

**TALK 2 (MONDAY, SEPT. 18, 2017):  
THE RELATIVE FARGUES-FONTAINE CURVE, PART I**

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CONTENTS

1. Geometerization of local Langlands	1
1.1. The quotient by Frobenius-twisted conjugacy	1
1.2. Introducing analytic geometry	3
1.3. How are the two approaches related?	3
2. The relative curve in the equal characteristic case	4
2.1. Interlude: perfectoids	4
2.2. Creating the relative curve (equal characteristic case)	6
2.3. The punctured open unit disc	7
References	10

1. GEOMETERIZATION OF LOCAL LANGLANDS

1.1. **The quotient by Frobenius-twisted conjugacy.** Let  $\mathbf{K}$  be a local field. Classically (and very roughly speaking), local Langlands aims to match homomorphisms

$$(1.1) \quad \sigma : \mathrm{Gal}(\mathbf{K}) \rightarrow \check{G}$$

with (irreducible) representations of the locally compact group  $G(\mathbf{K})$ .

However, there are multiple indications that the RHS is not quite the ring thing to consider. Here is a very rough indication as to what one may want to replace it by.

1.1.1. First, we consider the case when  $\mathbf{K}$  is of characteristic  $p$ , i.e.,

$$\mathbf{K} = \mathbb{F}_p((z)).$$

We consider the prestack quotient

$$G((z))/\mathrm{Ad}_{G((z))}^\varphi,$$

i.e., the quotient of  $G((z))$  by the action of  $G((z))$  given by

$$(1.2) \quad g \star g' = g \cdot g' \cdot \varphi(g^{-1}),$$

where  $\varphi$  denotes the geometric Frobenius.

For a test-scheme  $S$  over  $\mathrm{Spec}(\mathbb{F}_p)$ , we can think of

$$\mathrm{Hom}(S, G((z))/\mathrm{Ad}_{G((z))}^\varphi)$$

as the groupoid of *shtukas* on the formal punctures disc (with coordinate  $z$ ), parameterized by  $S$ .

This is a quotient is an ind-scheme that is *not* of ind-finite type by a group ind-scheme, so the resulting algebro-geometric object is quite unwieldy.

1.1.2. Yet, suppose we can make sense of the derived category

$$D(G((z))/\mathrm{Ad}_{G((z))}^\varphi)$$

of  $\ell$ -adic sheaves.

The functor of taking the fiber at  $1 \in G((z))$  defines a functor

$$D(G((z))/\mathrm{Ad}_{G((z))}^\varphi) \rightarrow G(\mathbf{K})\text{-mod}$$

(because  $G(\mathbf{K})$  is the stabilizer of 1 under the action (1.2)).

But the datum of an object of  $D(G((z))/\mathrm{Ad}_{G((z))}^\varphi) \rightarrow G(\mathbf{K})\text{-mod}$  carries quite a bit more information.

1.1.3. The idea that  $D(G((z))/\mathrm{Ad}_{G((z))}^\varphi)$  is the “right” substitute of  $G(\mathbf{K})\text{-mod}$  was originally suggested by V. Lafforgue.

A conceptual but heuristic explanation of why this is indeed the natural thing to do may be found in [Ga, Sect. 4]. It follows into the paradigm “classical Langlands” is the (categorical) trace of Frobenius on geometric Langlands.

1.1.4. Points of  $G((z))/\mathrm{Ad}_{G((z))}^\varphi$  with coefficients in  $\overline{\mathbb{F}}_p$  are called *isocrystals*; the set of isomorphism classes of isocrystals is denoted  $B(G)$ .

Inside  $B(G)$  one singles out a subset denoted  $B(G)_{\mathrm{basic}}$ . For  $b \in B(G)_{\mathrm{basic}}$ , the group  $J_b$  of its automorphisms, i.e.,

$$J_b := \{g \in G((z)) \mid g \cdot b = b \cdot \varphi(g)\}$$

is an inner form of  $G$ ; in this way we obtain all *extended pure inner forms* of  $G$ .

Thus, taking the fiber at an  $\overline{\mathbb{F}}_p$ -point of  $G((z))/\mathrm{Ad}_{G((z))}^\varphi$  corresponding to  $b \in B(G)_{\mathrm{basic}}$ , we obtain a functor

$$D(G((z))/\mathrm{Ad}_{G((z))}^\varphi) \rightarrow J_b(\mathbf{K})\text{-mod}.$$

Thus, the datum of an object of  $D(G((z))/\mathrm{Ad}_{G((z))}^\varphi)$  remembers not only the representation of  $G(\mathbf{K})$  but of  $J(\mathbf{K})$  for all extended pure inner forms  $J$  of  $G$ .

1.1.5. So, a refinement of local Langlands would consist of associating to a Galois representation  $\sigma$  as in (1.1) an object

$$\mathcal{F}_\sigma \in D(G((z))/\mathrm{Ad}_{G((z))}^\varphi),$$

whose fiber at the unit isocrystal recovers the corresponding  $G(\mathbf{K})$ -representation, and whose fibers at other basic isocrystals would recover the representations of its inner forms.

Moreover, one can imagine that one could use the nearby cycles functor to formulate a Hecke eigen-property of  $\mathcal{F}_\sigma$  with respect to  $\sigma$ . I.e., unlike the usual local Langlands, we will have a direct relation between  $\sigma$  and the automorphic object attached to it.

1.1.6. There are (at least) two major difficulties associated with realizing the above idea:

(I) It is really not clear how to work with  $G((z))/\mathrm{Ad}_{G((z))}^\varphi$  so as to have a manageable category  $D(G((z))/\mathrm{Ad}_{G((z))}^\varphi)$ , and especially the notion of nearby cycles (the latter in order to formulate the Hecke eigen-property).

(II) We would like a geometric theory also for  $\mathbf{K}$  of characteristic 0. One can imagine using Witt vectors to define a prestack over  $\mathbb{F}_p$  that would replace  $G((z))/\mathrm{Ad}_{G((z))}^\varphi$  (so that its  $\overline{\mathbb{F}}_p$ -points will be  $G(\mathbf{K}^{\mathrm{unr}})/\mathrm{Ad}_{G(\mathbf{K}^{\mathrm{unr}})}^\varphi$ , where  $\mathbf{K}^{\mathrm{unr}} \supset \mathbf{K}$  is the maximal unramified extension); probably this can be done by the methods of [Zhu]. However, it is hard to imagine how one could make sense of nearby cycles, because the latter would have to combine the geometries of equal and mixed characteristics.

**1.2. Introducing analytic geometry.** The idea of the Fargues-Scholze can be interpreted as changing the paradigm as follows:

When working with prestacks over  $\mathbb{F}_p$ , we are led to considering the (parameterized) formal punctured disc

$$(1.3) \quad \mathcal{D}_S := S \widehat{\times} \mathcal{D}_z := \mathrm{Spec}(R((z))), \quad S = \mathrm{Spec}(R), \quad \mathcal{D}_z = \mathrm{Spec}(\mathbb{F}_p((z)))$$

This formal disc is necessarily considered as a *scheme*, and as such it is not easy to manipulate. The problem is that  $S$  is too “skinny” to have a richer structure on the disc.

1.2.1. The idea is to replace *algebraic* geometry over  $\mathbb{F}_p$  by some sort of *analytic* geometry, whereby our test objects are no longer affine schemes  $\mathrm{Spec}(R)$  over  $\mathbb{F}_p$ , but *affinoid perfectoids*  $\mathrm{Spa}(R, R^+)$  (whatever they are), and this allows to replace the parameterized formal punctured disc (1.3) by an appropriately defined *punctured open unit disc*  $\mathbb{D}_S^{(0,1)} =: Y_S$ .

This  $Y_S$  (and its quotient by the action of the Frobenius, denoted  $X_S$ ) are actual geometric objects, and one can take bundles on them, consider Hecke correspondences, etc.

1.2.2. We will have a *pre-diamond*  $\mathrm{Bun}_G$ , i.e., a functor

$$(\mathrm{Perfctd}_{\mathbb{F}_p}^{\mathrm{aff}})^{\mathrm{op}} \rightarrow \mathrm{Groupoids},$$

that assigns to  $S$  the groupoid of  $G$ -bundles on  $X_S$ .

Ultimately, the Fargues-Scholze program aims to construct

$$\mathcal{F}_\sigma \in D(\mathrm{Bun}_G),$$

that satisfies the Hecke property with respect to  $\sigma$ .

The main feature of the Fargues-Scholze program is that it is equally applicable when  $\mathbf{K}$  is a local field of characteristic 0.

**1.3. How are the two approaches related?** Let us be very optimistic and imagine that both of the above programs have been carried. So, we have two versions of  $\mathcal{F}_\sigma$ : one is an object of  $D(G((z))/\mathrm{Ad}_{G((z))}^\varphi)$ , and the other is an object of  $D(\mathrm{Bun}_G)$ .

How are they supposed to be related?

1.3.1. In order to streamline the exposition, let us replace the category  $\text{Sch}_{/\mathbb{F}_p}^{\text{aff}}$  of affine schemes over  $\mathbb{F}_q$  by the category  $\text{Sch}_{/\mathbb{F}_p, \text{perf}}^{\text{aff}}$  of perfect schemes (this is not supposed to have any effect on  $\ell$ -adic sheaves. Correspondingly, we will replace the category

$$\text{PreStk}_{/\mathbb{F}_p} = \text{Funct}(\text{Sch}_{/\mathbb{F}_p}^{\text{aff}}, \text{Groupoid})$$

by

$$\text{PreStk}_{/\mathbb{F}_p, \text{perf}} = \text{Funct}(\text{Sch}_{/\mathbb{F}_p, \text{perf}}^{\text{aff}}, \text{Groupoid}).$$

1.3.2. The category  $\text{Sch}_{/\mathbb{F}_p, \text{perf}}^{\text{aff}}$  embeds into  $\text{PreDiam}_{/\mathbb{F}_p}$ , because it makes sense to map an object of  $\text{Perfctd}_{/\mathbb{F}_p}^{\text{aff}}$  to an object of  $\text{Sch}_{/\mathbb{F}_p, \text{perf}}^{\text{aff}}$  (just map the algebra of functions on affine the scheme to the algebra of functions on the affinoid perfectoid).

The operation of left Kan extension of the above functor

$$\text{Sch}_{/\mathbb{F}_p, \text{perf}}^{\text{aff}} \rightarrow \text{PreDiam}_{/\mathbb{F}_p}$$

along the Yoneda embedding  $\text{Sch}_{/\mathbb{F}_p, \text{perf}}^{\text{aff}} \hookrightarrow \text{PreStk}_{/\mathbb{F}_p, \text{perf}}$  defines a (fully faithful) functor

$$\text{PreStk}_{/\mathbb{F}_p, \text{perf}} \hookrightarrow \text{PreDiam}_{/\mathbb{F}_p},$$

which admits a right adjoint (which is also a left inverse)

$$(1.4) \quad \text{PreDiam}_{/\mathbb{F}_p} \rightarrow \text{PreStk}_{/\mathbb{F}_p, \text{perf}},$$

given by restriction.

We call the above functor (1.4) the functor of *skeleton*, and denote it by  $\text{Sk}$ .

1.3.3. It will follow from the construction that

$$\text{Sk}(\text{Bun}_G) \simeq G((z))/\text{Ad}_{G((z))}^\varphi,$$

up to perfection.

In particular, we obtain a map in  $\text{PreDiam}_{/\mathbb{F}_p}$

$$(1.5) \quad G((z))/\text{Ad}_{G((z))}^\varphi \rightarrow \text{Bun}_G.$$

1.3.4. Therefore, if the original program of constructing  $\mathcal{F}_\sigma$  as an object of  $D(G((z))/\text{Ad}_{G((z))}^\varphi)$  is realized, one would want to compare it with the pullback of the Fargues-Scholze

$$\mathcal{F}_\sigma \in D(\text{Bun}_G)$$

along (1.5).

Now, the conjecture would be that the former equals the pullback of the other.

## 2. THE RELATIVE CURVE IN THE EQUAL CHARACTERISTIC CASE

2.1. **Interlude: perfectoids.** Our analytic geometry looks a lot like algebraic geometry, but with a different category of test objects: the latter are affinoid perfectoids over  $\mathbb{F}_p$ . We will now make a short digression and explain what they are.

A good source for what follows is Chapters 1 and 4 in [Fe], and references therein.

2.1.1. Here are some definitions:

A *Huber ring* is a topological ring that contains an open subring  $A_0$  with a finitely generated ideal  $I \subset A_0$ , such that the induced topology on  $A_0$  is  $I$ -adic.

For a subset of a Huber ring it makes sense to ask whether it is *bounded*. We say that  $a \in A$  is *power-bounded* if the set  $\{a^n, n \in \mathbb{N}\}$  is bounded. We let

$$A^{\text{p-bdd}} \subset A$$

denote the set of power-bounded elements. We let

$$A^{\text{t-nilp}} \subset A^{\text{p-bdd}}$$

denote the subset of topologically nilpotent elements.

A *Huber pair* is  $(A, A^+)$ , where  $A$  is a Huber ring, and  $A^+ \subset A$  is an open subring, contained and integrally closed in  $A^{\text{p-bdd}}$ .

2.1.2. We will take a minimalist approach to analytic geometry. We define the category of *affinoid adic spaces* to be the opposite to that of Huber pairs. We will use the notation

$$(A, A^+) \rightsquigarrow \text{Spa}(A, A^+).$$

For us, an analytic space will be just a functor

$$\{\text{Affinoid adic spaces}\}^{\text{op}} \rightarrow \text{Groupoids}.$$

2.1.3. A *Tate ring* is a Huber ring for which there exists an element  $\varpi \in A^{\text{t-nilp}} \cap A^\times$ . (Such an element is called a *pseudo-uniformizer*.)

The existence of a pseudo-uniformizer is what makes the ring  $R$  not skinny.

2.1.4. When working over  $\mathbb{F}_p$ , we define a *perfectoid ring* to be a topological ring  $R$  such that

- $R$  is a complete Tate ring;
- the set  $A^{\text{p-bdd}}$  is itself bounded;
- $R$  is perfect (i.e., Frobenius is bijective).

A typical example of a perfectoid ring is

$$\overline{\mathbb{F}}_p((t^{\frac{1}{p^\infty}})) := \left( \left( \bigcup_n \overline{\mathbb{F}}_p[[t^{\frac{1}{p^n}}]] \right)^\wedge \right)_t,$$

where the subscript  $t$  means localization with respect to  $t$ .

A quasi-uniformizer is given by  $\varpi = t$ . Another example, containing the previous one, is  $\overline{\mathbb{F}}_p((t))^\wedge$ .

In this talk we will only need perfectoid rings *over*  $\mathbb{F}_p$ .

2.1.5. When  $R$  is not over  $\mathbb{F}_p$ , the definition is a bit more elaborate. One replaces the last condition by the following: there exists a quasi-uniformizer  $\varpi \in R$  such that

- $\frac{p}{\varpi^p} \in R^{\text{p-bdd}}$ ,
- Raising to the power  $p$  defines an isomorphism  $R/\varpi \rightarrow R/\varpi^p$ .

Here are the most typical examples:

- (a)  $R = \mathbb{Q}_p^{\text{cycl}}$ , i.e.,  $(\mathbb{Z}_p(\mu_{p^\infty}))^\wedge_p$ , where the subscript  $p$  means localization with respect to  $p$ ;
- (b)  $R = \mathbb{C}_p$ , i.e.,  $(\overline{\mathbb{Z}}_p^\wedge)_p$ .

In both these examples, a quasi-uniformizer can be taken to be  $\varpi = p$ .

2.1.6. The examples of perfectoid rings that we have above are actually perfectoid *fields*.

If  $\mathbf{F}$  is a perfectoid field, we let  $\mathbf{F}^0$  denote the subring  $\mathbf{F}^{\text{p-bdd}}$ . We have  $\mathbf{F} = (\mathbf{F}^0)_{\varpi}$ .

For example, for  $\mathbf{F} = \mathbb{F}_p((t^{\frac{1}{p^\infty}}))$ , we have  $\mathbf{F}^0 = \mathbb{F}_p[[t^{\frac{1}{p^\infty}}]]$  and for  $\mathbf{F} = \mathbb{Q}_p^{\text{cycl}}$ , we have  $\mathbf{F}^0 = \mathbb{Z}_p^{\text{cycl}}$ .

Here is a typical example of a perfectoid ring that is not a field. Start with a perfectoid field  $\mathbf{F}$ . Set

$$R^+ := \mathbf{F}^0 \langle s_1^{\frac{1}{p^\infty}}, \dots, s_m^{\frac{1}{p^\infty}} \rangle,$$

i.e., the completion in the topology coming from  $\mathbf{F}^0$  of

$$\mathbf{F}^0[s_1^{\frac{1}{p^\infty}}, \dots, s_m^{\frac{1}{p^\infty}}] = \bigcup_n \mathbf{F}^0[s_1^{\frac{1}{n}}, \dots, s_m^{\frac{1}{n}}],$$

and set

$$R := (R^+)_{\varpi}.$$

2.1.7. The category of of affinoid perfectoids, denoted  $\text{Perfctd}^{\text{aff}}$  is by definition the full subcategory of affinoid adic spaces  $(R, R^+)$  with  $R$  perfectoid ring.

**2.2. Creating the relative curve (equal characteristic case).** As was mentioned above, the sought-for prediamond  $\text{Bun}_G$  is supposed to assign to

$$S = \text{Spa}(R, R^+) \in \text{Perfctd}_{/\mathbb{F}_p}^{\text{aff}}$$

the groupoid of  $G$ -bundles on a certain geometric object  $X_S$ .

We will now indicate what this  $X_S$  (for the actual definition, see Sect. 2.3 below). We will first consider the case of the local field  $\mathbf{K} = \mathbb{F}_p((z))$ , which would later motivate the construction for  $\mathbf{K} = \mathbb{Q}_p$ .

A good reference for what follows is Chapter 10 in [Fe].

2.2.1. We start with a perfectoid affinoid  $S = \text{Spa}(R, R^+)$  over  $\mathbb{F}_p$ . Consider the topological ring

$$A_{\text{inf}, S} := R^+[[z]]$$

over  $\mathbb{F}_p[[z]]$ .

Tautologically,  $A_{\text{inf}, S}$  has the following universal property: for a ring  $R'$  over  $\mathbb{F}_p[[z]]$ , complete in the  $z$ -adic topology, the datum of a map of  $\mathbb{F}_p[[z]]$ -algebras

$$A_{\text{inf}, S} \rightarrow R'$$

is equivalent to the datum of a homomorphism of  $\mathbb{F}_p$ -algebras

$$R^+ \rightarrow R'.$$

*Remark 2.2.2.* The above forgetful functor

$$\{\mathbb{F}_p[[z]]\text{-algebras complete in the } z\text{-adic topology}\} \rightarrow \{\mathbb{F}_p\text{-algebras}\}$$

in the equal characteristic analog of the functor of tilt:

$$\{\mathbb{Z}_p\text{-algebra complete in the } p\text{-adic topology}\} \rightarrow \{\text{perfect } \mathbb{F}_p\text{-algebras}\},$$

to be discussed later.

In the equal characteristic case, the above functor has an obvious left adjoint

$$R^+ \mapsto R^+[[z]].$$

We will soon see what replaces this left adjoint in the mixed characteristic case.

2.2.3. We consider the analytic space

$$\mathrm{Spa}(A_{\mathrm{inf},S}, A_{\mathrm{inf},S})$$

and we will remove from it the closed subspace defined by the ideal generated by  $z \cdot \varpi$ .

By definition, this means that we are considering a subfunctor on affinoid adic spaces represented by  $\mathrm{Spa}(A_{\mathrm{inf},S}, A_{\mathrm{inf},S})$  that sends  $\mathrm{Spa}(R_1, R_1^+)$  to the set of maps

$$A_{\mathrm{inf},S} \rightarrow R_1^+$$

such that  $z$  is invertible in  $R_1^+$  and  $\varpi$  is invertible in  $R_1$ .

The resulting analytic space is the sought-for  $Y_S$ .

Note that  $Y_S$  is *not affinoid*. Rather, in Sect. 2.3 we will describe  $Y_S$  as the union of affinoid spaces  $Y_S^I$ , taken over closed intervals

$$I \subset (0, 1),$$

to be thought of closed (relative over  $S$ ) annuli corresponding to points  $|z| \in I$ .

2.2.4. By transport of structure, the Frobenius automorphism  $\varphi_S$  of the pair  $(R, R^+)$  induces an endomorphism of  $A_{\mathrm{inf}}$ , and hence on  $Y_S$ . In fact, it will map the affinoid  $Y_S^I$  isomorphically to the affinoid  $Y_S^{I^{\frac{1}{p}}}$ , where for  $I = (a, b)$  we have  $I^{\frac{1}{p}} := (a^{\frac{1}{p}}, b^{\frac{1}{p}})$ .

This implies that the resulting action of  $\mathbb{Z}$  on  $Y_S$  is *properly discontinuous* in a suitable sense, and we can form the analytic space quotient

$$X_S := Y_S / \varphi_S.$$

It is covered by affinoid adic spaces  $Y_S^I$  for  $I$  such that  $I \cap I^p = \emptyset$ .

This is the desired analytic quotient of the (relative over  $S$ ) punctured open unit disc by the action of the Frobenius.

2.2.5. We have a theory of vector bundles over analytic spaces. Using the Tannakian formalism, we can thus make sense of  $G$ -bundles on  $X_S$ . Thus, we obtain the sought-for groupoid

$$\mathrm{Hom}(S, \mathrm{Bun}_G).$$

**2.3. The punctured open unit disc.** In the equal characteristic case,  $Y_S$  is just the punctured open unit disc over  $S$ , denoted below by  $\mathbb{D}_S^{(0,1)}$ . We will now explain what it is.

2.3.1. Let  $(R, R^+)$  be a Tate pair, i.e.,  $R$  is a Tate ring, and  $R^+ \subset R$  an open subring contained in  $R^{\mathrm{p-bdd}}$ . For motivational purposes, we might as well take  $R$  to be a local field  $\mathbf{F}$  (not to be confused with “our” local field  $\mathbf{K}$ !) and  $R^+ := \mathcal{O}_{\mathbf{F}}$ . Denote  $S = \mathrm{Spa}(R, R^+)$ .

Perhaps, the most basic disc-like object associated to  $S$  is the *closed unit disc* over  $S$  of radius 1, denoted  $\mathbb{D}_S^{[0,1]}$ . By definition.

$$\mathbb{D}^{[0,1]} = \mathrm{Spa}(R\langle z \rangle, R^+\langle z \rangle),$$

where  $R^+\langle z \rangle$  is the completion of  $R^+[z]$ , i.e., the set of

$$\sum_{n \geq 0} r_n \cdot z^n, \quad r_n \rightarrow 0 \text{ in } R^+,$$

and  $R\langle z \rangle = (R^+\langle z \rangle)_{\varpi}$ .

To get a feel for why  $\mathbb{D}_S^{[0,1]}$  is indeed the unit disc, let us describe its points with coefficients in some non-archimedean valued field  $\mathbf{C}$ . Unwinding the definitions, a datum of such a point is

a homomorphism  $R \rightarrow \mathbf{C}$  (so that  $R^+$  automatically maps to  $\mathcal{O}_{\mathbf{C}}$ ), and an element  $c \in \mathbf{C}$  with  $|c| \leq 1$ . The latter condition is what ensures the convergence of the series

$$\sum_{n \geq 0} r_n \cdot c^n \in \mathbf{C} \text{ for } \sum_{n \geq 0} r_n \cdot z^n \in R\langle z \rangle.$$

2.3.2. In a similar way one defines the closed disc  $\mathbb{D}_S^{[0, \rho]}$  of radius  $\rho$  for any  $\rho \in \mathbb{R}^{>0}$ .

Namely, it is defined in the same way, modulo replacing  $R^+\langle z \rangle$  by  $R^+\langle z \rangle^\rho$ , where the latter consists of

$$\sum_{n \geq 0} r_n \cdot z^n, \quad |r_n| \cdot \rho^n \rightarrow 0,$$

where  $r \mapsto |r|$  is the norm on  $R$ .

Note that any  $\rho' \leq \rho$  defines a semi-norm  $|\cdot|_{\rho'}$  on  $R\langle z \rangle^\rho$ :

$$|\sum_{n \geq 0} r_n \cdot z^n|_{\rho'} := \sup_n |r_n| \cdot (\rho')^n.$$

2.3.3. Similar definitions apply when we have several variables  $z_1, \dots, z_k$ . I.e., we can create the closed multi-disc with radii  $(\rho_1, \dots, \rho_k)$ :

$$\text{Spa}(R\langle z_1, \dots, z_k \rangle^{\rho_1, \dots, \rho_k}, R^+\langle z_1, \dots, z_k \rangle^{\rho_1, \dots, \rho_k}).$$

2.3.4. For a closed interval  $I = [\rho_1, \rho_2] \subset \mathbb{R}^{>0}$  one defines the closed annulus  $\mathbb{D}_S^I$  to be  $\text{Spa}(R\langle z \rangle_I, R^+\langle z \rangle^I)$ , where

$$R\langle z \rangle^I = R\langle z_1, z_2 \rangle_{\rho_1^{-1}, \rho_2} / z_1 \cdot z_2 - 1,$$

and similarly for  $R^+\langle z \rangle_I$ .

However, we can also define  $R\langle z \rangle^I$  (resp.,  $R^+\langle z \rangle^I$ ) differently, and this will be of use for us later:

Say for simplicity that  $\rho_2 \leq 1$ . Then  $R\langle z \rangle^I$  is the completion of  $(R\langle z \rangle)_z$  with respect to the semi-norms  $|\cdot|_{\rho_1}$  and  $|\cdot|_{\rho_2}$  (equivalently, we can take the completion with respect to the family of semi-norms  $|\cdot|_{\rho'}$  for all  $\rho_1 \leq \rho' \leq \rho_2$ ).

Note also that if  $\rho_2 < 1$ , we have a natural map

$$(R^+[[z]])_{z \cdot \varpi} \rightarrow R\langle z \rangle^I,$$

and  $R\langle z \rangle^I$  identifies the completion of  $(R^+[[z]])_{z \cdot \varpi}$  with respect to the above semi-norms. (Note that for  $\rho < 1$ , the corresponding semi-norm  $|\cdot|_{\rho}$  continuously extends from  $R\langle z \rangle$  to  $(R^+[[z]])_{z \cdot \varpi}$ .)

2.3.5. We can now introduce other versions of the unit disc:

The open unit disc  $\mathbb{D}_S^{(0,1)}$  (resp., punctured unit disc  $\mathbb{D}_S^{(0,1]}$ , open punctured unit disc  $\mathbb{D}_S^{(0,1)}$ ) is defined as the colimit of the closed annuli  $\mathbb{D}_S^I$  over  $I \subset [0, 1)$  (resp.,  $I \subset (0, 1]$ ,  $I \subset (0, 1)$ ).



2.3.6. The corresponding spaces of global functions are defined to be the limits of functions on the  $\mathbb{D}_S^I$ 's that comprise our version of the disc.

Note that we have maps

$$\begin{aligned} R^+[[z]] &\mapsto \Gamma(\mathbb{D}_S^{(0,1)}, \mathcal{O}_{\mathbb{D}_S^{(0,1)}}^+), \quad (R^+[[z]])_{\varpi} \mapsto \Gamma(\mathbb{D}_S^{(0,1)}, \mathcal{O}_{\mathbb{D}_S^{(0,1)}}), \\ (R^+\langle z \rangle)_z &\mapsto \Gamma(\mathbb{D}_S^{(0,1]}, \mathcal{O}_{\mathbb{D}_S^{(0,1]}}^+), \quad (R\langle z \rangle)_z \mapsto \Gamma(\mathbb{D}_S^{(0,1]}, \mathcal{O}_{\mathbb{D}_S^{(0,1]}}^+), \end{aligned}$$

and what is of most of relevance for us, the maps

$$(R^+[[z]])_z \simeq R^+(\langle z \rangle) \rightarrow \Gamma(\mathbb{D}_S^{(0,1)}, \mathcal{O}_{\mathbb{D}_S^{(0,1)}}^+), \quad (R^+[[z]])_{z \cdot \varpi} \simeq (R(\langle z \rangle))_{\varpi} \rightarrow \Gamma(\mathbb{D}_S^{(0,1)}, \mathcal{O}_{\mathbb{D}_S^{(0,1)}}).$$

2.3.7. It also follows that  $\Gamma(\mathbb{D}_S^{(0,1)}, \mathcal{O}_{\mathbb{D}_S^{(0,1)}})$  can be described as the completion of  $(R^+[[z]])_{z \cdot \varpi}$  with respect to the family of semi-norms  $|\cdot|_{\rho}$  for  $0 < \rho < 1$  introduced above (or we can take the  $\rho$ 's belonging to two sequences, one converging to 1 and another to 0).

Note also that the algebra of global functions on each individual  $\mathbb{D}_S^I$  can be defined as the completion of  $(R^+[[z]])_{z \cdot \varpi}$  with respect to the same family of semi-norms.

2.3.8. *Action of the Frobenius.* Assume that  $S$  is over  $\mathbb{F}_p$ , and let us comment on the action of the Frobenius  $\phi_S$  on  $Y_S$ .

The action of  $\phi_S$  on an element

$$f \sum_{n \geq 0} r_n \cdot z^n \in (R^+[[z]])_{z \cdot \varpi}$$

is given by

$$\sum_{n \geq 0} (r_n)^p \cdot z^n.$$

I.e.,

$$|\phi_S(f)|_{\rho} = (|\phi_S(f)|_{\rho^{\frac{1}{p}}})^p.$$

It follows that the action of  $\phi_S$  on  $Y_S$  is such that it maps each

$$Y_S^I := \mathbb{D}_S^I$$

to the corresponding  $Y_S^{I^{\frac{1}{p}}}$ . I.e., this action ‘‘expands’’ the radii.

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