On Brauer Groups of Lubin-Tate Spectra I

Michael Hopkins and Jacob Lurie

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Contents

Chapter 1

Introduction

Let A be an associative ring spectrum. We will say that A is a *Morava K-theory* if it satisfies the following conditions (i) and (ii):

(i) The homotopy ring $\pi_* A$ is isomorphic to a Laurent polynomial ring $\kappa[t^{\pm 1}]$, where κ is a perfect field of characteristic $p > 0$ and $\deg(t) = 2$.

It follows from (i) that the cohomology $A^0(\mathbf{C}\mathrm{P}^\infty)$ is (non-canonically) isomorphic to a power series ring $\kappa[[e]]$, so that the formal spectrum $\mathbf{G}_0 = \operatorname{Spf} A^0(\mathbf{C}\mathrm{P}^\infty)$ can be regarded as a 1-dimensional formal group over κ.

(ii) The formal group $\mathbf{G}_0$ has finite height (that is, it is not isomorphic to the formal additive group).

Lubin and Tate have shown that condition (ii) implies that the formal group $\mathbf{G}_0$ (which is defined over κ) admits a universal deformation $\mathbf{G}$ (which is defined over a complete local ring R with residue field κ). We will refer to R as the *Lubin-Tate ring* of the pair $(\kappa, \mathbf{G}_0)$; it is non-canonically isomorphic to a power series ring $W(\kappa)[[v_1, \ldots, v_{n-1}]]$. Applying the Landweber exact functor theorem to the pair $(R, \mathbf{G})$, one deduces that there is an essentially unique cohomology theory E satisfying the following conditions:

(i') The homotopy ring $\pi_* E$ is isomorphic to a Laurent polynomial ring $R[t^{\pm 1}]$.

(ii') The formal spectrum $\operatorname{Spf} E^0(\mathbf{C}\mathrm{P}^\infty)$ is isomorphic to $\mathbf{G}$ (as a formal group over R).

We then have the following result:

Theorem 1.0.1 (Goerss-Hopkins-Miller)**.** *The cohomology theory E is (representable by) a commutative ring spectrum, which is unique up to a contractible space of choices and depends functorially on the pair $(\kappa, \mathbf{G}_0)$.*

We will refer to the commutative ring spectrum E of Theorem 1.0.1 as the *Lubin-Tate spectrum associated to A* (it is also commonly referred to as *Morava E-theory*). One can show that there is an essentially unique E-algebra structure on the ring spectrum A which is compatible with the identification between $\mathbf{G}_0$ and the special fiber of $\mathbf{G}$. For many purposes, it is useful to think of the Morava K-theory A as the "residue field" of the Lubin-Tate spectrum E (in the same way that $\kappa \simeq \pi_0 A$ is the residue field of the Lubin-Tate ring $R \simeq \pi_0 E$). However, this heuristic has the potential to be misleading, for two (related) reasons:

(a) Morava K-theories A can *never* be promoted to commutative ring spectra (in fact, if the field κ has characteristic 2, Morava K-theories are not even homotopy commutative).

(b) As an associative ring spectrum, the Morava K-theory A cannot be recovered from the Lubin-Tate spectrum E.

To elaborate on (b), it is useful to introduce some terminology.

Definition 1.0.2. Let E be a Lubin-Tate spectrum, so that $R = \pi_0 E$ is a complete regular local ring with maximal ideal $\mathfrak{m} \subseteq R$. We will say that an E-algebra A is *atomic* if the unit map $E \to A$ induces an isomorphism $(\pi_* E)/\mathfrak{m}(\pi_* E) \simeq \pi_* A$.

If A is a Morava K-theory and E is its associated Lubin-Tate spectrum, then A can be regarded as an atomic E-algebra. Conversely, if E is a Lubin-Tate spectrum and A is an atomic E-algebra, then A is a Morava K-theory whose associated Lubin-Tate spectrum can be identified with E. Using Definition 1.0.2, we can rephrase (a) and (b) as follows:

(a') If E is a Lubin-Tate spectrum, then atomic E-algebras are never commutative.

(b') If E is a Lubin-Tate spectrum, then not all atomic E-algebras are equivalent (at least as E-algebras; one can show that they are all equivalent as E-modules).

Motivated by (b'), we ask the following:

Question 1.0.3. Let E be a Lubin-Tate spectrum. Can one classify the atomic E-algebras, up to equivalence?

Remark 1.0.4. Question 1.0.3 is essentially equivalent to the problem of classifying Morava K-theories up to equivalence *as associative ring spectra*. Note that the datum of a Morava K-theory is equivalent to the data of a triple $(\kappa, \mathbf{G}_0, A)$, where κ is a perfect field of characteristic $p > 0$, $\mathbf{G}_0$ is a one-dimensional formal group of finite height over κ, and A is an atomic algebra over the Lubin-Tate spectrum of the pair $(\kappa, \mathbf{G}_0)$.

Before describing our approach to Question 1.0.3, let us consider a similar problem in a more familiar setting.

Definition 1.0.5. Let K be a field. We will say that a K-algebra A is an *Azumaya algebra* if $0 < \dim_K(A) < \infty$ and the actions of A on itself by left and right multiplication induce an isomorphism $A \otimes_K A^{\mathrm{op}} \to \mathrm{End}_K(A)$.

We say that Azumaya algebras A and B are *Morita equivalent* if the tensor product $A \otimes_K B^{\mathrm{op}}$ is isomorphic to a matrix ring $\mathrm{End}_K(V)$, for some vector space V over K. We let $\mathrm{Br}(K)$ denote the set of Morita equivalence classes of Azumaya algebras over K. If A is an Azumaya algebra over K, we let $[A] \in \mathrm{Br}(K)$ denote the equivalence class of A. We refer to $\mathrm{Br}(K)$ as the *Brauer group* of K.

The essential features of Definition 1.0.5 can be summarized as follows:

(*i*) For any field K, the set $\mathrm{Br}(K)$ can be equipped with the structure of an abelian group, whose addition law satisfies the formulae

$$[A] + [B] = [A \otimes_K B] \qquad 0 = [K] \qquad -[A] = [A^{\mathrm{op}}].$$

(*ii*) Let D be a *central division algebra* over K: that is, a finite-dimensional K-algebra whose center is K in which every nonzero element is invertible. Then D is an Azumaya algebra over K.

(*iii*) The construction $D \mapsto [D]$ induces a bijection

$$\{\text{Central division algebras over } K\}/\text{Isomorphism} \to \mathrm{Br}(K).$$

It follows from (*i*), (*ii*) and (*iii*) that the problem of classifying central algebras over K (up to isomorphism) has more structure than one might naively expect: the collection of isomorphism classes has the structure of an abelian group. We would like to apply similar ideas to the analysis of Question 1.0.3: roughly speaking, we want to think of atomic E-algebras as analogous to "division algebras" over E, and organize them into some sort of Brauer group.

Definition 1.0.6. Let E be a Lubin-Tate spectrum. We will say that an E-algebra A is an *Azumaya algebra* if it is nonzero, dualizable as an E-module spectrum, and the natural map $A \otimes_E A^{\mathrm{op}} \to \mathrm{End}_E(A)$ is a homotopy equivalence.

We say that Azumaya algebras A and B are *Morita equivalent* if the relative smash product $A \otimes_E B^{\mathrm{op}}$ is equivalent to $\mathrm{End}_E(V)$ for some dualizable E-module V. We let $\mathrm{Br}(E)$ denote the set of Morita equivalence classes of Azumaya algebras over E. We will refer to $\mathrm{Br}(E)$ as the *Brauer group of E*.

Remark 1.0.7. The Brauer group $\mathrm{Br}(E)$ was introduced by Baker, Richter and Szymik in [3]. For atomic E-algebras, the Azumaya condition has appeared in the work in Angeltveit ([2]).

As the terminology suggests, the Brauer group $\mathrm{Br}(E)$ can be regarded as an (abelian) group: just as in classical algebra, it comes equipped with an addition law which satisfies the formulae

$$[A] + [B] = [A \otimes_E B] \qquad 0 = [E] \qquad -[A] = [A^{\mathrm{op}}].$$

However, the homotopy-theoretic analogues of *(ii)* and *(iii)* are not as strong:

(1) An atomic E-algebra A need not be an Azumaya algebra. For example, if the residue field κ has characteristic different from 2, then there exist atomic E-algebras which are homotopy commutative; such algebras are never Azumaya.

(2) Not every Azumaya algebra over E is Morita equivalent to an atomic E-algebra. For example, the Lubin-Tate spectrum E itself is not Morita equivalent to an atomic E-algebra.

Because of (1), we cannot completely rephrase Question 1.0.3 in terms of the Brauer group $\mathrm{Br}(E)$. We therefore restrict our attention to a slightly less ambitious problem:

Question 1.0.8. Let E be a Lubin-Tate spectrum. Can one classify the atomic Azumaya algebras over E, up to equivalence?

It is not difficult to show that atomic Azumaya algebras are equivalent (as E-algebras) if and only if they are Morita equivalent (Proposition 10.1.1). It follows that the construction $A \mapsto [A]$ induces a monomorphism of sets

$$\theta : \{\text{Atomic Azumaya algebras over } E\}/\text{Equivalence} \hookrightarrow \mathrm{Br}(E).$$

Consequently, we can break Question 1.0.8 into two parts:

Question 1.0.9. Describe the Brauer group $\mathrm{Br}(E)$ of a Lubin-Tate spectrum E.

Question 1.0.10. Describe the image of the map θ (as a subset of $\mathrm{Br}(E)$).

Our primary goal in this paper is to address Questions 1.0.9 and 1.0.10. For simplicity, let us assume that the field κ has characteristic different from 2. Our main results can be summarized as follows:

Theorem 1.0.11. *Let E be a Lubin-Tate spectrum, let $\mathfrak{m}$ denote the maximal ideal in the Lubin-Tate ring $R = \pi_0 E$, and assume that the residue field $\kappa = R/\mathfrak{m}$ has characteristic $\neq 2$. Then the Brauer group $\mathrm{Br}(E)$ is isomorphic to a direct product $\mathrm{BW}(\kappa) \times \mathrm{Br}'(E)$, where $\mathrm{BW}(\kappa)$ is the Brauer-Wall group of κ (see §2.8) and $\mathrm{Br}'(E)$ is the inverse limit of a tower of abelian groups*

$$\cdots \to \mathrm{Br}'_4 \to \mathrm{Br}'_3 \to \mathrm{Br}'_2 \xrightarrow{\rho_2} \mathrm{Br}'_1 \to \mathrm{Br}'_0$$

where:

(a) *The group Br'_0 is isomorphic to $\mathfrak{m}^2/\mathfrak{m}^3$.*

(b) *For $k > 0$, the transition map $\mathrm{Br}'_k \to \mathrm{Br}'_{k-1}$ fits into a short exact sequence of abelian groups*
$$0 \to \mathfrak{m}^{k+2}/\mathfrak{m}^{k+3} \to \mathrm{Br}'_k \to \mathrm{Br}'_{k-1} \to 0.$$

(c) *Let x be an element of $\mathrm{Br}(E)$ having image $x' \in \mathrm{BW}(\kappa)$ and $x'' \in \mathrm{Br}_0 \simeq \mathfrak{m}^2/\mathfrak{m}^3$. Let us identify x'' with a quadratic form q on the Zariski tangent space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. Then x is representable by an atomic Azumaya algebra over E if and only if the quadratic form q is nondegenerate and x' is represented by the Clifford algebra Cl_q (as an element of the Brauer-Wall group $\mathrm{BW}(\kappa)$.*

Warning 1.0.12. In the statement of Theorem 1.0.11, the projection map $\mathrm{Br}(E) \to \mathrm{BW}(\kappa)$ and the isomorphisms

$$\ker(\mathrm{Br}'_k \to \mathrm{Br}'_{k-1}) \simeq \mathfrak{m}^{k+2}/\mathfrak{m}^{k+3}$$

are not quite canonical: they depend on choosing a nonzero element of $(\pi_2 E)/\mathfrak{m}(\pi_2 E)$. We refer the reader to the body of this paper for coordinate-independent statements (and for extensions to the case where κ has characteristic 2).

Let us now summarize our approach to Theorem 1.0.11. Our first goal is to give a precise definition of the Brauer group $\mathrm{Br}(E)$. In §2, we associate a Brauer group $\mathrm{Br}(\mathcal{C})$ to a symmetric monoidal ∞-category $\mathcal{C}$ (Definition 2.3.1). This notion simultaneously generalizes the classical Brauer group of a field K (obtained by taking $\mathcal{C}$ to be the category Vect_K of vector spaces over K), the classical Brauer-Wall group of a field K (obtained by taking $\mathcal{C}$ to be the category of $\mathbf{Z}/2\mathbf{Z}$-graded vector spaces over K), and the Brauer group $\mathrm{Br}(E)$ of interest to us in this paper (obtained by taking $\mathcal{C}$ to be the ∞-category $\mathrm{Mod}_E^{\mathrm{loc}}$ of $K(n)$-local E-module spectra). We adopt this general point of view not merely for the sake of generality, but in service of proving Theorem 1.0.11. Theorem 1.0.11 implies the existence of an inverse limit diagram of abelian groups

$$\mathrm{Br}(E) \to \cdots \mathrm{Br}_4 \to \mathrm{Br}_3 \to \mathrm{Br}_2 \to \mathrm{Br}_1 \to \mathrm{Br}_0,$$

where $\mathrm{Br}_k = \mathrm{BW}(\kappa) \times \mathrm{Br}'_k$. We will see that every term in this diagram can be conveniently realized as the Brauer group of a suitable symmetric monoidal ∞-category, and every map in the diagram is obtained by functoriality.

In §4 we introduce an ∞-category Syn_E which we refer to as the *∞-category of synthetic E-modules*. Our definition of the ∞-category Syn_E is inspired by the "resolution model category" technique introduced by Dwyer-Kan-Stover. The ∞-category Syn_E is equipped with a fully faithful symmetric monoidal embedding $\mathrm{Mod}_E^{\mathrm{loc}} \hookrightarrow \mathrm{Syn}_E$. The essential image of this embedding contains all dualizable objects of Syn_E and therefore induces an isomorphism of Brauer groups $\mathrm{Br}(E) = \mathrm{Br}(\mathrm{Mod}_E^{\mathrm{loc}}) \to \mathrm{Br}(\mathrm{Syn}_E)$. The unit object $\mathbf{1} \in \mathrm{Syn}_E$ comes equipped with a Postnikov filtration

$$\cdots \to \mathbf{1}^{\leqslant 4} \to \mathbf{1}^{\leqslant 3} \to \mathbf{1}^{\leqslant 2} \to \mathbf{1}^{\leqslant 1} \to \mathbf{1}^{\leqslant 0},$$

and the abelian groups Br_k described above can be realized as the Brauer groups of the ∞-categories $\mathrm{Syn}_{\mathbf{1}^{\leqslant k}} = \mathrm{Mod}_{\mathbf{1}^{\leqslant k}}(\mathrm{Syn}_E)$.

To understand the group Br_0, we need to analyze the *heart* $\mathrm{Syn}_E^{\heartsuit}$ of the ∞-category Syn_E. This is an abelian category, which we will refer to as the *category of Milnor modules*. In §6, we will carry out a detailed analysis of the category $\mathrm{Syn}_E^{\heartsuit}$. Our main result is that, if the field κ has characteristic different from 2, then there is a symmetric monoidal equivalence of $\mathrm{Syn}_E^{\heartsuit}$ with the category of $\mathbf{Z}/2\mathbf{Z}$-graded modules over the exterior algebra $\bigwedge^*(\mathfrak{m}/\mathfrak{m}^2)^\vee$ (Proposition 6.9.1). Using this equivalence together with a purely algebraic analysis (which we carry out in §5), we obtain a (not quite canonical) isomorphism of abelian groups $\mathrm{Br}_0 \simeq \mathrm{BW}(\kappa) \times \mathfrak{m}^2/\mathfrak{m}^3$ (Remark 6.9.4).

In §8, we prove the bulk of Theorem 1.0.11 by establishing that the canonical map $\mathrm{Br}(E) \to \varprojlim \mathrm{Br}_k$ is an isomorphism and analyzing the transition maps $\mathrm{Br}_k \to \mathrm{Br}_{k-1}$ (see Theorem 8.0.5). To carry this out, we will need to understand the relationship between (Azumaya) algebras over $\mathbf{1}^{\leqslant k}$ and (Azumaya) algebras over $\mathbf{1}^{\leqslant k-1}$ in the ∞-category Syn_E of synthetic E-modules. This is a deformation-theoretic problem which can be reduced to the calculation of certain Hochschild cohomology groups, which we compute in §7.

In §10, we turn to the study of atomic E-algebras. In particular, we prove that atomic E-algebras are equivalent if and only if they are Morita equivalent (Proposition 10.1.1) and explain how the classification of atomic E-algebras relates to the algebraic analysis of §6 and the obstruction theory of §8. We then combine these results to prove part (c) of Theorem 1.0.11 (see Theorem 10.3.1).

To complete the proof of Theorem 1.0.11, it will suffice to show that the composite map

$$\mathrm{Br}(E) \to \mathrm{Br}_0 \simeq \mathrm{BW}(\kappa) \times \mathfrak{m}^2/\mathfrak{m}^3 \to \mathrm{BW}(\kappa)$$

admits a section. We will prove this in §9 by constructing a subgroup $\mathrm{Br}^\flat(E) \subseteq \mathrm{Br}(E)$ which maps isomorphically to the Brauer-Wall group $\mathrm{BW}(\kappa)$ (see Theorem 9.3.1).

Concretely, the subgroup $\mathrm{Br}^{\flat}(E)$ will consist of those Brauer classes which can be represented by an Azumaya algebra A for which $\pi_* A$ is a free module over $\pi_*(E)$.

Remark 1.0.13. Theorem 1.0.11 does not provide a completely satisfying answer to Question 1.0.9: it computes the Brauer group $\mathrm{Br}(E)$ only up to filtration (thus giving a rough sense of how large it is), but does not describe the extensions which appear. We will return to this problem in a sequel to this paper.

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Chapter 2

Brauer Groups

In this section, we define the Brauer group $\mathrm{Br}(\mathcal{C})$ of a symmetric monoidal ∞-category $\mathcal{C}$ (Definition 2.3.1) and discuss several examples.

2.1 Morita Equivalence

Throughout this section, we fix a symmetric monoidal ∞-category $\mathcal{C}$ satisfying the following condition:

$(*)$ The ∞-category $\mathcal{C}$ admits geometric realizations of simplicial objects, and the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects.

Definition 2.1.1. Let A and B be associative algebra objects of $\mathcal{C}$. We will say that A and B are *Morita equivalent* if there exists a $\mathcal{C}$-linear equivalence $\mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{LMod}_B(\mathcal{C})$.

Our first goal is to characterize those algebras $A \in \mathrm{Alg}(\mathcal{C})$ which are *Morita trivial*: that is, which are Morita equivalent to the unit object $\mathbf{1} \in \mathrm{Alg}(\mathcal{C})$.

Definition 2.1.2. Let M be an object of $\mathcal{C}$. We will say that M is *full* if the construction $X \mapsto M \otimes X$ determines a conservative functor from $\mathcal{C}$ to itself.

Proposition 2.1.3. *Let A be an associative algebra object of $\mathcal{C}$ and let M be a left A-module, so that the construction $(X \in \mathcal{C}) \mapsto (M \otimes X \in \mathrm{LMod}_A(\mathcal{C}))$ determines a functor $T : \mathcal{C} \to \mathrm{LMod}_A(\mathcal{C})$. Then T is an equivalence of ∞-categories if and only if the following conditions are satisfied:*

(i) *The object M is dualizable in $\mathcal{C}$.*

(*ii*) *The left action of A on M induces an equivalence $A \simeq \mathrm{End}(M)$.*

(*iii*) *The object M is full (Definition 2.1.2).*

Corollary 2.1.4. *Let A be an associative algebra object of $\mathcal{C}$. The following conditions are equivalent:*

(*a*) *The algebra A is Morita equivalent to the unit algebra $\mathbf{1} \in \mathrm{Alg}(\mathcal{C})$.*

(*b*) *There exists an equivalence $A \simeq \mathrm{End}(M)$, where $M \in \mathcal{C}$ is full and dualizable.*

Corollary 2.1.5. *Let M and N be full dualizable objects of $\mathcal{C}$, and suppose that there exists an equivalence $\alpha : \mathrm{End}(M) \simeq \mathrm{End}(N)$ in $\mathrm{Alg}(\mathcal{C})$. Then there exists an invertible object $L \in \mathcal{C}$ and an equivalence $u : L \otimes M \simeq N$ such that α factors as a composition $\mathrm{End}(M) \simeq \mathrm{End}(L \otimes M) \xrightarrow{u} \mathrm{End}(N)$.*

Proof. Using Proposition 2.1.3, we obtain a commutative diagram of $\mathcal{C}$-linear equivalences

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C} \\ \downarrow{\scriptstyle N\otimes\bullet} & & \downarrow{\scriptstyle M\otimes\bullet} \\ \mathrm{LMod}_{\mathrm{End}(N)}(\mathcal{C}) & \longrightarrow & \mathrm{LMod}_{\mathrm{End}(M)}(\mathcal{C}), \end{array}$$

where the bottom horizontal map is given by restriction of scalars along α. We conclude by observing that F is given by tensor product with an object $L \in \mathcal{C}$, which is invertible by virtue of the fact that F is an equivalence. □

Proof of Proposition 2.1.3. Assume first that T is an equivalence; we will show that conditions (*i*), (*ii*) and (*iii*) are satisfied. We begin with condition (*i*). Using the essential surjectivity of T to choose equivalence of left A-modules $A \simeq M \otimes N = T(N)$, for some object $N \in \mathcal{C}$. Let $c : \mathbf{1} \to M \otimes N$ be the composition of this equivalence with the unit map $\mathbf{1} \to A$. The action of A on M determines a morphism of left A-modules

$$T(N \otimes M) \simeq T(N) \otimes M \simeq A \otimes M \to M = T(\mathbf{1}).$$

Since T is full, we can assume that this map has the form $T(e)$, for some morphism $e : N \otimes M \to \mathbf{1}$ in the ∞-category $\mathcal{C}$. We claim that e and c determine a duality between M and N: that is, that the composite maps

$$\alpha : M \simeq \mathbf{1} \otimes M \xrightarrow{c \otimes \mathrm{id}} M \otimes N \otimes M \xrightarrow{\mathrm{id} \otimes e} M \otimes \mathbf{1} \simeq M$$

$$\beta : N \simeq N \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes c} N \otimes M \otimes N \xrightarrow{e \otimes \mathrm{id}} \mathbf{1} \otimes N \simeq N$$

are homotopic to the identity maps on M and N, respectively. The existence of a homotopy $\alpha \simeq \mathrm{id}_M$ follows immediately from the definition of e. To verify that β is homotopic to the identity, it will suffice (by virtue of the faithfulness of T) to show that $T(\beta)$ is homotopic to the identity on $T(N) \simeq A$. It now suffices to observe that $T(\beta)$ can be identified with the composition $A \simeq A \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes u} A \otimes A \xrightarrow{m} A$, where $u : \mathbf{1} \to A$ is the unit map and $m : A \otimes A \to A$ is the multiplication on A. This completes the proof of (i).

To verify conditions (ii) and (iii), let $G' : \mathcal{C} \to \mathcal{C}$ denote the functor given by $G(X) = M \otimes X$. We then have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathrm{LMod}_A(\mathcal{C}) \\ & {\scriptstyle G}\searrow \quad \swarrow{\scriptstyle G'} & \\ & \mathcal{C}, & \end{array}$$

where G' is the forgetful functor. Condition (i) guarantees that G admits a left adjoint $F : \mathcal{C} \to \mathcal{C}$, given by $F(Y) = M^\vee \otimes Y$. Note that G' also admits a left adjoint F', given by $F'(Y) = A \otimes Y$. The diagram σ induces a natural transformation of functors $\gamma : G' \circ F' \to G \circ F$. Unwinding the definitions, we can restate conditions (ii) and (iii) as follows:

(ii') The natural transformation γ is an equivalence.

(iii') The functor G is conservative.

We now observe that if T is an equivalence, then assertion (ii') is automatic and assertion (iii') follows from the observation that G' is conservative.

Conversely, suppose that (i), (ii), and (iii) are satisfied. Applying Corollary HA.4.7.4.16 (and Remark HA.4.7.4.17) to the diagram σ, we deduce that T is an equivalence. $\square$

2.2 Azumaya Algebras

Throughout this section, we continue to assume that $\mathcal{C}$ is a symmetric monoidal ∞-category satisfying the following:

$(*)$ The ∞-category $\mathcal{C}$ admits geometric realizations of simplicial objects, and the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects.

Definition 2.2.1. Let A be an associative algebra object of $\mathcal{C}$. We will say that A is an *Azumaya algebra* if there exists an associative algebra object $B \in \mathcal{C}$ such that $A \otimes B$ is Morita equivalent to the unit object $\mathbf{1} \in \mathcal{C}$ (which we identify with the initial object of $\mathrm{Alg}(\mathcal{C})$).

Proposition 2.2.2. *Let A be an associative algebra object of $\mathcal{C}$. The following conditions are equivalent:*

(a) *The algebra A is Azumaya.*

(b) *The construction $X \mapsto A \otimes X$ induces an equivalence of ∞-categories $\mathcal{C} \to {}_A\mathrm{BMod}_A(\mathcal{C})$.*

Corollary 2.2.3. *Let A be an associative algebra object of $\mathcal{C}$. Then A is an Azumaya algebra if and only if it satisfies the following conditions:*

(i) *The algebra A is dualizable when regarded as an object of $\mathcal{C}$.*

(ii) *The left and right actions of A on itself induce an equivalence $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A)$.*

(iii) *The algebra A is full when regarded as an object of $\mathcal{C}$.*

Corollary 2.2.4. *Let A and B be Azumaya algebras in $\mathcal{C}$. Then the tensor product $A \otimes B$ is an Azumaya algebra.*

Proof of Proposition 2.2.2. Let $\mathcal{C}\mathrm{at}_\infty^\sigma$ denote the subcategory of $\mathcal{C}\mathrm{at}_\infty$ spanned by those ∞-categories which admit geometric realizations and those functors which preserve geometric realizations. Then the Cartesian product endows $\mathcal{C}\mathrm{at}_\infty^\sigma$ with the structure of a symmetric monoidal ∞-category. Moreover, $\mathcal{C}\mathrm{at}_\infty^\sigma$ is presentable and the Cartesian product preserves small colimits separately in each variable.

Let us regard $\mathcal{C}$ as a commutative algebra object of $\mathcal{C}\mathrm{at}_\infty^\sigma$, and set $\mathrm{Mod}_{\mathcal{C}}^\sigma = \mathrm{Mod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_\infty^\sigma)$. More informally, $\mathrm{Mod}_{\mathcal{C}}^\sigma$ is the ∞-category whose objects are ∞-categories $\mathcal{M}$ which are left-tensored over $\mathcal{C}$, for which $\mathcal{M}$ admits geometric realizations of simplicial objects and the action $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ preserves geometric realizations of simplicial objects, and the morphisms in $\mathrm{Mod}_{\mathcal{C}}^\sigma$ are $\mathcal{C}$-linear functors which commute with geometric realizations.

For each algebra object $A \in \mathrm{Alg}(\mathcal{C})$, we can regard the ∞-category $\mathrm{LMod}_A(\mathcal{C})$ as an object of $\mathrm{Mod}_{\mathcal{C}}^\sigma$. The bimodule ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$ can be identified with the tensor product (in $\mathrm{Mod}_{\mathcal{C}}^\sigma$) of $\mathrm{LMod}_A(\mathcal{C})$ and $\mathrm{LMod}_{A^{\mathrm{op}}}(\mathcal{C})$, where A^{op} denotes the opposite algebra of A. Moreover, the functor

$$\rho : \mathcal{C} \to {}_A\mathrm{BMod}_A(\mathcal{C}) \simeq \mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathrm{LMod}_{A^{\mathrm{op}}}(\mathcal{C}) \qquad X \mapsto X \otimes A$$

exhibits $\mathrm{LMod}_A(\mathcal{C})$ as a dual of $\mathrm{LMod}_{A^{\mathrm{op}}}(\mathcal{C})$ in the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathcal{C}}^\sigma$. Consequently, the functor ρ is an equivalence if and only if $\mathrm{LMod}_A(\mathcal{C})$ is an invertible object of $\mathrm{Mod}_{\mathcal{C}}^\sigma$: that is, if and only if A is an Azumaya algebra. □

Remark 2.2.5 (The Center of an Azumaya Algebras)**.** Suppose that the ∞-category $\mathcal{C}$ is presentable and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits in each variable. Then, to every associative algebra object $A \in \mathrm{Alg}(\mathcal{C})$, we can associate an $\mathbb{E}_2$-algebra object $\mathfrak{Z}(A) \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C})$, called the *center of* A, which is universal among those $\mathbb{E}_2$-algebras for which A an be promoted to an algebra object of the monoidal ∞-category $\mathrm{LMod}_{\mathfrak{Z}(A)}(\mathcal{C})$ (see Theorem HA.5.3.1.14). As an algebra object of $\mathcal{C}$, the center $\mathfrak{Z}(A)$ classifies endomorphisms of A as an object of the bimodule ∞-category ${}_A\mathrm{BMod}_A(\mathcal{C})$ (regarded as an ∞-category tensored over $\mathcal{C}$); see Theorem HA.4.4.1.28 .

In the special case where A is an Azumaya algebra object of $\mathcal{C}$, there exists a $\mathcal{C}$-linear equivalence $\mathcal{C} \to {}_A\mathrm{BMod}_A(\mathcal{C})$ which carries the unit object $\mathbf{1}$ to A. Consequently, we can identify the center $\mathfrak{Z}(A)$ with the endomorphism algebra $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq \mathbf{1}$ as an algebra object of $\mathcal{C}$. It follows that the unit map $\mathbf{1} \to \mathfrak{Z}(A)$ is an equivalence of associative algebra objects of $\mathcal{C}$, and therefore also an equivalence of $\mathbb{E}_2$-algebra objects of $\mathcal{C}$.

2.3 The Brauer Group

Throughout this section, we continue to assume that $\mathcal{C}$ is a symmetric monoidal ∞-category satisfying the following:

($*$) The ∞-category $\mathcal{C}$ admits geometric realizations of simplicial objects, and the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects.

Definition 2.3.1. We let $\mathrm{Br}(\mathcal{C})$ denote the set of Morita equivalence classes of Azumaya algebras $A \in \mathrm{Alg}(\mathcal{C})$. We will refer to $\mathrm{Br}(\mathcal{C})$ as the *Brauer group of* $\mathcal{C}$. If A is an Azumaya algebra, we let $[A]$ denote the Morita equivalence class of A in $\mathrm{Br}(\mathcal{C})$.

Proposition 2.3.2. *There exists a unique abelian group structure on the set* $\mathrm{Br}(\mathcal{C})$ *satisfying the following condition: for every pair of Azumaya algebras* $A, B \in \mathrm{Alg}(\mathcal{C})$*, we have* $[A \otimes B] = [A] + [B]$ *in* $\mathrm{Br}(\mathcal{C})$*.*

Proof. Let $\mathrm{Mod}^{\sigma}_{\mathcal{C}}$ be as in the proof of Proposition 2.2.2. Then $\mathrm{Mod}^{\sigma}_{\mathcal{C}}$ is a symmetric monoidal ∞-category. Let G denote the collection of isomorphism classes of invertible objects of $\mathrm{Mod}^{\sigma}_{\mathcal{C}}$, so that the tensor product on $\mathrm{Mod}^{\sigma}_{\mathcal{C}}$ endows G with the structure of an abelian group (which we will denote additively). The construction $[A] \mapsto \mathrm{LMod}_A(\mathcal{C})$ determines an injective map $\rho : \mathrm{Br}(\mathcal{C}) \to G$ satisfying $\rho([A \otimes B]) = \rho([A]) + \rho([B])$. It follows that the image of ρ is closed under addition. Moreover, $\rho([\mathbf{1}])$ is the unit element of G (given by the ∞-category $\mathcal{C}$, regarded as a module over itself). Using the identity $\rho([A^{\mathrm{op}}]) \simeq -\rho([A])$, we conclude that the image of ρ is a subgroup of G, so there is a unique abelian group structure on $\mathrm{Br}(\mathcal{C})$ for which the map ρ is a group homomorphism. □

Remark 2.3.3 (The Brauer Space)**.** For every symmetric monoidal ∞-category $\mathcal{C}$, let $\mathrm{Pic}(\mathcal{C})$ denote the subcategory of $\mathcal{C}$ spanned by the invertible objects and equivalences between them. Then $\mathrm{Pic}(\mathcal{C})$ is a grouplike $\mathbb{E}_\infty$-space, so that $\pi_0 \mathrm{Pic}(\mathcal{C})$ has the structure of an abelian group. If $\mathcal{C}$ admits geometric realizations and the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations, then we can identify $\mathcal{C}$ with a commutative algebra object of the ∞-category $\mathcal{C}\mathrm{at}_\infty^\sigma$ (as in the proof of Proposition 2.2.2). Let $\mathcal{E}$ denote the full subcategory of $\mathrm{Mod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_\infty^\sigma)$ spanned by those ∞-categories of the form $\mathrm{LMod}_A(\mathcal{C})$, where A is an associative algebra object of $\mathcal{C}$. We let $\mathbf{Br}(\mathcal{C})$ denote the space $\mathrm{Pic}(\mathcal{E})$ of invertible objects of $\mathcal{E}$. Then $\mathbf{Br}(\mathcal{C})$ is a nonconnected delooping of $\mathrm{Pic}(\mathcal{C})$: it is equipped with canonical equivalences

$$\Omega \, \mathbf{Br}(\mathcal{C}) \simeq \mathrm{Pic}(\mathcal{C}) \qquad \pi_0 \, \mathbf{Br}(\mathcal{C}) \simeq \mathrm{Br}(\mathcal{C}).$$

Remark 2.3.4. Since $\mathbf{Br}(\mathcal{C})$ is an infinite loop space, the homotopy groups $\pi_* \mathbf{Br}(\mathcal{C})$ can be regarded as a graded module over the ring $\pi_*(S)$ (where S denotes the sphere spectrum). In particular, the unique nonzero element $\eta \in \pi_1(S)$ induces a map $\mathrm{Br}(\mathcal{C}) = \pi_0 \, \mathbf{Br}(\mathcal{C}) \xrightarrow{\eta} \pi_1 \, \mathbf{Br}(\mathcal{C}) \simeq \pi_0 \, \mathrm{Pic}(\mathcal{C})$. Concretely, this map is given by the formation of Hochschild homology: if A is an Azumaya algebra of $\mathcal{C}$, then it carries the Brauer class $[A]$ to the equivalence class of the tensor product $A \otimes_{A \otimes A^{\mathrm{op}}} A$ (which is an invertible object of $\mathcal{C}$.

2.4 Functoriality

We now study the extent to which the Brauer group $\mathrm{Br}(\mathcal{C})$ of Definition 2.3.1 depends functorially on $\mathcal{C}$.

Proposition 2.4.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. Suppose that $\mathcal{C}$ and $\mathcal{D}$ admit geometric realizations of simplicial objects, and that the tensor product functors*

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

preserve geometric realizations. Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor which satisfies the following condition:

- $(*)$ *If $C \in \mathcal{C}$ is full and dualizable, then $F(C) \in \mathcal{D}$ is full (note that $F(C)$ is automatically dualizable, since the functor F is symmetric monoidal).*

Then:

- (a) *The functor F carries Azumaya algebras in $\mathcal{C}$ to Azumaya algebras in $\mathcal{D}$.*
- (b) *There is a unique group homomorphism* $\mathrm{Br}(F) : \mathrm{Br}(\mathcal{C}) \to \mathrm{Br}(\mathcal{D})$ *satisfying* $\mathrm{Br}(F)[A] = [F(A)]$.

Proof. Assertion (a) follows from Corollary 2.2.3 (note that conditions (i) and (ii) of Corollary 2.2.3 are preserved by any symmetric monoidal functor). To prove (b), we first observe that if A and B are Azumaya algebra objects of $\mathcal{C}$ satisfying $[A] = [B]$ in $\mathrm{Br}(\mathcal{C})$, then we have $A \otimes B^{\mathrm{op}} \simeq \mathrm{End}(M)$ where M is a full dualizable object of $\mathcal{C}$. We then obtain equivalences

$$F(A) \otimes F(B)^{\mathrm{op}} \simeq F(A \otimes B^{\mathrm{op}}) \simeq F(\mathrm{End}(M)) \simeq \mathrm{End}(F(M)),$$

so that $[F(A)] = [F(B)]$ in $\mathrm{Br}(\mathcal{D})$. It follows that there is a unique map of sets $\mathrm{Br}(F) : \mathrm{Br}(\mathcal{C}) \to \mathrm{Br}(\mathcal{D})$ satisfying $\mathrm{Br}(F)[A] = [F(A)]$. Since F commutes with tensor products, the map $\mathrm{Br}(F)$ is a group homomorphism. $\square$

Remark 2.4.2. In the situation of Proposition 2.4.1, it is not necessary to assume that the functor F preserves geometric realizations of simplicial objects.

2.5 Example: The Brauer Group of a Field

Let κ be a field and let Vect_κ denote the category of vector spaces over κ. We regard Vect_κ as equipped with the symmetric monoidal structure given by the usual tensor product $\otimes_\kappa$. Then:

- An object $V \in \mathrm{Vect}_\kappa$ is full (in the sense of Definition 2.1.2) if and only if $V \neq 0$.
- An object $V \in \mathrm{Vect}_\kappa$ is dualizable if and only if V is finite-dimensional as a vector space over κ.

It follows from Corollary 2.2.3 that that a κ-algebra A is an Azumaya algebra if and only if $0 < \dim_\kappa(A) < \infty$ and the natural map $A \otimes_\kappa A^{\mathrm{op}} \to \mathrm{End}_\kappa(A)$ is an isomorphism. In this case, the class of Azumaya algebras admits several other characterizations. The following results are well-known:

Proposition 2.5.1. *Let A be an algebra over a field κ. The following conditions are equivalent:*

(a) *The algebra A is Azumaya: that is, $0 < \dim_\kappa(A) < \infty$ and the natural map $A \otimes_\kappa A^{\mathrm{op}} \to \mathrm{End}_\kappa(A)$ is an isomorphism.*

(b) *The algebra A is central simple: that is, $\dim_\kappa(A) < \infty$, the unit map $\kappa \to A$ is an isomorphism from κ to the center of A, and for every two-sided ideal $I \subseteq A$ we have either $I = 0$ or $I = A$.*

(c) *The algebra A is isomorphic to a matrix ring $M_n(D)$, where $n > 0$ and D is a central division algebra over κ.*

Proposition 2.5.2. *Let D and D' be central division algebras over the same field κ. Then matrix algebras $M_n(D)$ and $M_{n'}(D')$ are Morita equivalent if and only if D and D' are isomorphic.*

Definition 2.5.3. Let κ be a field. We let $\mathrm{Br}(\kappa)$ denote the Brauer group of the category Vect_κ. We will refer to $\mathrm{Br}(\kappa)$ as the *Brauer group of κ.*

Combining Propositions 2.5.1 and 2.5.2, we obtain the following:

Corollary 2.5.4. *Let κ be a field. Then the construction $D \mapsto [D]$ induces an isomorphism of sets*

$$\{\textit{Central division algebras over } \kappa\}/\textit{Isomorphism} \to \mathrm{Br}(\kappa).$$

Remark 2.5.5. The Brauer group of a field κ admits a natural description in the language of Galois cohomology. If κ^{sep} denotes a separable closure of κ, then there is a canonical isomorphism $\mathrm{Br}(\kappa) \simeq \mathrm{H}^2(\mathrm{Gal}(\kappa^{\mathrm{sep}}/\kappa); \overline{\kappa}^\times)$.

2.6 Example: The Brauer Group of a Commutative Ring

Let R be a commutative ring. We let $\mathrm{Mod}_R^\heartsuit$ denote the abelian category of (discrete) R-modules, equipped with the symmetric monoidal structure given by tensor product over R. Then:

- An object $M \in \mathrm{Mod}_R^\heartsuit$ is dualizable if and only if it is a projective R-module of finite rank.
- A dualizable object $M \in \mathrm{Mod}_R^\heartsuit$ is full if and only if the rank of M is positive (when regarded as a locally constant function on the affine scheme $\mathrm{Spec}\, R$).

Definition 2.6.1. Let R be a commutative ring. We let $\mathrm{Br}(R)$ denote the Brauer group of the symmetric monoidal category $\mathrm{Mod}_R^\heartsuit$. We refer to $\mathrm{Br}(R)$ as the *Brauer group of R.*

Example 2.6.2. When R is a field, then the Brauer group $\mathrm{Br}(R)$ of Definition 2.6.1 specializes to the Brauer group of Definition 2.5.3.

Remark 2.6.3. If $\phi : R \to R'$ is a homomorphism of commutative rings, then extension of scalars along ϕ carries full dualizable objects of $\mathrm{Mod}_R^\heartsuit$ to full dualizable objects of $\mathrm{Mod}_{R'}^\heartsuit$. It follows that ϕ induces a homomorphism of Brauer groups $\mathrm{Br}(R) \to \mathrm{Br}(R')$ (Proposition 2.4.1).

We will need the following result of Grothendieck (see Corollary I.6.2 of [4]):

Proposition 2.6.4. *Let R be a Henselian local ring with residue field κ. Then the map $\mathrm{Br}(R) \to \mathrm{Br}(\kappa)$ of Remark 2.6.3 is an isomorphism.*

2.7 Example: The Brauer Group of a Connective Ring Spectrum

Let R be a connective $\mathbb{E}_\infty$-ring and let Mod_R^c denote the ∞-category of connective R-modules. We regard Mod_R^c as a symmetric monoidal ∞-category (via the relative smash product over R). Then:

- An object $M \in \mathrm{Mod}_R^c$ is dualizable if and only if it is a projective R-module of finite rank.
- A dualizable object $M \in \mathrm{Mod}_R^c$ is full if and only if the rank of M is positive (when regarded as a locally constant function on $\mathrm{Spec}\, R$).

Definition 2.7.1. Let R be a connective $\mathbb{E}_\infty$-ring. We let $\mathrm{Br}(R)$ denote the Brauer group of the symmetric monoidal category Mod_R^c. We refer to $\mathrm{Br}(R)$ as the *Brauer group of R*.

Beware that we now have two *different* definitions for the Brauer group of a commutative ring R: one given by Definition 2.6.1 (in terms of the abelian category $\mathrm{Mod}_R^\heartsuit$), and one given by Definition 2.7.1 (in terms of the ∞-category Mod_R^c). Fortunately, there is little danger of confusion:

Proposition 2.7.2. *Let R be a commutative ring. Then the symmetric monoidal functor* $\pi_0 : \mathrm{Mod}_R^c \to \mathrm{Mod}_R^\heartsuit$ *induces an isomorphism of Brauer groups* $\mathrm{Br}(\mathrm{Mod}_R^c) \to \mathrm{Br}(\mathrm{Mod}_R^\heartsuit)$.

Proof. The functor $\pi_0 : \mathrm{Mod}_R^c \to \mathrm{Mod}_R^\heartsuit$ induces an equivalence from the full subcategory of Mod_R^c spanned by the dualizable objects to the full subcategory of $\mathrm{Mod}_R^\heartsuit$ spanned by the dualizable objects (moreover, a dualizable R-module is full as an object of Mod_R^c if and only if it is full as an object of $\mathrm{Mod}_R^\heartsuit$. □

Remark 2.7.3 (Functoriality)**.** If $\phi : R \to R'$ is a morphism of $\mathbb{E}_\infty$-rings, then extension of scalars along ϕ carries full dualizable objects of Mod_R^c to full dualizable objects of $\mathrm{Mod}_{R'}^c$. It follows that ϕ induces a homomorphism of Brauer groups $\mathrm{Br}(R) \to \mathrm{Br}(R')$ (Proposition 2.4.1).

Our next result shows that the Brauer groups of Definition 2.7.1 do not capture any more than their algebraic counterparts:

Proposition 2.7.4. *Let R be a connective* $\mathbb{E}_\infty$*. Then the canonical map* $R \to \pi_0 R$ *induces an isomorphism of Brauer groups* $\mathrm{Br}(R) \to \mathrm{Br}(\pi_0 R)$.

Proposition 2.7.4 will be useful us in §9.1, for the purpose of comparing the Brauer group of a Lubin-Tate spectrum E (in the sense of Definition 1.0.6) with the Brauer group of its residue field (see Proposition 9.1.5). The proof of Proposition 2.7.4 is also of interest, since it highlights (in a substantially simpler setting) some of the ideas which will be used to analyze $\mathrm{Br}(E)$ in §8.

Lemma 2.7.5. *Let R be a connective $\mathbb{E}_\infty$-ring and let A be an Azumaya algebra object of the ∞-category Mod^c_R. Suppose that there exists an isomorphism $\alpha_0 : \pi_0 A \simeq \mathrm{End}_{\pi_0 R}(M_0)$ for some finitely generated projective $\pi_0 R$-module M_0. Then α_0 can be lifted to an equivalence $A \simeq \mathrm{End}_R(M)$, for some finitely generated projective R-module M.*

Proof. We will deduce Lemma 2.7.5 from the following:

($*$) The module M_0 belongs to the essential image of the extension-of-scalars functor

$$\mathrm{LMod}^c_A \to \mathrm{LMod}^c_{\pi_0 A} \qquad M \mapsto (\pi_0 A) \otimes_A M.$$

Assume that ($*$) is satisfied, so that we can write $M_0 = (\pi_0 A) \otimes_A M$. Since A is flat over R, we also have an equivalence $M_0 \simeq (\pi_0 R) \otimes_R M$. It follows that M is a locally free R-module of finite rank. Moreover, the action of A on M is classified by a map of flat R-modules $e : A \to \mathrm{End}_R(M)$ which induces an equivalence on π_0. It follows that e is an equivalence, which proves Lemma 2.7.5.

It remains to prove ($*$). Note that we can identify LMod^c_A with the inverse limit of the tower of ∞-categories $\{\mathrm{LMod}^c_{\tau_{\leqslant n} A}\}_{n \geqslant 0}$ (see Proposition SAG.**??**). It will therefore suffice to show that we can extend M_0 to a compatible sequence of objects $\{M_n \in \mathrm{LMod}^c_{\tau_{\leqslant n} A}\}_{n \geqslant 0}$. Assume that $n > 0$ and that the module M_{n-1} has been constructed. Theorem HA.7.4.1.26 implies that $\tau_{\leqslant n} R$ can be realized as a square-zero extension of $\tau_{\leqslant n-1} R$ by $N = \Sigma^n(\pi_n R)$: that is, there exists a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} \tau_{\leqslant n} R & \longrightarrow & \tau_{\leqslant n-1} R \\ \downarrow & & \downarrow d \\ \tau_{\leqslant n-1} R & \xrightarrow{d_0} & (\tau_{\leqslant n-1} R) \oplus \Sigma N. \end{array}$$

Set $A' = A \otimes_R (\tau_{\leqslant n-1} R \oplus \Sigma N)$, so that we have a pullback diagram of $\mathrm{Op}\,1$-algebras

$$\begin{array}{ccc} \tau_{\leqslant n} A & \longrightarrow & \tau_{\leqslant n-1} A \\ \downarrow & & \downarrow \phi \\ \tau_{\leqslant n-1} A & \xrightarrow{\psi} & A' \end{array}$$

and therefore a pullback diagram of ∞-categories σ :

$$\begin{array}{ccc} \mathrm{LMod}^c_{\tau_{\leqslant n}A} & \longrightarrow & \mathrm{LMod}^c_{\tau_{\leqslant n-1}A} \\ \downarrow & & \downarrow \phi^* \\ \mathrm{LMod}^c_{\tau_{\leqslant n-1}A} & \xrightarrow{\psi^*} & \mathrm{LMod}^c_{A'} \end{array}$$

(see Proposition SAG.??). Set $R' = (\tau_{\leqslant n-1}R) \oplus \Sigma N$, so that $K = \phi^* M_{n-1}$ and $K' = \psi^* M_{n-1}$ are finitely generated projective R'-modules and we have equivalences

$$\mathrm{End}_{R'}(K) \leftarrow A' \rightarrow \mathrm{End}_{R'}(K')).$$

Applying Corollary 2.1.5, we see that these equivalences are determined by an identification $K' \simeq L \otimes_{R'} K$ for some invertible R'-module L. Note that L becomes trivial after extending scalars along the projection map $R' \rightarrow \tau_{\leqslant n-1}R$. In particular, we have a canonical isomorphism $\pi_0 L \simeq \pi_0 R$. It follows that L is (non-canonically) equivalent to R', so that K and K' are equivalent objects of the ∞-category $\mathrm{LMod}^c_{A'}$. Invoking the fact that σ is a pullback diagram of ∞-categories, we deduce that M_{n-1} can be lifted to an object of $M_n \in \mathrm{LMod}^c_{\tau_{\leqslant n}A}$, as desired. □

Lemma 2.7.6. *Let R be a connective $\mathbb{E}_\infty$-ring and let A and B be Azumaya algebra objets of the ∞-category Mod^c_R. Suppose that there exists an $(\pi_0 A)$-$(\pi_0 B)$ bimodule M_0 that determines a Morita equivalence between $\pi_0 A$ and $\pi_0 B$ (as Azumaya algebra objects of $\mathrm{Mod}^c_{\pi_0 R}$). Then M_0 can be lifted to an A-B bimodule M which determines a Morita equivalence between A and B.*

Proof. Apply Lemma 2.7.5 to the tensor product $A \otimes_R B^{\mathrm{op}}$. □

Lemma 2.7.7. *Let R be a connective $\mathbb{E}_\infty$-ring and let A and B be Azumaya algebra objets of the ∞-category Mod^c_R. Then every isomorphism $\alpha_0 : \pi_0 A \simeq \pi_0 B$ (in the category of $\pi_0 R$-algebras) can be lifted to an equivalence $\alpha : A \simeq B$ (in the ∞-category of connective R-algebras).*

Proof. Set $M_0 = \pi_0 B$, which we regard as a left $(\pi_0 A)$-module via the isomorphism α_0 and a right $(\pi_0 B)$-module in the tautological way. Then M_0 is a bimodule which determines a Morita equivalence between $\pi_0 A$ and $\pi_0 B$. Applying Lemma 2.7.6, we can lift M_0 to an A-B bimodule M which determines a Morita equivalence from A to B. The unit element $1 \in \pi_0 B \simeq \pi_0 M$ determines a right B-module map $\rho : B \rightarrow M$. By construction, ρ induces an isomorphism on π_0. Since the domain and codomain of ρ are flat R-modules, it follows that ρ is an equivalence. The left action of A on M classifies an R-algebra map

$$\alpha : A \rightarrow \mathrm{End}_{\mathrm{RMod}_B}(M) \simeq \mathrm{End}_{\mathrm{RMod}_B}(B) = B.$$

which induces the isomorphism α_0 after applying the functor π_0. Since the domain and codomain of α are flat R-modules, it follows that α is an equivalence. $\square$

Lemma 2.7.8. *Let R be a connective $\mathbb{E}_\infty$-ring. Then every Azumaya algebra object A_0 of $\mathrm{Mod}^c_{\pi_0 R}$ can be lifted to an Azumaya algebra object A of Mod^c_R.*

Proof. We proceed as in the proof of Lemma 2.7.5. Using Proposition SAG.**??** , we can identify $\mathrm{Alg}(\mathrm{Mod}^c_R)$ with the inverse limit of the tower of ∞-categories

$$\{\mathrm{Alg}(\mathrm{Mod}^c_{\tau_{\leqslant n} R})\}_{n \geqslant 0}.$$

It will therefore suffice to show that we can extend A_0 to a compatible sequence of algebra objects $\{A_n \in \mathrm{Alg}(\mathrm{Mod}^c_{\tau_{\leqslant n} R})\}_{n \geqslant 0}$. Assume that $n > 0$ and that the algebra A_{n-1} has been constructed. Let R' be as in the proof of Lemma 2.7.5, so that we have a pullback diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Alg}(\mathrm{Mod}^c_{\tau_{\leqslant n} R}) & \longrightarrow & \mathrm{Alg}(\mathrm{Mod}^c_{\tau_{\leqslant n-1} R}) \\ \downarrow & & \downarrow{\scriptstyle d^*} \\ \mathrm{Alg}(\mathrm{Mod}^c_{\tau_{\leqslant n-1} R}) & \xrightarrow{d_0^*} & \mathrm{Alg}(\mathrm{Mod}^c_{R'}). \end{array}$$

To show that A_{n-1} can be lifted to an algebra object $A_n \in \mathrm{Alg}(\mathrm{Mod}^c_{\tau_{\leqslant n} R})$, it will suffice to show that $d^* A_{n-1}$ and $d_0^* A_{n-1}$ are equivalent as algebras over R'. By construction, they become equivalent after extension of scalars along the projection map $R' \to \tau_{\leqslant n-1} R$. In particular, $\pi_0(d^* A_{n-1})$ and $\pi_0(d_0^* A_{n-1})$ are isomorphic (as algebras over the commutative ring $\pi_0 R$). The desired result now follows from Lemma 2.7.7. $\square$

Proof of Proposition 2.7.4. Let R be a connective $\mathbb{E}_\infty$-ring and let $u : \mathrm{Br}(R) \to \mathrm{Br}(\pi_0 R)$ be the homomorphism given by extension of scalars along the map $R \to \pi_0 R$. We wish to show that u is an isomorphism. The injectivity of u follows from Lemma 2.7.5, and the surjectivity follows from Lemma 2.7.8. $\square$

2.8 Example: The Brauer-Wall Group of a Field

Let κ be a field. We let $\mathrm{Vect}^{\mathrm{gr}}_\kappa$ denote the category of $(\mathbf{Z}/2\mathbf{Z})$-graded vector spaces over κ: that is, vector spaces V equipped with a decomposition as a direct sum $V = V_0 \oplus V_1$. If V and W are $(\mathbf{Z}/2\mathbf{Z})$-graded vector spaces over κ, then we can regard the tensor product $V \otimes_\kappa W$ as equipped with the $(\mathbf{Z}/2\mathbf{Z})$-grading described by the formulae

$$(V \otimes_\kappa W)_0 = (V_0 \otimes_\kappa W_0) \oplus (V_1 \otimes_\kappa W_1)$$

$$(V \otimes_\kappa W)_1 = (V_0 \otimes_\kappa W_1) \oplus (V_1 \otimes_\kappa W_0).$$

We will regard $\mathrm{Vect}_\kappa^{\mathrm{gr}}$ as a symmetric monoidal category via the *Koszul sign rule*: for $(\mathbf{Z}/2\mathbf{Z})$-graded vector spaces V and W, the symmetry constraint $\sigma_{V,W} : V \otimes_\kappa W \simeq W \otimes_\kappa V$ is given by $\sigma_{V,W}(v \otimes w) = (-1)^{ij}(w \otimes v)$ for $v \in V_i$, $w \in W_j$.

As in §2.5, it is easy to see that:

- An object $V \in \mathrm{Vect}_\kappa^{\mathrm{gr}}$ is full (in the sense of Definition 2.1.2) if and only if $V \neq 0$.
- An object $V \in \mathrm{Vect}_\kappa^{\mathrm{gr}}$ is dualizable if and only if V is finite-dimensional as a vector space over κ.

Definition 2.8.1. Let A be a $(\mathbf{Z}/2\mathbf{Z})$-graded algebra over κ, which we regard as an associative algebra object of $\mathrm{Vect}_\kappa^{\mathrm{gr}}$. We will say that A is a *graded Azumaya algebra over* κ if it is an Azumaya algebra object of $\mathrm{Vect}_\kappa^{\mathrm{gr}}$, in the sense of Definition 2.2.1.

Warning 2.8.2. A graded Azumaya algebra over κ need not remain an Azumaya algebra over κ when the grading is ignored. Using Corollary 2.2.3, we see that A is a graded Azumaya algebra if and only if $0 < \dim_\kappa(A) < \infty$ and the canonical map $\rho^{\mathrm{gr}} : A \otimes_\kappa A \to \mathrm{End}_\kappa(A)$ is an isomorphism, where ρ is given by the formula $\rho^{\mathrm{gr}}(x \otimes y)(z) = (-1)^{jk} xzy$ for $x \in A_i$, $y \in A_j$, and $z \in A_k$. By contrast, A is an Azumaya algebra (in the ungraded sense) if and only if $0 < \dim_\kappa(A) < \infty$ and the map $\rho : A \otimes_\kappa A \to \mathrm{End}_\kappa(A)$ is an isomorphism, where ρ is given by $\rho(x \otimes y)(z) = xzy$.

Example 2.8.3. Let κ be a field of characteristic $\neq 2$, let a be an invertible element of κ, and define

$$\kappa(\sqrt{a}) = \kappa[x]/(x^2 - a).$$

Then $\kappa(\sqrt{a})$ is either a quadratic extension field of κ (if a is not a square) or is isomorphic to the product $\kappa \times \kappa$ (if a is a square). The decomposition $\kappa(\sqrt{a}) \simeq \kappa \oplus \kappa\sqrt{a}$ exhibits $\kappa(\sqrt{a})$ as a graded Azumaya algebra over κ. However, $\kappa(\sqrt{a})$ is not an Azumaya algebra over κ in the ungraded sense (since the center of $\kappa(\sqrt{a})$ is larger than κ).

Remark 2.8.4. Let $A = \kappa(\sqrt{a})$ be as in Example 2.8.3. Then the opposite algebra A^{op} (formed in the symmetric monoidal category $\mathrm{Vect}_\kappa^{\mathrm{gr}}$) can be identified with $\kappa(\sqrt{-a})$.

Definition 2.8.5. Let κ be a field. We let $\mathrm{BW}(\kappa)$ denote the Brauer group of the category $\mathrm{Vect}_\kappa^{\mathrm{gr}}$. We will refer to $\mathrm{BW}(\kappa)$ as the *Brauer-Wall group of* κ.

Example 2.8.6 (Clifford Algebras)**.** Let V be a vector space over κ and let $q : V \to \kappa$ be a quadratic form. We define the *Clifford algebra* $\mathrm{Cl}_q(V)$ to be the κ-algebra generated by V, subject to the relations $x^2 = q(x)$ for $x \in V$. We can regard $\mathrm{Cl}_q(V)$ as a $(\mathbf{Z}/2\mathbf{Z})$-graded algebra over κ, where the generators $x \in V$ are homogeneous of degree 1. Then:

- The Clifford algebra $\mathrm{Cl}_q(V)$ is a graded Azumaya algebra if and only if the quadratic form q is nondegenerate.

- The construction $(V, q) \mapsto [\mathrm{Cl}_q(V)]$ induces a group homomorphism $W(\kappa) \to \mathrm{BW}(\kappa)$, where $W(\kappa)$ denotes the Witt group of quadratic spaces over κ.

We now briefly review the structure of the Brauer-Wall group $\mathrm{BW}(\kappa)$. For a more detailed discussion, we refer the reader to [5].

Proposition 2.8.7. *Let A be a graded Azumaya algebra over κ. Then exactly one of the following assertions holds:*

(a) *The graded Azumaya algebra A is also an Azumaya algebra over κ.*

(b) *The characteristic of κ is different from 2 and A is isomorphic to a tensor product $B \otimes_\kappa \kappa(\sqrt{a})$, where B is an Azumaya algebra over κ (regarded as a graded Azumaya which is concentrated in degree zero) and $\kappa(\sqrt{a})$ is defined as in Example 2.8.3.*

Proof. The dimension of an Azumaya algebra over κ is always a square, so (a) and (b) cannot both occur. Assume that A is not an Azumaya algebra; we will show that (b) is satisfied. Note that the field κ must have characteristic different from 2 (otherwise, the forgetful functor $\mathrm{Vect}_\kappa^{\mathrm{gr}} \to \mathrm{Vect}_\kappa$ is symmetric monoidal and carries Azumaya algebras to Azumaya algebras).

Let σ denote the involution of A which is the identity on A_0 and multiplication by (-1) on A_1. Then σ is an algebra automorphism of A, and therefore carries the radical of A to itself. It follows that $E = \{f \in \mathrm{End}_\kappa(A) : f(I) \subseteq I\}$ is a $(\mathbf{Z}/2\mathbf{Z})$-graded subalgebra of $\mathrm{End}_\kappa(A)$. Note that the map $\rho^{\mathrm{gr}} : A \otimes_\kappa A \to \mathrm{End}_\kappa(A)$ factors through E. Our assumption that A is a graded Azumaya algebra guarantees that ρ^{gr} is an isomorphism, so that $E = \mathrm{End}_\kappa(A)$. It follows that $I = 0$, so that the algebra A is semisimple.

Let Z be the center of A. Then Z is invariant under the automorphism σ, and therefore inherits a grading $Z \simeq Z_0 \oplus Z_1$. Note that if z is an element of Z_0, then $z \otimes 1 - 1 \otimes z$ is annihilated by ρ^{gr}. Since the map ρ^{gr} is an isomorphism, it follows that $Z_0 = \kappa$ consists only of scalars. Since condition (a) is *not* satisfied, the center Z must be larger than κ. We can therefore choose some nonzero element $x \in Z_1$. Set $a = x^2 \in \kappa$. The element a must be nonzero (otherwise x would belong to the radical of A), so that x is invertible. We can therefore write

$$A = A_0 \oplus A_1 = A_0 \oplus A_0 x \simeq A_0 \otimes_\kappa \kappa(\sqrt{a}).$$

Using the centrality of x, we see that this identification is an isomorphism of $(\mathbf{Z}/2\mathbf{Z})$-graded algebras. To complete the proof, it will suffice to show that the algebra A_0 is central simple over κ. Note that any central element $z \in A_0$ is also central in A (since it also commutes with x), and therefore belongs to $Z_0 = \kappa$. Moreover, we can write $I = J + Jx$, where J is the radical of A_0; consequently, the vanishing of I implies the vanishing of J. □

Proposition 2.8.8. *Let κ be a field of characteristic $\neq 2$. Then:*

(1) *The inclusion functor* $\mathrm{Vect}_\kappa \hookrightarrow \mathrm{Vect}_\kappa^{\mathrm{gr}}$ *induces a monomorphism of Brauer groups*

$$\iota : \mathrm{Br}(\kappa) = \mathrm{Br}(\mathrm{Vect}_\kappa) \to \mathrm{Br}(\mathrm{Vect}_\kappa^{\mathrm{gr}}) = \mathrm{BW}(\kappa).$$

(2) *There exists a unique surjective group homomorphism* $\epsilon : \mathrm{BW}(\kappa) \to \mathbf{Z}/2\mathbf{Z}$ *with the following property: if A is a graded Azumaya algebra, then* $\epsilon([A]) = 0$ *if and only if A is an Azumaya algebra.*

(3) *The composition $\epsilon \circ \iota$ vanishes, and the homology* $\ker(\epsilon)/\operatorname{Im}(\iota)$ *is canonically isomorphic to $\kappa^\times/\kappa^{\times 2}$.*

It follows from Proposition 2.8.8 that if the field κ has characteristic $\neq 2$, then the Brauer-Wall group $\mathrm{BW}(\kappa)$ admits a composition series whose successive quotients are $\mathrm{Br}(\kappa)$, $\kappa^\times/\kappa^{\times 2}$, and $\mathbf{Z}/2\mathbf{Z}$. In general, it is a nontrivial extension of those groups:

Example 2.8.9. Let $\mathbf{R}$ be the field of real numbers. Then we have isomorphisms $\mathrm{Br}(\mathbf{R}) \simeq \mathbf{R}^\times / \mathbf{R}^{\times 2} \simeq \mathbf{Z}/2\mathbf{Z}$. The Brauer-Wall group $\mathrm{BW}(\mathbf{R})$ is isomorphic to $\mathbf{Z}/8\mathbf{Z}$.

Remark 2.8.10. If κ is a field of characteristic 2, then the maps $\iota : \mathrm{Br}(\kappa) \to \mathrm{BW}(\kappa)$ and $\epsilon : \mathrm{BW}(\kappa) \to \mathbf{Z}/2\mathbf{Z}$ are still well-defined. However, the map ϵ is identically zero, and the map ι is split injective (with a left inverse given by the forgetful functor $\mathrm{Vect}_\kappa^{\mathrm{gr}} \to \mathrm{Vect}_\kappa$). In this case, one can show that the Brauer-Wall group $\mathrm{BW}(\kappa)$ splits as a direct sum $\mathrm{BW}(\kappa) \simeq \mathrm{Br}(\kappa) \oplus \mathrm{H}^1_{\mathrm{et}}(\operatorname{Spec}\kappa, \mathbf{Z}/2\mathbf{Z})$, where the étale cohomology group $\mathrm{H}^1_{\mathrm{et}}(\operatorname{Spec}\kappa, \mathbf{Z}/2\mathbf{Z})$ can be described concretely as the cokernel of the Artin-Schreier map

$$\kappa \xrightarrow{x \mapsto x - x^2} \kappa.$$

Proof of Proposition 2.8.8. The well-definedness of the map $\iota : \mathrm{Br}(\kappa) \to \mathrm{BW}(\kappa)$ is a special case of Proposition 2.4.1. To complete the proof of (1), it will suffice to show that ι is injective. Let A be an Azumaya algebra over κ and suppose that $\iota([A])$ vanishes in the Brauer-Wall group $\mathrm{BW}(\kappa)$. Then there exists a finite-dimensional $(\mathbf{Z}/2\mathbf{Z})$-graded vector space V and an isomorphism of $(\mathbf{Z}/2\mathbf{Z})$-graded algebras $A \simeq \mathrm{End}_\kappa(V)$. It follows that $[A]$ vanishes in the Brauer group $\mathrm{Br}(\kappa)$, as desired.

We now prove (2). Note that if A is a graded Azumaya algebra over κ, then Proposition 2.8.7 implies that we can write $\dim_\kappa(A) = 2^{e(A)} d_A^2$, where

$$e(A) = \begin{cases} 0 & \text{if } A \text{ is an Azumaya algebra} \\ 1 & \text{otherwise} \end{cases}$$

and d_A is a positive integer. If A and B are Morita equivalent, then the product

$$\dim_\kappa(A)\dim_\kappa(B) = \dim_\kappa(A \otimes_\kappa B^{\mathrm{op}})$$

is a perfect square (since $A \otimes_\kappa B^{\mathrm{op}}$ is isomorphic to a matrix ring), and therefore $e(A) = e(B)$. It follows that there is a unique map of sets $\epsilon : \mathrm{BW}(\kappa) \to \mathbf{Z}/2\mathbf{Z}$ satisfying $\epsilon([A]) = e(A)$ for every graded Azumaya algebra A. From the identity $\dim_\kappa(A \otimes_\kappa B) = \dim_\kappa(A)\dim_\kappa(B)$, we deduce that

$$e(A \otimes_\kappa B) \equiv e(A) + e(B) \pmod 2,$$

so that ϵ is a group homomorphism. The surjectivity of ϵ follows from the observation that not every graded Azumaya algebra is an Azumaya algebra (Example 2.8.3).

We now prove (3). The vanishing of $\epsilon \circ \iota$ follows immediately from the definitions. For each element $a \in \kappa^\times$, set $Q(a) = [\kappa(\sqrt{a})] \in \mathrm{BW}(\kappa)$. Note that $Q(a)$ depends only on the residue class of a modulo $\kappa^{\times 2}$, so we can regard Q as a function from $\kappa^\times/\kappa^{\times 2}$ to the Brauer-Wall group $\mathrm{BW}(\kappa)$. Note that $\epsilon(Q(a)) = 1$ for each $a \in \kappa^\times$, so that $Q(a) - Q(1)$ belongs to the kernel $\ker(\epsilon)$.

To complete the proof, it will suffice to show the following:

- (i) The construction $a \mapsto Q(a) - Q(1)$ induces a group homomorphism $\lambda : \kappa^\times/\kappa^{\times 2} \to \ker(\epsilon)/\operatorname{Im}(\iota)$.
- (ii) The homomorphism λ is surjective.
- (iii) The homomorphism λ is injective.

To prove (i), we must show that for every pair of elements $a, b \in \kappa^\times$, we have

$$Q(ab) - Q(1) \equiv (Q(a) - Q(1)) + (Q(b) - Q(1) \pmod{\operatorname{Im}(\iota)}). \tag{2.1}$$

Using Remark 2.8.4, we obtain an identity $Q(-1) = -Q(1)$, so we can rewrite (2.1) as an identity

$$Q(ab) \equiv Q(a) + Q(b) + Q(-1) \pmod{\operatorname{Im}(\iota)} \tag{2.2}$$

Note that the right hand side of (2.2) is represented by the graded Azumaya algebra A generated by anticommuting odd variables x, y, and z satisfying $x^2 = a$, $y^2 = b$, and $z^2 = -1$. We now observe that $t = xyz$ is a central odd element of A satisfying $t^2 = ab$, so that the proof of Proposition 2.8.7 supplies an isomorphism of graded algebras $A \simeq A_0 \otimes_\kappa \kappa(\sqrt{ab})$ which witnesses the equality (2.2).

To prove (ii), it will suffice to show that every element $u \in \mathrm{BW}(\kappa)$ satisfying $\epsilon(u) = 1$ can be written as $\iota(\overline{u}) + Q(a)$, for some element $a \in \kappa^\times/\kappa^{\times 2}$. This follows immediately from Proposition 2.8.7.

To prove *(iii)*, suppose we are given an element $a \in \kappa^{\times}$ satisfying $\lambda(a) = 0$, so that $Q(a) - Q(1) = Q(a) + Q(-1)$ belongs to the image $\mathrm{Im}(\iota)$; we wish to show that a is a square. The class $Q(a) + Q(-1)$ is represented by the graded Azumaya algebra A generated by anticommuting odd elements x and y satisfying $x^2 = a$ and $y^2 = -1$. Our assumption that $[A] \in \mathrm{Im}(\iota)$ implies that we can choose an Azumaya algebra B over κ (which we regard as a graded Azumaya algebra which is concentrated in degree zero) for which the tensor product $A \otimes_{\kappa} B$ is Morita-trivial: that is, it is isomorphic to $\mathrm{End}_{\kappa}(V)$ for some $(\mathbf{Z}/2\mathbf{Z})$-graded vector space V over κ. Write $\dim_{\kappa}(B) = d^2$, so that V has dimension $2d$ over κ. Note that multiplication by $x \in A$ induces an automorphism of V which shifts degrees, so we must have $\dim_{\kappa}(V_0) = \dim_{\kappa}(V_1) = d$. It follows that the action of B on V_0 induces an isomorphism $B \simeq \mathrm{End}_{\kappa}(V_0)$, so that $[B] = 0$ in $\mathrm{Br}(\kappa)$. We may therefore replace B by κ and thereby reduce to the case $d = 1$.

Fix a nonzero element $v \in V_0$. Then xv and yv are nonzero elements of the 1-dimensional vector space V_1, so we can write $xv = cyv$ for some scalar $c \in \kappa$. We now compute

$$av = x^2 v = x(cyv) = -cy(xv) = -cy(cyv) = -c^2 y^2 v = c^2 v,$$

so that $a \in \kappa^{\times 2}$ as desired. □

Remark 2.8.11. Let κ be a field of characteristic different from 2. The proof of Proposition 2.8.8 shows that the Brauer-Wall group $\mathrm{BW}(\kappa)$ is generated (as an abelian group) by the image of the map $\iota : \mathrm{Br}(\kappa) \to \mathrm{BW}(\kappa)$ together with elements of the form $[\kappa(\sqrt{a})]$, where $\kappa(\sqrt{a})$ is defined as in Example 2.8.3.

2.9 The Brauer Group of a Lubin-Tate Spectrum

We now introduce the main object of interest in this paper. Let κ be a perfect field of characteristic $p > 0$, let $\mathbf{G}_0$ be a formal group of height $n < \infty$ over κ, and let E denote the associated Lubin-Tate spectrum. We let Mod_E denote the ∞-category of E-module spectra, and we let $\mathrm{Mod}_E^{\mathrm{loc}}$ denote the full subcategory of Mod_E spanned by the $K(n)$-local E-module spectra.

Remark 2.9.1. Let $R = \pi_0 E$ be the Lubin-Tate ring and let $\mathfrak{m} \subseteq R$ be the maximal ideal of R. An object $M \in \mathrm{Mod}_E$ belongs to the subcategory $\mathrm{Mod}_E^{\mathrm{loc}}$ if and only if, for every element $x \in \mathfrak{m}$, the homotopy limit of the tower

$$\cdots \to M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M$$

is contractible.

We will regard Mod_E as a symmetric monoidal ∞-category with respect to the formation of smash products relative to E, which we will denote by

$$\otimes_E : \mathrm{Mod}_E \times \mathrm{Mod}_E \to \mathrm{Mod}_E .$$

Let $L : \mathrm{Mod}_E \to \mathrm{Mod}_E^{\mathrm{loc}}$ denote a left adjoint to the inclusion functor. The localization functor L is compatible with the smash product $\otimes_E$ (in other words, the collection of $K(n)$-local equivalences is closed under smash products). It follows that there is an essentially unique symmetric monoidal structure on the ∞-category $\mathrm{Mod}_E^{\mathrm{loc}}$ for which the localization functor $L : \mathrm{Mod}_E \to \mathrm{Mod}_E^{\mathrm{loc}}$ is symmetric monoidal. We will denote the underlying tensor product by

$$\widehat{\otimes}_E : \mathrm{Mod}_E^{\mathrm{loc}} \times \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Mod}_E^{\mathrm{loc}} .$$

Concretely, it is given by the formula $M \widehat{\otimes}_E N = L(M \otimes_E N)$.

Definition 2.9.2. Let E be a Lubin-Tate spectrum. We let $\mathrm{Br}(E)$ denote the Brauer group of the symmetric monoidal ∞-category $\mathrm{Mod}_E^{\mathrm{loc}}$. We will refer to $\mathrm{Br}(E)$ as the *Brauer group of E*.

Warning 2.9.3. The terminology of Definition 2.9.2 has the potential to cause some confusion: it would be more accurate to refer to $\mathrm{Br}(E)$ as the *$K(n)$-local Brauer group of E* (this is the terminology used in [3]). We can also consider the Brauer group $\mathrm{Br}(\mathrm{Mod}_E)$ of the ∞-category Mod_E of *all* E-modules. However, this turns out to be less interesting: we will see later that $\mathrm{Br}(\mathrm{Mod}_E)$ can be identified with a subgroup of $\mathrm{Br}(E)$ (Proposition 9.2.1; see also Conjecture 9.4.1).

Let us now make Definition 2.9.2 a little bit more explicit by describing the Azumaya algebras of $\mathrm{Mod}_E^{\mathrm{loc}}$. The following result is standard:

Proposition 2.9.4. *Let M be an E-module spectrum. The following conditions are equivalent:*

(1) *The E-module M is perfect: that is, it is a dualizable object of the ∞-category Mod_E.*

(2) *The E-module M is a dualizable object of $\mathrm{Mod}_E^{\mathrm{loc}}$.*

(3) *The homotopy groups $\pi_0(K(n) \otimes_E M)$ and $\pi_1(K(n) \otimes_E M)$ are finite-dimensional vector spaces over κ, where $K(n)$ is an atomic E-algebra.*

(4) *The homotopy groups $\pi_0 M$ and $\pi_1 M$ are finitely generated modules over the Lubin-Tate ring $R = \pi_0 E$.*

Remark 2.9.5. It follows from Proposition 2.9.4 that if M is a dualizable object of $\mathrm{Mod}_E^{\mathrm{loc}}$, then the construction $N \mapsto M \otimes_E N$ preserves $K(n)$-local objects. Consequently, we do not need to distinguish between the smash product $M \otimes_E N$ and the completed smash product $M \widehat{\otimes}_E N$.

Proposition 2.9.6. *Let M be a $K(n)$-local E-module spectrum. Then M is a full object of $\mathrm{Mod}_E^{\mathrm{loc}}$ (in the sense of Definition 2.1.2) if and only if M is nonzero.*

Proof. The "only if" direction is obvious. Conversely, suppose that M is nonzero; we wish to show that the functor $\bullet \widehat{\otimes}_E M$ is conservative. Equivalently, we wish to show that if $N \in \mathrm{Mod}_E^{\mathrm{loc}}$ is nonzero, then the tensor product $N \widehat{\otimes}_E M$ is nonzero. Let $K(n)$ be an atomic E-algebra, so that $K(n) \otimes_E M$ and $N \otimes_E K(n)$ are nonzero. Since every (left or right) $K(n)$-module can be decomposed as a sum of (possibly shifted) copies of $K(n)$ (Proposition 3.6.3), it follows that

$$(N \otimes_E K(n)) \otimes_{K(n)} (K(n) \otimes_E M) \simeq N \otimes_E K(n) \otimes_E M \simeq K(n) \otimes_E (M \otimes_E N)$$

is also nonzero, so that $M \otimes_E N$ must be nonzero as desired. □

Corollary 2.9.7. *Let A be an E-algebra. Then A is an Azumaya algebra object of $\mathrm{Mod}_E^{\mathrm{loc}}$ if and only if it is nonzero, the homotopy groups $\pi_0 A$ and $\pi_1 A$ are finitely generated modules over $R = \pi_0 E$, and the natural map $A \otimes_E A^{\mathrm{op}} \to \mathrm{End}_E(A)$ is an equivalence.*

Proof. Combine Proposition 2.9.4, Remark 2.9.5, Proposition 2.9.6, and Corollary 2.2.3. □

Warning 2.9.8. Corollary 2.9.7 does *not* imply that every Azumaya algebra object A of $\mathrm{Mod}_E^{\mathrm{loc}}$ is also an Azumaya algebra object of Mod_E: beware that A need not be full as an object of Mod_E. This is exactly what happens in the case of greatest interest to us: we will see that there are plenty of examples of atomic Azumaya algebras in $\mathrm{Mod}_E^{\mathrm{loc}}$, but atomic E-algebras are never full as objects of Mod_E.

Chapter 3

Thom Spectra and Atomic Algebras

Let E be a Lubin-Tate spectrum and let $\mathfrak{m} \subseteq \pi_0 E$ denote the maximal ideal. We will say that an E-algebra A is *atomic* if the unit map $\pi_* E \to \pi_* A$ is a surjection, whose kernel is the graded ideal $\mathfrak{m}(\pi_* E)$ (Definition 1.0.2). In this section, we review some standard facts about atomic E-algebras:

(a) Atomic E-algebras always exist (Proposition 3.5.1).

(b) If A and A' are atomic E-algebras, then A and A' are equivalent as E-modules (Corollary 3.6.6).

(c) If the residue field $\kappa = (\pi_0 E)/\mathfrak{m}$ has characteristic different from 2, then there exists an atomic E-algebra A whose multiplication is homotopy commutative (Proposition 3.5.2).

To prove assertions (a) and (c), it will be convenient to introduce a general procedure for producing E-algebras as *Thom spectra* (Construction 3.1.5). As we will see, every atomic E-algebra can be realized as as the Thom spectrum of a *polarized* torus (Proposition 3.4.6). More generally, Thom spectra associated to polarized tori provide a useful tool for investigating the structure of the Brauer group $\mathrm{Br}(E)$, which we will exploit in the sequel to this paper.

3.1 Thom Spectra

We begin by reviewing the theory of Thom spectra from the ∞-categorical perspective. For more details, we refer the reader to [1].

Let R be an $\mathbb{E}_\infty$-ring, which we regard as fixed throughout this section. We let $\mathrm{Pic}(R) = \mathrm{Pic}(\mathrm{Mod}_R)$ denote the subcategory of Mod_R whose objects are invertible R-modules and whose morphisms are equivalences (see Remark 2.3.3). Note that $\mathrm{Pic}(R)$ is closed under tensor products in Mod_R, and therefore inherits the structure of a symmetric monoidal ∞-category: in other words, it can be regarded as an $\mathbb{E}_\infty$-space.

Notation 3.1.1. The $\mathbb{E}_\infty$-space $\mathrm{Pic}(R)$ is grouplike, and can therefore be identified with the 0th space of a connective spectrum. In particular, $\mathrm{Pic}(R)$ admits a canonical connected delooping, which we will denote by $\mathrm{BPic}(R)$. The space $\mathrm{BPic}(R)$ can be identified with a connected component of the Brauer space $\mathbf{Br}(\mathrm{Mod}_R)$ appearing in Remark 2.3.3.

Notation 3.1.2. Let $\mathcal{S}_{/\mathrm{Pic}(R)}$ denote the ∞-category whose objects are pairs (X, Q), where X is a Kan complex equipped with a map $Q : X \to \mathrm{Pic}(R)$. Note that, since $\mathrm{Pic}(R)$ is a commutative algebra object of the ∞-category $\mathcal{S}$ of spaces, the ∞-category $\mathcal{S}_{/\mathrm{Pic}(R)}$ inherits the structure of a symmetric monoidal ∞-category (see Theorem HA.2.2.2.4). Concretely, the tensor product on $\mathcal{S}_{/\mathrm{Pic}(R)}$ is given by

$$(X, Q_X) \otimes (Y, Q_Y) = (X \times Y, Q_{X\times Y}) \qquad Q_{X\times Y}(x, y) = Q_X(x) \otimes_R Q_Y(y).$$

Let R be an $\mathbb{E}_\infty$-ring. The construction

$$(L \in \mathrm{Pic}(R)) \mapsto ((*, L) \in \mathcal{S}_{/\mathrm{Pic}(R)})$$

determines a symmetric monoidal functor $f : \mathrm{Pic}(R) \to \mathcal{S}_{/\mathrm{Pic}(R)}$. Since the ∞-category $\mathcal{S}_{/\mathrm{Pic}(R)}$ admits small colimits (and the tensor product on $\mathcal{S}_{/\mathrm{Pic}(R)}$ preserves small colimits separately in each variable), the functor f admits an essentially unique extension to a colimit-preserving symmetric monoidal functor $F : \mathcal{P}(\mathrm{Pic}(R)) \to \mathcal{S}_{/\mathrm{Pic}(R)}$, where $\mathcal{P}(\mathrm{Pic}(R)) = \mathrm{Fun}(\mathrm{Pic}(R)^{\mathrm{op}}, \mathcal{S})$ denotes the ∞-category of $\mathcal{S}$-valued presheaves on $\mathrm{Pic}(R)$ (regarded as a symmetric monoidal ∞-category with respect to Day convolution; see §HA.4.8.1 for more details). It is not difficult to see that the functor F is an equivalence of ∞-categories. Invoking the universal property of $\mathcal{P}(\mathrm{Pic}(R))$, we obtain the following result:

Proposition 3.1.3. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admit small colimits and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then composition with the functor $f : \mathrm{Pic}(R) \to \mathcal{S}_{/\mathrm{Pic}(R)}$ described above induces an equivalence of ∞-categories*

$$\mathrm{LFun}^{\otimes}(\mathcal{S}_{/\mathrm{Pic}(R)}, \mathcal{C}) \to \mathrm{Fun}^{\otimes}(\mathrm{Pic}(R), \mathcal{C}).$$

Here $\mathrm{Fun}^{\otimes}(\mathrm{Pic}(R), \mathcal{C})$ denotes the ∞-category of symmetric monoidal functors from $\mathrm{Pic}(R)$ to $\mathcal{C}$, while $\mathrm{LFun}^{\otimes}(\mathcal{S}_{/\mathrm{Pic}(R)}, \mathcal{C})$ denotes the ∞-category of colimit-preserving symmetric monoidal functors from $\mathcal{S}_{/\mathrm{Pic}(R)}$ to $\mathcal{C}$.

Corollary 3.1.4. *There is an essentially unique symmetric monoidal functor* Th : $\mathcal{S}_{/\operatorname{Pic}(R)} \to \operatorname{Mod}_R$ *with the following properties:*

(i) *The functor* Th *commutes with small colimits.*

(ii) *The diagram of symmetric monoidal ∞-categories*

$$\begin{array}{ccccc} & & \mathcal{S}_{/\operatorname{Pic}(R)} & & \\ & \overset{f}{\nearrow} & & \overset{\mathrm{Th}}{\searrow} & \\ \operatorname{Pic}(R) & & \xrightarrow{\quad i \quad} & & \operatorname{Mod}_R \end{array}$$

commutes up to homotopy, where f is defined as above and i denotes the inclusion map.

Construction 3.1.5 (Thom Spectra)**.** Let Th : $\mathcal{S}_{/\operatorname{Pic}(R)} \to \operatorname{Mod}_R$ be the functor of Corollary 3.1.4. We will refer to Th as the *Thom spectrum functor*. Given an object $(X, Q) \in \mathcal{S}_{/\operatorname{Pic}(R)}$, we will refer to $\operatorname{Th}(X, Q)$ as the *Thom spectrum of X with respect to Q*.

Remark 3.1.6. Let (X, Q) be an object of $\mathcal{S}_{/\operatorname{Pic}(R)}$. Then the Thom spectrum $\operatorname{Th}(X, Q)$ can be identified with the colimit, formed in the ∞-category Mod_R, of the composite functor $X \xrightarrow{Q} \operatorname{Pic}(R) \hookrightarrow \operatorname{Mod}_R$.

Remark 3.1.7 (Cap Products)**.** Let X be a space. For any map $Q : X \to \operatorname{Pic}(R)$, the Thom spectrum $\operatorname{Th}(X, Q)$ carries an action of the function spectrum R^X. If X is finite (which is the only case of interest to us), then the action map $R^X \otimes_R \operatorname{Th}(X, Q) \to \operatorname{Th}(X, Q)$ is dual to the map

$$\operatorname{Th}(Q, X) \to (R \otimes \Sigma^\infty_+ X) \otimes_R \operatorname{Th}(X, Q) \simeq \operatorname{Th}(X \times X, \pi^* Q)$$

induced by the diagonal map $\delta : X \to X \times X$, where $\pi : X \times X \to X$ denotes the projection onto the second factor.

Example 3.1.8. Let X be a space, let $Q : X \to \operatorname{Pic}(R)$ be a map, and let u be an invertible element in the ring $\pi_0 R^X$. Then cap product with u induces an automorphism of the Thom spectrum $\operatorname{Th}(X, Q)$. Concretely, this automorphism is obtained using the functoriality of the construction $(X, Q) \mapsto \operatorname{Th}(X, Q)$ (note that u can be identified with a homotopy from the map $Q : X \to \operatorname{Pic}(R)$ to itself).

3.2 Polarizations

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section. Our goal is to single out a special class of Thom spectra (in the sense of Construction 3.1.5) which will be useful for studying the Brauer group $\mathrm{Br}(E)$.

Definition 3.2.1. A *lattice* is a free abelian group of finite rank. If Λ is a lattice, we let $K(\Lambda, 1)$ denote the associated Eilenberg-MacLane space, which we regard as a group object of the ∞-category $\mathcal{S}$ of spaces. A *polarization of* Λ is a map $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ in the ∞-category of group objects of $\mathcal{S}$. If Q is a polarization of Λ, then we can regard $(K(\Lambda, 1), Q)$ as an associative algebra object of the ∞-category $\mathcal{S}_{/\mathrm{Pic}(E)}$ of Notation 3.1.2. We let $\mathrm{Th}_Q \in \mathrm{Alg}_E$ denote the Thom spectrum $\mathrm{Th}(K(\Lambda, 1), Q)$, which we regard as an associative algebra object of Mod_E.

Variant 3.2.2 (Reduced Thom Spectra)**.** Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. We define the *reduced Thom spectrum* $\mathrm{Th}_Q^{\mathrm{red}}$ to be the cofiber of the unit map $E \to \mathrm{Th}_Q$, which we regard as an object of Mod_E.

Remark 3.2.3. Let Λ be a lattice equipped with a polarization $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$. Then the Thom spectrum Th_Q is equipped with an action of the function spectrum $E^{K(\Lambda,1)}$ via cap products (see Remark 3.1.7). Note that the homotopy ring $\pi_* E^{K(\Lambda,1)}$ can be identified with the exterior algebra $(\pi_* E) \otimes_{\mathbf{Z}} \bigwedge^*_{\mathbf{Z}} \Lambda^\vee$, where each element of the dual lattice $\Lambda^\vee$ is regarded as homogeneous of degree (-1). In particular, each element $\lambda^\vee \in \Lambda^\vee$ induces a map $D_{\lambda^\vee} : \mathrm{Th}_Q \to \Sigma \mathrm{Th}_Q$, whose square is nullhomotopic.

Remark 3.2.4 (Conjugate Polarizations)**.** Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization, so that we can regard the pair $(K(\Lambda, 1), Q)$ as an associative algebra object of the ∞-category $\mathcal{S}_{/\mathrm{Pic}(E)}$. We can identify the *opposite* algebra $(K(\Lambda, 1), Q)^{\mathrm{op}}$ with a pair $(K(\Lambda, 1), \overline{Q})$, where $\overline{Q} : K(\Lambda, 1) \to \mathrm{Pic}(E)$ is some other polarization. We will refer to $\overline{Q}$ as the *conjugate* of the polarization Q. Beware that although Q and $\overline{Q}$ are always homotopic as maps of spaces, they are usually not homotopic as morphisms of group objects of $\mathcal{S}$ (see Example 3.2.13). Note that we have a canonical equivalence of E-algebras $\mathrm{Th}_{\overline{Q}} \simeq \mathrm{Th}_Q^{\mathrm{op}}$.

Remark 3.2.5. Let Λ be a lattice. Then $K(\Lambda, 1)$ is a finite space. It follows that, for any polarization $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$, the Thom spectrum Th_Q can be written as a finite colimit of invertible objects of Mod_E (Remark 3.1.6). In particular, Th_Q is dualizable as an E-module, and is therefore belongs to $\mathrm{Mod}_E^{\mathrm{loc}}$.

Remark 3.2.6. Let Λ be a lattice. Then we can identify polarizations of Λ with maps $K(\Lambda, 2) \to \mathrm{BPic}(E)$ in the ∞-category of pointed spaces, where $\mathrm{BPic}(E)$ is the space described in Notation 3.1.1.

We close this section by introducing some useful invariants of a polarization Q.

Notation 3.2.7. Let Λ be a lattice. Since $K(\Lambda, 2)$ is an $\mathbb{E}_\infty$-space, the unreduced suspension spectrum $\Sigma^\infty_+ K(\Lambda, 2)$ inherits the structure of an $\mathbb{E}_\infty$-ring. Note that we have canonical decompositions

$$\Sigma^\infty_+ K(\Lambda, 2) \simeq S \oplus \Sigma^\infty K(\Lambda, 2)$$

$$\pi_2 \Sigma^\infty_+ K(\Lambda, 2) \simeq \pi_2 S \oplus \pi_2 \Sigma^\infty K(\Lambda, 2) \simeq \pi_2 S \oplus \Lambda,$$

where S denotes the sphere spectrum. This decomposition yields a map $\rho_1 : \Lambda \to \pi_2 \Sigma^\infty_+ K(\Lambda, 2)$. Using the multiplication on $\pi_* \Sigma^\infty_+ K(\Lambda, 2)$, we can extend ρ_1 to a family of maps

$$\rho_m : \mathrm{Sym}^m(\Lambda) \to \pi_{2m} \Sigma^\infty_+ K(\Lambda, 2).$$

Construction 3.2.8 (The Coefficients of a Polarization)**.** Since $\mathrm{Pic}(E)$ is a grouplike $\mathbb{E}_\infty$-space, there is an essentially unique connective spectrum $\mathfrak{pic}(E)$ equipped with an equivalence of $\mathbb{E}_\infty$-spaces $\mathrm{Pic}(E) \simeq \Omega^\infty \mathfrak{pic}(E)$. Note that the homotopy groups of $\mathfrak{pic}(E)$ are given by

$$\pi_n \mathfrak{pic}(E) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{if } n = 0 \\ (\pi_0 E)^\times & \text{if } n = 1 \\ \pi_{n-1} E & \text{if } n > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let Λ be a lattice equipped with a polarization Q, which we can identify with a map of pointed spaces $K(\Lambda, 2) \to \mathrm{BPic}(E)$ or a map of spectra $\Sigma^\infty K(\Lambda, 2) \to \Sigma\, \mathfrak{pic}(E)$. For each positive integer m, we define the *mth coefficient of* Q to be the map of abelian groups $c^Q_m : \mathrm{Sym}^m(\Lambda) \to \pi_{2m} \mathrm{BPic}(E)$ given by the composition

$$\mathrm{Sym}^m(\Lambda) \xrightarrow{\rho_m} \pi_{2m} \Sigma^\infty_+ K(\Lambda, 2) \to \pi_{2m} \Sigma^\infty K(\Lambda, 2) \xrightarrow{Q} \pi_{2m} \Sigma\, \mathfrak{pic}(E) = \pi_{2m} \mathrm{BPic}(E),$$

where ρ_m is the map defined in Notation 3.2.7.

Example 3.2.9. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. Then the 1st coefficient c^Q_1 (in the sense of Construction 3.2.8) can be identified with the group homomorphism

$$\Lambda \simeq \pi_1 K(\Lambda, 1) \xrightarrow{\pi_1(Q)} \pi_1 \mathrm{Pic}(E) \simeq (\pi_0 E)^\times$$

determined by Q at the level of fundamental groups.

Remark 3.2.10. Let Λ be a lattice equipped with a polarization $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$. From the Example 3.2.9, it follows that the 1st coefficient c_1^Q depends only on Q as a map of spaces $K(\Lambda, 1) \to \mathrm{Pic}(E)$, rather than as a map of group objects in $\mathcal{S}$. Beware that this is not true for the higher coefficients.

Example 3.2.11. Let Λ be a lattice of rank 2 with basis $\lambda_0, \lambda_1 \in \Lambda$, and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. For $i \in \{0, 1\}$, let $\rho_i : K(\Lambda, 1) \to K(\Lambda, 1)$ be the (pointed) map given by projection of Λ onto the summand $\mathbf{Z}\lambda_i$, and let Q_i denote the pullback of Q along ρ_i. Using the multiplicativity of Q, we obtain canonical equivalences

$$Q_0 \otimes_E Q_1 \simeq Q \simeq Q_1 \otimes_E Q_0$$

in the ∞-category $\mathrm{Fun}(K(\Lambda, 1), \mathrm{Pic}(E))$. Beware that the composition of these equivalences is usually *not* the canonical equivalence $Q_0 \otimes_E Q_1 \simeq Q_1 \otimes_E Q_0$ given by the symmetry constraint on the symmetric monoidal ∞-category Mod_E. Instead, the two maps differ by multiplication by

$$1 + c_2^Q(\lambda_0\lambda_1) \in \pi_0 E \oplus \pi_2 E \simeq \pi_0 E^{K(\Lambda,2)},$$

where c_2^Q is the second coefficient of Q (see Construction 3.2.8).

Remark 3.2.12. Let Λ be a lattice. One can show that a polarization of Λ is determined, up to homotopy, by its coefficients $\{c_m^Q\}_{m>0}$. Beware, however, that not every collection of maps $\{f_m : \mathrm{Sym}^m(\Lambda) \to \pi_{2m}\, \mathrm{BPic}(E)\}_{m>0}$ can be realized as the coefficients of a polarization of Λ.

Example 3.2.13 (Coefficients of the Conjugate Polarization)**.** Let Λ be a lattice and let Q be a polarization of Λ, which we will identify with a map of pointed spaces $K(\Lambda, 2) \to \mathrm{BPic}(E)$. Let $\overline{Q}$ denote the conjugate of the polarization Q (in the sense of Remark 3.2.6). Then we can also identify $\overline{Q}$ with a map of pointed spaces from $K(\Lambda, 2)$ to $\mathrm{BPic}(E)$, which is characterized (up to homotopy) by the requirement that the diagram

$$\begin{array}{ccc} K(\Lambda, 2) & \xrightarrow{Q} & \mathrm{BPic}(E) \\ \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle -1} \\ K(\Lambda, 2) & \xrightarrow{\overline{Q}} & \mathrm{BPic}(E) \end{array}$$

commutes up to homotopy. Note that the left vertical map induces $(-1)^k$ on the cohomology groups $\mathrm{H}^{2k}(K(\Lambda, 2); M)$, while the right vertical map induces (-1) on each homotopy group of $\mathrm{BPic}(E)$. It follows that the coefficients of Q and $\overline{Q}$ are related by the formula $c_m^{\overline{Q}} = (-1)^{m-1} c_m^Q$.

3.3 Nonsingular Polarizations

Let E be a Lubin-Tate spectrum and let $\mathfrak{m} \subseteq \pi_0 E$ denote the maximal ideal. Our goal in this section is to give a concrete criterion for the vanishing of the Thom spectrum Th_Q of a polarization $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ (Proposition 3.3.2).

Definition 3.3.1. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. We will say that Q is *nonsingular* if the first coefficient $c_1^Q : \Lambda \to (\pi_0 E)^\times$ factors through the subgroup $1 + \mathfrak{m} \subseteq (\pi_0 E)^\times$.

Proposition 3.3.2. *Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. Then Q is nonsingular (in the sense of Definition 3.3.1) if and only if the Thom spectrum Th_Q is nonzero.*

The proof of Proposition 3.3.2 will require some preliminary remarks.

Notation 3.3.3. Let $\rho : \Lambda' \to \Lambda$ be a homomorphism of lattices and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. We let $Q[\rho]$ denote the polarization of Λ' given by the composition

$$K(\Lambda', 1) \xrightarrow{\rho} K(\Lambda, 1) \xrightarrow{Q} \mathrm{Pic}(E).$$

By construction, ρ can be promoted to a map $(K(\Lambda', 1), Q[\rho]) \to (K(\Lambda, 1), Q)$ between algebra objects of the ∞-category $\mathcal{S}_{/\mathrm{Pic}(E)}$, and therefore induces a morphism $\mathrm{Th}_{Q[\rho]} \to \mathrm{Th}_Q$ in the ∞-category Alg_E.

Now suppose that $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ is a polarization of a lattice Λ and that we are given an element $\lambda \in \Lambda$. Then we can identify λ with a group homomorphism $\mathbf{Z} \to \Lambda$ (given by $n \mapsto n\lambda$). We let $Q[\lambda]$ denote the induced polarization of $\mathbf{Z}$, so that obtain a morphism $\mathrm{Th}_{Q[\lambda]} \to \mathrm{Th}_Q$ as above.

Remark 3.3.4. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. Suppose we are given a finite collection of lattice homomorphisms $\{\rho_i : \Lambda_i \to \Lambda\}_{1 \leqslant i \leqslant n}$ which induce an isomorphism $(\bigoplus \rho_i) : \bigoplus \Lambda_i \to \Lambda$. Then the canonical maps $(K(\Lambda_i, 1), Q[\rho_i]) \to (K(\Lambda, 1), Q)$ induce an equivalence

$$(K(\Lambda_1, 1), Q[\rho_1]) \otimes \cdots \otimes (K(\Lambda_n, 1), Q[\rho_n]) \to (K(\Lambda, 1), Q)$$

in the ∞-category $\mathcal{S}_{/\mathrm{Pic}(E)}$. It follows that the composite map

$$\mathrm{Th}_{Q[\rho_1]} \otimes_E \otimes \cdots \otimes_E \mathrm{Th}_{Q[\rho_n]} \to \mathrm{Th}_Q \otimes_E \cdots \otimes_E \mathrm{Th}_Q \xrightarrow{m} \mathrm{Th}_Q$$

is an equivalence of E-modules; here m is induced by the multiplication on Th_Q.

Warning 3.3.5. In the situation of Remark 3.3.4, the equivalence

$$\mathrm{Th}_{Q[\rho_1]} \otimes_E \otimes \cdots \otimes_E \mathrm{Th}_{Q[\rho_n]} \simeq \mathrm{Th}_Q$$

generally depends on the ordering of the factors (since the multiplication on Th_Q is not commutative). Beware also that this equivalence is usually not a map of algebra objects of Mod_E.

Example 3.3.6. Let Λ be a lattice of rank 2 with basis $\lambda_0, \lambda_1 \in \Lambda$, and let $\lambda_0^\vee, \lambda_1^\vee \in \Lambda^\vee$ denote the dual basis for $\Lambda^\vee$. Let α denote the composition

$$\mathrm{Th}_Q \simeq \mathrm{Th}_{Q[\lambda_0]} \otimes_E \mathrm{Th}_{Q[\lambda_1]} \simeq \mathrm{Th}_{Q[\lambda_1]} \otimes_E \mathrm{Th}_{Q[\lambda_0]} \simeq \mathrm{Th}_Q,$$

where the outer equivalences are supplied by Remark 3.3.4 and the inner map is given by the symmetry constraint on Mod_E. Combining Examples 3.1.8 and 3.2.11, we deduce that α is homotopic to the map $\mathrm{id} + c_2^Q(\lambda_0, \lambda_1) D_{\lambda_0^\vee} D_{\lambda_1^\vee}$, where the maps $D_{\lambda_0^\vee}, D_{\lambda_1^\vee} : \mathrm{Th}_Q \to \Sigma \mathrm{Th}_Q$ are defined in Remark 3.2.3.

Remark 3.3.7. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. For each $\lambda \in \Lambda$, the induced polarization $Q[\lambda] : K(\mathbf{Z}, 1) \simeq S^1 \to \mathrm{Pic}(E)$ is classified (as a pointed map) by the element $c_1^Q(\lambda) \in (\pi_0 E)^\times \simeq \pi_1 \mathrm{Pic}(E)$. Using Remark 3.1.6, we obtain a canonical fiber sequence

$$E \xrightarrow{c_1^Q(\lambda)-1} E \xrightarrow{e} \mathrm{Th}_{Q[\lambda]}$$

in the ∞-category Mod_E, where e is the unit map. In particular, we see that $\mathrm{Th}_{Q[\lambda]}$ is nonzero if and only if $c_1^Q(\lambda)$ belongs to the subgroup $1 + \mathfrak{m} \subseteq (\pi_0 E)^\times$.

Variant 3.3.8. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. For each $\lambda \in \Lambda$, the cofiber sequence

$$E \xrightarrow{c_1^Q(\lambda)-1} E \xrightarrow{e} \mathrm{Th}_{Q[\lambda]}$$

determines a canonical identification $\mathrm{Th}^{\mathrm{red}}_{Q[\lambda]} = \mathrm{cofib}(e) \simeq \Sigma E$.

Proof of Proposition 3.3.2. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. Suppose first that the Thom spectrum Th_Q is nonzero. For each element $\lambda \in \Lambda$, we have a morphism $\mathrm{Th}_{Q[\lambda]} \to \mathrm{Th}_Q$ in Alg_E, so that $\mathrm{Th}_{Q[\lambda]}$ is also nonzero. Applying Remark 3.3.7, we deduce that $c_1^Q(\lambda)$ belongs to the subgroup $1 + \mathfrak{m} \subseteq (\pi_0 E)^\times$. Allowing λ to vary, we deduce that Q is a nonsingular polarization.

We now prove the converse. Let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization, and choose a basis $\lambda_1, \lambda_2, \ldots, \lambda_n$ for the lattice Λ. Then each $c_1^Q(\lambda_i)$ belongs to the subgroup $1 + \mathfrak{m} \subseteq (\pi_0 E)^\times$, so each of the Thom spectra $\mathrm{Th}_{Q[\lambda_i]}$ is nonzero and therefore a full object of $\mathrm{Mod}_E^{\mathrm{loc}}$ (Proposition 2.9.6). Applying Remark 3.3.4, we deduce that the Thom spectrum

$$\mathrm{Th}_Q \simeq \mathrm{Th}_{Q[\lambda_1]} \otimes_E \cdots \otimes_E \mathrm{Th}_{Q[\lambda_n]}$$

is also full (and therefore nonzero). □

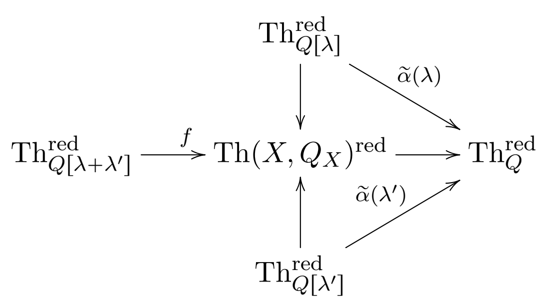
We conclude with a few observations which will be useful later for recognizing when the Thom spectrum construction yields Azumaya algebras.

Construction 3.3.9 (The Map $\widetilde{\alpha}$)**.** Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization. By functoriality, each element λ in Λ determines an E-algebra map $\mathrm{Th}_{Q[\lambda]} \to \mathrm{Th}_Q$, and therefore an E-module map $\mathrm{Th}^{\mathrm{red}}_{Q[\lambda]} \to \mathrm{Th}^{\mathrm{red}}_Q$. Under the canonical identification $\mathrm{Th}^{\mathrm{red}}_{Q[\lambda]} \simeq \Sigma E$ supplied by Variant 3.3.8, the homotopy class of this map determines an element $\widetilde{\alpha}(\lambda) \in \pi_1 \mathrm{Th}^{\mathrm{red}}_Q$.

Proposition 3.3.10. *Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. Then the elements $\widetilde{\alpha}(\lambda) \in \pi_1 \mathrm{Th}^{red}_Q$ satisfy the cocycle formulae*

$$\widetilde{\alpha}(0) = 0 \qquad \widetilde{\alpha}(\lambda + \lambda') = \widetilde{\alpha}(\lambda) + c_1^Q(\lambda)\widetilde{\alpha}(\lambda') \qquad \widetilde{\alpha}(-\lambda) = -c_1^Q(\lambda)^{-1}\widetilde{\alpha}(\lambda).$$

Proof. We will prove the identity $\widetilde{\alpha}(\lambda+\lambda') = \widetilde{\alpha}(\lambda) + c_1^Q(\lambda)\widetilde{\alpha}(\lambda')$; the other two identities follow as a formal consequence. Let $X = S^1 \amalg_* S^1$ be a wedge of two circles, so that λ and λ' determine a map $u : X \to K(\Lambda, 1)$. Let Q_X denote the restriction of Q to X, so that we can form the Thom spectrum $\mathrm{Th}(X, Q_X)$ and the reduced Thom spectrum $\mathrm{Th}(X, Q_X)^{\mathrm{red}}$ (given by the cofiber of the map $E \to \mathrm{Th}(X, Q_X)$ determined by the base point of X). We then have a commutative of E-modules



where the vertical maps are given by the two inclusions of S^1 into X, the map f is induced by the loop $S^1 \to X$ given by concatenating the two inclusions, and the horizontal composition is homotopic to $\widetilde{\alpha}(\lambda + \lambda')$. Since the formation of Thom spectra preserves colimits, the vertical maps induce an equivalence of E-modules $g : \mathrm{Th}^{\mathrm{red}}_{Q[\lambda]} \oplus \mathrm{Th}^{\mathrm{red}}_{Q[\lambda']} \to \mathrm{Th}(X, Q_X)^{\mathrm{red}}$. It will therefore suffice to observe that the composite map

$$\Sigma E \simeq \mathrm{Th}^{\mathrm{red}}_{Q[\lambda+\lambda']} \xrightarrow{f} \mathrm{Th}(X, Q_X)^{\mathrm{red}} \xrightarrow{g^{-1}} \mathrm{Th}^{\mathrm{red}}_{Q[\lambda]} \oplus \mathrm{Th}^{\mathrm{red}}_{Q[\lambda']} \simeq \Sigma E \oplus \Sigma E$$

is homotopic to $(\mathrm{id}, c_1^Q(\lambda))$. □

3.4 Atomic Polarizations

Let E be a Lubin-Tate spectrum with maximal ideal $\mathfrak{m} \subseteq \pi_0 E$ and residue field $\kappa = (\pi_0 E)/\mathfrak{m}$. In this section, we will supply a criterion which can be used to test whether the Thom spectrum Th_Q of a polarization $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ is an atomic E-algebra, in the sense of Definition 1.0.2 (Proposition 3.4.4). We also show that every atomic E-algebra arises in this way (Proposition 3.4.6).

Construction 3.4.1. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization, so that the first coefficient c_1^Q determines a group homomorphism $\Lambda \to 1 + \mathfrak{m} \subseteq (\pi_0 E)^\times$. We let $\overline{c}_1^Q : \kappa \otimes \Lambda \to \mathfrak{m}/\mathfrak{m}^2$ be the unique κ-linear map for which the diagram

$$\begin{CD} \Lambda @>{c_1^Q}>> 1+\mathfrak{m} \\ @VVV @VVV \\ @. (1+\mathfrak{m})/(1+\mathfrak{m}^2) \\ @VVV @VV{\sim}V \\ \kappa \otimes \Lambda @>{\overline{c}_1^Q}>> \mathfrak{m}/\mathfrak{m}^2 \end{CD}$$

commutes.

Definition 3.4.2. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. We will say that Q is *atomic* if it is nonsingular and the map

$$\overline{c}_1^Q : \kappa \otimes \Lambda \to \mathfrak{m}/\mathfrak{m}^2$$

of Construction 3.4.1 is an isomorphism of vector spaces over κ.

Remark 3.4.3. Let Λ be a lattice equipped with a basis $(\lambda_1, \ldots, \lambda_n)$, and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. Then Q is atomic (in the sense of Definition 3.4.2) if and only if the elements $c_1^Q(\lambda_i) - 1$ form a regular system of parameters for the local ring $\pi_0 E$. In particular, this is possible only when the rank of Λ (as an abelian group) coincides with the height of the Lubin-Tate spectrum E.

The terminology of Definition 3.4.2 is motivated by the following:

Proposition 3.4.4. *Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of Λ. If Q is atomic (in the sense of Definition 3.4.2), then the Thom spectrum Th_Q is an atomic E-algebra (in the sense of Definition 1.0.2).*

Remark 3.4.5. We will prove later that the converse of Proposition 3.4.4 is also true: if the Thom spectrum Th_Q is atomic, then the polarization Q is atomic (Corollary 6.6.12).

Proof of Proposition 3.4.4. Let $\{\lambda_i\}_{1 \leqslant i \leqslant n}$ be a basis for the lattice Λ. Our assumption that Q is atomic guarantees that we can write $c_1^Q(\lambda_i) = 1 + x_i$, where $\{x_i\}_{1 \leqslant i \leqslant n}$ is a regular system of parameters for the local ring $\pi_0 E$ (Remark 3.4.3). For $0 \leqslant m \leqslant n$, let $A(m)$ denote the E-module given by the formula

$$A(m) = \mathrm{Th}_{Q[\lambda_1]} \otimes_E \cdots \otimes \mathrm{Th}_{Q[\lambda_m]} .$$

We will prove the following assertion for $0 \leqslant m \leqslant n$:

$(*_m)$ The unit map $E \to A(m)$ induces an isomorphism $(\pi_* E)/(x_1, \ldots, x_m) \to \pi_* A(m)$.

We proceed by induction on m, the case $m = 0$ being obvious. Assume that $0 < m \leqslant n$ and that assertion $(*_{m-1})$ is true. Applying Remark 3.3.7, we deduce the existence of a fiber sequence of E-modules

$$A(m-1) \xrightarrow{x_m} A(m-1) \to A(m).$$

Using the regularity of the sequence $x_1, \ldots, x_m$ together with $(*_{m-1})$, we deduce that multiplication by x_m induces a monomorphism from $\pi_* A(m-1)$ to itself, and therefore a short exact sequence

$$0 \to \pi_* A(m-1) \xrightarrow{x_m} \pi_* A(m-1) \to \pi_* A(m) \to 0,$$

from which we immediately deduce $(*_m)$.

Combining $(*_n)$ with Remark 3.3.4, we deduce that the Thom spectrum Th_Q is atomic. □

We now show that every atomic E-algebra arises as a Thom spectrum.

Proposition 3.4.6. *Let A be an atomic E-algebra. Then there exists a lattice Λ, an atomic polarization $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$, and an equivalence of E-algebras $A \simeq \mathrm{Th}_Q$.*

Proof. Let $\mathrm{Pic}_A(E)$ denote the subcategory of $(\mathrm{Mod}_E)_{/A}$ whose objects are E-module maps $u : L \to A$ where L is invertible and u extends to an equivalence of A-modules $u_A : A \otimes_E L \simeq A$, and whose morphisms are equivalences (note that, since A is atomic, the map u_A is an equivalence if and only if u is nonzero). Using the algebra structure on A, we can regard $(\mathrm{Mod}_E)_{/A}$ as a monoidal ∞-category (see Theorem HA.2.2.2.4). The subcategory $\mathrm{Pic}_A(E)$ is closed under tensor products, and therefore inherits a monoidal structure. If Λ is a lattice and $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ is a polarization, then the following data are equivalent:

(a) Lifts of Q to a map $K(\Lambda, 1) \to \mathrm{Pic}_A(E)$ (as group objects of $\mathcal{S}$).

(b) Morphisms of E-algebras $\mathrm{Th}_Q \to A$.

Moreover, suppose we are given the data of a polarization $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ and a map $\phi : \mathrm{Th}_Q \to A$. If Q is atomic, then ϕ is a morphism of atomic E-algebras (Proposition 3.4.4), and is therefore automatically an equivalence. It will therefore suffice to prove the following:

($*$) There exists a lattice Λ with basis $\lambda_1, \ldots, \lambda_n$ and a map $\rho : K(\Lambda, 2) \to \mathrm{BPic}_A(E)$ for which the composite map

$$f : \Lambda = \pi_2 K(\Lambda, 2) \to \pi_2 \,\mathrm{BPic}_A(E) \to \pi_2 \,\mathrm{BPic}(E) \simeq (\pi_0 E)^\times$$

has the property that $\{f(\lambda_i) - 1\}$ is a regular system of parameters for the local ring $\pi_0 E$.

The proof of ($*$) proceeds by obstruction theory. Unwinding the definitions, we see that the homotopy groups of $\mathrm{BPic}_A(E)$ are given by the formula

$$\pi_n \,\mathrm{BPic}_A(E) = \begin{cases} 1 + \mathfrak{m} & \text{if } n = 2 \\ \mathfrak{m}\pi_{n-2}E & \text{if } n > 2. \end{cases}$$

Consequently, if $x_1, \ldots, x_n$ is any regular system of parameters for E, then $\rho_2 : K(\mathbf{Z}^n, 2) \to \tau_{\leqslant 2} \,\mathrm{BPic}_A(E)$ inducing the group homomorphism

$$\mathbf{Z}^n = \pi_2 K(\mathbf{Z}^n, 2) \to \pi_2(\tau_{\leqslant 2} \,\mathrm{BPic}_A(E)) = 1 + \mathfrak{m}$$

$(a_1, \ldots, a_n) \mapsto \prod (1 + x_i)^{a_i}$. To show that ρ_2 can be lifted to a map $\rho : K(\mathbf{Z}^n, 2) \to \mathrm{BPic}_A(E)$, it will suffice to verify the vanishing of a sequence of obstructions

$$o_k \in \mathrm{H}^{k+1}(K(\mathbf{Z}^n, 2), \pi_k \,\mathrm{BPic}_A(E)) \simeq \mathrm{H}^{k+1}(K(\mathbf{Z}^n, 2), \mathfrak{m}(\pi_{k-2}E))$$

for $k \geqslant 3$. These obstructions automatically vanish, since the groups $\pi_{k-2}E$ are trivial when k is odd, while the cohomology groups $\mathrm{H}^{k+1}(K(\mathbf{Z}^n, 2), M)$ are trivial for any abelian group M when k is even. $\square$

3.5 Existence of Atomic Algebras

We now apply the theory of Thom spectra to prove the following:

Proposition 3.5.1. *For every Lubin-Tate spectrum E, there exists an atomic E-algebra A.*

Proof. We proceed as in the proof of Proposition 3.4.6. By virtue of Proposition 3.4.4, it will suffice to prove the following:

($*$) There exists a lattice Λ with basis $\lambda_1, \ldots, \lambda_n$ and a map of pointed spaces $\rho : K(\Lambda, 2) \to \mathrm{BPic}(E)$ for which the map

$$f : \Lambda = \pi_2 K(\Lambda, 2) \to \pi_2 \, \mathrm{BPic}(E) \simeq (\pi_0 E)^{\times}$$

has the property that $\{f(\lambda_i) - 1\}$ is a regular system of parameters for the local ring $\pi_0 E$.

The proof of ($*$) proceeds by obstruction theory. If $x_1, \ldots, x_n$ is any regular system of parameters for the local ring $\pi_0 E$, then there is an essentially unique map $\rho_2 : K(\mathbf{Z}^n, 2) \to \tau_{\leqslant 2} \, \mathrm{BPic}(E)$ which induces the group homomorphism

$$\mathbf{Z}^n = \pi_2 K(\mathbf{Z}^n, 2) \to \pi_2(\tau_{\leqslant 2} \, \mathrm{BPic}(E)) = 1 + \mathfrak{m}.$$

To show that ρ_2 can be lifted to a map $\rho : K(\mathbf{Z}^n, 2) \to \mathrm{BPic}_A(E)$, it will suffice to verify the vanishing of a sequence of obstructions

$$o_k \in \mathrm{H}^{k+1}(K(\mathbf{Z}^n, 2), \pi_k \, \mathrm{BPic}(E)) \simeq \mathrm{H}^{k+1}(K(\mathbf{Z}^n, 2), \pi_{k-2} E)$$

for $k \geqslant 3$. These obstructions automatically vanish, since the groups $\pi_{k-2} E$ are trivial when k is odd, while the cohomology groups $\mathrm{H}^{k+1}(K(\mathbf{Z}^n, 2), M)$ are trivial for any abelian group M when k is even. □

We will also need the following variant of Proposition 3.5.1.

Proposition 3.5.2. *Let E be a Lubin-Tate spectrum with residue field κ. If the characteristic of κ is different from 2, then there exists an atomic E-algebra A which is homotopy commutative.*

Warning 3.5.3. If κ has characteristic 2, then atomic E-algebras are *never* homotopy commutative.

Warning 3.5.4. In the situation of Proposition 3.5.1, we cannot arrange that A is a commutative algebra object of the ∞-category Mod_E, even if the residue field κ has odd characteristic.

Remark 3.5.5. If the residue field of E has characteristic different from 2, then the atomic commutative algebra object of $\mathrm{hMod_E}$ whose existence is asserted by Proposition 3.5.2 is unique up to unique isomorphism; see Corollary 6.9.2.

Proof of Proposition 3.5.2. Let $x_1, \ldots, x_n$ be a regular system of parameters for the maximal ideal $\mathfrak{m} \subseteq \pi_0 E$. For $1 \leqslant i \leqslant n$, we can choose a polarization $Q_i : K(\mathbf{Z}, 1) \to \mathrm{Pic}(E)$ with $c_1^{Q_i}(1) = 1 + x_i$ (this follows by obstruction theory, as in the proof of Proposition 3.5.1). Let $\overline{Q}_i$ denote the conjugate polarization (see Construction 3.4.1). We let $Q'_i : K(\mathbf{Z}, 1) \to \mathrm{Pic}(E)$ denote the polarization given by $Q'_i = Q_i \otimes \overline{Q}_i$ and let $A(i)$ denote the Thom spectrum $\mathrm{Th}_{Q'_i}$. Note that, as an E-module, we can identify $A(i)$ with the cofiber of the map from E to itself given by multiplication by

$$c_1^{Q'_i}(1) - 1 = c_1^{Q_i}(1) c_1^{\overline{Q_i}}(1) - 1 = (x_i + 1)^2 - 1 = 2x_i + x_i^2.$$

Since $2x_i + x_i^2$ is not a zero divisor in the ring $\pi_0 E$, it follows that the homotopy group $\pi_1 A(i)$ vanishes.

By construction, the polarization Q'_i is conjugate to itself, so there exists an equivalence of E-algebras $\alpha : A(i) \simeq A(i)^{\mathrm{op}}$. Note that we have a fiber sequence

$$\mathrm{Map}_{\mathrm{Mod}_E}(\Sigma E, A(i)^{\mathrm{op}}) \to \mathrm{Map}_{\mathrm{Mod}_E}(A(i), A(i)^{\mathrm{op}}) \to \mathrm{Map}_{\mathrm{Mod}_E}(E, A(i)^{\mathrm{op}}).$$

Since the homotopy group $\pi_1 A(i)^{\mathrm{op}}$ vanishes, the first term of this fiber sequence is connected, so the map

$$\pi_0 \mathrm{Map}_{\mathrm{Mod}_E}(A(i), A(i)^{\mathrm{op}}) \to \pi_0 \mathrm{Map}_{\mathrm{Mod}_E}(E, A(i)^{\mathrm{op}})$$

is a monomorphism. It follows that, as a morphism of E-modules, α is homotopic to the identity.

Set $A = A(1) \otimes_E \cdots \otimes_E A(n)$. Our assumption that κ has characteristic different from 2 guarantees that the elements $\{2x_i + x_i^2\}_{1 \leqslant i \leqslant n}$ is also a regular system of parameters for the local ring $\pi_0 E$, so that A is an atomic E-algebra (this follows Proposition 3.4.4, or more directly from the proof of Proposition 3.4.4). By construction, the identity map from A to itself can be promoted to an equivalence of E-algebras $A \simeq A^{\mathrm{op}}$. In particular, the multiplication on A is homotopy commutative, as desired. □

3.6 Atomic and Molecular Modules

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section. We now show that although atomic objects of Alg_E need not be equivalent as *algebras* over E, they are always equivalent as modules over E (Corollary 3.6.6). For later use, it will be convenient to establish a slightly stronger result, which applies to E-algebras which are only homotopy associative.

Definition 3.6.1. Let A be an associative algebra object of the homotopy category $\mathrm{hMod_E}$. We will say that A is *atomic* if the unit map $E \to A$ induces an isomorphism $(\pi_* E)/\mathfrak{m} \to \pi_* A$, where $\mathfrak{m} \subseteq \pi_0 E$ denotes the maximal ideal.

Remark 3.6.2. An algebra object $A \in \mathrm{Alg}_E = \mathrm{Alg}(\mathrm{Mod}_E)$ is atomic in the sense of Definition 1.0.2 if and only if it is atomic in the sense of Definition 3.6.1.

Proposition 3.6.3. *Let E be a Lubin-Tate spectrum, let $A \in \mathrm{Alg}(\mathrm{hMod_E})$ be atomic, and let M be a left A-module object of the homotopy category $\mathrm{hMod_E}$. Then M is equivalent (as a left A-module) to a coproduct of copies of A and the suspension ΣA.*

Proof. Let M be a left A-module object of $\mathrm{hMod_E}$. Then we can regard $\pi_0 M$ and $\pi_1 M$ as vector spaces over the field $\kappa = \pi_0 A$, which admit bases $\{u_i\}_{i \in I}$ and $\{v_j\}_{j \in J}$. The elements $\{u_i, v_j\}$ determine a map of left A-modules $(\bigoplus_{i \in I} A) \oplus (\bigoplus_{j \in J} \Sigma A) \to M$, which induces an isomorphism on homotopy groups and is therefore an isomorphism in $\mathrm{hMod_E}$. □

Corollary 3.6.4. *Let E be a Lubin-Tate spectrum, let $A \in \mathrm{Alg}(\mathrm{hMod_E})$ be atomic, and let M be an E-module. The following conditions are equivalent:*

(*a*) *The module M is equivalent to a coproduct of copies of A and ΣA.*

(*b*) *The E-module M admits the structure of left A-module in the homotopy category $\mathrm{hMod_E}$.*

Remark 3.6.5. In the situation of Corollary 3.6.4, suppose that A is an algebra object of the ∞-category Mod_E. Then we can replace (b) by the following apparently strongly condition:

(b') The E-module M admits the structure of a left A-module object of Mod_E.

The implications $(a) \Rightarrow (b') \Rightarrow (b)$ are obvious, and the implication $(b) \Rightarrow (a)$ follows from Proposition 3.6.3.

Corollary 3.6.6. *Let E be a Lubin-Tate spectrum and let $A, A' \in \mathrm{Alg}(\mathrm{hMod_E})$ be atomic. Then A and A' are isomorphic as objects of Mod_E.*

Proof. Let B denote the relative smash product $A \otimes_E A'$. Using Proposition 3.6.3, we can choose a decomposition

$$B \simeq (\bigoplus_{i \in I} A) \oplus (\bigoplus_{j \in J} \Sigma A)$$

of left A-modules in $\mathrm{hMod_E}$. For each $i \in I$, let $e_i : B \to A$ denote the projection onto the ith factor. Since $B \neq 0$, the unit element $1 \in \pi_0 B$ is nonzero. It follows that we can choose an index $i \in I$ for which the map $e_i : B \to A$ does not annihilate 1. It follows that the composite map $A' \to B \xrightarrow{e_i} A$ is nonzero on homotopy groups. Using the assumption that A' and A are atomic, we deduce that the composite map $\pi_* A' \to \pi_* B \xrightarrow{e_i} \pi_* A$ is an isomorphism, so that A and A' are equivalent as E-modules. □

Warning 3.6.7. In the situation of Corollary 3.6.6, A and A' need not be isomorphic as associative algebra objects of $\mathrm{hMod_E}$.

Definition 3.6.8. Let E be a Lubin-Tate spectrum. We will say that an E-module M is *atomic* if there exists an equivalence $M \simeq A$, where $A \in \mathrm{Alg}(\mathrm{hMod_E})$ is atomic.

Remark 3.6.9. It follows from Proposition 3.5.1 and Corollary 3.6.6 that, up to equivalence, there exists a unique atomic E-module. We will sometimes refer to this E-module as the *Morava K-theory associated to E*, and denote it by $K(n)$ (where n is the height of E).

Let E be a Lubin-Tate spectrum and let M be an E-module. If M is atomic, then $\pi_* M$ is isomorphic to the quotient $(\pi_* E)/\mathfrak{m}(\pi_* E)$ as a graded module over the homotopy ring $\pi_* E$ (here $\mathfrak{m}$ denotes the maximal ideal of the local ring $\pi_0 E$). However, the converse is false in general:

Counterexample 3.6.10. Let E be a Lubin-Tate spectrum of height $n = 3$ and let $x_0, x_1, x_2 \in \pi_0 E$ be a regular system of parameters. For $0 \leqslant i \leqslant 2$, let $A(i)$ denote the cofiber of the map $x_i : E \to E$, so that we have cofiber sequences

$$E \to A(i) \xrightarrow{\delta(i)} \Sigma E.$$

Set $A = A(0) \otimes_E A(1) \otimes_E A(2)$, and let $\widetilde{A}$ denote the fiber of the map

$$A \xrightarrow{\delta(0) \otimes \delta(1) \otimes \delta(2)} (\Sigma E) \otimes_E (\Sigma E) \otimes_E (\Sigma E) \simeq \Sigma^3 E.$$

We then have a fiber sequence of E-modules $\Sigma^2 E \xrightarrow{\rho} \widetilde{A} \to A$ which induces a short exact sequence of abelian groups

$$0 \to \pi_0 E \to \pi_2 \widetilde{A} \xrightarrow{\phi} \pi_2 A \to 0$$

(since the homotopy groups of E and A are concentrated in even degrees). Because the groups $\mathrm{Ext}^m_{\pi_0 E}(\pi_2 A, \pi_0 E)$ vanish for $m \in \{0, 1\}$, this sequence splits uniquely: that is, the map ϕ has a unique section $\psi : \pi_2 A \to \pi_2 \widetilde{A}$. Set $\rho' = \rho + \psi(u) \in \pi_2 \widetilde{A}$, where u is a nonzero element of $\pi_2 A$, and let M denote the cofiber of the map $\rho' : \Sigma^2 E \to \widetilde{A}$. Then $\pi_* M$ is isomorphic to the quotient $(\pi_* E)/\mathfrak{m}(\pi_* E)$. However, M is not an atomic E-module: for example, one can show that the tautological map $M \to M \otimes_E A$ induces the zero map on homotopy groups, so that M cannot admit the structure of a (unital) algebra object of $\mathrm{hMod_E}$.

We will also need to consider some generalizations of Definition 3.6.8.

Definition 3.6.11. Let E be a Lubin-Tate spectrum and let M be an E-module. We will say that M is *molecular* if it is equivalent to direct sum of finitely many atomic E-modules and suspensions of atomic E-modules. We let $\mathrm{Mod}_E^{\mathrm{mol}}$ denote the full subcategory of Mod_E spanned by the molecular E-modules.

Variant 3.6.12. Let E be a Lubin-Tate spectrum and let M be an E-module. We will say that M is *quasi-molecular* if it is equivalent to a (not necessarily finite) direct sum of atomic E-modules and suspensions of atomic E-modules.

Example 3.6.13. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization. Then the Thom spectrum Th_Q is a molecular E-module if and only if the map

$$\overline{c}_1^Q : \kappa \otimes \Lambda \to \mathfrak{m}/\mathfrak{m}^2$$

of Construction 3.4.1 is an epimorphism.

Remark 3.6.14. Let M be an E-module. Then M is molecular if and only if it is both quasi-molecular and perfect (see Proposition 2.9.4).

Remark 3.6.15. Let M be a molecular E-module. Then M is a dualizable object of Mod_E; let us denote its dual by $M^\vee$. The module $M^\vee$ is quasi-molecular (this follows from the criterion of Corollary 3.6.4: note that if M has the structure of a left A-module object of $\mathrm{hMod}_{\mathrm{E}}$, then $M^\vee$ has the structure of a left A^{op}-module object of $\mathrm{hMod}_{\mathrm{E}}$) and perfect, and is therefore also molecular (Remark 3.6.14).

Chapter 4

Synthetic E-Modules

Let E be a Lubin-Tate spectrum of height n and let $\mathrm{Mod}_E^{\mathrm{loc}}$ denote the ∞-category of $K(n)$-local E-modules. In this section, we will construct a (symmetric monoidal) embedding of $\mathrm{Mod}_E^{\mathrm{loc}}$ into a larger ∞-category Syn_E, which we refer to as *the ∞-category of synthetic E-modules.*

4.1 The ∞-category Syn_E

We begin by introducing some definitions.

Definition 4.1.1. Let E be a Lubin-Tate spectrum, let $\mathrm{Mod}_E^{\mathrm{mol}} \subseteq \mathrm{Mod}_E$ be the ∞-category of molecular E-modules, and let $\mathcal{S}$ denote the ∞-category of spaces. A *synthetic E-module* is a functor $X : (\mathrm{Mod}_E^{\mathrm{mol}})^{\mathrm{op}} \to \mathcal{S}$ which preserves finite products. We let Syn_E denote the full subcategory of $\mathrm{Fun}((\mathrm{Mod}_E^{\mathrm{mol}})^{\mathrm{op}}, \mathcal{S})$ spanned by the synthetic E-modules.

The ∞-category Syn_E is an example of a *nonabelian derived ∞-category*, in the sense of §HTT.5.5.8 . Its formal properties can be summarized as follows:

Proposition 4.1.2. *Let E be a Lubin-Tate spectrum. Then:*

(1) *The ∞-category Syn_E is presentable. In particular, Syn_E admits small colimits.*

(2) *The inclusion functor $\mathrm{Syn}_E \hookrightarrow \mathrm{Fun}((\mathrm{Mod}_E^{\mathrm{mol}})^{\mathrm{op}}, \mathcal{S})$ preserves small sifted colimits. In other words, sifted colimits in Syn_E are computed "pointwise."*

(3) *The Yoneda embedding $j : \mathrm{Mod}_E^{\mathrm{mol}} \hookrightarrow \mathrm{Fun}((\mathrm{Mod}_E^{\mathrm{mol}})^{\mathrm{op}}, \mathcal{S})$ factors through Syn_E. Moreover, the functor $j : \mathrm{Mod}_E^{\mathrm{mol}} \to \mathrm{Syn}_E$ preserves finite coproducts.*

(4) *Let $\mathcal{C}$ be an ∞-category which admits small sifted colimits, and let $\mathrm{Fun}_\Sigma(\mathrm{Syn}_E, \mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathrm{Syn}_E, \mathcal{C})$ spanned by those functors which preserve small*

sifted colimits. Then composition with j induces an equivalence of ∞-categories $\mathrm{Fun}_{\Sigma}(\mathrm{Syn}_E, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Mod}_E^{\mathrm{mol}}, \mathcal{C})$.

(5) *Let $\mathcal{C}$ be an ∞-category which admits small colimits, and let* $\mathrm{LFun}(\mathrm{Syn}_E, \mathcal{C})$ *denote the full subcategory of* $\mathrm{Fun}(\mathrm{Syn}_E, \mathcal{C})$ *spanned by those functors which preserve small colimits. Then composition with j induces a fully faithful functor* $\mathrm{LFun}(\mathrm{Syn}_E, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Mod}_E^{\mathrm{mol}}, \mathcal{C})$, *whose essential image consists of those functors* $\mathrm{Mod}_E^{\mathrm{mol}} \to \mathcal{C}$ *which preserve finite coproducts.*

Proof. Combine Propositions HTT.5.5.8.10 and HTT.5.5.8.15. □

Remark 4.1.3. Assertion (4) of Proposition 4.1.2 can be summarized informally by saying that the ∞-category Syn_E is obtained from $\mathrm{Mod}_E^{\mathrm{mol}}$ by freely adjoining (small) sifted colimits.

Remark 4.1.4. Let E be a Lubin-Tate spectrum. Then the ∞-category of molecular E-modules $\mathrm{Mod}_E^{\mathrm{mol}}$ is additive (since it is a subcategory of a stable ∞-category which is closed under finite direct sums). It follows that Syn_E is a Grothendieck prestable ∞-category (see Proposition SAG.**??**). In particular, the functor $\Sigma^{\infty} : \mathrm{Syn}_E \to \mathrm{Sp}(\mathrm{Syn}_E)$ is a fully faithful embedding. Here we can identify $\mathrm{Sp}(\mathrm{Syn}_E)$ with the full subcategory of $\mathrm{Fun}((\mathrm{Mod}_E^{\mathrm{mol}})^{\mathrm{op}}, \mathrm{Sp})$ spanned by the additive functors

Remark 4.1.5 (The Structure of Syn_E)**.** Let E be a Lubin-Tate spectrum and let M be a molecular E-module for which both $\pi_0 M$ and $\pi_1 M$ are nonzero (for example, we could take $M = K \oplus \Sigma K$, where K is an atomic E-module). Then every molecular E-module can be obtained as a retract of M^k, for some integer $k \gg 0$. It follows that the image of M under the Yoneda embedding $j : \mathrm{Mod}_E^{\mathrm{mol}} \to \mathrm{Syn}_E$ is a compact projective generator for the ∞-category Syn_E of synthetic E-modules. We therefore obtain an equivalence of ∞-categories $\mathrm{Syn}_E \simeq \mathrm{LMod}_A^{\mathrm{c}}$, where $A = \mathrm{End}_{\mathrm{Syn}_E}(j(M))$ is the (connective) ring spectrum classifying endomorphisms of $j(M)$. Unwinding the definitions, we can identify A with the connective cover of the endomorphism algebra $\mathrm{End}_E(M)$.

This explicit description of Syn_E will not be particularly useful for us: it is not canonical (since it depends on a choice of the module M), and does not behave well with respect to the symmetric monoidal structure of §4.4.

4.2 The Restricted Yoneda Embedding

We now investigate the relationship between modules and synthetic modules over a Lubin-Tate spectrum E.

Construction 4.2.1 (Restricted Yoneda Embedding)**.** Let E be a Lubin-Tate spectrum and let M be an E-module. We let $\mathrm{Sy}[M] : (\mathrm{Mod}_E^{\mathrm{mol}})^{\mathrm{op}} \to \mathcal{S}$ denote the functor given informally by the formula $\mathrm{Sy}[M](N) = \mathrm{Map}_{\mathrm{Mod}_E}(M, N)$. It follows immediately from the definitions that $\mathrm{Sy}[M]$ is a synthetic E-module: that is, it preserves finite products. We will refer to $\mathrm{Sy}[M]$ as *the synthetic E-module associated to M*.

The construction $M \mapsto \mathrm{Sy}[M]$ determines a functor $\mathrm{Mod}_E \to \mathrm{Syn}_E$, which we will denote by Sy and refer to as *the restricted Yoneda embedding.*

Remark 4.2.2. When restricted to the full subcategory $\mathrm{Mod}_E^{\mathrm{mol}}$ of molecular E-modules, the functor $M \mapsto \mathrm{Sy}[M]$ coincides with the usual Yoneda embedding $j : \mathrm{Mod}_E^{\mathrm{mol}} \to \mathrm{Syn}_E \subseteq \mathrm{Fun}((\mathrm{Mod}_E^{\mathrm{mol}})^{\mathrm{op}}, \mathcal{S})$.

We begin by recording a few elementary properties of the construction $M \mapsto \mathrm{Sy}[M]$.

Proposition 4.2.3. *Let E be a Lubin-Tate spectrum and let* $\mathrm{Sy} : \mathrm{Mod}_E \to \mathrm{Syn}_E$ *be the restricted Yoneda embedding. Then:*

(1) *The functor* Sy *preserves small filtered colimits.*

(2) *The functor* Sy *preserves small limits.*

(3) *The functor* Sy *preserves small coproducts.*

Proof. Assertion (1) follows from the observation that every molecular E-module M is a compact object of Mod_E (see Proposition 2.9.4), and assertion (2) is immediate. To prove (3), we note that (2) implies that the functor Sy preserves finite products. Since the ∞-categories Mod_E and Syn_E are both additive (Remark 4.1.4), it follows that Sy preserves finite coproducts. Combining this observation with (1), we conclude that Sy preserves all small coproducts. □

Proposition 4.2.4. *Let E be a Lubin-Tate spectrum of height n and let $\alpha : M \to N$ be a morphism of E-modules. Then the induced map* $\mathrm{Sy}[M] \to \mathrm{Sy}[N]$ *is an equivalence of synthetic E-modules if and only if α is a $K(n)$-equivalence.*

Proof. Let $K(n)$ denote an atomic E-module, and let $K(n)^\vee$ denote its E-linear dual. The following assertions are equivalent:

(a) The map $\mathrm{Sy}[M] \to \mathrm{Sy}[N]$ is an equivalence of synthetic E-modules.

(b) For every molecular E-module P, the map of spaces

$$\mathrm{Map}_{\mathrm{Mod}_E}(P, M) \to \mathrm{Map}_{\mathrm{Mod}_E}(P, N))$$

is a homotopy equivalence.

(c) The map of spaces $\mathrm{Map}_{\mathrm{Mod}_E}(K(n)^\vee, M) \to \mathrm{Map}_{\mathrm{Mod}_E}(K(n)^\vee, N)$ is a homotopy equivalence.

(d) For every integer $m \leqslant 0$, composition with α induces an isomorphism of abelian groups

$$\mathrm{Ext}^m_E(K(n)^\vee, M) \to \mathrm{Ext}^m_E(K(n)^\vee, N).$$

(e) For every integer m, composition with α induces an isomorphism of abelian groups

$$\mathrm{Ext}^m_E(K(n)^\vee, M) \to \mathrm{Ext}^m_E(K(n)^\vee, N).$$

(f) The map α induces an isomorphism of homotopy groups

$$\pi_*(K(n) \otimes_E M) \to \pi_*(K(n) \otimes_E N).$$

(g) The map α is a $K(n)$-local equivalence: that is, the induced map $K(n) \otimes_E M \to K(n) \otimes_E N$ is an equivalence.

The implications $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Leftarrow (e) \Leftrightarrow (f) \Leftrightarrow (g)$ are easy. The implication $(c) \Rightarrow (b)$ follows from the fact that every molecular E-module can be obtained as a direct sum of $K(n)^\vee$ and $\Sigma K(n)^\vee$, and the implication $(e) \Rightarrow (d)$ follows from the periodicity of E. $\square$

It follows from Proposition 4.2.4 that the restricted Yoneda embedding $\mathrm{Sy} : \mathrm{Mod}_E \to \mathrm{Syn}_E$ factors (up to homotopy) through the $K(n)$-localization functor $L : \mathrm{Mod}_E \to \mathrm{Mod}_E^{\mathrm{loc}}$. For this reason, we will generally confine our attention to the restriction $\mathrm{Sy}|_{\mathrm{Mod}_E^{\mathrm{loc}}}$, which (by slight abuse of notation) we will also denote by Sy.

Proposition 4.2.5. *Let E be a Lubin-Tate spectrum. Then the restricted Yoneda embedding* $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Syn}_E$ *is a fully faithful embedding. Its essential image consists of those synthetic E-modules $X : (\mathrm{Mod}_E^{\mathrm{mol}})^{\mathrm{op}} \to \mathcal{S}$ which satisfy the following additional condition:*

($*$) *For every molecular E-module N, the canonical map $X(\Sigma N) \to \Omega X(N)$ is a homotopy equivalence.*

Remark 4.2.6. The restricted Yoneda embedding $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Syn}_E$ is not essentially surjective. For example, if M is an E-module, then the construction

$$(N \in \mathrm{Mod}_E^{\mathrm{mol}}) \mapsto (\pi_0 \mathrm{Map}_{\mathrm{Mod}_E}(N, M) \in \mathcal{S}\mathrm{et} \subseteq \mathcal{S})$$

determines a synthetic E-module, which we will denote by $\mathrm{Sy}^\heartsuit[M]$ (see Notation 6.1.7). The synthetic E-module $\mathrm{Sy}^\heartsuit[M]$ never belongs to the essential image of the restricted

Yoneda embedding, except in the trivial case where M is $K(n)$-acyclic (in which case we have $\mathrm{Sy}[M] \simeq \mathrm{Sy}^{\heartsuit}[M] \simeq 0$). The truncation $\mathrm{Sy}^{\heartsuit}[M]$ is an example of a *Milnor module* (Definition 6.1.1): that is, it is a discrete object of the ∞-category Syn_E. We will carry out a detailed study of Milnor modules in §6.

Proof of Proposition 4.2.5. Let $\mathcal{C}$ denote the full subcategory of Syn_E spanned by those functors which satisfy condition $(*)$. Note that $\mathcal{C}$ contains $\mathrm{Sy}[M]$ for every E-module M and is closed under limits in Syn_E. Moreover, for each object $X \in \mathcal{C}$, condition $(*)$ supplies a canonical equivalence $\Omega \circ X \simeq X \circ \Sigma$. Since the suspension functor Σ induces an equivalence of $\mathrm{Mod}_E^{\mathrm{mol}}$ with itself, it follows that the functor $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence of ∞-categories. Applying Proposition HA.1.4.2.11 , we deduce that the ∞-category $\mathcal{C}$ is stable.

Let $F : \mathrm{Mod}_E \to \mathcal{C}$ denote the functor given by $F(M) = \mathrm{Sy}[M]$. The functor Sy preserves small limits (Proposition 4.2.3), so that F also preserves small limits. Since domain and codomain of F are stable ∞-categories, it follows that F also preserves finite colimits (beware that this property is not shared by the functor $\mathrm{Sy} : \mathrm{Mod}_E \to \mathrm{Syn}_E$). Moreover, the functor F also preserves filtered colimits (Proposition 4.2.3), and so preserves all small colimits.

We next prove the following:

(a) Let M and N be E-modules, where $N \in \mathrm{Mod}_E^{\mathrm{loc}}$. Then the canonical map

$$\theta_{M,N} : \mathrm{Map}_{\mathrm{Mod}_E}(M, N) \to \mathrm{Map}_{\mathrm{Syn}_E}(\mathrm{Sy}[M], \mathrm{Sy}[N]) = \mathrm{Map}_{\mathcal{C}}(F(M), F(N))$$

is a homotopy equivalence.

To prove (a), let us regard the E-module N as fixed. We will say that an E-module M is *good* if the map $\theta_{M,N}$ is a homotopy equivalence. Using Yoneda's lemma, we see that every molecular E-module is good. Because the functor F preserves small colimits, we conclude that the collection of good E-modules is closed under small colimits. Let $\mathcal{E} \subseteq \mathrm{Mod}_E$ denote the smallest full subcategory which contains $\mathrm{Mod}_E^{\mathrm{mol}}$ and is closed under small colimits, so that every object of $\mathcal{E}$ is good. Applying Corollary HTT.5.5.2.9 , we deduce that the inclusion $\mathcal{E} \hookrightarrow \mathrm{Mod}_E$ admits a right adjoint: that is, every object $M \in \mathrm{Mod}_E$ fits into a fiber sequence $M' \to M \to M''$, where $M' \in \mathcal{E}$ and $\mathrm{Map}_{\mathrm{Mod}_E}(P, M'')$ is contractible for each $P \in \mathcal{E}$. In particular, the synthetic E-module $\mathrm{Sy}[M'']$ vanishes. Applying Proposition 4.2.4, we deduce that M'' is $K(n)$-acyclic. We therefore have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Mod}_E}(M, N) & \xrightarrow{\theta_{M,N}} & \mathrm{Map}_{\mathrm{Syn}_E}(\mathrm{Sy}[M], \mathrm{Sy}[N]) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Mod}_E}(M', N) & \xrightarrow{\theta_{M',N}} & \mathrm{Map}_{\mathrm{Syn}_E}(\mathrm{Sy}[M'], \mathrm{Sy}[N]) \end{array}$$

where the vertical maps are homotopy equivalences (by virtue of our assumption that N belongs to $\mathrm{Mod}_E^{\mathrm{loc}}$). Since the map $\theta_{M',N}$ is a homotopy equivalence, we conclude that $\theta_{M,N}$ is also a homotopy equivalence. This completes the proof of (a).

Note that the functor $F : \mathrm{Mod}_E \to \mathcal{C}$ factors as a composition

$$\mathrm{Mod}_E \xrightarrow{L} \mathrm{Mod}_E^{\mathrm{loc}} \xrightarrow{\overline{F}} \mathcal{C}$$

where $\overline{F} = F|_{\mathrm{Mod}_E^{\mathrm{loc}}}$. The functor $\overline{F}$ also preserves small colimits, and (a) guarantees that $\overline{F}$ is faithful. Using Corollary HTT.5.5.2.9 , we deduce that $\overline{F}$ admits a right adjoint $\overline{G} : \mathcal{C} \to \mathrm{Mod}_E^{\mathrm{loc}}$. Note that for every object $X \in \mathcal{C}$ and every molecular E-module M, we have canonical homotopy equivalences

$$X(M) \simeq \mathrm{Map}_{\mathcal{C}}(\overline{F}(M), X) \simeq \mathrm{Map}_{\mathrm{Mod}_E^{\mathrm{loc}}}(M, \overline{G}(X)).$$

It follows that $\overline{G}$ is conservative, so that $\overline{F}$ and $\overline{G}$ are mutually inverse equivalences of ∞-categories. □

4.3 Hypercoverings

We now study a special class of colimits in Mod_E which are preserved by the restricted Yoneda embedding $\mathrm{Sy} : \mathrm{Mod}_E \to \mathrm{Syn}_E$.

Definition 4.3.1. Let E be a Lubin-Tate spectrum, let A be an atomic E-algebra, and let $\overline{M}_\bullet$ be an augmented simplicial object of Mod_E. We will say that $\overline{M}_\bullet$ is *A-split* if $A \otimes_E \overline{M}_\bullet$ is a split augmented simplicial object of the ∞-category LMod_A (see Definition HA.4.7.3.2).

Remark 4.3.2. Let A be an atomic E-algebra and let $\overline{M}_\bullet$ be an augmented simplicial object of Mod_E. If $\overline{M}_\bullet$ is A-split, then $\overline{M}_\bullet \otimes_E N$ is also A-split, for every E-module N.

Proposition 4.3.3. *Let E be a Lubin-Tate spectrum, let A be an atomic E-algebra, and let M be an E-module. Then there exists an A-split augmented simplicial object $\overline{M}_\bullet$ of Mod_E such that $\overline{M}_{-1} = M$ and $\overline{M}_n$ is quasi-molecular for $n \geqslant 0$.*

Proof. Let $G : \mathrm{Mod}_E \to \mathrm{LMod}_A$ be the functor given by $G(N) = A \otimes_E N$. Since A is dualizable as an E-module, the functor G admits a left adjoint $F : \mathrm{LMod}_A \to \mathrm{Mod}_E$, given concretely by the formula $F(X) = A^\vee \otimes_A X$. Set $U = F \circ G$, so that U is a comonad on Mod_E. Then the construction $\overline{M}_k = U^{k+1}(M)$ determines an augmented simplicial object with the desired properties (note that the functor F carries each object of LMod_A to a quasi-molecular object of Mod_E). □

Proposition 4.3.4. *Let E be a Lubin-Tate spectrum and let $\overline{M}_\bullet$ be an augmented simplicial object of Mod_E. Suppose that there exists an atomic E-algebra A such that $\overline{M}_\bullet$ is A-split. Then $\mathrm{Sy}[\overline{M}_\bullet]$ is a colimit diagram in the ∞-category Syn_E.*

Proof. We prove a stronger assertion: for every molecular E-module N, the augmented simplicial space $\mathrm{Sy}[\overline{M}_\bullet](N) = \mathrm{Map}_{\mathrm{Mod}_E}(N, \overline{M}_\bullet)$ is a colimit diagram. Since the category $\mathbf{\Delta}^{\mathrm{op}}$ is sifted, the collection of those objects $N \in \mathrm{Mod}_E$ which satisfy this condition is closed under finite coproducts. We may therefore assume without loss of generality that $N \simeq \Sigma^m A^\vee$, where m is an integer and $A^\vee$ denotes the E-linear dual of A. In this case, we have an equivalence $\mathrm{Map}_{\mathrm{Mod}_E}(N, \overline{M}_\bullet) \simeq \Omega^{\infty+m}(A \otimes_E \overline{M}_\bullet)$, which is a split augmented simplicial object of $\mathcal{S}$ by virtue of our assumption that $\overline{M}_\bullet$ is A-split. □

Corollary 4.3.5. *Let E be a Lubin-Tate spectrum and let $L : \mathrm{Mod}_E \to \mathrm{Mod}_E^{\mathrm{loc}}$ denote a left adjoint to the inclusion functor. Let $\overline{M}_\bullet$ be an augmented simplicial object of Mod_E. If there exists an atomic E-algebra A for which $\overline{M}_\bullet$ is A-split, then $L\overline{M}_\bullet$ is a colimit diagram in the ∞-category $\mathrm{Mod}_E^{\mathrm{loc}}$.*

Proof. Combine Propositions 4.3.4 and 4.2.5. □

4.4 Smash Products of Synthetic E-Modules

Throughout this section, we let E denote a Lubin-Tate spectrum. We regard the ∞-category $\mathrm{Mod}_E^{\mathrm{loc}}$ as equipped with the symmetric monoidal structure given by the localized smash product $\widehat{\otimes}_E$ (see §2.9). Our goal in this section is to construct a compatible symmetric monoidal structure on the ∞-category Syn_E of synthetic E-modules. Our starting point is the following:

Proposition 4.4.1. *Let E be a Lubin-Tate spectrum. If M and N are molecular E-modules, then the relative smash product $M \otimes_E N$ is also molecular.*

Proof. Using the criterion of Corollary 3.6.4, we see that if M is quasi-molecular and N is an arbitrary E-module, then the relative smash product $M \otimes_E N$ is quasi-molecular. If M and N are perfect E-modules, then $M \otimes_E N$ is perfect. The desired result now follows from Remark 3.6.14. □

Corollary 4.4.2. *The symmetric monoidal structure on $\mathrm{Mod}_E^{\mathrm{loc}}$ restricts to a nonunital symmetric monoidal structure on the full subcategory $\mathrm{Mod}_E^{\mathrm{mol}}$. In other words, there is an essentially unique nonunital symmetric monoidal structure on the ∞-category $\mathrm{Mod}_E^{\mathrm{mol}}$ for which the inclusion $\mathrm{Mod}_E^{\mathrm{mol}} \hookrightarrow \mathrm{Mod}_E^{\mathrm{loc}}$ has the structure of a nonunital symmetric monoidal functor.*

Warning 4.4.3. The nonunital symmetric monoidal structure on $\mathrm{Mod}_E^{\mathrm{mol}}$ cannot be promoted to a symmetric monoidal structure (note that the module E is not molecular).

Applying the constructions of §HA.4.8.1 to the ∞-category of synthetic E-modules $\mathrm{Syn}_E = \mathcal{P}_\Sigma(\mathrm{Mod}_E^{\mathrm{mol}})$ we obtain the following result:

Proposition 4.4.4. *Let E be a Lubin-Tate spectrum. Then there is an essentially unique nonunital symmetric monoidal structure on the ∞-category Syn_E with the following features:*

(a) *The Yoneda embedding $j : \mathrm{Mod}_E^{\mathrm{mol}} \hookrightarrow \mathrm{Syn}_E$ is a nonunital symmetric monoidal functor.*

(b) *The tensor product functor $\mathrm{Syn}_E \times \mathrm{Syn}_E \to \mathrm{Syn}_E$ preserves small colimits separately in each variable.*

Moreover, this nonunital symmetric monoidal structure is characterized by the following universal property:

(c) *Let $\mathcal{C}$ be a presentable ∞-category equipped with a nonunital symmetric monoidal structure. Assume that the tensor product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable, and let $\mathrm{LFun}^{\otimes}_{\mathrm{nu}}(\mathrm{Syn}_E, \mathcal{C})$ denote the ∞-category of nonunital symmetric monoidal functors from Syn_E to $\mathcal{C}$ which preserve small colimits. Then composition with j induces a fully faithful embedding $\mathrm{LFun}^{\otimes}_{\mathrm{nu}}(\mathrm{Syn}_E, \mathcal{C}) \to \mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathrm{Mod}_E^{\mathrm{mol}}, \mathcal{C})$, whose essential image is spanned by those nonunital symmetric monoidal functors $F : \mathrm{Mod}_E^{\mathrm{mol}} \to \mathcal{C}$ which preserve finite coproducts.*

Notation 4.4.5. Let E be a Lubin-Tate spectrum. We let $\wedge : \mathrm{Syn}_E \times \mathrm{Syn}_E \to \mathrm{Syn}_E$ denote the tensor product functor underlying the nonunital symmetric monoidal structure of Proposition 4.4.4.

Applying part (c) of Proposition 4.4.4 in the case $\mathcal{C} = \mathrm{Mod}_E^{\mathrm{loc}}$, we deduce the following:

Corollary 4.4.6. *There is an essentially unique nonunital symmetric monoidal functor $F : \mathrm{Syn}_E \to \mathrm{Mod}_E^{\mathrm{loc}}$ with the following properties:*

(a) *The functor F preserves small colimits.*

(b) *The composite functor $\mathrm{Mod}_E^{\mathrm{mol}} \xrightarrow{j} \mathrm{Syn}_E \xrightarrow{F} \mathrm{Mod}_E^{\mathrm{loc}}$ is equivalent to the inclusion (as nonunital symmetric monoidal functors from $\mathrm{Mod}_E^{\mathrm{mol}}$ to $\mathrm{Mod}_E^{\mathrm{loc}}$).*

Unwinding the definitions, we see that the functor F of Corollary 4.4.6 can be identified with the left adjoint of the restricted Yoneda embedding $\mathrm{Sy} : \mathrm{Mod}_E^{K} \hookrightarrow \mathrm{Syn}_E$ of Construction 4.2.1. It follows formally that Sy inherits the structure of a *lax* nonunital symmetric monoidal functor from $\mathrm{Mod}_E^{\mathrm{loc}}$ to Syn_E. However, we can say more:

Proposition 4.4.7. *The lax nonunital symmetric monoidal functor* $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Syn}_E$ *is a nonunital symmetric monoidal functor. In other words, for every pair of objects* $M, N \in \mathrm{Mod}_E^{\mathrm{loc}}$*, the canonical map* $\rho_{M,N} : \mathrm{Sy}[M] \wedge \mathrm{Sy}[N] \to \mathrm{Sy}[M \widehat{\otimes}_E N]$ *is an equivalence of synthetic* E*-modules.*

Proof. Let A be an atomic E-algebra (see Proposition 3.5.1). Applying Proposition 4.3.3, we can choose a A-split augmented simplicial E-module $\overline{M}_\bullet$, where $\overline{M}_{-1} = M$ and $\overline{M}_n$ is quasi-molecular for each $n \geqslant 0$. Note that $\overline{M}_\bullet \otimes_E N$ is also A-split (Remark 4.3.2). Applying Proposition 4.3.4, we deduce that the augmented simplicial objects $\mathrm{Sy}[\overline{M}_\bullet]$ and $\mathrm{Sy}[\overline{M}_\bullet \otimes_E N] \simeq \mathrm{Sy}[\overline{M}_\bullet \widehat{\otimes}_E N]$ are colimit diagrams in the ∞-category Syn_E. Consequently, we can identify $\rho_{M,N}$ with a colimit of morphisms of the form $\rho_{\overline{M}_n,N}$ for $n \geqslant 0$. We may therefore replace M by $\overline{M}_n$ and thereby reduce to the case where M is quasi-molecular. Note that for fixed N, the construction $M \mapsto \rho_{M,N}$ commutes with filtered colimits. Since every quasi-molecular E-module can be written as a filtered colimit of molecular E-modules, we may further reduce to the case where M is molecular. Applying the same argument with the roles of M and N reversed, we can reduce to the case where N is also molecular. In this case, the desired result follows immediately from the definitions. □

We now construct a unit with respect to the tensor product on Syn_E.

Notation 4.4.8. Let E be a Lubin-Tate spectrum. We let $\mathbf{1}$ denote the synthetic E-module given by $\mathrm{Sy}[E]$.

Lemma 4.4.9. *Let* E *be a Lubin-Tate spectrum. Then the construction* $X \mapsto \mathbf{1} \wedge X$ *determines a functor* $\mathrm{Syn}_E \to \mathrm{Syn}_E$ *which is homotopic to the identity.*

Proof. Since precomposition with the Yoneda embedding $j : \mathrm{Mod}_E^{\mathrm{mol}} \to \mathrm{Syn}_E$ induces a fully faithful embedding $\mathrm{LFun}(\mathrm{Syn}_E, \mathrm{Syn}_E) \to \mathrm{Fun}(\mathrm{Mod}_E^{\mathrm{mol}}, \mathrm{Syn}_E)$, it will suffice to show that the functors j and $\mathbf{1} \wedge j$ are equivalent: that is, to show that there is an equivalence $\mathrm{Sy}[M] \simeq \mathrm{Sy}[E] \wedge \mathrm{Sy}[M]$ depending functorially on $M \in \mathrm{Mod}_E^{\mathrm{mol}}$. This is a special case of Proposition 4.4.7. □

Proposition 4.4.10. *Let* E *be a Lubin-Tate spectrum. Then the* ∞*-category* Syn_E *admits an essentially unique symmetric monoidal structure extending the nonunital symmetric monoidal structure of Corollary 4.4.6. Moreover, the unit object of* Syn_E *can be identified with* $\mathbf{1} = \mathrm{Sy}[E]$.

Proof. Combine Lemma 4.4.9 with Corollary HA.5.4.4.7 . □

Variant 4.4.11. Let E be a Lubin-Tate spectrum. Then the nonunital symmetric monoidal functor $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Syn}_E$ can be promoted, in an essentially unique way, to a symmetric monoidal functor.

Proof. By virtue of Corollary HA.5.4.4.7, it suffices to observe that the functor Sy carries the unit object $E \in \mathrm{Mod}_E^{\mathrm{loc}}$ to the unit object $\mathbf{1} = \mathrm{Sy}[E]$ of Syn_E. □

4.5 Truncated Synthetic E-Modules

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section. In §4.2, we proved that the restricted Yoneda embedding $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Syn}_E$ induces an equivalence of the ∞-category $\mathrm{Mod}_E^{\mathrm{loc}}$ with a full subcategory of the ∞-category Syn_E of synthetic E-modules (Proposition 4.2.5). In this section, we consider some other full subcategories of Syn_E which will play an important in our calculation of the Brauer group $\mathrm{Br}(E)$.

Definition 4.5.1. Let $n \geqslant 0$ be an nonnegative integer. We will say that a synthetic E-module X is *n-truncated* if, for every molecular E-module M, the space $X(M)$ is n-truncated (that is, the homotopy groups $\pi_m(X(M), x)$ vanish for $m > n$ and every choice of base point $x \in X(M)$). We let $\mathrm{Syn}_E^{\leqslant n}$ denote the full subcategory of Syn_E spanned by the n-truncated synthetic E-modules.

Remark 4.5.2. A synthetic E-module X is n-truncated in the sense of Definition 4.5.1 if and only if it is n-truncated when viewed as an object of the ∞-category Syn_E: that is, if and only if the mapping space $\mathrm{Map}_{\mathrm{Syn}_E}(Y, X)$ is n-truncated, for every synthetic E-module Y.

Remark 4.5.3. Let $n \geqslant 0$ be an integer. Then the inclusion functor $\mathrm{Syn}_E^{\leqslant n} \hookrightarrow \mathrm{Syn}_E$ admits a left adjoint $\tau_{\leqslant n} : \mathrm{Syn}_E \to \mathrm{Syn}_E^{\leqslant n}$, given concretely by the formula $(\tau_{\leqslant n} X)(M) = \tau_{\leqslant n} X(M)$.

Remark 4.5.4. Let M be an E-module and let n be a nonnegative integer. Then the associated synthetic E-module $\mathrm{Sy}[M] \in \mathrm{Syn}_E$ is n-truncated if and only if it is zero (this follows immediately from the criterion of Proposition 4.2.5.

Remark 4.5.5. We will carry out a detailed analysis of the category $\mathrm{Syn}_E^{\leqslant 0}$ in §6.

We now study the composite functor

$$(\tau_{\leqslant n} \circ \mathrm{Sy}) : \mathrm{Mod}_E \to \mathrm{Syn}_E^{\leqslant n},$$

where n is a nonnegative integer. Unlike the functor $M \mapsto \mathrm{Sy}[M]$ itself, the composite functor $\tau_{\leqslant n} \circ \mathrm{Sy}$ is not fully faithful. However, it is close to being fully faithful provided we restrict our attention to *quasi-molecular* E-modules.

Remark 4.5.6. Using Proposition 4.2.3, we see that the functor

$$(\tau_{\leqslant n} \circ \mathrm{Sy}) : \mathrm{Mod}_E \to \mathrm{Syn}_E^{\leqslant n}$$

preserves filtered colimits and arbitrary coproducts.

Proposition 4.5.7. *Let $n \geqslant 0$ be an integer and let $M, N \in \mathrm{Mod}_E$. If M is quasi-molecular, then the canonical map*

$$\theta : \mathrm{Map}_{\mathrm{Mod}_E}(M, N) \to \mathrm{Map}_{\mathrm{Syn}_E^{\leqslant n}}(\tau_{\leqslant n}\,\mathrm{Sy}[M], \tau_{\leqslant n}\,\mathrm{Sy}[N])$$

exhibits the mapping space $\mathrm{Map}_{\mathrm{Syn}_E^{\leqslant n}}(\tau_{\leqslant n}\,\mathrm{Sy}[M], \tau_{\leqslant n}\,\mathrm{Sy}[N])$ as an n-truncation of $\mathrm{Map}_{\mathrm{Mod}_E}(M, N)$.

Proof. Write M as a coproduct $\bigoplus M_\alpha$, where each M_α is molecular. Using Remark 4.5.6, we can write θ as a product of maps

$$\theta_\alpha : \mathrm{Map}_{\mathrm{Mod}_E}(M_\alpha, N) \to \mathrm{Map}_{\mathrm{Syn}_E^{\leqslant n}}(\tau_{\leqslant n}\,\mathrm{Sy}[M_\alpha], \tau_{\leqslant n}\,\mathrm{Sy}[N]).$$

It will therefore suffice to show that each of the maps θ_α exhibits

$$\mathrm{Map}_{\mathrm{Syn}_E^{\leqslant n}}(\tau_{\leqslant n}\,\mathrm{Sy}[M_\alpha], \tau_{\leqslant n}\,\mathrm{Sy}[N]) \simeq \mathrm{Map}_{\mathrm{Syn}_E}(\mathrm{Sy}[M_\alpha], \tau_{\leqslant n}\,\mathrm{Sy}[N])$$

as an n-truncation of the mapping space $\mathrm{Map}_{\mathrm{Mod}_E}(M_\alpha, N)$. This follows immediately from the definition of the synthetic E-module $\tau_{\leqslant n}\,\mathrm{Sy}[N]$. □

We now prove a dual version of Proposition 4.5.7, which is a bit less formal.

Proposition 4.5.8. *Let $n \geqslant 0$ be an integer and let $M, N \in \mathrm{Mod}_E$. If N is quasi-molecular, then the canonical map*

$$\theta : \mathrm{Map}_{\mathrm{Mod}_E}(M, N) \to \mathrm{Map}_{\mathrm{Syn}_E^{\leqslant n}}(\tau_{\leqslant n}\,\mathrm{Sy}[M], \tau_{\leqslant n}\,\mathrm{Sy}[N])$$

exhibits the mapping space $\mathrm{Map}_{\mathrm{Syn}_E^{\leqslant n}}(\tau_{\leqslant n}\,\mathrm{Sy}[M], \tau_{\leqslant n}\,\mathrm{Sy}[N])$ as an n-truncation of $\mathrm{Map}_{\mathrm{Mod}_E}(M, N)$.

Proof. Fix an atomic algebra $A \in \mathrm{Alg}(\mathrm{Mod}_E)$ (Proposition 3.5.1). Using Proposition 4.3.3, we can choose a A-split augmented simplicial E-module $\overline{M}_\bullet$ where $\overline{M}_{-1} = M$ and $\overline{M}_n$ is quasi-molecular for $n \geqslant 0$. Let $M_\bullet$ be the underlying simplicial object of $\overline{M}_\bullet$. Then the map θ fits into a commutative diagram

$$\begin{array}{ccc}
\tau_{\leqslant n}\,\mathrm{Map}_{\mathrm{Mod}_E}(M, N) & \xrightarrow{\theta} & \mathrm{Map}_{\mathrm{Syn}_E}(\tau_{\leqslant n}\,\mathrm{Sy}[M], \tau_{\leqslant n}\,\mathrm{Sy}[N]) \\
\downarrow & & \downarrow \\
\mathrm{Tot}\,\tau_{\leqslant n}\,\mathrm{Map}_{\mathrm{Mod}_E}(M_\bullet, N) & \longrightarrow & \mathrm{Tot}\,\mathrm{Map}_{\mathrm{Syn}_E}(\tau_{\leqslant n}\,\mathrm{Sy}[M_\bullet], \tau_{\leqslant n}\,\mathrm{Sy}[N]).
\end{array}$$

The bottom horizontal map is a homotopy equivalence by virtue of Proposition 4.5.7. We will complete the proof by showing that the left and right vertical maps are also homotopy equivalences. For the right vertical map, this follows from Proposition 4.3.4. To prove that the left vertical map is a homotopy equivalence, it will suffice to show that the augmented cosimplicial space $\tau_{\leqslant n} \operatorname{Map}_{\operatorname{Mod}_E}(\overline{M}_\bullet, N)$ is a limit diagram in the ∞-category $\mathcal{S}$. In fact, we claim that it is a split augmented cosimplicial object of $\mathcal{S}$. To see this, note that our assumption that N is quasi-molecular guarantees that we can promote N to a left A-module object of Mod_E. We therefore have an equivalence of augmented cosimplicial spaces

$$\tau_{\leqslant n} \operatorname{Map}_{\operatorname{Mod}_E}(\overline{M}_\bullet, N) \simeq \tau_{\leqslant n} \operatorname{Map}_{\operatorname{LMod}_A}(A \otimes_E \overline{M}_\bullet, N);$$

the desired result now follows from our assumption that $\overline{M}_\bullet$ is A-split. $\square$

Chapter 5

Representations of Exterior Algebras

Let E be a Lubin-Tate spectrum, let Syn_E denote the ∞-category of synthetic E-modules, and let $\mathrm{Syn}_E^{\heartsuit} \subseteq \mathrm{Syn}_E$ denote the full subcategory spanned by the discrete objects. In §6, we will prove that $\mathrm{Syn}_E^{\heartsuit}$ is equivalent to the abelian category of $(\mathbf{Z}/2\mathbf{Z})$-graded modules over an exterior algebra $\bigwedge^*(V)$, where $V \simeq (\mathfrak{m}/\mathfrak{m}^2)^\vee$ is the Zariski tangent space to the Lubin-Tate ring $\pi_0 E$ at its maximal ideal $\mathfrak{m} \subseteq \pi_0 E$ (Theorem 6.6.6). To prove this result (and to make effective use of it), we will need some purely algebraic facts about (graded) modules over exterior algebras, which we have collected in this section.

5.1 Conventions

We begin by establishing some conventions.

Notation 5.1.1. Throughout this section, we let K_* denote a commutative graded ring with the following properties:

(a) Every nonzero homogeneous element of K_* is invertible.

(b) The graded component K_n is nonzero if and only if n is even.

We let $\kappa = K_0$ denote the subring of K_* consisting of elements of degree zero. It follows from (a) and (b) that κ is a field, and that K_* is isomorphic to a Laurent polynomial ring $\kappa[t^{\pm 1}]$ where t is homogeneous of degree 2. Beware that this isomorphism is not canonical (it depends on the choice of a nonzero element $t \in K_2$).

Example 5.1.2. Let E be a Lubin-Tate spectrum and let $\mathfrak{m} \subseteq \pi_0 E$ be the maximal ideal. Then the graded ring $K_* = (\pi_* E)/\mathfrak{m}(\pi_* E)$ satisfies the requirements of Notation 5.1.1. For our ultimate applications, this is the example of interest.

Notation 5.1.3. We let $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ denote the abelian category of $\mathbf{Z}$-graded modules over K_*. We will refer to the objects of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ simply as *graded K_*-modules.* Note that the construction $V \mapsto (V_0 \oplus V_1)$ determines an equivalence of categories $\mathrm{Mod}^{\mathrm{gr}}_{K_*} \to \mathrm{Vect}^{\mathrm{gr}}_{\kappa}$; here $\mathrm{Vect}^{\mathrm{gr}}_{\kappa}$ is the category of $(\mathbf{Z}/2\mathbf{Z})$-graded vector spaces over κ (see §2.8).

Notation 5.1.4. Let V be a graded K_*-module. For every integer n, we let $V[n]$ denote the graded K_*-module given by the formula $V[n]_m = V_{m-n}$. Note that if n is even, we have a canonical isomorphism $V[n] \simeq V \otimes_{\kappa} K_{-n}$.

If V and W are graded K_*-modules, then the tensor product $V \otimes_{K_*} W$ inherits the structure of a graded K_*-module. The construction $(V, W) \mapsto V \otimes_{K_*} W$ determines a monoidal structure on the category $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$. We will regard $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ as a symmetric monoidal category by enforcing the usual Koszul sign rule: for graded K_*-modules V and W, the symmetry constraint $\sigma_{V,W} : V \otimes_{K_*} W \simeq W \otimes_{K_*} V$ is given by $\sigma_{V,W}(v \otimes w) = (-1)^{ij}(w \otimes v)$ for $v \in V_i$, $w \in W_j$.

Remark 5.1.5. Suppose we are given a nonzero element $t \in K_2$, which determines an isomorphism of graded rings $\kappa[t^{\pm 1}] \simeq K_*$. It follows that there is a unique ring homomorphism $\theta : K_* \to \kappa$ which is the identity on κ and satisfies $\theta(t) = 1$. In this case, extension of scalars along θ induces an equivalence of symmetric monoidal categories $F : \mathrm{Mod}^{\mathrm{gr}}_{K_*} \to \mathrm{Vect}^{\mathrm{gr}}_{\kappa}$. Note that the underlying equivalence of categories (obtained by ignoring monoidal structures) is isomorphic to the functor $V \mapsto V_0 \oplus V_1$ of Notation 5.1.3.

Warning 5.1.6. Let $F : \mathrm{Mod}^{\mathrm{gr}}_{K_*} \to \mathrm{Vect}^{\mathrm{gr}}_{\kappa}$ be the equivalence of categories described in Notation 5.1.3. Then F does not *canonically* admit the structure of a monoidal functor. It is not hard to see that promoting F to a monoidal functor is equivalent to choosing a nonzero element $t \in K_2$ (with the inverse equivalence given by the construction of Remark 5.1.5).

5.2 Exterior Algebras

Throughout this section, we let K_* denote a graded ring satisfying the requirements of Notation 5.1.1 and we let $\kappa = K_0$ be the underlying field.

Notation 5.2.1. Let V be a vector space over κ. We let $\bigwedge^*_{\kappa}(V)$ denote the exterior algebra on V. We regard $\bigwedge^*_{\kappa}(V)$ as a $\mathbf{Z}$-graded Hopf algebra over κ, where each element $v \in V$ is a primitive element of degree (-1). We let $\bigwedge^*_{K_*}(V)$ denote the tensor product

$K_* \otimes_\kappa \bigwedge^*(V)$, which we regard as a $\mathbf{Z}$-graded Hopf algebra over K_* (in other words, as a Hopf algebra object of the abelian category $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$).

Remark 5.2.2 (Duality)**.** In the situation of Notation 5.2.1, suppose that the vector space V is finite-dimensional. In this case, the exterior algebra $\bigwedge^*_{K_*}(V)$ is a dualizable object of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$. It follows that the dual of $\bigwedge^*_{K_*}(V)$ inherits the structure of a Hopf algebra object of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$. In fact, there is a canonical Hopf algebra isomorphism $\bigwedge^*_{K_*}(V)^\vee \simeq \bigwedge^*_{K_*}(V')$, where $V' = \mathrm{Hom}_\kappa(V, K_{-2})$. This isomorphism is uniquely determined by the requirement that the composite map

$$\begin{aligned} V \otimes_\kappa V' &\to \bigwedge\nolimits^*_{K_*}(V) \otimes_{K_*} \bigwedge\nolimits^*_{K_*}(V') \\ &\simeq \bigwedge\nolimits^*_{K_*}(V) \otimes_{K_*} \bigwedge\nolimits^*_{K_*}(V)^\vee \\ &\to K_* \end{aligned}$$

coincides with the tautological pairing $V \otimes_\kappa V' \to K_{-2}$.

Remark 5.2.3. Let t be a nonzero element of K_2, and let $F : \mathrm{Mod}^{\mathrm{gr}}_{K_*} \to \mathrm{Vect}^{\mathrm{gr}}_\kappa$ be the symmetric monoidal equivalence of Remark 5.1.5. For every vector space V over κ, we have a canonical isomorphism $F(\bigwedge^*_{K_*}(V)) \simeq \bigwedge^*_\kappa$ in the category of $(\mathbf{Z}/2\mathbf{Z})$-graded Hopf algebras over κ.

In the situation of Notation 5.2.1, the bialgebra structure on $\bigwedge^*_{K_*}(V)$ is unique in the following sense:

Proposition 5.2.4. *Let H be a bialgebra object of* $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$. *Suppose that there exists an isomorphism of graded K_*-algebras $H \simeq \bigwedge^*_{K_*}(V)$, for some finite-dimensional vector space V over κ. Then there also exists an isomorphism of bialgebras $H \simeq \bigwedge^*_{K_*}(V)$.*

Remark 5.2.5. In the statement of Proposition 5.2.4, we do not assume *a priori* that the comultiplication on H is (graded) commutative: this is part of the conclusion.

Proof of Proposition 5.2.4. Since V is finite-dimensional, the existence of an algebra isomorphism $\alpha : \bigwedge^*_{K_*}(V) \to H$ guarantees that H finite-dimensional in each degree. Let $H^\vee$ denote the dual of H (in the symmetric monoidal category $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$), so that $H^\vee$ inherits the structure of a graded bialgebra over K_*. Let $\epsilon : H \to K_*$ be the counit map, let I be the kernel of ϵ, and let I/I^2 be the space of indecomposable elements of H. Note that there is a unique graded K_*-algebra homomorphism $\bigwedge^*_{K_*}(V) \to K_*$ (which annihilates each element of V). Consequently, α induces isomorphisms

$$\bigwedge\nolimits^{>0}_{K_*}(V) \simeq I \qquad K_* \otimes_\kappa V \simeq I/I^2,$$

in the category of graded K_*-modules, where we regard V as concentrated in degree (-1). Let $W \subseteq H^\vee$ be the subspace of primitive elements, so that we have a canonical isomorphism $W \simeq K_* \otimes_\kappa V^\vee$. It follows that W is concentrated in odd degrees. For every pair of elements $w, w' \in W$, the supercommutator $ww' + w'w \in H^\vee$ is a primitive element of even degree, and therefore vanishes. It follows that the inclusion $W \hookrightarrow H^\vee$ extends to a graded algebra homomomorphism $\beta : \bigwedge^*_{K_*}(W_{-1}) \to H^\vee$. Since W consists of primitive elements, the map β is a bialgebra homomorphism. Set $W' = \mathrm{Hom}_\kappa(W_{-1}, K_{-2})$. Passing to duals, we obtain a bialgebra homomorphism

$$\beta^\vee : H \to \bigwedge\nolimits^*_{K_*}(W_{-1})^\vee \simeq \bigwedge\nolimits^*_{K_*}(W')$$

(where the second isomorphism is supplied by Remark 5.2.2). By construction, the composite map $\bigwedge^*_{K_*}(V) \xrightarrow{\alpha} H \xrightarrow{\beta^\vee} \bigwedge^*_{K_*}(W')$ induces an isomorphism on indecomposables and is therefore an isomorphism. Since α is an isomorphism, it follows that $\beta^\vee$ is also an isomorphism. □

Definition 5.2.6. Let V be a vector space over κ. We let $\mathcal{M}(V)$ denote the abelian category of $\mathbf{Z}$-graded modules over the exterior algebra $\bigwedge^*_{K_*}(V)$.

Remark 5.2.7. Let V be a vector space over κ. Unwinding the definitions, we see that the datum of an object $M \in \mathcal{M}(V)$ is equivalent to the datum of a pair $(M_*, \{d_v\}_{v\in V})$, where M_* is a graded K_*-module and $\{d_v\}_{v\in V}$ is a collection of K_*-linear maps $d_v : M_* \to M_{*-1}$ satisfying the identities

$$d_v^2 = 0 \qquad d_{v+v'} = d_v + d_{v'} \qquad d_{\lambda v} = \lambda d_v.$$

Remark 5.2.8. Let V be a vector space over κ. The Hopf algebra structure on the exterior algebra $\bigwedge^*_{K_*}(V)$ determines a symmetric monoidal structure on the category $\mathcal{M}(V)$. We will denote the underlying tensor product functor by

$$\otimes_{K_*} : \mathcal{M}(V) \times \mathcal{M}(V) \to \mathcal{M}(V).$$

Concretely, it is described by the formula

$$(M_*, \{d_v\}_{v\in V}) \otimes_{K_*} (M'_*, \{d'_v\}_{v\in V}) = (M_* \otimes_{K_*} M'_*, \{d''_v\}_{v\in V}),$$

where d''_v is given by the graded Leibniz rule $d''_v(x \otimes y) = (d_v(x) \otimes y) + (-1)^i(x \otimes d'_v(y))$ for $x \in M_i$.

Remark 5.2.9. Let V be a vector space over κ. We let $\mathrm{Alg}(\mathcal{M}(V))$ denote the category of associative algebra objects of $\mathcal{M}(V)$ (where $\mathcal{M}(V)$ is equipped with the symmetric monoidal structure of Remark 5.2.8). Concretely, we can identify objects of $\mathrm{Alg}(\mathcal{M}(V))$ with pairs $(A_*, \{d_v\}_{v\in V})$, where A_* is a graded K_*-algebra and $\{d_v\}_{v\in V}$ is a collection of K_*-linear derivations $A_* \to A_{*-1}$ satisfying the identities

$$d_v^2 = 0 \qquad d_{v+v'} = d_v + d_{v'} \qquad d_{\lambda v} = \lambda d_v.$$

5.3 Clifford Algebras

Throughout this section, we let K_* denote a graded ring satisfying the requirements of Notation 5.1.1 and we let $\kappa = K_0$ be the underlying field. For each vector space V over κ, we let $\mathcal{M}(V)$ denote the abelian category of Definition 5.2.6.

Definition 5.3.1. Let V be a finite-dimensional vector space over κ. We will say that an object $M \in \mathcal{M}(V)$ is *atomic* if, as a module over the exterior algebra $\bigwedge^*_{K_*}(V)$, it is freely generated by a single homogeneous element of degree $\dim_\kappa(V)$.

We will say that an algebra $A \in \mathrm{Alg}(\mathcal{M}(V))$ is *atomic* if it is atomic when regarded as an object of $\mathcal{M}(V)$. We let $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$ denote the full subcategory of $\mathrm{Alg}(\mathcal{M}(V))$ spanned by the atomic algebra objects of $\mathcal{M}(V)$.

Our goal in this section is to show that the category $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$ has a very simple structure (Corollary 5.3.6).

Construction 5.3.2 (Clifford Algebras)**.** Let V be a finite-dimensional vector space over κ and let $q : V^\vee \to K_2$ be a quadratic form. We let $\mathrm{Cl}_q(V^\vee)$ denote the quotient of the free K_*-algebra generated by $K_* \otimes V^\vee$, subject to the relations $w^2 = q(w)$ for each $w \in V^\vee$. We regard $\mathrm{Cl}_q(V^\vee)$ as a graded K_*-algebra, where each generator $w \in V^\vee$ is homogeneous of degree 1. We refer to $\mathrm{Cl}_q(V^\vee)$ as the *Clifford algebra of q*.

For each $v \in V$, we let $d_v : \mathrm{Cl}_q(V^\vee)_* \to \mathrm{Cl}_q(V^\vee)_{*-1}$ denote the unique K_*-linear derivation satisfying the identity $d_v(w) = \langle v, w\rangle$, for $v \in V$ and $w \in V^\vee$ (here $\langle v, w\rangle \in \kappa$ denotes the scalar obtained by evaluating w on v). An elementary calculation shows that $d_v^2 = 0$ for each $v \in V$, so we can regard $(\mathrm{Cl}_q(V^\vee), \{d_v\}_{v\in V})$ as an algebra object of the category $\mathcal{M}(V)$.

Remark 5.3.3. Let $q : V^\vee \to K_2$ be as in Construction 5.3.2, and let t be a nonzero element of K_2. Then the symmetric monoidal equivalence $\mathrm{Mod}^{\mathrm{gr}}_{K_*} \to \mathrm{Vect}^{\mathrm{gr}}_\kappa$ of Notation 5.1.3 carries $\mathrm{Cl}_q(V^\vee)$ to the usual $(\mathbf{Z}/2\mathbf{Z})$-graded Clifford algebra of the quadratic form $t^{-1}q : V^\vee \to \kappa$ (see Example 2.8.6).

Remark 5.3.4. Let $q : V^\vee \to K_2$ be as in Construction 5.3.2. The following conditions are equivalent:

- (a) The quadratic form q is nondegenerate: that is, the associated bilinear form $b(x, y) = q(x + y) - q(x) - q(y)$ induces an isomorphism $V^\vee \to K_2 \otimes_\kappa V$.
- (b) The Clifford algebra $\mathrm{Cl}_q(V^\vee)$ is an Azumaya algebra object of the symmetric monoidal category $\mathcal{M}(V)$.

This follows immediately from Remark 5.3.3 and Example 2.8.6.

Proposition 5.3.5. *Let V be a finite-dimensional vector space over κ. Then:*

(1) *For each quadratic form* $q : V^\vee \to \pi_2 K$*, the Clifford algebra* $\mathrm{Cl}_q(V^\vee)$ *is an atomic algebra object of* $\mathcal{M}(V)$*.*

(2) *Let* A *be an atomic algebra object of* $\mathcal{M}(V)$*. Then there exists an isomorphism* $A \simeq \mathrm{Cl}_q(V^\vee)$*, for some quadratic form* $q : V^\vee \to \pi_2 K$*.*

(3) *Let* $q, q' : V^\vee \to \pi_2 K$ *be quadratic forms. Then there exists a morphism from* $\mathrm{Cl}_q(V^\vee)$ *to* $\mathrm{Cl}_{q'}(V^\vee)$ *(as algebra objects of* $\mathcal{M}(V)$*) if and only if* $q = q'$*. If such a morphism exists, then it is unique (and is an isomorphism).*

Corollary 5.3.6. *Let* V *be a finite-dimensional vector space over* κ*. Then the construction* $q \mapsto \mathrm{Cl}_q(V^\vee)$ *induces an equivalence of categories from the set of quadratic forms* $q : V^\vee \to K_2$ *(which we regard as a category having only identity morphisms) to the category* $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$ *of atomic algebras in* $\mathcal{M}(V)$*.*

Proof of Proposition 5.3.5. We first prove (1). Let $q : V^\vee \to K_2$ be a quadratic form. Let $\{v_i\}_{1 \leqslant i \leqslant n}$ be a basis for V and $\{v_i^\vee\}_{1 \leqslant i \leqslant n}$ the dual basis for $V^\vee$. For $1 \leqslant i \leqslant n$, let d_{v_i} denote the associated derivation of $\mathrm{Cl}_q(V^\vee)$ (see Construction 5.3.2). For each $I = \{i_1 < i_2 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$, we define

$$v_I^\vee = v_{i_1}^\vee \cdots v_{i_k}^\vee \in \mathrm{Cl}_q(V^\vee) \qquad d_I = d_{v_{i_1}} d_{v_{i_2}} \cdots d_{v_{i_k}}.$$

Note that the elements $v_I^\vee$ form a basis for $\mathrm{Cl}_q(V^\vee)$ as a module over K_*. A simple calculation shows that $d_I v_{\{1,\ldots,n\}}^\vee = \pm v_{\overline{I}}^\vee$, where $\overline{I} = \{1, \ldots, n\} - I$ denotes the complement of I. It follows that the element $v_{\{1,\ldots,n\}}^\vee \in \mathrm{Cl}_q(V^\vee)_n$ freely generates $\mathrm{Cl}_q(V^\vee)$ as a module over $\bigwedge_{K_*}^*(V)$, so that $\mathrm{Cl}_q(V^\vee)$ is atomic.

We next prove (2). Let $A = (A_*, \{d_v\}_{v \in V})$ be an atomic algebra object of $\mathcal{M}(V)$. We will abuse notation by identifying K_* with its image in A. For each element $w \in V^\vee$, set $w^\perp = \{v \in V : \langle v, w \rangle = 0\}$ and let A^w denote the subalgebra of A consisting of those elements which are annihilated by the derivations $\{d_v\}_{v \in w^\perp}$. Our assumption that A is atomic guarantees that there is a unique element $\overline{w} \in A_1^w$ such that $d_v(\overline{w}) = \langle v, w \rangle \in \kappa$, and that $A^w \simeq K_* \oplus K_* \overline{w}$. In particular, the element $\overline{w}^2 \in A_2^w$ belongs to K_2. The map $(w \in V^\vee) \mapsto (\overline{w}^2 \in K_2)$ determines a quadratic form $q : V^\vee \to K_2$. By construction, there is a unique graded K_*-algebra homomorphism $\rho : \mathrm{Cl}_q(V^\vee) \to A_*$ satisfying $\rho(w) = \overline{w}$ for $w \in V^\vee$. For each $v \in V$, let us abuse notation by using the symbol d_v to also denote the corresponding derivation of $\mathrm{Cl}_q(V^\vee)$, so that the maps $\rho \circ d_v$ and $d_v \circ \rho$ are K_*-linear derivations of $\mathrm{Cl}_q(V^\vee)$ into A_*. By construction, these derivations agree on $V^\vee \subseteq \mathrm{Cl}_q(V^\vee)_1$. Since $V^\vee$ generates $\mathrm{Cl}_q(V^\vee)$ as an algebra over K_*, it follows that $\rho \circ d_v = d_v \circ \rho$ for each $v \in V$: that is, ρ is a morphism of algebra objects of $\mathcal{M}(V)$.

Let $x \in A_n$ freely generate A as a module over $\bigwedge_{K_*}^*(V)$. Let $I = \{1, \ldots, n\}$ and write $\rho(v_I^\vee) = \eta x$ for some $\eta \in \bigwedge_{K_*}^*(V)$. We then have $d_I(\eta x) = \rho(d_I v_I^\vee) = \rho(\pm 1) \neq 0$.

It follows that η is not annihilated by the product $v_1 v_2 \cdots v_n \in \bigwedge^*_{K_*}(V)$ and is therefore invertible, so that ρ is an isomorphism. This completes the proof of (2).

We now prove (3). Let $q, q' : V^\vee \to K_2$ be quadratic forms and let w be an element of $V^\vee$. Note that, when regarded as an element of either $\mathrm{Cl}_q(V^\vee)$ or $\mathrm{Cl}_{q'}(V^\vee)$, w is characterized by property that $d_v(w) = \langle v, w \rangle$ for each $v \in V$. It follows that any morphism $\rho : \mathrm{Cl}_q(V^\vee) \to \mathrm{Cl}_{q'}(V^\vee)$ in $\mathrm{Alg}(\mathcal{M}(V))$ must restrict to the identity on $V^\vee$. It follows immediately that such a morphism ρ is uniquely determined, and can exist only if $q = q'$. □

5.4 The Fiber Functor of an Atomic Algebra

Throughout this section, we let K_* denote a graded ring satisfying the requirements of Notation 5.1.1 and we let $\kappa = K_0$ be the underlying field.

Definition 5.4.1. Let V be a finite-dimensional vector space. We will say that a $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal functor $F : \mathcal{M}(V) \to \mathrm{Mod}^{\mathrm{gr}}_{K_*}$ is a *fiber functor* if it preserves small limits and colimits. The collection of fiber functors $F : \mathcal{M}(V) \to \mathrm{Mod}^{\mathrm{gr}}_{K_*}$ forms a category which we will denote by $\mathcal{F}\mathrm{ib}$.

Our goal in this section is to show that the category of fiber functors $\mathcal{F}\mathrm{ib}$ can be identified with the category $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$ studied in §5.3 (Proposition 5.4.4).

Construction 5.4.2. Let V be a vector space over κ. For each associative algebra object $A = (A_*, \{d_v\}_{v \in V})$ of the category $\mathcal{M}(V)$, we define a functor $\mu_A : \mathcal{M}(V) \to \mathrm{Mod}^{\mathrm{gr}}_{K_*}$ by the formula

$$\mu_A(M)_n = \mathrm{Hom}_{\mathcal{M}(V)}(K_*[n], M \otimes_{K_*} A).$$

Note that if A is an algebra object of $\mathrm{Mod}^{\mathrm{gr}}_{\bigwedge^*_{K_*}(V)}$, then the multiplication on A determines natural maps

$$\mu_A(M) \otimes_{K_*} \mu_A(N) \to \mu_A(M \otimes_{K_*} N)$$

which endow μ_A with the structure of a lax monoidal functor.

Proposition 5.4.3. *Let V be a finite-dimensional vector space over κ and let $A \in \mathrm{Alg}(\mathcal{M}(V))$. The following conditions are equivalent:*

(*a*) *The lax monoidal functor μ_A of Construction 5.4.2 is monoidal.*

(*b*) *The algebra A is atomic (in the sense of Definition 5.3.1).*

Proof of Proposition 5.4.3. Assume first that condition (a) is satisfied. Then there exists an isomorphism $\mu_A(K_*) \simeq K_*$, so that $A \neq 0$. Let $\bigwedge^*_{K_*}(V)^\vee$ denote the K_*-linear dual of $\bigwedge^*_{K_*}(V)$. For any $M \in \mathcal{M}(V)$, we have canonical isomorphisms

$$\begin{aligned}
\mu_A(\bigwedge\nolimits^*_{K_*}(V)^\vee \otimes_{K_*} M)_n &\simeq \operatorname{Hom}_{\mathcal{M}(V)}(K_*[n], \bigwedge\nolimits^*_{K_*}(V)^\vee \otimes_{K_*} M \otimes_{K_*} A) \\
&\simeq \operatorname{Hom}_{\mathcal{M}(V)}(\Lambda^*_{K_*}(V)[n], M \otimes_{K_*} A) \\
&\simeq (M \otimes_{K_*} A)_n.
\end{aligned}$$

Taking $M = K_*$, we obtain a canonical isomorphism $\mu_A(\Lambda^*_{K_*}(V)^\vee) \simeq A$. Since the functor μ_A is monoidal, we have A-linear isomorphisms

$$\begin{aligned}
M \otimes_{K_*} A &\simeq \mu_A(\bigwedge\nolimits^*_{K_*}(V)^\vee \otimes_{K_*} M) \\
&\simeq \mu_A(\bigwedge\nolimits^*_{K_*}(V)^\vee) \otimes_{K_*} \mu_A(M) \\
&\simeq A \otimes_{K_*} \mu_A(M).
\end{aligned}$$

In particular, if M is finite-dimensional in each degree, then $A \otimes_{K_*} \mu_A(M)$ is finitely generated as an A-module, so that $\mu_A(M)$ is also finite-dimensional in each degree. Applying this observation in the special case $M = \bigwedge^*_{K_*}(V)^\vee$, we conclude that A is finite-dimensional in each degree.

If M is a graded K_*-module, we define

$$\dim_{K_*}(M) = \dim_\kappa(M_0) + \dim_\kappa(M_1) \in \mathbf{Z}_{\geqslant 0} \cup \{\infty\}.$$

For $M \in \mathcal{M}(V)$, the preceding calculation gives

$$\dim_{K_*}(M) \dim_{K_*}(A) = \dim_{K_*}(A) \dim_{K_*}(\mu_A(M)).$$

Since we have shown above that $0 < \dim_{K_*}(A) < \infty$, we can divide by $\dim_{K_*}(A)$ to obtain $\dim_{K_*}(M) = \dim_{K_*}(\mu_A(M))$.

We now show that the functor μ_A is exact. For every exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in the abelian category $\mathcal{M}(V)$, we evidently have an exact sequence of graded K_*-modules

$$0 \to \mu_A(M') \to \mu_A(M) \xrightarrow{u} \mu_A(M'');$$

we wish to show that u is surjective. Using a direct limit argument, we can reduce to the case where $\dim_{K_*}(M) < \infty$. In this case, the equality $\dim_{K_*}(M) = \dim_{K_*}(M') + \dim_{K_*}(M'')$ guarantees that $\dim_{K_*}(\mu_A(M)) = \dim_{K_*}(\mu_A(M')) + \dim_{K_*}(\mu_A(M''))$, so that the map u is surjective as desired.

Let $A^\vee$ denote the K_*-linear dual of A, which we regard as an object of $\mathcal{M}(V)$. Since $\dim_{K_*}(A) < \infty$, we have canonical isomorphisms $\mu_A(M)_n \simeq \mathrm{Hom}_{\mathcal{M}(V)}(A^\vee[n], M)$. Consequently, the exactness of the functor μ_A guarantees that $A^\vee$ is projective when regarded as a module over $\bigwedge^*_{K_*}(V)$. Note that every projective $\bigwedge^*_{K_*}(V)$-module is free, and the existence of an isomorphism $\mu_A(K_*) \simeq K_*$ guarantees that $A^\vee$ is freely generated by a homogeneous element of degree 0. It follows that $A \simeq \bigwedge^*_{K_*}(V)^\vee$ is freely generated by a homogeneous element of degree $n = \dim_\kappa(V)$. This completes the proof that $(a) \Rightarrow (b)$.

Now suppose that (b) is satisfied; we will prove (a). Note that (b) guarantees that A is dualizable as a graded K_*-module, and that the dual $A^\vee$ is isomorphic to $\bigwedge^*_{K_*}(V)$ (as a $\bigwedge^*_{K_*}(V)$-module). It follows that μ_A is isomorphic to the forgetful functor $\mathcal{M}(V) \to \mathrm{Mod}^{\mathrm{gr}}_{K_*}$ *as a functor* (but not necessarily as a lax monoidal functor). We first claim that μ_A preserves unit objects: that is, the canonical map $u : K_* \to \mu_A(K_*)$ is an isomorphism. Since $\mu_A(K_*)$ is abstractly isomorphic to K_*, it will suffice to show that u is nonzero. This is clear, since u is the unit map for an algebra structure on $\mu_A(K_*) \neq 0$.

To complete the proof that μ_A is monoidal, it will suffice to show that for every pair of objects $M, N \in \mathcal{M}(V)$, the canonical map

$$\theta_{M,N} : \mu_A(M) \otimes_{K_*} \mu_A(N) \to \mu_A(M \otimes_{K_*} N).$$

Since the construction $(M, N) \mapsto \theta_{M,N}$ commutes with filtered colimits, we may assume without loss of generality that $\dim_{K_*}(M), \dim_{K_*}(N) < \infty$. Because the construction $(M, N) \mapsto \theta_{M,N}$ is exact in each variable, we may further reduce to the case where M and N are *simple* modules over $\bigwedge^*_{K_*}(V)$. Moreover, the functor μ_A commutes with shifts, so we can reduce to the case $M \simeq N \simeq K_*$. In this case, we wish to show that the multiplication map $\mu_A(K_*) \otimes_{K_*} \mu_A(K_*) \to \mu_A(K_*)$ is an isomorphis. This follows from the fact that the unit map $K_* \to \mu_A(K_*)$ is an isomorphism. □

Proposition 5.4.4. *Let V be a finite-dimensional vector space over κ. Then the construction $A \mapsto \mu_A$ induces an equivalence of categories* $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V)) \to \mathcal{F}\mathrm{ib}$.

Proof. Let $F : \mathcal{M}(V) \to \mathrm{Mod}^{\mathrm{gr}}_{K_*}$ be an object of $\mathcal{F}\mathrm{ib}$. Since F is a $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear functor which preserves small limits, it is given by the formula $F(M)_n = \mathrm{Hom}_{\mathcal{M}(V)}(C[n], M)$ for some object $C \in \mathcal{M}(V)$. The lax monoidal structure on F exhibits C as a coalgebra object of $\mathcal{M}(V)$. Moreover, since F preserves colimits, the coalgebra C is a finitely generated projective module over $\bigwedge^*_{K_*}(V)$. Let $A = C^\vee$ be the K_*-linear dual of C, which we regard as an algebra object of $\mathcal{M}(V)$. Then $F \simeq \mu_A$, so our assumption that F is monoidal guarantees that A is atomic (Proposition 5.4.3). Unwinding the definitions, we see that the construction $F \mapsto A$ determines a homotopy inverse to the functor $\mu : \mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V)) \to \mathcal{F}\mathrm{ib}$. □

5.5 Extensions in $\mathcal{M}(V)$

Throughout this section, we let K_* denote a graded ring satisfying the requirements of Notation 5.1.1 and we let $\kappa = K_0$ be the underlying field. Our goal in this section is to compute some Ext-groups in the abelian category $\mathcal{M}(V)$, where V is a finite-dimensional vector space over κ.

Construction 5.5.1. Let V be a vector space over κ. For each element $w \in V^\vee$, let M_w denote the direct sum $K_*[-1] \oplus K_*$, whose elements we identify with pairs (x, y) for $x, y \in K_*$. We regard M_w as a $\bigwedge^*_{K_*}(V)$-module, where an element $v \in V$ acts on M_w via the formula $v(x, y) = \langle v, w \rangle (y, 0)$. We have an evident exact sequence

$$0 \to K_*[-1] \to M_w \to K_* \to 0$$

in the abelian category $\mathcal{M}(V)$, which determines an extension class

$$\gamma(w) \in \mathrm{Ext}^1_{\mathcal{M}(V)}(K_*, K_*[-1]).$$

Our next goal is to show that Construction 5.5.1 induces a vector space isomorphism

$$\gamma : V^\vee \to \mathrm{Ext}^1_{\mathcal{M}(V)}(K_*, K_*[-1]).$$

For later reference, it will be convenient to formulate a stronger version of this result.

Notation 5.5.2. Let V be a vector space over κ. For every pair of integers $i, j \in \mathbf{Z}$, we let $\mathcal{H}^{i,j}(V)$ denote the group $\mathrm{Ext}^i_{\mathcal{M}(V)}(K_*, K_*[j])$ (by convention, these groups vanish for $i < 0$). Note that the canonical isomorphisms $K_*[j] \otimes_{K_*} K_*[j']$ determine a multiplication map

$$\begin{aligned}
\mathcal{H}^{i,j}(V) \times \mathcal{H}^{i',j'}(V) &= \mathrm{Ext}^i_{\mathcal{M}(V)}(K_*, K_*[j]) \times \mathrm{Ext}^{i'}_{\mathcal{M}(V)}(K_*, K_*[j']) \\
&\to \mathrm{Ext}^{i+i'}_{\mathcal{M}(V)}(K_*, K_*[j+j']) \\
&= \mathcal{H}^{i+i',j+j'}(V)
\end{aligned}$$

which endows $\mathcal{H}^{*,*}(V)$ with the structure of a bigraded ring; moreover, it satisfies the bigraded commutative law

$$xy = (-1)^{ii'+jj'} yx \text{ for } x \in \mathcal{H}^{i,j}(V) \text{ and } y \in \mathcal{H}^{i',j'}(V)$$

Note that we have a canonical isomorphism of graded rings $\mathcal{H}^{0,*}(V) \simeq K_{-*}$.

Construction 5.5.3 (The Koszul Complex)**.** Let V be a finite-dimensional vector space over κ. For every pair of integers m and n, we let $\mathcal{K}^m_n(V)$ denote the graded vector space given by $\mathrm{Sym}^m(V^\vee) \otimes_\kappa \bigwedge^{n-m}(V^\vee)$. We regard the sum $\bigoplus_{m,n} \mathcal{K}^m_n(V)$ as a commutative differential (bi)graded algebra, whose differential d of bidegree $(1, 0)$ is given on generators by the identity map

$$\mathcal{K}^0_1(V) = \bigwedge^1(V^\vee) \xrightarrow{\mathrm{id}} \mathrm{Sym}^1(V^\vee) = \mathcal{K}^1_1(V).$$

We regard the pair $(\mathcal{K}^*_*, d)$ as a cochain complex

$$\mathcal{K}^0_*(V) \xrightarrow{d} \mathcal{K}^1_*(V) \xrightarrow{d} \mathcal{K}^2_*(V) \to \cdots$$

in the category of $\mathbf{Z}$-graded vector spaces over κ.

Lemma 5.5.4. *Let V be a finite-dimensional vector space over κ. Then the unit map $\kappa \to \mathcal{K}^*_*(V)$ is a quasi-isomorphism. In other words, the Koszul complex*

$$\mathcal{K}^0_*(V) \xrightarrow{d} \mathcal{K}^1_*(V) \xrightarrow{d} \mathcal{K}^2_*(V) \to \cdots$$

is an acyclic resolution of $\kappa \subseteq \mathcal{K}^0_*(V)$.

Proof. Decomposing the Koszul complex $\mathcal{K}^*_*(V)$ as a tensor product, we can further reduce to the case where V is one-dimensional, so that the exterior algebra $\bigwedge^*(V^\vee)$ can be identified with the algebra $A = \kappa[\epsilon]/(\epsilon^2)$. In this case, the Koszul complex $\mathcal{K}^*_*(V)$ can be identified with the acyclic chain complex $A \xrightarrow{\epsilon} A \xrightarrow{\epsilon} A \xrightarrow{\epsilon} \cdots$. □

Remark 5.5.5. Let V be a finite-dimensional vector space over κ. For each $n \geqslant 0$, we can identify the nth term of the Koszul complex $\mathcal{K}^n_*(V)$ with the graded vector space $\mathrm{Hom}_\kappa(\bigwedge^*(V), \mathrm{Sym}^n(V^\vee)[n])$, where $\bigwedge^*(V)$ is equipped with the grading where each element of V is homogeneous of degree (-1). In particular, each $\mathcal{K}^n_*(V)$ can be regarded as a graded module over $\bigwedge^*(V)$, and the differentials of the Koszul complex are $\bigwedge^*(V)$-linear.

Proposition 5.5.6. *Let V be a finite-dimensional vector space over κ. Then the map $\gamma : V^\vee \to \mathrm{Ext}^1_{\mathcal{M}(V)}(K_*, K_*[-1])$ of Construction 5.5.1 extends to a collection of maps*

$$\gamma_{i,j} : K_{-i-j} \otimes_\kappa \mathrm{Sym}^i(V^\vee) \to \mathrm{Ext}^i_{\mathcal{M}(V)}(K_*, K_*[j])$$

which induce an isomorphism of bigraded rings $K_* \otimes_\kappa \mathrm{Sym}^*(V^\vee) \to \mathrm{H}^{*,*}(V)$.

Proof. It follows from Lemma 5.5.4 (and Remark 5.5.5) that we can regard the chain complex

$$K_* \otimes_\kappa \mathcal{K}^0_*(V) \to K_* \otimes_\kappa \mathcal{K}^1_*(V) \to \cdots$$

as an acyclic resolution of K_* in the abelian category $\mathcal{M}(V)$. Moreover, for each object $M \in \mathcal{M}(V)$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{M}(V)}(M, K_* \otimes_\kappa \mathcal{K}^n_*(V)) \simeq \mathrm{Hom}_{\mathrm{Mod}^{\mathrm{gr}}_{K_*}}(M, K_* \otimes_\kappa \mathrm{Sym}^n(V^\vee)[n]).$$

It follows that each $K_* \otimes_\kappa \mathcal{K}^n_*(V)$ is an injective object of the abelian category $\mathcal{M}(V)$, so that the Ext-groups $\mathrm{Ext}^*_{\mathcal{M}(V)}(K_*, K_*[j])$ can be described as the cohomology of a cochain complex whose ith term is given by

$$\begin{aligned}\mathrm{Hom}_{\mathcal{M}(V)}(K_*, K_*[j] \otimes_\kappa \mathcal{K}^i_*(V)) &\simeq \mathrm{Hom}_{\mathrm{Mod}^{\mathrm{gr}}_{K_*}}(K_*[-j], K_* \otimes_\kappa \mathrm{Sym}^i(V^\vee)[i]) \\ &\simeq K_{-i-j} \otimes_\kappa \mathrm{Sym}^i(V^\vee).\end{aligned}$$

The differentials in this chain complex are trivial (which follows either by inspection or by considerations of degree), so we obtain isomorphisms $\gamma_{i,j} : K_{-i-j} \otimes_\kappa \mathrm{Sym}^i(V^\vee) \to \mathrm{Ext}^i_{\mathcal{M}(V)}(K_*, K_*[j])$. Since the unit map $K_* \to K^*_*(V)$ is a quasi-isomorphism of differential graded algebras, the isomorphisms $\gamma_{i,j}$ are multiplicative: that is, they give rise to an isomorphism of bigraded rings $K_* \otimes_\kappa \mathrm{Sym}^*(V^\vee) \simeq \mathcal{H}^{*,*}(V)$. We leave it to the reader to verify that the map $\gamma_{1,-1}$ agrees with the description given in Construction 5.5.1. □

5.6 Automorphisms of $\mathcal{M}(V)$

Throughout this section, we let K_* denote a graded ring satisfying the requirements of Notation 5.1.1 and we let $\kappa = K_0$ be the underlying field. Let V be a finite-dimensional vector space over κ. Our goal in this section is to analyze the automorphism group of $\mathcal{M}(V)$ as a $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal category. Our first step is to filter out those automorphisms which arise from automorphisms of the vector space V itself.

Definition 5.6.1. Let V be a vector space over κ and let $F : \mathcal{M}(V) \to \mathcal{M}(V)$ be a $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal functor which is exact. Then F induces a κ-linear map

$$DF : V^\vee \xrightarrow{\gamma} \mathrm{Ext}^1_{\mathcal{M}(V)}(K_*, K_*[-1]) \xrightarrow{F} \mathrm{Ext}^1_{\mathcal{M}(V)}(K_*, K_*[-1]) \xrightarrow{\gamma^{-1}} V^\vee,$$

where γ is the isomorphism of Construction 5.5.1. We will say that F is *normalized* if DF is the identity map.

We now refine Construction 5.4.2 to produce some examples of normalized functors.

Construction 5.6.2. Let V be a finite-dimensional vector space over κ and let $A = (A_*, \{d_v\}_{v \in V})$ be an atomic algebra object of $\mathcal{M}(V)$. For each $M \in \mathcal{M}(V)$, postcomposition with the map

$$-(\mathrm{id}_M \otimes_{K_*} d_v) : M \otimes_{K_*} A \to M \otimes_{K_*} A[1]$$

determines a map $\partial_v : \mu_A(M) \to \mu_A(M)[1]$ in $\mathcal{M}(V)$. We will regard the pair $(\mu_A(M), \{\partial_v\}_{v \in V})$ as an object of the category $\mathcal{M}(V)$, which we will denote by $\mu_A^\partial(M)$. It follows from Proposition 5.4.3 (and its proof) that the construction $M \mapsto \mu_A^\partial(M)$ determines a $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear monoidal exact functor from the category $\mathcal{M}(V)$ to itself.

Example 5.6.3. Let V be a finite-dimensional vector space over κ and suppose we are given a pair of quadratic forms $q, q' : V^\vee \to K_2$. Set $A = \mathrm{Cl}_q(V^\vee)$ and $A' = \mathrm{Cl}_{q'}(V^\vee)$. Then the tensor product $A \otimes_{K_*} A'$ can be identified with $\mathrm{Cl}_{q \oplus q'}(V^\vee \oplus V^\vee)$, where $q \oplus q' : V^\vee \oplus V^\vee \to K_2$ denotes the quadratic form given by the formula $(q \oplus q')(w \oplus w') = q(w) + q'(w')$. The antidiagonal embedding $(-\mathrm{id} \oplus \mathrm{id}) : V^\vee \to V^\vee \oplus V^\vee$ induces an isomorphism from $\mathrm{Cl}_{q+q'}(V^\vee)$ to $\mu_{A'}^\partial(A) \subseteq A \otimes_{K_*} A'$.

Proposition 5.6.4. *Let V be a finite-dimensional vector space over κ and let A be an atomic algebra object of $\mathcal{M}(V)$. Then the functor $\mu_A^\partial : \mathcal{M}(V) \to \mathcal{M}(V)$ of Construction 5.6.2 is normalized (in the sense of Definition 5.6.1).*

Proof. Write $A = (A_*, \{d_v\}_{v \in V})$. For $w \in V^\vee$, let M_w be as in Construction 5.5.1. Unwinding the definitions, we see that $\mu_A(M_w)$ can be identified with the collection of those elements $(x, y) \in A[-1] \oplus A$ satisfying

$$= d_v x + \langle v, w \rangle y = 0 \qquad d_v y = 0$$

for all $v \in V$ and $y \in A_i$. Using the exactness of the sequence

$$0 \to K_*[-1] \to \mu_A(M_w) \to K_* \to 0,$$

we see that $\mu_A(M_w)_1$ contains a unique element of the form $(x, 1)$; here x is an element of A_1 satisfying $d_v x = \langle v, w \rangle$ for $v \in V$. We then compute $\partial_v(x, 1) = -(-d_v(x), d_v(1)) = (\langle v, w \rangle, 0)$, so that $\mu_A(M_w)$ is isomorphic to M_w as an extension of K_* by $K_*[-1]$. □

Proposition 5.6.5. *Let V be a finite-dimensional vector space over the field κ, and let $\mathrm{Aut}_{\mathrm{nm}}^{\otimes}(\mathcal{M}(V))$ denote the category of normalized $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear monoidal equivalences of $\mathcal{M}(V)$ with itself. Then the functor $A \mapsto \mu_A^\partial$ of Construction 5.6.2 determines an equivalence of categories*

$$\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V)) \to \mathrm{Aut}_{\mathrm{nm}}^{\otimes}(\mathcal{M}(V)).$$

Proof. We first note that if $A \in \mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$, then A is isomorphic to $\bigwedge_{K_*}^*(V)^\vee$ as an object of $\mathcal{M}(V)$ (see the proof of Proposition 5.4.3), so that μ_A^∂ is isomorphic to the identity as a functor from the category $\mathcal{M}(V)$ to itself (though not necessarily as a *monoidal* functor). In particular, μ_A^∂ is an equivalence of categories. We have already seen that μ_A^∂ is a normalized $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear monoidal functor (Proposition

5.6.4), so the construction $A \mapsto \mu^{\partial}_A$ carries $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$ into $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$. Let $U : \mathcal{M}(V) \to \mathrm{Mod}^{\mathrm{gr}}_{K_*}$ be the forgetful functor and let $\mathcal{F}\mathrm{ib}$ be as in Definition 5.4.1. Then composition with U determines a functor $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V)) \to \mathcal{F}\mathrm{ib}$. It follows from Proposition 5.4.4 that the composition

$$\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V)) \to \mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V)) \to \mathcal{F}\mathrm{ib}$$

is an equivalence of categories. Note that the category $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$ is a groupoid (Corollary 5.3.6), so that the category $\mathcal{F}\mathrm{ib}$ is also a groupoid. We will complete the proof by showing that the forgetful functor $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V)) \to \mathcal{F}\mathrm{ib}$ is fully faithful.

Fix a pair of normalized $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal equivalences $F, F' : \mathcal{M}(V) \to \mathcal{M}(V)$ and let $\alpha : U \circ F' \to U \circ F$ be a $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal functor; we wish to show that α can be promoted uniquely to a $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal functor $\overline{\alpha} : F' \to F$. The uniqueness of $\overline{\alpha}$ is immediate. To prove existence, we can replace F by $F \circ F'^{-1}$ and thereby reduce to the case where $F' = \mathrm{id}$. Since the morphism α is an isomorphism (because $\mathcal{F}\mathrm{ib}$ is a groupoid), we can reduce to the case $U \circ F = U$ and $\alpha = \mathrm{id}$. In this case, the functor F is given by restriction of scalars along some map $f : \bigwedge^*_{K_*}(V) \to \bigwedge^*_{K_*}(V)$ of graded K_*-algebras. Because the functor F is monoidal, the map f is a Hopf algebra homomorphism, and therefore obtained by applying the functor $\bigwedge^*_{K_*}(\bullet)$ to some κ-linear map $f_0 : V \to V$. Our assumption that F is normalized guarantees that $f_0 = \mathrm{id}_V$, so that F is the identity functor as desired. □

Remark 5.6.6. Let V be a finite-dimensional vector space over κ and let $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$ be defined as in Proposition 5.6.5. Then $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$ is a monoidal category (with monoidal structure given by composition of functors). The construction

$$(F \in \mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V)), A \in \mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))) \mapsto (F(A) \in \mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V)))$$

determines an action of $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$ on the category $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$, and the equivalence $\mu^{\partial}_{\bullet} : \mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V)) \to \mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$ of Proposition 5.6.5 is $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$-equivariant. It follows that the action of $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$ on $\mathrm{Alg}^{\mathrm{atm}}(\mathcal{M}(V))$ is simply transitive.

Proposition 5.6.7. *Let V be a finite-dimensional vector space over κ, let* QF *denote the set of quadratic forms $q : V^\vee \to K_2$, and let $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$ be as in Proposition 5.6.5. Then the construction $q \mapsto \mu^{\partial}_{\mathrm{Cl}_q(V^\vee)}$ induces an equivalence of monoidal categories $Q \to \mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$. Here we regard the set* QF *as a category with only identity morphisms, with the monoidal structure given by addition of quadratic forms.*

Proof. It follows from Propositions 5.4.3 and 5.3.5 that the construction $q \mapsto \mu^{\partial}_{\mathrm{Cl}_q(V^\vee)}$ is an equivalence of categories. To show that this equivalence is monoidal, it will suffice to show that for every pair of quadratic forms $q, q' : V^\vee \to K_2$, the functors

$\mu^{\partial}_{\mathrm{Cl}_q(V^\vee)} \circ \mu^{\partial}_{\mathrm{Cl}_{q'}(V^\vee)}$ and $\mu^{\partial}_{\mathrm{Cl}_{q+q'}(V^\vee)}$ are isomorphic (as objects of $\mathrm{Aut}^{\otimes}_{\mathrm{nm}}(\mathcal{M}(V))$). Choose an isomorphism $\mu^{\partial}_{\mathrm{Cl}_q(V^\vee)} \circ \mu^{\partial}_{\mathrm{Cl}_{q'}(V^\vee)} \simeq \mu^{\partial}_{\mathrm{Cl}_{q''}(V^\vee)}$ for some other quadratic form $q'' : V^\vee \to K_2$; we wish to show that $q'' = q + q'$. To prove this, we invoke Example 5.6.3 to compute

$$\begin{aligned}
\mathrm{Cl}_{q''}(V^\vee) &\simeq \mu^{\partial}_{\mathrm{Cl}_{q''}(V^\vee)}(\mathrm{Cl}_0(V^\vee)) \\
&\simeq \mu^{\partial}_{\mathrm{Cl}_q(V^\vee)}(\mu^{\partial}_{\mathrm{Cl}_{q'}(V^\vee)}(\mathrm{Cl}_0(V^\vee))) \\
&\simeq \mu^{\partial}_{\mathrm{Cl}_q(V^\vee)}(\mathrm{Cl}_{q'}(V^\vee)) \\
&\simeq \mathrm{Cl}_{q+q'}(V^\vee),
\end{aligned}$$

so that $q'' = q + q'$ by virtue of Proposition 5.3.5. □

Remark 5.6.8. We can extend Proposition 5.6.7 to describe the category of *all* $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal equivalences of $\mathcal{M}(V)$ with itself: it is equivalent to the semidirect product $Q \rtimes \mathrm{Aut}_\kappa(V)$, regarded as a (monoidal) category having only identity morphisms.

Proposition 5.6.9. *Let V be a finite-dimensional vector space and let A be an atomic algebra object of $\mathcal{M}(V)$. Then the construction $M \mapsto A \otimes_{K_*} M$ induces an equivalence of categories $\mathrm{Mod}^{\mathrm{gr}}_{K_*} \to \mathrm{LMod}_A(\mathcal{M}(V))$.*

Proof. Let us regard A as a graded K_*-algebra equipped with a collection of derivations $\{d_v\}_{v \in V}$ of degree (-1) (see Remark 5.2.9). Unwinding the definitions, we see that $\mathrm{LMod}_A(\mathcal{M}(V))$ can be identified with the category of graded A^+-modules, where A^+ denotes the graded K_*-algebra generated by A and $\bigwedge^*_{K_*}(V)$ subject to the relations $va + (-1)^m av = d_v(a)$ for $a \in A_m$ and $v \in V$ (note that we have an isomorphism $A^+ \simeq A \otimes_{K_*} \bigwedge^*_{K_*}(V)$ of graded K_*-modules). Combining the left action of A on itself with the action of $\bigwedge^*_{K_*}(V)$ on A, we obtain a map of graded K_*-algebras $\rho : A^+ \to \mathrm{End}_{K_*}(A)$. Moreover, Proposition 5.6.9 is equivalent to the assertion that ρ is an isomorphism (Proposition 2.1.3).

Let $q : V^\vee \to \pi_2 K$ be the quadratic form which is identically zero and let $A' = \mathrm{Cl}_q(V^\vee)$ be as in Construction 5.3.2 (so that A' is an exterior algebra over K_* on the vector space $V^\vee$, placed in degree 1). Then A' is atomic (Proposition 5.3.5). Using Remark 5.6.6, we can choose a $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal equivalence of $\mathcal{M}(V)$ with itself which carries A to A'. We may therefore replace A by A' and thereby reduce to the case where $A = \mathrm{Cl}_q(V^\vee)$. In this case, the map ρ depends only on the vector space V. Moreover, the construction $V \mapsto \rho$ carries direct sums of vector spaces to tensor products of graded algebra homomorphisms. We may therefore reduce to proving that ρ is an isomorphism in the special case where $\dim_\kappa(V) = 1$, which follows by inspection. □

5.7 The Brauer Group of $\mathcal{M}(V)$

Throughout this section, we let K_* denote a graded ring satisfying the requirements of Notation 5.1.1 and we let $\kappa = K_0$ be the underlying field. Our goal in this section is to describe the Brauer group of the symmetric monoidal category $\mathcal{M}(V)$, where V is a finite-dimensional vector space over κ. We begin by treating the case $V = 0$:

Proposition 5.7.1. *The Brauer group* $\mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$ *is isomorphic to the Brauer-Wall group* $\mathrm{BW}(\kappa)$ *of the field* κ*.*

Proof. A choice of nonzero element $t \in K_2$ determines a symmetric monoidal equivalence of categories $\mathrm{Mod}^{\mathrm{gr}}_{K_*} \simeq \mathrm{Vect}^{\mathrm{gr}}_{\kappa}$ (see Remark 5.1.5), hence an isomorphism of Brauer groups $\mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \simeq \mathrm{Br}(\mathrm{Vect}^{\mathrm{gr}}_{\kappa}) = \mathrm{BW}(\kappa)$. □

Warning 5.7.2. The isomorphism $\mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \to \mathrm{BW}(\kappa)$ of Proposition 5.7.1 is not canonical: in general, it depends on the choice of nonzero element $t \in K_2$. Put another way, the canonical isomorphism

$$\mathrm{BW}(\kappa) \simeq \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{\kappa[t^{\pm 1}]})$$

determines an action of the multiplicative group $\kappa^{\times}$ on BW(κ (since $\kappa^{\times}$ acts on the graded ring $\kappa[t^{\pm 1}]$ by rescaling the generator t), and this action is generally nontrivial.

To extend the calculation of $\mathrm{Br}(\mathcal{M}(V))$ to the case where V is nontrivial, we will need the following algebraic fact.

Proposition 5.7.3. *Let A be an Azumaya algebra object of* $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ *and let $d : A[-1] \to A$ be a derivation of degree* (-1)*. Then there exists a unique element $a \in A_{-1}$ such that $dx = ax + (-1)^{n+1} xa$ for all $x \in A_n$.*

Proof. Let M denote the direct sum $A \oplus A[-1]$, which we regard as a left A-module object of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$. We endow M with the structure of a right A-module via the formula

$$(x, y)a = (xa + y(da), (-1)^i ya)$$

for $a \in A_i$. We have an evident exact sequence

$$0 \to A \to M \xrightarrow{u} A[-1] \to 0$$

in the category ${}_A\mathrm{BMod}_A(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$ of A-A bimodules. Let $s : A[-1] \to M$ be left A-module map satisfying $u \circ s = \mathrm{id}_{A[-1]}$. Then we can write $s = (s_0, \mathrm{id})$ for some left A-module map $s_0 : A[-1] \to A$, which we can identify with an element $a \in A_{-1}$. In this case, the map s is given concretely by the formula $s(y) = (-ya, y)$ for $y \in A_j$. Unwinding the definitions, we see that s is a map of A-A bimodules if and only if a satisfies the identity $dx = ax + (-1)^{n+1}xa$ for all $x \in A_n$. Consequently, Proposition 5.7.3 is equivalent to assertion that u admits a unique section (in the category ${}_A\mathrm{BMod}_A(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$)).

Let $F : \mathrm{Mod}^{\mathrm{gr}}_{K_*} \to {}_A\mathrm{BMod}_A(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$ be the functor given by $F(M) = A \otimes_{K_*} M$. Since A is an Azumaya algebra object of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$, the functor F is an equivalence of categories. We are therefore reduced to showing that the exact sequence

$$0 \to K_* \to F^{-1}M \xrightarrow{F^{-1}(u)} K_*[-1] \to 0$$

splits uniquely in the abelian category $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$. The existence of the splitting is now obvious (the category $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ is semisimple), and the uniqueness follows from the observation that there are no nonzero maps from $K_*[-1]$ to K_*. □

Let V be a vector space over κ and let $\overline{A}$ be an algebra object of $\mathcal{M}(V)$, which we write as $\overline{A} = (A, \{d_v\}_{v \in V})$ (see Remark 5.2.9). Using Corollary 2.2.3, we see that $\overline{A}$ is an Azumaya algebra object of $\mathcal{M}(V)$ if and only if A is an Azumaya algebra object of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$.

Proposition 5.7.4. *Let V be a vector space over κ and let $\overline{A} = (A, \{d_v\}_{v \in V})$ be an Azumaya algebra object of $\mathcal{M}(V)$. Then:*

- (a) *For each element $v \in V$, there exists a unique element $a_v \in A_{-1}$ such that $d_v(x) = a_v x + (-1)^{n+1} x a_v$ for all $x \in A_n$.*
- (b) *The construction $v \mapsto a_v$ determines a κ-linear map $V \to A_{-1}$.*
- (c) *The construction $v \mapsto a_v^2$ determines a quadratic form $q_{\overline{A}} : V \to K_{-2}$.*

Proof. Assertion (a) follows from Proposition 5.7.3, and assertion (b) is immediate from the uniqueness of a_v. To prove (c), we note that the equation $d_v^2 = 0$ guarantees that $q_A(v) = a_v^2$ belongs to the center of A. Since A is an Azumaya algebra object of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$, the even part of the center of A coincides with K_*. □

Remark 5.7.5. Proposition 5.7.4 admits a converse. Suppose that A is an Azumaya algebra object of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$, and that we are given a κ-linear map $\rho : V \to A_{-1}$ satisfying $\rho(v)^2 \in K_{-2} \subseteq A_{-2}$ for all $v \in V$. Then we can promote A to an Azumaya algebra object $\overline{A} = (A, \{d_v\}_{v \in V})$ of $\mathcal{M}(V)$ by defining $d_v(x) = a_v x + (-1)^{n+1} x a_v$ for $x \in A_n$.

Theorem 5.7.6. *Let* QF *denote the set of quadratic forms $q : V \to K_{-2}$. Then the construction $[\overline{A}] = [(A, \{d_v\}_{v \in V})] \mapsto ([A], q_{\overline{A}})$ induces an isomorphism of abelian groups $\phi : \mathrm{Br}(\mathcal{M}(V)) \simeq \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \times \mathrm{QF}$.*

Proof. We first show that ϕ is a well-defined group homomorphism. Note that the construction $[\overline{A}] \mapsto [A]$ determines a group homomorphism $\mathrm{Br}(\mathcal{M}(V)) \to \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$ (this a special case of Proposition 2.4.1). It will therefore suffice to show that the map $[\overline{A}] \mapsto q_{\overline{A}}$ determines a group homomorphism $\mathrm{Br}(\mathcal{M}(V)) \to \mathrm{QF}$. The proof proceeds in several steps:

(*i*) Let $\overline{M} = (M, \{d_v\}_{v \in V})$ be a nonzero dualizable object of $\mathcal{M}(V)$. Let $\mathrm{End}(M)$ denote the endomorphism ring of M as a graded K_*-module, so that each d_v can be identified with an element of $\mathrm{End}(M)_{-1}$. Then we can write $\mathrm{End}(\overline{M}) = (\mathrm{End}(M), \{D_v\}_{v \in V})$, where each D_v is the derivation of $\mathrm{End}(M)$ given by $D_v(f) = d_v \circ f + (-1)^{n+1} f \circ d_v$ for $f \in \mathrm{End}(M)_n$. The equation $d_v^2 = 0$ then shows that the quadratic form $q_{\mathrm{End}(\overline{M})} = 0$.

(*ii*) Let $\overline{A} = (A, \{d_v\}_{v \in V})$ and $\overline{B} = (B, \{d'_v\}_{v \in V})$ be Azumaya algebra objects of $\mathcal{M}(V)$ with associated quadratic forms $q_{\overline{A}}, q_{\overline{B}} : V \to K_{-2}$. For each $v \in V$, let $a_v \in A_{-1}$ be as the statement of Proposition 5.7.4, and define $b_v \in B_{-1}$ similarly. Unwinding the definitions, we can write $\overline{A} \otimes \overline{B} = (A \otimes_{K_*} B, \{d''_v\}_{v \in V})$, where $d''_v(x) = (a_v + b_v)x + (-1)^{n+1} x (a_v + b_v)$ for $x \in (A \otimes_{K_*} B)_n$. Since a_v and b_v anticommute, we obtain $q_{\overline{A} \otimes \overline{B}}(v) = (a_v + b_v)^2 = a_v^2 + b_v^2 = q_{\overline{A}}(v) + q_{\overline{B}}(v)$.

(*iii*) Let $\overline{A}$ be an Azumaya algebra object of $\mathcal{M}(V)$. We then compute

$$0 = q_{\mathrm{End}(\overline{A})} = q_{\overline{A} \otimes \overline{A}^{\mathrm{op}}} = q_{\overline{A}} + q_{\overline{A}^{\mathrm{op}}}$$

so that $q_{\overline{A}^{\mathrm{op}}} = -q_{\overline{A}}$.

(*iv*) Let $\overline{A}$ and $\overline{B}$ be Azumaya algebras in $\mathcal{M}(V)$ satisfying $[\overline{A}] = [\overline{B}]$. We then have $\overline{A} \otimes \overline{B}^{\mathrm{op}} \simeq \mathrm{End}(\overline{M})$ for some nonzero dualizable object $\overline{M} \in \mathcal{M}(V)$. We then have

$$0 = q_{\mathrm{End}(\overline{M})} = q_{\overline{A} \otimes \overline{B}^{\mathrm{op}}} = q_{\overline{A}} + q_{\overline{B}^{\mathrm{op}}} = q_{\overline{A}} - q_{\overline{B}},$$

so that $q_{\overline{A}} = q_{\overline{B}}$. It follows that the construction $[\overline{A}] \mapsto q_{\overline{A}}$ determines a well-defined map of sets $\mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{\Lambda}) \to \mathrm{QF}$.

(*v*) Combining (*iv*) and (*ii*), we deduce that the map $[\overline{A}] \mapsto q_{\overline{A}}$ is a group homomorphism, as desired.

We now complete the proof by showing that the group homomorphism

$$\phi : \mathrm{Br}(\mathcal{M}(V)) \simeq \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \times \mathrm{QF}$$

is an isomorphism of abelian groups. We first show that ϕ is surjective. Observe that every Azumaya algebra $A \in \mathrm{Alg}(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$ can be lifted to an Azumaya algebra

$\overline{A} = (A, \{d_v\}_{v \in V}) \in \mathrm{Alg}(\mathcal{M}(V))$ by setting $d_v = 0$ for each $v \in V$, so that $q_{\overline{A}} = 0$. To complete the proof of surjectivity, it will suffice to show that for every quadratic form $q : V \to K_{-2}$, there exists an Azumaya algebra $\overline{A} = (A, \{d_v\}_{v \in V})$ satisfying $q_{\overline{A}} = q$. For this, we let $\mathrm{Cl}_q(V)$ denote the Clifford algebra of q, regarded as a graded K_*-algebra where each element of V is homogeneous of degree -1. Let $A = \mathrm{End}(\mathrm{Cl}_q(V))$, and define derivations $\{d_v\}_{v \in V}$ by the formulae $(d_v f)(x) = vf(x) + (-1)^{n+1} f(vx)$ for $f \in \mathrm{End}(\mathrm{Cl}_q(V))_n$. Setting $\overline{A} = (A, \{d_v\}_{v \in V})$, a simple calculation gives $q_{\overline{A}} = q$.

We now show that the homomorphism ϕ is injective. Let $\overline{A} = (A, \{d_v\}_{v \in V})$ be an Azumaya algebra object of $\mathcal{M}(V)$ satisfying $\phi([\overline{A}]) = 0$. Then $[A] = 0 \in \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$, so we can write $A = \mathrm{End}(M)$ for some nonzero dualizable object $M \in \mathrm{Mod}^{\mathrm{gr}}_{K_*}$. Let $\{a_v \in A_{-1}\}_{v \in V}$ be as in Proposition 5.7.4, so that each a_v can be identified with a map $M[-1] \to M$. We then have $a_v^2 = q_{\overline{A}}(v) = 0$ for $v \in V$, so that $\overline{M} = (M, \{a_v\}_{v \in V})$ can be viewed as an object of $\mathcal{M}(V)$. Unwinding the definitions, we see $\overline{A}$ can be identified with the endomorphism algebra $\mathrm{End}(\overline{M})$, so that $[\overline{A}]$ vanishes in the Brauer group $\mathrm{Br}(\mathcal{M}(V))$. □

Chapter 6

Milnor Modules

Let E be a Lubin-Tate spectrum. Our goal in this paper is to understand the Brauer group $\mathrm{Br}(E)$. In this section, we introduce an abelian group $\mathrm{BM}(E)$ which we call the *Brauer-Milnor group of* E, and construct a group homomorphism $\mathrm{Br}(E) \to \mathrm{BM}(E)$. Roughly speaking, the abelian group $\mathrm{BM}(E)$ captures the "purely algebraic" part of the Brauer group $\mathrm{Br}(E)$. More precisely, we can describe $\mathrm{BM}(E)$ as the Brauer group the abelian category $\mathrm{Syn}_E^\heartsuit$ of *discrete* synthetic E-modules, which we will refer to as *Milnor modules*. Our main results can be summarized as follows:

(a) The abelian category of Milnor modules $\mathrm{Syn}_E^\heartsuit$ is abstractly equivalent, as a monoidal category, to the category $\mathcal{M}(V)$ introduced in Definition 5.2.6; here $V = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ denotes the Zariski tangent space to the Lubin-Tate ring $\pi_0 E$ (Theorem 6.6.6).

(b) If the residue field κ of E has characteristic $\neq 2$, then there exists a *canonical* equivalence $\mathrm{Syn}_E^\heartsuit \simeq \mathcal{M}(V)$ which is *symmetric* monoidal (Proposition 6.9.1).

(c) If the field κ has characteristic $\neq 2$, then by combining (b) with the calculation of §5.7 we obtain an isomorphism

$$\mathrm{BM}(E) \simeq \mathrm{BW}(\kappa) \times \mathfrak{m}^2/\mathfrak{m}^3$$

(which is not quite canonical: it depends on a choice of nonzero element $t \in (\pi_2 E)/\mathfrak{m}(\pi_2 E)$).

6.1 The Abelian Category $\mathrm{Syn}_E^\heartsuit$

We begin by introducing some terminology.

Definition 6.1.1. Let E be a Lubin-Tate spectrum and let Syn_E denote the ∞-category of synthetic E-modules. A *Milnor module* is a discrete object of Syn_E. We let $\mathrm{Syn}_E^{\heartsuit}$ denote the full subcategory of Syn_E spanned by the Milnor modules.

Remark 6.1.2. The ∞-category $\mathrm{Syn}_E^{\heartsuit}$ is identical to the ∞-category $\mathrm{Syn}_E^{\leqslant 0}$ defined in §4.5.

Remark 6.1.3. Since every object of $\mathrm{Syn}_E^{\heartsuit}$ is discrete, the ∞-category $\mathrm{Syn}_E^{\heartsuit}$ is equivalent to (the nerve of) its homotopy category $\mathrm{hSyn}_{\mathrm{E}}^{\heartsuit}$. Throughout this paper, we will abuse terminology by not distinguishing between the ∞-category $\mathrm{Syn}_E^{\heartsuit}$ and the ordinary category $\mathrm{hSyn}_{\mathrm{E}}^{\heartsuit}$.

Remark 6.1.4. Unwinding the definitions, we see that a Milnor module can be identified with a functor $X : (\mathrm{hMod}_{\mathrm{E}}^{\mathrm{mol}})^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ which preserves finite products.

Remark 6.1.5. According to Remark 4.1.4, the ∞-category Syn_E of synthetic E-modules is a Grothendieck prestable ∞-category. It follows that the category $\mathrm{Syn}_E^{\heartsuit}$ of Milnor modules is a Grothendieck abelian category. In fact, we can be more precise: the category $\mathrm{Syn}_E^{\heartsuit}$ can be identified with the abelian category of left modules over the associative ring $\pi_0 \mathrm{End}_E(M)$, where M is any molecular E-module for which $\pi_0 M \neq 0 \neq \pi_1 M$ (see Remark 4.1.5). The main goal of this section is to obtain a similar identification which is compatible with tensor products (see Theorem 6.6.6 and Proposition 6.9.1).

Variant 6.1.6. Let E be a Lubin-Tate spectrum of height n, let M be an atomic E-module. We will see that the category $\mathrm{Syn}_E^{\heartsuit}$ can be identified with the abelian category of *graded* modules over the graded ring $\pi_* \mathrm{End}_E(M)$ (see Corollary 6.4.13), which can itself be described as the exterior algebra on an n-dimensional vector space V (Proposition 6.5.1). Here we can think of V as having a basis $\{Q^i\}_{0 \leqslant i < n}$ which is dual to to a regular system of parameters $\{v_i\}_{0 \leqslant i < n}$ for the Lubin-Tate ring $\pi_0 E$. The operators $Q^i \in \mathrm{Ext}_E^*(M, M)$ can be viewed as analogues (in the setting of Morava K-theory) of the classical *Milnor operators* in the Steenrod algebra (which act on ordinary cohomology with coefficients in $\mathbf{Z}/p\mathbf{Z}$). The terminology of Definition 6.1.1 is motivated by this analogy.

Notation 6.1.7. Let M be an E-module. We let $\mathrm{Sy}^{\heartsuit}[M]$ denote the truncation $\tau_{\leqslant 0} \mathrm{Sy}[M]$, More concretely, $\mathrm{Sy}^{\heartsuit}[M]$ is the Milnor module given by the formula

$$\mathrm{Sy}^{\heartsuit}[M](N) = \pi_0 \mathrm{Map}_{\mathrm{Mod}_E}(N, M).$$

We regard the construction $M \mapsto \mathrm{Sy}^{\heartsuit}[M]$ as a functor from the ∞-category Mod_E of E-modules to the category $\mathrm{Syn}_E^{\heartsuit}$ of Milnor modules.

Proposition 6.1.8. *The category* $\mathrm{Syn}_E^\heartsuit$ *of Milnor modules admits an essentially unique symmetric monoidal structure for which the truncation functor*

$$\tau_{\leqslant 0} : \mathrm{Syn}_E \to \mathrm{Syn}_E^\heartsuit$$

is symmetric monoidal (where we regard Syn_E *as equipped with the symmetric monoidal structure constructed in §4.4.*

Proof. This is a special case of Definition HA.2.2.1.6 (note that the truncation functor $\tau_{\leqslant 0}$ is compatible with the symmetric monoidal structure on Syn_E). □

Notation 6.1.9. For the remainder of this paper, we will regard the category $\mathrm{Syn}_E^\heartsuit$ as equipped with the symmetric monoidal structure of Proposition 6.1.8. We will denote the tensor product functor on $\mathrm{Syn}_E^\heartsuit$ by

$$\boxtimes : \mathrm{Syn}_E^\heartsuit \times \mathrm{Syn}_E^\heartsuit \to \mathrm{Syn}_E^\heartsuit$$

and we will denote the unit object of $\mathrm{Syn}_E^\heartsuit$ by $\mathbf{1}^\heartsuit$. Concretely, the functor $\boxtimes$ is given by the formula $M \boxtimes N = \tau_{\leqslant 0}(M \wedge N)$.

Note that the symmetric monoidal structure on the truncation functor $\tau_{\leqslant 0}$ determines canonical isomorphisms

$$\tau_{\leqslant 0}(X \wedge Y) \simeq (\tau_{\leqslant 0}X) \boxtimes (\tau_{\leqslant 0}Y)$$

for synthetic E-modules X and Y, and an isomorphism $\mathbf{1}^\heartsuit \simeq \tau_{\leqslant 0}\mathbf{1} = \mathrm{Sy}^\heartsuit[E]$.

Remark 6.1.10. It follows from Variant 4.4.11 that we can regard the functor $\mathrm{Sy}^\heartsuit : \mathrm{Mod}_E \to \mathrm{Syn}_E^\heartsuit$ as a symmetric monoidal functor. Similarly, the restriction of $\mathrm{Sy}^\heartsuit$ to the full subcategory $\mathrm{Mod}_E^{\mathrm{loc}}$ is also symmetric monoidal.

Definition 6.1.11. Let E be a Lubin-Tate spectrum. We let $\mathrm{BM}(E)$ denote the Brauer group of the symmetric monoidal category $\mathrm{Syn}_E^\heartsuit$ of Milnor modules. We will refer to $\mathrm{BM}(E)$ as the *Brauer-Milnor group* of E.

6.2 Atomic and Molecular Milnor Modules

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section.

Definition 6.2.1. Let X be a Milnor module. We will say that X is

- *atomic* if it is isomorphic to $\mathrm{Sy}^\heartsuit[M]$, where $M \in \mathrm{Mod}_E$ is atomic.
- *molecular* if it is isomorphic to $\mathrm{Sy}^\heartsuit[M]$, where $M \in \mathrm{Mod}_E$ is molecular.

- *quasi-molecular* if it is isomorphic to $\mathrm{Sy}^{\heartsuit}[M]$, where $M \in \mathrm{Mod}_E$ is quasi-molecular.

Remark 6.2.2. Let M be an atomic E-module (so that M is uniquely determined up to equivalence). Using the fact that the construction $N \mapsto \mathrm{Sy}^{\heartsuit}[N]$ commutes with coproducts (Proposition 4.2.3), we conclude:

- A Milnor module X is atomic if and only if it is isomorphic to $\mathrm{Sy}^{\heartsuit}[M]$.
- A Milnor module X is molecular and only if it is isomorphic to a direct sum of finitely many objects of the form $\mathrm{Sy}^{\heartsuit}[M]$ and $\mathrm{Sy}^{\heartsuit}[\Sigma M]$.
- A Milnor module X is quasi-molecular and only if it is isomorphic to a direct sum of objects of the form $\mathrm{Sy}^{\heartsuit}[M]$ and $\mathrm{Sy}^{\heartsuit}[\Sigma M]$.

Remark 6.2.3. A Milnor module X is quasi-molecular if and only if it projective as an object of the abelian category $\mathrm{Mod}_E^{\heartsuit}$. Moreover, the abelian category $\mathrm{Mod}_E^{\heartsuit}$ has enough projective objects: that is, every Milnor module X fits into an exact sequence $0 \to X' \to P \to X \to 0$ where P is quasi-molecular.

Remark 6.2.4. Let X be a molecular Milnor module. Then X is a dualizable object of $\mathrm{Syn}_E^{\heartsuit}$, and the dual $X^{\vee}$ is also molecular (see Remark 3.6.15).

Using Propositions 4.5.7 and 4.5.8, we obtain the following:

Proposition 6.2.5. *Let $\{M_i\}_{i \in I}$ be a finite collection of E-modules and let $N \in \mathrm{Mod}_E$. If either N or some M_i is quasi-molecular, then the canonical map*

$$\pi_0 \operatorname{Map}_{\mathrm{Mod}_E}(\bigotimes_{i \in I} M_i, N) \to \operatorname{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\boxtimes_{i \in I} \mathrm{Sy}^{\heartsuit}[M_i], \mathrm{Sy}^{\heartsuit}[N])$$

is a bijection.

Corollary 6.2.6. *Let M be an object of $\mathrm{Mod}_E^{\mathrm{loc}}$. Then M is quasi-molecular (in the sense of Variant 3.6.12) if and only if the Milnor module $\mathrm{Sy}^{\heartsuit}[M]$ is quasi-molecular (in the sense of Definition 6.2.1).*

Proof. It follows immediately from the definition that if M is quasi-molecular, then $\mathrm{Sy}^{\heartsuit}[M]$ is quasi-molecular. For the converse, suppose that $\mathrm{Sy}^{\heartsuit}[M]$ is quasi-molecular. Then there exists a quasi-molecular E-module N and an isomorphism of Milnor modules $\alpha : \mathrm{Sy}^{\heartsuit}[M] \simeq \mathrm{Sy}^{\heartsuit}[N]$. Applying Proposition 6.2.5, we can assume that α is obtained from a morphism of E-modules $\overline{\alpha} : M \to N$. We will complete the proof by showing that $\overline{\alpha}$ is an equivalence. By virtue of Proposition 4.2.5, it will suffice to show that $\overline{\alpha}$ induces an equivalence of synthetic E-modules $\mathrm{Sy}[M] \to \mathrm{Sy}\, N$. Let $n \geqslant 0$ be an integer; we will show that $\overline{\alpha}$ induces an isomorphism of Milnor modules $\pi_n \mathrm{Sy}[M] \to \pi_n \mathrm{Sy}[N]$. When $n = 0$, this follows from our assumption that α is an isomorphism. The general case follows by induction on n, since the synthetic E-modules $\mathrm{Sy}[M]$ and $\mathrm{Sy}[N]$ both satisfy condition $(*)$ of Proposition 4.2.5. □

Corollary 6.2.7. *Let M be an object of $\mathrm{Mod}_E^{\mathrm{loc}}$. Then M is molecular (atomic) if and only if the Milnor module $\mathrm{Sy}^{\heartsuit}[M]$ is molecular (atomic).*

Corollary 6.2.8. *Let $\mathcal{O}$ be an ∞-operad, let $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{qmol}}(\mathrm{hMod_E})$ denote the full subcategory of $\mathrm{Alg}(\mathrm{hMod_E})$ spanned by those $\mathcal{O}$-algebras A such that $A(X) \in \mathrm{hMod_E}$ is quasi-molecular for each $X \in \mathcal{O}$, and define $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{qmol}}(\mathrm{Syn}_E^{\heartsuit}) \subseteq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Syn}_E^{\heartsuit})$ similarly. Then the construction $A \mapsto \mathrm{Sy}^{\heartsuit}[A]$ induces an equivalence of categories $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{qmol}}(\mathrm{hMod_E}) \to \mathrm{Alg}_{\mathcal{O}}^{\mathrm{qmol}}(\mathrm{Syn_E^{\heartsuit}})$.*

Example 6.2.9. Applying Corollary 6.2.8 in the case where $\mathcal{O}$ is the trivial ∞-operad, we conclude that the construction $M \mapsto \mathrm{Sy}^{\heartsuit}[M]$ induces an equivalence from the homotopy category of quasi-molecular E-modules to the full subcategory of $\mathrm{Syn}_E^{\heartsuit}$ spanned by the quasi-molecular objects (this also follows directly from either Proposition 4.5.7 or Proposition 4.5.8).

Example 6.2.10. Applying Corollary 6.2.8 in the case where $\mathcal{O}$ is the associative ∞-operad, we obtain an equivalence between the following:

- The category of quasi-molecular associative algebras in the homotopy category $\mathrm{hMod_E}$.
- The category of quasi-molecular associative algebras in the category $\mathrm{Syn}_E^{\heartsuit}$

Note that we have a similar equivalences for the categories of atomic and molecular associative algebras.

Example 6.2.11. Applying Corollary 6.2.8 in the case where $\mathcal{O}$ is the commutative ∞-operad, we obtain an equivalence between the following:

- The category of quasi-molecular commutative algebras in the homotopy category $\mathrm{hMod_E}$.
- The category of quasi-molecular commutative algebras in the category $\mathrm{Syn}_E^{\heartsuit}$

Note that we have a similar equivalences for the categories of atomic and molecular commutative algebras (beware that if the

6.3 Constant Milnor Modules

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section. Let E_* denote the graded commutative ring $\pi_* E$, let $\mathfrak{m} \subseteq E_0 = \pi_0 E$ be the maximal ideal, and let K_* denote the graded commutative ring $E_*/\mathfrak{m}E_*$. Note that K_* satisfies the hypotheses of Notation 5.1.1 (that is, K_* is noncanonically isomorphic to a ring of

Laurent polynomials $\kappa[t^{\pm 1}]$, where $\kappa = K_0$ is the residue field of E and t is homogeneous of degree 2).

We let $\mathrm{Mod}^{\mathrm{gr}}_{E_*}$ denote the abelian category of graded E_*-modules, and we let $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ denote the abelian category of graded K_*-modules. We regard $\mathrm{Mod}^{\mathrm{gr}}_{E_*}$ and $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ as symmetric monoidal categories by means of the usual Koszul sign rule, so that the construction $M \mapsto \pi_* M$ determines a lax symmetric monoidal functor $\mathrm{Mod}_E \to \mathrm{Mod}^{\mathrm{gr}}_{E_*}$.

Definition 6.3.1. Let M_* be a graded module over the graded ring E_*. We will say that M_* is *free* if there exists a collection of homogeneous elements $\{x_\alpha \in M_*\}$ which freely generate M_* as a module over E_*. We let $\mathrm{Mod}^{\mathrm{fr}}_{E_*}$ denote the full subcategory of $\mathrm{Mod}^{\mathrm{gr}}_{E_*}$ spanned by the *free* graded E-modules.

Remark 6.3.2. In the setting of Definition 6.3.1, we could replace the graded ring E_* with the graded ring K_*. However, the resulting notion would be vacuous: since every nonzero homogeneous element of K_* is invertible, every graded K_*-module is automatically free.

Remark 6.3.3. Let $\mathfrak{m} \subseteq E_0 = \pi_0 E$ denote the maximal ideal. Then the construction $M_* \mapsto M_*/\mathfrak{m}M_*$ determines a symmetric monoidal functor $\mathrm{Mod}^{\mathrm{gr}}_{E_*} \to \mathrm{Mod}^{\mathrm{gr}}_{K_*}$. We make the following observations:

(a) Every object of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ can be written as a quotient $M_*/\mathfrak{m}M_*$, where $M_* \in \mathrm{Mod}^{\mathrm{gr}}_{E_*}$ is free.

(b) If M_* and N_* are graded E_*-modules for which M_* is free, then the canonical map
$$\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{gr}}_{E_*}}(M_*, N_*) \to \mathrm{Hom}_{\mathrm{Mod}^{\mathrm{gr}}_{K_*}}(M_*/\mathfrak{m}M_*, N_*/\mathfrak{m}N_*)$$
is a surjection whose kernel can be identified with $\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{gr}}_{E_*}}(M_*, \mathfrak{m}N_*)$.

Combining (a) and (b), we obtain an equivalence $\mathrm{Mod}^{\mathrm{gr}}_{K_*} \simeq \mathcal{C}$, where the category $\mathcal{C}$ can be described as follows:

- The objects of $\mathcal{C}$ are free graded E_*-modules.
- If M_* and N_* are free graded E_*-modules, then the set $\mathrm{Hom}_{\mathcal{C}}(M_*, N_*)$ is the cokernel of the inclusion map $\mathrm{Hom}_{\mathrm{Mod}^{\mathrm{gr}}_{E_*}}(M_*, \mathfrak{m}N_*) \hookrightarrow \mathrm{Hom}_{\mathrm{Mod}^{\mathrm{gr}}_{E_*}}(M_*, N_*)$.

Definition 6.3.4. Let M be an E-module. We will say that M is *free* if $\pi_* M$ is free when regarded as a graded E_*-module, in the sense of Definition 6.3.1. We let $\mathrm{Mod}^{\mathrm{fr}}_{E}$ denote the full subcategory of Mod_E spanned by the free E-modules.

Remark 6.3.5. An E-module M is free if and only if it can be written as a coproduct of E-modules of the form E and ΣE.

Remark 6.3.6. Let M and N be E-modules. If M is free, then the canonical map $\pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(M, N) \to \operatorname{Hom}_{\operatorname{Mod}^{\mathrm{gr}}_{E_*}}(\pi_* M, \pi_* N)$ is bijective. Consequently, the construction $M \mapsto \pi_* M$ induces an equivalence of categories $\mathrm{hMod}^{\mathrm{fr}}_{\mathrm{E}} \simeq \operatorname{Mod}^{\mathrm{fr}}_{\mathrm{E}_*}$. Moreover, this equivalence is symmetric monoidal.

Definition 6.3.7. Let X be a Milnor module. We will say that X is *constant* if it is isomorphic to $\mathrm{Sy}^{\heartsuit}[M]$, where $M \in \operatorname{Mod}_E$ is free. We let Syn^{c}_E denote the full subcategory of $\mathrm{Syn}^{\heartsuit}_E$ spanned by the constant Milnor modules.

Remark 6.3.8. Since the construction $M \mapsto \mathrm{Sy}^{\heartsuit}[M]$ commutes with coproducts, an object $X \in \mathrm{Syn}^{\heartsuit}_E$ is constant if and only if it is isomorphic to a coproduct of objects of the form $\mathbf{1}^{\heartsuit} = \mathrm{Sy}^{\heartsuit}[E]$ and $\mathrm{Sy}^{\heartsuit}[\Sigma E]$.

Remark 6.3.9. Since the collection of free objects of Mod_E is closed under tensor products and the functor $M \mapsto \mathrm{Sy}^{\heartsuit}_E$ is symmetric monoidal, the full subcategory $\mathrm{Syn}^{c}_E \subseteq \mathrm{Syn}^{\heartsuit}_E$ is closed under the tensor product functor $\boxtimes : \mathrm{Syn}^{\heartsuit}_E \times \mathrm{Syn}^{\heartsuit}_E \to \mathrm{Syn}^{\heartsuit}_E$.

The following result describes the structure of the category Syn^{c}_E:

Proposition 6.3.10. *Let M and N be free E-modules. Then the canonical map*

$$\rho : \pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(M, N) \to \operatorname{Hom}_{\mathrm{Syn}^{c}_E}(\mathrm{Sy}^{\heartsuit}[M], \mathrm{Sy}^{\heartsuit}[N])$$

is surjective, and its kernel is the image of the canonical map

$$\operatorname{Hom}_{\operatorname{Mod}^{\mathrm{gr}}_{E_*}}(\pi_* M, \mathfrak{m}\pi_* N) \hookrightarrow \operatorname{Hom}_{\operatorname{Mod}^{\mathrm{gr}}_{E_*}}(\pi_* M, \pi_* N) \simeq \pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(M, N).$$

Combining Proposition 6.3.10 with Remark 6.3.3, we obtain the following:

Corollary 6.3.11. *There is an essentially unique equivalence of symmetric monoidal categories $\Phi : \mathrm{Syn}^{c}_E \simeq \operatorname{Mod}^{\mathrm{gr}}_{K_*}$ for which the diagram*

$$\begin{array}{ccc} \operatorname{Mod}^{\mathrm{fr}}_E & \xrightarrow{\pi_*} & \operatorname{Mod}^{\mathrm{fr}}_{E_*} \\ \downarrow{\scriptstyle \mathrm{Sy}^{\heartsuit}} & & \downarrow \\ \mathrm{Syn}^{c}_E & \xrightarrow{\Phi} & \operatorname{Mod}^{\mathrm{fr}}_{K_*} \end{array}$$

commutes up to equivalence.

Proof of Proposition 6.3.10. Without loss of generality, we may assume that $M \simeq E$. Let us identify the domain of ρ with $\pi_0 N$. We wish to show that ρ is a surjection with kernel $\ker(\rho) = \mathfrak{m}(\pi_0 N)$. We first show that the kernel of ρ contains $\mathfrak{m}(\pi_0 N)$. Choose an element $x \in \mathfrak{m}$ and an element $y \in \pi_0 N$; we wish to show that $\rho(xy) = 0$. In other words, we wish to show that the composite map

$$\mathrm{Sy}^{\heartsuit}[E] \xrightarrow{x} \mathrm{Sy}^{\heartsuit}[E] \xrightarrow{y} \mathrm{Sy}^{\heartsuit}[N]$$

vanishes in the abelian category $\mathrm{Syn}_E^{\heartsuit}$. In fact, we claim that x induces the zero map from $\mathrm{Sy}^{\heartsuit}[E]$ to itself. For this, it suffices to show that for every molecular E-module M, multiplication by x annihilates the abelian group $\pi_0 \mathrm{Map}_{\mathrm{Mod}_E}(M, E) \simeq \pi_0 M^{\vee}$, which follows from the fact that $M^{\vee}$ is also molecular (Remark 3.6.15).

We now establish the reverse inclusion $\ker(\rho) \subseteq \mathfrak{m}(\pi_0 N)$. Suppose we are given an element $y \in \pi_0 N \simeq \pi_0 \mathrm{Map}_{\mathrm{Mod}_E}(E, N)$ with the property that the induced map $\mathrm{Sy}^{\heartsuit}[E] \to \mathrm{Sy}^{\heartsuit}[N]$ vanishes. Let K be an atomic E-module and let $K^{\vee}$ denote its E-linear dual, so that the induced map $\mathrm{Sy}^{\heartsuit}[E](K^{\vee}) \xrightarrow{y} \mathrm{Sy}^{\heartsuit}[E](K^{\vee})$ vanishes. Unwinding the definitions (and using our assumption that N is free), we deduce that the composite map

$$\pi_0(K \otimes_E E) \xrightarrow{y} \pi_0(K \otimes_E N) \simeq (\pi_0 N)/\mathfrak{m}(\pi_0 N)$$

vanishes, so that $y \in \mathfrak{m}(\pi_0 N)$ as desired.

We now show that ρ is surjective. Let $\alpha : \mathrm{Sy}^{\heartsuit}[E] \to \mathrm{Sy}^{\heartsuit}[N]$ be a natural transformation. Evaluating α on $K^{\vee}$, we obtain a map

$$\alpha_{K^{\vee}} : \kappa = \mathrm{Sy}^{\heartsuit}[E](K^{\vee}) \to \mathrm{Sy}^{\heartsuit}[N](K^{\vee}) \simeq (\pi_0 N)/\mathfrak{m}(\pi_0 N),$$

which carries the unit element $1 \in \kappa$ to some element $\overline{y} \in (\pi_0 N)/\mathfrak{m}(\pi_0 N)$. We will complete the proof by showing that $\alpha = \rho(\overline{y})$. To prove this, we can replace α by $\alpha - \rho(\overline{y})$ and thereby assume that $\alpha_{K^{\vee}}(1) = 0$. We will complete the proof by showing that $\alpha = 0$: that is, the induced map $\alpha_P : \mathrm{Sy}^{\heartsuit}[E](P) \to \mathrm{Sy}^{\heartsuit}[N](P)$ is vanishes for every molecular E-module P. Choose an element $z \in \mathbf{1}^{\heartsuit}(P)$, which we can identify with an element of $\pi_0 P^{\vee}$. Since P is molecular, we can choose a map $f : K \to P^{\vee}$ which carries the unit element of $\pi_0 K$ to z. In this case, the vanishing of $\alpha_P(z)$ follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Sy}^{\heartsuit}[E](K^{\vee}) & \xrightarrow{\alpha_{K^{\vee}}} & \mathrm{Sy}^{\heartsuit}[N](K^{\vee}) \\ \downarrow{\scriptstyle \mathrm{Sy}^{\heartsuit}[E](f^{\vee})} & & \downarrow{\scriptstyle \mathrm{Sy}^{\heartsuit}[N](f^{\vee})} \\ \mathrm{Sy}^{\heartsuit}[E](P) & \xrightarrow{\alpha_P} & \mathrm{Sy}^{\heartsuit}[N](P). \end{array}$$

□

6.4 The Structure of $\mathrm{Syn}_E^{\heartsuit}$

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section. Let E_* denote the graded commutative ring $\pi_* E$, let $\mathfrak{m} \subseteq E_0 = \pi_0 E$ be the maximal ideal, and let K_* denote the graded commutative ring $E_*/\mathfrak{m}E_*$.

Construction 6.4.1 (Enrichment in Graded K_*-Modules)**.** Let $\Phi : \mathrm{Mod}_{K_*}^{\mathrm{gr}} \simeq \mathrm{Syn}_E^{c} \subseteq \mathrm{Syn}_E^{\heartsuit}$ be the symmetric monoidal functor of Corollary 6.3.11. If X and Y are Milnor modules, we let $\underline{\mathrm{Hom}}(X,Y)_*$ denote a graded K_*-module with the following universal property: for every graded K_*-module M_*, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Mod}_{K_*}^{\mathrm{gr}}}(M_*, \underline{\mathrm{Hom}}(X,Y)_*) \simeq \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\Phi(M_*) \boxtimes X, Y).$$

Note that the construction $(X,Y) \mapsto \underline{\mathrm{Hom}}(X,Y)_*$ determines an enrichment of the category of Milnor modules $\mathrm{Syn}_E^{\heartsuit}$ over the symmetric monoidal category $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$.

Remark 6.4.2. Let X be a Milnor module. For every integer n, we let $X[n]$ denote the Milnor module given by the tensor product $X \boxtimes \mathrm{Sy}^{\heartsuit}[\Sigma^n E]$. We will refer to $X[n]$ as the *n-fold shift of* X. Note that if Y is another Milnor module, then the graded K_*-module $\underline{\mathrm{Hom}}(X,Y)_*$ can be described concretely by the formula $\underline{\mathrm{Hom}}(X,Y)_n = \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(X[n], Y)$.

Warning 6.4.3. If $X \in \mathrm{Syn}_E^{\heartsuit} \simeq \mathrm{Syn}_E$ is a Milnor module, then the n-fold shift $X[n]$ should not be confused with the n-fold suspension $\Sigma^n X$ (which we can regard as a *non-discrete* synthetic E-module).

Proposition 6.4.4. *Let M and N be E-modules. If M or N is quasi-molecular, then the canonical map*

$$\rho : \pi_* \underline{\mathrm{Map}}_E(M,N) \to \underline{\mathrm{Hom}}(\mathrm{Sy}^{\heartsuit}[M], \mathrm{Sy}^{\heartsuit}[N])_*$$

is an isomorphism of graded E_*-modules.

Proof. Replacing M by a suitable suspension, we can reduce to proving that the canonical map

$$\pi_0 \underline{\mathrm{Map}}_E(M,N) \to \underline{\mathrm{Hom}}(\mathrm{Sy}^{\heartsuit}[M], \mathrm{Sy}^{\heartsuit}[N])_0$$

is bijective, which follows from either Proposition 4.5.7 (if M is quasi-molecular) or Proposition 4.5.8 (if N is quasi-molecular). □

Warning 6.4.5. Proposition 6.4.4 does not necessarily hold without the assumption that either M or N quasi-molecular: note that the codomain of ρ is a graded K_*-module, but the domain of ρ need not be annihilated by the maximal ideal $\mathfrak{m} \subseteq \pi_0 E$.

Notation 6.4.6. Let X be a Milnor module. We let $\operatorname{End}(X)_*$ denote the mapping object $\underline{\operatorname{Hom}}(X, X)_*$ of Construction 6.4.1. Then $\operatorname{End}(X)_*$ is an associative algebra object of the category $\operatorname{Mod}^{\mathrm{gr}}_{K_*}$: that is, it is a graded algebra over K_*.

Example 6.4.7. Let M be an E-module, and let $\operatorname{End}_E(M) \in \operatorname{Alg}(\operatorname{Mod}_E)$ denote the associative E-algebra classifying endomorphisms of M. Then we have a canonical map of graded E_*-algebras $\pi_* \operatorname{End}_E(M) \to \operatorname{End}(\operatorname{Sy}^\heartsuit[M])_*$, which is an isomorphism if M is quasi-molecular (Proposition 6.4.4).

We now elaborate on the characterization of Milnor modules given in Variant 6.1.6.

Construction 6.4.8. Let A be a Milnor module. We let $\Gamma_A : \operatorname{Syn}^\heartsuit_E \to \operatorname{Mod}^{\mathrm{gr}}_{K_*}$ denote the functor given by $\Gamma_A(X) = \underline{\operatorname{Hom}}(\mathbf{1}^\heartsuit, A \boxtimes X)_*$.

Remark 6.4.9. Construction 6.4.8 makes sense if A is *any* Milnor module. However, we will primarily be interested in the case where A is an associative algebra object of $\operatorname{Syn}^\heartsuit_E$ (as suggested by our notation).

Remark 6.4.10. Let A be a dualizable object of $\operatorname{Syn}^\heartsuit_E$, with dual $A^\vee$. Then the functor $\Gamma_A : \operatorname{Syn}^\heartsuit_E \to \operatorname{Mod}^{\mathrm{gr}}_{K_*}$ is given by the formula $\Gamma_A(X) = \underline{\operatorname{Hom}}(A^\vee, X)_*$. It follows that the functor Γ_A admits a left adjoint F, given concretely by the formula $F(V) = A^\vee \boxtimes V$ (here we abuse notation by identifying $\operatorname{Mod}^{\mathrm{gr}}_{K_*}$ with the full subcategory $\operatorname{Syn}^c_E \subseteq \operatorname{Syn}^\heartsuit_E$ by means of Corollary 6.3.11).

Example 6.4.11. Let M be a molecular E-module and set $A = \operatorname{Sy}^\heartsuit[M]$. Then the functor Γ_A is given concretely by the formula

$$\Gamma_A(X)_n = \underline{\operatorname{Hom}}(A^\vee, X)_n = \underline{\operatorname{Map}}_0(\operatorname{Sy}^\heartsuit[\Sigma^n M^\vee], X) = X(\Sigma^n M^\vee).$$

Proposition 6.4.12. *Let A be a nonzero molecular object of $\operatorname{Syn}^\heartsuit_E$. Then the functor Γ_A induces an equivalence from the category of Milnor modules to the category of graded left modules over $\operatorname{End}(A)_*$ (see Notation 6.4.6).*

Proof of Proposition 6.4.12. Let $\mathcal{C} = \operatorname{LMod}_{\operatorname{End}(A)_*}(\operatorname{Mod}^{\mathrm{gr}}_{K_*})$ denote the abelian category of graded left modules over $\operatorname{End}(A)_*$, so we can promote Γ_A to a functor $G : \operatorname{Syn}^\heartsuit_E \to \mathcal{C}$. Since A is nonzero and molecular, we can write $A = \operatorname{Sy}^\heartsuit[M]$, where M is a nonzero molecular E-module. Using Example 6.4.11, we see that the functor G can be described concretely by the formula $G(X)_n = X(\Sigma^n M^\vee)$. From this, we deduce the following:

(i) The functor G commutes with small colimits.

(ii) The functor G is conservative (since every molecular E-module is a retract of a direct sum of modules of the form $\Sigma^n M^\vee$).

The functor G admits a left adjoint $F : \mathcal{C} \to \mathrm{Syn}_E^{\heartsuit}$, which we can describe concretely by the formula $F(N_*) = A \boxtimes_{\mathrm{End}(A)_*} N_*$ (here we abuse notation by identifying graded K_*-modules with their image under the equivalence $\Phi : \mathrm{Mod}_{K_*}^{\mathrm{gr}} \simeq \mathrm{Syn}_E^c$ of Corollary 6.3.11). To complete the proof, it will suffice to show that F is fully faithful: that is, that the unit map $u : N_* \to (G \circ F)(N_*)$ is an isomorphism for every object $N_* \in \mathcal{C}$. Since the functors F and G both preserve small colimits, the collection of those objects $N_* \in \mathcal{C}$ for which u is an isomorphism is closed under small colimits. We may therefore assume without loss of generality that M_* is a free left $\mathrm{End}(A)_*$-module on a single homogeneous generator. In this case, the desired result follows immediately from the definitions. □

Combining Proposition 6.4.12 with Example 6.4.7, we obtain the following:

Corollary 6.4.13. *Let M be a nonzero molecular E-module. Then the construction $X \mapsto \{X(\Sigma^n M^\vee)\}_{n \in \mathbf{Z}}$ determines an equivalence from the category of Milnor modules to the category of graded left modules over $\pi_* \mathrm{End}_E(M)$.*

6.5 Endomorphisms of Atomic E-Modules

Throughout this section, we fix a Lubin-Tate spectrum E. Let $\mathfrak{m}$ denote the maximal ideal of $\pi_0 E$, let K_* denote the graded ring $(\pi_* E)/\mathfrak{m}(\pi_* E)$, and let $\kappa = (\pi_0 E)/\mathfrak{m}$ denote the residue field of E. Suppose we are given an atomic E-module M. It follows from Corollary 6.4.13 that the category of Milnor modules $\mathrm{Syn}_E^{\heartsuit}$ can be identified with the category of modules over the graded ring $\pi_* \mathrm{End}_E(M)$. Our goal in this section is to describe $\pi_* \mathrm{End}_E(M)$ more explicitly.

Proposition 6.5.1. *Let E be a Lubin-Tate spectrum of height n and let M be an atomic E-module. Then there exists an n-dimensional vector space V over κ and an isomorphism of graded K_*-algebras $\gamma : \bigwedge_{K_*}^*(V) \simeq \pi_* \mathrm{End}_E(K)$ (see Notation 5.2.1).*

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ be a system of parameters for the regular local ring $\pi_0 E$. For $0 \leqslant m < n$, let P_m denote the cofiber of the map $v_i : E \to E$ (formed in the ∞-category Mod_E), so that we have a canonical fiber sequence

$$E \xrightarrow{f_m} P_m \xrightarrow{g_m} \Sigma E.$$

Since each v_m is a regular element of $\pi_* E$, we have canonical isomorphisms $\pi_* P_m \simeq (\pi_* E)/v_m(\pi_* E)$. In particular, the graded $\pi_* E$-module $\pi_* P_m$ is concentrated in even degrees and annihilated by v_m.

Let $B_E^{=m}$ denote the graded ring $\pi_* \mathrm{End}_E(P_m)$. Using the fiber sequence

$$\mathrm{End}_E(P_m) \to P_m \xrightarrow{v_i} P_m$$

together with our calculation of $\pi_* P_m$, we conclude that $B_E^{=m}$ is isomorphic to a free $(\pi_* E)/v_m(\pi_* E)$-module on generators $1 = \mathrm{id}_{P_m}$ and β_m, where $\beta_m \in \pi_{-1} \mathrm{End}_E(P_m)$ is given by the composition

$$\Sigma^{-1} P_m \xrightarrow{\Sigma^{-1}(g_m)} E \xrightarrow{f_m} P_m.$$

From this description, we immediately deduce that $\beta_m^2 = 0$.

For $0 \leqslant m \leqslant n$, set $Q_m = P_0 \otimes_E \cdots \otimes_E P_{m-1}$, and set $B_E^{\leqslant m} = \pi_* \mathrm{End}_E(Q_m)$. For $k \leqslant m$, we abuse notation by identifying the element $\beta_k \in (B_E^{=k})_{-1}$ with its image in $(B_E^{\leqslant m})_{-1}$. We will establish the following claim for each $0 \leqslant m \leqslant n$:

$(*_m)$ The graded ring $B_E^{\leqslant m}$ is isomorphic to an exterior algebra over the commutative ring $(\pi_* E)/(v_0, \ldots, v_{m-1})$ on generators $\beta_0, \ldots, \beta_{m-1}$.

Note that the statement of Proposition 6.5.1 follows immediately from assertion $(*_n)$ (since Q_n is atomic and therefore equivalent to M). We will prove $(*_m)$ by induction on m, the case $m = 0$ being trivial. To carry out the inductive step, let us suppose that $m > 0$ and that assertion $(*_{m-1})$ holds. There is a canonical equivalence of E-modules $\mathrm{End}_E(Q_{m-1}) \otimes_E \mathrm{End}_E(P_m) \simeq \mathrm{End}_E(Q_m)$, which yields a convergent spectral sequence $\mathrm{Tor}_*^{\pi_* E}(B_E^{\leqslant m-1}, B_E^{=m}) \Rightarrow B_E^{\leqslant m}$. Since the elements $v_0, \ldots, v_{m-1}$ form a regular sequence in $\pi_* E$, our inductive hypothesis guarantees that the groups $\mathrm{Tor}_s^{\pi_* E}(B_E^{\leqslant m-1}, B_E^{=m})$ vanish for $s > 0$. Consequently, the spectral sequence degenerates to yield an isomorphism of graded rings $B_E^{\leqslant m} \simeq B_E^{\leqslant m-1} \otimes_{\pi_* E} B_E^{=m}$, from which $(*_m)$ follows immediately. □

Corollary 6.5.2. *Let E be a Lubin-Tate spectrum of height n. Then the group*

$$\mathrm{Ext}^1_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}, \mathbf{1}^{\heartsuit}[-1])$$

is an n-dimensional vector space over the residue field κ of E.

Proof. Combine Proposition 6.5.1, Corollary 6.4.13, and Proposition 5.5.6. □

Our next goal is to obtain a more intrinsic description of the vector space

$$\mathrm{Ext}^1_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}, \mathbf{1}^{\heartsuit}[-1])$$

and, by extension, the vector space V appearing in Proposition 6.5.1).

Proposition 6.5.3. *Let x be an element of the maximal ideal $\mathfrak{m}$ and let M_x denote the fiber of the map $x : E \to E$, so that we have a fiber sequence of E-modules $\Sigma^{-1} E \to M_x \to E$. Then the induced sequence*

$$0 \to \mathrm{Sy}^{\heartsuit}[\Sigma^{-1} E] \to \mathrm{Sy}^{\heartsuit}[M_x] \to \mathrm{Sy}^{\heartsuit}[E] \to 0$$

is exact (in the abelian category of Milnor modules).

Proof. Let N be a molecular E-module; we wish to show that the sequence of abelian groups

$$0 \to \pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(N, \Sigma^{-1}E) \to \pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(N, M_x) \to \pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(N, E) \to 0$$

is exact. Equivalently, we wish to show that the boundary maps

$$\pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(N, \Sigma^{-1}E) \xrightarrow{x} \pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(N, \Sigma^{-1}E)$$

$$\pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(N, E) \xrightarrow{x} \pi_0 \operatorname{Map}_{\operatorname{Mod}_E}(N, E)$$

both vanish. To see this, we note that $N^\vee$ is also a molecular E-module (Remark 3.6.15), so the homotopy groups of $N^\vee$ are annihilated by the maximal ideal $\mathfrak{m} \subseteq \pi_0 E$. □

Theorem 6.5.4. *There exists a unique vector space isomorphism*

$$\psi : \mathfrak{m}/\mathfrak{m}^2 \to \operatorname{Ext}^1_{\operatorname{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit, \mathbf{1}^\heartsuit[-1])$$

with the following property: for every element $x \in \mathfrak{m}$ having image $\overline{x} \in \mathfrak{m}/\mathfrak{m}^2$, $\psi(\overline{x})$ is the extension class of the exact sequence $0 \to \mathbf{1}^\heartsuit[-1] \to \operatorname{Sy}^\heartsuit[M_x] \to \mathbf{1}^\heartsuit \to 0$ of Proposition 6.5.3.

Proof. It is not difficult to show that the construction $x \mapsto M_x$ induces a $(\pi_0 E)$-linear map

$$\overline{\psi} : \mathfrak{m} \to \operatorname{Ext}^1_{\operatorname{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit, \mathbf{1}^\heartsuit[-1]).$$

Since the codomain of $\overline{\psi}$ is annihilated by the maximal ideal $\mathfrak{m} \subseteq \pi_0 E$, we see that $\overline{\psi}$ descends to a map of vector spaces

$$\psi : \mathfrak{m}/\mathfrak{m}^2 \to \operatorname{Ext}^1_{\operatorname{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit, \mathbf{1}^\heartsuit[-1]).$$

It follows from Corollary 6.5.2 that the domain and codomain of ψ are vector spaces of the same (finite) dimension over κ. Consequently, to show that ψ is an isomorphism, it will suffice to show that ψ is injective.

Choose an element $x \in \mathfrak{m}$, and suppose that the exact sequence $0 \to \mathbf{1}^\heartsuit[-1] \to \operatorname{Sy}^\heartsuit[M_x] \to \mathbf{1}^\heartsuit \to 0$ splits (in the abelian category $\operatorname{Syn}_E^\heartsuit$); we wish to show that x belongs to $\mathfrak{m}^2$. Suppose otherwise. Then we can choose a regular system of parameters $v_0, \ldots, v_{n-1}$ for the local ring $\pi_0 E$ which contains x. The proof of Proposition 6.5.1 then shows that there exists an atomic E-module M which factors as a tensor product $M_x \otimes_E N$, for some auxiliary E-module N. We then obtain

$$\begin{aligned} \operatorname{Sy}^\heartsuit[M] &\simeq \operatorname{Sy}^\heartsuit[M_x] \boxtimes \operatorname{Sy}^\heartsuit[N] \\ &\simeq \operatorname{Sy}^\heartsuit[N] \oplus \operatorname{Sy}^\heartsuit[N][1]. \end{aligned}$$

It follows that the endomorphism ring $\pi_* \operatorname{End}_E(M) \simeq \operatorname{End}(\operatorname{Sy}^\heartsuit M)_*$ contains an invertible element of degree 1, contradicting Proposition 6.5.1. □

6.6 The Monoidal Structure of $\mathrm{Syn}_E^\heartsuit$

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section. Our next goal is to promote the equivalence of Proposition 6.4.12 to an equivalence of *monoidal* categories.

Remark 6.6.1. Let A be an associative algebra object of $\mathrm{Syn}_E^\heartsuit$ and let $\Gamma_A : \mathrm{Syn}_E^\heartsuit \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$ be as in Construction 6.4.8. Then Γ_A can be written as a composition

$$\mathrm{Syn}_E^\heartsuit \xrightarrow{A\boxtimes} \mathrm{Syn}_E^\heartsuit \xrightarrow{\Gamma} \mathrm{Mod}_{K_*}^{\mathrm{gr}}$$

where the first functor is lax monoidal (since A is an associative algebra object of $\mathrm{Syn}_E^\heartsuit$) and Γ is right adjoint to the inclusion $\mathrm{Syn}_E^c \subseteq \mathrm{Syn}_E^\heartsuit$ (and therefore inherits the structure of a lax symmetric monoidal functor). It follows that we can regard Γ_A as a lax monoidal functor from $\mathrm{Syn}_E^\heartsuit$ to $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$.

Remark 6.6.2. In the situation of Remark 6.6.1, suppose that A is a *commutative* algebra object of the category $\mathrm{Syn}_E^\heartsuit$. Then we can regard Γ_*^A as a lax symmetric monoidal functor from $\mathrm{Syn}_E^\heartsuit$ to $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$.

Our goal is to prove the following analogue of Proposition 5.4.3:

Proposition 6.6.3. *Let A be an associative algebra object of $\mathrm{Syn}_E^\heartsuit$. If A is atomic, then the lax monoidal functor $\Gamma_A : \mathrm{Syn}_E^\heartsuit \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$ is monoidal.*

The proof of Proposition 6.6.3 will make use of the following:

Lemma 6.6.4. *Let $A \in \mathrm{Alg}(\mathrm{hMod_E})$ be atomic, and let M and N be E-modules. If M is perfect, then the multiplication on A induces an isomorphism*

$$\theta_{M,N} : \pi_* \underline{\mathrm{Map}}_E(M, A) \otimes_{\pi_* A} \pi_* \underline{\mathrm{Map}}_E(N, A) \to \pi_* \underline{\mathrm{Map}}_E(M \otimes_E N, A)$$

of graded modules over $\pi_* A$.

Proof. Let us regard $N \in \mathrm{Mod}_E$ as fixed. The collection of those E-module spectra M for which $\theta_{M,N}$ is an isomorphism is closed under retracts and extensions. Consequently, to show that $\theta_{M,N}$ is an equivalence for all perfect E-modules M, it will suffice to show that the map $\theta_{E,N}$ is an equivalence. Unwinding the definitions, we see that $\theta_{E,N}$ can be identified with the endomorphism of $\underline{\mathrm{Map}}_E(N, A)$ given by postcomposition with the composite map

$$A \simeq E \otimes_E A \xrightarrow{e \otimes \mathrm{id}} A \otimes_E A \xrightarrow{m} A$$

, where $e : E \to A$ is the unit map. Since the multiplication on A is unital, we conclude that $\theta_{E,N}$ is an isomorphism. □

Proof of Proposition 6.6.3. Let $A \in \mathrm{Alg}(\mathrm{Syn}_E^\heartsuit)$ be atomic. We first show that the functor Γ_A preserves unit objects: that is, the canonical map $K_* \to \underline{\mathrm{Hom}}(\mathbf{1}^\heartsuit, A)_*$ is an isomorphism. Using Example 6.2.10, we can choose an isomorphism $A \simeq \mathrm{Sy}^\heartsuit[\overline{A}]$, where $\overline{A}$ is an atomic algebra object of the homotopy category $\mathrm{hMod_E}$. In this case, the desired result follows from Proposition 6.2.5.

We now complete the proof by showing that, for every pair of objects $X, Y \in \mathrm{Syn}_E^\heartsuit$, the canonical map $\theta_{X,Y} : \Gamma_A(X) \otimes_{K_*} \Gamma_A(Y) \to \Gamma_A(X \boxtimes Y)$ is an isomorphism. Note that for fixed $Y \in \mathrm{Syn}_E^\heartsuit$, the collection of those objects X for which $\theta_{X,Y}$ is an isomorphism is closed under small colimits. We may therefore assume without loss of generality that $X = \mathrm{Syn}^\heartsuit[M^\vee]$, where M is a molecular E-module. Similarly, we may assume that $Y = \mathrm{Sy}^\heartsuit[N]$, where $N \in \mathrm{Mod}_E$ is molecular. In this case, the desired conclusion is a special case of Lemma 6.6.4. □

Our next goal is to apply Proposition 6.6.3 to construct an equivalence of *monoidal* categories $\mathrm{Syn}_E^\heartsuit \simeq \mathcal{M}(V)$, where $V = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ denotes the Zariski tangent space of the Lubin-Tate ring and $\mathcal{M}(V)$ is the category of graded modules over the exterior algebra $\bigwedge^*_{K_*}(V)$ (as in Definition 5.2.6). It will be useful to formulate a more precise statement.

Definition 6.6.5. Let $F : \mathrm{Syn}_E^\heartsuit \to \mathcal{M}(V)$ be an equivalence of $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear monoidal categories. Then F induces a κ-linear isomorphism

$$DF : \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\psi} \mathrm{Ext}^1_{\mathrm{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit[1], \mathbf{1}^\heartsuit) \xrightarrow{F} \mathrm{Ext}^1_{\mathcal{M}(V)}(K_*[1], K_*) \xrightarrow{\gamma^{-1}} \mathfrak{m}/\mathfrak{m}^2,$$

where ψ is the isomorphism of Theorem 6.5.4 and γ is the isomorphism of Proposition 5.5.6. We will say that F is *normalized* if DF is the identity map $\mathrm{id}_{\mathfrak{m}/\mathfrak{m}^2}$.

Our main result can now be stated as follows:

Theorem 6.6.6. *There exists a normalized* $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$*-linear monoidal equivalence of categories* $F : \mathrm{Syn}_E^\heartsuit \to \mathcal{M}(V)$.

Corollary 6.6.7. *The tensor product functor* $\boxtimes : \mathrm{Syn}_E^\heartsuit \times \mathrm{Syn}_E^\heartsuit \to \mathrm{Syn}_E^\heartsuit$ *is exact in each variable.*

Corollary 6.6.8. *The fully faithful embedding* $\mathrm{Mod}^{\mathrm{gr}}_{K_*} \simeq \mathrm{Syn}_E^c \hookrightarrow \mathrm{Syn}_E^\heartsuit$ *induces an isomorphism on Picard groups. In other words, every invertible Milnor module is isomorphic either to the unit object* $\mathbf{1}^\heartsuit$ *or to its shift* $\mathbf{1}^\heartsuit[1]$.

Proof. By virtue of Theorem 6.6.6, it suffices to observe that any invertible object M of the symmetric monoidal category $\mathcal{M}(V)$ is isomorphic either to K_* or to the shift $K_*[1]$. □

Remark 6.6.9. Let $F : \mathrm{Syn}_E^{\heartsuit} \to \mathcal{M}(V)$ be as in Theorem 6.6.6, let M be a Milnor module, and regard $F(M)$ as a graded module over the exterior algebra $\bigwedge^*_{K_*}(V)$. Then:

(a) The Milnor module M is quasi-molecular if and only if $F(M)$ is a free module over $\bigwedge^*_{K_*}(V)$ on homogeneous generators.

(b) The Milnor module M is molecular if and only if $F(M)$ is a free module over $\bigwedge^*_{K_*}(V)$ on finitely many homogeneous generators.

(c) The Milnor module M is constant (in the sense of Definition 6.3.7) if and only if V acts trivially on $F(M)$.

Corollary 6.6.10. *Let M and N be nonzero Milnor modules. If $M \boxtimes N$ is constant, then M and N are constant.*

Corollary 6.6.11. *Let M be a quasi-molecular Milnor module. Then M is an injective object of the abelian category* $\mathrm{Syn}_E^{\heartsuit}$.

Corollary 6.6.12. *Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. If the Thom spectrum* Th_Q *is an atomic E-algebra, then the polarization Q is atomic (in the sense of Definition 3.4.2).*

Proof. It follows from Proposition 3.3.2 that Q is nonsingular, and therefore induces a map

$$\overline{c}_1^Q : \kappa \otimes \Lambda \to \mathfrak{m}/\mathfrak{m}^2$$

(see Construction 3.4.1). We wish to show that $\overline{c}_1^Q$ is an isomorphism. Choose a basis $\lambda_1, \ldots, \lambda_m$ for the lattice Λ. Let A_Q denote the Milnor module $\mathrm{Sy}^{\heartsuit}[\mathrm{Th}_Q]$, and define $A_{Q[\lambda_i]}$ for $1 \leqslant i \leqslant m$ similarly. Using Remarks 3.3.4 and 6.1.10, we obtain an isomorphism of Milnor modules

$$A_Q \simeq A_{Q[\lambda_1]} \boxtimes \cdots \boxtimes A_{Q[\lambda_m]}$$

(beware that this isomorphism does not necessarily respect the algebra structures on both sides).

Write $c_1^Q(\lambda_i) = 1 + x_i$ for $x_i \in \mathfrak{m}$. Using Proposition 6.5.3, we see that each of the cofiber sequences $E \xrightarrow{x_i} E \xrightarrow{\mathrm{Th}}_{Q[\lambda_i]}$ determines a short exact sequence of Milnor modules

$$0 \to \mathbf{1}^{\heartsuit} \to A_{Q[\lambda_i]} \to \mathbf{1}^{\heartsuit}[1] \to 0,$$

classified by an element of $\mathrm{Ext}^1_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[1], \mathbf{1}^{\heartsuit})$, which corresponds to the element $\overline{c}_1^Q(\lambda_i)$ under the isomorphism $\psi : \mathfrak{m}/\mathfrak{m}^2 \simeq \mathrm{Ext}^1_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}, \mathbf{1}^{\heartsuit}[-1])$ of Theorem 6.5.4.

Set $V = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ and let $F : \mathrm{Syn}_E^\heartsuit$ be a normalized $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear equivalence of monoidal categories (which exists by virtue of Theorem 6.6.6). The condition that F is normalized guarantees that each $F(A_{Q[\lambda_i]})$ can be identified with the extension of $K_*[1]$ by K_* classified by $\overline{x}_i = \overline{c}^1_Q(\lambda_i)$: that is, with the shifted exterior algebra $\Lambda^*_{K_*}(\kappa)[1]$, regarded as an object of $\mathcal{M}(V)$ by means of the homomorphism

$$\bigwedge\nolimits^*_{K_*}(V) \to \bigwedge\nolimits^*_{K_*}(\kappa)$$

induced by the linear map $V \to \kappa$ given by evaluation at $\overline{x}_i$. It follows that

$$F(A_Q) \simeq F(A_{Q[\lambda_1]}) \otimes_{K_*} \cdots \otimes_{K_*} F(A_{Q[\lambda_m]})$$

can be identified with the shifted exterior algebra $\bigwedge^*_{K_*}(\kappa \otimes \Lambda^\vee)[m]$, regarded as an object of $\mathcal{M}(V)$ by means of the homomorphism $\bigwedge^*_{K_*}(V) \to \bigwedge^*_{K_*}(\kappa \otimes \Lambda^\vee)$ given by the κ-linear dual of $\overline{c}^Q_1$. Note that if Th_Q is an atomic E-algebra, then $F(A_Q)$ is an atomic object of $\mathcal{M}(V)$ (see Remark 6.6.9), so that $\overline{c}^Q_1$ is an isomorphism as desired. □

The proof of Theorem 6.6.6 will require some preliminaries.

Construction 6.6.13 (The Bialgebra Structure on $\mathrm{End}_*(A)$)**.** Let A be an atomic algebra object of $\mathrm{Syn}_E^\heartsuit$, let $\Gamma_A : \mathrm{Syn}_E^\heartsuit \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$ be as in Construction 6.4.8, and let $F : \mathrm{Mod}_{K_*}^{\mathrm{gr}} \to \mathrm{Syn}_E^\heartsuit$ be a left adjoint to Γ_A (given concretely by the formula $F(V) = A^\vee \boxtimes V$; see Remark 6.4.10). Then the composition $\Gamma_A \circ F : \mathrm{Mod}_{K_*}^{\mathrm{gr}} \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$ is a monad T on the category $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$, given concretely by the formula $T(V) = \mathrm{End}_*(A) \otimes_{K_*} V$ (see Proposition 6.4.12).

Since the functor Γ_A is monoidal, the functor F inherits a colax monoidal structure for which the unit and counit maps

$$\mathrm{id}_{\mathrm{Mod}_{K_*}^{\mathrm{gr}}} \to \Gamma_A \circ F \qquad F \circ \Gamma_A \to \mathrm{id}_{\mathrm{Syn}_E^\heartsuit}$$

are natural transformations of colax monoidal functors. In particular, the functor $T = \Gamma_A \circ F$ inherits a colax monoidal structure for which the unit and multiplication maps

$$\mathrm{id} \to T \qquad T \circ T \to T$$

are colax monoidal natural transformations. It follows that the endomorphism algebra $\mathrm{End}_*(A) = T(K_*)$ can be regarded as an associative coalgebra object of $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$, for which the unit and multiplication maps

$$K_* \to \mathrm{End}_*(A) \qquad \mathrm{End}_*(A) \otimes_{K_*} \mathrm{End}_*(A) \to \mathrm{End}_*(A)$$

are morphisms of coalgebras: that is, $\mathrm{End}_*(A)$ has the structure of an (associative and coassociative) bialgebra object of the category $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$.

Remark 6.6.14. In the situation of Construction 6.6.13, the coalgebra structure on $\mathrm{End}_*(A)$ can be described more explicitly as follows. Set $V = \underline{\mathrm{Hom}}(A \boxtimes A, A)_*$, so that V inherits a left action of the graded ring $\mathrm{End}_*(A)$ (via postcomposition with endomorphisms of A) and a commuting right action of the tensor product $(\mathrm{End}_*(A) \otimes_{K_*} \mathrm{End}_*(A))$ (via precomposition with endomorphisms of each factor of A). It follows from Proposition 6.6.3 that V is freely generated as an $(\mathrm{End}_*(A) \otimes \mathrm{End}_*(A))$-module by the multiplication map $m : A \boxtimes A \to A$ (which we can regard as an element of V_0). Consequently, the left action of $\mathrm{End}_*(A)$ on V is classified by a homomorphism of graded K_*-algebras $\Delta : \mathrm{End}_*(A) \to \mathrm{End}_*(A) \otimes \mathrm{End}_*(A)$. More concretely, an equation $\Delta(f) = \sum f_i' \otimes f_i''$ in the graded vector space $\mathrm{End}_*(A) \otimes_{K_*} \mathrm{End}_*(A)$ is equivalent to the equation $f \circ m = \sum m \circ (f_i' \boxtimes f_i'')$ in the graded vector space $\underline{\mathrm{Hom}}(A \boxtimes A, A)_*$.

Remark 6.6.15. Let A be an atomic algebra object of $\mathrm{Syn}_E^{\heartsuit}$, and regard $\mathrm{End}_*(A)$ as a graded bialgebra over $K*$ (see Construction 6.6.13). If V_* and W_* are graded left $\mathrm{End}_*(A)$-modules, then the tensor product $V_* \otimes_{K_*} W_*$ inherits the structure of a left $\mathrm{End}_*(A)$-module by means of the comultiplication

$$\Delta : \mathrm{End}_*(A) \to \mathrm{End}_*(A) \otimes_{K_*} \mathrm{End}_*(A).$$

In the special case where $V_* = \Gamma_A(X)$ and $W_* = \Gamma_A(Y)$ for $X, Y \in \mathrm{Syn}_E^{\heartsuit}$, the isomorphism $\Gamma_A(X) \otimes_{K_*} \Gamma_A(Y) \simeq \Gamma_A(X \boxtimes Y)$ supplied by Proposition 6.6.3 is $\mathrm{End}_*(A)$-linear. In other words, we can regard the functor $\Gamma_A : \mathrm{Syn}_E^{\heartsuit} \simeq \mathrm{LMod}^{\mathrm{gr}}_{\mathrm{End}_*(A))}$ of Proposition 6.4.12 as an equivalence of monoidal categories, where the monoidal structure on $\mathrm{LMod}^{\mathrm{gr}}_{\mathrm{End}_*(A)}$ is obtained from the comultiplication on $\mathrm{End}_*(A)$.

Proof of Theorem 6.6.6. Let A be an atomic algebra object of $\mathrm{Syn}_E^{\heartsuit}$. Combining Propositions 6.5.1 and 5.2.4, we can choose a bialgebra isomorphism $\mathrm{End}_*(A) \simeq \bigwedge^*_{K_*}(W)$, where W is some finite-dimensional vector space over κ. Applying Proposition 6.4.12, we obtain an equivalence of monoidal categories $F : \mathrm{Syn}_E^{\heartsuit} \simeq \mathcal{M}(W)$, hence an isomorphism of vector spaces

$$DF : \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\psi} \mathrm{Ext}^1_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[1], \mathbf{1}^{\heartsuit}) \xrightarrow{F} \mathrm{Ext}^1_{\mathcal{M}(W)}(K_*[1], K_*) \simeq W^{\vee}.$$

Composing F with the monoidal equivalence $\mathcal{M}(W) \simeq \mathcal{M}((\mathfrak{m}/\mathfrak{m}^2)^{\vee})$ determined by DF, we can reduce to the case where $W = (\mathfrak{m}/\mathfrak{m}^2)^{\vee}$ and F is normalized. □

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6.7 Milnor Modules Associated to a Polarization

Fix a Lubin-Tate spectrum E with residue field κ. Let $\mathfrak{m} \subseteq \pi_0 E$ denote the maximal ideal, and let K_* denote the graded ring $(\pi_* E)/\mathfrak{m}(\pi_* E)$.

Construction 6.7.1 (The Milnor Module of a Polarization)**.** Let Λ be a lattice, let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization, and let Th_Q denote the Thom spectrum of Q (see Definition 3.2.1). We let A_Q denote the Milnor module $\mathrm{Sy}[\mathrm{Th}_Q]$ associated to the Thom spectrum Th_Q, and we let A_Q^{red} denote the Milnor module $\mathrm{Sy}[\mathrm{Th}_Q^{\mathrm{red}}]$ of the reduced Thom spectrum $\mathrm{Th}_Q^{\mathrm{red}}$ (see Variant 3.2.2). We regard A_Q as an associative algebra object of the abelian category $\mathrm{Syn}_E^{\heartsuit}$ of Milnor modules.

Our goal in this section is to relate the quadratic coefficient c_2^Q of a polarization $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ (in the sense of Construction 3.2.8) to the algebra structure on the Milnor module A_Q. Roughly speaking, our main result (Proposition 6.7.15) articulates the idea that c_2^Q measures the noncommutativity of the algebra A_Q. This result will be used in §6.8 to characterize those polarizations Q for which the Thom spectrum Th_Q is an Azumaya algebra, but otherwise plays no role in this paper.

Remark 6.7.2. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization. Then the Thom spectrum Th_Q is nonzero (Proposition 3.3.2), so the algebra $A_Q \in \mathrm{Syn}_E^{\heartsuit}$ is also nonzero. It follows that the unit map $e : \mathbf{1}^{\heartsuit} \to A_Q$ is nonzero, and therefore a monomorphism (since $\mathbf{1}^{\heartsuit}$ is a simple object of the abelian category $\mathrm{Syn}_E^{\heartsuit}$). The fiber sequence of spectra $E \to \mathrm{Th}_Q \to \mathrm{Th}_Q^{\mathrm{red}}$ determines a long exact sequence

$$\mathbf{1}^{\heartsuit} \xrightarrow{e} A_Q \to A_Q^{\mathrm{red}} \to \mathbf{1}^{\heartsuit}[1] \xrightarrow{e[1]} A_Q[1]$$

in the abelian category $\mathrm{Syn}_E^{\heartsuit}$. Using the injectivity of e, we obtain a short exact sequence $0 \to \mathbf{1}^{\heartsuit} \xrightarrow{e} A_Q \to A_Q^{\mathrm{red}} \to 0$: that is, we can identify A_Q^{red} with the cokernel of the unit map $e : \mathbf{1}^{\heartsuit} \to A_Q$.

Example 6.7.3. Let $\Lambda = \mathbf{Z}$ and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. Then we have a canonical isomorphism $A_Q^{\mathrm{red}} \simeq \mathrm{Sy}[\Sigma E] = \mathbf{1}^{\heartsuit}[1]$ (see Variant 3.3.8). Beware that this isomorphism depends on the choice of identification of Λ with $\mathbf{Z}$.

Example 6.7.4. Let $\Lambda = \mathbf{Z}$ and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be the constant map taking the value $E \in \mathrm{Pic}(E)$. Then the Thom spectrum Th_Q is equipped with an augmentation $\epsilon : \mathrm{Th}_Q \to E$, and therefore splits as a direct sum of E (via the unit map $E \to \mathrm{Th}_Q$) and $\mathrm{fib}(\epsilon) \simeq \Sigma E$. It follows that A_Q splits as a direct sum $A_Q \simeq \mathbf{1}^{\heartsuit} \oplus \mathbf{1}^{\heartsuit}[1]$ in the abelian category $\mathrm{Syn}_E^{\heartsuit}$, where $\mathbf{1}^{\heartsuit}[1]$ is the kernel of an algebra map $A_Q \to \mathbf{1}^{\heartsuit}$ and is therefore closed under multiplication. Since $\mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[1] \boxtimes \mathbf{1}^{\heartsuit}[1], \mathbf{1}^{\heartsuit}[1]) \simeq 0$, it follows that the direct sum decomposition $A_Q \simeq \mathbf{1}^{\heartsuit} \oplus \mathbf{1}^{\heartsuit}[1]$ exhibits A_Q as the trivial square zero extension of $\mathbf{1}^{\heartsuit}$ by the module $\mathbf{1}^{\heartsuit}[1]$.

Construction 6.7.5 (Derivations of A_Q)**.** Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization. For each element $\lambda^{\vee}$ of the dual lattice $\Lambda^{\vee}$, the map $D_{\lambda^\vee} : \mathrm{Th}_Q \to \Sigma \mathrm{Th}_Q$ of Remark 3.2.3 induces a map of Milnor modules $d_{\lambda^\vee} : A_Q \to A_Q[1]$.

Proposition 6.7.6. *Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization. For each $\lambda^\vee$ in the dual lattice $\Lambda^\vee$, the map $d_{\lambda^\vee} : A_Q \to A_Q[1]$ is a derivation (of degree (-1)) in the category $\mathrm{Syn}_E^\heartsuit$: that is, it satisfies the equation*

$$d_{\lambda^\vee} \circ m = m \circ (d_{\lambda^\vee} \boxtimes \mathrm{id}) + m \circ (\mathrm{id} \boxtimes d_{\lambda^\vee}) \in \mathrm{Hom}_{\mathrm{Syn}_E^\heartsuit}(A_Q \boxtimes A_Q, A_Q[1]),$$

where m denotes the multiplication on A_Q.

Proof. We let Q^+ denote the composite map $K(\Lambda \times \mathbf{Z}, 1) \to K(\Lambda, 1) \xrightarrow{Q} \mathrm{Pic}(E)$, which we regard as a polarization of the lattice $\Lambda \times \mathbf{Z}$. Let Th_{Q^+} denote the Thom spectrum of Q^+, which we regard as an algebra object of Mod_E. Since the formation of Thom spectra is symmetric monoidal, we can identify Th_{Q^+} with the tensor product $\mathrm{Th}_Q \otimes_E \mathrm{Th}_{Q_0}$ as objects of Alg_E, where $Q_0 : K(\mathbf{Z}, 1) \to \mathrm{Pic}(E)$ is the constant map taking the value $E \in \mathrm{Pic}(E)$. Combining this observation with the analysis of Example 6.7.4, we obtain a canonical isomorphism $A_{Q^+} \simeq A_Q \oplus A_Q[1]$ in $\mathrm{Alg}(\mathrm{Syn}_E^\heartsuit)$, where the right hand side denotes the trivial square-zero extension of A_Q by the shift $A_Q[1]$ (regarded as a bimodule over A_Q).

Let $\lambda^\vee$ be an element of the dual lattice $\Lambda^\vee$. Then the construction $(\lambda \in \Lambda) \mapsto ((\lambda, \langle \lambda, \lambda^\vee \rangle) \in \Lambda \times \mathbf{Z})$ determines a section of the projection map $\Lambda \times \mathbf{Z} \to \Lambda$, and therefore induces a map of Thom spectra $\mathrm{Th}_Q \to \mathrm{Th}_{Q^+}$, hence a map of Milnor modules

$$\phi : A_Q \to A_{Q^+} \simeq A_Q \oplus A_Q[1].$$

Using the description of the cap product given in Remark 3.1.7, we see that ϕ is given by $(\mathrm{id}, d_{\lambda^\vee})$. Since ϕ is a morphism of algebra objects, it follows that $d_{\lambda^\vee}$ is a derivation. $\square$

Remark 6.7.7. In the situation of Construction 6.7.5, the derivations $d_{\lambda^\vee}$ satisfy the equations

$$d_{\lambda^\vee + \lambda'^\vee} = d_{\lambda^\vee} + d_{\lambda'^\vee} \qquad d_{\lambda^\vee}^2 = 0.$$

These relations follow immediately from the analogous assertions for the maps $D_{\lambda^\vee} : \mathrm{Th}_Q \to \Sigma \mathrm{Th}_Q$ of Remark 3.2.3.

Construction 6.7.8. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. For each $\lambda \in \Lambda$, we let $\alpha(\lambda) \in \mathrm{Hom}_{\mathrm{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit[1], A_Q^{\mathrm{red}})$ denote the composition of the isomorphism $\mathbf{1}^\heartsuit[1] \simeq A_{Q[\lambda]}^{\mathrm{red}}$ of Example 6.7.3 with the natural map $A_{Q[\lambda]}^{\mathrm{red}} \to A_Q^{\mathrm{red}}$.

In the situation of Construction 6.7.8, the map α is given by the composition

$$\Lambda \xrightarrow{\tilde{\alpha}} \pi_1 \mathrm{Th}_Q^{\mathrm{red}} \to \mathrm{Hom}_{\mathrm{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit[1], A_Q^{\mathrm{red}}),$$

where $\widetilde{\alpha}$ is defined as in Construction 3.3.9. Applying Proposition 3.3.10, we obtain the following:

Proposition 6.7.9. *Let Λ be a lattice and let $Q : K(\Lambda,1) \to \mathrm{Pic}(E)$ be a nonsingular polarization. Then the map*

$$\alpha : \Lambda \to \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}, A_Q^{red})$$

of Construction 6.7.8 is a group homomorphism.

Remark 6.7.10. In the situation of Proposition 6.7.9, suppose that Q fails to be nonsingular. Then $\mathrm{Th}_Q \simeq 0$ (Proposition 3.3.2), so Remark 6.7.2 supplies a canonical equivalence $A_Q^{\mathrm{red}} \simeq \mathbf{1}^{\heartsuit}[1]$. Using this equivalence, we can identify α with the function $\Lambda \to \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[1], \mathbf{1}^{\heartsuit}[1]) \simeq \kappa$ given by the composition

$$\Lambda \xrightarrow{c_1^Q} (\pi_0 E)^{\times} \xrightarrow{x \mapsto x-1} \pi_0 E \to \kappa.$$

In this case, α need not be a group homomorphism.

Remark 6.7.11. Let Λ be a lattice, let $Q : K(\Lambda,1) \to \mathrm{Pic}(E)$ be a nonsingular polarization, and let $\lambda^{\vee} \in \Lambda^{\vee}$ be an element of the dual lattice. Then the derivation $d_{\lambda^{\vee}} : A_Q \to A_Q[1]$ automatically annihilates the unit map $\mathbf{1}^{\heartsuit} \to A_Q$, and therefore factors through a map $A_Q^{\mathrm{red}} \to A_Q[1]$. For any element $\lambda \in \Lambda$, the composite map

$$\mathbf{1}^{\heartsuit}[1] \xrightarrow{\alpha(\lambda)} A_Q^{\mathrm{red}} \xrightarrow{d_{\lambda^{\vee}}} A_Q[1]$$

is obtained by multiplying the unit map $\mathbf{1}^{\heartsuit} \to A_Q$ by the integer $\langle \lambda, \lambda^{\vee} \rangle$ (this follows from the functoriality of Construction 6.7.8).

Definition 6.7.12. Let Λ be a lattice, let $Q : K(\Lambda,1) \to \mathrm{Pic}(E)$ be a polarization of Λ, and let c_2^Q be the second coefficient of Q (Construction 3.2.8), which we regard as a map $\mathrm{Sym}^2(\Lambda) \to \pi_4 \,\mathrm{BPic}(E) \simeq \pi_2 E$. We let $b^Q : \Lambda \times \Lambda \to \pi_2 K$ denote the symmetric bilinear form given by the composition

$$\Lambda \times \Lambda \to \mathrm{Sym}^2(\Lambda) \xrightarrow{c_2^Q} \pi_2 E \to \pi_2 K.$$

We now describe the bilinear form b^Q more explicitly in terms of the algebra $A_Q \in \mathrm{Syn}_E^{\heartsuit}$.

Construction 6.7.13 (The Commutator Bracket)**.** Let Λ be a lattice and let $Q : K(\Lambda,1) \to \mathrm{Pic}(E)$ be a nonsingular polarization, so that A_Q is an associative algebra object of $\mathrm{Syn}_E^{\heartsuit}$. Let $m : A_Q \boxtimes A_Q \to A_Q$ denote the multiplication on A_Q, and let

$m^{\mathrm{op}} : A_Q \boxtimes A_Q \to A_Q$ be the opposite multiplication (that is, the composition of m with the automorphism of $A_Q \boxtimes A_Q$ given by swapping the two factors). We regard the difference $m - m^{\mathrm{op}}$ as a morphism $[\bullet, \bullet] : A_Q \boxtimes A_Q \to A_Q$ in the abelian category $\mathrm{Syn}_E^{\heartsuit}$, which we will refer to as the *commutator bracket*. Note that the commutator bracket annihilates the subobjects

$$\mathbf{1}^{\heartsuit} \boxtimes A_Q, A_Q \boxtimes \mathbf{1}^{\heartsuit} \subseteq A_Q \boxtimes A_Q,$$

and therefore factors uniquely through a map $A_Q^{\mathrm{red}} \boxtimes A_Q^{\mathrm{red}} \to A_Q$, which we will denote also by $[\bullet, \bullet]$.

Remark 6.7.14. In the situation of Construction 6.7.13, suppose we are given a monoidal functor $F : \mathrm{Syn}_E^{\heartsuit} \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$. Then $B = F(A_Q)$ inherits the structure of a graded K_*-algebra, and the commutator bracket map $[\bullet, \bullet] : A_Q \boxtimes A_Q \to A_Q$ determines a map $s : B \otimes_{K_*} B \to B$. If the functor F is symmetric monoidal, then we can identify s with the "super-commutator": that is, it is given by the formula $s(x \otimes y) = xy - (-1)^{ij} yx$ for $x \in B_i$ and $y \in B_j$. Beware that if F is not assumed to be symmetric monoidal, then it is not possible to describe the map s using only the algebra structure of B.

We can now formulate our main result:

Proposition 6.7.15. *Let Λ be a lattice, let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization, let $\alpha : \Lambda \to \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[1], A_Q^{red})$ be as in Construction 6.7.8, and let $b^Q : \Lambda \times \Lambda \to \pi_2 K$ be the bilinear form of Definition 6.7.12. Then the diagram*

$$\begin{array}{ccc}
\Lambda \times \Lambda & \xrightarrow{\quad b^Q \quad} & \pi_2 K \\
\downarrow{\scriptstyle \alpha \times \alpha} & & \downarrow \\
\mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[1], A_Q^{red}) \times \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[1], A_Q^{red}) & & \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[2], \mathbf{1}^{\heartsuit}) \\
\downarrow{\scriptstyle \boxtimes} & & \downarrow \\
\mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[2], A_Q^{red} \boxtimes A_Q^{red}) & \xrightarrow{[\bullet,\bullet]} & \mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[2], A_Q)
\end{array}$$

commutes.

Remark 6.7.16. In the situation of Proposition 6.7.15, suppose we are given an exact $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear monoidal functor $F : \mathrm{Syn}_E^{\heartsuit} \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$, and set $B = F(A_Q)$. The exact sequence of Milnor modules

$$0 \to \mathbf{1}^{\heartsuit} \to A_Q \to A_Q^{\mathrm{red}} \to 0$$

then determines an exact sequence of graded K_*-modules $0 \to K_* \to B \to F(A_Q^{\mathrm{red}}) \to 0$. For each $\lambda \in \Lambda$, we can identify $F(\alpha(\lambda))$ with an element of $F(A_Q^{\mathrm{red}})$ which is homogeneous of degree 1, which can be lifted uniquely to an element $a(\lambda) \in B_1$. In this case, Proposition 6.7.15 supplies a formula $s(a(\lambda) \otimes a(\lambda')) = b^Q(\lambda, \lambda')$ in B, where both sides are homogeneous of degree 2 (here $s : B \otimes_{K_*} B \to B$ is the map of Remark 6.7.14). If the functor F is assumed to be *symmetric* monoidal, then we can rewrite this formula as $a(\lambda)a(\lambda') + a(\lambda')a(\lambda) = b^Q(\lambda, \lambda')$.

Proof of Proposition 6.7.15. Choose elements $\lambda, \lambda' \in \Lambda$; we wish to prove the identity

$$[\alpha(\lambda), \alpha(\lambda')] = b^Q(\lambda, \lambda')$$

in the abelian group $\mathrm{Hom}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}[2], A_Q)$. By functoriality, we can reduce to the case where λ and λ' form a basis for Λ. Let $\lambda^\vee$ and $\lambda'^\vee$ be the dual basis for $\Lambda^\vee$. Let $m, m^{\mathrm{op}} : A_{Q[\lambda]} \boxtimes A_{Q[\lambda']} \to A_Q$ denote the isomorphisms induced by the multiplication on A_Q and its opposite, respectively. Using Example 3.3.6, we see that m^{op} is given by composing m with the map $\mathrm{id} - b^Q(\lambda, \lambda') d_{\lambda^\vee} d_{\lambda'^\vee}$. It follows that the composite map

$$A_{Q[\lambda]} \boxtimes A_{Q[\lambda']} \to A_Q \boxtimes A_Q \xrightarrow{[\bullet,\bullet]} A_Q$$

is equal to the the composition

$$A_{Q[\lambda]} \boxtimes A_{Q[\lambda']} \xrightarrow{m} A_Q \xrightarrow{d_{\lambda^\vee} d_{\lambda'^\vee}} A_Q[2] \xrightarrow{b^Q(\lambda,\lambda')} A_Q.$$

Using the fact that $d_{\lambda^\vee}$ and $d_{\lambda'^\vee}$ are derivations which vanish on $A_{Q[\lambda']}$ and $A_{Q[\lambda]}$, respectively, we can rewrite this composition as

$$A_{Q[\lambda]} \boxtimes A_{Q[\lambda']} \xrightarrow{d_{\lambda^\vee} \boxtimes d_{\lambda'^\vee}} A_{Q[\lambda]}[1] \boxtimes A_{Q[\lambda']}[1] \xrightarrow{m} A_Q[2] \xrightarrow{b^Q(\lambda,\lambda')} A_Q.$$

Combining this observation with Remark 6.7.11, we see that the restriction of the map $[\bullet, \bullet] : A_Q^{\mathrm{red}} \boxtimes A_Q^{\mathrm{red}} \to A_Q$ to $A_{Q[\lambda]}^{\mathrm{red}} \boxtimes A_{Q[\lambda']}^{\mathrm{red}}$ is given by the composition

$$A_{Q[\lambda]}^{\mathrm{red}} \boxtimes A_{Q[\lambda']}^{\mathrm{red}} \simeq \mathbf{1}^{\heartsuit}[1] \boxtimes \mathbf{1}^{\heartsuit}[1] \xrightarrow{b^Q(\lambda,\lambda')} \mathbf{1}^{\heartsuit} \to A_Q,$$

from which we conclude that $[\alpha(\lambda), \alpha(\lambda')] = b^Q(\lambda, \lambda')$ as desired. □

6.8 Nondegenerate Polarizations

Fix a Lubin-Tate spectrum E with residue field κ. Let $\mathfrak{m} \subseteq \pi_0 E$ denote the maximal ideal, and let K_* denote the graded ring $(\pi_* E)/\mathfrak{m}(\pi_* E)$.

Definition 6.8.1. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. We will say that Q is *nondegenerate* if it is nonsingular and the bilinear form $b^Q : \Lambda \times \Lambda \to \pi_2 K$ of Definition 6.7.12 is nondegenerate (that is, the bilinear form b^Q induces a vector space isomorphism $\kappa \otimes_{\mathbf{Z}} \Lambda \to (\pi_2 K) \otimes_{\mathbf{Z}} \Lambda^\vee$.

Our goal in this section is to prove the following:

Theorem 6.8.2. *Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization. Then Q is nondegenerate if and only if the Thom spectrum Th_Q is an Azumaya algebra over E.*

The proof of Theorem 6.8.2 will require some preliminaries.

Notation 6.8.3. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization. For each $\lambda \in \Lambda$, we let $[\alpha(\lambda), \bullet] : A_Q \to A_Q[-1]$ denote the map given by the composition

$$A_Q \simeq \mathbf{1}^\heartsuit[1] \boxtimes A_Q[-1] \xrightarrow{\alpha(\lambda) \boxtimes \mathrm{id}} A_Q^{\mathrm{red}} \boxtimes A_Q[-1] \xrightarrow{[\bullet,\bullet]} A_Q[-1].$$

Proposition 6.8.4. *Let Λ be a lattice, let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a nonsingular polarization For each $\lambda \in \Lambda$, the map $[\alpha(\lambda), \bullet] : A_Q \to A_Q[-1]$ of Notation 6.8.3 can be identified with the image of λ under the composite map*

$$\Lambda \xrightarrow{u} (\pi_2 K) \otimes_{\mathbf{Z}} \Lambda^\vee \xrightarrow{v} (\pi_2 K) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathrm{Syn}_E^\heartsuit}(A_Q, A_Q[1]) \simeq \mathrm{Hom}_{\mathrm{Syn}_E^\heartsuit}(A_Q, A_Q[-1]),$$

where u is the map classifying the bilinear form $b^Q : \Lambda \times \Lambda \to \pi_2 K$ of Definition 6.7.12, and v is induced by the homomorphism $\lambda^\vee \mapsto d_{\lambda^\vee}$.

Proof. Let $D : A_Q \to A_Q[-1]$ denote the difference between $[\alpha(\lambda), \bullet]$ and $(v \circ u)(\lambda)$; we wish to show that $D = 0$. Choose a basis $\lambda_1, \ldots, \lambda_n$ for the lattice Λ, so that the multiplication on A_Q induces an isomorphism $A_{Q[\lambda_1]} \boxtimes \cdots \boxtimes A_{Q[\lambda_n]} \to A_Q$. Since D is a derivation, it will suffice to show that D vanishes on each $A_{Q[\lambda_i]}$. Factoring D as a composition $A_Q \to A_Q^{\mathrm{red}} \xrightarrow{D^{\mathrm{red}}} A_Q[-1]$, we are reduced to showing that the composition $\mathbf{1}^\heartsuit[1] \xrightarrow{\alpha(\lambda_i)} A_Q^{\mathrm{red}} \to D^{\mathrm{red}} A_Q[-1]$ vanishes, which follows immediately from Proposition 6.7.15 and Remark 6.7.11. □

Proof of Theorem 6.8.2. Let $Q : K(\Lambda, 1) \to \mathrm{Pic}(E)$ be a polarization of a lattice Λ. We wish to show that Q is nondegenerate if and only if the Thom spectrum Th_Q is an Azumaya algebra. Without loss of generality we may assume that Q is nonsingular (Proposition 3.3.2). In this case, Th_Q is nonzero and dualizable as an E-module (Remark 3.2.5). It will therefore suffice to prove that the following assertions are equivalent:

(1) The left and right actions of A_Q on itself induce an isomorphism $\xi : A_Q \boxtimes A_Q^{\mathrm{op}} \to \mathrm{End}(A_Q)$ in the abelian category $\mathrm{Syn}_E^{\heartsuit}$.

(2) The bilinear form b^Q of Definition 6.7.12 is nondegenerate.

Using Theorem 6.6.6, we can choose an exact $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear monoidal functor $F : \mathrm{Syn}_E^{\heartsuit} \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$ (beware that if the characteristic of κ is 2, we cannot necessarily arrange that F is *symmetric* monoidal). Set $A' = F(A_Q)$ and $A'^{\mathrm{red}} = F(A_Q^{\mathrm{red}})$, so that we can regard A' as a graded K_*-algebra. The map α of Construction 6.7.8 determines a group homomorphism $\alpha' : \Lambda \to \mathrm{Hom}_{K_*}(K_*[1], A'^{\mathrm{red}}) = A_1'^{\mathrm{red}} \simeq A_1$. Let $\xi' = F(\xi)$, which we regard as a morphism of graded K_*-modules $A' \otimes_{K_*} A' \to \mathrm{End}_{K_*}(A')$. Since F is a monoidal functor, the restriction of ξ' to the first tensor factor $A' \simeq A' \otimes_{K_*} K_* \subseteq A' \otimes_{K_*} A'$ is an algebra homomorphism, which classifies the left action of A' on itself (beware that since F is not necessarily *symmetric* monoidal, we cannot assume that ξ' is an algebra homomorphism, or that the restriction of ξ' to the second factor classifies the right action of A' on itself). Since ξ is an algebra homomorphism, the image of ξ is a subalgebra of $\mathrm{End}(A_Q)$. It follows that $\mathrm{Im}(\xi') = F(\mathrm{Im}(\xi))$ is also a subalgebra of $\mathrm{End}_{K_*}(A')$.

Choose a basis $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$. For $1 \leqslant i \leqslant n$, set $A'(i) = F(A_{Q[\lambda_i]})$ and $a_i = \lambda(\alpha_i)$. It follows from Remark 3.3.4 that the multiplication on A' induces an isomorphism

$$A'(1) \otimes_{K_*} \cdots \otimes_{K_*} A'(n) \to A'.$$

It follows that A' is freely generated, as a module over K_*, by the ordered products $a_I = a_{i_1} \cdots a_{i_m} \in A'_m$, where $I = \{i_1 < \cdots < i_m\}$ ranges over all subsets of $\{1, \ldots, n\}$. It follows that the construction $(\lambda \in \Lambda) \mapsto \alpha'(\lambda) \otimes 1 - 1 \otimes \alpha'(\lambda)$ induces a monomorphism of vector spaces $\kappa \otimes_{\mathbf{Z}} \Lambda \to (A' \otimes_{K_*} A')_1$. Consequently, if the map ξ' is an isomorphism, then the map

$$(\lambda \in \Lambda) \mapsto \xi'(\alpha'(\lambda) \otimes 1 - 1 \otimes \alpha'(\lambda)) = F([\alpha(\lambda, \bullet)])$$

induces a monomorphism $\kappa \otimes_{\mathbf{Z}} \Lambda \to \mathrm{Hom}_{K_*}(A', A'[-1])$. It follows from Proposition 6.8.4 that this map factors through the map $u : \kappa \otimes_{\mathbf{Z}} \Lambda \to (\pi_2 K) \otimes_{\mathbf{Z}} \Lambda^{\vee}$ determined by the bilinear form b^Q, so that b^Q is nondegenerate. This shows that (1) implies (2).

We now complete the proof by showing that (2) $\Rightarrow$ (1). Assume that the bilinear form b^Q is nondegenerate; we wish to show that ξ is an isomorphism in $\mathrm{Syn}_E^{\heartsuit}$. Let $\lambda_1^{\vee}, \ldots, \lambda_n^{\vee} \in \Lambda^{\vee}$ be the dual basis of $\lambda_1, \ldots, \lambda_n \in \Lambda$. For $1 \leqslant i \leqslant n$, let $d'_i : A' \to A'[1]$ be the image under the functor F of the derivation $d_{\lambda_i^{\vee}} : A_Q \to A_Q[1]$. Then each d'_i is a derivation of the graded K_*-algebra A', which is homogeneous of degree 1. Using the calculation of Remark 6.7.11, we compute $d'_i(a_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$ For each subset

$I = \{i_1 < i_2 < \cdots < i_m\} \subseteq \{1, \ldots, n\}$, set $d'_I = d'_{i_m} \circ \cdots \circ d'_{i_1} \in \mathrm{End}_{K_*}(A')_{-|I|}$. Note that for $J \subseteq \{1, \ldots, n\}$, we have

$$d'_I(a_J) = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J, |I| \geqslant |J|. \end{cases}$$

Using the nondegeneracy of b^Q and Proposition 6.8.4, we conclude that each $d^\vee_{\lambda_i}$ belongs to the image of the map ξ, so that d'_i belongs to the image of the map ξ'. Note that the domain and codomain of ξ' are free K_*-modules of the same rank. Consequently, to show that ξ' is an isomorphism, it will suffice to show that ξ' is surjective. Let f be a homogeneous element of $\mathrm{End}_{K_*}(A')$; we wish to show that f belongs to the image of the map ξ'. If $f = 0$, there is nothing to prove. Otherwise, there exists a subset $I \subseteq \{1, \ldots, n\}$ such that $f(a_I) \neq 0$. Choose I such that $m = |I|$ is as small as possible. We proceed by descending induction on m. For every homogeneous element x of A', let $l_x \in \mathrm{End}_{K_*}(A')$ be the map given by left multiplication by x. Define $f' \in \mathrm{End}_{K_*}(A')$ by the formula $f' = \sum_{J \subseteq \{1,\ldots,m\}} l_{f(a_J)} d'_J$. Then f' is a homogeneous element of $\mathrm{End}_{K_*}(A')$ (of the same degree as f). Moreover, since the image of ξ' is a subalgebra of A' which contains each d'_i and each l_x, the endomorphism f' belongs to the image of ξ'. Consequently, to show that f belongs to the image of ξ', it will suffice to show that $f - f'$ belongs to the image of ξ'. This follows from our inductive hypothesis, since $(f - f')(a_J) = 0$ whenever $|J| \leqslant m$. □

6.9 The Case of an Odd Prime

Let E be a Lubin-Tate spectrum, let $\mathfrak{m} \subseteq \pi_0 E$ be the maximal ideal, and let $V = (\mathfrak{m}/\mathfrak{m}^2)$ denote the Zariski tangent space of the Lubin-Tate ring $\pi_0 E$. If the residue field $\kappa = (\pi_0 E)/\mathfrak{m}$ has characteristic $\neq 2$, then we can promote the equivalence Theorem 6.6.6 to a *symmetric* monoidal functor.

Proposition 6.9.1. *Suppose that the residue field κ has characteristic $\neq 2$. Then there exists a normalized $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear symmetric monoidal equivalence of categories $F : \mathrm{Syn}^{\heartsuit}_E \simeq \mathcal{M}(V)$. Moreover, the equivalence F is unique (up to unique isomorphism).*

Proof. Using Proposition 3.5.1, we can choose an atomic commutative algebra $\overline{A}$ in the homotopy category $\mathrm{hMod_E}$. Let $A = \mathrm{Sy}^{\heartsuit}[\overline{A}]$ denote the associated Milnor module, so that Construction 6.4.8 supplies a symmetric monoidal equivalence $\Gamma_A : \mathrm{Syn}^{\heartsuit}_E \to \mathrm{Mod}^{\mathrm{gr}}_{\mathrm{End}_*(A)}$ (see Remark 6.6.2). Arguing as in the proof of Theorem 6.6.6, we can identify $\mathrm{Mod}^{\mathrm{gr}}_{\mathrm{End}_*(A)}$ with the category $\mathcal{M}(V)$ so that the functor F is normalized. This proves the existence of F.

Now suppose that $F' : \mathrm{Syn}_E^{\heartsuit} \to \mathcal{M}(V)$ is another normalized $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear symmetric monoidal equivalence. Then $F' \circ F^{-1}$ is a normalized symmetric monoidal $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear equivalence of the ∞-category $\mathcal{M}(V)$ with itself. Using Proposition 5.6.5, we can choose an isomorphism $F' \circ F^{-1} \simeq \mu_B^{\partial}$ for some atomic algebra $B \in \mathrm{Alg}(\mathcal{M}(V))$. Write $B = \mathrm{Cl}_q(V^{\vee})$ for some quadratic form $q : V^{\vee} = \mathfrak{m}/\mathfrak{m}^2 \to K_2$ (Proposition 5.3.5). Using our assumption that the functor $F' \circ F^{-1}$ is *symmetric* monoidal, we deduce that B is a commutative algebra object of $\mathcal{M}(V)$, so that the quadratic form q vanishes. It follows that there is a unique isomorphism of $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$-linear symmetric monoidal functors $\mu_B^{\partial} \simeq \mathrm{id}$, and therefore a unique isomorphism $F \simeq F'$. □

Corollary 6.9.2. *Suppose that the residue field κ has characteristic $\neq 2$. Then there exists a unique atomic commutative algebra object of the homotopy category* $\mathrm{hMod}_{\mathrm{E}}$ *(up to unique isomorphism).*

Proof of Corollary 6.9.2. Using Proposition 6.9.1, we are reduced to showing that the category $\mathcal{M}(V)$ contains a unique atomic commutative algebra object (up to unique isomorphism). Using Proposition 5.3.5, we see that every atomic algebra object of $\mathcal{M}(V)$ is isomorphic (in a unique way) to a Clifford algebra $\mathrm{Cl}_q(V^{\vee})$; such an algebra is commutative if and only if $q = 0$. □

Using Corollary 6.9.2, we can give a concrete description of the Brauer-Milnor group $\mathrm{BM}(E)$ at odd primes:

Proposition 6.9.3. *Let E be a Lubin-Tate spectrum whose residue field has characteristic different from 2. Then there is a canonical isomorphism $\rho : \mathrm{BM}(E) \simeq \mathrm{Br}(\mathrm{Mod}_{K_*}^{\mathrm{gr}}) \times \mathrm{QF}$, where* QF *denotes the set of quadratic forms $q : (\mathfrak{m}/\mathfrak{m}^2)^{\vee} \to K_{-2}$.*

Proof. Combine Theorem 5.7.6 with Proposition 6.9.1. □

Remark 6.9.4. A choice of nonzero element $t \in K_2$ determines an isomorphism of Brauer groups $\mathrm{Br}(\mathrm{Mod}_{K_*}^{\mathrm{gr}}) \simeq \mathrm{BW}(\kappa)$, and an isomorphism $\mathrm{QF} \simeq \mathfrak{m}^2/\mathfrak{m}^3$. In this case, Proposition 6.9.3 supplies an isomorphism

$$\mathrm{BM}(E) \simeq \mathrm{BW}(\kappa) \times \mathfrak{m}^2/\mathfrak{m}^3.$$

Beware that this isomorphism depends on the choice of t.

Remark 6.9.5. Let E be a Lubin-Tate spectrum of arbitrary residue characteristic. The fully faithful embedding $\mathrm{Mod}_{K_*}^{\mathrm{gr}} \to \mathrm{Syn}_E^{c}$ constructed in §6.3 induces a monomorphism of Brauer groups

$$\iota : \mathrm{Br}(\mathrm{Mod}_{K_*}^{\mathrm{gr}}) \to \mathrm{Br}(\mathrm{Syn}_E^{\heartsuit}) = \mathrm{BM}(E),$$

whose domain can be identified with the Brauer-Wall group $\mathrm{BW}(\kappa)$ (by choosing a nonzero element $t \in \pi_2 K$). Using Remark 2.3.3, we see that ι fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) & \xrightarrow{\iota} & \mathrm{BM}(E) \\ \downarrow \eta & & \downarrow \eta \\ \pi_0 \operatorname{Pic}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) & \longrightarrow & \pi_0 \operatorname{Pic}(\mathrm{Syn}^{\heartsuit}_E). \end{array}$$

Here the bottom vertical map is an isomorphism (Corollary 6.6.8), with both groups being isomorphic to $\mathbf{Z}/2\mathbf{Z}$. Using the classical theory of the Brauer-Wall group, one can show that the left vertical map is surjective when κ has odd characteristic, but vanishes when κ has characteristic 2. However, one can show that the right vertical map is *always* surjective. Consequently, it is possible to view the failure of Proposition 6.9.3 in characteristic 2 as a feature, rather than a bug. The Brauer-Wall group exhibits some degenerate behavior over fields of characteristic 2 that is not shared by the Brauer-Milnor group, so we cannot expect to reduce the latter to the former.

By virtue of Theorem 6.8.2, the Thom spectrum construction $Q \mapsto \mathrm{Th}_Q$ can be used to produce many examples of Azumaya algebras over E. Using the isomorphism of Proposition 6.9.3, we can describe their images in the Brauer-Morava group $\mathrm{BM}(E)$.

Proposition 6.9.6. *Assume that the residue field κ of E has characteristic different from 2. Let Λ be a lattice and let $Q : K(\Lambda, 1) \to \operatorname{Pic}(E)$ be a nondegenerate polarization, so that A_Q is an Azumaya algebra object of $\mathrm{Syn}^{\heartsuit}_E$ representing a Brauer class $[A_Q] \in \mathrm{BM}(E)$. Then $\rho([A_Q]) = (x, q)$, where:*

- *(a) The class $x \in \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$ is represented by the Clifford algebra $\mathrm{Cl}_u(\kappa \otimes_{\mathbf{Z}} \Lambda)$ of the quadratic form $u : (\kappa \otimes_{\mathbf{Z}} \Lambda) \to \pi_2 K$ associated to the bilinear form $b^Q : \Lambda \times \Lambda \to \pi_2 K$ of Definition 6.7.12 (so that $u(\lambda) = \frac{1}{2} b^Q(\lambda, \lambda)$ for each $\lambda \in \Lambda$).*
- *(b) The quadratic form $q : (\mathfrak{m}/\mathfrak{m}^2)^\vee \to \pi_{-2} K$ is given by the composition*

$$(\mathfrak{m}/\mathfrak{m}^2)^\vee \xrightarrow{\xi} \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \kappa) \to \theta \pi_{-2} K \otimes_{\mathbf{Z}} \Lambda \xrightarrow{u} \pi_{-2} K,$$

where θ is the isomorphism induced by the (nondegenerate) bilinear form b^Q and ξ is the dual of the map $\overline{c}^Q_1 : (\kappa \otimes_{\mathbf{Z}} \Lambda) \to \mathfrak{m}/\mathfrak{m}^2$ appearing in Construction 3.4.1.

Proof. Set $V = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ and let $H = \Lambda^*_{K_*}(V)$ be denote the exterior algebra over V. Since the characteristic of κ is different from 2, there is an essentially unique normalized symmetric monoidal equivalence $F : \mathrm{Syn}^{\heartsuit}_E \to \mathrm{Mod}^{\mathrm{gr}}_H$. Let $A' = F(A_Q)$ and $\alpha' : \Lambda \to A'_1$ be as in the proof of Proposition 6.8.2. Using Remark 6.7.16, we see that $\alpha'(\lambda)\alpha'(\lambda') + \alpha'(\lambda') + \alpha'(\lambda) = b^Q(\lambda, \lambda')$ for $\lambda, \lambda' \in \Lambda$, so that α' induces a graded

K_*-algebra homomorphism $\mathrm{Cl}_u(\kappa \otimes_{\mathbf{Z}} \Lambda) \to A'$. The analysis of Proposition 6.8.2 shows that this map is an isomorphism, so that $x = [A'] = [\mathrm{Cl}_u(\kappa \otimes_{\mathbf{Z}} \Lambda)]$, which proves (a).

To prove (b), we need to analyze the action of the Hopf algebra H on A'. For each $v \in V$, let $\partial_v : A' \to A'[1]$ denote the derivation (of degree -1) determined by v. Note that, for each $\lambda \in \Lambda$, the element $\partial_v(\alpha'(\lambda))$ is an element of A' which is homogeneous of degree zero which belongs to the image of the canonical map $F(A_{Q[\lambda]}) \to F(A_Q) = A'$. It follows that $\partial_v(\alpha'(\lambda)) = c_{v,\lambda}$ for some scalar $c_{v,\lambda} \in \kappa$. Our assumption that F is normalized implies that $c_{v,\lambda} = \xi(v)(\lambda)$, so that the derivation ∂_v is given by (super)-commutation with the element $\theta(\xi(v)) \in A_{-1}$. Unwinding the definitions, we obtain $q(v) = \theta(\xi(v))^2 = u(\theta(\xi(v)))$, which proves (b). $\square$

Chapter 7

Hochschild Cohomology

Let E be a Lubin-Tate spectrum and let $\mathrm{Syn}_E^{\heartsuit}$ denote the category of Milnor modules studied in §6. In this section, we associate to each algebra object $A \in \mathrm{Alg}(\mathrm{Syn}_E^{\heartsuit})$ a bigraded ring $\mathrm{HC}^{*,*}(A)$, which we call the *Hochschild cohomology* of A (Definition 7.2.1). Our main goals are to show that the Hochschild cohomology groups $\mathrm{HC}^{*,*}(A)$ control the problem of lifting A to an associative algebra object of Syn_E (see §7.3), and to compute $\mathrm{HC}^{*,*}(A)$ in the case where A is an Azumaya algebra (Proposition 7.2.7).

7.1 Digression: Modules in Syn_E and $\mathrm{Syn}_E^{\heartsuit}$

Let E be a Lubin-Tate spectrum and let Syn_E denote the ∞-category of synthetic E-modules. We can regard Syn_E as a tool for relating questions about the homotopy theory of E-module spectra to more concrete questions about linear algebra. More precisely, we have shown that the ∞-category Syn_E has two features:

(a) It contains the stable ∞-category $\mathrm{Mod}_E^{\mathrm{loc}}$ of $K(n)$-local E-modules as a full subcategory (Proposition 4.2.5).

(b) The heart of Syn_E is the abelian category $\mathrm{Syn}_E^{\heartsuit}$ of Milnor modules, which can be identified with the category of graded modules over an exterior algebra (Theorem 6.6.6).

To make effective use of (a) and (b), we would like to know to what extent the ∞-category Syn_E is determined by its heart $\mathrm{Syn}_E^{\heartsuit}$. It is not true that Syn_E can be identified with (the connective part of) the derived category of the abelian category $\mathrm{Syn}_E^{\heartsuit}$: for example, the ∞-category Syn_E is not $\mathbf{Z}$-linear. However, we will show that the next best thing is true: for every associative algebra A of $\mathrm{Syn}_E^{\heartsuit}$, the ∞-category $\mathrm{LMod}_A(\mathrm{Syn}_E)$ can be identified with (the connective part of) the derived category of the abelian category

$\mathrm{LMod}_A(\mathrm{Syn}_E^{\heartsuit})$ (Proposition 7.1.4). To prove this, we need to compare the symmetric monoidal structures on Syn_E and $\mathrm{Syn}_E^{\heartsuit}$.

Lemma 7.1.1. *Let M be an E-module. Then the canonical map $\theta : \mathrm{Sy}[M] \wedge \mathbf{1}^{\heartsuit} \to \mathrm{Sy}^{\heartsuit}[M]$ is an equivalence of synthetic E-modules.*

Proof. The map θ fits into a commutative diagram of cofiber sequences

$$\begin{array}{ccccc} \mathrm{Sy}[M] \wedge \Sigma\Omega\mathbf{1} & \longrightarrow & \mathrm{Sy}[M] \wedge \mathbf{1} & \longrightarrow & \mathrm{Sy}[M] \wedge \mathbf{1}^{\heartsuit} \\ \downarrow{\scriptstyle \theta''} & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle \theta} \\ \Sigma\Omega\,\mathrm{Sy}[M] & \longrightarrow & \mathrm{Sy}[M] & \longrightarrow & \mathrm{Sy}^{\heartsuit}[M]. \end{array}$$

Using the left exactness of the functor $M \mapsto \mathrm{Sy}^{\heartsuit}[M]$, we see that θ'' can be identified with the suspension of the canonical map $\mathrm{Sy}[M] \wedge \mathrm{Sy}[\Sigma^{-1}E] \to \mathrm{Sy}[\Sigma^{-1}M]$. Applying Proposition 4.4.7, we conclude that θ' and θ'' are equivalences, so that θ is also an equivalence. □

Proposition 7.1.2. *The inclusion functor*

$$\mathrm{Syn}_E^{\heartsuit} \simeq \mathrm{Mod}_{\mathbf{1}^{\heartsuit}}(\mathrm{Syn}_E^{\heartsuit}) \hookrightarrow \mathrm{Mod}_{\mathbf{1}^{\heartsuit}}(\mathrm{Syn}_E)$$

is symmetric monoidal. In other words, if X and Y are Milnor modules, then the relative smash product $X \wedge_{\mathbf{1}^{\heartsuit}} Y \in \mathrm{Syn}_E$ is discrete (and can therefore be identified with the Milnor module $X \boxtimes Y = \pi_0(X \wedge_{\mathbf{1}^{\heartsuit}} Y)$).

Proof. Let X and Y be Milnor modules; we will show that the Milnor modules $\pi_n(X \wedge_{\mathbf{1}^{\heartsuit}} Y)$ vanish for $n > 0$. Our proof proceeds by induction on n. Choose an exact sequence of Milnor modules $0 \to X' \to P \to X \to 0$, where P is quasi-molecular (Remark 6.2.3). We then have a long exact sequence of homotopy groups

$$\pi_n(P \wedge_{\mathbf{1}^{\heartsuit}} Y) \to \pi_n(X \wedge_{\mathbf{1}^{\heartsuit}} Y) \to \pi_{n-1}(X' \wedge_{\mathbf{1}^{\heartsuit}} Y) \to \pi_{n-1}(P \wedge_{\mathbf{1}^{\heartsuit}} Y).$$

Consequently, it will suffice to prove the following:

(i) The groups $\pi_n(P \wedge_{\mathbf{1}^{\heartsuit}} Y)$ vanish for $n > 0$.

(ii) The canonical map $\pi_0(X' \wedge_{\mathbf{1}^{\heartsuit}} Y) \to \pi_0(P \wedge_{\mathbf{1}^{\heartsuit}} Y$ is a monomorphism (in the abelian category $\mathrm{Syn}_E^{\heartsuit}$).

Assertion (ii) follows immediately from Corollary 6.6.7. To prove (i), we can replace X by P and thereby reduce to the case where $X = \mathrm{Sy}^{\heartsuit}[M]$, where M is a quasi-molecular

E-module. Similarly, we may assume that $Y = \mathrm{Sy}^{\heartsuit}[N]$, where N is a quasi-molecular E-module. In this case, we apply Lemma 7.1.1 to compute

$$\begin{aligned} X \wedge_{\mathbf{1}^{\heartsuit}} Y &\simeq (\mathrm{Sy}[M] \wedge \mathbf{1}^{\heartsuit}) \wedge_{\mathbf{1}^{\heartsuit}} (\mathrm{Sy}[N] \wedge \mathbf{1}^{\heartsuit}) \\ &\simeq \mathrm{Sy}[M] \wedge \mathrm{Sy}[N] \wedge \mathbf{1}^{\heartsuit} \\ &\simeq \mathrm{Sy}[M \otimes_E N] \wedge \mathbf{1}^{\heartsuit} \\ &\simeq \mathrm{Sy}^{\heartsuit}[M \otimes_E N]. \end{aligned}$$

□

Notation 7.1.3. Let A be an associative algebra object of the ∞-category of synthetic E-modules Syn_E. We let Syn_A denote the ∞-category $\mathrm{LMod}_A(\mathrm{Syn}_E)$ of left A-module objects of Syn_E, and we let $\mathrm{Syn}_A^{\heartsuit} = \mathrm{LMod}_A(\mathrm{Syn}_E^{\heartsuit})$ denote the full subcategory of Syn_A spanned by the discrete objects.

Proposition 7.1.4. *Let A be an algebra object of the abelian category $\mathrm{Syn}_E^{\heartsuit}$. Then the inclusion functor $\mathrm{Syn}_A^{\heartsuit} \hookrightarrow \mathrm{Syn}_A$ extends to an equivalence of ∞-categories $\mathcal{D}(\mathrm{Syn}_A^{\heartsuit})_{\geqslant 0} \simeq \mathrm{Syn}_A$.*

Proof. By construction, the ∞-category Syn_E admits compact projective generators given by $\mathrm{Sy}[M]$, where M is a molecular E-module. It follows that Syn_A admits compact projective generators given by $A \wedge \mathrm{Sy}[M]$, where M is a molecular E-module. By virtue of Proposition HA.1.3.3.7 , it will suffice to show that each of the synthetic E-modules $A \wedge \mathrm{Sy}[M]$ is discrete. Using the equivalence

$$A \wedge \mathrm{Sy}[M] \simeq A \wedge_{\mathbf{1}^{\heartsuit}} (\mathbf{1}^{\heartsuit} \wedge \mathrm{Sy}[M])$$

and invoking Proposition 7.1.2, we can reduce to the case where $A = \mathbf{1}^{\heartsuit}$, in which case the desired result follows from Lemma 7.1.1. □

7.2 Hochschild Cohomology of Milnor Modules

Throughout this section, we fix a Lubin-Tate spectrum E. Let $\mathfrak{m} \subseteq \pi_0 E$ be the maximal ideal, let K_* denote the graded ring $(\pi_* E)/\mathfrak{m}(\pi_* E)$, and let $\mathrm{Syn}_E^{\heartsuit}$ denote the category of Milnor modules. For every pair of algebras $A, B \in \mathrm{Alg}(\mathrm{Syn}_E^{\heartsuit})$, we let ${}_A\mathrm{BMod}_B(\mathrm{Syn}_E^{\heartsuit})$ denote the abelian category of A-B bimodule objects of $\mathrm{Syn}_E^{\heartsuit}$.

Definition 7.2.1. [Hochschild Cohomology] Let A be an associative algebra in the category $\mathrm{Syn}_E^{\heartsuit}$. For every pair of integers $i, j \in \mathbf{Z}$, we define

$$\mathrm{HC}^{i,j}(A) = \mathrm{Ext}^i_{{}_A\mathrm{BMod}_A(\mathrm{Syn}_E^{\heartsuit})}(A, A[j]).$$

We will refer to $\mathrm{HC}^{i,j}(A)$ as the *Hochschild cohomology groups of A*.

Remark 7.2.2. Let A be an algebra object of $\mathrm{Syn}_E^{\heartsuit}$. Then the Yoneda product on $\mathrm{Ext}^*_{{}_A\mathrm{BMod}_A(\mathrm{Syn}_E^{\heartsuit})}(\bullet, \bullet)$ endows $\mathrm{HC}^{*,*}(A)$ with the structure of a bigraded ring. Moreover, it satisfies the bigraded commutative law $xy = (-1)^{ii'+jj'}yx \in \mathrm{HC}^{i+i',j+j'}(A)$ for $x \in \mathrm{HC}^{i,j}(A)$ and $y \in \mathrm{HC}^{i',j'}(A)$.

Remark 7.2.3 (Periodicity)**.** Let A be an algebra object of $\mathrm{Syn}_E^{\heartsuit}$. Then the graded ring K_* acts on $\mathrm{HC}^{*,*}(A)$ by means of maps $K_m \times \mathrm{HC}^{i,j}(A) \to \mathrm{HC}^{i,j-m}(A)$. We therefore obtain periodicity isomorphisms

$$(\pi_{2m}K) \otimes_{\kappa} \mathrm{HC}^{i,j}(A) \simeq \mathrm{HC}^{i,j-2m}(A).$$

Remark 7.2.4 (Functoriality of Hochschild Cohomology)**.** Let A and B be algebra objects of $\mathrm{Syn}_E^{\heartsuit}$. Using Corollary 6.6.7, we see that the construction $M \mapsto M \boxtimes B$ determines an exact functor of abelian categories

$$F : {}_A\mathrm{BMod}_A(\mathrm{Syn}_E^{\heartsuit}) \to {}_{A\boxtimes B}\mathrm{BMod}_{A\boxtimes B}(\mathrm{Syn}_E^{\heartsuit}).$$

In particular, for every pair of bimodules $M, N \in {}_A\mathrm{BMod}_A(\mathrm{Syn}_E^{\heartsuit})$, we get a canonical map

$$\mathrm{Ext}^*_{{}_A\mathrm{BMod}_A(\mathrm{Syn}_E^{\heartsuit})}(M, N) \to \mathrm{Ext}^*_{{}_{A\boxtimes B}\mathrm{BMod}_{A\boxtimes B}(\mathrm{Syn}_E^{\heartsuit})}(M \boxtimes B, N \boxtimes B).$$

Taking $M = A$ and $N = A[j]$, we obtain maps of Hochschild cohomology groups $\mathrm{HC}^{i,j}(A) \to \mathrm{HC}^{i,j}(A \boxtimes B)$. It is easy to see that these maps are compatible with composition, and therefore yield a bigraded ring homomorphism

$$\mathrm{HC}^{*,*}(A) \to \mathrm{HC}^{*,*}(A \boxtimes B).$$

Using the results of §7.1, we can give an alternative description of the Hochschild cohomology groups $\mathrm{HC}^{*,*}(A)$.

Proposition 7.2.5. *Let A and B be algebra objects of $\mathrm{Syn}_E^{\heartsuit}$. Then the inclusion functor*

$${}_A\mathrm{BMod}_B(\mathrm{Syn}_E^{\heartsuit}) \hookrightarrow {}_A\mathrm{BMod}_B(\mathrm{Syn}_{\mathbf{1}^{\heartsuit}})$$

extends to an equivalence of ∞-categories $\mathcal{D}({}_A\mathrm{BMod}_B(\mathrm{Syn}_E^{\heartsuit}))_{\geqslant 0} \simeq {}_A\mathrm{BMod}_B(\mathrm{Syn}_{\mathbf{1}^{\heartsuit}})$.

Proof. Combine Propositions 7.1.4 and 7.1.2. □

Corollary 7.2.6. *Let A and B be algebra objects of $\mathrm{Syn}_E^{\heartsuit}$. Then, for every pair of objects $M, N \in {}_A\mathrm{BMod}_B(\mathrm{Syn}_E^{\heartsuit})$, the canonical map*

$$\mathrm{Ext}^*_{{}_A\mathrm{BMod}_B(\mathrm{Syn}_E^{\heartsuit})}(M, N) \to \mathrm{Ext}^*_{{}_A\mathrm{BMod}_B(\mathrm{Syn}_{\mathbf{1}^{\heartsuit}})}(M, N)$$

is an isomorphism of graded abelian groups. In particular, we have canonical isomorphisms

$$\mathrm{HC}^{i,j}(A) \to \mathrm{Ext}^i_{{}_A\mathrm{BMod}_A(\mathrm{Syn}_{\mathbf{1}^{\heartsuit}})}(A, A[j]).$$

We now compute Hochschild cohomology in some particularly simple examples.

Proposition 7.2.7. *Let E be a Lubin-Tate spectrum and let*

$$\psi : \mathfrak{m}/\mathfrak{m}^2 \to \mathrm{Ext}^1_{\mathrm{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit, \mathbf{1}^\heartsuit[-1]) \to \mathrm{HC}^{1,-1}(\mathbf{1}^\heartsuit)$$

be the isomorphism of Theorem 6.5.4. Then ψ extends to an isomorphism of bigraded rings

$$K_* \otimes_\kappa \mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2) \simeq \mathrm{HC}^{*,*}(\mathbf{1}^\heartsuit)$$
$$K_m \otimes_\kappa \mathrm{Sym}^n(\mathfrak{m}/\mathfrak{m}^2) \simeq \mathrm{HC}^{n,-m-n}(\mathbf{1}^\heartsuit).$$

Proof. Combine Theorem 6.6.6 with Proposition 5.5.6. □

Let A and B be algebra objects of the category $\mathrm{Mod}_E^\heartsuit$. If A is an Azumaya algebra, then the extension of scalars functor

$${}_B\mathrm{BMod}_B(\mathrm{Syn}_E^\heartsuit) \to {}_{A\boxtimes B}\mathrm{BMod}_{A\boxtimes B}(\mathrm{Syn}_E^\heartsuit) \qquad M \mapsto M \boxtimes A$$

is an equivalence of categories, and therefore induces an isomorphism of bigraded rings $\mathrm{HC}^{*,*}(B) \to \mathrm{HC}^{*,*}(A \boxtimes B)$. Combining this observation with Proposition 7.2.7, we obtain the following:

Corollary 7.2.8. *Let A be an Azumaya algebra object of $\mathrm{Syn}_E^\heartsuit$. Then we have canonical isomorphisms*

$$\mathrm{HC}^{i,j}(A) \simeq K_{-i-j} \otimes_\kappa (\mathfrak{m}^i/\mathfrak{m}^{i+1}),$$

which determine an isomorphism of bigraded rings $\mathrm{HC}^{*,*}(A) \simeq K_* \otimes_\kappa \mathrm{Sym}^*(\mathfrak{m}/\mathfrak{m}^2)$.

7.3 Obstruction Theory

Let E be a Lubin-Tate specturm, which we regard as fixed throughout this section. We would like to analyze the structure of the ∞-category Syn_E by bootstrapping from the description of the abelian category $\mathrm{Syn}_E^\heartsuit$ given in §6. Our strategy will to study the relationships between the ∞-categories $\tau_{\leqslant n}\mathrm{Syn}_E$ of n-truncated synthetic E-modules as n varies.

Notation 7.3.1. Let n be a nonnegative integer. We let $\mathbf{1}^{\leqslant n}$ denote the synthetic E-module $\tau_{\leqslant n}\mathbf{1}$. More concretely, $\mathbf{1}^{\leqslant n}$ is the functor which assigns to each molecular E-module M the n-truncated space

$$\mathbf{1}^{\leqslant n}(M) = \tau_{\leqslant n}\mathrm{Map}_{\mathrm{Mod}_E}(M, E) = \tau_{\leqslant n}\Omega^\infty M^\vee.$$

Note that we have a tower

$$\mathbf{1} \to \cdots \to \mathbf{1}^{\leqslant 3} \to \mathbf{1}^{\leqslant 2} \to \mathbf{1}^{\leqslant 1} \to \mathbf{1}^{\leqslant 0} = \mathbf{1}^\heartsuit$$

in the ∞-category Syn_E.

Notation 7.3.2. For each $n \geqslant 0$, we can regard $\mathbf{1}^{\leqslant n}$ as a commutative algebra object of the symmetric monoidal ∞-category Syn_E. We let $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ denote the ∞-category $\mathrm{Mod}_{\mathbf{1}^{\leqslant n}}(\mathrm{Syn}_E)$ (see Notation 7.1.3). Note that the forgetful functor $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}} \to \mathrm{Syn}_E$ induces an equivalence on n-truncated objects (in other words, every n-truncated synthetic E-module can be regarded as a module over $\mathbf{1}^{\leqslant n}$ in an essentially unique way).

The following observation will be useful for comparing the ∞-categories $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ and $\tau_{\leqslant n} \mathrm{Syn}_E$.

Proposition 7.3.3. *Let n be a nonnegative integer and let X be a dualizable object of the symmetric monoidal ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Then X is n-truncated.*

Proof. Let $X^\vee$ be a dual of X in the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. For each object $Y \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}}(Y, X) \simeq \mathrm{Map}_{\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}}(Y \wedge_{\mathbf{1}^{\leqslant n}} X^\vee, \mathbf{1}^{\leqslant n}).$$

Since $\mathbf{1}^{\leqslant n}$ is an n-truncated object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$, it follows that the mapping space $\mathrm{Map}_{\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}}(Y, X)$ is n-truncated. □

Corollary 7.3.4. *Let X be a dualizable object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Then, for $0 \leqslant m \leqslant n$, the canonical map $\rho : X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant m} \to \tau_{\leqslant m} X$ is an equivalence.*

Proof. It is clear that ρ exhibits $\tau_{\leqslant m} X$ as an m-truncation of $X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant m}$. Consequently, it will suffice to show that the relative smash product $X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant m}$ is m-truncated. This follows from Proposition 7.3.3, since $X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant m}$ is dualizable as a module over $\mathbf{1}^{\leqslant m}$. □

Proposition 7.3.5. *Let n be a positive integer. Then the commutative algebra $\mathbf{1}^{\leqslant n}$ is a square-zero extension of $\mathbf{1}^{\leqslant n-1}$ by the module $\Sigma^n \mathbf{1}^\heartsuit[-n]$. In other words, there exists a pullback diagram σ :*

$$\begin{array}{ccc} \mathbf{1}^{\leqslant n} & \longrightarrow & \mathbf{1}^{\leqslant n-1} \\ \downarrow & & \downarrow d \\ \mathbf{1}^{\leqslant n-1} & \xrightarrow{d_0} & \mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1} \mathbf{1}^\heartsuit[-n]. \end{array}$$

in the ∞-category $\mathrm{CAlg}(\mathrm{Syn}_E)$ of commutative algebra objects of Syn_E. Here d_0 denotes the tautological map from $\mathbf{1}^{\leqslant n-1}$ to the trivial square zero extension $\mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1} \mathbf{1}^\heartsuit[-n]$, and d is some other section of the projection map $\mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1} \mathbf{1}^\heartsuit[-n] \to \mathbf{1}^{\leqslant n-1}$.

Proof. This is a special case of Theorem HA.7.4.1.26 . □

Proposition 7.3.6. *Let n be a positive integer and set $\mathcal{C} = \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n]}$. Then the pullback diagram σ of Proposition 7.3.5 induces a pullback diagram of symmetric monoidal ∞-categories τ :*

$$\begin{array}{ccc} \mathrm{Syn}_{\mathbf{1}^{\leqslant n}} & \longrightarrow & \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \\ \downarrow & & \downarrow {\scriptstyle d^*} \\ \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} & \xrightarrow{d_0^*} & \mathcal{C}\,. \end{array}$$

Proof. The diagram τ determines a functor

$$F : \mathrm{Syn}_{\mathbf{1}^{\leqslant n}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \times_{\mathcal{C}} \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}\,.$$

Let us identify objects of the codomain of F with triples (X, Y, α), where $X, Y \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ and $\alpha : d^*Y \simeq d_0^* X$ is an equivalence. The functor F admits a right adjoint G, given by the construction $G(X, Y, \alpha) = X \times_{d_0^* X} Y$. We first claim that the unit map $\mathrm{id} \to G \circ F$ is an equivalence: that is, for every object $X \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$, the diagram σ_X:

$$\begin{array}{ccc} X & \longrightarrow & X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1} \\ \downarrow & & \downarrow \\ X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1} & \longrightarrow & X \wedge_{\mathbf{1}^{\leqslant n}} (\mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n]) \end{array}$$

is a pullback square in the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. This is clear: the diagram σ_X is a pushout square (since σ is a pushout square and the relative smash product functor $\wedge_{\mathbf{1}^{\leqslant n}}$ preserves colimits in each variable), and therefore also a pullback square (since the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ is prestable). This proves that the functor F is fully faithful.

To complete the proof, it will suffice to show that the functor G is conservative. Let u be a morphism in the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \times_{\mathcal{C}} \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ for which $G(u)$ is an equivalence in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$; we wish to show that u is an equivalence. An easy calculation shows that G is right exact, so that $G(\mathrm{cofib}(u)) \simeq \mathrm{cofib}(G(u)) \simeq 0$. We will complete the proof by showing that $\mathrm{cofib}(u) \simeq 0$. Suppose otherwise, and write $\mathrm{cofib}(u) = (X, Y, \alpha)$. Then there exists some smallest integer k such that either $\pi_k X \neq 0$ or $\pi_k Y \neq 0$. Without loss of generality, we may suppose that $\pi_k Y \neq 0$ and $\pi_i X \simeq 0$ for $0 \leqslant i < k$. It then follows that the projection map $\pi_k(X \times_{d_0^* X} Y) \to \pi_k Y$ is an epimorphism, so that $G(X, Y, \alpha) \neq 0$ and we obtain a contradiction. □

Corollary 7.3.7. *Let n be a nonnegative integer. Then:*

(*a*) *An object $X \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ is zero if and only if $X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\heartsuit}$ is zero.*

(*b*) *An object $X \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ is dualizable if and only if $X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\heartsuit} \in \mathrm{Syn}_{\mathbf{1}^{\heartsuit}}$ is dualizable.*

(*c*) *A morphism $X \to Y$ in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ is an equivalence if and only if the induced map $X \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\heartsuit} \to Y \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\heartsuit}$ is an equivalence.*

(*d*) *An associative algebra object $A \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})$ is an Azumaya algebra if and only if $A \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\heartsuit}$ is an Azumaya algebra object of $\mathrm{Syn}_{\mathbf{1}^{\heartsuit}}$.*

Proof. Assertions (*a*) and (*b*) follow from Proposition 7.3.6 using induction on n. Assertion (*c*) follows from (*a*) (since a morphism $\alpha : X \to Y$ is an equivalence if and only if $\mathrm{cofib}(\alpha) \simeq 0$). To prove (*d*), let $A \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})$ and set $A_0 = A \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\heartsuit}$. Then A is full if and only if A_0 is full (this follows from (*a*) and Lemma 8.1.5) and A is dualizable if and only if A_0 is dualizable (this follows from (*b*)). If these conditions are satisfied, then the canonical map $A \wedge \mathbf{1}^{\leqslant n} A^{\mathrm{op}} \to \mathrm{End}(A)$ is an equivalence if and only if the canonical map $A_0 \wedge_{\mathbf{1}^{\heartsuit}} A_0^{\mathrm{op}} \to \mathrm{End}(A_0)$ is an an equivalence (by virtue of (*c*)). Using the criterion of Corollary 2.2.3, we deduce that A is Azumaya if and only if A_0 is Azumaya. □

Corollary 7.3.8. *The functor $\mathrm{Sy}^{\heartsuit} : \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Syn}_E^{\heartsuit}$ is conservative.*

Proof. Let $f : M \to N$ be a morphism in $\mathrm{Mod}_E^{\mathrm{loc}}$ for which the induced map $\mathrm{Sy}^{\heartsuit}[M] \to \mathrm{Sy}^{\heartsuit}[N]$ is an isomorphism of Milnor modules. Combining Corollary 7.3.7 with Lemma 7.1.1, we deduce that f induces an equivalence $\mathbf{1}^{\leqslant n} \wedge \mathrm{Sy}[M] \to \mathbf{1}^{\leqslant n} \wedge \mathrm{Sy}[N]$ for every integer n. Allowing n to vary, we deduce that $\mathrm{Sy}[f]$ is an equivalence of synthetic E-modules, so that f is an equivalence by virtue of Proposition 4.2.5. □

We now introduce some notation which will be useful for exploiting Proposition 7.3.6.

Construction 7.3.9. Fix an integer $n > 0$ and let

$$d, d_0 : \mathbf{1}^{\leqslant n-1} \to \mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n]$$

be as in Proposition 7.3.5. Then d and d_0 induce symmetric monoidal functors

$$d^*, d_0^* : \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n]}$$

(given by extension of scalars along d and d_0, respectively). These functors admit lax symmetric monoidal right adjoints

$$d_*, d_{0*} : \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n] \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$$

(given by restriction of scalars along d and d_0, respectively). We let

$$\Theta = (d_* \circ d_0^*) : \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$$

denote the composition of d_* with d_0^*, which we regard as a lax symmetric monoidal functor from the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ to itself. Note that the projection map $\mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n] \to \mathbf{1}^{\leqslant n-1}$ induces a lax symmetric monoidal natural transformation $\phi : \Theta \to \mathrm{id}$.

Remark 7.3.10. The functor Θ of Construction 7.3.9 can be described more informally by the formula

$$\Theta(M) = M \oplus \Sigma^{n+1}(M \wedge_{\mathbf{1}^{\leqslant n-1}} \mathbf{1}^{\heartsuit}[-n]).$$

However, this formula is a bit misleading: it really describes the composition of Θ with the forgetful functor $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \to \mathrm{Syn}_E$. In order to regard Θ as a functor from the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ to itself, one needs to understand the derivation

$$d : \mathbf{1}^{\leqslant n-1} \to \mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n]$$

which appears in Proposition 7.3.5 (the nontriviality of these derivations encode the contrast between the homotopy-theoretic character of the ∞-category Syn_E and the purely algebraic character of the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\heartsuit}} \simeq \mathcal{D}(\mathrm{Syn}_E^{\heartsuit})_{\geqslant 0}$).

In the special case where M is a dualizable object of $\mathrm{Mod}_{\mathbf{1}^{\leqslant n-1}}$, we can identify the tensor product $M \wedge_{\mathbf{1}^{\leqslant n-1}} \mathbf{1}^{\heartsuit}[-n]$ with $(\pi_0 M)[-n]$ (Corollary 7.3.4. In this case, we obtain an equivalence of synthetic E-modules $\Theta(X) \simeq X \oplus \Sigma^{n+1}(\pi_0 X)[-n]$.

Remark 7.3.11. Let n be a positive integer and suppose we are given objects $X, Y \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. The following data are equivalent:

(*i*) Morphisms $\alpha : d^* Y \to d_0^* X$ in the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1} \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n]}$.

(*ii*) Morphisms $\beta : Y \to \Theta(X)$ in the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$.

Moreover, a morphism $\alpha : d^* Y \to d_0^* X$ is an equivalence if and only if the corresponding morphism $\beta : Y \to \Theta(X)$ has the property that the composite map $Y \xrightarrow{\beta} \Theta(X) \xrightarrow{\phi} X$ is an equivalence in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. It follows that the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \times_{\mathcal{C}} \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ appearing in the proof of Proposition 7.3.6 can be identified with the ∞-category of pairs (X, s), where X is an object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ and $s : X \to \Theta(X)$ is a section of the map $\phi : \Theta(X) \to X$.

7.4 Lifting Associative Algebras

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section. Our goal is to apply the analysis of §7.3 to study the problem of lifting associative algebra objects of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ to associative algebra objects of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$, where n is a positive integer.

Notation 7.4.1. Let $n > 0$ and let A be an associative algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. We let $\mathrm{Lift}(A)$ denote the set $\pi_0(\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}) \times_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})} \{A\})$, which parametrizes equivalence classes of objects $\overline{A} \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})$ equipped with an equivalence $A \simeq \overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$.

Remark 7.4.2. Let $n > 0$ and let A be an associative algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. Combining Proposition 7.3.6 with Remark 7.3.11, we obtain a canonical homotopy equivalence

$$\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}) \times_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})} \{A\}) \simeq \mathrm{Map}_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})}(A, \Theta(A)).$$

In particular, we can identify $\mathrm{Lift}(A)$ with the set of homotopy classes of sections of the canonical map $\phi_A : \Theta(A) \to A$ described in Construction 7.3.9 (taken in the ∞-category of associative algebra objects of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$).

Our next goal is to obtain a homological description of the space $\mathrm{Lift}(A)$.

Proposition 7.4.3. *Let A be an associative algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ which is dualizable as an object $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. Then the map $\phi_A : \Theta(A) \to A$ appearing in Construction 7.3.9 exhibits $\Theta(A)$ as a square-zero extension of A by the $\Sigma^{n+1}(\pi_0 A)[-n]$ (which we regard as an A-A bimodule object of the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$).*

Proof. Combine Theorem HA.7.4.1.26 with Remark 7.3.10. □

Notation 7.4.4 (The Cotangent Complex)**.** Let $n > 0$ and let A be a dualizable associative algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. Set $A_0 = \pi_0 A \simeq A \wedge_{\mathbf{1}^{\leqslant n-1}} \mathbf{1}^{\heartsuit}$ (see Corollary 7.3.4). Let $L_A \in {}_A\mathrm{BMod}_A(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})$ denote the absolute cotangent complex of A (regarded as an associative algebra object of the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$), and define $L_{A_0} \in {}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_{\mathbf{1}^{\heartsuit}})$ similarly, so that L_{A_0} can be identified with the image of L_A under the extension of scalars functor $M \mapsto M \wedge_{\mathbf{1}^{\leqslant n-1}} \mathbf{1}^{\heartsuit}$. Using Theorem HA.7.3.5.1 (together with Proposition 7.1.2), we see that the cotangent complex L_{A_0} is a discrete object of ${}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_{\mathbf{1}^{\heartsuit}})$, which fits into a short exact sequence of bimodules

$$0 \to L_{A_0} \to A_0 \boxtimes A_0 \xrightarrow{m} A_0 \to 0.$$

where m is the multiplication on A_0.

Remark 7.4.5 (Classification of Lifts)**.** Let $n > 0$, let A be a dualizable associative algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$, and set $A_0 = \pi_0 A$. Then the square-zero extension $\phi : \Theta(A) \to A$ of Proposition 7.4.3 is classified by an element

$$\begin{aligned} o(A) &\in \mathrm{Ext}^{n+2}_{{}_A\mathrm{BMod}_A(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})}(L_A, A_0[-n]) \\ &\simeq \mathrm{Ext}^{n+2}_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_{\mathbf{1}^{\heartsuit}})}(L_{A_0}, A_0[-n]) \\ &\simeq \mathrm{Ext}^{n+2}_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})}(L_{A_0}, A_0[-n]) \end{aligned}$$

(where the second isomorphism is supplied by Corollary 7.2.6). Using Remark 7.4.2, we deduce:

(*a*) The obstruction $o(A)$ vanishes if and only if the set $\mathrm{Lift}(A)$ is nonempty.

(*b*) If the set $\mathrm{Lift}(A)$ is nonempty, then it forms a torsor for the abelian group

$$\mathrm{Ext}^{n+1}_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})}(L_{A_0}, A_0[-n]).$$

Construction 7.4.6 (Obstructions in Hochschild Cohomology)**.** Let $n > 0$, let A be a dualizable associative algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$, and set $A_0 = \pi_0 A$. Using the short exact sequence $0 \to L_{A_0} \to A_0 \boxtimes A_0 \to A_0 \to 0$, we obtain a boundary map

$$\begin{aligned} e : \mathrm{Ext}^{n+1}_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})}(L_{A_0}, A_0[-n]) &\to \mathrm{Ext}^{n+2}_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})}(A_0, A_0[-n]) \\ &= \mathrm{HC}^{n+2,-n}(A_0) \end{aligned}$$

Suppose we are given a pair of lifts $\overline{A}, \overline{A}' \in \mathrm{Lift}(A)$. Then the set $\mathrm{Lift}(A)$ is nonempty, and is therefore a torsor for the abelian group $\mathrm{Ext}^{n+1}_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})}(L_{A_0}, A_0[-n])$. It follows that there is a unique element $g \in \mathrm{Ext}^{n+1}_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})}(L_{A_0}, A_0[-n])$ which carries $\overline{A}$ to $\overline{A}'$. We let $\delta(\overline{A}, \overline{A}')$ denote the element $e(g) \in \mathrm{HC}^{n+2,-n}(A_0)$.

In the special case where A is an Azumaya algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$, the Milnor module A_0 is an Azumaya algebra object of $\mathrm{Syn}_E^{\heartsuit}$. In this case, Corollary 7.2.8 supplies a canonical isomorphism

$$\mathrm{HC}^{n+2,-n}(A_0) \simeq (\pi_{-2} K) \otimes_\kappa (\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3}).$$

In this case, we will abuse notation by identifying $\delta(\overline{A}, \overline{A}')$ with its image under this isomorphism.

Remark 7.4.7 (Functoriality)**.** Let $n > 0$, and suppose we are given dualizable algebra objects $A, B \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})$ together with lifts $\overline{A}, \overline{A}' \in \mathrm{Lift}(A)$ and $\overline{B} \in \mathrm{Lift}(B)$. Then we can regard $\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}$ and $\overline{A}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}$ as elements of $\mathrm{Lift}(A \wedge_{\mathbf{1}^{\leqslant n-1}} B)$. Then we have an equality

$$\delta(\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}, \overline{A}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}) = \theta(\delta(\overline{A}, \overline{A}')) \in \mathrm{HC}^{n+2,-n}(\pi_0 A \boxtimes \pi_0 B),$$

where $\theta : \mathrm{HC}^{n+2,-n}(\pi_0 A) \to \mathrm{HC}^{n+2,-n}(\pi_0 A \boxtimes \pi_0 B)$ is the map on Hochschild cohomology described in Remark 7.2.4.

In particular, if A and B are Azumaya algebras, then we have an equality $\delta(\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}, \overline{A}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}) = \delta(\overline{A}, \overline{A}')$ in the abelian group $(\pi_{-2} K) \otimes_\kappa (\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3})$.

7.5 Digression: Molecular Objects of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$

Let E be a Lubin-Tate spectrum. In §6.2 we introduced the notion of a *molecular* Milnor module $M \in \tau_{\leqslant 0} \mathrm{Syn}_E$ (see Definition 6.2.1). In this section, we consider a generalization of this notion to the setting of n-truncated synthetic E-modules, for any nonnegative integer n.

Definition 7.5.1. Let n be a nonnegative integer and let $X \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. We will say that X is *molecular* if it is dualizable (as an object of the symmetric monoidal ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$) and the Milnor module $\pi_0 X$ is molecular (in the sense of Definition 6.2.1).

Remark 7.5.2. Let X be a molecular object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Then X is n-truncated (Proposition 7.3.3). It follows that the $\mathbf{1}^{\leqslant n}$-module structure on X is unique up to a contractible space of choices; see Notation 7.3.2.

Warning 7.5.3. The terminology of Definition 7.5.1 is potentially ambiguous: if X is a molecular object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$, then it is usually not molecular when regarded as an object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant m}}$ for $m \geqslant n$.

Remark 7.5.4. Let X be an object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant 0}}$. Then X is molecular in the sense of Definition 7.5.1 if and only if it discrete and molecular in the sense of Definition 6.2.1.

Proposition 7.5.5. *Let n be a nonnegative integer and let $X \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ be molecular. Then there exists a molecular E-module M and an equivalence $X \simeq \tau_{\leqslant n} \mathrm{Sy}[M]$. Moreover, the module M is unique up to equivalence.*

Proof. Our assumption that X is molecular guarantees that we can choose a molecular E-module M and an isomorphism of Milnor modules $\alpha_0 : \mathrm{Sy}^{\heartsuit}[M] \simeq \pi_0 X$. Note that we can identify α_0 with an element of $(\pi_0 X)(M) = \pi_0(X(M))$, which we can identify with a homotopy class of maps $\alpha : \tau_{\leqslant n} \mathrm{Sy}[M] \to X$ in the ∞-category $\tau_{\leqslant n} \mathrm{Syn}_E \simeq \tau_{\leqslant n} \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Since $\tau_{\leqslant n} \mathrm{Sy}[M]$ and X are dualizable objects of the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$, Corollary 7.3.4 implies that α_0 can be obtained from α by applying the extension of scalars functor $N \mapsto N \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\heartsuit}$. It follows from Corollary 7.3.7 that α is an equivalence. This proves the existence of M. For the uniqueness, it suffices to observe that a molecular E-module M is determined, up to equivalence, by the Milnor module $\mathrm{Sy}^{\heartsuit}[M]$ (Corollary 6.2.8). $\square$

Corollary 7.5.6. *Let n be a positive integer and let $X \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ be molecular. Then there exists a tiny object $\overline{X} \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ and an equivalence $X \simeq \overline{X} \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$.*

Proposition 7.5.7. *Let n be a nonnegative integer and let $X, Y \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. If X is dualizable and Y is molecular, then $X \wedge_{\mathbf{1}^{\leqslant n}} Y$ is molecular.*

Proof. It is clear that $X \wedge_{\mathbf{1}^{\leqslant n}} Y$ is a dualizable object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. It will therefore suffice to show that $\pi_0(X \wedge_{\mathbf{1}^{\leqslant n}} Y) \simeq (\pi_0 X) \boxtimes (\pi_0 Y)$ is a molecular Milnor module. By virtue of Theorem 6.6.6, there exists an equivalence of monoidal categories $F : \mathrm{Syn}_E^{\heartsuit} \simeq \mathcal{M}(V)$, where V is a finite-dimensional vector space over κ. Observe that a Milnor module M is molecular if and only if $F(M)$ is a finitely generated module over the exterior algebra $H = \bigwedge^*_{K_*}(V)$. We now complete the proof by observing that if $F(\pi_0 X)$ is finitely generated H-module and $F(\pi_0 Y)$ is a finitely generated free H-module, then the tensor product $F(\pi_0 X) \otimes_{K_*} F(\pi_0 Y) \simeq F((\pi_0 X) \boxtimes (\pi_0 Y))$ is also a finitely generated free H-module. □

7.6 Lifting Molecular Algebras

Let E be a Lubin-Tate spectrum, let $\mathfrak{m} \subseteq \pi_0 E$ be the maximal ideal, and let κ denote the residue field $(\pi_0 E)/\mathfrak{m}$. In this section, we specialize the deformation-theoretic ideas of §7.4 to study the problem of lifting *molecular* algebras.

Proposition 7.6.1. *Let n be a positive integer, let A be a molecular Azumaya algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$, and set $A_0 = \pi_0 A$. Then:*

(a) *The set* $\mathrm{Lift}(A)$ *is nonempty.*

(b) *The boundary map*

$$\mathrm{Ext}^{n+1}_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})}(L_{A_0}, A_0[-n]) \xrightarrow{e} \mathrm{HC}^{n+2,-n}(A_0)$$

appearing in Construction 7.4.6 is an isomorphism. Consequently, we can regard the set $\mathrm{Lift}(A)$ *as a torsor for the abelian group*

$$\mathrm{HC}^{n+2,-n}(A_0) \simeq (\pi_{-2} K) \otimes_{\kappa} (\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3}).$$

Proof. Let $\mathcal{C}$ denote the abelian category ${}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})$. Combining the short exact sequence $0 \to L_{A_0} \to A_0 \boxtimes A_0 \to A_0 \to 0$ appearing in Construction 7.4.6 with the calculation

$$\mathrm{Ext}^*_{\mathcal{C}}(A_0 \boxtimes A_0, A_0[-n]) \simeq \mathrm{Ext}^*_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}, A_0[-n]),$$

we obtain a long exact sequence

$$\mathrm{Ext}^{*-1}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}, A_0[-n]) \to \mathrm{Ext}^{*-1}_{\mathcal{C}}(L_{A_0}, A_0[-n]) \to \mathrm{HC}^{*,-n}(A_0) \to \mathrm{Ext}^*_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}, A_0[-n])$$

Since A is molecular, the object $A_0[-n] \in \mathrm{Syn}_E^{\heartsuit}$ is injective (Corollary 6.6.11), so the boundary maps

$$\mathrm{Ext}^{*-1}_{\mathcal{C}}(L_{A_0}, A_0[-n]) \to \mathrm{HC}^{*,-n}(A_0)$$

are isomorphisms for $* > 0$. This proves (b). To prove (a), it will suffice (by virtue of Remark 7.4.5) to show that the Hochschild cohomology group $\mathrm{HC}^{n+3,-n}(A_0)$ vanishes, which is a special case of Corollary 7.2.8. $\square$

Corollary 7.6.2. *Let n be a positive integer and suppose we are given Azumaya algebras $A, B \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})$ together with a lift $\overline{B} \in \mathrm{Lift}(B)$. If A is molecular, then the construction $\overline{A} \mapsto \overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}$ induces a bijection $\rho : \mathrm{Lift}(A) \to \mathrm{Lift}(A \wedge_{\mathbf{1}^{\leqslant n-1}} B)$.*

Proof. Note that $A \wedge_{\mathbf{1}^{\leqslant n-1}} B$ is also molecular (Proposition 7.5.7). Using Proposition 7.6.1, we can regard $\mathrm{Lift}(A)$ and $\mathrm{Lift}(A \wedge_{\mathbf{1}^{\leqslant n-1}} B)$ as torsors for the abelian group $G = K_{-2} \otimes_{\kappa} (\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3})$. It follows from Remark 7.4.7 that ρ is G-equivariant and therefore bijective. $\square$

Remark 7.6.3 (Lifting Algebra Automorphisms)**.** In the situation of Proposition 7.6.1, Corollary 7.2.8 also guarantees the vanishing of the group

$$\mathrm{HC}^{n+1,-n}(A_0) \simeq \mathrm{Ext}^n_{{}_{A_0}\mathrm{BMod}_{A_0}(\mathrm{Syn}_E^{\heartsuit})}(L_{A_0}, A_0[-n]).$$

It follows that every connected component of the space

$$\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}) \times_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})} \{A\}$$

is simply connected. In particular, for each $\overline{A} \in \mathrm{Lift}(A)$, the canonical map $\pi_0 \mathrm{Aut}(\overline{A}) \to \pi_0 \mathrm{Aut}(A)$ is surjective: here $\mathrm{Aut}(\overline{A})$ denotes the subspace of $\mathrm{Map}_{\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})}(\overline{A}, \overline{A})$ spanned by the equivalences, and $\mathrm{Aut}(A)$ is defined similarly.

Chapter 8

The Calculation of $\mathrm{Br}(E)$

Let E be a Lubin-Tate spectrum, which we regard as fixed throughout this section. Let $\mathfrak{m} \subseteq \pi_0 E$ be the maximal ideal and let K_* denote the graded ring $(\pi_* E)/\mathfrak{m}(\pi_* E)$. Our goal is to compute the Brauer group $\mathrm{Br}(E)$, at least up to filtration.

Notation 8.0.4. Let A be a commutative algebra object of Syn_E. We let $\mathrm{Br}(A)$ denote the Brauer group of the symmetric monoidal ∞-category $\mathrm{Syn}_A = \mathrm{Mod}_A(\mathrm{Syn}_E)$.

We will obtain information the Brauer group $\mathrm{Br}(E)$ by analyzing the tower of symmetric monoidal ∞-categories

$$\mathrm{Mod}_E^{\mathrm{loc}} \xrightarrow{\mathrm{Sy}} \mathrm{Syn}_E \to \cdots \to \mathrm{Syn}_{\mathbf{1}^{\leqslant 2}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant 1}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant 0}} \xrightarrow{\pi_0} \mathrm{Syn}_E^{\heartsuit}.$$

Our principal results can be summarized as follows:

Theorem 8.0.5. (1) *The restricted Yoneda embedding* $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \hookrightarrow \mathrm{Syn}_E$ *induces an isomorphism of Brauer groups* $\mathrm{Br}(E) \to \mathrm{Br}(\mathrm{Syn}_E)$.

(2) *The functor* $\pi_0 : \mathrm{Syn}_{\mathbf{1}^{\leqslant 0}} \to \mathrm{Syn}_E^{\heartsuit}$ *induces an isomorphism of Brauer groups* $\mathrm{Br}(\mathbf{1}^{\leqslant 0}) \to \mathrm{Br}(\mathrm{Syn}_E^{\heartsuit}) = \mathrm{BM}(E)$.

(3) *The canonical maps* $\mathrm{Syn}_E \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ *induce an isomorphism of abelian groups* $\mathrm{Br}(\mathrm{Syn}_E) \to \varprojlim \mathrm{Br}(\mathbf{1}^{\leqslant n})$.

(4) *For each $n > 0$, the extension of scalars functor*

$$\mathrm{Syn}_{\mathbf{1}^{\leqslant n}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \qquad M \mapsto M \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$$

induces a homomorphism of Brauer groups $\mathrm{Br}(\mathbf{1}^{\leqslant n}) \to \mathrm{Br}(\mathbf{1}^{\leqslant n-1})$ *which fits into a short exact sequence*

$$0 \to K_{-2} \otimes_{\kappa} (\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3}) \to \mathrm{Br}(\mathbf{1}^{\leqslant n}) \to \mathrm{Br}(\mathbf{1}^{\leqslant n-1}) \to 0.$$

The remainder of this section is to devoted to the proof of Theorem 8.0.5. We will establish each assertion in turn (see Propositions 8.1.1, 8.2.1, 8.3.1, and 8.4.1). The most difficult part of the argument will be the proof of (4): this will require the theory of Hochschild cohomology developed in §7.

8.1 Comparison of $\mathrm{Mod}_E^{\mathrm{loc}}$ with Syn_E

We begin by establishing the first assertion of Theorem 8.0.5, which we restate for the reader's convenience:

Proposition 8.1.1. *The Yoneda embedding* $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \hookrightarrow \mathrm{Syn}_E$ *induces an isomorphism of Brauer groups* $\mathrm{Br}(E) = \mathrm{Br}(\mathrm{Mod}_E^{\mathrm{loc}}) \to \mathrm{Br}(\mathrm{Syn}_E)$*.*

The proof of Proposition 8.0.5 will require some preliminary observations.

Lemma 8.1.2. *Let X and Y be nonzero Milnor modules. Then the tensor product $X \boxtimes Y$ is nonzero.*

Proof. By virtue of Proposition 6.6.3, there exists a conservative monoidal functor $\mathrm{Syn}_E^{\heartsuit} \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$. We are therefore reduced to the observation that if M and N are nonzero objects of $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$, then the tensor product $M \otimes_{K_*} N$ is also nonzero. □

Lemma 8.1.3. (*i*) *Let X and Y be nonzero synthetic E-modules. Then the smash product $X \wedge Y$ is also nonzero.*

(*ii*) *Let $n \geqslant 0$ and let X and Y be nonzero objects of* $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$*. Then the relative smash product $X \wedge_{\mathbf{1}^{\leqslant n}} Y$ is nonzero.*

Remark 8.1.4. The proof of Proposition 8.1.1 will use only part (*i*) of Lemma 8.1.3. However, part (*ii*) will be useful later in this section.

Proof of Lemma 8.1.3. We will prove (*i*); the proof of (*ii*) is similar. Let m be the smallest positive integer such that $\pi_m X$ is nonzero. Since the ∞-category Syn_E is prestable, we can write $X = \Sigma^m X_0$, where $\pi_0 X_0 \neq 0$. We then have $X \wedge Y \simeq \Sigma^m (X_0 \wedge Y)$. Consequently, to show that $X \wedge Y$ is nonzero, it will suffice to show that $X_0 \wedge Y$ is nonzero. We may therefore replace X by X_0 and thereby reduce to the case where $\pi_0 X \neq 0$. Similarly, we may assume that $\pi_0 Y \neq 0$. In this case, we have $\pi_0(X \wedge Y) \simeq \pi_0 X \boxtimes \pi_0 Y \neq 0$ by virtue of Lemma 8.1.2. □

Lemma 8.1.5. (*i*) *An object $X \in \mathrm{Syn}_E$ is full (in the sense of Definition 2.1.2) if and only if it is nonzero.*

(*ii*) *For $n \geqslant 0$, an object $X \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ is full if and only if it is nonzero.*

Proof. The "only if" direction is obvious. Conversely, suppose that $X \neq 0$. Let $\alpha : Y \to Z$ be a morphism in Syn_E and suppose that the induced map $\alpha_X : X \wedge Y \to X \wedge Z$ is an equivalence. We then have $X \wedge \mathrm{cofib}(\alpha) \simeq \mathrm{cofib}(\alpha_X) \simeq 0$. Since X is nonzero, Lemma 8.1.3 guarantees that $\mathrm{cofib}(\alpha) \simeq 0$. Because the ∞-category Syn_E is prestable, it follows that α is an equivalence. □

Lemma 8.1.6. *Let X be a synthetic E-module. The following conditions are equivalent:*

(1) *The synthetic E-module X is dualizable (as an object of the symmetric monoidal ∞-category Syn_E).*

(2) *There exists a perfect E-module M and an equivalence $X \simeq \mathrm{Sy}[M]$.*

Proof. The implication (2) ⇒ (1) is immediate (note that the functor $M \mapsto \mathrm{Sy}[M]$ is symmetric monoidal by Variant 4.4.11, and therefore carries dualizable objects to dualizable objects). Conversely, suppose that (1) is satisfied. We will show that $X \simeq \mathrm{Sy}[M]$ for some $M \in \mathrm{Mod}_E^{\mathrm{loc}}$. Applying the same argument to the dual $X^\vee$, we can write $X^\vee = \mathrm{Sy}[N]$ for some $N \in \mathrm{Mod}_E^{\mathrm{loc}}$. Using the fact that the functor $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \to \mathrm{Syn}_E$ is fully faithful (Proposition 4.2.5) and symmetric monoidal (Variant 4.4.11), we conclude that M and N are mutually dual objects of $\mathrm{Mod}_E^{\mathrm{loc}}$, so that M is perfect (Proposition 2.9.4).

To show that X belongs to the essential image of the functor Sy, it will suffice to verify that it satisfies condition $(*)$ of Proposition 4.2.5. That is, we must show that for every molecular E-module M, the canonical map

$$\mathrm{Map}_{\mathrm{Syn}_E}(\mathrm{Sy}[\Sigma M], X) \to \mathrm{Map}_{\mathrm{Syn}_E}(\Sigma\,\mathrm{Sy}[M], X)$$

is a homotopy equivalence. Equivalently, we must show that the canonical map

$$\mathrm{Map}_{\mathrm{Syn}_E}(\mathrm{Sy}[\Sigma M] \wedge X^\vee, \mathbf{1}) \to \mathrm{Map}_{\mathrm{Syn}_E}((\Sigma\,\mathrm{Sy}[M]) \wedge X^\vee, \mathbf{1})$$

is a homotopy equivalence. Let $F : \mathrm{Syn}_E \to \mathrm{Mod}_E^{\mathrm{loc}}$ be a left adjoint to the functor Sy; we claim that canonical map $F(\Sigma\,\mathrm{Sy}[M] \wedge X^\vee) \to F(\mathrm{Sy}[\Sigma M] \wedge X^\vee)$ is an equivalence. This is clear, since the functor F is nonunital symmetric monoidal (see Corollary 4.4.6) and induces an equivalence $F(\Sigma\,\mathrm{Sy}[M]) \to F(\mathrm{Sy}[\Sigma M])$. □

Proof of Proposition 8.1.1. Using Lemma 8.1.5, we see that the restricted Yoneda embedding $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \hookrightarrow \mathrm{Syn}_E$ carries full objects of $\mathrm{Mod}_E^{\mathrm{loc}}$ to full objects of Syn_E, and therefore induces a group homomorphism $\theta : \mathrm{Br}(E) = \mathrm{Br}(\mathrm{Mod}_E^{\mathrm{loc}}) \to \mathrm{Br}(\mathrm{Syn}_E)$ (see Proposition 2.4.1). We first claim that θ is injective. Let A be an Azumaya algebra object of $\mathrm{Mod}_E^{\mathrm{loc}}$, and suppose that $\theta([A]) = [\mathrm{Sy}[A]]$ vanishes in $\mathrm{Br}(\mathrm{Syn}_E)$. Then there exists an equivalence $\alpha : \mathrm{Sy}[A] \simeq \mathrm{End}(X)$ in $\mathrm{Alg}(\mathrm{Syn}_E)$, where X is a nonzero dualizable object of Syn_E. Using Lemma 8.1.6, we can choose an equivalence $X \simeq \mathrm{Sy}[M]$, where M is a perfect E-module (and thus a dualizable object of $\mathrm{Mod}_E^{\mathrm{loc}}$). Since the functor Sy is fully faithful (Proposition 4.2.5) and symmetric monoidal (Variant 4.4.11), we can lift α to an equivalence $\overline{\alpha} : A \simeq \mathrm{End}(M)$ in $\mathrm{Alg}(\mathrm{Mod}_E^{\mathrm{loc}})$, so that $[A]$ vanishes in $\mathrm{Br}(E) = \mathrm{Br}(\mathrm{Mod}_E^{\mathrm{loc}})$.

We now complete the proof by showing that θ is surjective. Let B be an Azumaya algebra object of Syn_E. Using Lemma 8.1.6, we deduce that there is an equivalence $\beta : B \simeq \mathrm{Sy}[A]$ for some dualizable object $A \in \mathrm{Mod}_E^{\mathrm{loc}}$. Because the functor Sy is symmetric monoidal and fully faithful, there exists an essentially unique algebra structure on A for which β can be promoted to an equivalence of algebras. The canonical map $A \otimes_E A^{\mathrm{op}} \to \mathrm{End}(A)$ becomes an equivalence after applying the conservative functor $\mathrm{Sy} : \mathrm{Mod}_E^{\mathrm{loc}} \hookrightarrow \mathrm{Syn}_E$, and is therefore an equivalence. Applying Corollary 2.2.3, we deduce that A is an Azumaya algebra, so that $[B] = [\mathrm{Sy}[A]] = \theta([A])$ belongs to the image of θ. □

8.2 Comparison of $\mathrm{Syn}_{\mathbf{1}^\heartsuit}$ with $\mathrm{Syn}_E^\heartsuit$

We now prove the second part of Theorem 8.0.5, which we formulate as follows:

Proposition 8.2.1. *The inclusion functor* $\iota : \mathrm{Syn}_E^\heartsuit \hookrightarrow \mathrm{Syn}_{\mathbf{1}^\heartsuit}$ *induces an isomorphism of Brauer groups* $\mathrm{BM}(E) = \mathrm{Br}(\mathrm{Syn}_E^\heartsuit) \to \mathrm{Br}(\mathbf{1}^\heartsuit)$.

Proof. It follows from Proposition 7.1.2 that the functor ι is symmetric monoidal, and from Lemma 8.1.5 that ι carries full objects of $\mathrm{Syn}_E^\heartsuit$ to full objects of $\mathrm{Syn}_{\mathbf{1}^\heartsuit}$. Applying Proposition 2.4.1, we deduce that ι induces a group homomorphism $\rho : \mathrm{BM}(E) = \mathrm{Br}(\mathrm{Syn}_E^\heartsuit) \to \mathrm{Br}(\mathbf{1}^\heartsuit)$. Note that if A is an Azumaya algebra object of $\mathrm{Syn}_E^\heartsuit$ and $\rho([A])$ vanishes in $\mathrm{Br}(\mathbf{1}^\heartsuit)$, then we can choose an equivalence $A \simeq \mathrm{End}(M)$ for some nonzero dualizable object $M \in \mathrm{Syn}_{\mathbf{1}^\heartsuit}$. The dualizability of M implies that M is discrete (Proposition 7.3.3), so that $[A]$ vanishes in the group $\mathrm{BM}(E)$. This shows that ρ is injective. To prove surjectivity, it suffices to observe that every Azumaya algebra $A \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^\heartsuit})$ is discrete (Proposition 7.3.3), and can therefore also be regarded as an Azumaya algebra object of the category $\mathrm{Syn}_E^\heartsuit$ of Milnor modules (this is immediate from the criterion of Corollary 2.2.3). □

Remark 8.2.2. Let n be a nonnegative integer. Then the ∞-category $\tau_{\leqslant n} \mathrm{Syn}_E$ of n-truncated synthetic E-modules admits an essentially unique symmetric monoidal structure for which the truncation functor $\tau_{\leqslant n} : \mathrm{Syn}_E \to \tau_{\leqslant n} \mathrm{Syn}_E$ is symmetric monoidal; concretely, the tensor product on $\tau_{\leqslant n} \mathrm{Syn}_E$ is given by the construction

$$(X, Y) \mapsto \tau_{\leqslant n}(X \wedge Y).$$

It is not difficult to show (by a variant on the proof of Proposition 8.2.1) that the truncation functor $\tau_{\leqslant n}$ induces an isomorphism of Brauer groups

$$\mathrm{Br}(\mathbf{1}^{\leqslant n}) = \mathrm{Br}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}) \to \mathrm{Br}(\tau_{\leqslant n}\,\mathrm{Syn}_E)$$

(in the case $n = 0$, this is the inverse of the isomorphism appearing in Proposition 8.2.1). The essential observation is that every Azumaya algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ is dualizable, and therefore n-truncated (Proposition 7.3.3). Since we do not need this result, we leave further details to the reader.

8.3 Passage to the Inverse Limit

It follows from Lemma 8.1.5 that the extension-of-scalars functors

$$\begin{array}{cc} \mathrm{Syn}_E \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n}} & \mathrm{Syn}_{\mathbf{1}^{\leqslant m}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n}} \\ X \mapsto X \wedge \mathbf{1}^{\leqslant n} & X \mapsto X \wedge_{\mathbf{1}^{\leqslant m}} \mathbf{1}^{\leqslant n} \end{array}$$

carry full objects to full objects. Applying Proposition 2.4.1, we obtain a diagram of Brauer groups

$$\mathrm{Br}(\mathrm{Syn}_E) \to \cdots \to \mathrm{Br}(\mathbf{1}^{\leqslant 3}) \to \mathrm{Br}(\mathbf{1}^{\leqslant 2}) \to \mathrm{Br}(\mathbf{1}^{\leqslant 1}) \to \mathrm{Br}(\mathbf{1}^{\leqslant 0}).$$

Our goal in this section is to prove the third part of Theorem 8.0.5, which we restate as follows:

Proposition 8.3.1. *Let $\theta : \mathrm{Br}(\mathrm{Syn}_E) \to \varprojlim \mathrm{Br}(\mathbf{1}^{\leqslant n})$ be the group homomorphism determined by the diagram above. Then θ is an isomorphism of abelian groups.*

The proof of Proposition 8.3.1 will require some preliminaries.

Lemma 8.3.2. *Let n be a nonnegative integer and let L be an invertible object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Then L is equivalent either to $\mathbf{1}^{\leqslant n} = \tau_{\leqslant n}\,\mathrm{Sy}[E]$ or to $\tau_{\leqslant n}\,\mathrm{Sy}[\Sigma E]$.*

Proof. Note that any invertible object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ is dualizable, and therefore n-truncated (Proposition 7.3.3). We proceed by induction on n, beginning with the case $n = 0$. If $L \in \mathrm{Syn}_{\mathbf{1}^{\leqslant 0}}$ is invertible, then L is invertible when regarded as a Milnor module. Using Theorem 6.6.6, we can choose an equivalence of monoidal categories $\mathrm{Syn}_E^{\heartsuit} \simeq \mathcal{M}(V)$, where V is a finite-dimensional vector space over the residue field $\kappa = (\pi_0 E)/\mathfrak{m}$. We observe that $\mathcal{M}(V)$ is equipped with a monoidal forgetful functor $\mathcal{M}(V) \to \mathrm{Mod}_{K_*}^{\mathrm{gr}}$, and that every invertible object of $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$ is isomorphic either to the unit object K_* or to its shift $K_*[1]$. Moreover, since K_* is concentrated in even degrees, every action of the exterior algebra $\bigwedge^*_{K_*}(V)$ on K_* or $K_*[1]$ is automatically trivial on V. It follows that every invertible object of $\mathcal{M}(V)$ is isomorphic either to the unit object K_* or $K_*[1]$, which proves Lemma 8.3.2 in the case $n = 0$.

We now carry out the inductive step. Suppose that $n > 0$ and that L is an invertible object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Our inductive hypothesis guarantees that $L \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$ is equivalent either to $\tau_{\leqslant n-1}\,\mathrm{Sy}[E]$ or to $\tau_{\leqslant n-1}\,\mathrm{Sy}[\Sigma E]$. Without loss of generality, we may assume that $L_0 = L \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$ is equivalent to $\mathbf{1}^{\leqslant n-1}$. Let $\Theta : \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ be the functor described in Construction 7.3.9. Using Proposition 7.3.6 and Remark 7.3.11, we can identify lifts of L_0 to an object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ with sections of the canonical map $\phi : \Theta(L_0) \to L_0$ in the ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. Since the set of such lifts is nonempty, the map ϕ admits a section; it therefore exhibits $\Theta(L_0)$ as a direct sum $L_0 \oplus \Sigma^{n+1}\mathbf{1}^{\heartsuit}[-n]$. We can therefore identify the set of equivalence classes of lifts of L_0 with the group

$$\mathrm{Ext}^{n+1}_{\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}}(\mathbf{1}^{\leqslant n-1}, \mathbf{1}^{\heartsuit}[-n]) \simeq \mathrm{Ext}^{n+1}_{\mathrm{Syn}_E^{\heartsuit}}(\mathbf{1}^{\heartsuit}, \mathbf{1}^{\heartsuit}[-n]).$$

We now observe that this group vanishes (see Proposition 7.2.7). □

Lemma 8.3.3. *Let n be a positive integer, let A be a molecular Azumaya algebra object of the symmetric monoidal ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$, and let $x \in \mathrm{Br}(\mathbf{1}^{\leqslant n})$ be an element whose image in $\mathrm{Br}(\mathbf{1}^{\leqslant n-1})$ coincides with $[A]$. Then $x = [\overline{A}]$ for some molecular Azumaya algebra object $\overline{A}$ of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ satisfying $A \simeq \overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$.*

Proof. Write $x = [\overline{B}]$ for some Azumaya algebra $\overline{B} \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$, and set $B = \overline{B} \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$. Then $[A] = [B]$ in $\mathrm{Br}(\mathbf{1}^{\leqslant n-1})$. It follows that there exists an equivalence of algebras

$$A \wedge_{\mathbf{1}^{\leqslant n-1}} B^{\mathrm{op}} \simeq \mathrm{End}(M)$$

for some nonzero dualizable object $M \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. Let N be an atomic E-module. Replacing $\overline{B}$ by $\overline{B} \wedge \mathrm{Sy}[\mathrm{End}_E(N)]$, we can arrange that M is molecular. In this case, we can apply Corollary 7.5.6 to lift M to a molecular object $\overline{M} \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Then $\mathrm{End}(\overline{M})$ can be regarded as an element of the set $\mathrm{Lift}(A \wedge_{\mathbf{1}^{\leqslant n-1}} B^{\mathrm{op}})$ (see Notation 7.4.1). Invoking Corollary 7.6.2, we deduce that there exists an equivalence

$$\mathrm{End}(\overline{M}) \simeq \overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}^{\mathrm{op}}$$

for some $\overline{A} \in \mathrm{Lift}(A)$. Then $\overline{A}$ is an Azumaya algebra satisfying $[\overline{A}] = x$, as desired. □

Proof of Proposition 8.3.1. We first show that the homomorphism $\theta : \mathrm{Br}(\mathrm{Syn}_E) \to \varprojlim \mathrm{Br}(\mathbf{1}^{\leqslant n})$ is injective. Suppose that A is an Azumaya algebra object of Syn_E and that $\theta([A]) = 0$ in $\varprojlim \mathrm{Br}(\mathbf{1}^{\leqslant n})$; we wish to show that $[A] = 0$. For each $n \geqslant 0$, we can choose a full dualizable object $M_n \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ and an equivalence $A \wedge \mathbf{1}^{\leqslant n} \simeq \mathrm{End}(M_n)$ (where the endomorphism object is formed in the symmetric monoidal ∞-category $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$). We therefore have equivalences

$$\beta_n : \mathrm{End}(M_n) \simeq \mathrm{End}(M_{n+1} \wedge_{\mathbf{1}^{\leqslant n+1}} \mathbf{1}^{\leqslant n})$$

in the ∞-categories $\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})$. Using Corollary 2.1.5, we see that each β_n is induced by an equivalence

$$M_n \simeq (M_{n+1} \wedge_{\mathbf{1}^{\leqslant n+1}} \mathbf{1}^{\leqslant n}) \wedge_{\mathbf{1}^{\leqslant n}} L_n$$

for some invertible object $L_n \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Using Lemma 8.3.2, we can assume $L_n = \mathbf{1}^{\leqslant n} \wedge \mathrm{Sy}[\Sigma^{k_n} E]$ for some integers $k_n \in \{0,1\}$. Replacing each M_n by $M_n \wedge \mathrm{Sy}[\Sigma^{k_0+\cdots+k_{n-1}} E]$, we can arrange that each L_n is trivial, so that each β_n is induces by an equivalence $M_n \simeq M_{n+1} \wedge_{\mathbf{1}^{\leqslant n+1}} \mathbf{1}^{\leqslant n}$. In this case, we can regard $M = \{M_n\}_{n \geqslant 0}$ as a nonzero dualizable object of the ∞-category $\mathrm{Syn}_E \simeq \varprojlim \tau_{\leqslant n} \mathrm{Syn}_E$. We then have an equivalence $A \simeq \mathrm{End}(M)$, so that $[A] = 0$ in $\mathrm{Br}(\mathrm{Syn}_E)$.

We now prove that θ is surjective. Suppose we are given an element x of $\varprojlim \mathrm{Br}(\mathbf{1}^{\leqslant n})$, which we can identify with a compatible sequence of elements $\{x_n \in \mathrm{Br}(\mathbf{1}^{\leqslant n})\}_{n \geqslant 0}$. Write $x_0 = [A_0]$ for some Azumaya algebra $A_0 \in \mathrm{Syn}_{\mathbf{1}^{\leqslant 0}}$. Without loss of generality, we may assume that A_0 is molecular. Invoking Lemma 8.3.3 repeatedly, we can choose Azumaya algebras $A_n \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})$ satisfying

$$[A_n] = x_n \qquad A_{n-1} \simeq A_n \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}.$$

We can therefore identify each A_{n-1} with the truncation $\tau_{\leqslant n-1} A_n$ (Corollary 7.3.4), so that $A = \{A_n\}_{n \geqslant 0}$ can be regarded as an algebra object of the ∞-category $\mathrm{Syn}_E \simeq \varprojlim \tau_{\leqslant n} \mathrm{Syn}_E$. It follows immediately that A is an Azumaya algebra satisfying $\theta([A]) = x$ in $\varprojlim \mathrm{Br}(\mathbf{1}^{\leqslant n})$. $\square$

8.4 Comparison of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ with $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$

Let us now fix an integer $n > 0$. We saw in §8.3 that the extension-of-scalars functor

$$\mathrm{Syn}_{\mathbf{1}^{\leqslant n}} \to \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}} \qquad M \mapsto M \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$$

induces a homomorphism of Brauer groups $\rho : \mathrm{Br}(\mathbf{1}^{\leqslant n}) \to \mathrm{Br}(\mathbf{1}^{\leqslant n-1})$. The final assertion of Theorem 8.0.5 is a consequence of the following more precise result:

Proposition 8.4.1. *The homomorphism $\rho : \mathrm{Br}(\mathbf{1}^{\leqslant n}) \to \mathrm{Br}(\mathbf{1}^{\leqslant n-1})$ is surjective. Moreover, there is a unique isomorphism $\xi : \ker(\rho) \simeq K_{-2} \otimes_\kappa (\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3})$ which satisfies the following condition:*

(∗) *Let B be an Azumaya algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ and suppose we are given lifts $\overline{B}, \overline{B}' \in \mathrm{Lift}(B)$ (see Notation 7.4.1). Then $\xi([\overline{B}] - [\overline{B}']) = \delta(\overline{B}, \overline{B}')$, where δ is defined as in Construction 7.4.6.*

Proof. We first show that ρ is surjective. Fix an element $x \in \mathrm{Br}(\mathbf{1}^{\leqslant n-1})$; we wish to show that x belongs to the image of ρ. Write $x = [A]$ for some Azumaya algebra A in

$\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. Let N be an atomic E-module. Replacing A by $A \wedge \mathrm{Sy}[\mathrm{End}_E(N)]$, we can arrange that A is molecular. In this case, Proposition 7.6.1 implies that there exists a lift $\overline{A} \in \mathrm{Lift}(A)$. Then $\overline{A}$ is an Azumaya algebra object of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ and $x = [A] = \rho([\overline{A}])$.

We will carry out the construction of ξ in several steps. Let us first fix a molecular Azumaya algebra A in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ satisfying $[A] = 0$ in $\mathrm{Br}(\mathbf{1}^{\leqslant n-1})$ (this can be achieved by setting $A = \mathrm{Sy}[\mathrm{End}_E(N)] \wedge \mathbf{1}^{\leqslant n-1}$, where N is an atomic E-module as above). We now argue as follows:

(1) Every element of $\ker(\rho)$ can be written as $[\overline{A}]$, for some $\overline{A} \in \mathrm{Lift}(A)$. This is a special case of Lemma 8.3.3.

(2) If $\overline{A}, \overline{A}' \in \mathrm{Lift}(A)$ are elements satisfying $[\overline{A}] = [\overline{A}']$ in $\mathrm{Br}(\mathbf{1}^{\leqslant n})$, then $\overline{A} = \overline{A}'$ (as element of $\mathrm{Lift}(A)$). To prove this, note that the equality $[\overline{A}] = [\overline{A}']$ guarantees the existence of an equivalence $\overline{A}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{A}^{\mathrm{op}} \simeq \mathrm{End}(\overline{M})$ for some nonzero dualizable object $\overline{M} \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Set $M = \overline{M} \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$, so that we have equivalences

$$\mathrm{End}(M) \simeq A \otimes_{\mathbf{1}^{\leqslant n-1}} A^{\mathrm{op}} \simeq \mathrm{End}(A)$$

in the ∞-category $\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})$. Applying Corollary 2.1.5, we deduce that there exists an invertible object $L \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ satisfying $A \simeq L \wedge_{\mathbf{1}^{\leqslant n-1}} M$. Using Lemma 8.3.2, we can lift L to an invertible object $\overline{L} \in \mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Replacing $\overline{M}$ by $\overline{L} \wedge_{\mathbf{1}^{\leqslant n}} \overline{M}$, we can arrange that there exists an equivalence $M \simeq A$ in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$. Since A is molecular, this guarantees the existence of an equivalence $\overline{M} \simeq \overline{A}$ in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ (Corollary 7.5.6). Because $\overline{A}$ is an Azumaya algebra, we obtain an equivalence

$$\overline{\alpha} : \overline{A}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{A}^{\mathrm{op}} \simeq \mathrm{End}(\overline{M}) \simeq \mathrm{End}(\overline{A}) \simeq \overline{A} \otimes_{\mathbf{1}^{\leqslant n}} \overline{A}^{\mathrm{op}}$$

in the ∞-category $\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})$. Let α denote the image of $\overline{\alpha}$ in $\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}})$, which we regard as an automorphism of $\mathrm{End}(A)$. Using Remark 7.6.3, we can lift α to an automorphism $\overline{\alpha}'$ of $\mathrm{End}(\overline{A})$ in the ∞-category $\mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})$. Replacing $\overline{\alpha}$ by $\overline{\alpha}'^{-1} \circ \overline{\alpha}$, we can reduce to the case where α is homotopic to the identity. It then follows that $\overline{A}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{A}^{\mathrm{op}}$ and $\overline{A} \otimes_{\mathbf{1}^{\leqslant n}} \overline{A}^{\mathrm{op}}$ represent the same element of $\mathrm{Lift}(A \wedge_{\mathbf{1}^{\leqslant n-1}} A^{\mathrm{op}})$. Applying Corollary 7.6.2, we deduce that $\overline{A}$ and $\overline{A}'$ represent the same element of $\mathrm{Lift}(A)$, as desired.

(3) It follows from (1) and (2) that the construction $\overline{A} \mapsto [\overline{A}]$ induces a bijection of sets $b : \mathrm{Lift}(A) \to \ker(\rho)$. Using Proposition 7.6.1, we can regard $\mathrm{Lift}(A)$ as a torsor for the group $G = K_{-2} \otimes_{\kappa} (\mathfrak{m}^{n+2}/\mathfrak{m}^{n+3})$. It follows that there is a unique bijection $\xi_A : \ker(\rho) \to G$ such that $\xi_A(0) = 0$ and $\xi_A \circ b$ is a map of G-torsors. Concretely, ξ_A is given by the formula $\xi_A([\overline{A}]) = \delta(\overline{A}, \overline{A}_0)$, where δ is defined as in Construction 7.4.6 and $\overline{A}_0$ denotes the unique element of $\mathrm{Lift}(A)$ satisfying $[\overline{A}_0] = 0 \in \mathrm{Br}(\mathbf{1}^{\leqslant n})$.

(4) We now claim that the bijection $\xi_A : \ker(\rho) \to G$ does not depend on the choice of A. To prove this, suppose that we are given some other molecular Azumaya algebra B of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ satisfying $[B] = 0 \in \mathrm{Br}(\mathbf{1}^{\leqslant n-1})$; we will show that $\xi_A(x) = \xi_B(x)$ for each $x \in \ker(\rho)$. Using (1), we can choose lifts

$$\overline{A}, \overline{A}_0 \in \mathrm{Lift}(A) \qquad \overline{B}, \overline{B}_0 \in \mathrm{Lift}(B)$$

satisfying $[\overline{A}] = [\overline{B}] = x$ and $[\overline{A}_0] = [\overline{B}_0] = 0$ in $\mathrm{Br}(\mathbf{1}^{\leqslant n})$. Then

$$[\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0] = x = [\overline{A}_0 \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}],$$

so (2) guarantees that $\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0$ and $\overline{A}_0 \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}$ represent the same element of $\mathrm{Lift}(A \wedge_{\mathbf{1}^{\leqslant n-1}} B)$. Using Corollary 7.6.2, we compute

$$\begin{aligned}
\xi_A(x) &= \delta(\overline{A}, \overline{A}_0) \\
&= \delta(\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0, \overline{A}_0 \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0) \\
&= \delta(\overline{A}_0 \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}, \overline{A}_0 \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0) \\
&= \delta(\overline{B}, \overline{B}_0) \\
&= \xi_B(x).
\end{aligned}$$

(5) It follows from (4) that there exists a unique bijection $\xi : \ker(\rho) \to G$ such that $\xi = \xi_A$ for every molecular Azumaya algebra A of $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ satisfying $[A] = 0$. We claim that ξ is a group homomorphism. Choose elements $x, y \in \mathrm{Br}(\mathbf{1}^{\leqslant n})$ satisfying $\rho(x) = \rho(y) = 0$; we wish to show that $\xi(x + y) = \xi(x) + \xi(y)$ in $\mathrm{Br}(\mathbf{1}^{\leqslant n})$. Choose Azumaya algebras $\overline{A}, \overline{B} \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})$ such that $x = [\overline{A}]$ and $y = [\overline{B}]$. Set $A = \overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$ and $B = \overline{B} \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$. Without loss of generality, we can assume that A and B are molecular. Using (1), we can choose lifts $\overline{A}_0 \in \mathrm{Lift}(A)$ and $\overline{B}_0 \in \mathrm{Lift}(B)$ satisfying $[\overline{A}_0] = [\overline{B}_0] = 0 \in \mathrm{Br}(\mathbf{1}^{\leqslant n})$. Set $C = A \wedge_{\mathbf{1}^{\leqslant n-1}} B$. Using Corollary 7.6.2, we compute

$$\begin{aligned}
\xi(x+y) &= \xi([\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}]) \\
&= \xi_C([\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}] \\
&= \delta(\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}, \overline{A}_0 \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0) \\
&= \delta(\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}, \overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0) + \delta(\overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0, \overline{A}_0 \wedge_{\mathbf{1}^{\leqslant n}} \overline{B}_0) \\
&= \delta(\overline{A}, \overline{A}_0) + \delta(\overline{B}, \overline{B}_0) \\
&= \xi_A([\overline{A}]) + \xi_B([\overline{B}]) \\
&= \xi([\overline{A}]) + \xi([\overline{B}]) \\
&= \xi(x) + \xi(y).
\end{aligned}$$

(6) We now show that the homomorphism ξ satisfies condition $(*)$. Let B be an arbitrary Azumaya algebra in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$, and suppose that we are given lifts $\overline{B}, \overline{B'} \in \mathrm{Lift}(B)$. We wish to show that $\xi([\overline{B}] - [\overline{B}']) = \delta(\overline{B}, \overline{B}')$ in the abelian group G. To prove this, choose a molecular Azumaya algebra in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n-1}}$ such that $[C] = -[B]$ in $\mathrm{Br}(\mathbf{1}^{\leqslant n-1})$. Using Proposition 7.6.1, we can lift C to an Azumaya algebra $\overline{C}$ in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Set $A = B \wedge_{\mathbf{1}^{\leqslant n-1}} C$, so that A is a molecular Azumaya algebra whose Brauer class $[A] \in \mathrm{Br}(\mathbf{1}^{\leqslant n-1})$ vanishes. Using (1), we can lift A to an Azumaya algebra $\overline{A}$ in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$ satisfying $[\overline{A}] = 0 \in \mathrm{Br}(\mathbf{1}^{\leqslant n})$. Using (5) and Corollary 7.6.2, we compute

$$\begin{aligned}
\xi([\overline{B}'] - [\overline{B}]) &= \xi([\overline{B}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{C}]) - \xi([\overline{B} \wedge_{\mathbf{1}^{\leqslant n}} \overline{C}]) \\
&= \xi_A([\overline{B}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{C}]) - \xi_A([\overline{B} \wedge_{\mathbf{1}^{\leqslant n}} \overline{C}) \\
&= \delta(\overline{B}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{C}, \overline{A}) - \delta(\overline{B} \wedge_{\mathbf{1}^{\leqslant n}} \overline{C}, \overline{A}) \\
&= \delta(\overline{B}' \wedge_{\mathbf{1}^{\leqslant n}} \overline{C}, \overline{B} \wedge_{\mathbf{1}^{\leqslant n}} \overline{C}) \\
&= \delta(\overline{B}', \overline{B}).
\end{aligned}$$

(7) We now complete the proof by showing that the homomorphism ξ is unique. Suppose that $\xi' : \ker(\rho) \to G$ is some other group homomorphism satisfying condition $(*)$. We will show that $\xi(x) = \xi'(x)$ for each $x \in \ker(\rho)$. To prove this, write $x = [\overline{A}]$ for some molecular Azumaya algebra $\overline{A}$ in $\mathrm{Syn}_{\mathbf{1}^{\leqslant n}}$. Set $A = \overline{A} \wedge_{\mathbf{1}^{\leqslant n}} \mathbf{1}^{\leqslant n-1}$. Using (1), we can choose $\overline{A}_0 \in \mathrm{Lift}(A)$ such that $[\overline{A}_0] = 0 \in \mathrm{Br}(\mathbf{1}^{\leqslant n})$. We then compute

$$\begin{aligned}
\xi(x) &= \xi([\overline{A}]) - \xi([\overline{A}_0]) \\
&= \delta(\overline{A}, \overline{A}_0) \\
&= \xi'([\overline{A}]) - \xi'([\overline{A}_0]) \\
&= \xi'(x).
\end{aligned}$$

□

Chapter 9

Subgroups of $\mathrm{Br}(E)$

Let E be a Lubin-Tate spectrum and let P be some property of E-modules which satisfies the following requirement:

$(*)$ If M and N are E-modules having the property P, then the tensor product $M \otimes_E N$ also has the property P. Moreover, the Lubin-Tate spectrum E has the property P.

In this case, we let $\mathrm{Br}^P(E)$ denote the subset of $\mathrm{Br}(E)$ spanned by those Brauer classes which can be represented by an Azumaya algebra A having the property P. It follows immediately from $(*)$ that $\mathrm{Br}^P(E)$ is a subgroup of $\mathrm{Br}(E)$. In this section, we will study subgroups

$$\mathrm{Br}^\flat(E) \subseteq \mathrm{Br}^{\mathrm{fr}}(E) \subseteq \mathrm{Br}^{\mathrm{full}}(E) \subseteq \mathrm{Br}(E)$$

which can be defined by this procedure:

- The subgroup $\mathrm{Br}^\flat(E)$ consists of those element of $\mathrm{Br}(E)$ which have the form $[A]$, where A is an Azumaya algebra which is flat as an E-module: that is, equivalent to a sum of (finitely many) copies of E.
- The subgroup $\mathrm{Br}^{\mathrm{fr}}(E)$ consists of those elements of $\mathrm{Br}(E)$ which have the form $[A]$, where A is an Azumaya algebra which is free in the sense of Definition 6.3.4: that is, equivalent to a sum of copies of E and its suspension ΣE.
- The subgroup $\mathrm{Br}^{\mathrm{full}}(E)$ consists of those elements of $\mathrm{Br}(E)$ which have the form $[A]$, where A is an Azumaya algebra which is full (in the sense of Definition 2.1.2)) when regarded as an object of the ∞-category Mod_E (for a more concrete characterization, see Proposition 9.2.2).

Remark 9.0.2. Let P be as above, and let $\mathcal{C}$ denote the full subcategory of Mod_E spanned by those E-modules having the property P. Condition $(*)$ is equivalent to the requirement that $\mathcal{C}$ is a symmetric monoidal subcategory of Mod_E. Roughly speaking, we can think of the subgroup $\mathrm{Br}^P(E) \subseteq \mathrm{Br}(E)$ defined above as a Brauer group of the ∞-category $\mathcal{C}$. Beware, however, that $\mathcal{C}$ might not fit the general paradigm of §2 (in the examples of interest to us, the ∞-category $\mathcal{C}$ does not admit geometric realizations of simplicial objects).

Warning 9.0.3. Let P be as above, and let x be an element of $\mathrm{Br}^P(E)$. The condition that x belongs to $\mathrm{Br}^P(E)$ guarantees that there exists *some* Azumaya algebra A representing x which satisfies the property P. However, it does not guarantee that *every* Azumaya algebra representing x has the property P (in the examples of interest to us, this stronger property is never satisfied).

9.1 The Subgroup $\mathrm{Br}^{\mathrm{fr}}(E)$

Let E be a Lubin-Tate spectrum with maximal ideal $\mathfrak{m}$ and residue field κ. Our first goal is to describe the group $\mathrm{Br}^{\mathrm{fr}}(E) \subseteq \mathrm{Br}(E)$ of Morita equivalence classes of Azumaya algebras A which are *free* over E. Our main result can be stated as follows:

Theorem 9.1.1. *There is a unique isomorphism of abelian groups $u : \mathrm{Br}^{\mathrm{fr}}(E) \to \mathrm{Br}(\kappa)$ with the following property: if A is an Azumaya algebra over E which is free as an E-module, then $u([A]) = [(\pi_0 A)/\mathfrak{m}(\pi_0 A)]$.*

To prove Theorem 9.1.1, it will be convenient to compare both $\mathrm{Br}^{\mathrm{fr}}(E)$ and $\mathrm{Br}(\kappa)$ with an auxiliary object: the Brauer group of the connective cover $\tau_{\geqslant 0} E$, in the sense of Definition 2.7.1.

Proposition 9.1.2. *The tautological map of $\mathbb{E}_\infty$-rings $\tau_{\geqslant 0} E \to \kappa$ induces an isomorphism of Brauer groups $\mathrm{Br}(\tau_{\geqslant 0} E) \to \mathrm{Br}(\kappa)$.*

Proof. Combine Propositions 2.7.4 and 2.6.4 (note that $\pi_0 E$ is a complete local Noetherian ring, and therefore Henselian). □

Let $L : \mathrm{Mod}_E \to \mathrm{Mod}_E^{\mathrm{loc}}$ be a left adjoint to the inclusion functor. Note that the construction

$$M \mapsto L(E \otimes_{\tau_{\geqslant 0} E} M)$$

determines a symmetric monoidal functor $\mathrm{Mod}^c_{\tau_{\geqslant 0} E} \to \mathrm{Mod}_E^{\mathrm{loc}}$, which carries full dualizable objects to full dualizable objects. Applying Proposition 2.4.1, we obtain a homomorphism of Brauer groups $\gamma : \mathrm{Br}(\tau_{\geqslant 0} E) \to \mathrm{Br}(\mathrm{Mod}_E)$.

Proposition 9.1.3. *The image of* $\gamma : \mathrm{Br}(\tau_{\geqslant 0}E) \to \mathrm{Br}(E)$ *is the subgroup* $\mathrm{Br}^{\mathrm{fr}}(E) \subseteq \mathrm{Br}(E)$.

Proof. Unwinding the definitions, we see that an element $x \in \mathrm{Br}(E)$ belongs to the image of γ if and only if $x = [E \otimes_{\tau_{\geqslant 0}E} A_0]$ for some Azumaya algebra object A_0 of the symmetric monoidal ∞-category $\mathrm{Mod}^c_{\tau_{\geqslant 0}E}$. In this case, A_0 is a free $\tau_{\geqslant 0}E$-module of finite rank, so that $A = E \otimes_{\tau_{\geqslant 0}E} A_0$ is a free E-module of finite rank. Conversely, suppose that $x = [A]$, where A is an Azumaya algebra which is free of finite rank over E. Then $A \simeq E \otimes_{\tau_{\geqslant 0}E} A_0$, where A_0 is the connective cover of A. To show that x belongs to the image of ξ, it will suffice to show that A_0 is an Azumaya algebra object of $\mathrm{Mod}^c_{\tau_{\geqslant 0}E}$. By virtue of Corollary 2.2.3, it will suffice to show that the multiplication on A_0 induces an equivalence $e : A_0 \otimes_{\tau_{\geqslant 0}E} A_0^{\mathrm{op}} \to \mathrm{End}_{\tau_{\geqslant 0}E}(A_0)$. Note that the domain and codomain of e are free modules of finite rank over $\tau_{\geqslant 0}E$. Consequently, to show that e is an equivalence, it will suffice to show that e becomes an equivalence after extending scalars along the map $\tau_{\geqslant 0}E \to E$. This follows from our assumption that A is Azumaya. □

Our next goal is to show that the map γ of Proposition 9.1.3 is injective. This is a consequence of the following algebraic assertion:

Lemma 9.1.4. *The canonical isomorphism* $\kappa \simeq \mathrm{End}_{\mathrm{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit)$ *determines a fully faithful symmetric monoidal functor*

$$F : \mathrm{Vect}_\kappa \to \mathrm{Syn}_E^\heartsuit \qquad F(V) = V \otimes_\kappa \mathbf{1}^\heartsuit$$

which induces a monomorphism of Brauer groups $\iota_0 : \mathrm{Br}(\kappa) \to \mathrm{Br}(\mathrm{Syn}_E^\heartsuit) = \mathrm{BM}(E)$.

Proof. The first assertion follows immediately from Proposition 6.3.10 and the well-definedness of ι_0 follows from Proposition 2.4.1. To show that ι_0 is injective, it will suffice to show that if M is a dualizable Milnor module and $\mathrm{End}(M)$ belongs to the essential image of F, then either M or $M[1]$ belongs to the essential image of F. To prove this, choose a $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear equivalence of monoidal categories $G : \mathrm{Syn}_E^\heartsuit \simeq \mathcal{M}(V)$ (Theorem 6.6.6), for some finite-dimensional vector space V over κ. Then we can identify $G(M)$ with a graded module over the exterior algebra $\bigwedge^*_{K_*}(V)$ which is of finite rank over K_*. Our assumption that $\mathrm{End}(M)$ belongs to the image of F guarantees that $G(\mathrm{End}(M)) = G(M) \otimes_{K_*} G(M)^\vee$ is concentrated in even degrees. Replacing M by $M[1]$ if necessary, we can assume that $G(M)$ is also concentrated in even degrees. It follows that the action of V on $G(M)$ must be trivial, so that $G(M)$ is a direct sum of copies of $K_* = G(\mathbf{1}^\heartsuit)$ and therefore M belongs to the essential image of F. □

Proposition 9.1.5. *The homomorphism γ of Proposition 9.1.2 fits into a commutative diagram of Brauer groups*

$$\begin{array}{ccc} \mathrm{Br}(\tau_{\geqslant 0}E) & \xrightarrow{\gamma} & \mathrm{Br}(E) \\ \downarrow & & \downarrow \\ \mathrm{Br}(\kappa) & \xrightarrow{\iota_0} & \mathrm{BM}(E) \end{array}$$

where ι_0 is the monomorphism of Lemma 9.1.4 and the left vertical map is the isomorphism of Proposition 9.1.4.

Proof. Let A be an Azumaya algebra object of the symmetric monoidal ∞-category $\mathrm{Mod}^c_{\tau_{\geqslant 0}E}$, and let $A_0 = \kappa \otimes_{\tau_{\geqslant 0}E} A$ be the induced Azumaya algebra over κ. To prove Proposition 9.1.5, it will suffice to show that $F(A_0)$ and $\mathrm{Sy}^{\heartsuit}[E \otimes_{\tau_{\geqslant 0}E} A]$ are isomorphic (as Azumaya algebra objects of $\mathrm{Syn}^{\heartsuit}_E$), where F is the symmetric monoidal functor of Lemma 9.1.4. This follows from the commutativity of the diagram of symmetric monoidal ∞-categories σ :

$$\begin{array}{ccc} \mathrm{Mod}^{\mathrm{fr}}_{\tau_{\geqslant 0}E} & \longrightarrow & \mathrm{Mod}_E \\ \downarrow & & \downarrow \\ \mathrm{Vect}_\kappa & \xrightarrow{F} & \mathrm{Syn}^{\heartsuit}_E, \end{array}$$

where $\mathrm{Mod}^{\mathrm{fr}}_{\tau_{\geqslant 0}E}$ denotes the full subcategory of $\mathrm{Mod}^c_{\tau_{\geqslant 0}E}$ spanned by the *free* modules over $\tau_{\geqslant 0}E$ (that is, those modules which are direct sums of copies of $\tau_{\geqslant 0}E$). The commutativity of σ is implicit in the construction of F (see §6.3). □

Corollary 9.1.6. *The map $\gamma : \mathrm{Br}(\tau_{\geqslant 0}E) \to \mathrm{Br}^{\mathrm{fr}}(E)$ is an isomorphism.*

Proof of Theorem 9.1.1. Proposition 9.1.2 and Corollary 9.1.6 supply isomorphisms

$$\mathrm{Br}(\kappa) \leftarrow \mathrm{Br}(\tau_{\geqslant 0}E) \xrightarrow{\gamma} \mathrm{Br}^{\mathrm{fr}}(E).$$

Note that if A is an Azumaya algebra over E which is free as an E-module, then this isomorphism carries the element $[A] \in \mathrm{Br}^{\mathrm{fr}}(E)$ to the class of the Azumaya algebra $\kappa \otimes_{\tau_{\geqslant 0}E} \tau_{\geqslant 0}A \simeq (\pi_0 A)/\mathfrak{m}(\pi_0 A)$. □

9.2 The Subgroup $\mathrm{Br}^{\mathrm{full}}(E)$

Let E be a Lubin-Tate spectrum. In this paper, we have defined the Brauer group $\mathrm{Br}(E)$ to be the Brauer group of the ∞-category $\mathrm{Mod}^{\mathrm{loc}}_E$ of $K(n)$-local E-modules, where n is the height of E. One can also consider the ∞-category Mod_E of *all* E-modules. However, this gives rise to a smaller Brauer group:

Proposition 9.2.1. *Let $L : \mathrm{Mod}_E \to \mathrm{Mod}_E^{\mathrm{loc}}$ be a left adjoint to the inclusion functor. Then L induces a group homomorphism $\alpha : \mathrm{Br}(\mathrm{Mod}_E) \to \mathrm{Br}(\mathrm{Mod}_E^{\mathrm{loc}}) = \mathrm{Br}(E)$. Moreover, α induces an isomorphism from $\mathrm{Br}(\mathrm{Mod}_E)$ to the subgroup $\mathrm{Br}^{\mathrm{full}}(E)$ consisting of those elements of $\mathrm{Br}(E)$ which have the form $[A]$, where A is an Azumaya algebra over E which is full when regarded as an object of Mod_E.*

Proof. To show that α is well-defined, it will suffice to show that the functor L carries full dualizable objects of Mod_E to full objects of $\mathrm{Mod}_E^{\mathrm{loc}}$ (Proposition 2.4.1). Note that if $M \in \mathrm{Mod}_E$ is dualizable, then M already belongs to $\mathrm{Mod}_E^{\mathrm{loc}}$, so we have an equivalence $LM \simeq M$ (Proposition 2.9.4). If M is full as an object of Mod_E, then it is necessarily nonzero, so that $LM \simeq M$ is a full object of $\mathrm{Mod}_E^{\mathrm{loc}}$ by virtue of Proposition 2.9.6. This proves that α is well-defined.

Note that every Azumaya algebra object A of Mod_E is also an Azumaya algebra object of $\mathrm{Mod}_E^{\mathrm{loc}}$, and the converse holds if and only if A is full as an object of Mod_E. It follows that the image of α is the subgroup $\mathrm{Br}^{\mathrm{full}}(E) \subseteq \mathrm{Br}(E)$.

We now complete the proof by showing that α is injective. Suppose that A is an Azumaya algebra object of Mod_E and that $\alpha([A])$ vanishes in $\mathrm{Br}(\mathrm{Mod}_E^{\mathrm{loc}})$; we wish to show that $[A]$ vanishes in $\mathrm{Br}(\mathrm{Mod}_E)$. The vanishing of $\alpha([A])$ guarantees that we can identify A with $\mathrm{End}(M)$ for some dualizable object $M \in \mathrm{Mod}_E^{\mathrm{loc}}$. Proposition 2.9.4 guarantees that M is also dualizable as an object of Mod_E. Moreover, since $A = M \otimes_E M^\vee$ is a full object of Mod_E, the module M must also be full as an object of Mod_E. Applying Corollary 2.1.4, we deduce that $[A]$ vanishes in $\mathrm{Br}(\mathrm{Mod}_E)$. □

The following result gives a concrete criterion for testing the fullness of dualizable objects of Mod_E:

Proposition 9.2.2. *Let M be a perfect E-module. The following conditions are equivalent:*

- (*a*) *The module M is full (as an object of the ∞-category Mod_E).*
- (*b*) *For every nonzero element $x \in \pi_0 E$, multiplication by x induces a nonzero map $\pi_* M \xrightarrow{x} \pi_* M$.*

The proof of Proposition 9.2.2 will require some preliminaries.

Lemma 9.2.3. *Let M be a nonzero E-module. Then there exists a prime ideal $\mathfrak{p} \subseteq \pi_0 E$ and a regular system of parameters $x_1, \ldots, x_k$ for the local ring $(\pi_0 E)_{\mathfrak{p}}$ for which the tensor product $M \otimes_E A$ is nonzero, where $A = \otimes_{1 \leqslant i \leqslant k} \mathrm{cofib}(x_i : E_{\mathfrak{p}} \to E_{\mathfrak{p}})$.*

Proof. Let $\mathfrak{p}$ be minimal among those prime ideals of $\pi_0 E$ for which the localization $M_{\mathfrak{p}}$ is nonzero (such a prime ideal must exist by virtue of our assumption that M is nonzero). Choose elements $x_1, \ldots, x_k \in \mathfrak{p}$ which form a regular system of parameters for the local ring $(\pi_0 E)_{\mathfrak{p}}$. For $0 \leqslant j \leqslant k$, let $A(i)$ denote the tensor product $\bigotimes_{1 \leqslant i \leqslant j} \operatorname{cofib}(x_i : E_{\mathfrak{p}} \to E_{\mathfrak{p}})$ (formed in the ∞-category of modules over the localization $E_{\mathfrak{p}}$), so that $A(0) = E_{\mathfrak{p}}$ by convention. We will prove the following assertion $0 \leqslant i \leqslant k$:

$(*_i)$ The tensor product $A(i) \otimes_E M$ is nonzero.

Note that $(*_0)$ follows from our assumption that the localization $M_{\mathfrak{p}}$ is nonzero, and $(*_k)$ implies Lemma 9.2.3. We will complete the proof by showing that $(*_{i-1})$ implies $(*_i)$. Note that we have a cofiber sequence

$$A(i-1) \otimes_E M \xrightarrow{x_i} A(i-1) \otimes_E M \to A(i) \otimes_E M.$$

Consequently, if $(*_i)$ is not satisfied, then multiplication by x_i induces an equivalence from $A(i-1) \otimes_E M$ to itself, so that the tautological map $e : A(i-1) \otimes_E M \to A(i-1) \otimes_E M[x_i^{-1}]$ is an equivalence. Assumption $(*_{i-1})$ guarantees that the domain of e is nonzero. It follows that the codomain of e is also nonzero, so that the localization $M_{\mathfrak{p}}[x_i^{-1}]$ is nonzero. It follows that there exists a prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$ which does not contain x_i for which the localization $M_{\mathfrak{q}}$ is nonzero, contradicting our assumption that $\mathfrak{p}$ is minimal. □

Lemma 9.2.4. *Let M be an E-module spectrum. The following conditions are equivalent:*

(1) *The module M is a full object of Mod_E: that is, the functor $N \mapsto M \otimes_E N$ is conservative.*

(2) *For every nonzero E-module N, the tensor product $M \otimes_E N$ is nonzero.*

(3) *For every prime ideal $\mathfrak{p} \subseteq \pi_0 E$ and every regular system of parameters $x_1, \ldots, x_k$ for the local ring $(\pi_0 E)_{\mathfrak{p}}$, the tensor product $M \otimes_E A$ is nonzero, where A is defined as in Lemma 9.2.3.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious, and the implication (2) ⇒ (1) follows from the stability of the ∞-category Mod_E. We will complete the proof by showing that (3) ⇒ (2). Let N be a nonzero E-module; we wish to show that $M \otimes_E N$ is nonzero. Applying Lemma 9.2.3, we can choose a prime ideal $\mathfrak{p} \subseteq \pi_0 E$ and a regular system of parameters $x_1, \ldots, x_k$ for the local ring $(\pi_0 E)_{\mathfrak{p}}$ for which the tensor product $A \otimes_E N$ is nonzero, where A is defined as in Lemma 9.2.3. We note that A admits the structure of an algebra over the localization $E_{\mathfrak{p}}$ (for example, it can be obtained by a variant of the Thom spectrum construction studied in §3). By construction, $\pi_* A$ is isomorphic to $\kappa(\mathfrak{p})[t^{\pm 1}]$, where $\kappa(\mathfrak{p})$ denotes the residue field of the local ring $(\pi_0 E)_{\mathfrak{p}}$ and the element t has degree 2. It follows that every (left or right) A-module spectrum can be decomposed as a direct sum of copies of A and the suspension ΣA. Assumption

(3) guarantees that the tensor product $M \otimes_E A$ is nonzero. Consequently, we can assume that $M \otimes_E A$ contains $\Sigma^i A$ as a direct summand (as a right A-module) and $A \otimes_E N$ contains $\Sigma^j A$ as a direct summand (as a left A-module), for some $i, j \in \{0, 1\}$. It then follows that the tensor product

$$M \otimes_E A \otimes_E N \simeq (M \otimes_E A) \otimes_A (A \otimes_E N)$$

contains $\Sigma^{i+j} A$ as a direct summand (in the ∞-category Mod_E). In particular, $M \otimes_E A \otimes_E N$ is nonzero, so that $M \otimes_E N$ is also nonzero. □

Proof of Proposition 9.2.2. Let M be a perfect E-module. If M is full and $x \in \pi_0 E$ is nonzero, then the localization $M[x^{-1}] \simeq M \otimes_E E[x^{-1}]$ must be nonzero, so the abelian group $\pi_* M[x^{-1}] \simeq (\pi_* M)[x^{-1}]$ is likewise nonzero, which shows that $(a) \Rightarrow (b)$.

Conversely, suppose that (b) is satisfied. We will show that M satisfies condition (3) of Proposition 9.2.4. Fix a prime ideal $\mathfrak{p} \subseteq \pi_0 E$ and a regular system of parameters $x_1, \ldots, x_k$ for the local ring $(\pi_0 E)_{\mathfrak{p}}$. Note that $\pi_* M$ is a finitely generated module over the Noetherian ring $\pi_* E$ (Proposition 2.9.4) which is not annihilated by any element of $(\pi_0 E) - \mathfrak{p}$. It follows that the localization $M_{\mathfrak{p}}$ is a nonzero (perfect) module over the localization $E_{\mathfrak{p}}$. For $0 \leqslant i \leqslant k$, define $A(i)$ as in the proof of Lemma 9.2.3, and set $M(i) = A(i) \otimes_E M$. We will prove the following assertion $0 \leqslant i \leqslant k$:

($*_i$) Each homotopy group of $M(i)$ is a nonzero finitely generated module over the commutative ring $\pi_0 E_{\mathfrak{p}}$.

Note that assertion ($*_0$) is obvious (since $M(0) = M_{\mathfrak{p}}$), and assertion ($*_k$) will complete the proof of (a) (by virtue of Proposition 9.2.4). It will therefore suffice to show that ($*_{i-1}$) implies ($*_i$). For this, we note that we have a fiber sequence

$$M(i-1) \xrightarrow{x_i} M(i-1) \to M(i)$$

which yields a long exact sequence of homotopy groups

$$\pi_m M(i-1) \xrightarrow{x_i} \pi_m M(i-1) \to \pi_m M(i) \to \pi_{m-1} M(i-1) \to \pi_{m-1} M(i-1).$$

Here the outer terms are finitely generated modules over $\pi_0 E_{\mathfrak{p}}$ (by virtue of assumption ($*_{i-1}$)), so the middle term is as well (since the ring $\pi_0 E_{\mathfrak{p}}$ is Noetherian). Moreover, if we choose m so that $\pi_m M(i-1)$ is nonzero, then $\pi_m M(i)$ is also nonzero (the map $x_i : \pi_m M(i-1) \to \pi_m M(i-1)$ cannot be surjective, by Nakayama's Lemma). □

9.3 The Subgroup $\mathrm{Br}^\flat(E)$

Let E be a Lubin-Tate spectrum and let κ be the residue field of E. Our goal in this section is to show that the subgroup $\mathrm{Br}^\flat(E) \subseteq \mathrm{Br}(E)$ (defined in the introduction to §9) is isomorphic to the Brauer-Wall group $\mathrm{BW}(\kappa)$ (defined in §2.8). More precisely, we have the following result:

Theorem 9.3.1. *The composite map*

$$\mathrm{Br}^{\flat}(E) \hookrightarrow \mathrm{Br}(E) \to \mathrm{BM}(E)$$

is a monomorphism, whose image coincides with the image of the monomorphism $\iota : \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \to \mathrm{BM}(E)$ *be the monomorphism* ι *of Remark 6.9.5.*

Corollary 9.3.2. *The Brauer group* $\mathrm{Br}^{\flat}(E)$ *is isomorphic to the Brauer-Wall group* $\mathrm{BW}(\kappa)$.

Proof. Combine Theorem 9.3.1 with Proposition 5.7.1. □

Warning 9.3.3. The isomorphism $\mathrm{Br}^{\flat}(E) \simeq \mathrm{BW}(\kappa)$ of Corollary 9.3.2 is not quite canonical: it depends on a choice of nonzero element $t \in K_2$ (see Warning 5.7.2).

Remark 9.3.4. Suppose that the residue field κ of E has characteristic different from 2. It follows from Theorem 9.3.1 and Proposition 6.9.3 that Brauer group $\mathrm{Br}(E)$ splits as a direct sum $\mathrm{Br}^{\flat}(E) \oplus \mathrm{Br}'(E)$, where $\mathrm{Br}'(E)$ is the kernel of the composite map

$$\mathrm{Br}(E) \to \mathrm{BM}(E) \xrightarrow{\rho} \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \times \mathrm{QF} \to \mathrm{QF}$$

(here ρ is the isomorphism of Proposition 6.9.3). Using Theorem 8.0.5, we deduce that $\mathrm{Br}'(E)$ can be obtained as the inverse limit of a tower of surjective group homomorphisms

$$\cdots \mathrm{Br}'(\mathbf{1}^{\leqslant 3}) \xrightarrow{\rho_3} \mathrm{Br}'(\mathbf{1}^{\leqslant 2}) \xrightarrow{\rho_2} \mathrm{Br}'(\mathbf{1}^{\leqslant 1}) \xrightarrow{\rho_1} \mathrm{Br}'(\mathbf{1}^{\leqslant 0}) \xrightarrow{\rho_0} 0,$$

where we have canonical isomorphisms $\ker(\rho_i) \simeq K_{-2} \otimes_{\kappa} (\mathfrak{m}^{i+2}/\mathfrak{m}^{i+3})$ After making a choice of nonzero element $t \in K_2$, we obtain the (slightly less canonical) description of $\mathrm{Br}(E)$ given in Theorem 1.0.11.

Theorem 9.3.1 is an immediate consequence of the following three assertions:

Proposition 9.3.5. *There exists a commutative diagram of Brauer groups*

$$\begin{array}{ccc} \mathrm{Br}^{\flat}(E) & \longrightarrow & \mathrm{Br}(E) \\ \downarrow & & \downarrow \\ \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) & \xrightarrow{\iota} & \mathrm{BM}(E), \end{array}$$

where the upper horizontal map is the canonical inclusion, the bottom horizontal map is the monomorphism of Remark 6.9.5, and the right vertical map is induced by the functor $\mathrm{Syn}^{\heartsuit} : \mathrm{Mod}^{\mathrm{loc}}_{E} \to \mathrm{Syn}^{\heartsuit}_{E}$.

Proposition 9.3.6. *The image of the composite map* $\mathrm{Br}^\flat(E) \hookrightarrow \mathrm{Br}(E) \to \mathrm{BM}(E)$ *contains the image of the monomorphism* $\iota : \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \to \mathrm{BM}(E)$.

Proposition 9.3.7. *The map* $\mathrm{Br}(E) \to \mathrm{BM}(E)$ *is a monomorphism when restricted to the subgroup* $\mathrm{Br}^\flat(E)$.

Proposition 9.3.5 follows immediately from the construction of the map ι (by definition, if A is an E-algebra having the property that $\pi_* A$ is a free module over $\pi_* E$, then the Milnor module $\mathrm{Syn}^\heartsuit[A]$ is constant in the sense of Definition 6.3.7).

Proof of Proposition 9.3.6. For simplicity, let us assume that the field κ has characteristic different from 2 (for the characteristic 2 case, we refer the reader to Remark 9.3.8 below). Choose a nonzero element $t \in \pi_2 E$, so that we can identify $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$ with the category $\mathrm{Vect}^{\mathrm{gr}}_\kappa$ of $(\mathbf{Z}/2\mathbf{Z})$-graded vector spaces and the Brauer group $\mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*})$ with the Brauer-Wall group $\mathrm{BW}(\kappa)$ of the field κ. According to Remark 2.8.11, the group $\mathrm{BW}(\kappa)$ is generated by the image of $\mathrm{Br}(\kappa)$ together with elements of the form $[\kappa(\sqrt{a})]$ for $a \in \kappa^\times$. Using Theorem 9.1.1, we are reduced to showing that $\iota([\kappa(\sqrt{a})])$ belongs to the image of α_0, for each $a \in \kappa^\times$. We will prove this by establishing the following more precise assertion:

- $(*)$ There exists a full Azumaya algebra $A \in \mathrm{Alg}_E$ such that $\mathrm{Sy}^\heartsuit[A]$ is isomorphic to the tensor product $\mathbf{1}^\heartsuit \otimes_\kappa \kappa(\sqrt{a})$ (as an associative algebra object of the category $\mathrm{Syn}^\heartsuit_E$ of Milnor modules).

To prove $(*)$, choose an element $\overline{a} \in (\pi_0 E)^\times$ representing a, and $\mathrm{Pic}(E)$ denote the Picard space of E (see §3.1). Unwinding the definitions, we see that the truncation $\tau_{\leqslant 1} \mathrm{Pic}(E)$ can be identified with the groupoid $\mathcal{C}$ of free graded $(\pi_* E)$-modules (up to isomorphism, this category has two objects, given by $\pi_* E$ and its shift $\pi_* \Sigma E$). The tensor product of E-modules endows $\mathrm{Pic}(E)$ with the structure of an $\mathbb{E}_\infty$-space and $\mathcal{C}$ with the structure of a symmetric monoidal category. Note that the data of a monoidal functor $\mathbf{Z}/2\mathbf{Z} \to \tau_{\leqslant 1} \mathrm{Pic}(E)$ is equivalent to the data of an object $L \in \mathcal{C}$ equipped with an isomorphism $e : L^{\otimes 2} \simeq \pi_* E$. In particular, we can choose a monoidal functor corresponding to the object $L = \pi_*(\Sigma E)$, where e is the isomorphism $\pi_*(\Sigma^2 E) \to \pi_* E$ given by multiplication by $\overline{a}t$. This monoidal functor determines a map of classifying spaces $Q_0 : \mathrm{B}\,\mathbf{Z}/2\mathbf{Z} \to \tau_{\leqslant 2} \mathrm{BPic}(E)$. Since the homotopy groups $\pi_n \mathrm{BPic}(E)$ are uniquely 2-divisible for $n > 2$, we can lift Q_0 to a map $Q : \mathrm{B}\,\mathbf{Z}/2\mathbf{Z} \to \mathrm{BPic}(E)$. Let A denote the Thom spectrum of the induced map $\Omega(Q) : \mathbf{Z}/2\mathbf{Z} \to \mathrm{Pic}(E)$. Unwinding the definitions, we see that A can be identified with the direct sum $E \oplus \Sigma E$, and that we have a canonical isomorphism $\pi_* A \simeq (\pi_* E)[x]/(x^2 - \overline{a}t)$ where x is homogeneous of degree 1. It is now easy to see that A satisfies the requirements of $(*)$. $\square$

Remark 9.3.8. Proposition 9.3.6 remains valid when κ has characteristic 2, but requires a different proof. As above, we can choose a nonzero element $t \in K_2$, which supplies an isomorphism $\mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \simeq \mathrm{BW}(\kappa)$. Let A_0 be a graded Azumaya algebra over κ. The assumption that κ has characteristic 2 guarantees that $[A_0]$ is annihilated by the map $\mathrm{BW}(\kappa) \to \mathrm{BW}(\kappa')$, for some finite Galois extension κ' of κ (see Remark 2.8.10). In this case, we can choose an isomorphism $\kappa' \otimes_\kappa A_0 \simeq \mathrm{End}_{\kappa'}(V)$, where V is a $(\mathbf{Z}/2\mathbf{Z})$-graded vector space over κ. It follows that there exists a semilinear action of the Galois group $G = \mathrm{Gal}(\kappa'/\kappa)$ on the algebra $\mathrm{End}_{\kappa'}(V)$, whose algebra of invariants can be identified with A_0. In this case, we can regard κ' as the residue field of a Lubin-Tate spectrum E' which is étale over E, and we can lift V to an E'-module M such that $\pi_* M$ is a finitely generated free module over $\pi_* E'$. The action of the Galois group G on $\mathrm{End}_{\kappa'}(V)$ can then be lifted to an action of G on $\mathrm{End}_{E'}(M)$ (in the ∞-category of E'-algebras), whose (homotopy) fixed point algebra $A = \mathrm{End}_{E'}(M)^G$ is an Azumaya algebra over E. It is then easy to check that the Brauer class $[A]$ belongs to $\mathrm{Br}^\flat(E)$ which is a preimage of $\iota([A_0])$ under the map $\mathrm{Br}^\flat(E) \hookrightarrow \mathrm{Br}(E) \to \mathrm{BM}(E)$.

Proof of Proposition 9.3.7. Let x belong to the kernel of the map $\mathrm{Br}^\flat(E) \to \mathrm{BM}(E)$; we wish to show that x vanishes. Write $x = [A]$, where A is an Azumaya algebra for which $\pi_* A$ is a free module over $\pi_* E$. Then the Milnor module $\mathrm{Sy}^\heartsuit[A]$ can be identified with $\mathrm{End}(M)$, where M is a nonzero dualizable object of $\mathrm{Syn}_E^\heartsuit$. Our assumption that $\pi_* A$ is free over $\pi_* E$ guarantees that the Milnor module $\mathrm{Sy}^\heartsuit[A]$ is constant. Applying Corollary 6.6.10, we deduce that M is constant. Replacing M by $M[1]$ if necessary, we may assume that M contains $\mathbf{1}^\heartsuit$ as a direct summand. Let $\overline{e} : M \to M$ denote the associated projection map. Then we can identify $\overline{e}$ with an element of $\mathrm{Hom}_{\mathrm{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit, \mathrm{End}(M)) \simeq (\pi_0 A)/\mathfrak{m}(\pi_0 A)$. Because $\pi_0 E$ is a Henselian local ring (and $\pi_0 A$ is a finite algebra over $\pi_0 E$), we can lift $\overline{e}$ to an idempotent element $e \in \pi_0 A$. This idempotent determines a decomposition $A \simeq N \oplus N'$ in the ∞-category of left A-modules, where $\pi_* N \simeq (\pi_* A)e$ and $\pi_* N' \simeq (\pi_* A)(1-e)$. The left action of A on N endows $\mathrm{Sy}^\heartsuit[N]$ with the structure of a left module over $\mathrm{Sy}^\heartsuit[A]$. By construction, this module is isomorphic to M. Consequently, the map $A \to \mathrm{End}_E(N)$ induces an isomorphism of Milnor modules, and is therefore an equivalence (Corollary 7.3.8). It follows that $x = [A]$ vanishes in $\mathrm{Br}^\flat(E) \subseteq \mathrm{Br}(E)$, as desired. □

9.4 Comparison of $\mathrm{Br}^\flat(E)$ and $\mathrm{Br}^{\mathrm{full}}(E)$

Let E be a Lubin-Tate spectrum with residue field κ.

Conjecture 9.4.1. If the residue field κ has characteristic different from 2, then the inclusion $\mathrm{Br}^\flat(E) \subseteq \mathrm{Br}^{\mathrm{full}}(E)$ is an equality. In other words, every Azumaya algebra A

over E which is full as an E-module is Morita equivalent to an Azumaya algebra B such that $\pi_* B$ is free as a module over $\pi_* E$.

As partial evidence for Conjecture 9.4.1, we offer the following:

Proposition 9.4.2. *Assume that κ has characteristic different from 2 and let $\rho : \mathrm{BM}(E) \to \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \times \mathrm{QF}$ be the isomorphism of Proposition 6.9.3. Then the composite map*

$$\mathrm{Br}^{\mathrm{full}}(E) \to \mathrm{Br}(E) \to \mathrm{BM}(E) \xrightarrow{\rho} \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \times \mathrm{QF} \to \mathrm{QF}$$

vanishes.

Remark 9.4.3. Assume that κ has characteristic different from 2. If Conjecture 9.4.1 is satisfied, then Proposition 9.4.2 follows immediately from Proposition 9.3.5. Conversely, Propositions 9.4.2 and 9.3.6 guarantee that $\mathrm{Br}^{\mathrm{full}}(E)$ and $\mathrm{Br}^{\flat}(E)$ have the same image in the Brauer-Milnor group $\mathrm{BM}(E)$. It follows that Conjecture 9.4.1 is equivalent to the assertion that the composite map $\mathrm{Br}^{\mathrm{full}}(E) \to \mathrm{Br}(E) \to \mathrm{BM}(E)$ is a monomorphism (see Proposition 9.3.7).

Proof of Proposition 9.4.2. Let $\mathfrak{m} \subseteq \pi_0 E$ denote the maximal ideal, let K_* denote the graded ring $(\pi_* E)/\mathfrak{m}(\pi_* E)$, and set $V = (\mathfrak{m}/\mathfrak{m}^2)^\vee$ so that QF is the set of quadratic forms $q : V \to K_{-2}$. Let A be a full Azumaya algebra over E, and let η denote the class of the Azumaya algebra $\mathrm{Sy}^{\heartsuit}[A]$ in the Brauer-Milnor group $\mathrm{BM}(E)$, so that we can write $\rho(\eta) = (\eta_0, q)$ for some $q \in \mathrm{QF}$; we wish to prove that $q = 0$. The associated bilinear form of q determines a linear map

$$\lambda : V \to \mathrm{Hom}_\kappa(V, K_{-2}) = (\mathfrak{m}/\mathfrak{m}^2) \otimes_\kappa K_{-2}.$$

Since the characteristic κ is different from 2, it will suffice to show that the map λ vanishes. Assume otherwise. Then we can choose an element $v \in V$ such that $\lambda(v) \neq 0$. Write $\lambda(v) = t^{-1}\overline{x}$, where t is nonzero element of K_2 and $\overline{x}$ is the residue class of some element $x \in \mathfrak{m} - \mathfrak{m}^2$.

Let $F : \mathrm{Syn}^{\heartsuit}_E \to \mathcal{M}(V)$ be the normalized $\mathrm{Mod}^{\mathrm{gr}}_{K_*}$-linear equivalence of symmetric monoidal categories of Proposition 6.9.1, and set $B = F(\mathrm{Sy}^{\heartsuit}[A])$. Write $B = (B_*, \{d_w\}_{w \in V})$, where B^* is a graded K_*-algebra equipped with derivations $d_w : B_* \to B_{*-1}$. For each element $w \in V$, Proposition 5.7.4 supplies a unique element $b_w \in B_{-1}$ satisfying the identity $d_w(b) = b_w b + (-1)^{\deg(b)+1} b b_w$. In particular, we have

$$d_w(b_v) = b_w b_v + b_v b_w = \langle w, \lambda(v) \rangle = t^{-1} w(\overline{x}) \tag{9.1}$$

Let $M_{\overline{x}}$ be the $\bigwedge^*_{K_*}(V)$-module described in Construction 5.5.1, so that we have an exact sequence

$$0 \to K_*[-1] \to M_{\overline{x}} \to K_* \to 0.$$

Tensoring with B, we obtain an exact sequence

$$0 \to B[-1] \to M_{\overline{x}} \otimes_{K_*} B \xrightarrow{u} B \to 0 \tag{9.2}$$

Note that we can identify $M_{\overline{x}} \otimes_{K_*} B$ with $B[-1] \oplus B$ as a graded K_*-module, with action of $\bigwedge^*_{K_*}(V)$ given by the formula

$$w(b', b) = (-d_w b' + w(\overline{x})b, d_w b).$$

An elementary calculation using (9.1) shows that the construction $b \mapsto (t^{-1} b_v b, b)$ determines a section of u in the category $\mathcal{M}(V)$: that is, the exact sequence (9.2) splits (in the abelian category $\mathcal{M}(V)$) Shifting and invoking our assumption that F is normalized, we deduce that the exact sequence of Milnor modules

$$0 \to \mathrm{Sy}^{\heartsuit}[A] \to \mathrm{Sy}^{\heartsuit}[\mathrm{cofib}(x : A \to A)] \to \mathrm{Sy}^{\heartsuit}[\Sigma A] \to 0$$

splits. In particular, the Milnor module $\mathrm{Sy}^{\heartsuit}[A]$ is isomorphic to a direct summand of $\mathrm{Sy}^{\heartsuit}[\mathrm{cofib}(x : A \to A)]$.

Since x does not belong to $\mathfrak{m}^2$, we can extend x to a regular system of parameters $x, y_1, \ldots, y_m$ for the local ring $\pi_0 E$. Set $B = \bigotimes_{1 \leqslant i \leqslant m} \mathrm{cofib}(E \xrightarrow{y_i} E)$. Using the above argument (and the fact that the functor $\mathrm{Sy}^{\heartsuit}$ is symmetric monoidal), we deduce that the Milnor module $\mathrm{Sy}^{\heartsuit}[A \otimes_E B]$ is a direct summand of

$$\mathrm{Sy}^{\heartsuit}[\mathrm{cofib}(x : A \to A) \otimes_E B] \simeq \mathrm{Sy}^{\heartsuit}[A \otimes_E \mathrm{cofib}(x : B \to B)].$$

By construction, the cofiber $\mathrm{cofib}(x : B \to B)$ is an atomic E-module, so that $A \otimes_E \mathrm{cofib}(x : B \to B)$ is a quasi-molecular E-module. It follows that the Milnor module $\mathrm{Sy}^{\heartsuit}[A \otimes_E \mathrm{cofib}(x : B \to B)]$ is quasi-molecular, and therefore the direct summand $\mathrm{Sy}^{\heartsuit}[A \otimes_E B]$ is quasi-molecular. Applying Corollary 6.2.6, we deduce that E-module $A \otimes_E B$ is a quasi-molecular. In particular, the localization

$$(A \otimes_E B)[x^{-1}] \simeq A \otimes_E B[x^{-1}]$$

vanishes. This contradicts our assumption that A is full, since $B[x^{-1}]$ is a nonzero E-module. $\square$

Chapter 10

Atomic Azumaya Algebras

Let κ be a field. Then every element of $\mathrm{Br}(\kappa)$ has the form $[D]$, where D is a central division algebra over κ. Moreover, the division algebra D is unique up to isomorphism. In this section, we establish a weak analogue for the Brauer group $\mathrm{Br}(E)$ of a Lubin-Tate spectrum E: if an element $x \in \mathrm{Br}(E)$ can be represented by an *atomic* Azumaya algebra A, then A is determined up to equivalence (Proposition 10.1.1). We also characterize those elements $x \in \mathrm{Br}(E)$ which admit such representatives, at least when the residue field of E has characteristic $\neq 2$ (Theorem 10.3.1). Beware, however, that not every element of $\mathrm{Br}(E)$ has this property (see Example 10.1.6).

10.1 Atomic Elements of $\mathrm{Br}(E)$

Let E be a Lubin-Tate specturm, which we regard as fixed throughout this section.

Proposition 10.1.1. *Let $A, B \in \mathrm{Alg}_E$ be atomic E-algebras. Then A and B are equivalent (as objects of the ∞-category Alg_E) if and only if they are Morita equivalent (in the sense of Definition 2.1.1).*

Corollary 10.1.2. *Let $A, B \in \mathrm{Alg}_E$ be atomic Azumaya algebras. Then A and B are equivalent (as objects of the ∞-category Alg_E) if and only if the Brauer classes $[A]$ and $[B]$ are identical (as elements of the abelian group $\mathrm{Br}(E)$).*

Definition 10.1.3. Let x be an element of $\mathrm{Br}(E)$. We will say that x is *atomic* if we can write $x = [A]$, where A is an atomic Azumaya algebra over E.

Remark 10.1.4. It follows from Corollary 10.1.2 that the construction $A \mapsto [A]$ determines a bijection of sets

$$\{\text{Atomic Azumaya algebras over } E\}/\text{equivalence} \simeq \{\text{Atomic elements of } \mathrm{Br}(E)\}.$$

Proof of Proposition 10.1.1. The "only if" direction is obvious. For the converse, assume that A and B are atomic E-algebras which are Morita equivalent. It follows that there exists a $\mathrm{Mod}_E^{\mathrm{loc}}$-linear equivalence of ∞-categories

$$\lambda : \mathrm{LMod}_A = \mathrm{LMod}_A(\mathrm{Mod}_E^{\mathrm{loc}}) \to \mathrm{LMod}_B(\mathrm{Mod}_E^{\mathrm{loc}}) \simeq \mathrm{LMod}_B \,.$$

Note that the ring $\kappa = \pi_0 A$ is a field. In particular, it contains no idempotents other than 0 and 1. It follows that A is indecomposable as a left A-module spectrum: that is, it cannot be written as direct sum $M \oplus N$ where M and N are both nonzero. Because the functor λ is an equivalence, it follows that $\lambda(B)$ is an indecomposable object of LMod_B. Our assumption that B is atomic guarantees that every B-module can be decomposed as a direct sum of modules of the form B and ΣB (Proposition 3.6.3). It follows that $\lambda(A)$ is equivalent either to B or ΣB (as an object of LMod_B). Composing λ with the suspension functor Σ if necessary, we may assume that there exists an equivalence $\lambda(A) \simeq B$.

Let $\mathcal{E} = (\mathrm{Mod}_{\mathrm{Mod}_E^{\mathrm{loc}}}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Mod}_E^{\mathrm{loc}}/})$ denote the ∞-category whose objects are pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is a presentable ∞-category equipped with an action of $\mathrm{Mod}_E^{\mathrm{loc}}$ and $C \in \mathcal{C}$ is a distinguished object (which we can identify with a $\mathrm{Mod}_E^{\mathrm{loc}}$-linear functor $\lambda : \mathrm{Mod}_E^{\mathrm{loc}} \to \mathcal{C}$). Our assumption that $\lambda(A) \simeq B$ guarantees that λ can be promoted to an equivalence $\overline{\lambda} : (\mathrm{LMod}_A, A) \simeq (\mathrm{LMod}_B, B)$ in the ∞-category $\mathcal{E}$. According to Theorem HA.4.8.5.5 , the construction $R \mapsto (\mathrm{LMod}_R(\mathrm{Mod}_E^{\mathrm{loc}}), R)$ determines a fully faithful embedding $\mathrm{Alg}(\mathrm{Mod}_E^{\mathrm{loc}}) \hookrightarrow \mathcal{E}$. It follows that $\overline{\lambda}$ can be lifted to an equivalence $A \simeq B$ in the ∞-category $\mathrm{Alg}(\mathrm{Mod}_E^{\mathrm{loc}}) \subseteq \mathrm{Alg}_E$. □

Remark 10.1.5. The proof of Proposition 10.1.1 does not require the full strength of our assumption that A is atomic: it is sufficient to assume that B is atomic and that the ring $\pi_0 A$ does not contain idempotent elements different from 0 and 1.

Example 10.1.6. The identity element $0 \in \mathrm{Br}(E)$ is not atomic. In other words, there does not exist an atomic Azumaya algebra B satisfying $[E] = [B]$ in $\mathrm{Br}(E)$. This follows immediately from Remark 10.1.5 (since $\pi_0 E$ is an integral domain, and B is not equivalent to E as an E-algebra).

10.2 Atomic Elements of $\mathrm{BM}(E)$

Let E be a Lubin-Tate spectrum and let x be an element of the Brauer group $\mathrm{Br}(E)$. Our goal in this section is to show that the question of whether or not x is atomic (in the sense of Definition 10.1.3) depends only on the image of x in the Brauer-Milnor group $\mathrm{BM}(E)$ (Proposition 10.2.5). We begin by establishing some purely algebraic analogues of the results of §10.1.

Lemma 10.2.1. *Let B be an atomic algebra object of the abelian category of Milnor modules $\mathrm{Syn}_E^\heartsuit$, and let M be a left B-module object (in the abelian category $\mathrm{Syn}_E^\heartsuit$). Then M can be decomposed as a direct sum of modules of the form B and $B[1]$.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of $\pi_0 E$ and let K_* denote the graded ring $(\pi_* E)/\mathfrak{m}(\pi_* E)$. Combining Proposition 5.6.9 with Theorem 6.6.6, we obtain an equivalence of categories $\mathrm{LMod}_B(\mathrm{Syn}_E^\heartsuit) \simeq \mathrm{Mod}_{K_*}^{\mathrm{gr}}$. It now suffices to observe that every graded K_*-module can be decomposed as a direct sum of copies of K_* and $K_*[1]$. □

Proposition 10.2.2. *Let A and B be atomic algebra objects of $\mathrm{Syn}_E^\heartsuit$. Then A and B are isomorphic (as algebra objects of $\mathrm{Syn}_E^\heartsuit$) if and only if they are Morita equivalent (in the sense of Definition 2.1.1).*

Proof. We proceed as in the proof of Proposition 10.1.1. The "only if" direction is obvious. To prove the converse, assume that A and B are Morita equivalent: that is, there exists a $\mathrm{Syn}_E^\heartsuit$-linear equivalence of categories

$$\lambda : \mathrm{LMod}_A(\mathrm{Syn}_E^\heartsuit) \to \mathrm{LMod}_B(\mathrm{Syn}_E^\heartsuit).$$

Note that the endomorphism ring of A in the abelian category of left A-modules can be identified with $\mathrm{Hom}_{\mathrm{Syn}_E^\heartsuit}(\mathbf{1}^\heartsuit, A) \simeq \kappa$, where κ is the residue field of E (see Proposition 6.4.4). It follows that A is an indecomposable object of the abelian category $\mathrm{LMod}_A(\mathrm{Syn}_E^\heartsuit)$, so that $\lambda(A)$ is an indecomposable object of $\mathrm{LMod}_B(\mathrm{Syn}_E^\heartsuit)$. Invoking Lemma 10.2.1, we deduce that $\lambda(A)$ is isomorphic to either B or $B[1]$. Replacing λ by $\lambda[-1]$ if necessary, we may assume that $\lambda(A)$ is isomorphic to B.

Let $\mathcal{E} = (\mathrm{Mod}_{\mathrm{Syn}_E^\heartsuit}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Syn}_E^\heartsuit /})$ denote the ∞-category whose objects are pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is a presentable ∞-category equipped with an action of $\mathrm{Syn}_E^\heartsuit$ and $C \in \mathcal{C}$ is a distinguished object (which we can identify with a $\mathrm{Syn}_E^\heartsuit$-linear functor $\lambda : \mathrm{Syn}_E^\heartsuit \to \mathcal{C}$). Our assumption that $\lambda(A) \simeq B$ guarantees that λ can be promoted to an equivalence $\overline{\lambda} : (\mathrm{LMod}_A(\mathrm{Syn}_E^\heartsuit), A) \simeq (\mathrm{LMod}_B(\mathrm{Syn}_E^\heartsuit), B)$ in the ∞-category $\mathcal{E}$. According to Theorem HA.4.8.5.5 , the construction $R \mapsto (\mathrm{LMod}_R(\mathrm{Syn}_E^\heartsuit), R)$ determines a fully faithful embedding $\mathrm{Alg}(\mathrm{Syn}_E^\heartsuit) \hookrightarrow \mathcal{E}$. It follows that $\overline{\lambda}$ can be lifted to an isomorphism $A \simeq B$ in $\mathrm{Alg}(\mathrm{Syn}_E^\heartsuit)$. □

Corollary 10.2.3. *Let A and B be atomic Azumaya algebra objects of the abelian category $\mathrm{Syn}_E^\heartsuit$. Then A and B are isomorphic (as algebra objects of $\mathrm{Syn}_E^\heartsuit$) if and only if the Brauer classes $[A]$ and $[B]$ are equal (as elements of the Brauer-Milnor group $\mathrm{BM}(E)$).*

Definition 10.2.4. Let E be a Lubin-Tate spectrum. We will say that an element $x \in \mathrm{BM}(E)$ is *atomic* if we can write $x = [A]$, where A is an atomic Azumaya algebra object of $\mathrm{Syn}_E^\heartsuit$.

Proposition 10.2.5. *Let x be an element of the Brauer group $\mathrm{Br}(E)$ and let $\overline{x}$ denote the image of x in the Brauer-Milnor group $\mathrm{BM}(E)$. Then x is atomic (in the sense of Definition 10.1.3) if and only if $\overline{x}$ is atomic (in the sense of Definition 10.2.4).*

Proof. The "only if" direction is clear: if we can write $x = [A]$ for some atomic Azumaya algebra $A \in \mathrm{Alg}_E$, then $\mathrm{Sy}^{\heartsuit}[A]$ is an atomic Azumaya algebra in $\mathrm{Syn}_E^{\heartsuit}$ satisfying $\overline{x} = [\mathrm{Sy}^{\heartsuit}[A]]$. Conversely, suppose that $\overline{x}$ is atomic, so we can write $\overline{x} = [\overline{A}]$ for some atomic Azumaya algebra $\overline{A} \in \mathrm{Alg}(\mathrm{Syn}_E^{\heartsuit})$. For $n \geqslant 0$, let x_n denote the image of x in $\mathrm{Br}(\mathbf{1}^{\leqslant n})$ (see §8). Applying Lemma 8.3.3 repeatedly, we can choose a compatible sequence of Azumaya algebras $\{A_n \in \mathrm{Alg}(\mathrm{Syn}_{\mathbf{1}^{\leqslant n}})\}_{n \geqslant 0}$ satisfying $A_0 = \overline{A}$ and $[A_n] = x_n \in \mathrm{Br}(\mathbf{1}^{\leqslant n})$. Then $A = \{A_n\}_{n \geqslant 0}$ can be identified with an Azumaya algebra object of the ∞-category $\mathrm{Syn}_E \simeq \varprojlim \tau_{\leqslant n} \mathrm{Syn}_E$. In particular, A is dualizable as a synthetic E-module, so we can write $A = \mathrm{Sy}[B]$ for some essentially unique object $B \in \mathrm{Alg}(\mathrm{Mod}_E^{\mathrm{loc}})$ (Lemma 8.1.6). It follows immediately that B is an Azumaya algebra (as in the proof of Proposition 8.1.1). The equivalence $\overline{A} \simeq \mathrm{Sy}^{\heartsuit}[B]$ shows that $\mathrm{Sy}^{\heartsuit}[B]$ is an atomic algebra object of $\mathrm{Syn}_E^{\heartsuit}$, so that B is an atomic E-module (Corollary 6.2.7). Using the injectivity of the map $\mathrm{Br}(E) \to \varprojlim \mathrm{Br}(\mathbf{1}^{\leqslant n})$ (Theorem 8.0.5), we deduce that $[B] = x \in \mathrm{Br}(E)$, so that x is atomic as desired. □

10.3 The Case of an Odd Prime

Let E be a Lubin-Tate spectrum, let $\mathfrak{m} \subseteq \pi_0 E$ be the maximal ideal, and let K_* denote the graded ring $(\pi_* E)/\mathfrak{m}(\pi_* E)$. If the residue field $\kappa = (\pi_0 E)/\mathfrak{m}$ has odd characteristic, then we can use the isomorphism of Proposition 6.9.3 to explicitly describe the atomic elements of the Brauer-Milnor group $\mathrm{BM}(E)$.

Theorem 10.3.1. *Assume that κ has characteristic $\neq 2$, and let*

$$\rho : \mathrm{BM}(E) \simeq \mathrm{Br}(\mathrm{Mod}_{K_*}^{\mathrm{gr}}) \times \mathrm{QF}$$

denote the isomorphism of Proposition 6.9.3; here QF denotes the set of quadratic forms $q : (\mathfrak{m}/\mathfrak{m}^2)^\vee \to K_{-2}$. Let x be an element of the Brauer-Milnor group $\mathrm{BM}(E)$ and write $\rho(x) = (\overline{x}, q)$. Then x is atomic if and only if the following conditions are satisfied:

- (a) *The quadratic form q is nondegenerate. In particular, q induces an isomorphism $(\mathfrak{m}/\mathfrak{m}^2)^\vee \to (\pi_{-2} K) \otimes_\kappa (\mathfrak{m}/\mathfrak{m}^2)$, under which we can identify q with a quadratic form $\widehat{q} : \mathfrak{m}/\mathfrak{m}^2 \to \pi_2 K$.*

- (b) *We have $\overline{x} = [\mathrm{Cl}_{\widehat{q}}(\mathfrak{m}/\mathfrak{m}^2)] \in \mathrm{Br}(\mathrm{Mod}_{K_*}^{\mathrm{gr}})$, where $\mathrm{Cl}_{\widehat{q}}(\mathfrak{m}/\mathfrak{m}^2)$ denotes the Clifford algebra of $\widehat{q}$ (see Construction 5.3.2).*

Proof. Let V denote the vector space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$ and let $F : \mathrm{Syn}_E^\heartsuit \simeq \mathcal{M}(V)$ be the symmetric monoidal equivalence of categories supplied by Proposition 6.9.1. Suppose that x is atomic, so we can write $x = [A]$ for some atomic Azumaya algebra $A \in \mathrm{Syn}_E^\heartsuit$. Then $F(A)$ is an atomic algebra object of $\mathcal{M}(V)$ (in the sense of Definition 5.3.1), and is therefore isomorphic to the Clifford algebra $\mathrm{Cl}_{\widehat{q}'}(V^\vee)$ for some quadratic form $\widehat{q}' : V^\vee \to \pi_2 K$ (Proposition 5.3.5). Let $b : V^\vee \times V^\vee \to \pi_2 K$ be the bilinear form associated to $\widehat{q}'$ (given by the formula $b(x,y) = \widehat{q}'(x+y) - \widehat{q}'(x) - \widehat{q}'(y)$). Note that we have the identity $b(x,y) = xy + yx \in \mathrm{Cl}_{\widehat{q}'}(V^\vee)_2$. Consequently, if $x \in V^\vee$ belongs to the kernel of b (meaning that $b(x,y) = 0$ for all $y \in V^\vee$), then x belongs to the graded center of the Clifford algebra $\mathrm{Cl}_{\widehat{q}'}(V^\vee)$. Our assumption that A is an Azumaya algebra guarantees that $\mathrm{Cl}_{\widehat{q}'}(V^\vee)$ is an Azumaya algebra object of $\mathrm{Mod}_{K_*}^{\mathrm{gr}}$, so its graded center coincides with K_*. It follows that the bilinear form b is nondegenerate, and induces an isomorphism $\xi : (\pi_{-2}K) \otimes_\kappa V^\vee \to V$. Using this isomorphism, we can identify $\widehat{q}'$ with a quadratic form $q' : V^\vee \to \pi_{-2}K$. To verify (a) and (b), it will suffice to show that $q = q'$ (so that $\widehat{q} = \widehat{q}'$).

Fix an element $v \in V$; we claim that $q(v) = q'(v)$. Let us regard λ as a primitive element of the Hopf algebra $\bigwedge_{K_*}^*(V)$, so that it determines a derivation

$$d_v : \mathrm{Cl}_{\widehat{q}'}(V^\vee) \to \mathrm{Cl}_{\widehat{q}'}(V^\vee)$$

of degree -1, characterized by the identity $d_v(\lambda) = \lambda(v)$ for $\lambda \in V^\vee$. Choose a nonzero element $t \in \pi_{-2}K$. According to Proposition 5.7.4, there exists a unique element $a_v \in \mathrm{Cl}_{\widehat{q}'}(V^\vee)_1$ satisfying

$$d_v(y) = t(a_v y + (-1)^{n+1} y a_v) \tag{10.1}$$

for every element $y \in \mathrm{Cl}_{\widehat{q}'}(V^\vee)$ which is homogeneous of degree n. Note that, to establish that (10.1) holds for *all* y, it suffices to consider the case where $y \in V^\vee$ (since both sides of (10.1) can be regarded as derivations of the Clifford algebra $\mathrm{Cl}_{\widehat{q}'}(V^\vee)$ of degree (-1)). Consequently, the element a_v is characterized by requirement that the formula

$$\lambda(v) = d_v(\lambda) = t(a_v\lambda + \lambda a_v) = tb(t^{-1}\xi^{-1}(v), \lambda)$$

holds for all $\lambda \in V^\vee$. We therefore have $a_v = t^{-1}\xi^{-1}(v) \in V^\vee$. Unwinding the definitions, we obtain

$$q(v) = (ta_v)^2 = t^2\widehat{q}'(a_v) = t^2\widehat{q}(t^{-1}\xi^{-1}(v)) = q'(v),$$

as desired. This completes the proof of the "only if" assertion of Theorem 10.3.1.

For the converse, suppose that (a) and (b) are satisfied. Condition (a) guarantees that $B = \mathrm{Cl}_{\widehat{q}}(V^\vee)$ is an Azumaya algebra object of $\mathcal{M}(V)$ (see Remark 5.3.4). Using Proposition 6.9.1, we can choose an isomorphism $B \simeq F(A)$, where A is an atomic Azumaya algebra object of $\mathrm{Syn}_E^\heartsuit$. Using assumption (b) and the preceding calculation, we see that $\rho([A]) = (\overline{x}, q)$. Since the map ρ is an isomorphism (Proposition 6.9.3), we deduce that $[A] = x \in \mathrm{BM}(E)$, so that x is atomic. □

Remark 10.3.2. In the situation of Theorem 10.3.1, we have an isomorphism of Clifford algebras

$$\mathrm{Cl}_{\widehat{q}}(\mathfrak{m}/\mathfrak{m}^2) \simeq \mathrm{Cl}_q(V),$$

where $V = (\mathfrak{m}/\mathfrak{m}^2)^\vee$, and $\mathrm{Cl}_q(V)$ denotes the graded K_*-algebra generated by V (whose elements we regard as homogeneous of degree (-1)) subject to the relations $v^2 = q(v)$ for $v \in V$.

Corollary 10.3.3. *Let E be a Lubin-Tate spectrum whose residue field κ has characteristic $\neq 2$ and let x be an element of the Brauer group $\mathrm{Br}(E)$ having image $(\overline{x}, q)$ under the composite map*

$$\mathrm{Br}(E) \to \mathrm{BM}(E) \xrightarrow{\rho} \mathrm{Br}(\mathrm{Mod}^{\mathrm{gr}}_{K_*}) \times Q,$$

where ρ is the isomorphism of Proposition 6.9.3. Then x is atomic if and only if the quadratic form q is nondegenerate and $\overline{x}$ is represented by the Clifford algebra $\mathrm{Cl}_q((\mathfrak{m}/\mathfrak{m}^2)^\vee)$ of Remark 10.3.2.

Proof. Combine Theorem 10.3.1, Remark 10.3.2, and Proposition 10.2.5. □

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