{S,s})\}_{s\in S} : \{M_s\}_{s\in S} \to \{M_s\}_{s\in S}$ is a left unit with respect to composition (the fact that it is also a right unit follows from Corollary 1.3.6).
- In the construction of Definition 8.4.1, the envelope $\mathrm{Env}(\mathcal{M})$ is realized as a full subcategory of the larger category $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$, which is well-defined even if we do not assume that $\mathcal{M}$ satisfies axioms (A), (B), and (C). However, axioms (A) and (C) are needed to construct certain *objects* of the category $\mathrm{Stone}_{\mathcal{M}}^{\mathrm{op}}$ (Proposition 4.2.8), and these objects span the full subcategory $\mathrm{Env}(\mathcal{M})$ that we are interested in.

We now give another description of the category $\mathrm{Env}(\mathcal{M})$.

Proposition 8.4.7. *Let $\mathcal{M}$ be an ultracategory which is locally small (that is, for every pair of objects $M, N \in \mathcal{M}$, the collection of morphisms* $\mathrm{Hom}_{\mathcal{M}}(M,N)$ *is small). Then:*

(1) *Let $M \in \mathcal{M}$ be an object and let $h^M : \mathcal{M} \to \mathrm{Set}$ denote the functor corepresented by M, given by the formula $h^M(N) = \mathrm{Hom}_{\mathcal{M}}(M, N)$. Then h^M admits a unique right ultrastructure.*
(2) *The category $\mathrm{Env}(\mathcal{M})^{\mathrm{op}}$ is equivalent to the smallest full subcategory of $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathrm{Set})$ which contains each of the objects corepresentable functors h^M (equipped with the right ultrastructures described by (1)) and is closed under small coproducts.*

Proof. Let $h^{\underline{M}} : \mathrm{Env}(\mathcal{M}) \to \mathrm{Set}$ denote the functor represented by the object $\underline{M} \in \mathrm{Env}(\mathcal{M})$. Then $h^{\underline{M}}$ preserves small products, and therefore determines a right ultrastructure on the restriction $h^M \simeq h^{\underline{M}}|_{\mathcal{M}}$ (see Proposition 8.1.10). This proves the existence statement of (1). To prove uniqueness, we observe that any other right ultrastructure on h^M can be obtained by applying the same construction to some other product-preserving functor $F : \mathrm{Env}(\mathcal{M}) \to \mathrm{Set}$ equipped with an isomorphism $e : h^M \simeq F|_{\mathcal{M}}$ (Proposition 8.3.1). By Yoneda's lemma, the map e extends uniquely to a natural transformation $\overline{e} : h^{\underline{M}} \to F$ of set-valued functors on $\mathrm{Env}(\mathcal{M})$. Since e is an isomorphism and the functors $h^{\underline{M}}$ and F both commute with products, it follows that $\overline{e}$ is an isomorphism. This proves the uniqueness statement of (1).

We now prove (2). Using Proposition 8.3.1 again, we can identify $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathrm{Set})$ with the category $\mathrm{Fun}^{\Pi}(\mathrm{Env}(\mathcal{M}), \mathrm{Set})$ of product-preserving functors from $\mathrm{Env}(\mathcal{M})$ to the category of sets. We now observe that the Yoneda embedding $\mathrm{Env}(\mathcal{M})^{\mathrm{op}} \to \mathrm{Fun}^{\Pi}(\mathrm{Env}(\mathcal{M}), \mathrm{Set})$ is a fully faithful embedding which preserves small coproducts, and therefore induces an equivalence from $\mathrm{Env}(\mathcal{M})^{\mathrm{op}}$ to the full subcategory of $\mathrm{Fun}^{\Pi}(\mathrm{Env}(\mathcal{M}), \mathrm{Set})$ generated under coproducts by objects of the form $h^{\overline{M}}$. □

Remark 8.4.8. In theory, Proposition 8.4.7 supplies a construction of the envelope $\mathrm{Env}(\mathcal{M})$ which is independent of both Definition 8.4.1 and Remark 8.4.5. However, our proof of Proposition 8.4.7 depends on an assumption that $\mathrm{Env}(\mathcal{M})$ already exists. Without the universal property of Proposition 8.3.1, it is not clear how to work with the category of right ultrafunctors $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathrm{Set})$ (for example, to show that coproducts of corepresentable right ultrafunctors exist in $\mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathrm{Set})$).

In the situation of Proposition 8.4.7, one can give a similar description of the larger category $\mathrm{Comp}_{\mathcal{M}} \supseteq \mathrm{Env}(\mathcal{M})^{\mathrm{op}}$.

Proposition 8.4.9. *Let $\mathcal{M}$ be an ultracategory which is locally small. Then the construction $(X, \mathcal{O}_X) \mapsto \mathrm{Hom}_{\mathrm{Comp}_{\mathcal{M}}}(\bullet, (X, \mathcal{O}_X))$ determines a fully faithful embedding*

$$\mathrm{Comp}_{\mathcal{M}} \xrightarrow{\theta} \mathrm{Fun}^{\Pi}(\mathrm{Env}(\mathcal{M}), \mathrm{Set}) \simeq \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathrm{Set}).$$

Proof. Let $(Y, \mathcal{O}_Y)$ and $(Z, \mathcal{O}_Z)$ be objects of $\mathrm{Comp}_{\mathcal{M}}$ and let $u : \theta(Y, \mathcal{O}_Y) \to \theta(Z, \mathcal{O}_Z)$ be a natural transformation of functors from $\mathrm{Env}(\mathcal{M})$ to Set. For every morphism $\xi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in the category $\mathrm{Comp}_{\mathcal{M}}$, where $(X, \mathcal{O}_X)$ belongs to $\mathrm{Env}(\mathcal{M})^{\mathrm{op}} \subseteq \mathrm{Comp}_{\mathcal{M}}$, we let $u(\xi) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ denote the image of ξ under u. We wish to show that there is a unique morphism $(f, \alpha) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ in $\mathrm{Comp}_{\mathcal{M}}$ satisfying $u(\xi) = (f, \alpha) \circ \xi$ for all ξ as above.

For each point $x \in X$, let $\xi_x : (\{y\}, \mathcal{O}_{Y,y}) \hookrightarrow (Y, \mathcal{O}_Y)$ be the canonical map, so that $u(\xi_y) : (\{y\}, \mathcal{O}_{Y,y}) \to (Z, \mathcal{O}_Z)$ determines a point $f(y) \in Z$ and a morphism $\alpha_y : \mathcal{O}_{Z,f(y)} \to \mathcal{O}_{Y,y}$ in the ultracategory $\mathcal{M}$; we regard $\{\alpha_y\}_{y \in Y}$ as a natural transformation of functors $\mathcal{O}_Z \circ f \to \mathcal{O}_Y$. We will prove the following:

($*$) The pair (f, α) is a morphism from $(Y, \mathcal{O}_Y)$ to $(Z, \mathcal{O}_Z)$ in the category $\mathrm{Comp}_{\mathcal{M}}$. That is, the function f is continuous and α is a natural transformation of left ultrafunctors.

Assume ($*$) for the moment. We will complete the proof of Proposition 8.4.9 by showing that (f, α) is the unique morphism in $\mathrm{Comp}_{\mathcal{M}}$ satisfying $u(\xi) = (f, \alpha) \circ \xi$ for all $\xi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ with $(X, \mathcal{O}_X)$ in $\mathrm{Env}(\mathcal{M})^{\mathrm{op}}$. Uniqueness is clear: by construction, (f, α) is characterized by the requirement that we have an equality $u(\xi_y) = (f, \alpha) \circ \xi_y$ for each $y \in Y$. To show that the equality $u(\xi) = (f, \alpha) \circ \xi$ holds in general, we can decompose $(X, \mathcal{O}_X)$ as a coproduct and thereby reduce to the case where $(X, \mathcal{O}_X) = \underline{M}$ for some object $M \in \mathcal{M}$. In this case, the map ξ factors uniquely as a composition

$$\underline{M} \xrightarrow{\xi'} (\{y\}, \mathcal{O}_{Y,y}) \xrightarrow{\xi_y} (Y, \mathcal{O}_Y)$$

for some point $xy \in Y$, so we can replace ξ by ξ_y in which case the desired identity holds by construction.

It remains to prove $(*)$. Fix a set S and a map $g : S \to Y$. Set $(X, \mathcal{O}_X) = \coprod_{s \in S} \underline{\mathcal{O}_{Y,f(s)}}$, where the coproduct is taken in $\mathrm{Comp}_{\mathcal{M}}$, so that we can amalgamate the maps $\xi_{g(s)}$ to a single map $\xi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. Then $u(\xi) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ is a morphism in $\mathrm{Comp}_{\mathcal{M}}$. Using Proposition 4.2.9, we can identify X with the Stone-Čech compactification βS, and $\mathcal{O}_X$ with the left ultrafunctor of Proposition 4.2.8. Consequently, the morphism $u(\xi)$ associates to each ultrafilter $\mu \in \beta S$ a point $h(\mu) \in Z$ and a morphism $\alpha'_\mu : \mathcal{O}_{Z,h(\mu)} \to \mathcal{O}_{X,\mu} = \int_S \mathcal{O}_{Y,f(s)}\, d\mu$ in the ultracategory $\mathcal{M}$. From the functoriality of u, we deduce the following:

(a) For each ultrafilter μ on S, the point $h(\mu) \in Z$ is given by $f(\int_S g(s) d\mu)$, and the map α'_μ by the composition
$$\mathcal{O}_{Z,h(\mu)} \xrightarrow{\alpha_{\int_S g(s) d\mu}} \mathcal{O}_{Y, \int_S g(s) d\mu} \xrightarrow{\sigma_\mu} \int_S \mathcal{O}_{Y,g(s)}\, d\mu,$$
where σ_μ is determined by the left ultrastructure on the functor $\mathcal{O}_Y$.

In particular, we can identify $u(\xi) : (X, \mathcal{O}_X) = \coprod_{s \in S} \underline{\mathcal{O}_{Y,f(s)}} \to (Z, \mathcal{O}_Z)$ given by amalgamating the maps $u(\xi_{g(s)})$ for $s \in S$. This yields a different description of $h(\mu)$ and α'_μ:

(b) For each ultrafilter μ on S, the point $h(\mu) \in \mathbf{Z}$ is given by $\int_S (f \circ g)(s) d\mu$, and the map α'_μ by the composition
$$\mathcal{O}_{Z,h(\mu)} = \mathcal{O}_{Z, \int_S (f\circ g)(s) d\mu} \xrightarrow{\sigma'_\mu} \int_S \mathcal{O}_{Z,(f\circ g)(s)}\, d\mu \xrightarrow{\int_S \alpha_{g(s)} d\mu} \int_S \mathcal{O}_{Y,g(s)}\, d\mu,$$
where σ'_μ is determined by the left ultrastructure on the functor $\mathcal{O}_Z$.

It follows from (a) and (b) that we have $f(\int_S g(s) d\mu) = \int_S (f \circ g)(s) d\mu$ for every map $g : S \to X$ and every ultrafilter μ on S. That is, f is a morphism of ultrasets, and is therefore continuous (Theorem 3.1.5). Moreover, (a) and (b) also imply the commutativity of the diagram
$$\begin{tikzcd}
\mathcal{O}_{Z,h(z)} \arrow[r, "\sigma'_\mu"] \arrow[d, "\alpha_{\int_S g(s) d\mu}"] & \int_S \mathcal{O}_{Z,(f\circ g)(s)}\, d\mu \arrow[d, "\int_S \alpha_{g(s)} d\mu"] \\
\mathcal{O}_{Y, \int_S g(s) d\mu} \arrow[r, "\sigma_\mu"] & \int_S \mathcal{O}_{Y,g(s)}\, d\mu,
\end{tikzcd}$$
so that α is a natural transformation of left ultrafunctors. □

Example 8.4.10. Let $\mathcal{M} = \{*\}$ denote a category with a single object and a single morphism. In this case, the fully faithful embedding $\mathrm{Comp}_{\mathcal{M}} \hookrightarrow \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{M}, \mathrm{Set})$ is an equivalence of categories; essential surjectivity follows from Example 8.1.2 (and Theorem 3.1.5).

8.5. **Digression: Categories with Unique Factorization.** Let $\mathcal{E}$ be an ultracategory envelope, let $\mathcal{M}^+$ be a category which admits small products, and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory which admits categorical ultraproducts in $\mathcal{M}^+$. To prove Theorem 8.2.6, we must show that every right ultrafunctor $F : \mathcal{E}^{\mathrm{cc}} \to \mathcal{M}$ admits an essentially unique extension to a functor $F^+ : \mathcal{E} \to \mathcal{M}^+$ which preserves small products. At the level of objects, it is clear what we need to do: since every object $X \in \mathcal{E}$ factors as a product of coconnected objects $\{X_s\}_{s \in S}$, the functor F^+ must satisfy $F^+(X) \simeq \prod_{s \in S} F(X_s)$. However, to see that this construction is functorial, it will be important to know that the factorization $X \simeq \prod_{s \in S} X_s$ is essentially unique (Proposition 8.5.5). For the proof, we will not need the full strength of our assumption that $\mathcal{E}$ is an ultracategory envelope.

Definition 8.5.1. Let $\mathcal{E}$ be a category. We will say that $\mathcal{E}$ *has unique factorization* if it satisfies the following axioms:

($E1$) The category $\mathcal{E}$ admits small products.

($E2$) Every object $X \in \mathcal{E}$ can be written as a (small) product $\prod_{s \in S} X_s$, where each factor X_s is a coconnected object of $\mathcal{E}$.

Example 8.5.2. Every ultracategory envelope is a category with unique factorization.

Example 8.5.3. Let $\mathcal{E}$ be any category which admits small products, and let $\mathcal{E}' \subseteq \mathcal{E}$ be the smallest full subcategory which contains all coconnected objects of $\mathcal{E}$ and is closed under products. Then $\mathcal{E}'$ is a category with unique factorization.

Example 8.5.4. The category Set admits small coproducts, and a set S is connected (as an object of Set) if and only if it consists of a single element. Moreover, every set S can be written as a coproduct $\coprod_{s \in S}\{s\}$ of connected objects of Set. It follows that the opposite category $\mathrm{Set}^{\mathrm{op}}$ has unique factorization.

However, $\mathrm{Set}^{\mathrm{op}}$ is not an ultracategory envelope: if μ is an ultrafilter on a set S, then the ultraproduct $\int_S \{s\} d\mu$ (computed in the category $\mathrm{Set}^{\mathrm{op}}$) can be described concretely as the the subset of S given the intersection $\bigcap_{\mu(S_0)} S_0$. If μ is a nonprincipal ultrafilter, then this intersection is the empty set, which is not a coconnected object of $\mathrm{Set}^{\mathrm{op}}$.

Our goal in this section is to prove the following result, which will be needed in the proof of Theorem 8.2.6.

Proposition 8.5.5. *Let $\mathcal{E}$ be a category with unique factorization (Definition 8.5.1). Suppose we are given families of coconnected objects $\{X_s\}_{s \in S}$ and $\{Y_t\}_{t \in T}$, and let $f : \prod_{s \in S} X_s \to \prod_{t \in T} Y_t$ be an isomorphism. Then there exists a bijection $\rho : S \simeq T$ and a collection of isomorphisms $f_s : X_s \simeq Y_{\rho(s)}$ such that f is the the product map $\prod_{s \in S} X_s \xrightarrow{\{f_s\}} \prod_{s \in S} Y_{\rho(s)} \simeq \prod_{t \in T} Y_t$.*

The proof of Proposition 8.5.5 will require a few preliminaries.

Lemma 8.5.6. *Let $\mathcal{E}$ be a category with unique factorization containing morphisms $u : X \to X'$ and $v : Y \to V'$, and suppose that the product map $(u \times v) : X \times Y \to X' \times Y'$ is an isomorphism. Then u and v are isomorphisms.*

Proof. Fix an object $Z \in \mathcal{E}$; we will show that composition with u and v induce bijections

$$\phi : \mathrm{Hom}_{\mathcal{E}}(X', Z) \to \mathrm{Hom}_{\mathcal{E}}(X, Z) \qquad \psi : \mathrm{Hom}_{\mathcal{E}}(Y', Z) \to \mathrm{Hom}_{\mathcal{E}}(Y, Z).$$

Writing Z as a product of coconnected objects, we may assume without loss of generality that Z is coconnected. In this case, the coproduct of the maps ϕ and ψ (in the category of sets) can be identified with the map

$$\mathrm{Hom}_{\mathcal{E}}(X' \times Y', Z) \to \mathrm{Hom}_{\mathcal{E}}(X \times Y, Z)$$

given by precomposition with $u \times v$, and is therefore bijective. □

Lemma 8.5.6 immediately implies the following slightly stronger assertion:

Lemma 8.5.7. *Let $\mathcal{E}$ be a category with unique factorization and let $\{u_s : X_s \to Y_s\}_{s \in S}$ be a collection of morphisms in $\mathcal{E}$. Suppose that the product map $\prod_{s \in S} X_s \to \prod_{s \in S} Y_s$ is an isomorphism. Then each u_s is an isomorphism.*

Lemma 8.5.8. *Let $\mathcal{E}$ be a category with unique factorization, let $\{X_s\}_{s \in S}$ be a collection of coconnected objects of $\mathcal{E}$ having product $X = \prod_{s \in S} X_s$. Then every direct factor X has the form $\prod_{s \in I} X_s$ for some subset $I \subseteq S$.*

Proof. Suppose we are given a pair of maps $f : X \to Y$ and $g : X \to Z$ which exhibit X as a product of Y and Z in the category $\mathcal{E}$. For each $s \in S$, let $p_s : X \to X_s$ be the projection map. Since each X_s is coconnected, composition with f and g induce a bijection

$$\mathrm{Hom}_{\mathcal{E}}(Y, X_s) \amalg \mathrm{Hom}_{\mathcal{E}}(Z, X_s) \to \mathrm{Hom}_{\mathcal{E}}(X, X_s).$$

In particular, each of the maps p_s factors uniquely either as a composition $X \xrightarrow{f} Y \xrightarrow{p_s^-} X_s$ or $X \xrightarrow{g} Z \xrightarrow{p_s^+} X_s$. Let $I \subseteq S$ be the collection of those indices s for which p_s factors through f. Then $\{p_s^-\}_{s \in I}$ and $\{p_s^+\}_{s \notin I}$ induce maps $u : Y \to \prod_{s \in I} X_s$ and $v : Z \to \prod_{s \notin I} X_s$, and the product map

$$(u \times v) : Y \times Z \to \prod_{s \in S} X_s$$

is an isomorphism. Applying Lemma 8.5.6, we deduce that u and v are isomorphisms. □

Remark 8.5.9. Let $\mathcal{E}$ be a category which admits finite products, so that $\mathcal{E}$ has a final object $\mathbf{1}$. Then an object $X \in \mathcal{E}$ is coconnected if and only if it satisfies the following pair of conditions:

- The set $\mathrm{Hom}_{\mathcal{E}}(\mathbf{1}, X)$ is empty.
- For every pair of objects $Y, Z \in \mathcal{E}$, the canonical map
$$\mathrm{Hom}_{\mathcal{E}}(Y, X) \amalg \mathrm{Hom}_{\mathcal{E}}(Z, X) \to \mathrm{Hom}_{\mathcal{E}}(Y \times Z, X)$$
is bijective. In other words, every $f : Y \times Z \to X$ factors (uniquely) through either Y or Z (but not both).

In particular, the final object $\mathbf{1} \in \mathcal{E}$ is not coconnected.

Lemma 8.5.10. *Let $\mathcal{E}$ be a category with unique factorization and let $\{X_s\}_{s \in S}$ be a collection of coconnected objects of $\mathcal{E}$ having product $X = \prod_{s \in S} X_s$. Then X is coconnected if and only if S is a singleton.*

Proof. We will show that if X is coconnected, then S is a singleton (the converse is immediate). We first note that S is nonempty (since the final object $\mathbf{1} \in \mathcal{E}$ is not coconnected; see Remark 8.5.9). If S has more than one element, then we can write $S = I \amalg J$ for nonempty subsets $I, J \subseteq S$. We then have projection maps $p : X \to X_I \prod_{s \in I} X_s$ and $q : X \to X_J = \prod_{s \in J} X_s$ which exhibit X as a product of X_I and X_J. Using the coconnectivity of X, we deduce that the identity map id_X factors (uniquely) through either p or q. Without loss of generality, we may assume that $\mathrm{id}_X = f \circ p$ for some map $f : X_I \to X$. For each $s \in J$, let $g_s : X_s \to \mathbf{1}$ be the projection map. Then the composition
$$X \simeq X_I \times \prod_{s \in J} X_s \xrightarrow{f \times \{g_s\}} X \times \prod_{s \in J} \mathbf{1} \simeq X$$
is the identity map id_X. It follows from Lemma 8.5.7 that each g_s is an isomorphism. This is a contradiction, since the objects X_s are coconnected and therefore cannot be final objects of $\mathcal{E}$. □

Proof of Proposition 8.5.5. Let $\mathcal{E}$ be a category with unique factorization and let $f : \prod_{s \in S} X_s \simeq \prod_{t \in T} Y_t$ be an isomorphism in $\mathcal{E}$, where each X_s and each Y_t is coconnected. For each $s_0 \in S$, the composition of f^{-1} with the projection map $\prod_{s \in S} X_s \to X_{s_0}$ exhibits X_{s_0} as a direct factor of $\prod_{t \in T} Y_t$. It follows from Lemma 8.5.8 that this factor must have the form $\prod_{t \in I} Y_t$, for some subset $I \subseteq T$. Since X_{s_0} is coconnected, we must have $I = \{\rho(s_0)\}$ for some element $\rho(s_0) \in T$ (Lemma 8.5.10). It follows that f fits into a commutative diagram

$$\begin{tikzcd}
\prod_{s \in S} X_s \arrow[r, "f"] \arrow[d] & \prod_{t \in T} Y_t \arrow[d] \\
X_{s_0} \arrow[r, "f_s"] & Y_{\rho(s_0)}
\end{tikzcd}$$

for some isomorphism $f_{s_0} : X_{s_0} \simeq Y_{\rho(s_0)}$. To complete the proof, it will suffice to show that ρ is bijective. This follows by the same analysis (with the roles of S and T reversed): for each element $t_0 \in T$, the composition $\prod_{s \in S} X_s \xrightarrow{f} \prod_{t \in T} Y_t \to Y_{t_0}$ exhibits Y_{t_0} as a coconnected direct factor of $\prod_{s \in S} X_s$, which is therefore of the form X_{s_0} for some unique element $s_0 \in S$. □

Remark 8.5.11. Let $\mathcal{E}$ be a category with unique factorization. Then, for every pair of objects $X, Y \in \mathcal{E}$, the projection maps $X \leftarrow X \times Y \to Y$ are epimorphisms in $\mathcal{E}$. In other words, for every object $Z \in \mathcal{E}$, the canonical maps
$$\mathrm{Hom}_{\mathcal{E}}(X, Z) \xrightarrow{i} \mathrm{Hom}_{\mathcal{E}}(X \times Y, Z) \xleftarrow{j} \mathrm{Hom}_{\mathcal{E}}(Y, Z).$$
To prove this, we can factor Z as a product of coconnected objects and thereby reduce to the case where Z is coconnected. In this case, i and j are inclusions of complementary summands (Remark 8.5.9).

8.6. **The Proof of Theorem 8.2.6.** We begin with some general observations concerning ultracategory envelopes.

Remark 8.6.1. Let $\mathcal{E}$ be an ultracategory envelope, and suppose we are given a collection of coconnected objects $\{X_s\}_{s\in S}$ indexed by a set S. For each ultrafilter μ on S, we let

$$\int_S X_s d\mu \simeq \varinjlim_{\mu(S_0)=1} \prod_{s\in S_0} X_s.$$

denote the categorical ultraproduct of Construction 1.2.2. For each subset $S_0 \subseteq S$, the canonical map $\prod_{s\in S} X_s \to \prod_{s\in S_0} X_s$ is projection onto a direct factor, and is therefore an epimorphism in $\mathcal{E}$ (Remark 8.5.11). Passing to the direct limit, we see that the map $q_\mu : \prod_{s\in S} X_s \twoheadrightarrow \int_S X_s d\mu$ of Notation 1.2.3 is also an epimorphism in $\mathcal{E}$.

Lemma 8.6.2. *Let $\mathcal{E}$ be an ultracategory envelope, let $\{X_s\}_{s\in S}$ be a collection of coconnected objects of $\mathcal{E}$, and let Y be another coconnected object of $\mathcal{E}$. Then composition with the maps $q_\mu : \prod_{s\in S} X_s \twoheadrightarrow \int_S X_s d\mu$ of Notation 1.2.3 induces a bijection*

$$\coprod_{\mu\in\beta S} \mathrm{Hom}_{\mathcal{E}^{\mathrm{cc}}}(\int_S X_s d\mu, Y) \to \mathrm{Hom}_{\mathcal{E}}(\prod_{s\in S} X_s, Y).$$

In other words, every morphism $\prod_{s\in S} X_s \to Y$ factors uniquely through $q_\mu : \prod_{s\in S} X_s \twoheadrightarrow \int_S X_s d\mu$ for some uniquely determined ultrafilter μ on S.

Proof. For each subset $S_0 \subseteq S$, set $X_{S_0} = \prod_{s\in S_0} X_s$. Our assumption that Y is coconnected guarantees that the map

$$\mathrm{Hom}_{\mathcal{E}}(X_{S_0}, Y) \coprod \mathrm{Hom}_{\mathcal{E}}(X_{S\smallsetminus S_0}, Y) \to \mathrm{Hom}_{\mathcal{E}}(X_S, Y)$$

is bijective. In particular, every map $f : X_S \to Y$ factors through exactly one of the projection maps $X_{S_0} \twoheadleftarrow X_S \twoheadrightarrow X_{S\smallsetminus S_0}$. It follows that the map

$$\mu : P(S) \to \{0,1\} \qquad \mu(S_0) = \begin{cases} 1 & \text{if } f \text{ factors through } X_S \twoheadrightarrow X_{S_0} \\ 0 & \text{otherwise.} \end{cases}$$

is an ultrafilter on S, which is uniquely determined by the requirement that f factors through the map $q_\mu : X_S \twoheadrightarrow \varprojlim_{\mu(S_0)=1} X_{S_0} = \int_S X_s d\mu$. □

From Lemma 8.6.2, we can immediately deduce a weak version of Theorem 8.2.6.

Proposition 8.6.3. *Let $\mathcal{E}$ be an ultracategory envelope, let $\mathcal{M}^+$ be a category, and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory which has ultraproducts in $\mathcal{M}^+$. Then the functor $\mathrm{Fun}'(\mathcal{E}, \mathcal{M}^+) \to \mathrm{Fun}^{\mathrm{RUlt}}(\mathcal{E}^{\mathrm{cc}}, \mathcal{M})$ of Theorem 8.2.6 is fully faithful.*

Proof. Let $F^+, G^+ : \mathcal{E} \to \mathcal{M}^+$ be functors which preserve small products and carry $\mathcal{E}^{\mathrm{cc}}$ into $\mathcal{M}$ and let $\alpha : F \to G$ be a natural transformation. We wish to show that if α is a natural transformation of right ultrafunctors, then it extends uniquely to a natural transformation between F^+ and G^+.

For each object $X \in \mathcal{E}$, choose a set $S(X)$ and a collection of maps $\{p_{X,s} : X \to X_s\}_{s\in S(X)}$ which exhibit X as a product of coconnected objects X_s of $\mathcal{E}$ (by virtue of Proposition 8.5.5, this product decomposition is essentially unique; however, we do not yet need to know this). Since the functor G^+ preserves products, there is a unique map $\alpha^+(X) : F^+(X) \to G^+(X)$ which fits into a commutative diagram

$$\begin{tikzcd} F^+(X) \arrow[r, "\{F^+(p_{X,s})\}_{s\in S(X)}"] \arrow[d, "\alpha^+(X)"] & \prod_{s\in S(X)} F(X_s) \arrow[d, "\prod_{s\in S(X)} \alpha(X_s)"] \\ G^+(X) \arrow[r, "\{G^+(p_{X,s})\}_{s\in S(X)}"] & \prod_{s\in S(X)} G(X_s). \end{tikzcd}$$

It is clear that if α extends to a natural transformation from F^+ to G^+, then the extension must be given by $X \mapsto \alpha^+(X)$. Moreover, from the uniqueness of $\alpha^+(X)$ (and the naturality of α) we see that $\alpha^+(X) = \alpha(X)$ when X is coconnected. It will therefore suffice to show that α^+ is a natural transformation. That is, we must show that for any morphism $f : X \to Y$ in the category $\mathcal{E}$, the left square in the diagram

$$\begin{tikzcd}
F^+(X) \arrow[r, "F^+(f)"] \arrow[d, "\alpha^+(X)"] & F^+(Y) \arrow[r, "\{F^+(p_{Y,t})\}_{t\in S(Y)}"] \arrow[d, "\alpha^+(X)"] & \prod_{t\in S(Y)} F(Y_t) \arrow[d, "\{\alpha(Y_t)\}"] \\
G^+(X) \arrow[r, "G^+(f)"] & G^+(Y) \arrow[r, "\{G^+(p_{Y,t})\}_{t\in S(Y)}"] & \prod_{t\in S(Y)} G(Y_t)
\end{tikzcd}$$

commutes. Since the right square commutes by construction and the horizontal maps on the right are bijective, it will suffice to show that the outer square commutes: that is, for each $t \in S(Y)$, we have a commutative diagram σ :

$$\begin{tikzcd}
F^+(X) \arrow[r, "F^+(f_t)"] \arrow[d, "\alpha^+(Y)"] & F(Y_t) \arrow[d, "\alpha(Y_t)"] \\
G^+(X) \arrow[r, "G^+(f_t)"] & G(Y_t),
\end{tikzcd}$$

where f_t denotes the composition $p_{Y,t} \circ f$. Using Lemma 8.6.2, we see that f_t factors (uniquely) as a composition $X \xrightarrow{q_\mu} \int_{S(X)} X_s d\mu \xrightarrow{f_0} Y_t$ for some ultrafilter μ on the set $S(X)$. Unwinding the definitions, we can identify σ with the outer rectangle in the diagram

$$\begin{tikzcd}
\prod_{s\in S(X)} F(X_s) \arrow[r, "q_\mu"] \arrow[d, "\prod_{s\in S(X)} \alpha(X_s)"] & \int_{S(X)} F(X_s) d\mu \arrow[r] \arrow[d, "\int_{S(X)} \alpha(X_s) d\mu"] & F(\int_{S(X)} X_s d\mu) \arrow[r, "F(f_0)"] \arrow[d, "\alpha(\int_{S(X)} X_s d\mu)"] & F(Y_t) \arrow[d, "\alpha(Y_t)"] \\
\prod_{s\in S(X)} G(X_s) \arrow[r, "q_\mu"] & \int_{S(X)} G(X_s) d\mu \arrow[r] & G(\int_{S(X)} X_s d\mu) \arrow[r, "G(f_0)"] & G(Y_t).
\end{tikzcd}$$

Here the left square commutes by the functoriality of the ultraproduct construction, the right square commutes by the naturality of α, and the middle square commutes by virtue of our assumption that α is a natural transformation of right ultrafunctors. □

To complete the proof of Theorem 8.2.6, it will suffice to prove the following:

Proposition 8.6.4. *Let $\mathcal{E}$ be an ultracategory envelope, let $\mathcal{M}^+$ be a category, and let $\mathcal{M} \subseteq \mathcal{M}^+$ be a full subcategory which has ultraproducts in $\mathcal{M}^+$. Let $F : \mathcal{E}^{cc} \to \mathcal{M}$ be a right ultrafunctor. Then there exists a functor $F^+ : \mathcal{E} \to \mathcal{M}^+$, which preserves small products and carries $\mathcal{E}^{cc}$ into $\mathcal{M}$, and an isomorphism $\alpha : F^+|_{\mathcal{E}^{cc}} \simeq F$ of right ultrafunctors from $\mathcal{E}^{cc}$ to $\mathcal{M}$.*

The proof of Proposition 8.6.4 requires a straightforward but somewhat lengthy construction. For the remainder of this section, we fix an ultracategory envelope $\mathcal{E}$, a category $\mathcal{M}^+$ which admits small products, a full subcategory $\mathcal{M} \subseteq \mathcal{M}^+$ which admits categorical ultraproducts in $\mathcal{M}^+$, and a right ultrafunctor $(F, \{\gamma_\mu\})$ from $\mathcal{E}^{cc}$ to $\mathcal{M}$.

Construction 8.6.5 (The Functor F^+ on Objects)**.** For each object $X \in \mathcal{E}$, choose a collection of maps $\{p_{X,s} : X \to X_s\}_{s\in S(X)}$ which exhibit X as a product of coconnected objects X_s of $\mathcal{E}$ (as in the proof of Proposition 8.6.3). We let $F^+(X)$ denote a product $\prod_{s\in S(X)} F(X_s)$, formed in the category $\mathcal{M}^+$. Note that when X is coconnected, then $S(X)$ is a singleton (Lemma 8.5.10) so we can arrange that $F^+(X)$ belongs to $\mathcal{M}$.

Notation 8.6.6. Let $f : X \to Y$ be a morphism in the category $\mathcal{E}$, where Y is coconnected. It follows from Lemma 8.6.2 that f factors uniquely as a composition

$$X \simeq \prod_{s \in S(X)} X_s \xrightarrow{q_{\mu_f}} \int_{S(X)} X_s d\mu_f \xrightarrow{f_0} Y$$

for some uniquely determined ultrafilter μ_f on $S(X)$. Let $\varphi_f : F^+(X) \to F(Y)$ denote the morphism in $\mathcal{M}^+$ given by the composition

$$F^+(X) = \prod_{s \in S(X)} F(X_s) \to \int_{S(X)} F(X_s) d\mu_f \xrightarrow{\gamma_{\mu_f}} F(\int_{S(X)} X_s d\mu_f) \xrightarrow{F(f_0)} F(Y).$$

Example 8.6.7. Let X be an object of $\mathcal{E}$ and take $f : X \to Y$ to be the projection map $p_{X,s_0} : X \to X_{s_0}$ for some $s_0 \in S(X)$. In this case, the map $\varphi_f : F^+(X) \to F(Y) = F(X_{s_0})$ of Notation 8.6.6 is defined to be the composition

$$F^+(X) = \prod_{s \in S(X)} F(X_s) \to \int_{S(X)} F(X_s) d\delta_{s_0} \xrightarrow{\gamma_{\delta_{s_0}}} F(\int_{S(X)} X_s d\delta_{s_0}) \xrightarrow{F(\epsilon_{S(X),s_0})} (F(X_{s_0})),$$

where $\epsilon_{S(X),s_0} : \int_{S(X)} X_s d\delta_{s_0} \simeq X_{s_0}$ is the isomorphism of Example 1.2.7. Using condition (1) of Definition 8.1.1, we can rewrite this composition as

$$F^+(X) = \prod_{s \in S(X)} F(X_s) \to \int_{S(X)} F(X_s) d\delta_{s_0} \xrightarrow{\epsilon_{S(X),s_0}} F(X_{s_0}),$$

which coincides with the projection map $\prod_{s \in S(X)} F(X_s) \to F(X_{s_0})$.

Remark 8.6.8. Suppose that we are given a composable pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the category $\mathcal{E}$, where both Y and Z are coconnected. Then the morphism $\varphi_{g \circ f} : F^+(X) \to F(Z)$ of Notation 8.6.6 factors as a composition $F(g) \circ \varphi_f$.

To define the functor F^+ on morphisms, we will prove the following:

Lemma 8.6.9. *Let $f : X \to Y$ be a morphism in $\mathcal{E}$. Then there is a unique morphism $F^+(f) : F^+(X) \to F^+(Y)$ in the category $\mathcal{M}^+$ with the following property:*

($*$) *For every morphism $g : Y \to Z$ in $\mathcal{E}$, where Z is coconnected, the diagram*

$$\begin{tikzcd} F^+(X) \arrow[rr, "F^+(f)"] \arrow[dr, "\varphi_{g \circ f}"'] & & F^+(Y) \arrow[dl, "\varphi_g"] \\ & F(Z) & \end{tikzcd}$$

commutes.

Proof. For each $s \in S(Y)$, let $f_s : X \to Y_s$ denote the composition $X \xrightarrow{f} Y \xrightarrow{p_{Y,s}} Y_s$. Using Lemma 8.6.2, we see that f_s factors as a composition

$$X = \prod_{t \in S(X)} X_t \xrightarrow{q_{\nu_s}} \int_{S(X)} X_t d\nu_s \xrightarrow{\overline{f}_s} Y_s,$$

for some uniquely determined ultrafilter ν_s on $S(X)$ and morphism $\overline{f}_s : \int_{S(X)} X_t d\nu_s \to Y_s$ between coconnected objects of $\mathcal{E}$. Let us define $F^+(f)$ to be the unique morphism from $F^+(X)$ to $F^+(Y)$ having the property that, for each $s \in S(Y)$, the composite map

$$F^+(X) \xrightarrow{F^+(f)} F^+(Y) = \prod_{s' \in S(Y)} F(Y_{s'}) \to F(Y_s)$$

coincides with φ_{f_s}. It follows from Example 8.6.7 that $F^+(f)$ is the unique morphism from $F^+(X)$ to $F^+(Y)$ which satisfies condition ($*$) in those cases where $g : Y \to Z$ is one of the projection maps $p_{Y,s} : Y \to Y_s$.

To complete the proof, we must show that the morphism $F^{+}(f)$ satisfies condition $(*)$ for *any* morphism $g : Y \to Z$, where Z is coconnected. Using Lemma 8.6.2, we can factor the morphism g as a composition

$$Y = \prod_{s \in S(Y)} Y_s \xrightarrow{q_\mu} \int_{S(Y)} Y_s d\mu \xrightarrow{g_0} Z$$

for some ultrafilter μ on $S(Y)$. By virtue of Remark 8.6.8, we may replace the map $g : Y \to Z$ with the epimorphism $q_\mu : \prod_{s \in S(Y)} Y_s \twoheadrightarrow \int_{S(Y)} Y_s d\mu$, and thereby reduce to the case where g_0 is the identity map. Let λ denote the ultrafilter on $S(X)$ given by $\int_{S(Y)} \nu_s d\mu$, so that $g \circ f$ is given by the composition

$$X = \prod_{t \in S(X)} X_t \xrightarrow{q_\lambda} \int_{S(X)} X_t d\lambda \xrightarrow{\Delta_{\mu,\nu_\bullet}} \int_{S(Y)} (\int_{S(X)} X_t d\nu_s) d\mu \xrightarrow{\int_{S(Y)} \overline{f}_s d\mu} \int_{S(Y)} Y_s d\mu,$$

where $\Delta_{\mu,\nu_\bullet}$ is the Fubini transformation of Notation 1.2.9. It follows that $\varphi_{g\circ f}$ is given by the clockwise composition in the diagram

$$\begin{tikzcd}
\prod_{t \in S(X)} F(X_t) \arrow[r, "q_\lambda"] \arrow[d, "\{q_{\nu_s}\}_{s \in S(Y)}"] & \int_{S(X)} F(X_t) d\lambda \arrow[r, "\gamma_\lambda"] \arrow[d, "\Delta_{\mu,\nu_\bullet}"] & F(\int_{S(X)} X_t d\lambda) \arrow[dd, "F(\Delta_{\mu,\nu_\bullet})"] \\
\prod_{s \in S(Y)} \int_{S(X)} F(X_t) d\nu_s \arrow[r, "q_\lambda"] \arrow[d, "\prod_{s \in S(Y)} \gamma_{\nu_s}"] & \int_{S(Y)} (\int_{S(X)} F(X_t) d\nu_s) d\mu \arrow[d, "\int_{S(Y)} \gamma_{\nu_s} d\mu"] & \\
\prod_{s \in S(Y)} F(\int_{S(X)} X_t d\nu_s) \arrow[r, "q_\lambda"] \arrow[d, "\prod_{s \in S(Y)} F(\overline{f}_s)"] & \int_{S(Y)} F(\int_{S(X)} X_t d\nu_s) d\mu \arrow[r, "\gamma_\mu"] \arrow[d, "\int_{S(Y)} F(\overline{f}_s) d\mu"] & F(\int_{S(Y)} (\int_{S(X)} X_t d\nu_s) d\mu) \arrow[d, "F(\int_{S(Y)} \overline{f}_s d\mu)"] \\
\prod_{s \in S(Y)} F(Y_s) \arrow[r, "q_\lambda"] & \int_{S(Y)} F(Y_s) d\mu \arrow[r, "\gamma_\mu"] & F(\int_{S(Y)} Y_s d\mu),
\end{tikzcd}$$

while the counterclockwise composition coincides with $\varphi_g \circ F^{+}(f)$. It will therefore suffice to show that the diagram commutes. Here the commutativity is immediate except for the rectangle in the upper right, which commutes by virtue of the compatibility of the maps γ with the ultrapower diagonals in the categories $\mathcal{E}^{\mathrm{cc}}$ and $\mathcal{M}$ (condition (2) of Definition 8.1.1). $\square$

Let $f : X \to Y$ and $g : Y \to Z$ be a pair of morphisms in the category $\mathcal{E}$. It follows immediately from the definitions that the composite map $F^{+}(X) \xrightarrow{F^{+}(f)} F^{+}(Y) \xrightarrow{F^{+}(g)} F^{+}(Z)$ satisfies condition $(*)$ of Lemma 8.6.9, and therefore coincides with $F^{+}(g \circ f)$. Similarly, for each object $X \in \mathcal{E}$, the identity map $\mathrm{id}_{F^{+}(X)}$ satisfies condition $(*)$ of Lemma 8.6.9, and therefore coincides with $F^{+}(\mathrm{id}_X)$. Consequently, we can regard Construction 8.6.5 and Lemma 8.6.9 as defining a functor $F^{+} : \mathcal{E} \to \mathcal{M}^{+}$.

Notation 8.6.10. For each coconnected object X of $\mathcal{E}$, let $\alpha_X : F^{+}(X) \to F(X)$ denote the morphism φ_{id_X} of Notation 8.6.6.

Lemma 8.6.11. *The construction $X \mapsto \alpha_X$ of Notation 8.6.10 determines a natural isomorphism of functors $F^{+}|_{\mathcal{E}^{\mathrm{cc}}} \simeq F$.*

Proof. We first note that X is a coconnected object of $\mathcal{E}$, then the set $S(X)$ has a single element s (Lemma 8.5.10) and the projection map $p_{X,s} : X \to X_s$ is an isomorphism. It follows from Example 8.6.7 and Remark 8.6.8 that the map $\alpha_X : F^{+}(X) = X_s \to X$ is the inverse isomorphism $p_{X,s}^{-1}$. To complete the proof, it will suffice to show α_X is natural in X: that is, for every morphism $f : X \to Y$ between coconnected objects of $\mathcal{E}$, we have a commutative diagram

$$\begin{tikzcd}
F^{+}(X) \arrow[r, "F^{+}(f)"] \arrow[d, "\alpha_X"] & F^{+}(Y) \arrow[d, "\alpha_Y"] \\
F(X) \arrow[r, "F(f)"] & F(Y).
\end{tikzcd}$$

The clockwise composition coincides with φ_f by virtue of condition $(*)$ of Lemma 8.6.9, and the counterclockwise composition coincides with φ_f by Remark 8.6.8. $\square$

Lemma 8.6.12. *The functor $F^+ : \mathcal{E} \to \mathcal{M}^+$ preserves small products.*

Proof. Since $\mathcal{E}$ is an ultracategory envelope, every object of $\mathcal{E}$ can be decomposed as a product of coconnected objects. It will therefore suffice to show that the functor F^+ preserves products of coconnected objects. Suppose we are given a collection of maps $\{u_i : X \to X_i\}_{i \in I}$ which exhibit X as a product of the objects X_i; we wish to show that the maps $F^+(u_i) : F^+(X) \to F^+(X_i)$ exhibit $F^+(X)$ as a product of the objects $\{F^+(X_i)\}_{i \in I}$. By virtue of Proposition 8.5.5, we may assume without loss of generality that $I = S(X)$ and each u_s is the projection map $p_{X,s} : X \to X_s$ chosen in Construction 8.6.5. In this case, we can identify $F^+(u_s)$ with the composition

$$F^+(X) = \prod_{s' \in S(X)} X_{s'} \to X_s \xrightarrow{\alpha_{X_s}^{-1}} F^+(X_s),$$

from which the desired result is immediate. $\square$

Proof of Proposition 8.6.4. The functor $F^+ : \mathcal{E} \to \mathcal{M}^+$ carries $\mathcal{E}^{\mathrm{cc}}$ into $\mathcal{M}$ by construction, and preserves small products by Lemma 8.6.12. Using Remark 1.4.10, we can regard $F^+|_{\mathcal{E}^{\mathrm{cc}}}$ as a right ultrafunctor from $\mathcal{E}^{\mathrm{cc}}$ to $\mathcal{M}$. To complete the proof, it will suffice to show that α is a natural isomorphism of right ultrafunctors. In other words, we wish to show that for every collection of coconnected objects $\{X_t\}_{t \in T}$ of $\mathcal{E}$ and every ultrafilter μ on T, the diagram

$$\begin{tikzcd}
\int_T F(X_t) d\mu \arrow[r, "\int_T \alpha_{X_t}^{-1} d\mu"] \arrow[d, "\gamma_\mu"] & \int_T F^+(X_t) d\mu \arrow[d, "\gamma'_\mu"] \\
F(\int_T X_t d\mu) \arrow[r, "\alpha^{-1}_{\int_T X_t d\mu}"] & F^+(\int_T X_t d\mu)
\end{tikzcd}$$

commutes, where γ'_μ is determined by the right ultrastructure structure on $F^+|_{\mathcal{E}^{\mathrm{cc,op}}}$.

Writing the ultraproduct $\int_T F^+(X_t) d\mu$ as a filtered colimit of products $\prod_{t \in T_0} F^+(X_t)$, we are reduced to proving an equality

$$\gamma'_\mu \circ \Big(\int_T \alpha_{X_t}^{-1} d\mu\Big) \circ q_\mu^{T_0} = \alpha^{-1}_{\int_T X_t d\mu} \circ \gamma_\mu \circ q_\mu^{T_0}$$

for each $T_0 \subseteq T$ with $\mu(T_0) = 1$. Set $Y = \prod_{t \in T_0} X_t$, where the product is formed in the category $\mathcal{E}$. Using Proposition 8.5.5, we can choose a bijection $\rho : S(Y) \simeq T_0$ such that each of the projection maps $Y \to \prod_{t \in T_0} X_t \to X_{\rho(s)}$ factors as a composition $Y \xrightarrow{p_{Y,s}} Y_s \xrightarrow{u_s} X_{\rho(s)}$, where u_s is an isomorphism. Since $\mu(T_0) = 1$, we can write μ as the pushforward $\rho_*(\mu_0)$ for some ultrafilter μ_0 on the set $S(Y)$. Consider the diagram

$$\begin{tikzcd}
\prod_{s \in S(Y)} F(Y_s) \arrow[r, "\prod_{s \in S(Y)} F(u_s)", "\sim"'] \arrow[d, "q_{\mu_0}"] & \prod_{t \in T_0} F(X_t) \arrow[r, "\prod_{t \in T_0} \alpha_{X_t}^{-1}"] \arrow[d, "q_\mu^{T_0}"] & \prod_{t \in T_0} F^+(X_t) \arrow[d, "q_\mu^{T_0}"] \\
\int_{S(Y)} F(Y_s) d\mu_0 \arrow[r, "\int_{S(Y)} F(u_s) d\mu_0", "\sim"'] \arrow[d, "\gamma_{\mu_0}"] & \int_T F(X_t) d\mu \arrow[r, "\int_T \alpha_{X_t}^{-1} d\mu"] \arrow[d, "\gamma_\mu"] & \int_T F^+(X_t) d\mu \arrow[d, "\gamma'_\mu"] \\
F(\int_S Y_s d\mu_0) \arrow[r, "F(\int_{S(Y)} u_s d\mu_0", "\sim"'] & F(\int_T X_t d\mu) \arrow[r, "\alpha^{-1}_{\int_T X_t d\mu}"] & F^+(\int_T X_t d\mu).
\end{tikzcd}$$

Note that the composition of the horizontal maps at the top of the diagram can be identified with the comparison isomorphism $F^+(Y) = F^+(\prod_{t \in T_0} Y_s) \to \prod_{t \in T_0} F^+(X_t)$. Invoking the definition of γ'_μ, we see that clockwise composition around the diagram yields the map $F^+(q_\mu^{T_0}) : F^+(Y) = F^+(\prod_{t \in T_0} X_t) \to F^+(\int_T X_t d\mu)$.

On the other hand, counterclockwise composition yields the morphism $\alpha^{-1}_{\int_T X_t d\mu} \circ \varphi_v$, where φ_v is defined as in Notation 8.6.6. Using condition $(*)$ of Lemma 8.6.9, we deduce that the outer square of this diagram commutes. Consequently, to show that the rectangle on the right side commutes, it will suffice to show that the two squares on the right commute. Taking inverses of the horizontal maps and noting that we can write $\mu = \rho_*(\mu_0) = \int_{S(Y)} \delta_{\rho(s)} d\mu_0$, we are reduced to proving the commutativity of the diagram

$$\begin{tikzcd}
\prod_{s \in S(Y)} F(Y_s) \arrow[d, "q_{\mu_0}"] & & \prod_{t \in T_0} F(X_s) \arrow[ll, "\prod F(u_s^{-1})"'] \arrow[d, "q_\mu^{T_0}"] \\
\int_{S(Y)} F(Y_s) d\mu_0 \arrow[d, "\gamma_{\mu_0}"] & \int_{S(Y)} F(X_{\rho(s)}) d\mu_0 \arrow[l, "\int_{S(Y)} F(u_s) d\mu_0"', "\sim"] \arrow[d, "\gamma_{\mu_0}"] & \int_T F(X_t) d\mu \arrow[l, "\Delta_{\mu_0,\rho}"', "\sim"] \arrow[d, "\gamma_\mu"] \\
F(\int_{S(Y)} Y_s d\mu_0) & F(\int_{S(Y)} X_{\rho(s)} d\mu_0) \arrow[l, "F(\int_{S(Y)} u_s d\mu_0)"', "\sim"] & F(\int_T X_t d\mu); \arrow[l, "F(\Delta_{\mu_0,\rho});"', "\sim"]
\end{tikzcd}$$

here $\Delta_{\mu_0,\rho}$ denotes the ultraproduct diagonal of Notation 1.3.3, so the lower right square commutes by virtue of our assumption that the maps $\{\gamma_\nu\}$ comprise a right ultrastructure on F. $\square$

In this appendix, we review some well-known concepts and results from category theory and sheaf theory which are needed in the body of this paper. Since these ideas are treated extensively elsewhere in the literature, our exposition is somewhat terse.

Appendix A. Category Theory

A.1. Regular Categories.

Definition A.1.1. Let $\mathcal{C}$ be a category which admits fiber products, and suppose we are given a morphism $f : X \to Y$ in $\mathcal{C}$. Let $X \times_Y X$ denote the fiber product of X with itself over Y, and let $\pi, \pi' : X \times_Y X \to X$ denote the projection maps onto the two factors. We will say that f is an *effective epimorphism* if it exhibits Y as a coequalizer of the maps $\pi, \pi' : X \times_Y X \rightrightarrows X$. In other words, f is an effective epimorphism if, for every object $Z \in \mathcal{C}$, composition with f induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \simeq \{u \in \mathrm{Hom}_{\mathcal{C}}(X, Z) : u \circ \pi = u \circ \pi'\}.$$

Remark A.1.2. Let $\mathcal{C}$ be a category which admits fiber products. Then every effective epimorphism is an epimorphism. In the category of sets, the converse is true: if $g : X \to Y$ is a surjective map of sets, then we can recover Y as the quotient of X by the equivalence relation $R = X \times_Y X = \{(x, x') : g(x) = g(x')\}$. However, this need not be true in a general category.

Definition A.1.3. Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *regular* if the following conditions are satisfied:

- $(R1)$ The category $\mathcal{C}$ admits finite limits.
- $(R2)$ Every morphism $f : X \to Z$ in $\mathcal{C}$ can be written as a composition $X \overset{g}{\twoheadrightarrow} Y \overset{h}{\hookrightarrow} Z$, where g is an effective epimorphism and h is a monomorphism.
- $(R3)$ The collection of effective epimorphisms in $\mathcal{C}$ is closed under pullbacks. That is, if we are given a pullback diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

in $\mathcal{C}$ where f is an effective epimorphism, the morphism f' is also an effective epimorphism.

In the situation of Definition A.1.3, the factorization demanded by $(R2)$ is depends functorially on the morphism $f : X \to Z$.

Proposition A.1.4. *Let $\mathcal{C}$ be a category which admits fiber products, and suppose we are given a commutative square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & U \arrow[d, "j"] \\ Y \arrow[r, "g"] \arrow[ur, dashed, "h"] & V. \end{tikzcd}$$

If p is an effective epimorphism and j is a monomorphism, then there exists a unique morphism $h : Y \to U$ (as indicated by the dotted arrow) satisfying $j \circ h = g$ and $h \circ p = f$.

Proof. Let $\pi, \pi' : X \times_Y X \rightrightarrows X$ be the projection maps. Then we have

$$j \circ f \circ \pi = h \circ p \circ \pi = h \circ p \circ \pi' = j \circ f \circ \pi'.$$

Since j is a monomorphism, it follows that $f \circ \pi = f \circ \pi'$. Our assumption that p is an effective epimorphism then guarantees that there is a unique morphism $h : Y \to U$ satisfying $h \circ p = f$. We will complete the proof by showing that $j \circ h = g$. Since p is an epimorphism (Remark A.1.2), this follows from the identity $j \circ h \circ p = j \circ f = g \circ p$. □

It follows from Proposition A.1.4 that if $\mathcal{C}$ is a category which admits pullbacks and satisfies axiom $(R2)$ of Definition A.1.3, then the collections of monomorphisms and effective epimorphisms comprise a *factorization system* on the category $\mathcal{C}$. In particular, we have the following consequences:

Corollary A.1.5. *Let $\mathcal{C}$ be a category which admits pullbacks and which satisfies condition $(R2)$ of Definition A.1.3. Then, for every morphism $f : X \to Z$ in $\mathcal{C}$, the factorization $X \overset{g}{\twoheadrightarrow} Y \overset{h}{\hookrightarrow} Z$ of condition $(R2)$ is unique up to (unique) isomorphism.*

Proof. Suppose we are given another factorization $X \overset{g'}{\twoheadrightarrow} Y' \overset{h'}{\hookrightarrow} Z$, where g' is an effective epimorphsim and h' is a monomorphism. Invoking Proposition A.1.4, we deduce that there exists a unique map $u : Y \to Y'$ for which the diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "f"'] & Y' \arrow[d, "g'"] \\ Y \arrow[r, "g"] \arrow[ur, "u"] & Z \end{tikzcd}$$

commutes. Similarly, there is a unique morphism $v : Y' \to Y$ for which the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] & Y \arrow[d, "g"] \\ Y' \arrow[r, "g'"] \arrow[ur, "v"] & Z. \end{tikzcd}$$

The uniqueness assertion of Proposition A.1.4 then guarantees that $u \circ v = \mathrm{id}_{Y'}$ and $v \circ u = \mathrm{id}_Y$. □

Notation A.1.6. In the situation of Corollary A.1.5, the monomorphism $h : Y \hookrightarrow Z$ exhibits Y as a subobject of Z, which we will denote by $\mathrm{Im}(f)$ and refer to as the *image of f*. It follows from Proposition A.1.4 that the image $\mathrm{Im}(f)$ depends functorially on the morphism f. More precisely, every commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ X' \arrow[r, "f'"] & Z' \end{tikzcd}$$

induces a map $v : \mathrm{Im}(f) \to \mathrm{Im}(f')$, which is determined by the requirement that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Im}(f) \arrow[r] \arrow[d, "v"] & Z \arrow[d] \\ X' \arrow[r] & \mathrm{Im}(f') \arrow[r] & Z' \end{tikzcd}$$

commutes.

Remark A.1.7. Let $\mathcal{C}$ be a category which admits pullbacks, satisfying condition $(R2)$ of Definition A.1.3. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then:

- (a) The morphism f is an effective epimorphism if and only if the induced map $\mathrm{Im}(f) \hookrightarrow Y$ is an isomorphism.
- (b) The morphism f is a monomorphism if and only if the induced map $X \twoheadrightarrow \mathrm{Im}(f)$ is an isomorphism.

Proposition A.1.8. *Let $\mathcal{C}$ be a category which admits pullbacks. Suppose that the collection of effective epimorphisms in $\mathcal{C}$ is closed under pullbacks (Definition A.1.3). Then the collection of effective epimorphisms in $\mathcal{C}$ is closed under composition.*

Proof. Let $f : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ be effective epimorphisms in $\mathcal{C}$; we wish to show that the composite map $g \circ f : X \to Z$ is also an effective epimorphism. Let $\pi, \pi' : X \times_Z X \to X$ denote the projection maps, and suppose we are given a morphism $u : X \to C$ satisfying $u \circ \pi = u \circ \pi'$. We wish to show that there exists a unique morphism $w : Z \to C$ such that $u = w \circ g \circ f$. The uniqueness of w is clear (since f and g are epimorphisms; see Remark A.1.2). To prove existence, we first observe that u coequalizes the projection maps $X \times_Y X \rightrightarrows X$. Invoking our assumption that f is an effective epimorphism, we deduce that there is a unique map $v : Y \to C$ satisfying $u = v \circ f$. It will therefore suffice to prove that we can write $v = w \circ g$ for some map $w : Z \to C$. Since g is an effective epimorphism, this is equivalent to showing that the map v coequalizes the projection maps $\overline{\pi}, \overline{\pi}' : Y \times_Z Y \to Y$. Let $F : X \times_Z X \to Y \times_Z Y$ denote the map induced by F, so that we have equalities

$$v \circ \overline{\pi} \circ F = v \circ f \circ \pi = u \circ \pi = u \circ \pi' = v \circ f \circ \pi' = v \circ \overline{\pi}' \circ F.$$

It will therefore suffice to show that the morphism F is an epimorphism. In fact, we can write F as a composition of morphisms

$$X \times_Z X \to X \times_Z Y \to Y \times_Z Y,$$

each of which is a epimorphism because it is a pullback of f (hence an effective epimorphism). □

Remark A.1.9. One can also prove the following variant of Proposition A.1.8: if $\mathcal{C}$ is a category with fiber products which satisfies axiom $(R2)$ of Definition A.1.3, then the collection of effective epimorphisms in $\mathcal{C}$ is closed under composition. This follows from the observation that the collections of monomorphisms and effective epimorphisms determine a factorization system on $\mathcal{C}$ (and are therefore closed under composition).

Corollary A.1.10. *Let $\mathcal{C}$ be a regular category, and suppose we are given a pair of effective epimorphisms $f : C \twoheadrightarrow D$ and $f' : C' \twoheadrightarrow D'$ in $\mathcal{C}$. Then the product map $(f \times f') : C \times C' \to D \times D'$ is also an effective epimorphism.*

Proof. The product map $f \times f'$ can be written as a composition

$$C \times C' \xrightarrow{f \times \mathrm{id}_{C'}} D \times C' \xrightarrow{\mathrm{id}_D \times f'} D \times D'.$$

Each of these maps is an effective epimorphism (the first because it is a pullback of f, and the second because it is a pullback of f'). Since the collection of effective epimorphisms in $\mathcal{C}$ is closed under composition (Proposition A.1.8), it follows that $f \times f'$ is also an effective epimorphism. □

Proposition A.1.11. *Let $\mathcal{C}$ be a category which admits pullbacks and satisfies condition $(R2)$ of Definition A.1.3. Then $\mathcal{C}$ satisfies condition $(R3)$ of Definition A.1.3 if and only if it satisfies the following variant:*

$(*)$ *The formation of images in $\mathcal{C}$ is compatible with pullback. That is, for every pullback diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ X' \arrow[r, "f'"] & Z' \end{tikzcd}$$

in $\mathcal{C}$, the induced diagram τ :

$$\begin{tikzcd} \mathrm{Im}(f) \arrow[r] \arrow[d] & Z \arrow[d] \\ \mathrm{Im}(f') \arrow[r] & Z' \end{tikzcd}$$

is also a pullback square.

Proof. Form a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & \mathrm{Im}(f') \times_{Z'} Z \arrow[r, "h"] \arrow[d] & Z \arrow[d] \\ X' \arrow[r, "g'"] & \mathrm{Im}(f') \arrow[r, "h'"] & Z' \end{tikzcd}$$

where g' is an effective epimorphism, h' is a monomorphism, and the right square is a pullback. Then h is also a monomorphism. Since the outer rectangle is a pullback, it follows that the left square is also a pullback. If condition $(R3)$ is satisfied, then g is an effective epimorphism. It follows that we can identify τ with the right square of the preceding diagram, so that τ is a pullback square and $(*)$ is also satisfied. For the converse, suppose that $(*)$ is satisfied and that we are given a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

where f is an effective epimorphism. Then the monomorphism $\mathrm{Im}(f) \hookrightarrow Y$ is an isomorphism (Remark A.1.7). Using $(R3)$, we conclude that the monomorphism $\mathrm{Im}(f') \hookrightarrow Y'$ is also an isomorphism, so that f' is an effective epimorphism as desired. □

A.2. **Exact Categories.**

Definition A.2.1. Let $\mathcal{C}$ be a category which admits finite limits and let X be an object of $\mathcal{C}$. We say that a subobject $R \subseteq X \times X$ is an *equivalence relation* on X if, for every object $Y \in \mathcal{C}$, the image of the induced map

$$\mathrm{Hom}_{\mathcal{C}}(Y, R) \to \mathrm{Hom}_{\mathcal{C}}(Y, X \times X) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X) \times \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

is an equivalence relation on the set $\mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Example A.2.2. Let $\mathcal{C}$ be a category which admits finite limits and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then the fiber product $X \times_Y X$ can be regarded as an equivalence relation on the object X.

Definition A.2.3. Let $\mathcal{C}$ be a category which admits finite limits and let X be an object of $\mathcal{C}$. We will say that an equivalence relation R on X is *effective* if there exists an effective epimorphism $f : X \twoheadrightarrow Y$ such that $R = X \times_Y X$ (as subobjects of $X \times X$).

Notation A.2.4. Let $\mathcal{C}$ be a category which admits finite limits, let X be an object of $\mathcal{C}$, and let R be an effective equivalence relation on X. Then there exists an effective epimorphism $f : X \to Y$ in $\mathcal{C}$ such that $R = X \times_Y X$. The assumption that f is an effective epimorphism then implies that it exhibits Y as the coequalizer of the diagram $R \rightrightarrows X$. In particular, Y is determined (up to unique isomorphism) by the

equivalence relation R; we will emphasize this dependence by denoting Y by X/R. It follows the construction $R \mapsto X/R$ induces a bijection

$$\begin{tikzcd} \{\text{Effective equivalence relations } R \subseteq X \times X\} \arrow[d, "\sim"] \\ \{\text{Effective epimorphisms } f : X \twoheadrightarrow Y\}/\text{isomorphism}; \end{tikzcd}$$

the inverse bijection carries an effective epimorphism $f : X \to Y$ to the equivalence relation $X \times_Y X$ of Example A.2.2.

Proposition A.2.5. *Let $\mathcal{C}$ be a regular category, let X be an object of $\mathcal{C}$, and let $R \subseteq X \times X$ be an equivalence relation on X. The following conditions are equivalent:*

(1) *The equivalence relation R is effective.*
(2) *There exists a morphism $f : X \to Y$ such that $R = X \times_Y X$ (as a subobject of $X \times X$).*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. For the converse, suppose that $R = X \times_Y X$ for some morphism $f : X \to Y$. Since $\mathcal{C}$ is regular, the morphism f factors as a composition $X \overset{g}{\twoheadrightarrow} \mathrm{Im}(f) \overset{h}{\hookrightarrow} Y$, where g is an effective epimorphism and h is a monomorphism. The desired conclusion follows from the observation that R can also be identified with the fiber product $X \times_{\mathrm{Im}(f)} X$. □

Definition A.2.6. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *exact* if it satisfies the following axioms:

($R1$) The category $\mathcal{C}$ admits finite limits.
($R2'$) Every equivalence relation on an object $X \in \mathcal{C}$ is effective.
($R3$) The collection of effective epimorphisms in $\mathcal{C}$ is closed under pullbacks. That is, if we are given a pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{C}$ where f is an effective epimorphism, the morphism f' is also an effective epimorphism.

Example A.2.7. The category of sets is exact.

Proposition A.2.8. *Let $\mathcal{C}$ be a category. If $\mathcal{C}$ is exact (in the sense of Definition A.2.6), then it is regular (in the sense of Definition A.1.3).*

The proof requires the following elementary observation:

Lemma A.2.9. *Let $\mathcal{C}$ be a category which admits fiber products. Suppose we are given a pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{C}$, where both f and f' are effective epimorphisms. If g' is an isomorphism, then g is also an isomorphism.

Proof. We have a commutative diagram

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X' \arrow[r] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \times_Y X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r] & Y \end{tikzcd}$$

where the rows are coequalizer diagrams and the left and middle vertical maps are isomorphisms. It follows that g is an isomorphism as well. □

Proof of Proposition A.2.8. We must show that $\mathcal{C}$ satisfies axiom $(R2)$ of Definition A.1.3. Let $f : X \to Z$ be a morphism in $\mathcal{C}$, and let $R = X \times_Z X$ be the equivalence relation of Example A.2.2. Since $\mathcal{C}$ is exact, the equivalence relation R is effective. We can therefore choose an effective epimorphism $g : X \twoheadrightarrow Y$ such that R coincides with $X \times_Y X$ (as subobjects of $X \times X$). Our assumption that g is an effective epimorphism guarantees that it exhibits Y as a coequalizer of the diagram $R \rightrightarrows X$. Consequently, there is a unique morphism $h : Y \to Z$ satisfying $h \circ g = f$. To complete the proof, it will suffice to show that h is a monomorphism: that is, that the diagonal map $\delta : Y \to Y \times_Z Y$ is an isomorphism. We have a commutative diagram of pullback squares

$$\begin{tikzcd} X \times_Y X \arrow[r, "\delta'"] \arrow[d] & X \times_Z X \arrow[r] \arrow[d] & X \times X \arrow[d, "g \times g"] \\ Y \arrow[r, "\delta"] & Y \times_Z Y \arrow[r] & Y \times Y. \end{tikzcd}$$

Since g is an effective epimorphism, axiom $(R3)$ of Definition A.2.6 and Proposition A.1.8 guarantee that the vertical maps in the above diagram are effective epimorphisms. Since $X \times_Z X$ and $X \times_Y X$ are both equal to R as subobjects of $X \times X$, the map δ' is an isomorphism. Applying Lemma A.2.9, we conclude that δ is an isomorphism, as desired. $\square$

A.3. **Extensive Categories.**

Definition A.3.1. Let $\mathcal{C}$ be a category which admits fiber products, and let $X, Y \in \mathcal{C}$ be objects which admit a coproduct $X \amalg Y$. We will say that $X \amalg Y$ is a *disjoint coproduct* of X and Y if the following pair of conditions is satisfied:

- Each of the maps $X \to (X \amalg Y) \leftarrow Y$ is a monomorphism.
- The fiber product $X \times_{X \amalg Y} Y$ is an initial object of $\mathcal{C}$.

Definition A.3.2. Let $\mathcal{C}$ be a category which admits finite limits. We will say that $\mathcal{C}$ is *extensive* if it satisfies the following conditions:

$(E1)$ The category $\mathcal{C}$ has finite coproducts, and coproducts in $\mathcal{C}$ are disjoint.

$(E2)$ The formation of finite coproducts in $\mathcal{C}$ is preserved by pullbacks. More precisely, for every morphism $f : X \to Y$ in $\mathcal{C}$, the pullback functor

$$f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X} \qquad f^*(U) = U \times_Y X$$

preserves finite coproducts.

Remark A.3.3. It is possible to define the notion of extensive category without assuming that $\mathcal{C}$ admits fiber products; in this case, condition $(E2)$ needs to be reformulated. For details, we refer the reader to [5].

Example A.3.4. The category of sets is extensive.

Remark A.3.5. Let $\mathcal{C}$ be an extensive category which admits fiber products. Then $\mathcal{C}$ has an initial object, which we will denote by $\varnothing$. For any morphism $f : C \to \varnothing$ in $\mathcal{C}$, axiom $(E2)$ of Definition A.3.2 guarantees that the pullback functor $f^* : \mathcal{C}_{/\varnothing} \to \mathcal{C}_{/C}$ preserves initial objects. In particular, the pullback $f^*(\varnothing) \simeq C$ is an initial object of $\mathcal{C}_{/C}$, so that C is an initial object of $\mathcal{C}$ (and f is automatically an isomorphism).

Remark A.3.6. Let $\mathcal{C}$ be an extensive category which admits fiber products, with initial object $\varnothing$. For every object $C \in \mathcal{C}$, there is a unique morphism $f : \varnothing \to C$ in $\mathcal{C}$. It follows from Remark A.3.5 that the fiber product $\varnothing \times_C \varnothing$ is also an initial object of $\mathcal{C}$, so that the relative diagonal map $\delta : \varnothing \to \varnothing \times_C \varnothing$ is an isomorphism. It follows that f is a monomorphism in $\mathcal{C}$. That is, the initial object $\varnothing$ can be regarded as a subobject of any other object $C \in \mathcal{C}$ (which is then a least element of the partially ordered set $\mathrm{Sub}(C)$).

We now collect some facts about extensive categories which will be useful in the body of this paper.

Proposition A.3.7. *Let $\mathcal{C}$ be an extensive category which admits fiber products. Then, for every pair of morphisms $f : C \to D$ and $f : C' \to D'$ in $\mathcal{C}$, the diagram*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & C \amalg C' \arrow[d, "f \amalg f'"] \\ D \arrow[r] & D \amalg D' \end{tikzcd}$$

is a pullback square.

Proof. Using axiom $(E2)$ of Definition A.3.2, we can identify the fiber product $(C \amalg C') \times_{(D \amalg D')} D$ with the coproduct of objects

$$X = C \times_{(D \amalg D')} D \simeq C \times_D (D \times_{(D \amalg D')} D)$$
$$Y = C' \times_{(D \amalg D')} D \simeq C' \times_{D'} (D' \times_{(D \amalg D')} D).$$

Since coproducts in $\mathcal{C}$ are disjoint, the canonical map $D \to D \amalg D'$ is a monomorphism, so the map $C \to X$ is an isomorphism. It will therefore suffice to show that Y is an initial object of $\mathcal{C}$. This follows from Remark A.3.6, since $(D' \times_{D \amalg D'} D$ is an initial object of $\mathcal{C}$. □

Proposition A.3.8. *Let $\mathcal{C}$ be an extensive category which admits fiber products and let $f : C \twoheadrightarrow D$ and $f' : C' \twoheadrightarrow D'$ be effective epimorphisms in $\mathcal{C}$. Then the induced map $(f \amalg f') : C \amalg C' \to D \amalg D'$ is also an effective epimorphism in $\mathcal{C}$.*

Proof. Set $R = C \times_D C$ and $R' = C' \times_{D'} C'$. Using Proposition A.3.7, we see that the canonical maps

$$R \to D \times_{D \amalg D'} ((C \amalg C') \times_{D \amalg D'} (C \amalg C'))$$
$$R' \to D' \times_{D \amalg D'} ((C \amalg C') \times_{D \amalg D'} (C \amalg C'))$$

are isomorphisms. Combining this with axiom $(E2)$, we obtain an isomorphism

$$R \amalg R' \to (C \amalg C') \times_{D \amalg D'} (C \amalg C').$$

Consequently, to show that $f \amalg f'$ is an effective epimorphism, it will suffice to show that the diagram

$$(R \amalg R') \rightrightarrows (C \amalg C') \to (D \amalg D')$$

is a coequalizer. This is clear, since f and f' are effective epimorphisms and the collection of coequalizer diagrams is closed under the formation of coproducts. □

Proposition A.3.9. *Let $\mathcal{C}$ be an extensive category which admits finite limits and let* Fin *denote the category of finite sets. Then there is an essentially unique functor $F : \mathrm{Fin} \to \mathcal{C}$ which preserves finite coproducts and finite limits, given on objects by the formula $F(S) = \coprod_{s \in S} \mathbf{1}$, where $\mathbf{1}$ is the final object of $\mathcal{C}$. More precisely, if we let* $\mathrm{Fun}^{\mathrm{lex},\amalg}(\mathrm{Fin}, \mathcal{C})$ *denote the full subcategory of* $\mathrm{Fun}(\mathrm{Fin}, \mathcal{C})$ *spanned by those functors which preserve finite limits and finite coproducts, then* $\mathrm{Fun}^{\mathrm{lex},\amalg}(\mathrm{Fin}, \mathcal{C})$ *is equivalent to the category $*$ having a single object and a single morphism.*

Proof. Let $\mathrm{Fun}^{\amalg}(\mathrm{Fin}, \mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathrm{Fin}, \mathcal{C})$ spanned by those functors which preserve finite coproducts. Note that a functor $F : \mathrm{Fin} \to \mathcal{C}$ belongs to $\mathrm{Fun}^{\amalg}(\mathrm{Fin}, \mathcal{C})$ if and only if it is a left Kan extension of its restriction to the full subcategory of $\mathcal{C}$ spanned by the final object $\mathbf{1}$. It follows that the construction $F \mapsto F(\mathbf{1})$ induces an equivalence $\mathrm{Fun}^{\amalg}(\mathrm{Fin}, \mathcal{C}) \to \mathcal{C}$. In particular, if we let $\mathrm{Fun}'(\mathrm{Fin}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Fin}, \mathcal{C})$ spanned by those functors which preserve finite coproducts and final objects, then $\mathrm{Fun}'(\mathrm{Fin}, \mathcal{C})$ is equivalent to the full subcategory $\{\mathbf{1}\} \subseteq \mathcal{C}$. In particular, the category $\mathrm{Fun}'(\mathrm{Fin}, \mathcal{C})$ contains an essentially unique functor F, given on objects by $F(S) = \coprod_{s \in S} \mathbf{1}$. To complete the proof, it will suffice to show that the functor F is left exact. Since F preserves final objects, it is sufficient to show that it preserves fiber products. Suppose we are given maps of finite sets $S_0 \to S \leftarrow S_1$; we wish to show that the canonical map

$$\theta : F(S_0 \times_S S_1) \to F(S_0) \times_{F(S)} F(S_1)$$

is an equivalence. Note that, as functors of S_0, both the domain and codomain of θ commute with finite coproducts. We may therefore assume without loss of generality that S_0 has a single element, having image

$s \in S$. Similarly, we may assume that S_1 has a single element having image $s' \in S$. We now consider two cases:

- If $s \neq s'$, then the fiber product $S_0 \times_S S_1$ is empty. It follows that the domain of θ is an initial object of $\mathcal{C}$. It will therefore suffice to show that the codomain of θ is also an initial object of $\mathcal{C}$. Applying Remark A.3.5 to the map
$$F(S_0) \times_{F(S)} F(S_1) \to F(\{s\}) \times_{F(S)} F(S \smallsetminus \{s\}),$$
we are reduced to showing that $F(\{s\}) \times_{F(S)} F(S \smallsetminus \{s\})$ is an initial object of $\mathcal{C}$, which follows from the disjointness of coproducts in $\mathcal{C}$.
- If $s = s'$, then we can identify θ with the relative diagonal of the morphism $u : F(\{s\}) \to F(S)$. Consequently, to show that θ is an isomorphism, it will suffice to show that u is a monomorphism. This follows from the disjointness of coproducts in $\mathcal{C}$, since u is the inclusion of a summand.

□

Proposition A.3.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be extensive categories which admit finite limits, and let $G : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits. The following conditions are equivalent:*

(1) *The functor G preserves finite coproducts.*
(2) *Let $F : \mathrm{Fin} \to \mathcal{C}$ be as in Proposition A.3.9. Then the composition $G \circ F$ preserves finite coproducts.*
(3) *The functor G preserves initial objects, and the canonical map $G(\mathbf{1}) \amalg G(\mathbf{1}) \to G(\mathbf{1} \amalg \mathbf{1})$ is an equivalence in $\mathcal{D}$ (here $\mathbf{1}$ denotes a final object of $\mathcal{C}$).*

Proof. The implications (1) ⇒ (2) ⇒ (3) are immediate. Assume that G satisfies (3); we wish to show that for every pair of objects $C, C' \in \mathcal{C}$, the canonical map $\theta : G(C) \amalg G(C') \to G(C \amalg C')$ is an equivalence. Choose maps $C \to \mathbf{1}$ and $C' \to \mathbf{1}'$, where $\mathbf{1}$ and $\mathbf{1}'$ are final objects of $\mathcal{C}$ (which we can take to be the same, but will distinguish notationally for the sake of clarity). We have a commutative diagram

$$\begin{tikzcd}
G(C) \amalg G(C') \arrow[r] \arrow[d, "\theta'"] & G(\mathbf{1}) \amalg G(\mathbf{1}') \arrow[d] \\
(G(\mathbf{1}) \times_{G(\mathbf{1} \amalg \mathbf{1}')} G(C \amalg C')) \amalg (G(\mathbf{1}') \times_{G(\mathbf{1} \amalg \mathbf{1}')} G(C \amalg C')) \arrow[r] \arrow[d, "\theta''"] & G(\mathbf{1}) \amalg G(\mathbf{1}') \arrow[d] \\
G(C \amalg C') \arrow[r] & G(\mathbf{1} \amalg \mathbf{1}')
\end{tikzcd}$$

and we wish to show that the left vertical composition $\theta = \theta'' \circ \theta'$ is an isomorphism. Note that the bottom square of this diagram is a pullback (since $\mathcal{D}$ is extensive) and the right vertical maps are isomorphisms (by virtue of assumption (3)), so the map θ'' is an isomorphism. It will therefore suffice to show that θ' is an isomorphism. Using our assumption that θ is left exact, we are reduced to showing that both squares in the diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & C \amalg C' \arrow[d] & C' \arrow[l] \arrow[d] \\
\mathbf{1} \arrow[r] & \mathbf{1} \amalg \mathbf{1}' & \mathbf{1}' \arrow[l]
\end{tikzcd}$$

are pullbacks, which follows from Proposition A.3.7. □

A.4. **Pretopoi and Models.**

Definition A.4.1. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a *pretopos* if it is exact and extensive.

Example A.4.2. The category of sets is a pretopos (combine Example A.2.7 with Example A.3.4).

Example A.4.3. Let $\mathcal{C}$ be a pretopos and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under the formation of finite limits and finite coproducts, and satisfies the following further condition:

($*$) For every object $X \in \mathcal{C}_0$ and every equivalence relation $R \subseteq X \times X$ which belongs to $\mathcal{C}_0$, the quotient X/R also belongs to $\mathcal{C}_0$.

Then $\mathcal{C}_0$ is a pretopos.

Example A.4.4. The category Fin of finite sets is a pretopos. This follows from Examples A.4.2 and A.4.3.

Definition A.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be pretopoi. We will say that a functor $F : \mathcal{C} \to \mathcal{D}$ is a *pretopos functor* if F preserves finite limits, finite coproducts, and carries effective epimorphisms in $\mathcal{C}$ to effective epimorphisms in $\mathcal{D}$. We let $\mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the pretopos functors.

If $\mathcal{C}$ is a pretopos, then a *model of* $\mathcal{C}$ is a pretopos functor $M : \mathcal{C} \to \mathrm{Set}$. We let $\mathrm{Mod}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ spanned by the models of $\mathcal{C}$.

A.5. **Distributive Lattices.**

Definition A.5.1. Let L be a partially ordered set. We say that L is a *lattice* if every finite subset of L has a least upper bound and a greatest lower bound. Equivalently, L is a lattice if it has a least element $0 \in L$, a largest element $1 \in L$, and every pair of elements $x, y \in L$ have a least upper bound $x \vee y$ (called the *join* of x and y) and a greatest lower bound $x \wedge y$ (called the *meet* of x and y).

We say that a lattice L is *distributive* if, for every triple of elements $x, y, z \in L$, we have the distributive law

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Example A.5.2. Let $\mathcal{C}$ be a regular extensive category. Then, for every object $X \in \mathcal{C}$, the partially ordered set $\mathrm{Sub}(X)$ of subobjects of X is a distributive lattice. The largest element of $\mathrm{Sub}(X)$ is the object X, the least element is the initial object $\varnothing$ of $\mathcal{C}$. Given a pair of subobjects $X_0, X_1 \subseteq X$, their join and meet are given by the formulae

$$X_0 \vee X_1 = \mathrm{Im}(X_0 \amalg X_1 \to X) \qquad X_0 \wedge X_1 = X_0 \times_X X_1.$$

The distributive law follows from the assumption that coproducts and images in $\mathcal{C}$ are compatible with the formation of pullbacks.

Definition A.5.3. . Let L and L' be lattices. A *lattice homomorphism* from L to L' is a function $\mu : L \to L'$ which preserves least upper bounds and greatest lower bounds of finite subsets: that is, μ satisfies the identities

$$\mu(0) = 0 \qquad \mu(x \vee y) = \mu(x) \vee \mu(y)$$
$$\mu(1) = 1 \qquad \mu(x \wedge y) = \mu(x) \wedge \mu(y).$$

If L is a distributive lattice, we let $\mathrm{Spec}(L)$ denote the set of all lattice homomorphisms $\mu : L \to \{0 < 1\}$. We refer to $\mathrm{Spec}(L)$ as the *spectrum of* L.

Example A.5.4. Every Boolean algebra B is a distributive lattice. Moreover, the spectrum of B as a Boolean algebra coincides with its spectrum as a distributive lattice.

Remark A.5.5. Let L be a distributive lattice. Then the spectrum $\mathrm{Spec}(L)$ can be equipped with a topology, having a basis of open sets of the form

$$U_x = \{\mu \in \mathrm{Spec}(L) : \mu(x) = 1\}$$

where x ranges over the elements of L. Moreover, the construction $x \mapsto U_x$ induces an isomorphism of distributive lattices

$$L \to \{\text{Quasi-compact open subsets } U \subseteq \mathrm{Spec}(L)\}.$$

Remark A.5.6. The construction $L \mapsto \mathrm{Spec}(L)$ is functorial: for every homomorphism of distributive lattices $\lambda : L \to L'$, composition with λ induces a continuous map of topological spaces $\mathrm{Spec}(L) \to \mathrm{Spec}(L')$. Moreover, formation of the spectrum is compatible with filtered colimits. If L is a distributive lattice which is given as the colimit of a filtered diagram $\{L_\alpha\}$, then the induced map

$$\mathrm{Spec}(L) \to \varprojlim \mathrm{Spec}(L_\alpha)$$

is a homeomorphism of topological spaces.

We will need the following:

Proposition A.5.7. *Let $\lambda : L \to L'$ be a homomorphism of distributive lattices. The following conditions are equivalent:*

(a) *The homomorphism λ is injective.*
(b) *The induced map of topological spaces $\mathrm{Spec}(L') \to \mathrm{Spec}(L)$ is surjective.*

Proof. Suppose first that (b) is satisfied, and let $x, y \in L$ satisfy $\lambda(x) = \lambda(y)$. We wish to show that $x = y$. We will prove this under the assumption that $x \leq y$ (the general case then follows by applying this argument with the pair (x, y) replaced by $(x \wedge y, x)$ and $(x \wedge y, y)$). Assume, for a contradiction, that $x \neq y$. Let $\mathcal{P}$ denote the collection of all subsets $I \subseteq L$ which are closed downward, closed under joins, and contain the element x, but do not contain y. Then $\mathcal{P}$ is nonempty (it contains the subset $L_{\leq x} = \{z \in L : z \leq x\}$). Applying Zorn's lemma, we conclude that $\mathcal{P}$ contains a maximal element $\mathfrak{p} \subseteq L$. Define $\mu : L \to \{0 < 1\}$ by the formula

$$\mu(z) = \begin{cases} 0 & \text{if } z \in \mathfrak{p} \\ 1 & \text{otherwise.} \end{cases}$$

We claim that μ is a lattice homomorphism. The compatibility of μ with the formation of joins follows from our assumption that $\mathfrak{p}$ is closed under joins, and the formula $\mu(1) = 1$ follows from our assumption that $y \notin \mathfrak{p}$ (so also $1 \notin \mathfrak{p}$, since $\mathfrak{p}$ is closed downward). To complete the proof, it will suffice to show that $\mu(z \wedge z') = \mu(z) \wedge \mu(z')$. Equivalently, we must show that if $z \wedge z'$ belongs to $\mathfrak{p}$, then either z or z' belongs to $\mathfrak{p}$. Assume otherwise. Then the sets

$$I = \{w \in L : (\exists v \in \mathfrak{p}) w \leq v \vee z\} \qquad I' = \{w' \in L : (\exists v' \in \mathfrak{p}) w' \leq v' \vee z'\}$$

are subsets of L which are downward closed, closed under joins, and properly contain $\mathfrak{p}$. It follows from the maximality of $\mathfrak{p}$ that both I and I' must contain y. That is, we have $y \leq v \vee z$ and $y \leq v' \vee z'$ for some $v, v' \in \mathfrak{p}$. Applying the distributive law, we obtain

$$y \leq (v \vee z) \wedge (v' \vee z') = (v \wedge v') \vee (v \wedge z') \vee (z \wedge v') \vee (z \wedge z') \in \mathfrak{p},$$

which is a contradiction. This completes the proof that μ is a lattice homomorphism. If (b) is satisfied, then we can write $\mu = \mu' \circ \lambda$, for some lattice homomorphism $\mu' : L' \to \{0 < 1\}$. The identity $\lambda(x) = \lambda(y)$ then shows that $\mu(x) = \mu(y)$, which is a contradiction.

We now show that (a) implies (b). Assume that λ is injective, and let $\mu : L \to \{0 < 1\}$ be a lattice homomorphism. Let $\mathcal{Q}$ denote the collection of all subsets $I \subseteq L'$ which are closed downward, closed under joins, contain $\lambda(x)$ for each element $x \in L$ satisfying $\mu(x) = 0$, and do not contain $\lambda(x)$ for elements $x \in L$ satisfying $\mu(x) = 1$. Our assumption that λ is injective guarantees that the set

$$\{y \in L' : (\exists x \in L)[\mu(x) = 0 \text{ and } y \leq \lambda(x)]\}$$

belongs to $\mathcal{Q}$. In particular, $\mathcal{Q}$ is nonempty. We can therefore choose a maximal element $\mathfrak{q} \in \mathcal{Q}$. Define $\mu' : L' \to \{0 < 1\}$ by the formula

$$\mu'(y) = \begin{cases} 0 & \text{if } y \in \mathfrak{q} \\ 1 & \text{otherwise.} \end{cases}$$

By construction, we have $\mu = \mu' \circ \lambda$. We will complete the proof by showing that μ' is a lattice homomorphism, so that μ is the image of μ' under the map $\mathrm{Spec}(L') \xrightarrow{\circ\lambda} \mathrm{Spec}(L)$. Arguing as above, we are reduced to showing that if we are given a pair of elements $y, y' \in L'$ such that $y \wedge y'$ belongs to $\mathfrak{q}$, then either y or y' belongs to $\mathfrak{q}$. Assume otherwise: then the sets

$$J = \{w \in L' : (\exists v \in \mathfrak{q}) w \leq v \vee y\} \qquad J' = \{w' \in L' : (\exists v' \in \mathfrak{q}) w' \leq v' \vee y'\}$$

are downward closed, closed under joins, and properly contain $\mathfrak{q}$. It follows from the maximality of $\mathfrak{q}$ that J must contain an element of the form $\lambda(x)$ where $\mu(x) = 1$, and J' must contain an element of the form

$\lambda(x')$ where $\mu(x') = 1$. We therefore have $\lambda(x) \leq v \vee y$ and $\lambda(x') \wedge v' \vee y'$, for some elements $v, v' \in \mathfrak{q}$. Using the distributive law and the fact that λ is a lattice homomorphism, we obtain

$$\begin{aligned} \lambda(x \wedge x') &= \lambda(x) \wedge \lambda(x') \\ &\leq (v \vee y) \wedge (v \vee y') \\ &= (v \wedge v') \vee (v \wedge y') \vee (y \wedge v') \vee (y \wedge y') \\ &\in \mathfrak{q}. \end{aligned}$$

Since $\mu(x \wedge x') = \mu(x) \wedge \mu(x') = 1$, this contradicts our assumption that $\mathfrak{q}$ belongs to $\mathcal{Q}$. □

Appendix B. Sheaf Theory

In this section, we review the theory of Grothendieck topologies and the associated sheaf theory.

B.1. **Grothendieck Topologies.**

Definition B.1.1. Let $\mathcal{C}$ be a category. A *sieve on* $\mathcal{C}$ is a full subcategory $\mathcal{C}^{(0)} \subseteq \mathcal{C}$ with the following property: for every morphism $f : C' \to C$ in $\mathcal{C}$, if C belongs to $\mathcal{C}^{(0)}$, then C' also belongs to $\mathcal{C}^{(0)}$. If $C \in \mathcal{C}$ is an object, then a *sieve on* C is a sieve on the overcategory $\mathcal{C}_{/C}$.

Let $f : D \to C$ be a morphism in the category $\mathcal{C}$. If $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ is a sieve on the object C, then we let $f^* \mathcal{C}^{(0)}_{/C}$ denote the sieve on D consisting of those morphisms $g : E \to D$ such that the composite morphism $(f \circ g) : E \to C$ belongs to $\mathcal{C}^{(0)}_{/C}$. We will refer to $f^* \mathcal{C}^{(0)}_{/C}$ as the *pullback* of the sieve $\mathcal{C}^{(0)}_{/C}$.

Remark B.1.2. Let $\mathcal{C}$ be a category containing an object C and let $\{f_i : C_i \to C\}_{i \in I}$ be a collection of morphisms having codomain C. Then there is a smallest sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ which contains each f_i: namely, the full subcategory of $\mathcal{C}_{/C}$ spanned by those morphisms $g : D \to C$ which factor as a composition $D \to C_i \xrightarrow{f_i} C$, for some $i \in I$. We will refer to $\mathcal{C}^{(0)}_{/C}$ as the *sieve generated by the morphisms* f_i. We say that a sieve $\mathcal{C}^{(0)}_{/C}$ is *finitely generated* if it is generated by a finite collection of morphisms in $\mathcal{C}$.

Definition B.1.3. Let $\mathcal{C}$ be a category. A *Grothendieck topology on* $\mathcal{C}$ is a procedure which assigns to each object $C \in \mathcal{C}$ a collection of sieves on C, which we refer to as *covering sieves*. This assignment is required to have the following properties:

- $(T1)$ For each object $C \in \mathcal{C}$, the sieve $\mathcal{C}_{/C}$ is a covering sieve on C.
- $(T2)$ For each morphism $f : D \to C$ in $\mathcal{C}$ and each covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ on C, the pullback $f^* \mathcal{C}^{(0)}_{/C}$ is a covering sieve on D.
- $(T3)$ Let $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ be a covering sieve on an object $C \in \mathcal{C}$ and let $\mathcal{C}^{(1)}_{/C}$ be another sieve having the property that for every morphism $f : D \to C$ belonging to $\mathcal{C}^{(0)}_{/C}$, the pullback $f^* \mathcal{C}^{(1)}_{/C}$ is a covering sieve on D. Then $\mathcal{C}^{(1)}_{/C}$ is a covering sieve on C.

If $\mathcal{C}$ is equipped with a Grothendieck topology, then we will say that a collection of morphisms $\{f_i : C_i \to C\}$ is a *covering* if it generates a covering sieve on C (see Remark B.1.2).

Example B.1.4. Let X be a topological space and let $\mathcal{U}(X)$ denote the partially ordered set of all open subsets of X, regarded as a category. Then we can equip $\mathcal{U}(X)$ with a Grothendieck topology, where a collection of morphisms $\{U_i \subseteq U\}_{i \in I}$ is a covering of an object $U \in \mathcal{U}(X)$ if $U = \bigcup_{i \in I} U_i$.

Example B.1.5. Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. We will say that a sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ is a *$\mathscr{F}$-covering* if, for every morphism $f : D \to C$ in $\mathcal{C}$, the canonical map

$$\mathscr{F}(D) \to \varprojlim_{E \in (f^* \mathcal{C}^{(0)}_{/C})} \mathscr{F}(E)$$

is bijective. The collection of $\mathscr{F}$-covering sieves determines a Grothendieck topology on $\mathcal{C}$: axioms $(T1)$ and $(T2)$ are immediate from the definition, and axiom $(T3)$ follows by an easy interchange of limits argument.

B.2. **Sheaves for a Grothendieck Topology.** Let $\mathcal{C}$ be a category equipped with a Grothendieck topology. We say that a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a *sheaf* if it satisfies the following condition: for each object $C \in \mathcal{C}$ and each covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$, the canonical map

$$\mathscr{F}(C) \to \varprojlim_{D \in (\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}} \mathscr{F}(D)$$

is a bijection.

Notation B.2.1. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology. We let $\mathrm{Shv}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors which are sheaves on $\mathcal{C}$.

Example B.2.2. Let X be a topological space and let $\mathcal{U}(X)$ denote the partially ordered set of open subsets of X. We let $\mathrm{Shv}(X)$ denote the category $\mathrm{Shv}(\mathcal{U}(X))$, where $\mathcal{U}(X)$ is endowed with the Grothendieck topology of Example B.1.4. We will refer to objects of $\mathrm{Shv}(X)$ as *sheaves on* X.

Remark B.2.3. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. Then $\mathscr{F}$ is a sheaf if and only if every covering sieve (for our Grothendieck topology on $\mathcal{C}$) is a $\mathscr{F}$-covering sieve (in the sense of Example B.1.5). In other words, the Grothendieck topology of Example B.1.5 is the *finest* Grothendieck on $\mathcal{C}$ with respect to which $\mathscr{F}$ is sheaf.

The following result is standard (see [11] for a textbook account):

Proposition B.2.4. *Let $\mathcal{C}$ be a small category equipped with a Grothendieck topology. Then the inclusion functor* $\mathrm{Shv}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ *admits a left adjoint* $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Shv}(\mathcal{C})$. *Moreover, the functor L preserves finite limits.*

We refer to the functor $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Shv}(\mathcal{C})$ as the *sheafification functor*.

Warning B.2.5. In §7, we will study sheaves on the category $\mathrm{Pro}(\mathcal{C})$ of pro-objects of a (small) pretopos $\mathcal{C}$. The category $\mathrm{Pro}(\mathcal{C})$ is not small, and Proposition B.2.4 does not apply to this situation. In order to sheafify a presheaf $\mathscr{F} : \mathrm{Pro}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Set}$ taking values in the category of small sets, one needs to allow sheaves which take non-small values. For us, this is irrelevant; we will have no need to sheafify any set-valued presheaves on $\mathrm{Pro}(\mathcal{C})$.

Definition B.2.6. Let $\mathcal{C}$ be a category. For each object $C \in \mathcal{C}$, we let $h_C : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ denote the functor represented by C, given on objects by the formula $h_C(D) = \mathrm{Hom}_{\mathcal{C}}(D, C)$. We say that a Grothendieck topology on $\mathcal{C}$ is *subcanonical* if each of the functors h_C is a sheaf. In this case, the construction $C \mapsto h_C$ determines a fully faithful embedding $h : \mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C})$, which we will refer to as *the Yoneda embedding*.

Remark B.2.7 (Sheafified Yoneda Embedding)**.** Let $\mathcal{C}$ be a small category equipped with a Grothendieck topology. For each object $C \in \mathcal{C}$, we let $\widetilde{h}_C$ denote the sheafification of the representable functor $h_C = \mathrm{Hom}_{\mathcal{C}}(\bullet, C)$. The functor $C \mapsto \widetilde{h}_C$ then determines a functor $\widetilde{h} : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$, which we will refer to as the *sheafified Yoneda embedding*. Beware that this terminology is somewhat misleading: if the Grothendieck topology on $\mathcal{C}$ is not subcanonical, then the functor $\widetilde{h} : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ is not fully faithful.

B.3. **Example: The Regular Topology.**

Definition B.3.1. Let $\mathcal{C}$ be a regular category (Definition A.1.3), and let $\mathcal{C}^{(0)}_{/C}$ be a sieve on an object $C \in \mathcal{C}$. We say that $\mathcal{C}^{(0)}_{/C}$ is a *regular covering sieve* if it contains an effective epimorphism $D \twoheadrightarrow C$.

Proposition B.3.2. *Let $\mathcal{C}$ be a regular category. Then the collection of regular covering sieves determines a Grothendieck topology on $\mathcal{C}$.*

Proof. Axiom $(T1)$ of Definition B.1.3 follows from the fact that each identity map $\mathrm{id}_C : C \to C$ is an effective epimorphism in $\mathcal{C}$, and axiom $(T2)$ from fact that effective epimorphisms in $\mathcal{C}$ are stable under pullback (axiom $(R3)$ of Definition A.1.3). Axiom $(T3)$ follows from the fact that the collection of effective epimorphisms in $\mathcal{C}$ is closed under composition (Proposition A.1.8). □

Definition B.3.3. Let $\mathcal{C}$ be a regular category. We will refer to the Grothendieck topology of Proposition B.3.2 as the *regular topology* on $\mathcal{C}$.

Remark B.3.4. Let $\mathcal{C}$ be a regular category. Then a collection of morphisms $\{f_i : C_i \to C\}_{i \in I}$ is a covering for the regular topology if and only if some $f_i : C_i \to C$ is an effective epimorphism. In particular, the regular topology on $\mathcal{C}$ is the *coarsest* Grothendieck topology for which effective epimorphism $D \twoheadrightarrow C$ generates a covering sieve.

Proposition B.3.5. *Let $\mathcal{C}$ be a regular category. Then a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf with respect to the regular topology if and only if it satisfies the following condition:*

$(*)$ *For every effective epimorphism $D \twoheadrightarrow C$ in $\mathcal{C}$, the diagram of sets*
$$\mathscr{F}(C) \to \mathscr{F}(D) \rightrightarrows \mathscr{F}(D \times_C D)$$
is an equalizer.

Corollary B.3.6. *Let $\mathcal{C}$ be a regular category. Then the regular topology on $\mathcal{C}$ is subcanonical. That is, for every object $E \in \mathcal{C}$, the representable functor $h_E(\bullet) = \mathrm{Hom}_{\mathcal{C}}(\bullet, E)$ is a sheaf for the regular topology.*

Proof. By virtue of Proposition B.3.5, it suffices to verify that for every effective epimorphism $D \twoheadrightarrow C$ in $\mathcal{C}$, the diagram of sets
$$\mathrm{Hom}_{\mathcal{C}}(C, E) \to \mathrm{Hom}_{\mathcal{C}}(D, E) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(D \times_C D, E)$$
is an equalizer. This is precisely the definition of an effective epimorphism. □

Proposition B.3.5 is a consequence of the following:

Lemma B.3.7. *Let $\mathcal{C}$ be a category which admits fiber products, let $f : D \to C$ be a morphism in $\mathcal{C}$, and let $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ be the sieve generated by f. Then the canonical map*
$$\varprojlim_{E \in \mathcal{C}^{(0),\mathrm{op}}_{/C}} \mathscr{F}(E) \to \mathrm{Eq}(\mathscr{F}(D) \rightrightarrows \mathscr{F}(D \times_C D))$$
is a bijection.

Proof. Fix an element $\eta \in \mathscr{F}(D)$. Unwinding the definitions, we see that η can be lifted to an element of the inverse limit $\varprojlim_{E \in \mathcal{C}^{(0),\mathrm{op}}_{/C}} \mathscr{F}(E)$ if and only if, for every pair of maps $u, v : E \to D$ in $\mathcal{C}$ satisfying $f \circ u = f \circ g$, the induced maps $\mathscr{F}(u), \mathscr{F}(v) : \mathscr{F}(D) \to \mathscr{F}(E)$ carry η to the same point of $\mathscr{F}(E)$ (moreover, the lifting is unique if it exists). To verify this condition, it suffices to treat the universal case where $E = D \times_C D$ and $u, v : E \to D$ are the projection maps. □

Proof of Proposition B.3.5. Let $\mathcal{C}$ be a regular category and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. Then $\mathscr{F}$ is a sheaf for the regular topology if and only if every regular covering sieve is a $\mathscr{F}$-covering sieve, in the sense of Example B.1.5 (Remark B.2.3). By virtue of Remark B.3.4, this is equivalent to the requirement that every effective epimorphism $D \twoheadrightarrow C$ in $\mathcal{C}$ generates a $\mathscr{F}$-covering sieve. According to Lemma B.3.7, this holds if and only if $\mathscr{F}$ satisfies condition $(*)$ of Proposition B.3.5. □

B.4. **Example: The Extensive Topology.**

Definition B.4.1. Let $\mathcal{C}$ be an extensive category which admits pullbacks, and let $\mathcal{C}^{(0)}_{/C}$ be a sieve on an object $C \in \mathcal{C}$. We say that $\mathcal{C}^{(0)}_{/C}$ is a *extensive covering sieve* if it contains a finite collection of maps $\{C_i \to C\}_{i \in I}$ which exhibit C as a coproduct of the collection $\{C_i\}_{i \in I}$.

Proposition B.4.2. *Let $\mathcal{C}$ be an extensive category which admits pullbacks. Then the collection of extensive covering sieves determines a Grothendieck topology on $\mathcal{C}$.*

Proof. Axioms $(T1)$ and $(T3)$ of Definition B.1.3 follow immediately from the definition, and $(T2)$ follows from axiom $(E2)$ of Definition B.4.3. □

Definition B.4.3. Let $\mathcal{C}$ be an extensive category which admits pullbacks. We will refer to the Grothendieck topology of Proposition B.4.2 as the *extensive topology* on $\mathcal{C}$.

Remark B.4.4. Let $\mathcal{C}$ be an extenstive category. Then a collection of morphisms $\{f_i : D_i \to C\}_{i \in I}$ is a covering for the extensive topology if and only C can be written as a finite coproduct $\coprod_{j \in J} C_j$, where each of the inclusion maps $C_j \hookrightarrow C$ factors through $f_i : D_i \to C$ for some i (which might depend on j).

In particular, the extensive topology on $\mathcal{C}$ is the *coarsest* topology having the property that, for every finite collection of objects $\{C_j\}_{j \in J}$ with coproduct $C = \coprod_{j \in J} C_j$, the inclusion maps $\{C_j \hookrightarrow C\}_{j \in J}$ comprise a covering.

Proposition B.4.5. *Let $\mathcal{C}$ be an extensive category which admits pullbacks. Then a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf with respect to the extensive topology if and only if it preserves finite products (that is, it carries finite coproducts in $\mathcal{C}$ to products of sets).*

Corollary B.4.6. *Let $\mathcal{C}$ be an extensive category which admits pullbacks. Then the extensive topology on $\mathcal{C}$ is subcanonical.*

We will deduce Proposition B.4.5 from the following:

Lemma B.4.7. *Let $\mathcal{C}$ be an extensive category which admits pullbacks and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor having the property that $\mathscr{F}(\varnothing)$ is a singleton (where $\varnothing$ denotes an initial object of $\mathcal{C}$). Let $\{C_i\}_{i \in I}$ be a finite collection of objects of $\mathcal{C}$ having coproduct $C = \coprod_{i \in I} C_i$, and let $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ be the sieve generated by the tautological maps $C_i \to C$. Then the canonical map*

$$\varprojlim_{D \in \mathcal{C}^{(0),\mathrm{op}}_{/C}} \mathscr{F}(D) \to \prod_{i \in I} \mathscr{F}(C_i)$$

is bijective.

Proof. Suppose we are given an element $\eta \in \prod_{i \in I} \mathscr{F}(C_i)$, given by a tuple $\{\eta_i \in \mathscr{F}(C_i)\}_{i \in I}$; we wish to show that η can be lifted uniquely to a point

$$\overline{\eta} = \{\overline{\eta}_D\}_{D \in \mathcal{C}^{(0)}_{/C}} \in \varprojlim_{D \in \mathcal{C}^{(0),\mathrm{op}}_{/C}} \mathscr{F}(D).$$

Uniqueness is clear: for any morphism $f : D \to C$ which belongs to the sieve $\mathcal{C}^{(0)}_{/C}$, we can choose a factorization of f as a composition $D \xrightarrow{f'} C_i \to C$, so that $\overline{\eta}_D$ must be given by the image of η_i under the map $\mathscr{F}(f') : \mathscr{F}(C_i) \to \mathscr{F}(D)$. To prove existence, it will suffice to show that this construction is independent of the factorization chosen. Since coproducts in $\mathcal{C}$ are disjoint, the tautological maps $C_i \to C$ are monomorphisms; consequently, the morphism $f' : D \to C_i$ is uniquely determined once i is fixed. Consequently, it will suffice to show that if we have a commutative diagram σ :

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "f''"] & C_i \arrow[d] \\ C_j \arrow[r] & C, \end{tikzcd}$$

then η_i and η_j have the same image in the set $\mathscr{F}(D)$. Note that the disjointness of coproducts in $\mathcal{C}$ guarantees that the fiber product $C_i \times_C C_j$ is an initial object of $\mathcal{C}$, so that D is also initial (Remark A.3.6). It follows that the set $\mathscr{F}(D)$ is a singleton, so the desired equality is automatic. □

Proof of Proposition B.4.5. Let $\mathcal{C}$ be an extensive category which admits pullbacks and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor; we wish to show that $\mathscr{F}$ is a sheaf for the extensive topology if and only if it preserves finite products. Note that, if $\varnothing$ denotes the initial object of $\mathcal{C}$, then the empty sieve is a covering of $\varnothing$. Consequently, if $\mathscr{F}$ is a sheaf for the extensive topology, then the set $\mathscr{F}(\varnothing)$ must be a singleton. Let us henceforth assume that this condition is satisfied. Then $\mathscr{F}$ is a sheaf for the extensive topology if and only

if every extensive covering sieve is a $\mathscr{F}$-covering sieve, in the sense of Example B.1.5 (Remark B.2.3). By virtue of Remark B.4.4, this is equivalent to the requirement that for every finite collection of objects $\{C_i\}_{i\in I}$ having coproduct C, the inclusion maps $\{C_i \hookrightarrow C\}_{i\in I}$ generate a $\mathscr{F}$-covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$. It follows from axiom $(E2)$ of Definition A.3.2 that the formation of this sieve is compatible with pullback, so we only need to check that the canonical map

$$\theta : \mathscr{F}(C) \to \varprojlim_{D \in (\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}} \mathscr{F}(D)$$

is a bijection. Using Lemma B.4.7, we can identify θ with the canonical map $\mathscr{F}(C) \to \prod_{i\in I} \mathscr{F}(C_i)$, so that $\mathscr{F}$ is a sheaf if and only if it commutes with finite products. □

B.5. **Example: The Coherent Topology.** We now combine the constructions of §B.3 and §B.4.

Definition B.5.1. Let $\mathcal{C}$ be a category which is regular and extensive. We will say that a sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ on an object $C \in \mathcal{C}$ is a *coherent covering sieve* if it contains a finite collection of morphisms $\{C_i \to C\}_{i\in I}$ for which the induced map $\coprod_{i\in I} C_i \to C$ is an effective epimorphism in $\mathcal{C}$.

Arguing as in the proofs Proposition B.3.2 and Proposition B.4.2, one obtains the following:

Proposition B.5.2. *Let $\mathcal{C}$ be a category which is regular and extensive. Then the collection of coherent covering sieves determines a Grothendieck topology on $\mathcal{C}$.*

Definition B.5.3. Let $\mathcal{C}$ be a regular extensive category. We will refer to the Grothendieck topology of Proposition B.5.2 as the *coherent topology* on $\mathcal{C}$.

Remark B.5.4. The coherent topology on a category $\mathcal{C}$ can be defined more generally when the category $\mathcal{C}$ is *coherent*; see [7].

Let $\mathcal{C}$ be a regular extensive category. The coherent topology on $\mathcal{C}$ is the coarsest Grothendieck topology which refines both the regular and extensive topologies on $\mathcal{C}$. In particular, a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf for the coherent topology if and only if it is a sheaf for both the regular and extensive topologies. Using Propositions B.3.5 and B.4.5, we obtain the following:

Proposition B.5.5. *Let $\mathcal{C}$ be a regular extensive category. Then a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf with respect to the coherent topology (Definition B.5.3) if and only if satisfies the following pair of conditions:*

1. *The functor $\mathscr{F}$ preserves finite products. That is, for every finite collection of objects $\{C_i\}_{i\in I}$ of $\mathcal{C}$, the canonical map $\mathscr{F}(\coprod_{i\in I} C_i) \to \prod_{i\in I} \mathscr{F}(C_i)$ is bijective.*
2. *For every effective epimorphism $D \twoheadrightarrow C$ in $\mathcal{C}$, the diagram of sets*

$$\mathscr{F}(C) \to \mathscr{F}(D) \rightrightarrows \mathscr{F}(D \times_C D)$$

is an equalizer.

Corollary B.5.6. *Let $\mathcal{C}$ be a regular extensive category. Then the coherent topology on $\mathcal{C}$ is subcanonical. That is, for every object $E \in \mathcal{C}$, the representable functor $h_E(\bullet) = \mathrm{Hom}_{\mathcal{C}}(\bullet, E)$ is a sheaf for the coherent topology.*

Remark B.5.7. Let $\mathcal{C}$ be a small category which is regular and extensive. Then the regular and extensive topologies on $\mathcal{C}$ are compatible in the following sense:

- If $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf for the extensive topology and $\widetilde{\mathscr{F}}$ is the sheafification of $\mathscr{F}$ with respect to the regular topology, then $\widetilde{\mathscr{F}}$ is also a sheaf for the extensive topology (that is, it commutes with finite products), and therefore also for the coherent topology.

In other words, if $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is any functor, then its sheafification for the coherent topology can be computed in two steps: sheafification for the extensive topology, followed by sheafification for the regular topology.

B.6. **Bases.** In practice, it is often useful to describe a sheaf by specifying its value only on a restricted class of objects.

Definition B.6.1. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology. We will say that a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is a *basis for* $\mathcal{C}$ if, for every object $C \in \mathcal{C}$, there exists a covering $\{f_i : D_i \to C\}_{i \in I}$, where the set I is small and each D_i belongs to $\mathcal{D}$.

Example B.6.2. Let X be a topological space and let $\mathcal{U}(X)$ be the partially ordered set of open subsets of X, endowed with the Grothendieck topology of Example B.1.4. Then a full subcategory $\mathcal{U}_0(X) \subseteq \mathcal{U}(X)$ is a basis in the sense of Definition B.6.1 if and only if it is a basis in the usual sense: that is, every open subset of X can be realized as a union of open sets belonging to $\mathcal{U}_0(X)$.

Proposition B.6.3. *Let $\mathcal{C}$ be a category equipped with a Grothendieck topology and let $\mathcal{D} \subseteq \mathcal{C}$ be a basis. Then there is a unique Grothendieck topology on the category $\mathcal{D}$ with the following property: a collection of morphisms $\{D_i \to D\}_{i \in I}$ in $\mathcal{D}$ is a covering if and only if it is a covering in $\mathcal{C}$.*

Proof. Let us say that a sieve $\mathcal{D}^{(0)}_{/D} \subseteq \mathcal{D}_{/D}$ is a *covering* if it contains a collection of morphisms $\{f_i : D_i \to D\}_{i \in I}$ which form a covering in $\mathcal{C}$. Axioms $(T1)$ and $(T3)$ of Definition B.1.3 follow immediately from the definitions (and do not require the assumption that $\mathcal{D}$ is a basis for $\mathcal{C}$). To verify $(T2)$, suppose we are given a covering sieve $\mathcal{D}^{(0)}_{/D}$ on an object $D \in \mathcal{D}$ and a morphism $f : D' \to D$ in the category $\mathcal{D}$; we wish to show that the pullback $f^* \mathcal{D}^{(0)}_{/D} \subseteq \mathcal{D}_{/D'}$ is a covering sieve on D'. Let $\mathcal{C}^{(0)}_{/D} \subseteq \mathcal{C}_{/D}$ denote the sieve generated by $\mathcal{D}^{(0)}_{/D}$, so that the pullback $f^* \mathcal{C}^{(0)}_{/D} \subseteq \mathcal{C}_{/D'}$ is a covering sieve. In other words, there exists a covering $\{g_i : C_i \to D'\}_{i \in I}$ in $\mathcal{C}$ such that each of the composite maps $f \circ g_i : C_i \to D$ belongs to the sieve $\mathcal{C}^{(0)}_{/D}$, which means there exists a commutative diagram

$$\begin{tikzcd} C_i \arrow[r] \arrow[d, "g_i"] & D_i \arrow[d, "g'_i"] \\ D' \arrow[r, "f"] & D \end{tikzcd}$$

where each $g'_i : D_i \to D$ belongs to the sieve $\mathcal{D}^{(0)}_{/D}$. Using our assumption that $\mathcal{D} \subseteq \mathcal{C}$ is a basis, we conclude that each of the objects C_i admits a covering $\{D'_{i,j} \to C_i\}_{j \in J_i}$, where the objects $D'_{i,j}$ belong to $\mathcal{D}$. Then the composite maps $\{D'_{i,j} \to C_i \xrightarrow{g_i} D'\}_{i \in I, j \in J_i}$ comprise a covering of D' by objects of the sieve $f^* \mathcal{D}^{(0)}_{/D}$. □

Proposition B.6.4. *Let $\mathcal{C}$ be a category equipped with a Grothendieck topology and let $\mathcal{D} \subseteq \mathcal{C}$ be a basis. Regard $\mathcal{D}$ as equipped with the Grothendieck topology of Proposition B.6.3. Then precomposition with the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ induces an equivalence of categories*

$$\mathrm{Shv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{D}) \qquad \mathscr{F} \mapsto \mathscr{F}|_{\mathcal{D}^{\mathrm{op}}}.$$

Corollary B.6.5. *Let X be a Stone space, let $\mathcal{U}(X)$ denote the partially ordered set of all open subsets of X, and let $\mathcal{U}_0(X) \subseteq \mathcal{U}(X)$ be the collection of all closed and open subsets of X. Then the construction $\mathscr{F} \mapsto \mathscr{F}|_{\mathcal{U}_0(X)}$ induces a fully faithful embedding*

$$\mathrm{Shv}(X) \subseteq \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}(\mathcal{U}_0(X)^{\mathrm{op}}, \mathrm{Set}),$$

whose essential image is spanned by those functors $\mathscr{F} : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathrm{Set}$ with the following property:

- $(*)$ *For every finite collection of pairwise disjoint closed and open subsets $\{U_i \subseteq X\}_{i \in I}$, the canonical map*
$$\mathscr{F}(\bigcup_{i \in I} U_i) \to \prod_{i \in I} \mathscr{F}(U_i)$$
is bijective.

Proof. Since X is a Stone space, the collection of closed and open subsets of X forms a basis for the topology of X. Applying Proposition B.6.4, we deduce that the restriction functor $\mathrm{Shv}(X) \to \mathrm{Fun}(\mathcal{U}_0(X)^{\mathrm{op}}, \mathrm{Set})$ is a fully faithful embedding whose essential image is spanned by those functors $\mathscr{F} : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathrm{Set}$ which are

sheaves with respect to the topology on $\mathcal{U}_0(X)$ given by open coverings. Note that if U is a closed and open subset of X, then any covering of U by closed and open subsets admits a refinement $\{U_i \subseteq U\}_{i \in I}$ where the index set I is finite and the sets U_i are pairwise disjoint. It follows that the topology of Proposition B.6.3 is the coarsest Grothendieck topology on $\mathcal{U}_0(X)$ for which every such collection $\{U_i \subseteq U\}_{i \in I}$ is a covering. It follows that a functor $\mathscr{F} : \mathcal{U}_0(X)^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf (with respect to the Grothendieck topology of Proposition B.6.3) if and only if, for every collection of pairwise disjoint closed and open subsets $U_i \subseteq X$ with union U, the inclusion maps $\{U_i \subseteq U\}$ are an $\mathscr{F}$-covering (in the sense of Example B.1.5). Unwinding the definitions, this is equivalent to the requirement that for each closed and open subset $V \subseteq U$, the canonical map

$$\mathscr{F}(V) \to \varprojlim_{W \subseteq U_i \cap V} \mathscr{F}(W) \simeq \prod_{i \in I} \mathscr{F}(U_i \cap V)$$

is bijective. To verify this condition, we may assume without loss of generality that $U = V$, in which case it is precisely the criterion of $(*)$. □

Proposition B.6.4 is a consequence of the following more precise assertion:

Proposition B.6.6. *Let $\mathcal{C}$ be a category equipped with a Grothendieck topology, let $\mathcal{D} \subseteq \mathcal{C}$ be a basis, and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. Then $\mathscr{F}$ is a sheaf if and only if it satisfies the following pair of conditions:*

- (a) *The restriction $\mathscr{F}|_{\mathcal{D}^{\mathrm{op}}} : \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf (with respect to the Grothendieck topology of Proposition B.6.3).*
- (b) *The functor $\mathscr{F}$ is a right Kan extension of its restriction $\mathscr{F}|_{\mathcal{D}^{\mathrm{op}}}$.*

Proof of Proposition B.6.4 from Proposition B.6.6. If the category $\mathcal{C}$ is small, then Proposition B.6.4 is an immediate consequence of Proposition B.6.6. However, in §7, we would like to apply Proposition B.6.4 in a situation where $\mathcal{C}$ is not small. In this case, there is a potential technicality: if $\mathscr{F}_0 \in \mathrm{Shv}(\mathcal{D})$ is a sheaf and $\mathscr{F}$ is a right Kan extension of $\mathscr{F}_0$ (given by the formula

$$\mathscr{F}(C) = \varprojlim_{D \in (\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/C})^{\mathrm{op}}} \mathscr{F}_0(D),$$

when $\mathscr{F}$ might potentially fail to be a set-valued sheaf because it values $\mathscr{F}(C)$ might not be small. However, this is impossible: Definition B.6.1 guarantees that there exists a covering $\{f_i : D_i \to C\}_{i \in I}$ of the object C by a *small* collection of objects of $\mathcal{D}$, so that $\mathscr{F}(C)$ can be identified with a subset of $\prod_{i \in I} \mathscr{F}_0(D_i)$ and is therefore small. □

Proof of Proposition B.6.6. Suppose first that $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf; we will show that it satisfies conditions (a) and (b) of Proposition B.6.6. Set $\mathscr{F}_0 = \mathscr{F}|_{\mathcal{D}^{\mathrm{op}}}$ and let $\widehat{\mathscr{F}}$ be a right Kan extension of $\mathscr{F}_0$, given by the formula

$$\widehat{\mathscr{F}}(C) = \varprojlim_{D \in (\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/C})^{\mathrm{op}}} \mathscr{F}_0(D)$$

(if the category $\mathcal{C}$ is not small, then the functor $\widehat{\mathscr{F}}$ might *a priori* take non-small values, but we will see in a moment that this is not the case). We have a canonical map $\theta : \mathscr{F} \to \widehat{\mathscr{F}}$ which is an isomorphism when restricted to $\mathcal{D}$. To prove (b), we must show that θ is an isomorphism. Fix an object $C \in \mathcal{C}$, let $\mathcal{D}_{/C}$ denote the fiber product $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/C}$, and let $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$ denote the sieve generated by $\mathcal{D}_{/C}$. Since $\mathcal{D}$ is a basis for the topology on $\mathcal{C}$, the sieve $\mathcal{C}^{(0)}_{/C}$ is covering. The map $\theta(C)$ fits into a commutative diagram

$$\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "\theta(C)"] \arrow[d] & \widehat{\mathscr{F}}(C) \arrow[d] \\
\varprojlim_{C' \in (\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}} \mathscr{F}(C') \arrow[r, "\theta'"] & \varprojlim_{C' \in (\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}} \widehat{\mathscr{F}}(C'),
\end{tikzcd}$$

where the left vertical map is bijective by virtue of our assumption that $\mathscr{F}$ is a sheaf, and the right vertical map is bijective by virtue of the fact that $\widehat{\mathscr{F}}$ is a right Kan extension of its restriction to $\mathcal{D}$. Since the sieve $\mathcal{C}^{(0)}_{/C}$ is generated by $\mathcal{D}_{/C}$ (and θ is an isomorphism when restricted to $\mathcal{D}$), we conclude that θ' is a monomorphism. It follows that $\theta(C)$ is a monomorphism. Applying the same argument to $\theta(C')$ for $C' \in \mathcal{C}^{(0)}_{/C}$, we conclude that θ' is a monomorphism whose image is the collection of elements $\eta \in \varprojlim_{C' \in (\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}} \widehat{\mathscr{F}}(C')$ having the property that, for each $C' \in (\mathcal{C}^{(0)}_{/C})$, the image of $\eta \in \widehat{\mathscr{F}}(C')$ belongs to the image of $\theta(C')$. Invoking again the fact that $\mathcal{C}^{(0)}_{/C}$ is generated by $\mathcal{D}_{/C}$, we see that it suffices to check this condition in the case where C' belongs to $\mathcal{D}_{/C}$, in which case it is automatic because $\theta(C')$ is bijective. It follows that θ' is a bijection, so that $\theta(C)$ is also a bijection. This completes the proof of (b).

We now verify (a). Let D be an object of $\mathcal{D}$ and let $\mathcal{D}^{(0)}_{/D}$ be a covering sieve on D; we wish to show that the canonical map $\rho : \mathscr{F}_0(D) \to \varprojlim_{D' \in (\mathcal{D}^{(0)}_{/D})^{\mathrm{op}}} \mathscr{F}_0(D')$ is bijective. Let $\mathcal{C}^{(0)}_{/D} \subseteq \mathcal{C}_{/D}$ denote the sieve generated by $\mathcal{D}^{(0)}_{/D}$. Our assumption that $\mathcal{D}^{(0)}_{/D}$ is covering for the Grothendieck topology of Proposition B.6.3 guarantees that $\mathcal{C}^{(0)}_{/C}$ is covering for the original Grothendieck topology on $\mathcal{C}$. The map ρ factors as a composition

$$\mathscr{F}_0(D) = \mathscr{F}(D) \xrightarrow{\rho'} \varprojlim_{C' \in (\mathcal{C}^{(0)}_{/D})^{\mathrm{op}}} \mathscr{F}(C') \xrightarrow{\rho''} \varprojlim_{D' \in (\mathcal{D}^{(0)}_{/D})^{\mathrm{op}}} \mathscr{F}_0(D'),$$

where ρ' is a bijection by virtue of our assumption that $\mathscr{F}$ is a sheaf, and the map ρ'' is bijective because $\mathscr{F}$ is a right Kan extension of its restriction to $\mathcal{D}^{\mathrm{op}}$. This completes the proof of (b).

We now prove the converse. Suppose that $\mathscr{F}$ satisfies conditions (a) and (b); we wish to show that $\mathscr{F}$ is a sheaf on $\mathcal{C}$. Fix a covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$. Set $\mathcal{D}_{/C} = \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/C}$ and $\mathcal{D}^{(0)}_{/C} = \mathcal{D} \times_{\mathcal{C}} \mathcal{C}^{(0)}_{/C}$. We wish to show that the upper vertical map in the diagram

$$\begin{tikzcd}
\mathscr{F}(C) \arrow[r] \arrow[d] & \varprojlim_{C' \in (\mathcal{C}^{(0)}_{/C})^{\mathrm{op}}} \mathscr{F}(C') \arrow[d] \\
\varprojlim_{D \in \mathcal{D}^{\mathrm{op}}_{/C}} \mathscr{F}(D) \arrow[r] & \varprojlim_{D \in (\mathcal{D}^{(0)}_{/C})^{\mathrm{op}}} \mathscr{F}(D)
\end{tikzcd}$$

is bijective. We conclude by observing that the vertical maps are bijective by virtue of (b), and the lower horizontal map by virtue of (a) (together with our assumption that $\mathcal{D}$ is a basis for $\mathcal{C}$). $\square$

Appendix C. Topos Theory

C.1. Grothendieck Topoi.

Definition C.1.1. A *Grothendieck topos* is a category $\mathcal{X}$ satisfying the following axioms:

(1) The category $\mathcal{X}$ is exact (Definition A.2.6).
(2) The category $\mathcal{X}$ admits small coproducts, and coproducts in $\mathcal{X}$ are disjoint (Definition A.3.1).
(3) The formation of small coproducts in $\mathcal{X}$ is compatible with pullback. That is, for every collection of objects $\{X_i\}_{i \in I}$ having coproduct $X = \coprod_{i \in I} X_i$ and every morphism $f : Y \to X$, the projection maps $\{X_i \times_X Y \to Y\}_{i \in I}$ exhibit Y as a coproduct of the objects $\{X_i \times_X Y\}_{i \in I}$.
(4) The category $\mathcal{X}$ is locally small, and there exists a small subcategory $\mathcal{X}_0$ which *generates* $\mathcal{X}$ in the sense that every object $X \in \mathcal{X}$ admits an effective epimorphism $\coprod_{i \in I} U_i \twoheadrightarrow X$, where each U_i belongs to $\mathcal{X}_0$.

Remark C.1.2. Every Grothendieck topos is a pretopos.

Example C.1.3. The category Set is a Grothendieck topos.

Example C.1.4. Let $\mathcal{C}$ be a small category and let $\mathcal{X}$ be a Grothendieck topos. Then the category $\mathrm{Fun}(\mathcal{C},\mathcal{X})$ is a Grothendieck topos. In particular, for any small category $\mathcal{C}$, the category of presheaves $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Set})$ is a Grothendieck topos.

Proposition C.1.5. *Let $\mathcal{X}$ be a Grothendieck topos and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a full subcategory. Suppose that the inclusion functor $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ admits a left adjoint $L : \mathcal{X} \to \mathcal{X}_0$ which preserves finite limits. Then $\mathcal{X}_0$ is also a Grothendieck topos.*

Remark C.1.6. In the situation of Proposition C.1.5, the existence of the left adjoint $L : \mathcal{X} \to \mathcal{X}_0$ guarantees that the subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ is closed under the formation of finite limits. In particular, the meaning of our assumption that L preserves finite limits does not depend on whether we regard L as an object of $\mathrm{Fun}(\mathcal{X},\mathcal{X}_0)$ or $\mathrm{Fun}(\mathcal{X},\mathcal{X})$.

Corollary C.1.7. *Let $\mathcal{C}$ be a small category equipped with a Grothendieck topology. Then the category of sheaves $\mathrm{Shv}(\mathcal{C})$ is a Grothendieck topos.*

Proof. Combine Proposition C.1.5, Example C.1.4, and Proposition B.2.4. □

Remark C.1.8. Proposition C.1.5 has a converse: every Grothendieck topos is equivalent to $\mathrm{Shv}(\mathcal{C})$, for some small category $\mathcal{C}$ equipped with a Grothendieck topology (Theorem C.4.1). In fact, this is often taken as the definition of a Grothendieck topos.

Remark C.1.9. In the situation of Definition C.1.1, condition (4) is automatically satisfied if the category $\mathcal{X}$ is accessible. Conversely, every Grothendieck topos is locally presentable as a category (and in particular accessible): this follows from the fact that every Grothendieck topos can be identified with $\mathrm{Shv}(\mathcal{C})$, for some small category $\mathcal{C}$ equipped with a Grothendieck topology (Theorem C.4.1).

Proof of Proposition C.1.5. Let $\mathcal{X}$ be a Grothendieck topos, let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a full subcategory, and let $L : \mathcal{X} \to \mathcal{X}_0$ be a left adjoint to the inclusion functor. Assume that the functor L preserves finite limits. We wish to show that $\mathcal{X}_0$ is a Grothendieck topos. We first show that $\mathcal{X}_0$ is exact: that is, that it satisfies axioms $(R1)$, $(R2')$, and $(R3)$ of Definition A.2.6. Axiom $(R1)$ follows from Remark C.1.6. To verify $(R2')$, suppose we are given an object $X \in \mathcal{X}_0$ and an equivalence relation $R \subseteq X \times X$ in the category $\mathcal{X}_0$. Without loss of generality, we may assume that $X = L\overline{X}$ for some object $\overline{X} \in \mathcal{X}$ and $R = L\overline{R}$ for some equivalence relation $\overline{R} \subseteq \overline{X} \times \overline{X}$ (for example, we can take $\overline{X} = X$ and $\overline{R} = R$). Since $\mathcal{X}$ satisfies axiom $(R2')$, we can choose a coequalizer diagram

$$\overline{R} \rightrightarrows \overline{X} \to \overline{X}/\overline{R}$$

in the category $\mathcal{X}$. Applying the functor L (which preserves all colimits which exist in $\mathcal{C}$, since it is a left adjoint), we obtain a coequalizer diagram

$$R \rightrightarrows X \to L(\overline{X}/\overline{R}).$$

To show that the equivalence relation R is effective in $\mathcal{X}_0$, it will suffice to show that the diagram σ :

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & L(\overline{X}/\overline{R}) \end{tikzcd}$$

is a pullback square in $\mathcal{X}_0$. This follows from our assumption that L preserves pullback squares, since σ is the image under L of the diagram

$$\begin{tikzcd} \overline{R} \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ \overline{X} \arrow[r] & \overline{X}/\overline{R}, \end{tikzcd}$$

which is a pullback diagram in $\mathcal{X}$ (since every equivalence relation in $\mathcal{X}$ is effective). This completes the proof of $(R2')$, and also gives the following characterization of effective epimorphisms in the category $\mathcal{X}_0$:

($*$) A morphism $f : X \to Y$ in $\mathcal{X}_0$ is an effective epimorphism (in the category $\mathcal{X}_0$) if it and only if it is isomorphic to a morphism of the form $L\overline{f} : L\overline{X} \to L\overline{Y}$, where $\overline{f} : \overline{X} \to \overline{Y}$ is an effective epimorphism in $\mathcal{X}$.

We now complete the proof that $\mathcal{X}_0$ is exact by verifying condition ($R3$) of Definition A.2.6. Suppose we are given a pullback square τ :

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{X}_0$, where f is an effective epimorphism in $\mathcal{X}_0$. We wish to show that $\overline{f}'$ is also an effective epimorphism in $\mathcal{X}_0$. By virtue of ($*$), we can assume that $f = L\overline{f}$ for some effective epimorphism $\overline{f} : \overline{X} \to \overline{Y}$ in $\mathcal{C}$. Using the assumption that L preserves pullbacks, we see that τ can be identified with the image under L of a pullback diagram

$$\begin{tikzcd} Y' \times_Y \overline{X} \arrow[r] \arrow[d, "\overline{f}'"] & \overline{X} \arrow[d, "\overline{f}"] \\ Y' \times_Y \overline{Y} \arrow[r] & \overline{Y} \end{tikzcd}$$

in the category $\mathcal{X}$. Our assumption that $\mathcal{X}$ is exact then guarantees that $\overline{f}'$ is an effective epimorphism in $\mathcal{X}$, so that $f' = L\overline{f}'$ is an effective epimorphism in $\mathcal{X}_0$ by virtue of ($*$).

Every small collection of objects $\{X_i\}_{i \in I}$ of the category $\mathcal{X}_0$ admit a coproduct in $\mathcal{X}_0$, which can be obtained by forming the coproduct $\coprod_{i \in I} X_i$ in the category $\mathcal{X}$ and then applying the functor L. We claim that for every pair of objects $X, Y \in \mathcal{X}_0$, their coproduct in $\mathcal{X}_0$ is disjoint. Without loss of generality, we may assume that $X = L\overline{X}$ and $Y = L\overline{Y}$ for some objects $\overline{X}, \overline{Y} \in \mathcal{X}$. We then have a commutative diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \overline{X} \arrow[d, "\overline{f}"] \\ \overline{Y} \arrow[r, "\overline{g}"] & \overline{X} \amalg \overline{Y}, \end{tikzcd}$$

and our assumption that $\mathcal{X}$ has disjoint coproduct guarantees that this diagram is a pullback square and that $\overline{f}$ and $\overline{g}$ are monomorphisms. Applying the functor L, we obtain a diagram

$$\begin{tikzcd} L\emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & L(\overline{X} \amalg \overline{Y}) \end{tikzcd}$$

which exhibits $L(\overline{X} \amalg \overline{Y})$ as a coproduct of X and Y in $\mathcal{C}_0$. Since L preserves fiber products, this diagram is a pullback square and f and g are monomorphisms. It follows that X and Y have a disjoint coproduct in the category $\mathcal{X}_0$.

We now show that the formation of small coproducts in $\mathcal{X}_0$ is compatible with pullbacks. We now complete the proof by showing that $\mathcal{C}_0$ satisfies the axiom ($E2$). Let $f : X \to Y$ be a morphism in $\mathcal{X}_0$ and suppose we are given a small collection of morphisms $\{Y_i \to Y\}_{i \in I}$. Let $\coprod_{i \in I} Y_i$ denote a coproduct of the collection $\{Y_i\}_{i \in I}$ in the category $\mathcal{X}$, so that $L(\coprod_{i \in I} Y_i)$ is a coproduct of the collection $\{Y_i\}_{i \in I}$ in the category $\mathcal{X}_0$. We wish to show that the pullback $X \times_Y L(\coprod_{i \in I} Y_i)$ can be identified with a coproduct of the collection $\{X \times_Y Y_i\}_{i \in I}$. In other words, we wish to show that the canonical map

$$\theta : L(\coprod_{i \in I}(X \times_Y Y_i)) \to X \times_Y L(\coprod_{i \in I} Y_i)$$

is an isomorphism. This is clear: we can identify θ with the image under L of the canonical map

$$\overline{\theta}: \coprod_{i\in I}(X \times_Y Y_i) \to X \times_Y \coprod_{i\in I} Y_i,$$

which is an isomorphism in $\mathcal{X}$ by virtue of our assumption that $\mathcal{X}$ is a Grothendieck topos.

We complete the proof by observing that if $\mathcal{X}$ is generated by a small collection of objects $\{X_i\}_{i\in I}$, then $\mathcal{X}_0$ is generated by the small collection of objects $\{LX_i\}_{i\in I}$. □

C.2. **Geometric Morphisms.**

Definition C.2.1. Let $\mathcal{X}$ and $\mathcal{Y}$ Grothendieck topoi. A *geometric morphism* from $\mathcal{X}$ to $\mathcal{Y}$ is a pretopos functor $F^*: \mathcal{Y} \to \mathcal{X}$ which preserves small coproducts. We let $\mathrm{Fun}^*(\mathcal{Y}, \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{Y}, \mathcal{X})$ spanned by the geometric morphisms from $\mathcal{X}$ to $\mathcal{Y}$.

More concretely, a geometric morphism from $\mathcal{X}$ to $\mathcal{Y}$ is a functor $F^*: \mathcal{Y} \to \mathcal{X}$ which preserves finite limits, small coproducts, and effective epimorphisms.

Example C.2.2. Let $\mathcal{X}$ be a Grothendieck topos and let $f: X \to Y$ be a morphism in $\mathcal{X}$. Then the pullback functor

$$\mathcal{X}_{/Y} \to \mathcal{X}_{/X} \qquad Z \mapsto Z \times_Y X$$

is a geometric morphism from $\mathcal{X}_{/X}$ to $\mathcal{X}_{/Y}$.

Example C.2.3. Let $\mathcal{C}$ be a small category equipped with a Grothendieck topology. Then the sheafification functor $L: \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Shv}(\mathcal{C})$ is a geometric morphism of topoi.

Definition C.2.1 has several other formulations. To establish their equivalence (Proposition C.2.9 below), we need some preliminaries.

Lemma C.2.4. *Let $\mathcal{X}$ be a Grothendieck topos containing an object X. Then:*

(1) *Every set of subobjects $\{X_i \subseteq X\}_{i\in I}$ has a least upper bound $\bigcup_{i\in I} X_i$ in the partially ordered set* $\mathrm{Sub}(X)$.
(2) *If $F^*: \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of topoi, then we have $F^*(\bigcup_{i\in I} X_i) = \bigcup_{i\in I}(F^*(X_i))$ in* $\mathrm{Sub}(F^*(X))$.
(3) *The collection of subobjects* $\mathrm{Sub}(X)$ *is small.*

Remark C.2.5. Let $\mathcal{X}$ be a Grothendieck topos containing an object X with a subobject $Y \subseteq X$. Applying assertion (2) of Lemma C.2.4 to the pullback functor $\bullet \times_X Y$ of Example C.2.2, we deduce that the formation of unions in $\mathrm{Sub}(X)$ satisfies the infinite distributive law $Y \cap (\bigcup_{i\in I} X_i) = \bigcup_{i\in I}(Y \cap X_i)$. In other words, the partially ordered set $\mathrm{Sub}(X)$ is a *frame*.

Proof of Lemma C.2.4. For assertion (1), we note that $\bigcup_{i\in I} X_i$ can be characterized as the image of the map $\coprod_{i\in I} X_i \to X$ (obtained by amalgamating the inclusions $X_i \hookrightarrow X$). Assertion (2) follows by construction, since geometric morphisms preserve the formation of coproducts and images. To prove (3), we note that that if $\{G_i\}_{i\in I}$ is a set of generators for $\mathcal{X}$, then a subobject $Y \subseteq X$ is determined (as an object of $\mathrm{Sub}(X)$) by the collection of subsets $\{\mathrm{Hom}_{\mathcal{X}}(G_i, Y) \subseteq \mathrm{Hom}_{\mathcal{X}}(G_i, X)\}_{i\in I}$. □

Lemma C.2.6. *Let $\mathcal{X}$ be a Grothendieck topos and suppose we are given a pair of morphisms $f, g: Y \to X$ in $\mathcal{X}$ having the same domain and codomain. Then:*

(1) *There is a smallest equivalence relation $R \subseteq X \times X$ such that the pair $(f, g): Y \to X \times X$ factors through R. We will refer to R as the equivalence relation generated by (f, g).*
(2) *If $F^*: \mathcal{X} \to \mathcal{Y}$ is a geometric morphism of topoi, then $F^*(R) \subseteq F^*(X) \times F^*(X)$ is the equivalence relation generated by the pair of morphisms $F^*(f), F^*(g): F^*(Y) \to F^*(X)$.*

Proof. We define an increasing sequence of subobjects

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq X \times X$$

as follows:

- We take R_0 to be the smallest subobject of $X \times X$ which contains the images of the maps
$$(f,g) : Y \to X \times X \qquad (g,f) : Y \to X \times X \qquad \delta : X \to X \times X,$$
where δ is the diagonal.
- For $n > 0$, let R_n denote the image of the map $R_{n-1} \times_X R_{n-1} \to X \times X$. That is, R_n is the smallest subobject of $X \times X$ with the property that, for every triple of maps $a, b, c : Z \to X$ such that $(a,b) : Z \to X \times X$ and $(b,c) : Z \to X \times X$ factor through R_{n-1}, then $(a,c) : Z \to X \times X$ factors through R_n.

Note that if $(f,g) : Y \to X \times X$ factors through some equivalence relation $E \subseteq X \times X$, then E must contain each R_n and must therefore contain the union $R = \bigcup_n R_n$. We will complete the proof by showing that R is an equivalence relation on X. It is immediate from the definition that R is reflexive (since R_0 contains the image of the diagonal $\delta : X \hookrightarrow X \times X$) and symmetric (each of the relations R_n is symmetric by construction). It will therefore suffice to show that R is transitive. Suppose we are given a triple of maps $a, b, c : Z \to X$ for which the maps
$$(a,b) : Z \to X \times X \qquad (b,c) : Z \to X \times X$$
factor through R. For each $n \geq 0$, let Z_n denote the subobject of Z given by the intersection $(a,b)^{-1} R_n \cap (b,c)^{-1} R_n$. By construction, each of the maps $(a,c) : Z \to X \times X$ carries Z_n into R_{n+1}, so that $Z_n \subseteq (a,c)^{-1}(R)$. Since the formation of unions of subobjects is compatible with pullback (by part (2) of Lemma C.2.4), we see that $Z = \bigcup_n Z_n$, so that $Z = (a,c)^{-1}(R)$ as desired. This completes the proof of (1). Assertion (2) follows by construction, since geometric morphisms preserve the formation of pullbacks, images, and unions of subobjects (Lemma C.2.4). □

Proposition C.2.7. *Let $\mathcal{X}$ be a Grothendieck topos. Then $\mathcal{X}$ admits small colimits. Moreover, every geometric morphism $F^* : \mathcal{X} \to \mathcal{Y}$ preserves small colimits.*

Proof. By definition, the category $\mathcal{X}$ admits small coproducts, and every geometric morphism $F^* : \mathcal{X} \to \mathcal{Y}$ preserves small coproducts. It will therefore suffice to show that every pair of morphisms $f, g : Y \to X$ admits a coequalizer in $\mathcal{X}$, which is preserved by every geometric morphism $F^* : \mathcal{X} \to \mathcal{Y}$. Let $R \subseteq X \times X$ be the equivalence relation generated by the pair (f,g) (Lemma C.2.6). Unwinding the definitions, we see that a morphism $h : X \to X'$ factors (uniquely) through X/R if and only if the fiber product $X \times_{X'} X \subseteq X \times X$ contains R, or equivalently that the map $(f,g) : Y \to X \times X$ factors through $X \times_{X'} X$. It follows that the quotient map $X \twoheadrightarrow X/R$ exhibits X/R as a coequalizer of the morphisms f and g. For every geometric morphism $F^* : \mathcal{X} \to \mathcal{Y}$, the induced map $F^*(X) \twoheadrightarrow F^*(X/R)$ exhibits $F^*(X/R)$ as the quotient of $F^*(X)$ by the equivalence relation
$$F^*(X) \times_{F^*(X/R)} F^*(X) \simeq F^*(X \times_{X/R} X) \simeq F^*(R) \subseteq F^*(X) \times F^*(X).$$
It follows from part (2) of Lemma C.2.6 that this is the equivalence relation generated by $F^*(f), F^*(g) : F^*(Y) \to F^*(X)$. □

Corollary C.2.8. *Let $\mathcal{X}$ be a Grothendieck topos. Then colimits in $\mathcal{X}$ are universal. That is, for every morphism $f : X \to Y$ in $\mathcal{X}$, the pullback functor*
$$\mathcal{X}_{/Y} \to \mathcal{X}_{/X} \qquad Z \mapsto Z \times_Y X$$
preserves small colimits.

Proof. Combine Example C.2.2 with Proposition C.2.7. □

Proposition C.2.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Grothendieck topoi, and let $F^* : \mathcal{X} \to \mathcal{Y}$ be a functor which preserves finite limits. The following conditions are equivalent:*

1. *The functor F^* is a geometric morphism from $\mathcal{Y}$ to $\mathcal{X}$: that is, it preserves small coproducts and effective epimorphisms.*
2. *The functor F^* preserves small colimits.*
3. *The functor F^* admits a right adjoint $F_* : \mathcal{Y} \to \mathcal{X}$.*

Proof. The implication (3) ⇒ (2) is immediate, and the reverse implication follows from the adjoint functor theorem. The implication (1) ⇒ (2) follows from Proposition C.2.7. We complete the proof by showing that (2) ⇒ (1). Suppose that $F^* : \mathcal{X} \to \mathcal{Y}$ preserves small colimits and finite limits and let $f : Y \twoheadrightarrow X$ is an effective epimorphism in $\mathcal{X}$; we wish to show that $F^*(f)$ is an effective epimorphism in $\mathcal{Y}$. By definition, the map f exhibits X as a coequalizer of the projection maps $Y \times_X Y \rightrightarrows Y$. Since F^* preserves finite limits, it follows that $F^*(f)$ exhibits $F^*(X)$ as the coequalizer of the projection maps $F^*(Y) \times_{F^*(X)} F^*(Y) \rightrightarrows F^*(Y)$, so that $F^*(f)$ is an effective epimorphism. This shows that (2) ⇒ (1). □

Remark C.2.10. In the situation of Proposition C.2.9, the functor $F^* : \mathcal{X} \to \mathcal{Y}$ and its right adjoint $F_* : \mathcal{Y} \to \mathcal{X}$ are equivalent data: either can be recovered (up to canonical isomorphism) from the other. It is common to emphasize the role of the functor F_*, and to refer to the functor $F_* : \mathcal{Y} \to \mathcal{X}$ as a *geometric morphism from $\mathcal{Y}$ to $\mathcal{X}$.*

C.3. **Diaconescu's Theorem.**

Theorem C.3.1 (Diaconescu)**.** *Let $\mathcal{C}$ be a small category which admits finite limits, and let $\mathcal{X}$ be a Grothendieck topos. Then composition with the Yoneda embedding $h : \mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ induces a fully faithful embedding*

$$\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X}).$$

The essential image of this embedding consists of those functors $f : \mathcal{C} \to \mathcal{X}$ which preserve finite limits.

Before giving the proof of Theorem C.3.1, let us describe some of its consequences. Let $\mathcal{C}$ be a small category which admits finite limits. Suppose that $\mathcal{C}$ is equipped with a Grothendieck topology, and let $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Shv}(\mathcal{C})$ be the sheafification functor. Then composition with L induces a fully faithful embedding

$$\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \xrightarrow{\circ L} \mathrm{Fun}^*(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \mathcal{X}),$$

whose essential image consists of those geometric morphisms $f^* : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathcal{X}$ having the property that the right adjoint $f_* : \mathcal{X} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ factors through the full subcategory $\mathrm{Shv}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$. Combining this observation with Theorem C.3.1, we obtain the following:

Corollary C.3.2. *Let $\mathcal{C}$ be a small category which admits finite limits. Suppose that $\mathcal{C}$ is equipped with a Grothendieck topology, and $\widetilde{h} : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ denote the sheafified Yoneda embedding (Remark B.2.7). Then, for any Grothendieck topos $\mathcal{X}$, composition with $\widetilde{h}$ induces a fully faithful embedding $\theta : \mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{X})$, whose essential image is spanned by those functors $f : \mathcal{C} \to \mathcal{X}$ which preserve finite limits and satisfy the following additional condition:*

- $(*)$ *For each object $X \in \mathcal{X}$, the functor $\mathrm{Hom}_{\mathcal{X}}(f(\bullet), X) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf on $\mathcal{C}$.*

Remark C.3.3. In the situation of Corollary C.3.2, condition $(*)$ is equivalent (under the assumption that f preserves finite limits) to the following more concrete assertion:

- $(*')$ For every covering $\{C_i \to C\}_{i \in I}$ in the category $\mathcal{C}$, the induced map $\coprod_{i \in I} f(C_i) \to f(C)$ is an effective epimorphism in the topos $\mathcal{X}$.

Corollary C.3.4. *Let $\mathcal{C}$ be a small regular category, endowed with the regular topology of Definition B.3.3. Let $\mathcal{X}$ be any Grothendieck topos. Then composition with the Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ induces a fully faithful embedding $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$, whose essential image is spanned by the* regular *functors $f : \mathcal{C} \to \mathcal{X}$: that is, functors which preserve finite limits and effective epimorphisms.*

Proof. Combine Corollary C.3.2 with Proposition B.3.5. □

Corollary C.3.5. *Let $\mathcal{C}$ be a small extensive category which admits finite limits, endowed with the extensive topology of Definition B.4.3. Let $\mathcal{X}$ be any Grothendieck topos. Then composition with the Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ induces a fully faithful embedding $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$, whose essential image is spanned by those functors $f : \mathcal{C} \to \mathcal{X}$: which preserve finite limits and finite coproducts.*

Proof. Combine Corollary C.3.2 with Proposition B.4.5. □

Corollary C.3.6. *Let $\mathcal{C}$ be a small regular extensive category, endowed with the coherent topology of Definition B.5.3. Let $\mathcal{X}$ be any Grothendieck topos. Then composition with the Yoneda embedding $h : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ induces a fully faithful embedding $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{X})$, whose essential image is spanned by those functors $f : \mathcal{C} \to \mathcal{X}$: which preserve finite limits, finite coproducts, and effective epimorphisms.*

In particular, if $\mathcal{C}$ is a pretopos, then composition with h induces an equivalence $\mathrm{Fun}^*(\mathrm{Shv}(\mathcal{C}), \mathcal{X}) \simeq \mathrm{Fun}^{\mathrm{Pretop}}(\mathcal{C}, \mathcal{X})$.

Proof. Combine Corollary C.3.2 with Proposition B.5.5. □

We now turn to the proof of Theorem C.3.1. We begin by recalling that for any small category $\mathcal{C}$, the presheaf category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is freely generated under small colimits by the image of the Yoneda embedding $h : \mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$. More precisely, suppose that $\mathcal{X}$ is *any* category which admits small colimits, and let $\mathrm{LFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \mathcal{X})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \mathcal{X})$ spanned by those functors which preserve small colimits. Then composition with the Yoneda embedding induces an equivalence of categories

$$\mathrm{LFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}), \mathcal{X}) \xrightarrow{\circ h} \mathrm{Fun}(\mathcal{C}, \mathcal{X});$$

the inverse equivalence is given by left Kan extension along h. To prove Theorem C.3.1, we must show that if $\mathcal{C}$ admits finite limits and $\mathcal{X}$ is a Grothendieck topos, then a colimit preserving functor $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathcal{X}$ preserves finite limits if and only if the composition $F \circ h : \mathcal{C} \to \mathcal{X}$ preserves finite limits. The "only if" direction is clear (since the Yoneda embedding $h : \mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ preserves small limits), and does not require the assumption that $\mathcal{X}$ is a Grothendieck topos. We can therefore reformulate Theorem C.3.1 as follows:

Proposition C.3.7. *Let $\mathcal{C}$ be a small category which admits finite limits, let $\mathcal{X}$ be a Grothendieck topos, and let $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathcal{X}$ be a functor which preserves small colimits. If the composite functor $\mathcal{C} \xrightarrow{h} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \xrightarrow{F} \mathcal{X}$ preserves finite limits, then F preserves finite limits.*

Proof. Let $f : \mathcal{C} \to \mathcal{X}$ denote the composition $F \circ h$. Since f and h preserve final objects, the functor F preserves final objects. It will therefore suffice to show that F preserves pullbacks. For every pair of maps

$$\mathscr{F}_0 \xrightarrow{\alpha} \mathscr{F} \xleftarrow{\beta} \mathscr{F}_1$$

in the presheaf category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, let $\theta_{\alpha,\beta}$ denote the natural map

$$F(\mathscr{F}_0 \times_{\mathscr{F}} \mathscr{F}_1) \to F(\mathscr{F}_0) \times_{F(\mathscr{F})} F(\mathscr{F}_1).$$

Let us say that a presheaf $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is *good* it it satisfies the following condition:

($*$) For every pair of objects $C, D \in \mathcal{C}$ equipped with maps $h_C \xrightarrow{\alpha} \mathscr{F} \xleftarrow{\beta} h_D$, the comparison map
$$\theta_{\alpha,\beta} : F(h_C \times_{\mathscr{F}} h_D) \to F(h_C) \times_{F(\mathscr{F})} F(h_D)$$
is an isomorphism in $\mathcal{X}$.

Note that, if we regard $\mathscr{F}$ and the map $\alpha : h_C \to \mathscr{F}$ as fixed, then the constructions

$$\mathscr{F}_1 \mapsto F(h_C \times_{\mathscr{F}} \mathscr{F}_1)$$
$$\mathscr{F}_1 \mapsto F(h_C) \times_{F(\mathscr{F})} F(\mathscr{F}_1)$$

carry colimits in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})_{/\mathscr{F}}$ to colimits in $\mathcal{X}$. Consequently, the collection of those objects $\mathscr{F}_1 \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})_{/\mathscr{F}}$ which $\theta_{\alpha,\beta}$ is an isomorphism is closed under small colimits. Since every object of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ can be realized as a colimit of representable functors, we see that every good object $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ satisfies the following stronger version of condition ($*$):

($*'$) For every object $C \in \mathcal{C}$ and every pair of morphisms $h_C \xrightarrow{\alpha} \mathscr{F} \xleftarrow{\beta} \mathscr{F}_1$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, the comparison map
$$\theta_{\alpha,\beta} : F(h_C \times_{\mathscr{F}} \mathscr{F}_1) \to F(h_C) \times_{F(\mathscr{F})} F(\mathscr{F}_1)$$
is an isomorphism in $\mathcal{X}$.

Applying the same argument with the roles of α and β reversed, we see that every good object $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ satisfies the following even stronger condition:

$(*'')$ For every pair of morphisms $\mathscr{F}_0 \xrightarrow{\alpha} \mathscr{F} \xleftarrow{\beta} \mathscr{F}_1$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, the comparison map

$$\theta_{\alpha,\beta} : F(h_C \times_{\mathscr{F}} \mathscr{F}_1) \to F(h_C) \times_{F(\mathscr{F})} F(\mathscr{F}_1)$$

is an isomorphism in $\mathcal{X}$.

To complete the proof of Proposition C.3.7, it will suffice to show that every object of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is good. We first treat the case where $\mathscr{F} = \coprod_{i \in I} h_{E_i}$ is a coproduct of representable functors. In this case, any map $\alpha : h_C \to \mathscr{F}$ arises from a morphism $C \to E_i$ in the category $\mathcal{C}$, and any morphism $\beta : h_D \to \mathscr{F}$ arises from a morphism $D \to E_j$ in the category $\mathcal{C}$. If $i \neq j$, then the domain and codomain of the map $\theta_{\alpha,\beta}$ are both initial objects of $\mathcal{X}$ (since coproducts are disjoint in both $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ and $\mathcal{X}$, and the functor F preserves initial objects), so that $\theta_{\alpha,\beta}$ is automatically an isomorphism. In the case $i = j$, we can identify $\theta_{\alpha,\beta}$ with the comparison map $f(C \times_{E_i} D) \to f(C) \times_{f(E_i)} f(D)$ which is an isomorphism by virtue of our assumption that f preserves finite limits.

Since the final object $\mathbf{1}$ of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ is a representable presheaf (representable by the final object of $\mathcal{C}$), it follows that the object $\mathbf{1}$ is good. Using $(*'')$, we see that the functor $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathcal{X}$ commutes with products.

Now suppose that $\mathscr{F}$ is an arbitrary presheaf on $\mathcal{C}$. Choose an effective epimorphism $u : \mathscr{F}' \twoheadrightarrow \mathscr{F}$, where $\mathscr{F}'$ is a coproduct of representable presheaves. Then u is a levelwise surjection, so any pair of maps $h_C \xrightarrow{\alpha} \mathscr{F} \xleftarrow{\beta} h_D$ as in $(*)$ can be written as compositions

$$h_C \xrightarrow{\overline{\alpha}} \mathscr{F}' \xrightarrow{u} \mathscr{F} \xleftarrow{u} \mathscr{F}' \xleftarrow{\overline{\beta}} h_D.$$

In this case, the comparison map $\theta_{\alpha,\beta}$ factors as a composition

$$\begin{array}{rcl} F(h_C \times_{\mathscr{F}} h_D) & \xrightarrow{\theta'} & F(h_C) \times_{F(\mathscr{F}')} F(\mathscr{F}' \times_{\mathscr{F}} h_D) \\ & \xrightarrow{\theta''} & F(h_C) \times_{F(\mathscr{F}')} F(\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}') \times_{F(\mathscr{F}')} F(h_D) \\ & \xrightarrow{\theta'''} & F(h_C) \times_{F(\mathscr{F}')} (F(\mathscr{F}') \times_{F(\mathscr{F})} F(\mathscr{F}')) \times_{F(\mathscr{F}')} F(h_D) \\ & \simeq & F(h_C) \times_{F(\mathscr{F})} F(h_D). \end{array}$$

Since $\mathscr{F}'$ is good, it follows from $(*'')$ that the maps θ' and θ'' are isomorphisms. Consequently, to show that $\mathscr{F}$ is good, it will suffice to show that the map θ''' is an isomorphism. For this, we show that the comparison map

$$\theta_{u,u} : F(\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}') \to F(\mathscr{F}') \times_{F(\mathscr{F})} F(\mathscr{F}')$$

is an isomorphism in $\mathcal{X}$.

Set $\mathscr{R} = \mathscr{F}' \times_{\mathscr{F}} \mathscr{F}'$, which we regard as a subobject of $\mathscr{F}' \times \mathscr{F}'$. Note that $\mathscr{F}' \times \mathscr{F}'$ can be written as a coproduct of representable presheaves, so that the comparison map

$$F(\mathscr{R}) \simeq F(\mathscr{R} \times_{\mathscr{F}' \times \mathscr{F}'} \mathscr{R}) \to F(\mathscr{R}) \times_{F(\mathscr{F}' \times \mathscr{F}')} F(\mathscr{R})$$

is an isomorphism. It follows that we can identify $F(\mathscr{R})$ with a subobject of $F(\mathscr{F}' \times \mathscr{F}') \simeq F(\mathscr{F}') \times F(\mathscr{F}')$. Since u is an effective epimorphism, we can identify $\mathscr{F}$ with the coequalizer of the projection maps $\mathscr{R} \rightrightarrows \mathscr{F}'$. Because the functor F preserves colimits, we obtain a coequalizer diagram

$$F(\mathscr{R}) \rightrightarrows F(\mathscr{F}') \xrightarrow{F(u)} F(\mathscr{F})$$

in the topos $\mathcal{X}$. It follows that the fiber product $F(\mathscr{F}') \times_{F(\mathscr{F})} F(\mathscr{F}')$ can be identified with the equivalence relation on $F(\mathscr{F}')$ generated by $F(\mathscr{R}) \subseteq F(\mathscr{F}') \times F(\mathscr{F}')$. (see the proof of Proposition C.2.7). To complete the proof that the $\theta_{u,u}$ is an isomorphism, it will suffice to show that $F(\mathscr{R})$ is already an equivalence relation on $F(\mathscr{F}')$. Reflexivity and symmetry are clear. To verify transitivity, we must show that the natural map

$$\rho : F(\mathscr{R}) \times_{F(\mathscr{F}')} F(\mathscr{R}) \to F(\mathscr{F}') \times F(\mathscr{F}')$$

factors through $F(\mathscr{R})$. Since $\mathscr{F}'$ is good, we can use $(*'')$ to identify the domain of ρ with $F(\mathscr{R} \times_{\mathscr{F}'} \mathscr{R}) \simeq F(\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}' \times_{\mathscr{F}} \mathscr{F}')$. The existence of the desired factorization is now obvious (it can be obtained by applying the functor F to the canonical map

$$\mathscr{F}' \times_{\mathscr{F}} \mathscr{F}' \times_{\mathscr{F}} \mathscr{F}' \xrightarrow{\mathrm{id} \times u \times \mathrm{id}} \mathscr{F}' \times_{\mathscr{F}} \mathscr{F} \times_{\mathscr{F}} \mathscr{F}'.$$

□

C.4. **Giraud's Theorem.** Corollary C.1.7 admits a converse:

Theorem C.4.1 (Giraud)**.** *Let $\mathcal{X}$ be a category. The following conditions are equivalent:*

(1) *There exists a small category $\mathcal{C}$ which admits finite limits, a Grothendieck topology on $\mathcal{C}$, and an equivalence of categories $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$.*
(2) *There exists a small category $\mathcal{C}$, a Grothendieck topology on $\mathcal{C}$, and an equivalence of categories $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$.*
(3) *There exists a small category $\mathcal{C}$ and a fully faithful embedding $\mathcal{X} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ which admits a left adjoint $L : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathcal{X}$ which preserves finite limits.*
(4) *The category $\mathcal{X}$ is a Grothendieck topos (in the sense of Definition C.1.1).*

The implication (1) ⇒ (2) is immediate, (2) ⇒ (3) follows from Proposition B.2.4, and the implication (3) ⇒ (4) from Proposition C.1.5. To show that (4) ⇒ (1), we need some preliminaries.

Construction C.4.2 (The Canonical Topology)**.** Let $\mathcal{X}$ be a Grothendieck topos. We will say that a sieve $\mathcal{X}^{(0)}_{/X}$ on an object $X \in \mathcal{X}$ is a *canonical covering sieve* if it contains a set of morphisms $\{U_i \to X\}_{i \in I}$ for which the induced map $\coprod_{i \in I} U_i \to X$ is an effective epimorphism in $\mathcal{X}$.

Remark C.4.3. If $\mathcal{X}$ is a Grothendieck topos, then a sieve $\mathcal{X}^{(0)}_{/X}$ is a canonical covering sieve if and only if there is no proper subobject $X' \subsetneq X$ such that each morphism $f : U \to X$ in $\mathcal{X}^{(0)}_{/X}$ factors through X' (if this condition is satisfied, then X can be realized as the join of the images $\mathrm{Im}(f)$ where $f : U \to X$ ranges over all morphisms belonging to $\mathcal{X}^{(0)}_{/X}$; note that the collection of such images is small by virtue of Lemma C.2.4.

Lemma C.4.4. *Let $\mathcal{X}$ be a Grothendieck topos and let $\{f_i : X_i \twoheadrightarrow Y_i\}_{i \in I}$ be a small collection of effective epimorphisms in $\mathcal{X}$. Then the coproduct map $f : \coprod_{i \in I} X_i \to \coprod_{i \in I} Y_i$ is also an effective epimorphism in $\mathcal{X}$.*

Proof. Set $Y = \coprod_{i \in I} Y_i$ and $U = \mathrm{Im}(f) \subseteq Y$. Since the formation of coproducts in $\mathcal{X}$ is pullback-stable, we can identify U with the coproduct of the inverse images $U_i = U \times_Y Y_i$. By construction, each of the maps $f_i : X_i \twoheadrightarrow Y_i$ factors through U_i. Since f_i is an effective epimorphism we conclude that $U_i = Y_i$, so that $U = \coprod_{i \in I} U_i = \coprod_{i \in I} Y_i = Y$. □

Proposition C.4.5. *Let $\mathcal{X}$ be a Grothendieck topos. Then the collection of canonical covering sieves determines a Grothendieck topology on $\mathcal{X}$.*

We refer to the Grothendieck topology of Proposition C.4.5 as the *canonical topology* on $\mathcal{X}$.

Proof of Proposition C.4.5. Axiom $(T1)$ of Definition B.1.3 is immediate, and $(T2)$ follows from the assumption that small coproducts and effective epimorphisms in $\mathcal{X}$ are stable under pullback. To prove $(T3)$, it suffices to show that for every effective epimorphism $f : \coprod_{i \in I} U_i \twoheadrightarrow X$ and collection of effective epimorphisms $\{g_i : \coprod_{j \in J_i} V_{i,j} \twoheadrightarrow U_i\}_{i \in I}$, the composite map

$$\coprod_{i \in I, j \in J_i} V_{i,j} \xrightarrow{\coprod_i g_i} \coprod_{i \in I} U_i \xrightarrow{f} X$$

is an effective epimorphism. This follows from Proposition A.1.8 and Lemma C.4.4. □

Remark C.4.6. Let $\mathcal{X}$ be a Grothendieck topos and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a full subcategory. Then $\mathcal{X}_0$ generates $\mathcal{X}$ (in the sense of Definition C.1.1 if and only if $\mathcal{X}_0$ is a basis for the canonical topology on $\mathcal{X}$ (Definition B.6.1). In this case, $\mathcal{X}_0$ inherits a Grothendieck topology (Proposition B.6.3), which we will refer to as the

restricted canonical topology on $\mathcal{X}_0$. Beware that this terminology is potentially confusing: the restricted canonical topology on $\mathcal{X}_0$ depends not only on $\mathcal{X}_0$ as an abstract category, but on its realization as a full subcategory of the topos $\mathcal{X}$.

Remark C.4.7. Let $\mathcal{X}$ be a Grothendieck topos and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a full subcategory which generates $\mathcal{X}$. Regard $\mathcal{X}$ as equipped with the canonical topology of Proposition C.4.5 and $\mathcal{X}_0$ with the restricted canonical topology of Remark C.4.6. Then composition with the inclusion $\mathcal{X}_0 \subseteq \mathcal{X}$ induces an equivalence of categories $\mathrm{Shv}(\mathcal{X}) \simeq \mathrm{Shv}(\mathcal{X}_0)$. This is a special case of Proposition B.6.4.

Proposition C.4.8. *Let $\mathcal{X}$ be a Grothendieck topos. Then the Yoneda embedding $h : \mathcal{X} \hookrightarrow \mathrm{Fun}(\mathcal{X}^{\mathrm{op}}, \mathrm{Set})$ induces an equivalence of $\mathcal{X}$ with the category $\mathrm{Shv}(\mathcal{X})$ of sheaves with respect to the canonical topology on $\mathcal{X}$.*

Proof. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a small full subcategory which generates $\mathcal{X}$. Then Remark C.4.7 supplies an equivalence $\mathrm{Shv}(\mathcal{X}) \simeq \mathrm{Shv}(\mathcal{X}_0)$, so that $\mathrm{Shv}(\mathcal{X})$ is a Grothendieck topos by virtue of Corollary C.1.7. We leave it to the reader to verify that for every object $X \in \mathcal{X}$, the representable presheaf $h_X = \mathrm{Hom}_{\mathcal{X}}(\bullet, X)$ is a sheaf for the canonical topology. We wish to prove the converse. Let $\mathscr{F}$ be an object of $\mathrm{Shv}(\mathcal{X})$; we wish to show that $\mathscr{F}$ is representable. Since $\mathcal{X}_0$ is small, we can choose a map $\alpha : \mathscr{F}' \to \mathscr{F}$, where $\mathscr{F}'$ is a small coproduct of sheaves representable by objects $X \in \mathcal{X}_0$, such that the map of sets $\mathscr{F}'(X) \to \mathscr{F}(X)$ is surjective for each $X \in \mathcal{X}_0$. Then the induced map $\mathscr{F}'|_{\mathcal{X}_0^{\mathrm{op}}} \to \mathscr{F}|_{\mathcal{X}_0^{\mathrm{op}}}$ is an effective epimorphism in the category of presheaves $\mathrm{Fun}(\mathcal{X}_0^{\mathrm{op}}, \mathrm{Set})$ and therefore also in the category of sheaves $\mathrm{Shv}(\mathcal{X}_0)$. Since the restriction functor $\mathrm{Shv}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{X}_0)$ is an equivalence of categories, it follows that α is an effective epimorphism in $\mathrm{Shv}(\mathcal{X})$. It follows from the definition of the canonical topology on $\mathcal{X}$ that the construction $X \mapsto h_X$ carries coproducts and effective epimorphisms in $\mathcal{X}$ to coproducts and effective epimorphisms in $\mathrm{Shv}(\mathcal{X})$. Consequently, the sheaf $\mathscr{F}' \in \mathrm{Shv}(\mathcal{X})$ is representable by an object $U \in \mathcal{X}$ (which might not belong to $\mathcal{X}_0$).

Let us now treat the special case where $\mathscr{F}$ is given as a subobject of a representable sheaf $h_X \in \mathrm{Shv}(\mathcal{X})$. In this case, we can identify α with a map of representable functors $h_U \to h_X$ which, by virtue of Yoneda's lemma, is induced by a morphism $\overline{\alpha} : U \to X$ in the category $\mathcal{X}$. Since the functor $Y \mapsto h_Y$ preserves finite limits and effective epimorphisms, it preserves the formation of images. Consequently, the sheaf $\mathscr{F} = \mathrm{Im}(\alpha : h_U \to h_X)$ can be identified with $h_{\mathrm{Im}(\overline{\alpha})}$, and is therefore representable as desired.

We now treat the general case. The effective epimorphism $\alpha : h_U \to \mathscr{F}$ exhibits $\mathscr{F}$ as the quotient of h_U by an equivalence relation $\mathscr{R} \subseteq h_U \times h_U \simeq h_{U \times U}$. The preceding argument shows that $\mathscr{R}$ is representable by a subobject $R \subseteq U \times U$, which is easily seen to be an equivalence relation. Since the functor $Y \mapsto h_Y$ preserves finite limits and effective epimorphisms, it preserves the formation of quotient by equivalence relations. We therefore obtain a canonical isomorphism $\mathscr{F} \simeq h_U / h_R \simeq h_{U/R}$, so that $\mathscr{F}$ is representable as desired. □

By definition, for any Grothendieck topos $\mathcal{X}$ we can select a small full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ which generates $\mathcal{X}$. Enlarging $\mathcal{X}_0$ if necessary, we can assume that $\mathcal{X}_0$ is closed under finite limits in $\mathcal{X}$. Consequently, the implication (4) ⇒ (1) of Theorem C.4.1 follows from Remark C.4.7 together with the following:

Corollary C.4.9. *Let $\mathcal{X}$ be a Grothendieck topos, and let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a full subcategory which generates $\mathcal{X}$. Then the restricted Yoneda embedding $X \mapsto h_X|_{\mathcal{X}_0^{\mathrm{op}}}$ induces an equivalence of categories $\mathcal{X} \to \mathrm{Shv}(\mathcal{X}_0)$, where $\mathcal{X}_0$ is equipped with the restricted canonical topology of Remark C.4.6.*

Proof. Combine Proposition C.4.8 with Remark C.4.7. □

C.5. **Coherent Topoi.** Let $\mathcal{X}$ be a Grothendieck topos. We will say that a set of morphisms $\{U_i \to X\}_{i \in I}$ is a *covering* if it is a covering for the canonical topology of Proposition C.4.5: that is, if the induced map $\coprod_{i \in I} U_i \to X$ is an effective epimorphism in $\mathcal{X}$.

Definition C.5.1. Let $\mathcal{X}$ be a Grothendieck topos. We say that an object $X \in \mathcal{X}$ is *quasi-compact* if every covering $\{U_i \to X\}_{i \in I}$ (for the canonical topology) has a finite subcovering: that is, there exists a finite subset $I_0 \subseteq I$ for which the maps $\{U_i \to X\}_{i \in I_0}$ are also a covering.

Proposition C.5.2. *Let $\mathcal{X}$ be a Grothendieck topos and let X be an object of $\mathcal{X}$. Suppose that X admits a covering $\{U_i \to X\}_{i \in I}$. If the set I is finite and each U_i is quasi-compact, then X is quasi-compact.*

Proof. Let $\{V_j \to X\}_{j\in J}$ be any covering of X. Then, for each $i \in I$, the collection of morphisms $\{U_i \times_X V_j \to U_i\}_{j\in J}$ is also a covering. Using our assumption that each U_i is quasi-compact, we deduce that there exists a finite subset $J_0 \subseteq J$ such that each of the collections $\{U_i \times_X V_j \to U_i\}_{j\in J_0}$ is a covering. We then have a commutative diagram

$$\begin{tikzcd} \coprod_{i\in I} \coprod_{j\in J_0} U_i \times_X V_j \arrow[r] \arrow[d] & \coprod_{i\in I} U_i \arrow[d] \\ \coprod_{j\in J_0} V_j \arrow[r] & X \end{tikzcd}$$

where the upper horizontal and right vertical map are effective epimorphisms (Proposition A.3.8). It follows that the lower horizontal map is also an effective epimorphism, so that $\{V_j \to X\}_{j\in J_0}$ is a covering of X. $\square$

Remark C.5.3. Let $\mathcal{X}$ be a Grothendieck topos containing a pair of objects X and Y. If the coproduct $X \amalg Y$ is quasi-compact, then X and Y are quasi-compact.

Proposition C.5.4. *Let $\mathcal{X}$ be a Grothendieck topos and let $\mathcal{X}_{\mathrm{qc}} \subseteq \mathcal{X}$ be the full subcategory spanned by the quasi-compact objects. Then $\mathcal{X}_{\mathrm{qc}}$ is essentially small (that is, it is equivalent to a small category).*

Proof. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be a small full subcategory which generates $\mathcal{X}$. Enlarging $\mathcal{X}_0$ if necessayr, we may assume that $\mathcal{X}_0$ is closed under finite coproducts. For each object $X \in \mathcal{X}$, we can choose a covering $\{U_i \to X\}$, where each U_i belongs to $\mathcal{X}_0$. If X is quasi-compact, this covering admits a finite subcover. We can therefore choose a single object $U \in \mathcal{X}_0$ and an effective epimorphism $U \twoheadrightarrow X$. It follows that X can be identified with the coequalizer of a diagram $R \rightrightarrows U$, for some equivalence relation $R \subseteq U \times U$. Since $\mathcal{X}_0$ is small and the collection $\mathrm{Sub}(U \times U)$ of subobjects of $U \times U$ is small (Lemma C.2.4), it follows that the collection of isomorphism classes of quasi-compact objects of $\mathcal{X}$ is small. $\square$

Definition C.5.5. Let $\mathcal{X}$ be a Grothendieck topos. We will say that an object $X \in \mathcal{X}$ is *quasi-separated* if, for every pair of morphisms $U \to X \leftarrow V$, where U and V are quasi-compact, the fiber product $U \times_X V$ is also quasi-compact.

Remark C.5.6. Let $\mathcal{X}$ be a Grothendieck topos and let $X \in \mathcal{X}$ be a quasi-separated object. Then every subobject $U \subseteq X$ is also quasi-separated.

Beware that the requirement of Definition C.5.5 is sometimes satisfied for uninteresting reasons. For example, if $\mathcal{X} = \mathrm{Shv}(\mathbb{R}^n)$ is the category of sheaves on the Euclidean space $\mathbb{R}^n$, then the only quasi-compact object of $\mathcal{X}$ is the initial object. In this case, every object of $\mathcal{X}$ is quasi-separated. For Definition C.5.5 to be meaningful, we need to ensure that there exists a good supply of quasi-compact objects.

Definition C.5.7. Let $\mathcal{X}$ be a Grothendieck topos. We will say that $\mathcal{X}$ is *coherent* if there exists a collection of objects $\mathcal{U}$ satisfying the following conditions:

- (a) The collection $\mathcal{U}$ generates $\mathcal{X}$: that is, every object $X \in \mathcal{X}$ admits a covering $\{U_i \to X\}$, where each U_i belongs to $\mathcal{U}$.
- (b) The collection $\mathcal{U}$ is closed under finite products. In particular, it contains a final object of $\mathcal{X}$.
- (c) Every object of $\mathcal{U}$ is quasi-compact and quasi-separated.

If these conditions are satisfied, then we say that an object $X \in \mathcal{X}$ is *coherent* if it is quasi-compact and quasi-separated. We let $\mathcal{X}_{\mathrm{coh}}$ denote the full subcategory of $\mathcal{X}$ spanned by the coherent objects.

Remark C.5.8. Let $\mathcal{X}$ be a coherent Grothendieck topos. Then the final object of $\mathcal{X}$ is quasi-separated. It follows that the collection of quasi-compact objects of $\mathcal{X}$ is closed under finite products.

Our next goal is to show that if $\mathcal{X}$ is a coherent Grothendieck topos, then we can take the full subcategory $\mathcal{U} \subseteq \mathcal{X}$ of Definition C.5.7 to be the category of coherent objects $\mathcal{X}_{\mathrm{coh}}$. To prove this, it suffices to show that the category of coherent objects $\mathcal{X}_{\mathrm{coh}}$ is closed under finite products. In fact, we have the following:

Proposition C.5.9. *Let $\mathcal{X}$ be a coherent Grothendieck topos. Then the full subcategory $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is closed under finite limits.*

To prove Proposition C.5.9, it will be convenient to employ another characterization of the class of quasi-separated objects.

Lemma C.5.10. *Let $\mathcal{X}$ be a coherent Grothendieck topos. Then an object $X \in \mathcal{C}$ is quasi-separated if and only if it satisfies the following condition:*

($*$) *For every quasi-compact object $U \in \mathcal{X}$ and every pair of morphisms $f, g : U \to X$, the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ is quasi-compact.*

Proof. Suppose first that X is quasi-separated, and that we are given a pair of morphisms $f, g : U \to X$; we wish to show that the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ is quasi-compact. Let $\mathcal{U}$ be as in Definition C.5.7, and choose a covering $\{U_i \to U\}_{i \in I}$ where each U_i belongs to $\mathcal{U}$. Since U is quasi-compact, we can assume that this covering is finite. Then f and g induce maps $f_i, g_i : U_i \to X$, having an equalizer $\mathrm{Eq}(U_i \rightrightarrows X) \simeq \mathrm{Eq}(U \rightrightarrows X) \times_U U_i$. It follows that $\mathrm{Eq}(U \rightrightarrows X)$ admits a finite covering by the objects $\mathrm{Eq}(U_i \rightrightarrows X)$. It will therefore suffice to show that each $\mathrm{Eq}(U_i \rightrightarrows X)$ is quasi-compact. In other words, we may replace U by U_i and thereby reduce to the case where $U \in \mathcal{U}$.

Unwinding the definitions, we see that the equalizer $\mathrm{Eq}(U \rightrightarrows X)$ can be identified with the fiber product

$$(U \times_X U) \times_{U \times U} U.$$

Note that the product $U \times U$ belongs to $\mathcal{U}$, and is therefore quasi-separated. Since U and and $U \times_X U$ are quasi-compact (the second by virtue of our assumption that X is quasi-separated), it follows that $\mathrm{Eq}(U \rightrightarrows X)$ is also quasi-compact.

We now prove the converse. Assume that condition ($*$) is satisfied; we wish to prove that X is quasi-separated. Choose quasi-compact objects $U, V \in \mathcal{X}$ equipped with maps $U \to X \leftarrow V$; we wish to show that the fiber product $U \times_X V$ is quasi-compact. Unwinding the definitions, we can identify $U \times_X V$ with the equalizer of a diagram $(U \times V) \rightrightarrows X$. The desired result now follows from ($*$), since $U \times V$ is a quasi-compact object of $\mathcal{X}$ (Remark C.5.8). ☐

Lemma C.5.11. *Let $\mathcal{X}$ be a coherent Grothendieck topos and let $X, Y \in \mathcal{X}$ be quasi-separated objects. Then the product $X \times Y$ is quasi-separated.*

Proof. Suppose we are given a pair of maps $U \to X \times Y \leftarrow V$, where U and V are quasi-compact. We wish to show that the fiber product $Z = U \times_{X \times Y} V$ is quasi-compact. This follows from Lemma C.5.8, since Z can be identified with the equalizer of a pair of maps $U \times_X V \rightrightarrows Y$. ☐

Proof of Proposition C.5.9. Let $\mathcal{X}$ be a coherent Grothendieck topos. Then the final object $\mathbf{1} \in \mathcal{X}$ is coherent (since $\mathbf{1}$ belongs to any full subcategory $\mathcal{U} \subseteq \mathcal{X}$ satisfying the requirements of Definition C.5.7). It will therefore suffice to show that for every diagram $U \to X \leftarrow V$ in $\mathcal{X}_{\mathrm{coh}}$, the fiber product $U \times_X V$ is coherent. The quasi-compactness of $U \times_X V$ follows from our assumption that U and V are quasi-compact and that X is quasi-separated. To show that $U \times_X V$ is quasi-separated, it will suffice to show that the product $U \times V$ is quasi-separated (Remark C.5.6), which is a special case of Lemma C.5.11. ☐

We now establish some further closure properties of coherent objects.

Proposition C.5.12. *Let $\mathcal{X}$ be a coherent Grothendieck topos. Then the full subcategory $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is closed under finite coproducts.*

Proof. Let $\{X_i\}_{i \in I}$ be a collection of coherent objects of $\mathcal{X}$ indexed by a finite set I, having coproduct $X = \coprod_{i \in I} X_i$. Then X is quasi-compact (Proposition C.5.2); we must show that it is also quasi-separated. Suppose we are given maps $U \to X \leftarrow V$, where U and V are quasi-compact; we wish to show that the fiber product $U \times_X V$ is quasi-compact. For each $i \in I$, set $U_i = U \times_X X_i$ and $V_i = V \times_X X_i$. Then each U_i is summand of U, hence quasi-compact (Remark C.5.3); similarly each V_i is quasi-compact. It follows that each of the fiber products $U_i \times_{X_i} V_i$ is quasi-compact, so that the coproduct

$$\coprod_{i \in I} U_i \times_{X_i} V_i \simeq U \times_X V$$

is quasi-compact by virtue of Proposition C.5.2. ☐

Proposition C.5.13. *Let $\mathcal{X}$ be a coherent Grothendieck topos. Suppose that we are given an effective epimorphism $f : X \twoheadrightarrow Y$ in $\mathcal{X}$. If X is coherent and the equivalence relation $R = X \times_Y X$ is quasi-compact, then Y is coherent.*

Proof. It follows from Proposition C.5.2 that Y is quasi-compact. We claim that Y is quasi-separated. Suppose we are given morphisms $U \xrightarrow{f} Y \xleftarrow{g} V$, where U and V are quasi-compact objects of $\mathcal{X}$; we wish to show that the fiber product $U \times_Y V$ is quasi-compact. Set $\widetilde{U} = U \times_Y X$, so that we have an effective epimorphism $\widetilde{U} \times_Y X \to U \times_Y X$. By virtue of Proposition C.5.2, we can replace U by $\widetilde{U}$ and thereby reduce to the case where f lifts to a map $\overline{f} : U \to X$. Similarly, we may assume that g lifts to a map $\overline{g} : V \to X$. In this case, we can identify $U \times_Y V$ with the fiber product $(U \times V) \times_{X \times X} R$. Since $X \times X$ is quasi-separated (Lemma C.5.11) and R is quasi-compact, we are reduced to showing that the product $U \times V$ is quasi-compact, which is a special case of Remark C.5.8. ☐

Corollary C.5.14. *Let $\mathcal{X}$ be a coherent Grothendieck topos. Then the full subcategory $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is an essentially small pretopos.*

Proof. The category $\mathcal{X}_{\mathrm{coh}}$ is essentially small by virtue of Proposition C.5.4. To show that it is a pretopos, it will suffice (by virtue of Example A.4.3) to show that the full subcategory $\mathcal{X}_{\mathrm{coh}} \subseteq \mathcal{X}$ is closed under the formation of finite limits, finite coproducts, and quotients by equivalence relations. This follows from Propositions C.5.9, C.5.12, and C.5.13. ☐

C.6. **Finitary Grothendieck Topologies.** We now provide some examples of coherent Grothendieck topoi.

Definition C.6.1. Let $\mathcal{C}$ be a category. We say that a Grothendieck topology on $\mathcal{C}$ is *finitary* if, for every collection of morphisms $\{f_i : U_i \to X\}_{i \in I}$ in $\mathcal{C}$ which cover X, there exists a finite subset $I_0 \subseteq I$ such that the collection of morphisms $\{f_i : U_i \to X\}_{i \in I_0}$ is also a covering of X.

Example C.6.2. Let $\mathcal{C}$ be a regular category (extensive category, regular and extensive category). Then the regular topology (extensive topology, coherent topology) on $\mathcal{C}$ is finitary.

Proposition C.6.3. *Let $\mathcal{C}$ be a small category which admits finite limits which is equipped with a finitary Grothendieck topology. Then the Grothendieck topos* $\mathrm{Shv}(\mathcal{C})$ *is coherent. Moreover, the sheafified Yoneda embedding $\widetilde{h} : \mathcal{C} \to \mathrm{Shv}(\mathcal{C})$ carries each object of $\mathcal{C}$ to a coherent object of* $\mathrm{Shv}(\mathcal{C})$.

Proof. Let $\mathcal{U} \subseteq \mathrm{Shv}(\mathcal{C})$ denote the full subcategory spanned by objects of the form $\widetilde{h}_C$, where $C \in \mathcal{C}$. Then $\mathcal{U}$ generates the topos $\mathrm{Shv}(\mathcal{C})$. Moreover, the functor $C \mapsto \widetilde{h}_C$ preserves finite limits, and therefore finite products. It follows that $\mathcal{U}$ is closed under finite products. We will complete the proof by showing that for each $C \in \mathcal{C}$, the sheaf $\widetilde{h}_C$ is quasi-compact and quasi-separated as an object of $\mathrm{Shv}(\mathcal{C})$.

We first verify quasi-compactness. Choose a covering $\{\mathscr{F}_i \to \widetilde{h}_C\}_{i \in I}$ in the Grothendieck topos $\mathrm{Shv}(\mathcal{C})$. Note that the identity map $\mathrm{id}_C : C \to C$ determines a section $s \in \widetilde{h}_C(C)$. It follows that there exists a covering $\{C_j \to C\}_{j \in J}$ in the category $\mathcal{C}$ such that, for each $j \in J$, the image $s_j \in \widetilde{h}_C(C_j)$ of s can be lifted to an element $\widetilde{s}_j \in \mathscr{F}_{i_j}(C_j)$ for some $i_j \in I$. Since the topology on $\mathcal{C}$ is finitary, we may assume without loss of generality that J is finite. Setting $I_0 = \{i_j\}_{j \in J} \subseteq I$, we deduce that $\{\mathscr{F}_i \to \widetilde{h}_C\}_{i \in I_0}$ is a finite subcover of $\{\mathscr{F}_i \to \widetilde{h}_C\}_{i \in I}$.

We now argue that each of the sheaves $\widetilde{h}_C$ is quasi-separated. Choose quasi-compact objects $\mathscr{F}, \mathscr{G} \in \mathrm{Shv}(\mathcal{C})$ equipped with maps $\mathscr{F} \to \widetilde{h}_C \leftarrow \mathscr{G}$; we wish to show that the fiber product $\mathscr{F} \times_{\widetilde{h}_C} \mathscr{G}$ is quasi-compact. Note that $\mathscr{F}$ admits a covering $\{\mathscr{F}_i \to \mathscr{F}\}_{i \in I}$, where each $\mathscr{F}_i$ belongs to $\mathcal{U}$. Since $\mathscr{F}$ is quasi-compact, we may assume that I is finite. Then $\{\mathscr{F}_i \times_{\widetilde{h}_C} \mathscr{G}\}_{i \in I}$ is a finite covering of $\mathscr{F} \times_{\widetilde{h}_C} \mathscr{G}$. It will therefore suffice to show that each $\mathscr{F}_i \times_{\widetilde{h}_C} \mathscr{G}$ is quasi-compact. Replacing $\mathscr{F}$ by $\mathscr{F}_i$, we are reduced to the case where $\mathscr{F}$ has the form $\widetilde{h}_D$ for some object $D \in \mathcal{C}$. In this case, the map $\mathscr{F} \to \widetilde{h}_C$ can be identified with an element of $\widetilde{h}_C(D)$. Passing to a covering of D (which we may also assume to be finite), we may assume that this element lies in the image of the map $h_C(D) \to \widetilde{h}_C(D)$. In other words, we may assume that the map $\mathscr{F} \to \widetilde{h}_C$ arises from applying the functor $\widetilde{h}_\bullet$ to a morphism $D \to C$ in the category $\mathcal{C}$. Similarly, we may assume that the map

$\mathscr{G} \to \widetilde{h}_C$ arises by applying $\widetilde{h}_\bullet$ to a morphism $E \to C$ in $\mathcal{C}$. In this case, the fiber product $\mathscr{F} \times_{\widetilde{h}_C} \mathscr{G}$ can be identified with $\widetilde{h}_{D \times_C E}$, which is quasi-compact by the first part of the argument. □

Proposition C.6.4. *Let $\mathcal{X}$ be a Grothendieck topos. The following conditions are equivalent:*

- (a) *The Grothendieck topos $\mathcal{X}$ is coherent (in the sense of Definition C.5.7).*
- (b) *There exists a small pretopos $\mathcal{C}$ and an equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is equipped with the coherent topology.*
- (c) *There exists a small category $\mathcal{C}$ which admits finite limits, a finitaryGrothendieck topology on $\mathcal{C}$, and an equivalence of categories $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$.*

Proof. The implication $(b) \Rightarrow (c)$ is immediate (Example C.6.2), and the implication $(c) \Rightarrow (a)$ follows from Proposition C.6.3. We will show that $(a) \Rightarrow (b)$. Assume that $\mathcal{X}$ is coherent, and let $\mathcal{X}_{\mathrm{coh}}$ denote the full subcategory of $\mathcal{X}$ spanned by the coherent objects. Then $\mathcal{X}_{\mathrm{coh}}$ is an essentially small pretopos (Corollary C.5.14). Let $\mathcal{C} \subseteq \mathcal{X}_{\mathrm{coh}}$ be a small full subcategory which is equivalent to $\mathcal{X}$. Then the subcategory $\mathcal{C}$ generates $\mathcal{X}$, so Corollary C.4.9 supplies an equivalence $\mathcal{X} \simeq \mathrm{Shv}(\mathcal{C})$, where $\mathcal{C}$ is equipped with the restricted canonical topology of Remark C.4.6. By definition, a collection of morphisms $\{C_i \to C\}_{i \in I}$ in $\mathcal{C}$ form a covering for the restricted canonical topology if and only if the induced map $\coprod_{i \in I} C_i \to C$ is an effective epimorphism in $\mathcal{X}$. Since C is a quasi-compact object of $\mathcal{X}$, this is equivalent to the requirement that there exists a finite subset $I_0 \subseteq I$ such that the map $\coprod_{i \in I_0} C_i \to C$ is an effective epimorphism in $\mathcal{X}$, or equivalently in the pretopos $\mathcal{C}$ itself. It follows that the restricted canonical topology on $\mathcal{C}$ coincides with the coherent topology of Definition B.5.3, so that $\mathcal{X}$ satisfies (b). □

The pretopos $\mathcal{C}$ appearing in part (b) of Proposition C.6.4 is actually determined (up to equivalence) by the Grothendieck topos $\mathcal{X}$: it can always be identified with the category $\mathcal{X}_{\mathrm{coh}}$ of coherent objects of $\mathcal{X}$. This is a consequence of the following:

Theorem C.6.5. *Let $\mathcal{C}$ be a small pretopos, which we endow with the coherent topology of Definition B.5.3. Then the Yoneda embedding $h : \mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C})$ induces an equivalence of categories $\mathcal{C} \simeq \mathrm{Shv}(\mathcal{C})_{\mathrm{coh}}$.*

Proof. It is clear that the Yoneda embedding $h : \mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C})$ is fully faithful, and Proposition C.6.3 guarantees that the essential image of h is contained in $\mathrm{Shv}(\mathcal{C})_{\mathrm{coh}}$. To establish Theorem C.6.5, we must prove the converse: that every coherent object $\mathscr{F} \in \mathrm{Shv}(\mathcal{C})$ is representable by an object of $\mathcal{C}$. Choose a covering $\{h_{X_i} \to \mathscr{F}\}_{i \in I}$ in $\mathrm{Shv}(\mathcal{C})$. Since $\mathscr{F}$ is quasi-compact, we can assume that I is finite. Setting $X = \coprod_{i \in I} X_i$, we can arrange that there is an effective epimorphism $\alpha : h_X \twoheadrightarrow \mathscr{F}$ for some object $X \in \mathcal{C}$.

We now complete the proof in the special case where there exists a monomorphism $\beta : \mathscr{F} \hookrightarrow h_Y$, for some object Y in $\mathcal{C}$. In this case, we can identify $\mathscr{F}$ with the image of the composite map $h_X \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} h_Y$. Since the Yoneda embedding is fully faithful, this composite map is induced by a morphism $u : X \to Y$ in the category $\mathcal{C}$. It follows from the definition of the coherent topology that the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Shv}(\mathcal{C})$ preserves finite limits and effective epimorphisms, and therefore commutes with the formation of images. In particular, we obtain an isomorphism $\mathscr{F} \simeq \mathrm{Im}(h_u) \simeq h_{\mathrm{Im}(u)}$, so that $\mathscr{F}$ is representable by an object of $\mathcal{C}$ as desired.

We now treat the general case. Let $\alpha : h_X \twoheadrightarrow \mathscr{F}$ be as above, and set $\mathscr{R} = h_X \times_{\mathscr{F}} h_X$. Then $\mathscr{R}$ is a coherent object of $\mathrm{Shv}(\mathcal{C})$ (Proposition C.5.9), and it can be realized as a subsheaf of $h_X \times h_X \simeq h_{X \times X}$. Applying the first part of the argument, we can write $\mathscr{R} = h_R$ for some subobject $R \subseteq X \times X$. Using the fact that $\mathscr{R}$ is an equivalence relation on h_X, it follows easily that R is an equivalence relation on X in the pretopos $\mathcal{C}$. Because $\mathcal{C}$ is exact, there exists an effective epimorphism $v : X \twoheadrightarrow X/R$ having the property that $R = X \times_{X/R} X$ (as subobjects of $X \times X$). Applying the Yoneda embedding, we obtain an effective epimorphism of sheaves $h_X \twoheadrightarrow h_{X/R}$ such that $h_X \times_{h_{X/R}} h_X \simeq h_R \simeq \mathscr{R}$. It follows that $h_{X/R}$ can be identified with the quotient of h_X by the equivalence relation $\mathscr{R}$, and is therefore isomorphic to $\mathscr{F}$. □

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