SYMMETRIC POWERS OF THE SPHERE

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M. J. HOPKINS

Contents

1. Nakaoka's computation and the length filtration

For a space X let

$$\mathrm{SP}^n(X) = X^n/\Sigma_n$$

be the n^{th} symmetric power of X. There is a natural transformation

$$S \times \mathrm{SP}^n(X) \to \mathrm{SP}^n(S \times X)$$

given by

$$(s, (x_1, \dots, x_n)) \mapsto ((s, x_1), \dots, (s, x_n)),$$

and in particular a map

$$\Sigma\, \mathrm{SP}^n(X) \to \mathrm{SP}^n(\Sigma X).$$

This implies that $\mathrm{SP}^n(-)$ induces a functor on spectra sending $X = \{X_k\}$ to

$$\mathrm{SP}^n(X) = \varinjlim_k \Sigma^{-k}\, \mathrm{SP}^n(X_k). \tag{1.1}$$

The symmetric power functor on spectra preserves cofibration sequences and filtered colimits, and so for any X one has

$$\mathrm{SP}^n(X) \approx \mathrm{SP}^n(S^0) \wedge X.$$

Remark 1.2. There is potential for confusion between the space S^0 and its suspension spectrum, also denoted S^0. Hopefully it will be clear from context which one is meant.

Nakaoka's Theorem concerns the symmetric powers $\mathrm{SP}^n(S^0)$ of the sphere spectrum. By the Dold-Thom-Kan Theorem, the infinite symmetric product is the Eilenberg-MacLane spectrum

$$\varinjlim \mathrm{SP}^n(S^0) = H\mathbb{Z}.$$

This gives a map

$$H^*(H\mathbb{Z}) \to H^*(\mathrm{SP}^n(S^0)). \tag{1.3}$$

where $H^*(-)$ indicates cohomology with coefficients in $\mathbb{Z}/p$. Nakaoka's Theorem concerns this map.

Theorem 1.4 (Nakaoka). *The symmetric powers of the sphere spectrum have the following properties.*

i) *If n is not a power of a prime, the map $\mathrm{SP}^{n-1}(S^0) \to \mathrm{SP}^n(S^0)$ is a weak equivalence.*

ii) *For all $n \geq 0$ the map* (1.3) *is surjective;*

iii) *When $n = p^k$, the kernel of* (1.3) *has a basis consisting of the monomials P^I, in which*

$$I = (\varepsilon_0, s_1, \cdots, \varepsilon_{m-1}, s_m, \varepsilon_m) \qquad s_i > 0 \quad \varepsilon \in \{0,1\},$$

which are admissible *in the sense that for all i, $s_i \geq p s_{i+1} + \varepsilon_i$ and having the property that either $m > k$ or $m = k$ and $\varepsilon_k = 1$.*

By part i), if n is not a power of a prime then $\mathrm{SP}^n(S^0)/\mathrm{SP}^{n-1}(S^0)$ is contractible, while for a prime p the map

$$\mathrm{SP}^{p^k}(S^0)/\mathrm{SP}^{p^{k-1}}(S^0) \to \mathrm{SP}^{p^k}(S^0)/\mathrm{SP}^{p^k-1}(S^0)$$

is a weak equivalence. The work of Mitchell-Priddy [3, 4] and Arone-Dwyer [1] concerns the homotopy type of this spectrum.

Let $\mathbf{A}_n$ be the subalgebra of the Steenrod algebra generated by $\{\mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^{n+1}-1}\}$ if $p = 2$ and by $\{\beta, \mathcal{P}^1, \ldots \mathcal{P}^{p^n-1}\}$ if $p > 2$. One consequence of Nakaoka's theorem is the following result of Welcher [5].

Theorem 1.5. *The cohomology $H^*(\mathrm{SP}^{p^k}(S^0)/\mathrm{SP}^{p^{k-1}}(S^0))$ is free over $\mathbf{A}_{k-1}$.*

The point of this first section is to expand on several useful details about the Steenrod algebra and to give a proof of Theorem 1.5, assuming Nakaoka's theorem.

Remark 1.6. It follows from the Adem relations that $\mathbf{A}_n$ is also generated by

$$\{\mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^n}\} \qquad p = 2$$

$$\{\beta, \mathcal{P}^1, \ldots \mathcal{P}^{p^{n-1}}\} \qquad p > 2.$$

It is slightly easier to prove Welcher's Theorem using the definition given in the text above.

1.1. Preliminaries on the Steenrod algebra.

For simplicity we will work at the prime 2. See §1.2.3 for an enumeration of the differences at odd primes.

1.1.1. *The Steenrod algebra.* The *Steenrod algebra* $\mathbf{A}$ is the free associative graded algebra over $\mathbf{F}_2$ generators Sq^i of degree i, for $i = 0, 1, 2, \ldots$ subject to the relation

$$\mathrm{Sq}^0 = 1$$

and the Adem relation that for $a < 2b$,

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j \leq a/2} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j .$$

The Steenrod algebra becomes a cocommutative Hopf algebra when equipped with the (Cartan) coproduct

$$\mathrm{Sq}^n \mapsto \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j .$$

It requires a little organization to show that this is compatible with the Adem relations, and I will assume it to be known. Probably the clearest purely algebraic development of the Steenrod algebra begins with the describing the dual of the Steenrod algebra as the algebraic group of automorphisms of the additive formal group, and deducing all of the properties from it. From that point of view most things are obvious, except for the Adem relations, which admit a conceptual explanation.

1.1.2. *The fundamental representation.* Let $R = H^*(\mathbf{RP}^\infty) = \mathbf{F}_2[x]$ with $|x| = 1$. One defines an action of the Steenrod algebra on R by setting

$$\mathrm{Sq}^i(x) = \begin{cases} x^2 & i = 1 \\ 0 & i > 1, \end{cases}$$

and extending to powers of x using the Cartan formula

$$\mathrm{Sq}^n(x\, y) = \sum_{i+j=n} \mathrm{Sq}^i(x)\, \mathrm{Sq}^j(y).$$

One must check that this is compatible with the Adem relations. Once one has done so this gives R the structure of an algebra over the Hopf algebra $\mathbf{A}$. We will refer to this as the *fundamental representation* of $\mathbf{A}$.

Several other useful representations can be built from R. Tensoring R with itself gives a representation of $\mathbf{A}$ on

$$R_n = R^{\otimes n} = \mathbf{F}_2[x_1, \dots, x_n].$$

One can easily check that there is a unique extension of the $\mathbf{A}$ action on $R^{\otimes n}$ to an action on

$$(x_1 \cdots x_n)^{-1}\mathbf{F}_2[x_1, \dots, x_n].$$

In fact Wilkerson proved?? that there is a unique extension of the $\mathbf{A}$ action to the graded field of fractions

$$K_n = \mathbf{F}_2(x_1, \dots, x_n).$$

We will also refer to R_n and K_n as *fundamental representations* of $\mathbf{A}$.

1.1.3. *Structure of the Steenrod algebra.* For a sequence $I = (r_1, \dots, r_n)$ of integers $r_i \geq 0$ write

$$\mathrm{Sq}^I = \mathrm{Sq}^{r_1} \cdots \mathrm{Sq}^{r_n} .$$

If for some i one has $r_i < 2r_{i+1}$ then the Adem relation can be applied at position i in Sq^I enabling one to write Sq^I as a linear combination of Sq^J for some other sequence J. If $r_i = 0$ for some i then $\mathrm{Sq}^I = \mathrm{Sq}^{I'}$ in which I' is the sequence gotten from I by dropping the term r_i. The sequences I for which Sq^I cannot be rewritten using these rules way are called *admissible sequences*

Definition 1.7. A sequence $(r_1, \dots, r_n)$ is called *admissible* if $r_n \geq 0$ and if for all $i < n$ $r_i \geq 2r_{i+1}$. An *admissible monomial* is an element of the Steenrod algebra of the form Sq^I with I admissibe.

One then has the following theorem

Theorem 1.8. *The admissible monomials form a basis of the Steenrod algebra.*

There are two steps to the proof of Theorem 1.8. One is to show that the set of admissible monomials spans **A** and the other is to show that it is linearly independent. To show that it spans one must show that repeatedly applying the Adem relations (and the relation $\mathrm{Sq}^0 = 1$) writes any Sq^I as a linear combination of admissible monomials. This is done by finding a quantity that decreases when an Adem relation is applied.

Definition 1.9. The *moment* of a sequence $I = (r_1, \dots, r_n)$ is the number

$$\mu(I) = r_1 + 2\,r_2 + \cdots + n\,r_n.$$

One can easily check by direct inspection that the result of applying an Adem relation to Sq^I leads to a linear combination of monomials Sq^J having the property that $\mu(J) < \mu(I)$. Since the moment is always non-negative this process must terminate. This shows that the admissible monomials span. To show that they are linearly independent one can work in the fundamental representation. See Lemma 1.22 below.

1.2. **Modules related to symmetric powers.** We now turn to several modules over the Steenrod algebra and describe an outline of the proof of Theorem 1.5. The details will be given in later sections. We will assume Theorem 1.8. What remains of the proof is checked as part of Lemma 1.22 below.

Remark 1.10. Each of the modules we consider is a sub-quotient of the Steenrod algebra, with a basis consisting of some subset of the admissible monomials. Because of this each module comes equipped with a canonical vector space homomorphism back to **A**. It is a misleading, but commmon practice to refer to these modules in terms of the corresponding vector subspaces.

1.2.1. *The length filtration.*

Definition 1.11. Let $I = (r_1, \dots, r_n)$ be a sequence with $r_i > 0$ for all i. The *length* of I is the number

$$\ell(I) = n.$$

The *length* of a sequence $J = (r_1, \dots, r_m)$ with $r_j \geq 0$ is defined to be $\ell(I)$ in which I is the sequence gotten from J by deleting all r_i with $r_i = 0$.

It is immediate from the definition that if I is a sequence of length n then the Adem relations write Sq^I as a linear combination of admissible monomials Sq^J with $\ell(J) \leq \ell(I)$.

Definition 1.12. The subspace $F_n \subset \mathbf{A}$ is the vector subspace spanned by the Sq^I with $\ell(I) \leq n$.

The following is straightforward

Proposition 1.13. *The subspace F_n has a basis consisting of admissible monomials Sq^I with $\ell(I) \leq n$.* □

Definition 1.14. The subspace $G_{n+1} \subset \mathbf{A}$ is the subspace with basis the admissible monomials Sq^I with $\ell(I) \geq (n+1)$.

We will see in Proposition 1.25 below that G_{n+1} is a left ideal. The quotient $\mathbf{A}/G_{n+1}$ is then a left $\mathbf{A}$ module and has a basis consisting of admissible monomials of length n. As a vector space it is isomorphic to F_n. It is common to use the symbol F_n to also denote this quotient module (see Remark 1.10).

Definition 1.15. The $\mathbf{A}$ module M_n is the quotient G_n/G_{n+1}.

The module M_n has basis the set of admissible monomials Sq^I with $\ell(I) = n$.

Expressing $\mathrm{Sq}^I \mathrm{Sq}^b$ as a linear combination of admissible monomials can be complicated, but when $b = 1$ it takes on a simple form. Indeed if $I = (r_1, \dots, r_n)$ is admissible, then

$$\mathrm{Sq}^I \mathrm{Sq}^1 = \begin{cases} 0 & r_n = 1 \\ \mathrm{Sq}^{(r_1,\dots,r_n,1)} & r_n > 1. \end{cases}$$

This implies that the sequence

$$\mathbf{A} \xrightarrow{\cdot \mathrm{Sq}^1} \mathbf{A} \xrightarrow{\cdot \mathrm{Sq}^1} \mathbf{A}$$

is exact, that the left ideal $J = \mathbf{A} \cdot \mathrm{Sq}^1 \subset \mathbf{A}$ has a vector space basis consisting of admissible monomials Sq^I ending with Sq^1, i.e. with $I = (r_1 \dots, r_m)$ and $r_m = 1$, and that that $\mathbf{A}/J$ has a basis consisting of admissible monomials Sq^I which do not end in 1, that is $I = (r_1, \dots, r_m)$ and $r_m > 1$. The map $\mathbf{A}/J \to H^*(H\mathbb{Z})$ is an isomorphism.

The vector space with basis the admissible monomials Sq^I with $I = (r_1, \dots, r_m)$, $m \le n$ and $r_m > 1$ (so, length at most n and not ending in Sq^1) will be denoted $F_n^{\mathbb{Z}}$. It gets the structure of a module over the Steenrod algebra when identified with the quotient

$$F_n/(F_{n-1} \mathrm{Sq}^1) = \mathbf{A}/\big(G_{n+1} + \mathbf{A} \cdot \mathrm{Sq}^1\big).$$

Right multiplicaton by Sq^1 gives an $\mathbf{A}$ module ismorphism

$$F_{n-1}^{\mathbb{Z}} \to \ker\{F_n \xrightarrow{\cdot \mathrm{Sq}^1} F_{n+1}\}.$$

Nakaoka's Theorem asserts that the inclusion

$$\mathrm{SP}^{2^n}(S^0) \to H\mathbb{Z}$$

gives an isomorphism

$$F_n^{\mathbb{Z}} \approx H^* \mathrm{SP}^{2^n}(S^0).$$

The quotient $M_n/(M_n \cdot \mathrm{Sq}^1)$ is denoted L_n. Nakaoka's Theorem identifies L_n with the cohomology of $\mathrm{SP}^{2^n}(S^0)/\mathrm{SP}^{2^{n-1}}(S^0)$, and as such, is the module we wish to show is $\mathbf{A}_{n-1}$ free. It is an $\mathbf{A}$ module having basis the admissible monomials Sq^I with $\ell(I) = n$ and which do not end in Sq^1. Right multiplication by Sq^1 gives an isomorphism of L_n with the submodule of M_{n+1} having basis the admissible monomials which *do* end in Sq^1, that is, the kernel of right multiplication by Sq^1

There is an exact sequence of $\mathbf{A}$ modules

$$0 \to L_n \xrightarrow{\mathrm{Sq}^1} M_{n+1} \to L_{n+1} \to 0. \tag{1.16}$$

We will show that this sequence splits. A relatively easy argument will show that M_{n+1} is free over $\mathbf{A}_{n-1}$, and so by the splitting above that L_n is projective over $\mathbf{A}_{n-1}$ hence free since projective modules over a connected graded ring which are bounded below are free.

The algebraic assertions above are all proved by examining the fundamental representation. For convenience here is a table summarizing the modules just described.

Name	vector space basis	realization as $\mathbf{A}$ module
G_{n+1}	admissible monomials of length greater than n	a left ideal
F_n	admissible monomials of length $\leq n$	$\mathbf{A}/G_{n+1}$
M_n	admissible monomials of length exactly n	G_n/G_{n+1}
$F_n^{\mathbb{Z}}$	admissible monomials in F_n not ending in Sq^1	$\mathbf{A}/\big(G_{n+1} + \mathbf{A}\cdot \mathrm{Sq}^1\big)$
L_n	admissible monomials of length exactly n,not ending in Sq^1	$\operatorname{coker}\{F_{n-1}^{\mathbb{Z}} \xrightarrow{\cdot \mathrm{Sq}^1} F_{n+1}\}$

And here are some exact sequences of $\mathbf{A}$ modules in which they sit.

$$0 \to M_n \to F_n \to F_{n-1} \to 0$$
$$0 \to L_n \to F_n^{\mathbb{Z}} \to F_{n-1}^{\mathbb{Z}} \to 0$$
$$0 \to L_n \xrightarrow{\cdot \mathrm{Sq}^1} M_{n+1} \to L_n \to 0.$$

1.2.2. *Applications of the fundamental representations.* We now examine the fundamental representation $(x_1 \cdots x_n)^{-1} R_n$ and, following Mitchell and Priddy [3], study the action of the Steenrod algebra on the class $U = (x_1 \cdots x_n)^{-1}$. We begin with a couple of observations.

Observation 1.17. For $w \in R_n$ if $r > |w|$ then $\mathrm{Sq}^r(w) = 0$.

The above property is called *unstability* and is part of the axiomatic development of Steenrod operations on the cohomology of spaces. It follows easily from the fact that the total squaring operation $\mathrm{Sq}_t(x) = \sum \mathrm{Sq}^i(x) t^i$ is a ring homomorphism and for the generator $x \in R$ one has

$$\mathrm{Sq}_t(x) = x + tx^2.$$

Observation 1.18. If $I = (r_1, \dots, r_n)$ is admissible then there is an inequality

$$\begin{aligned} r_1 > 2r_2 > r_2 + 2r_3 > \cdots > r_2 + \cdots + 2r_n \\ > r_2 + \cdots r_n. \end{aligned}$$

Armed with these, we can begin. Let $I = (r_1, \dots, r_k)$ be admissible and write $I' = (r_2, \dots, r_k)$.

Lemma 1.19. *If* $\mathrm{Sq}^{I'} U \in R_n$ *then* $\mathrm{Sq}^I U = 0$.

Proof: This is a direct consequence of unstability. The degree of $\mathrm{Sq}^{I'}$ is

$$r_2 + \cdots + r_k - n$$

which, since I is admissible, is strictly smaller than r_1 by Observation 1.18. □

For a monomial $m = x_1^{a_1} \cdots x_n^{a_n} \in U^{-1}R_n$ let

$$\nu(m) = \#\{a_i \mid a_i < 0\}$$

be the number of negative powers of x_i ocurring in m. For a sum of monomials $m = \sum m_i$ we set $\nu(m) = \max\{\nu(m_i)\}$. By definition, an element $m \in (x_1 \cdots x_n)^{-1}R_n$ lies in R_n if an only if $\nu(m) = 0$.

Lemma 1.20. *if $I = (r_1, \ldots, r_k)$ is an admissible sequence then*

$$\nu(\mathrm{Sq}^I(U)) \leq n - k.$$

Proof: The proof is by induction on k, the case $k = 1$ being immediate from the Cartan formula. Set $I' = (r_2, \ldots r_k)$. By induction $\mathrm{Sq}^{I'} U$ is a linear combination of monomials m with $\nu(m) \leq n - k + 1$. Write

$$m = a\, b$$

with $\nu(a) = 0$ and $\nu(b) = \nu(m)$ (so that b is a product of x_i^{-1}). Then a is an element of R_n whose degree, by Remark 1.18, is strictly smaller than r_1. It follows that $\mathrm{Sq}^{r_1}(a) = 0$. This means that

$$\mathrm{Sq}^{r_1}(a\, b) = \sum_{i>0} \mathrm{Sq}^{r_1 - i}(a)\, \mathrm{Sq}^i(b)$$

and the result follows from the case $k = 1$. □

Corollary 1.21. *If $I = (r_1, \ldots, r_m)$ is an admissible sequence with $m > n$ then $\mathrm{Sq}^I U = 0$.*

Proof: Write $I' = (r_2, \ldots, r_m)$ so that $\mathrm{Sq}^I = \mathrm{Sq}^{r_1} \mathrm{Sq}^{(I')}$. By Lemma 1.20 $\nu(\mathrm{Sq}^{I'} U) = 0$ so $\mathrm{Sq}^{I'} U$ is in R_n. The result now follows from Lemma 1.19. □

It is useful to be a bit more explicit. To do so, order the sequences $(a_1, \ldots, a_n)$ by setting

$$(a_1, \ldots, a_n) > (a_1', \ldots, a_n')$$

if $a_j > a_j'$ at the the smallest j for which $a_j \neq a_j'$. This would be "lexicographic" ordering if we were using the reverse of numerical inequality on the a_is. A simple direct computation shows

Lemma 1.22. *if $I = (r_1, \ldots, r_k)$ is an admissible sequence then*

$$\mathrm{Sq}^I(U) = \big(x_1^{r_1} \cdots x_k^{r_k} + \textit{lower terms}\big)U.$$

In particular, the set $\{\mathrm{Sq}^I(U) \mid I$ admissible $, \ell(I) \leq n\}$ is linearly independent. □

Proof: From the Cartan formula, it suffices to consider the case $k = n$. Write $J = (r_2, \dots, r_n)$. By Lemma 1.20, one has

$$\mathrm{Sq}^J U = m_1 x_1^{-1} + \cdots + m_n x_n^{-1} + m$$

with $m \in R_n$ (and so annihilated by Sq^{r_1}), and with

$$m_i = \mathrm{Sq}^J(Ux_i) = \mathrm{Sq}^J \big(\prod_{j \neq i} x_j\big)^{-1}.$$

By induction,

$$m_1 = x_2^{r_2 - 1} \cdots x_n^{r_n - 1}$$

and for $i > 1$,

$$m_i = x_1^{r_2 - 1} \cdots x_{i-1}^{r_{i-1} - 1} x_{i+1}^{r_i - 1} \cdots x_n^{r_n - 1} + \text{lower terms}.$$

Since $r_1 > 2r_2$ it follows that the highest term after applying Sq^{r_1} comes from $m_1 x_1^{-1}$ and that it is

$$x_1^{r_1 - 1} \cdots x_n^{r_n - 1}.$$

□

Combining Lemmas 1.20 and 1.22 gives the following improvement of Lemma 1.20.

Corollary 1.23. *If $I = (r_1, \dots, r_k)$ is an admissible sequence then*

$$\nu(\mathrm{Sq}^I(U)) = n - k.$$

We can now establish some of the main properties of our list of $\mathbf{A}$ modules.

Proposition 1.24. *The subspace $F_n \subset \mathbf{A}$ is a sub coalgebra and has a basis consisting of admissible monomials Sq^I with $\ell(I) \leq n$.*

Proof: The fact that it is a sub coalgebra follows immediately from the Cartan formula. The fact that the admissible monomials are linearly independent is Lemma 1.22. □

Proposition 1.25. *The left $\mathbf{A}$-module map*

$$\Phi : \mathbf{A} \to (x_1 \cdots x_n)^{-1} R_n$$
$$a \mapsto a \cdot U$$

has kernel G_{n+1} and takes the set of admissible monomials in F_n to a basis for the image. In particular G_{n+1} is a left ideal.

Proof: The fact that G_{n+1} is in the kernel of Φ is Corollary 1.21, and the assertion about F_n is Lemma 1.22. Together these imply that G_{n+1} is the entire kernel. □

One of the assertions of Proposition 1.25 is that the composite map $F_n \to \mathbf{A} \to \mathbf{A}/G_{n+1}$ is an isomorphism. As the right hand side is a left $\mathbf{A}$-module this gives the vector space F_n the structure of a left $\mathbf{A}$ module. As mentioned in Remark 1.10, the convention is to say that *F_n is an $\mathbf{A}$ module* under this structure. This is a little misleading as F_n is certainly not a *sub* $\mathbf{A}$-module of $\mathbf{A}$, but rather a quotient module. However, the subspace F_n is stable under the action of *some* Steenrod operations.

Lemma 1.26. *The subspace F_n is stable under Sq^a for $a < 2^n$.*

Proof: The proof is by induction on n. Let's start with $n = 1$. In that case the assertion is that $\mathrm{Sq}^1 \mathrm{Sq}^m$ has length at most 1. This is immediate from the Adem relations, which show it to be Sq^{m+1} if m is even, and 0 if m is odd. Suppose now that $I = (r_1, \ldots, r_n)$ is admissible and write $I' = (r_2, \ldots, r_n)$. If $a < 2^n$ then by the Adem relations

$$\mathrm{Sq}^a \mathrm{Sq}^I = \sum_{j \le a/2} \mathrm{Sq}^{a+r_1-j} \mathrm{Sq}^j \mathrm{Sq}^{I'}.$$

Since $j \le a/2 < 2^{n-1}$ the induction hypothesis applies to give that $\mathrm{Sq}^j \mathrm{Sq}^{I'} \in F_{n-1}$, and so $\mathrm{Sq}^{a+r_1-j} \mathrm{Sq}^j \mathrm{Sq}^{I'} \in F_n$. □

Let $\mathbf{A}_k \subset \mathbf{A}$ be the sub algebra generated by $\{\mathrm{Sq}^i \mid i \le 2^{k+1}-1\}$. It is immediate from the formula for the diagonal that $\mathbf{A}_k$ is a sub Hopf algebra of $\mathbf{A}$. It follows from the Adem relations that $\mathbf{A}_k$ is generated by $\{\mathrm{Sq}^{2^i} \mid 1 \le i \le k\}$.

Corollary 1.27. *The subspace $F_n \subset \mathbf{A}$ is a free $\mathbf{A}_{n-1}$ submodule.*

Proof: The fact that it is a submodule is immediate from Lemma 1.26. Since the composite $F_n \to \mathbf{A} \to \mathbf{A}/G_{n+1}$ is an isomorphism it follows that F_n is a retract of $\mathbf{A}$. But $\mathbf{A}$ is a free $\mathbf{A}_{n-1}$ module since $\mathbf{A}_{n-1} \to \mathbf{A}$ is an inclusion of Hopf algebras over a field. It follows that F_n is a projective $\mathbf{A}_{n-1}$-module, hence free since connected projective modules over connected graded rings over a field are automatically free. □

Let M_n be the kernel of the $\mathbf{A}$ map

$$\mathbf{A}/G_{n+1} \to \mathbf{A}/G_n.$$

As a vector space M_n has a basis consisting of admissible monomials Sq^I with $\ell(I) = n$.

Corollary 1.28. *The $\mathbf{A}$-module M_n is free over $\mathbf{A}_{n-2}$.*

Proof: By definition there is an exact sequence

$$0 \to M_n \to \mathbf{A}/G_{n+1} \to \mathbf{A}/G_n \to 0$$

and the middle and right terms are free over $\mathbf{A}_{n-2}$ by Corollary 1.27. □

Our next aim is to show that the short exact sequence

$$0 \to L_{n-1} \xrightarrow{\cdot \mathrm{Sq}^1} M_n \to L_n \to 0$$

splits.

Proposition 1.29. *For $I = (r_1, \ldots, r_n)$ admissible, the following are equivalent*

i) $r_n > 1$

ii) $\mathrm{Sq}^I U \in (x_1 \cdots x_n)$.

Proof: If $r_n = 1$ then by Lemma 1.22

$$\mathrm{Sq}^I U = x_1^{r_1-1} \cdots x_{n-1}^{r_n-1} + \text{lower terms}$$

which is not in $(x_1 \dots x_n)$. For the other implication it suffices, by symmetry, to show that $\mathrm{Sq}^I U$ is divisible by x_1. write $J = (r_2, \dots, r_n)$ and, as in the proof of Lemma 1.22 write

$$\mathrm{Sq}^J U = m_1 x_1^{-1} + \cdots + m_n x_n^{-1} + m$$

with $m \in R$. By induction, for $i > 1$ each m_i is divisible by x_1. It follows that

$$\mathrm{Sq}^I U \equiv \sum \mathrm{Sq}^i m_1 \,\mathrm{Sq}^{r_1 - i} x_1^{-1} = \sum \mathrm{Sq}^i m_1 x_1^{r_1 - i - 1} \mod (x_1).$$

Unstability implies that the terms in the sum with $i > |m_1|$ are zero. Since I is admissible we have

$$r_1 > r_2 + \cdots + r_n + r_n \geq r_2 + \cdots + r_n + 2 \geq |m_1| + 2.$$

□

Corollary 1.30. *The sequence of* $\mathbf{A}$ *modules*

$$0 \to L_{n-1} \xrightarrow{\cdot \mathrm{Sq}^1} M_n \to L_n \to 0$$

splits.

Proof: By Proposition 1.29, the map from the kernel of

$$M_n \xrightarrow{x \cdot U} R_n \to R_n/(x_1 \dots x_n)$$

to L_n is an isomorphism. □

Remark 1.31. In fact something stronger is true, and it seems can be proved along the same lines. However I haven't quite worked out the details. The assertion is that if Sq^I is an admissible monomial of length n and not ending in Sq^1 then $\mathrm{Sq}^I U$ is divisible by every linear combination of the x_i. In fact, for any admissible I of length n, one has $\mathrm{Sq}^I U = mU$ with m divisible by every linear expression in the x_i.

Corollary 1.32. *The* $\mathbf{A}$ *module* L_n *is free over* $\mathbf{A}_{n-1}$*.*

Proof: By Corollary 1.30 L_n is a summand of M_{n+1} which is free over A_{n-1} by Corollary 1.28. □

Corollary 1.32 was the mission of this first part of the lecture. I've added some material below in hopes of clarifying other aspects of the story.

1.2.3. *Modification for odd primes.* For $p > 2$ the Steenrod algebra is generated by mod p operations $\mathcal{P}^n$ of degree $2n(p-1)$ and the Bockstein β of degree 1 subject to the relation

$$\begin{aligned}
\mathcal{P}^0 &= 1\\
\beta^2 &= 0\\
\mathcal{P}^a\,\mathcal{P}^b &= \sum_{t=0}^{\lfloor a/p\rfloor}(-1)^{a+t}\binom{(p-1)(b-t)-1}{a-p\,t}\mathcal{P}^{a+b-t}\,\mathcal{P}^t \qquad a<p\,b\\
\mathcal{P}^a\beta\,\mathcal{P}^b &= \sum_{t=0}^{\lfloor a/p\rfloor}(-1)^{a+t}\binom{(p-1)(b-t)}{a-p\,t}\beta\,P^{a+b-t}\,\mathcal{P}^t\\
&\quad+\sum_{t=0}^{\lfloor (a-1)/p\rfloor}(-1)^{a+t-1}\binom{(p-1)(b-t)-1}{a-p\,t-1}\mathcal{P}^{a+b-t}\beta\,\mathcal{P}^t \qquad a\le b\ .
\end{aligned}$$

The Cartan formula takes the form

$$\begin{aligned}
\beta(xy) &= \beta(x)y+(-1)^{|x|}x\beta(y)\\
\mathcal{P}^n(x\,y) &= \sum \mathcal{P}^i(x)\,\mathcal{P}^{n-i}(y).
\end{aligned}$$

The fundamental representations are built from

$$R = E[x]\otimes \mathbf{F}_p[y]$$

with $|x|=1$, $|y|=2$ and with

$$\begin{aligned}
\beta(x) &= y\\
\mathcal{P}^i(x) &= 0 \qquad i>0\\
\mathcal{P}^1(y) &= y^p\\
\mathcal{P}^i(y) &= 0 \qquad i>1.
\end{aligned}$$

Let

$$R_n = R\otimes\cdots\otimes R \qquad (n\text{ times})$$

and let

$$\begin{aligned}
W &= y_1\ldots y_n\in R_n\\
U &= (x_1\cdots x_n)/W\in W^{-1}R_n.
\end{aligned}$$

As at the prime 2 one can establish the main properties of the length filtration by studying the cyclic $\mathbf{A}$ module generated by U.

To a sequence

$$I=(\varepsilon_0,s_1,\varepsilon_1,\ldots,s_k,\varepsilon_k) \tag{1.33}$$

with $\varepsilon_i\in\{0,1\}$ and $s_i\ge 1$ one associates the monomial

$$\mathcal{P}^I=\beta^{\varepsilon_0}\mathcal{P}^{s_1}\cdots\mathcal{P}^{s_k}\beta^{\varepsilon_k}.$$

A sequence (1.33) is *admissible* if for each i

$$s_i\ge ps_{i+1}+\varepsilon_{i+1}$$

and the corresponding $\mathcal{P}^I$ is an *admissible monomial*. The *moment* of I is

$$\mu(I)=\sum i(s_i+\varepsilon_i).$$

Using the moment and the fundamental representation one can show that the admissible monomials form a basis of the Steenrod algebra. The *length* $\ell(I)$ of (1.33) is k if $\varepsilon_k = 0$ and $(k+1)$ otherwise, and the length of a monomial $\mathcal{P}^I$ is defined to be the length of I. So for example, the length of β is 1, as is the length of $\beta\mathcal{P}^1$, while the length of $P^1\beta$ is 2.

1.3. **Symmetric powers.** Mitchell and Priddy introduce a filtration

$$D(0) \to D(1) \cdots \to H\mathbb{Z}/p \tag{1.34}$$

of the mod p Eilenberg-MacLane spectrum $H\mathbb{Z}/p$ analogous to the filtration of $H\mathbb{Z}$ by $\mathrm{SP}^{p^k}(S^0)$. One way of defining such a filtration would be by filtering $H\mathbb{Z}/p$ by the mapping cone of the degree p map

$$\mathrm{SP}^{p^{k-1}}(S^0) \xrightarrow{p} \mathrm{SP}^{p^{k-1}}(S^0). \tag{1.35}$$

The multiplicative properties work out cleaner if one compose (1.35) with the inclusion

$$\mathrm{SP}^{p^{k-1}}(S^0) \to \mathrm{SP}^{p^k}(S^0)$$

define $D(k)$ as a specific map Δ_p homotopic to this inclusion.

To describe this in more detail let's go back to the situation with spaces and think more carefully about the symmetric power functor. Thus let X be a *pointed* space and make

$$\mathrm{SP}^n(X) = X^n/\Sigma_n$$

into a pointed space by taking $(*, \cdots, *)$ as the base point. The map

$$(x_1, \dots, x_k) \mapsto (*, x_1, \dots, x_k)$$

defines a map

$$\mathrm{SP}^n(X) \to \mathrm{SP}^{n+1}(X),$$

and the sequence

$$\cdots \to \mathrm{SP}^n(X) \to \mathrm{SP}^{n+1}(X)$$

then becomes a filtered unital commutative monoid under concatenation, with $*$ as the unit. If X and Y are two spaces there are bilinear maps

$$\mathrm{SP}^k(X) \times \mathrm{SP}^\ell(Y) \to \mathrm{SP}^{k+\ell}(X \times Y)$$
$$\mathrm{SP}^k(X) \wedge \mathrm{SP}^\ell(Y) \to \mathrm{SP}^{k+\ell}(X \wedge Y)$$

sending $(\underline{x}, \underline{y})$, with

$$\underline{x} = (x_1, \dots, x_k)$$
$$\underline{y} = (y_1, \dots, y_\ell)$$

to the sequence

$$((x_1, y_2), (x_1, y_2), \cdots, (x_k, y_\ell)).$$

For a natural number ℓ let

$$\Delta_\ell : \mathrm{SP}^n(X) \to \mathrm{SP}^{n\ell}(X)$$

be the map sending an element $\underline{x} \in \mathrm{SP}^n(X)$ to its ℓ^{th} power

$$\Delta_\ell(\underline{x}) = (\underline{x}, \dots, \underline{x}).$$

By continuity, the construction of Δ_ℓ determines a natural transformation

$$\mathrm{SP}^n(-) \xrightarrow{\Delta_\ell} \mathrm{SP}^{n\ell}(-)$$

of functors of spectra.

Now the transformation Δ_ℓ can also be interpreted as as multiplication by the class (pt, ..., pt) in the pointed space $S^0 = \{\mathrm{pt}\}_+$, using the bilinear pairing

$$\mathrm{SP}^k(X) \wedge \mathrm{SP}^\ell(S^0) \to \mathrm{SP}^{k\ell}(X \wedge S^0)$$

The induced map on the suspension spectrum of X is constructed from smashing with S^{-n}, the corresponding maps

$$\mathrm{SP}^k(X) \wedge S^n \to \mathrm{SP}^k(X) \wedge \mathrm{SP}^\ell(S^n) \to \mathrm{SP}^{k\ell}(X \wedge S^n)$$

in which the map $S^n \to \mathrm{SP}^k(S^n)$ is induced by the diagonal map. This latter map is homotopic to the composition

$$S^n \xrightarrow{\ell} S^n = \mathrm{SP}^1(S^n) \to \mathrm{SP}^\ell(S^n)$$

from which we conclude

Corollary 1.36. *Suppose that X is a spectrum. The map of filtered spectra*

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{SP}^k(X) & \longrightarrow & \mathrm{SP}^{k+1}(X) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \Delta_\ell} & & \downarrow{\scriptstyle \Delta_\ell} & \\ \cdots \longrightarrow & \mathrm{SP}^{k\ell}(X) & \longrightarrow & \mathrm{SP}^{(k+1)\ell}(X) & \longrightarrow \cdots \end{array}$$

is naturally homotopic to the composition

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{SP}^k(X) & \longrightarrow & \mathrm{SP}^{k+1}(X) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \ell} & \\ \cdots \longrightarrow & \mathrm{SP}^k(X) & \longrightarrow & \mathrm{SP}^{k+1}(X) & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & \\ \cdots \longrightarrow & \mathrm{SP}^{k\ell}(X) & \longrightarrow & \mathrm{SP}^{(k+1)\ell}(X) & \longrightarrow \cdots \end{array}$$

in which the maps ι are the filtration inclusions. □

Definition 1.37. The spectrum $D(k)$ is the cofiber of the map

$$\Delta_p : \mathrm{SP}^{p^{k-1}}(S^0) \to \mathrm{SP}^{p^k}(S^0).$$

The filtration $D^0 = S^0 \to D(1) \to \cdots \to H\mathbb{Z}/p$ gives $H\mathbb{Z}/p$ the structure of a filtered (homotopy) commutative ring. The following result of Mitchell and Priddy [3, Proposition 4.3] is equivalent to Nakaoka's Theorem.

Proposition 1.38. *Let $\mathbf{A}_p = H^*(H\mathbb{Z}/p)$ be the mod p Steenrod algebra. The map $H^*(H\mathbb{Z}/2) \to H^*(D(k))$ induces an isomorphism*

$$\mathbf{A}/G_{k+1} \to H^*(D(k))$$

where $G_{k+1} \subset \mathbf{A}$ is the ideal generated by the admissible monomials P^I of length greater than or equal to $(k+1)$.

To see the equivalence with Nakaoka's theorem, note that by Corollary 1.36 the map Δ_p is zero in cohomology with $\mathbb{Z}/p$ coefficients. It follows that there is a short exact sequence

$$0 \to H^*(\Sigma\,\mathrm{SP}^{p^{k-1}}(S^0)) \to H^*(D(k)) \to H^*\,\mathrm{SP}^{p^k}(S^0) \to 0.$$

This receives a map from the short exact sequence (1.16) (and its analogue for $p > 2$).

From the diagram

$$\begin{array}{ccccc} \mathrm{SP}^{p^{k-2}}(S^0) & \longrightarrow & \mathrm{SP}^{p^{k-1}}(S^0) & \longrightarrow & \mathrm{SP}^{p^{k-1}}(S^0)/\,\mathrm{SP}^{p^{k-2}}(S^0) \\ \downarrow{\scriptstyle \Delta_p} & & \downarrow{\scriptstyle \Delta_p} & & \downarrow \\ \mathrm{SP}^{p^{k-1}}(S^0) & \longrightarrow & \mathrm{SP}^{p^{k}}(S^0) & \longrightarrow & \mathrm{SP}^{p^{k}}(S^0)/\,\mathrm{SP}^{p^{k-1}}(S^0) \\ \downarrow & & \downarrow & & \downarrow \\ D(k-1) & \longrightarrow & D(k) & \longrightarrow & D(k)/D(k-1) \end{array} \tag{1.39}$$

one gets a cofibration sequence

$$\mathrm{SP}^{p^k}(S^0)/\,\mathrm{SP}^{p^{k-1}}(S^0) \to D(k)/D(k-1) \to \Sigma\,\mathrm{SP}^{p^{k-1}}(S^0)/\,\mathrm{SP}^{p^{k-2}}(S^0). \tag{1.40}$$

From Proposition 1.36 one can work out that the upper right vertical map in (1.39) is null. This means that the cofibration sequence (1.40) splits, and there is a weak equivalence

$$D(k)/D(k-1) \approx \mathrm{SP}^{p^k}(S^0)/\,\mathrm{SP}^{p^{k-1}}(S^0) \vee \Sigma\,\mathrm{SP}^{p^{k-1}}(S^0). \tag{1.41}$$

This is a geometric version of the splitting of Corollary 1.30, and in fact combined with Nakaoka's Theorem this gives another proof of it.

Mitchell and Priddy use the notation

$$\begin{aligned} M(k) &= \Sigma^{-k} D(k)/D(k-1) \\ L(k) &= \Sigma^{-k}\,\mathrm{SP}^{p^k}(S^0)/\,\mathrm{SP}^{p^{k-1}}(S^0). \end{aligned}$$

and, up to suspension, write the splitting (1.41) as

$$M(k) \approx L(k) \vee L(k-1).$$

At $p = 2$, the cohomology of $D(1)$ is suspension of the fundamental representation $\mathbf{F}_2[x]\cdot x^{-1} \subset x^{-1}\mathbf{F}_2[x]$ and the cohomology of $D(1)\wedge\cdots\wedge D(1)$ is a suspension of the fundamental representation

$$R_k \cdot U \subset (x_1 \cdots x_k)^{-1} R_k.$$

The multiplication defines a map

$$D(1)\wedge\cdots\wedge D(1) \to D(k)$$

which motivates the use of the fundamental representation above.

1.4. **The Steinberg summands.** Mitchell and Priddy show that the Steinberg idempotent applied to the suspension spectrum of $B(\mathbb{Z}/p)^k_+$ splits off a summand weakly equivalent to $M(k) \approx L(k) \vee L(k-1)$. On the other hand Arone and Dwyer show that the Steinberg idempotent applied to

$$\mathrm{Thom}(B\mathbb{Z}/p^n;\bar\rho)$$

splits off a single copy of $L(k)$, where $\bar\rho$ is the reduced regular representation (of dimension $p^k - 1$). These are related by the maps

$$D(1)^{\wedge k} \to (\Sigma B\mathbb{Z}/p_+)^{\wedge k} \to \Sigma^k\,\mathrm{Thom}(B(\mathbb{Z}/p)^k;\bar\rho) \tag{1.42}$$

in which the first is the smash products of the map

$$D(1) \to \Sigma B\Sigma_{p+} \to \Sigma B\mathbb{Z}/p_+$$

and the second is the inclusion of the zero section. Applying the Steinberg idempotent to the middle and right terms of (1.42) gives the projection map

$$M(k) \to L(k).$$

In cohomology, the map

$$D(1)^{\wedge k} \to \Sigma^k \operatorname{Thom}(B(\mathbb{Z}/p)^k; \bar{\rho})$$

has image the ideal generated by the product of all the linear forms in the x_i for $p = 2$ and in the y_i for $p > 2$. This is the source of Remark 1.31.

2. Symmetric powers of the sphere spectrum

We now turn to work of Arone-Dwyer on the symmetric powers of the sphere spectrum. Throughout this section we will work with spaces, so the symbol S^ℓ will refer to the topological ℓ-sphere. More generally for a vector space V we will write S^V for the one point compactification of V and $S(V)$ for the unit sphere in V with respect to some choice of positive definite inner product.

Following Arone-Dwyer we will write

$$X^\diamondsuit = X \times [0,1] \amalg \{0,1\}/(x,0) \sim 0, (x,1) \sim 1$$

for the unreduced suspension of X, regarded as a pointed space with the equivalence class of 1 taken as the base point. Fix a homeomorphism $S(V)^\diamondsuit \approx S^V$ which is the identity on $S(V)$, under which 0 corresponds to the origin and 1 the point at ∞.

The quotient

$$\operatorname{SP}^n(S^\ell)/\operatorname{SP}^{n-1}(S^\ell) \tag{2.1}$$

is the orbit space

$$S^{n\ell}/\Sigma_n.$$

Write $\rho = \rho_n$ for the defining permutation representation of Σ_n on $\mathbb{R}^n$, and $\bar{\rho} = \bar{\rho}_n$ for the reduced regular representation on the subspace of vectors $(x_1, \dots, x_n) \in \mathbb{R}^n$ with $x_1 + \cdots + x_n = 0$. With this notation we have an equivalence

$$\operatorname{SP}^n(S^\ell)/\operatorname{SP}^{n-1}(S^\ell) \approx S^{\ell\rho} \approx S^n \wedge S^{\ell\bar{\rho}}/\Sigma.$$

From the expression (1.1) it follows that the quotient of the n^{th} symmytric power of the sphere spectrum by the $(n-1)^{\text{st}}$ is the suspension spectrum of

$$S(\infty\bar{\rho})^\diamondsuit/\Sigma_n.$$

We are therefore interested in the Σ_n equivariant homotopy type of $S(\infty\bar{\rho})$.

A *set partition* (or just *partition*) of $\{1, \dots, n\}$ is a set of disjoint, non empty subsets $I_j \subset \{1, \dots, n\}$ whose union is all of $\{1, \dots, n\}$

$$\{1, \dots, n\} = I_1 \amalg \cdots \amalg I_k.$$

The corresponding *partition subroup*

$$\Sigma_\lambda = \Sigma_{I_1} \times \cdots \Sigma_{I_k}$$

is the subgroup of Σ_n preserving the subset I_j. A set partition may be defined as an equivalence relation on $\{1, \dots, n\}$ in which the subset I_j are the equivalence classes.

For a G-space X and $x \in X$ let

$$G_x = \{g \in G \mid g \cdot x = x\} \subset G$$

be the isotropy group of x and let

$$\operatorname{Iso}(X) = \{G_x \mid x \in X\}$$

be the set of subgroups of G occurring as isotropy in X. Supplemented by a little further data, the equivariant homotopy type of a G-space X can be recovered from the set $\operatorname{Iso}(X)$ and the homotopy type of the fixed point spaces X^K for $K \in \operatorname{Iso}(X)$.

Proposition 2.2. *The set* $\operatorname{Iso}(S(\infty\bar{\rho}))$ *is the set of proper partition subgroups* Σ_λ *of* G*. For each proper partition partition subgroup* $\Sigma_\lambda \subset \Sigma_n$ *one has*

$$S(\infty\bar{\rho})^{\Sigma_\lambda} = S(\infty\bar{\rho}^{\Sigma_\lambda}) \sim *.$$

□

This suggests that the Σ_n-equivariant homotopy type of $S(\infty\bar{\rho})$ is characterized by a universal property.

Definition 2.3. A *collection* of subgroups of a group G is a set $\mathcal{C}$ of subgroups of G which is stable under conjugation: if $K \in \mathcal{C}$ and $g \in G$ then $gKg^{-1} \in \mathcal{C}$.

Remark 2.4. Don't confuse a *collection* with a family $\mathcal{F}$ which is a set of subgroups of G stable under conjugation and passing to subgroups.

Associated to a collection are two objects: the collection $\mathcal{C}$ regarded as a partially ordered set under inclusion, and the category $\mathcal{O}_{\mathcal{C}}$ of transitive G-sets X with $\operatorname{Iso}(X) \in \mathcal{C}$. The Grothendieck construction applied to the forgetful functor $\mathcal{O}_{\mathcal{C}} \to \mathbf{Sets}$ is the category $\tilde{\mathcal{O}}_{\mathcal{C}}$ of pairs (X, x) with X a transitive G-set with $\operatorname{Iso}(X) \in \mathcal{C}$ and $x \in X$, and in which a morphism from (X, x) to (Y, y) is an equivariant map $f : X \to Y$ with $f(x) = y$. The functor from $\tilde{\mathcal{O}}_{\mathcal{C}}$ to $\mathcal{C}$ sending (X, x) to G_x is an equivalence of categories. It follows that the G-equivariant map

$$|\tilde{\mathcal{O}}_{\mathcal{C}}| \to |\mathcal{C}|$$

is a weak equivalence of underying spaces, and so induces an equivalence

$$|\tilde{\mathcal{O}}_{\mathcal{C}}|_{hG} \to |\mathcal{C}|_{hG}$$

on homotopy orbit spaces.

Given a collection $\mathcal{C}$, for every G-space X there is a functorial map

$$X_{\mathcal{C}} \to X$$

characterized up to weak equivalence by the following two properties

(1) $\operatorname{Iso}(X_{\mathcal{C}}) \subset \mathcal{C}$

(2) For all $K \in \mathcal{C}$ the map on K-fixed points

$$X_{\mathcal{C}}^K \to X^K$$

is a weak equivalence.

The map $X_{\mathcal{C}} \to X$ is the counit of the adjunction between the category of G-spaces, thought of as product preserving contravariant functors on the category $\mathbf{Sets}^{G,\mathrm{tran}}$ of finite G-sets and the restriction to the full subcategory $\mathbf{Sets}_{\mathcal{C}}^{G,\mathrm{tran}} \subset$

$\mathbf{Sets}^{G,\mathrm{tran}}$ G-sets with isotropy in $\mathcal{C}$. The construction of $X_{\mathcal{C}}$ is given by left Kan extension, as the geometric realization of

$$\cdots \coprod_{S_0\to S_1\in \mathbf{Sets}_{\mathcal{C}}^{G,\mathrm{tran}}} S_0 \times \mathrm{Map}^G(S_1, X) \rightrightarrows \coprod_{S\in \mathbf{Sets}_{\mathcal{C}}^{G,\mathrm{tran}}} S \times \mathrm{Map}^G(S, X).$$

The verification of the properties above is straightforward.

When X is a point, the G-space $X_{\mathcal{C}}$ is denoted $E\mathcal{C}$. By the above, there is an equivariant equivalence $E\mathcal{C} \approx |\mathcal{O}_{\mathcal{C}}|$, and an equivariant map $E\mathcal{C} \to |\mathcal{C}|$ which is an equivalence of underlying spaces.

In the case $G = \Sigma_n$ there are two collections that concern us in this lecture. The collection $\tilde{\mathcal{P}}$ of proper partition subgroups of Σ_n and the collection $\mathcal{P}$ of non-trivial proper partition subgroups of Σ_n. The partially ordered set of non-trivial proper subgropus of Σ_n is the *partition poset*. Its realization $|\mathcal{P}|$ is denoted $\mathbf{P}_n$ and called the *partition complex*.

Remark 2.5. In the language of families of subgroups, the space $S(\infty\bar{\rho})$ is the space $E\mathcal{F}$ for the family of non-transitive subgroups. The space $S^{\infty\bar{\rho}}$ is the mapping cone of $E\mathcal{F} \to \mathrm{pt}$ and usually denoted $\tilde{E}\mathcal{F}$. It is one of the fundamental spaces occurring in the isotropy separation sequence for Σ_n.

Definition 2.6. The *singular space* of a G-space X is the subspace

$$\mathrm{Sing}\, X = \{x \in X \mid G_x \neq \{e\}\}$$

of points with non-trivial isotropy.

The singular space is useful for relating the orbit space X/G of a G-space to the homotopy orbit space X_{hG}. There is a coCartesian square

$$\begin{array}{ccc} (\mathrm{Sing}\, X)_{hG} & \longrightarrow & X_{hG} \\ \downarrow & & \downarrow \\ \mathrm{Sing}\, X/G & \longrightarrow & X/G\ . \end{array}$$

More generally if Y is a G-stable subspace of X containing $\mathrm{Sing}\, X$ then there is a homotopy coCartesian diagram

$$\begin{array}{ccc} Y_{hG} & \longrightarrow & X_{hG} \\ \downarrow & & \downarrow \\ Y/G & \longrightarrow & X/G. \end{array}$$

From Proposition 2.2 one concludes

Proposition 2.7. *There is an equivariant equivalence*

$$\mathrm{Sing}\, S(\infty\bar{\rho}) \approx E\mathcal{P}$$

and an equivariant map

$$\mathrm{Sing}\, S(\infty\bar{\rho}) \to \mathbf{P}_n$$

which is an equivalence of underlying spaces. □

Arone and Dwyer analyze the homotopy type of $S^{\infty\bar{\rho}}/\Sigma_n$ from the square

$$\begin{array}{ccc} ((\operatorname{Sing} S^{\infty\bar{\rho}}) \wedge S^{\bar{\rho}})_{\tilde{h}\Sigma_n} & \longrightarrow & (S^{\infty\bar{\rho}} \wedge S^{\bar{\rho}})_{\tilde{h}\Sigma_n} \\ \downarrow & & \downarrow \\ ((\operatorname{Sing} S^{\infty\bar{\rho}}) \wedge S^{\bar{\rho}})/\Sigma_n & \longrightarrow & (S^{\infty\bar{\rho}} \wedge S^{\bar{\rho}})/\Sigma_n \end{array} \tag{2.8}$$

in which, for a pointed G-space X, we are using the notation

$$X_{\tilde{h}G} = (EG \underset{G}{\times} X)/(EG \underset{G}{\times} *).$$

The square is coCartesian since

$$\operatorname{Sing}(S^{\infty\bar{\rho}} \wedge S^{\bar{\rho}}) \subset \operatorname{Sing}(S^{\infty\bar{\rho}}) \wedge S^{\bar{\rho}}.$$

We now analyze the terms in the square beginning with the upper left. From Proposition 2.8 there is an equivalence

$$(\operatorname{Sing}(S^{\infty\bar{\rho}}) \wedge S^{\bar{\rho}})_{\tilde{h}\Sigma_n} \approx (\mathbf{P}_n^{\diamondsuit} \wedge S^{\bar{\rho}})_{\tilde{h}\Sigma_n}.$$

Since $\infty + 1 = \infty$ we have

$$S^{\infty\bar{\rho}} \wedge S^{\bar{\rho}} = S^{\infty\bar{\rho}}.$$

The upper right term is therefore contractible, since the map $S^{\infty\bar{\rho}} \to \mathrm{pt}$ is a weak equivalence of underlying spaces, and the lower right term is

$$S^{\infty\bar{\rho}}/\Sigma_n.$$

We will see in Corollary 2.13 below that the lower left term is contractible (this was the whole point of putting in the extra $S^{\bar{\rho}}$). Granting this for the moment, we may, up to weak equivalence, rewrite the square (2.9) as

$$\begin{array}{ccc} (\mathbf{P}_n^{\diamondsuit} \wedge S^{\bar{\rho}})_{\tilde{h}\Sigma_n} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^{\infty\bar{\rho}}/\Sigma_n. \end{array}$$

This gives the main theorem of this lecture

Proposition 2.9. *There is an equivalence*

$$S^{\infty\bar{\rho}}/\Sigma_n \approx \Sigma(\mathbf{P}_n^{\diamondsuit} \wedge S^{\bar{\rho}})_{\tilde{h}\Sigma_n} \approx (\mathbf{P}_n^{\diamondsuit} \wedge S^{\rho})_{\tilde{h}\Sigma_n}$$

and so and equivalence

$$\mathrm{SP}^n(S^0)/\mathrm{SP}^{n-1}(S^0) \approx \Sigma^\infty(\mathbf{P}_n^{\diamondsuit} \wedge S^{\rho})_{\tilde{h}\Sigma_n}. \quad \square$$

We now turn to the contractibility of the lower left term in (2.9).

Lemma 2.10. *For $n \geq 2$, the orbit space $S(\bar{\rho}_n)/\Sigma_n$ is contractible, hence so is*

$$S^{\bar{\rho}_n}/\Sigma_n = S(\rho_n)^{\diamondsuit}/\Sigma_n.$$

Proof: The Σ_n-space $S(\rho_n)$ is equivariantly homeomorphic to the boundary of the simplex with vertices the standard basis vectors in $\mathbb{R}^n$. This is the set of points

$(s_1, \dots, s_n) \in \mathbb{R}^n$ having the properties that

$$0 \le s_i \le 1$$
$$s_1 + \cdots + s_n = 1$$
$$s_i = 0 \quad \text{for some } i.$$

Each Σ_n-orbit contains a unique point satisfying

$$0 = s_1 \le s_2 \le \cdots \le s_n \le 1.$$

The orbit space is homeomorphic to the standard $(n-2)$-simplex with vertices

$$\begin{aligned}&(0, \tfrac{1}{n-1}, \dots, \tfrac{1}{n-1})\\ &(0, 0, \tfrac{1}{n-2}, \dots, \tfrac{1}{n-2})\\ &\dots\\ &(0, \dots, 0, \tfrac{1}{2}, \tfrac{1}{2})\\ &(0, \dots, 0, 1).\end{aligned}$$

Alternatively, one can identify the boundary of $\Delta(n)$ with the nerve of the poset of proper subsets of $\{1, \dots, n\}$. The orbit through each flag contains a unique flag in the ordered set

$$\{\{1\} < \{1,2\} < \cdots < \{1, \dots, n-1\}\}$$

so the orbit space is the geometric realization of this ordered set which is an $(n-2)$-simplex. □

Corollary 2.11. *If $\Sigma_\lambda \subset \Sigma_n$ is a non-trivial partition subgroup then the orbit space $S^{\bar{\rho}_n}/\Sigma_\lambda$ is contractible.* □

Working through an equivariant cell decomposition one can then show

Corollary 2.12. *If X is a G-space and $\operatorname{Iso}(X)$ is contained in the collection $\mathcal{P}$ of non-trivial partition subgroups, then the orbit space*

$$\big(S^{\rho_n} \wedge X\big)/\Sigma_n$$

is contractible. □

The contractibility of $((\operatorname{Sing} S^{\infty\bar{\rho}}) \wedge S^{\bar{\rho}})/\Sigma_n$ is immediate from Corollary 2.13.

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Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: mjh@math.harvard.edu