# LECTURE 1: OVERVIEW

One of the principal aims of algebraic topology is to study topological spaces $X$ by means of algebraic invariants, such as their homology and cohomology groups. Ideally, one would like invariants which have both of the following features:

- $(a)$ They should be rich and interesting, with the potential to capture useful information about $X$.
- $(b)$ They should be simple and easy to work with, amenable to study by methods of algebra.

These requirements are in tension with one another: it is generally hard to satisfy both of them simultaneously. Nevertheless, there are some invariants which do a good job of reconciling $(a)$ with $(b)$. Let us consider some examples which arise in the study of rational homotopy theory. We begin by reviewing some definitions.

**Definition 1.** Let $f : X \to Y$ be a map between simply connected topological spaces. We will say that $f$ is a *rational homotopy equivalence* if the pullback map $f^* : \mathrm{H}^*(Y; \mathbf{Q}) \to \mathrm{H}^*(X; \mathbf{Q})$ is an isomorphism.

More generally, we say that a pair of (simply connected) topological spaces $X$ and $Y$ are *rationally homotopy equivalent* if they are related by a chain of rational homotopy equivalences.

Roughly speaking, the goal of rational homotopy theory is to classify (simply connected) topological spaces up to rational homotopy equivalence. It follows immediately from the definitions that if $X$ and $Y$ are rationally homotopy equivalent, then the rational cohomology rings $\mathrm{H}^*(X; \mathbf{Q})$ and $\mathrm{H}^*(Y; \mathbf{Q})$ are isomorphic. One can ask if the converse holds:

**Question 2.** Let $X$ and $Y$ be simply connected topological spaces, and suppose that the cohomology rings $\mathrm{H}^*(X; \mathbf{Q})$ and $\mathrm{H}^*(Y; \mathbf{Q})$ are isomorphic. Does it follow that $X$ and $Y$ are rationally homotopy equivalent?

The answer to Question 2 is "no" in general: it is possible for spaces $X$ and $Y$ to have the same rational cohomology ring "by accident" (that is, an isomorphism $\mathrm{H}^*(X; \mathbf{Q}) \simeq \mathrm{H}^*(Y; \mathbf{Q})$ which is not induced by a map of topological spaces, or a chain of such). Put another way, it is hopeless to try to reconstruct a space $X$ (or any approximation to $X$) from the information of the cohomology ring $\mathrm{H}^*(X; \mathbf{Q})$ alone.

One can attempt to remedy the situation by replacing the cohomology ring $\mathrm{H}^*(X; \mathbf{Q})$ by a somewhat richer invariant: the singular cochain complex $C^*(X; \mathbf{Q})$. Note that the cup product on cohomology is defined at the cochain level: that is, there are maps

$$\cup : C^s(X;\mathbf{Q}) \otimes_{\mathbf{Q}} C^t(X;\mathbf{Q}) \to C^{s+t}(X;\mathbf{Q})$$

which endow $C^*(X;\mathbf{Q})$ with the structure of a *differential graded algebra* over $\mathbf{Q}$. We can therefore regard the construction $X \mapsto C^*(X;\mathbf{Q})$ as a contravariant functor

$$\{\text{Topological spaces}\} \to \{\text{Differential graded algebras over } \mathbf{Q}\}.$$

Essentially by definition, this functor carries rational homotopy equivalences to quasi-isomorphisms between differential graded algebras. We can therefore pose the following refinement of Question 2:

**Question 3.** Let $X$ and $Y$ be simply connected topological spaces, and suppose that the differential graded algebras $C^*(X;\mathbf{Q})$ and $C^*(Y;\mathbf{Q})$ are quasi-isomorphic. Does it follow that $X$ and $Y$ are rationally homotopy equivalent?

Question 3 still has a negative answer, but the reason is more subtle. For any topological space $X$, the cohomology ring $\mathrm{H}^*(X;\mathbf{Q})$ is a graded-commutative algebra: that is, we have

$$x \cup x' = (-1)^{st} x' \cup x$$

for $x \in \mathrm{H}^s(X;\mathbf{Q})$ and $x' \in \mathrm{H}^t(X;\mathbf{Q})$. However, this commutativity is not visible at the cochain level. If $x$ and $x'$ are represented by cocycles $\overline{x} \in C^s(X;\mathbf{Q})$ and $\overline{x}' \in C^t(X;\mathbf{Q})$, then we can write

$$\overline{x} \cup \overline{x}' = (-1)^{st} \overline{x}' \cup \overline{x} + d\sigma(\overline{x}, \overline{x}'),$$

for some cochain $\sigma(\overline{x}, \overline{x}') \in C^{s+t-1}(X;\mathbf{Q})$, which can be explicitly chosen. If $\alpha : C^*(X;\mathbf{Q}) \to C^*(Y;\mathbf{Q})$ is a quasi-isomorphism of differential graded algebras, then the difference $\alpha(\sigma(\overline{x}, \overline{x}')) - \sigma(\alpha(\overline{x}), \alpha(\overline{x}'))$ is an $(s+t-1)$-cycle which represents a cohomology class $\eta \in \mathrm{H}^{s+t-1}(Y;\mathbf{Q})$. If this cohomology class does not vanish, then it is impossible for $\alpha$ to arise from a rational homotopy equivalence of spaces.

Put more informally: Question 3 has a negative answer because the differential graded algebra $C^*(X;\mathbf{Q})$ fails to encode information about the commutativity of the cup product on cohomology. To remedy this, Sullivan introduced a variant of the singular cochain complex $C^*(X;\mathbf{Q})$, in which multiplication is graded-commutative "on the nose":

**Construction 4** (Sullivan)**.** For each $n \geq 0$, let

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbf{R}_{\geq 0} | x_0 + \cdots + x_n = 1\}$$

denote the standard $n$-simplex and let

$$\Delta^n_+ = \{(x_0, \ldots, x_n) \in \mathbf{R} | x_0 + \cdots + x_n = 1\}$$

denote the affine space containing it. We will say that a differential form $\omega$ on $\Delta^n_+$ is *polynomial* it it belongs to the subalgebra of $\Omega^*(\Delta^n_+)$ generated (over the rational numbers) by the functions $x_i$ and their differentials $dx_i$.

Let $X$ be an arbitrary topological space and let $m \geq 0$ be an integer. A *singular $m$-form* on $X$ is function

$$\omega : \{\text{Continuous maps } \Delta^n \to X\} \to \{\text{Polynomial } m\text{-forms on } \Delta^n_+\}$$

$$\sigma \mapsto \omega_\sigma$$

which satisfies the following constraint: if $f : \Delta^{n'} \to \Delta^n$ is the map of simplices associated to a nondecreasing function $\{0 < 1 < \ldots < n'\} \to \{0 < 1 < \cdots < n\}$, then $\omega_{\sigma \circ f} = \omega_\sigma|_{\Delta^{n'}_+}$. We let $\Omega^m_{\mathrm{poly}}(X)$ denote the set of all singular $m$-forms on $X$.

If $\omega$ is a singular $m$-form on $X$, then we can define a singular $(m+1)$-form $d\omega$ on $X$ by the formula $(d\omega)_\sigma = d(\omega_\sigma)$. Using this differential, we can regard

$$0 \to \Omega^0_{\mathrm{poly}}(X) \to \Omega^1_{\mathrm{poly}}(X) \to \Omega^2_{\mathrm{poly}}(X) \to \cdots$$

as a chain complex of rational vector spaces, which we will refer to as the *polynomial de Rham complex of* $X$ and denote by $\Omega^*_{\mathrm{poly}}(X)$. We regard $\Omega^*_{\mathrm{poly}}(X)$ as a commutative differential graded algebra over $\mathbf{Q}$, with multiplication given by $(\omega \wedge \omega')_\sigma = \omega_\sigma \wedge \omega'_\sigma$.

The construction $X \mapsto \Omega^*_{\mathrm{poly}}(X)$ determines a contravariant functor

$$\{\text{Topological Spaces}\} \to \{\text{Commutative Differential Graded Algebras over } \mathbf{Q}\}.$$

It is closely related to the singular cochain functor $X \mapsto C^*(X;\mathbf{Q})$: one can show that integration of differential forms defines a quasi-isomorphism of chain complexes: $\int : \Omega^*_{\mathrm{poly}}(X) \to C^*(X;\mathbf{Q})$. In particular, the rational cohomology $\mathrm{H}^*(X;\mathbf{Q})$ can be computed by the chain complex $\Omega^*_{\mathrm{poly}}(X)$. We can therefore ask another variant of Question 2:

**Question 5.** Let $X$ and $Y$ be simply connected topological spaces, and suppose that $\Omega^*_{\mathrm{poly}}(X)$ and $\Omega^*_{\mathrm{poly}}(Y)$ are quasi-isomorphic (as *commutative* differential graded algebras). Does it follow that $X$ and $Y$ are rationally homotopy equivalent?

Perhaps surprisingly, the answer to Question 5 turns out to be "yes", at least if we impose some mild finiteness assumptions. To formulate this precisely, we need to introduce a bit of terminology.

**Notation 6.** Let Top denote the category of topological spaces and let $\mathrm{Top}^{\mathrm{sc}} \subseteq \mathrm{Top}$ be the full subcategory spanned by the simply connected topological spaces. For every field $k$, we let $\mathrm{Top}^{\mathrm{ft}(k)} \subseteq \mathrm{Top}^{\mathrm{sc}}$ denote the full subcategory spanned by those simply connected spaces $X$ for which the cohomology group $\mathrm{H}^n(X;k)$ is a finite-dimensional vector space over $k$, for each $n \geq 0$ (note that this condition depends only on the characteristic of $k$).

We let $\mathrm{h}\mathcal{S}^{\mathbf{Q}}$ denote the category obtained from $\mathrm{Top}^{\mathrm{sc}}$ by formally inverting all rational homotopy equivalences, and we let $\mathrm{h}\mathcal{S}^{\mathrm{ft}(\mathbf{Q})}$ denote the category obtained from $\mathrm{Top}^{\mathrm{ft}}$ by formally inverting all rational homotopy equivalences (one can show that this is a full subcategory of $\mathrm{h}\mathcal{S}^{\mathrm{ft}(\mathbf{Q})}$.

Let $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ denote the category of commutative differential graded algebras over $\mathbf{Q}$, and let $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ denote the category obtained from $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ by formally inverting all quasi-isomorphisms.

Since the construction $X \mapsto \Omega^*_{\mathrm{poly}}(X)$ carries rational homotopy equivalences to quasi-isomorphisms, it induces a functor of homotopy categories

$$\Omega^*_{\mathrm{poly}} : (\mathrm{h}\mathcal{S}^{\mathbf{Q}})^{\mathrm{op}} \to \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}.$$

We then have the following:

**Theorem 7** (Sullivan)**.** *The functor $\Omega^*_{\mathrm{poly}}$ determines a fully faithful embedding $(\mathrm{h}\mathcal{S}^{\mathrm{ft}(\mathbf{Q})})^{\mathrm{op}} \hookrightarrow \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$. Moreover, the essential image of this embedding consists of those commutative differential graded algebras $A^*$ which satisfy the following conditions:*

- (*i*) *The unit map $\mathbf{Q} \to \mathrm{H}^0(A^*)$ is an isomorphism.*
- (*ii*) *The cohomology groups $\mathrm{H}^n(A^*)$ vanish for $n = 1$ and $n < 0$.*
- (*iii*) *The cohomology groups $\mathrm{H}^n(A^*)$ are finite-dimensional (as vector spaces over $\mathbf{Q}$) for $n \geq 2$.*

It follows from Theorem 7 that the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$ does an excellent job of satisfying requirements (*a*) and (*b*) above.

- (*a*) By virtue of Theorem 7, we can say exactly what information about a topological space $X$ is captured by the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$: namely, the rational homotopy type of $X$ (at least if $X$ is rationally of finite type).
- (*b*) The polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$ is a commutative differential graded algebra over $\mathbf{Q}$, and therefore belongs to a class of objects which can be studied by purely algebraic techniques.

**Remark 8.** One might argue that the polynomial de Rham functor $\Omega^*_{\mathrm{poly}}$ fails to satisfy (*b*) for one of the following reasons:

1. Even when the topological space $X$ is relatively simple, the commutative differential graded algebra $\Omega^*_{\mathrm{poly}}(X)$ tends to be enormous (and hence not really suitable for calculations).
2. The *isomorphism* class of the algebra $\Omega^*_{\mathrm{poly}}(X)$ is not a rational homotopy invariant: only its quasi-isomorphism class is. In other words, Theorem 7 does not really allow us to reduce homotopy theory to algebra: instead, it reduces one kind of homotopy theory (having to do with topological spaces) to another (having to do with differential graded algebras).

However, there is a sense in which these objections "cancel out." Every commutative differential graded algebra $A^*$ satisfying the hypotheses of Theorem 7 admits a "minimal model" $B_*$, which is well-defined up to (non-unique) isomorphism and depends only on the quasi-isomorphism class of $A^*$. The minimal model of a polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$ is often a tractable object that one can use for concrete calculations. Moreover, Theorem 7 ensures that spaces $X, Y \in \mathrm{Top}^{\mathrm{ft}}$ are rationally homotopy equivalent if and only if the minimal models of $\Omega^*_{\mathrm{poly}}(X)$ and $\Omega^*_{\mathrm{poly}}(Y)$ are isomorphic.

We now review another approach to the study of rational homotopy theory (which historically predates Sullivan's work), due to Quillen. First, we need some more terminology.

**Notation 9.** Let $\mathrm{Top}_*$ denote the category of pointed topological spaces and let $\mathrm{Top}^{\mathrm{sc}}_* \subseteq \mathrm{Top}_*$ be the full subcategory spanned by the simply connected pointed spaces. We let $\mathrm{h}\mathcal{S}^{\mathbf{Q}}_*$ denote the category obtained from $\mathrm{Top}^{\mathrm{sc}}_*$ by formally inverting all rational homotopy equivalences. Let $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{Q}}$ denote the category of differential graded Lie algebras over $\mathbf{Q}$, and let $\mathrm{hLie}^{\mathrm{dg}}_{\mathbf{Q}}$ denote the category obtained from $\mathrm{Lie}^{\mathrm{dg}}_{\mathbf{Q}}$ by formally inverting all quasi-isomorphisms.

**Theorem 10** (Quillen)**.** *There is a fully faithful embedding $L_* : \mathrm{h}\mathcal{S}^{\mathbf{Q}}_* \hookrightarrow \mathrm{hLie}^{\mathrm{dg}}_{\mathbf{Q}}$, whose essential image consists of those differential graded Lie algebras $\mathfrak{g}_*$ which are* connected*: that is, for which the homology groups of the underlying chain complex vanish in degrees $\leq 0$.*

Let us contrast Theorem 7 with Theorem 10.

(*i*) Theorem 10 is about the rational homotopy theory of *pointed* spaces, and the base points play an essential role. In contrast, Sullivan's approach to rational homotopy theory does not require base points. There is an analogue of Theorem 7 for pointed spaces, where we replace $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ by the homotopy category of *augmented* commutative differential graded algebras over $\mathbf{Q}$. However, I do not know of an "unpointed" version of Theorem 10.

(*ii*) The equivalence of Theorem 7 requires that we work with spaces which are rationally of finite type (essentially, this is because the definition of cohomology involves forming a dual). However, Theorem 10 does not need any finiteness hypotheses.

(*iii*) From the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$, it is easy to read off information about the *cohomology* of the space $X$: the rational cohomology groups of $X$ are simply the cohomology groups of the underlying cochain complex $\Omega^*_{\mathrm{poly}}(X)$ (and the multiplication on $\Omega^*_{\mathrm{poly}}(X)$ encodes the cup product on cohomology). From the differential graded Lie algebra $L_*(X)$, one can instead read off information about the *rational homotopy groups*

$\pi_n^{\mathbf{Q}}(X) = \mathbf{Q} \otimes \pi_n(X)$: these are given by the homology groups of the underlying chain complex of $L_*(X)$ (and the Lie bracket on $L_*(X)$ encodes the Whitehead product on homotopy groups).

Note that point (*iii*) suggests that the constructions $\Omega^*_{\mathrm{poly}}(X)$ and $L_*(X)$ are related by some form of Eckmann-Hilton duality. One can articulate this using the Koszul duality between (differential graded) Lie algebras and commutative (differential graded) algebras. More precisely, there is a commutative diagram of homotopy categories

$$\begin{array}{ccc} & \mathrm{h}\mathcal{S}_*^{\mathbf{Q}} & \\ {\scriptstyle L_*}\swarrow & & \searrow{\scriptstyle \Omega^*_{\mathrm{poly}}} \\ \mathrm{hLie}^{\mathrm{dg}}_{\mathbf{Q}} & \xrightarrow{\mathrm{CE}} & (\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}})^{\mathrm{op}}, \end{array}$$

where the bottom horizontal map is obtained from the *Chevalley-Eilenberg functor* $\mathrm{CE} : \mathrm{Lie}^{\mathrm{dg}}_{\mathbf{Q}} \to (\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})^{\mathrm{op}}$ (which assigns to each Lie algebra $\mathfrak{g}_*$ an explicit chain complex $\mathrm{CE}(\mathfrak{g}_*)$ which computes the Lie algebra cohomology of $\mathfrak{g}_*$). This diagram (and its higher-chromatic analogues) will play an essential role throughout this seminar.

Like the polynomial de Rham complex of Construction 4, we can regard Quillen's functor $L_* : \mathrm{h}\mathcal{S}_*^{\mathbf{Q}} \hookrightarrow \mathrm{hLie}^{\mathrm{dg}}_{\mathbf{Q}}$ as an invariant of topological spaces which satisfies requirements $(a)$ and $(b)$ simultaneously: for any simply connected topological space $X$, we can regard $L_*(X)$ as a purely algebraic object which encodes the entire rational homotopy type of $X$. However, this is a great loss of information: the constructions $X \mapsto L_*(X)$ and $X \mapsto \Omega^*_{\mathrm{poly}}(X)$ are both rational homotopy invariants, and are therefore insensitive to "torsion phenemena" in homotopy theory. This seminar is motivated by the following:

**Question 11.** Can the algebraic models for rational homotopy types described above be extended to the non-rational setting?

Let us begin by describing one answer to Question 11, due to Mandell. In what follows, let $p$ be some fixed prime number. We say that a map of topological spaces $f : X \to Y$ is an $\mathbf{F}_p$*-cohomology equivalence* if the induced map

$$f^* : \mathrm{H}^*(Y; \mathbf{F}_p) \to \mathrm{H}^*(X; \mathbf{F}_p)$$

is an isomorphism. Note that this implies that $f^* : \mathrm{H}^*(Y; k) \to \mathrm{H}^*(X; k)$ is an isomorphism for *any* field $k$ of characteristic $p$. It follows that if spaces $X$ and $Y$ are $\mathbf{F}_p$-cohomology equivalent, then the differential graded algebras $C^*(X; k)$ and $C^*(Y; k)$ are quasi-isomorphic. The converse fails in general, for the same reason that Question 3 has a negative answer: the differential graded algebra structure on $C^*(X; k)$ fails to encode information about the commutativity of the cup product in cohomology. When $k = \mathbf{Q}$, this can be remedied by replacing $C^*(X;k)$ by the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$ of Construction 4. For fields of characteristic $p$, the same trick does not work: there is no way to replace $C^*(X;k)$ by a quasi-isomorphic differential graded algebra which is commutative and depends functorially on $X$. However, one can pursue another remedy: instead of working with commutative differential graded algebras (which are badly-behaved objects over fields of positive characteristic), one can regard the singular cochain complex $C^*(X;k)$ as an *$E_\infty$-algbebra over $k$*. One then has the following analogue of Theorem 7:

**Theorem 12.** *[Mandell] Let $\mathrm{h}\mathcal{S}^{\mathrm{ft}(\mathbf{F}_p)}$ denote the category obtained from $\mathrm{Top}^{\mathrm{ft}(\mathbf{F}_p)}$ by formally inverting the $\mathbf{F}_p$-cohomology equivalences. Then the construction $X \mapsto C^*(X;\overline{\mathbf{F}}_p)$ determines a fully faithful embedding from $\mathrm{h}\mathcal{S}^{\mathrm{ft}(\mathbf{F}_p)}$ to the homotopy category of $E_\infty$-algebras over $\overline{\mathbf{F}}_p$; here $\overline{\mathbf{F}}_p$ denotes an algebraic closure of $\mathbf{F}_p$.*

**Remark 13.** I do not know if there is an analogue of Theorem 12 for Quillen's model of rational homotopy theory. There is perhaps good reason to expect that there is not. Note that if $X$ is a pointed space of which is rationally of finite type, then passage from the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$ to the differential graded Lie algebra $L_*(X)$ is given concretely by the formation of André-Quillen cohomology. However, one can show that the $E_\infty$-algebras $C^*(X;\overline{\mathbf{F}}_p)$ are formally étale over $\overline{\mathbf{F}}_p$: that is, their (topological) André-Quillen cohomology vanishes.

Theorem 12 can be regarded as a $p$-adic version of Sullivan's approach to rational homotopy theory. However, we mention it only for culture: it is not the subject of this seminar. Instead, we will concern ourselves with a regime which is *intermediate* between rational and mod $p$ cohomology. Before we can elaborate, we need to review a bit of chromatic homotopy theory.

**Definition 14.** Let $\mathcal{C}$ be a triangulated category. We will say that a full subcategory $\mathcal{C}_0$ is *thick* if it satisfies the following conditions:

(1) For every exact triangle $C' \to C \to C'' \to C'[1]$ in $\mathcal{C}$, if any two of the objects $C, C', C''$ belong to $\mathcal{C}_0$, then so does the third.
(2) If a direct sum $C \oplus C'$ belongs to $\mathcal{C}_0$, then $C$ and $C'$ also belong to $\mathcal{C}_0$.
(3) The zero object $0 \in \mathcal{C}$ belongs to $\mathcal{C}_0$.

If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a thick subcategory, then one can form a new triangulated category $\mathcal{C}/\mathcal{C}_0$, called the *Verdier quotient of $\mathcal{C}$ by $\mathcal{C}_0$*.

**Example 15.** Let hSp denote the homotopy category of spectra, and let $\mathrm{hSp}^{\mathrm{fin}}_{(p)}$ denote the full subcategory of hSp spanned by the finite $p$-local spectra. Then $\mathrm{hSp}^{\mathrm{fin}}_{(p)}$ is a triangulated category. We let $\mathrm{hSp}^{\mathrm{fin}}_{\geq 1}$ denote the full subcategory of $\mathrm{hSp}^{\mathrm{fin}}_{(p)}$ spanned by those finite $p$-local spectra $X$ for which the homotopy groups

$\pi_* X$ are torsion (that is, for which the rationalization $X_{\mathbf{Q}}$ vanishes). Then $\mathrm{hSp}^{\mathrm{fin}}_{\geq 1}$ is a thick subcategory of $\mathrm{hSp}^{\mathrm{fin}}_{(p)}$.

Thick subcategories of $\mathrm{hSp}^{\mathrm{fin}}$ were classified by Hopkins and Smith. To state their classification, we first recall a definition: a finite $p$-local spectrum $X$ is said to be *of type* $\geq n$ if the smash product $X \wedge K(m)$ vanishes for $m < n$; here we adopt the convention that $K(0) = H\,\mathbf{Q}$ is the rational Eilenberg-MacLane spectrum. We let $\mathrm{hSp}^{\mathrm{fin}}_{\geq n}$ denote the full subcategory of $\mathrm{hSp}^{\mathrm{fin}}_{(p)}$ consisting of finite $p$-local spectra of type $\geq n$.

**Theorem 16** (Thick Subcategory Theorem)**.** *Every nonzero thick subcategory of* $\mathrm{hSp}^{\mathrm{fin}}_{(p)}$ *has the form* $\mathrm{hSp}^{\mathrm{fin}}_{\geq n}$ *for some* $n \geq 0$*. Moreover, we have*

$$\cdots \subsetneq \mathrm{hSp}^{\mathrm{fin}}_{\geq 4} \subsetneq \mathrm{hSp}^{\mathrm{fin}}_{\geq 3} \subsetneq \mathrm{hSp}^{\mathrm{fin}}_{\geq 2} \subsetneq \mathrm{hSp}^{\mathrm{fin}}_{\geq 1} \subsetneq \mathrm{hSp}^{\mathrm{fin}}_{\geq 0} = \mathrm{hSp}^{\mathrm{fin}}_{(p)}.$$

In what follows, it will be convenient to enlarge these triangulated categories (by removing finiteness conditions).

**Notation 17.** For each $n \geq 1$, we let $\mathrm{hSp}_{\geq n}$ denote the smallest thick subcategory of hSp which contains $\mathrm{hSp}^{\mathrm{fin}}_{\geq n}$ and is closed under (possibly infinite) direct sums. We let $\mathrm{hSp}_{T(n)}$ denote the Verdier quotient $\mathrm{hSp}_{\geq n} / \mathrm{hSp}_{\geq n+1}$. We refer to $\mathrm{hSp}_{T(n)}$ as the *$n$th telescopic localization of* hSp. In what follows, we will view $\mathrm{hSp}_{T(n)}$ as a full subcategory of hSp, where the inclusion $\mathrm{hSp}_{T(n)} \hookrightarrow \mathrm{hSp}$ has a left adjoint $L_{T(n)} : \mathrm{hSp} \to \mathrm{hSp}_{T(n)}$. By convention, we agree that $\mathrm{hSp}_{T(0)}$ denotes the homotopy category of *rational* spectra (which can also be described as the Verdier quotient $\mathrm{hSp}_{(p)} / \mathrm{hSp}_{\geq 1}$).

Roughly speaking, Theorem 16 implies that the homotopy category $\mathrm{hSp}_{(p)}$ of $p$-local spectra admits a filtration, whose successive quotients are given by the triangulated categories $\{\mathrm{Sp}_{T(n)}\}_{n \geq 0}$. *Chromatic homotopy theory* attempts to gain information about the stable homotopy category hSp in two steps:

1. First, analyze the individual categories $\mathrm{hSp}_{T(n)}$. One hopes that these objects are more tractable than hSp itself. For example, $\mathrm{hSp}_{T(0)}$ is equivalent to the homotopy category of *rational* spectra: that is, to the category of graded vector spaces over $\mathbf{Q}$.
2. Second, try to understand how the categories $\mathrm{hSp}_{T(n)}$ fit together inside of hSp.

In this seminar, we will attempt to apply this philosophy to the study of *unstable* homotopy theory. More precisely, we will attempt to carry out (1) by completing the following analogy:

$$\mathrm{hSp}_{T(0)} : \mathrm{hSp}_{T(n)} :: \text{Rational homotopy theory} : ???$$

Our ultimate target is the following recent theorem of Heuts:

**Theorem 18.** *Let $n > 0$ be an integer, and let $\mathrm{h}\mathcal{S}_*^{v_n}$ denote the $v_n$-periodic unstable homotopy category (we will define this in a moment). Then there exists an equivalence of categories*

$$\mathrm{h}\mathcal{S}_*^{v_n} \to \mathrm{hLie}(\mathrm{Sp}_{T(n)}).$$

*Here $\mathrm{hLie}(\mathrm{Sp}_{T(n)})$ denotes the homotopy category of Lie algebras in $T(n)$-local spectra.*

**Remark 19.** Theorem 18 should be regarded as a higher chromatic analogue of the Quillen model of rational homotopy theory: in the special case $n = 0$, we can identify $T(n)$-local spectra with chain complexes of vector spaces, so that $\mathrm{hLie}(\mathrm{Sp}_{T(n)})$ reduces to the theory of differential graded Lie algebras over $\mathbf{Q}$ (let us regard this as a heuristic for the moment; later in this seminar, we will make it precise). Note one contrast with Theorem 10: in Quillen's theory, only *connected* differential graded Lie algebras appear (at least if we restrict our attention to simply-connected spaces). The connectedness hypotheses has no analogue in Theorem 18.

Let us now give an outline of the proof of Theorem 18, highlighting some of the ideas we will meet along the way. We begin by attempting to describe the functor $\mathrm{h}\mathcal{S}_*^{v_n} \to \mathrm{hLie}(\mathrm{Sp}_{T(n)})$. First, we need to say something about the domain and codomain of this functor. The category $\mathrm{h}\mathcal{S}_*^{v_n}$ will be a higher chromatic analogue of the (pointed) rational homotopy category $\mathrm{h}\mathcal{S}_*^{\mathbf{Q}}$ defined above: in particular, it will be obtained from the homotopy category $\mathrm{h}\mathcal{S}_*$ of pointed spaces by inverting some morphisms (we could restrict our attention to simply connected spaces here, but it will turn out not to matter for $n > 0$). The category $\mathrm{hLie}(\mathrm{Sp}_{T(n)})$ consists of $T(n)$-local spectra equipped with a Lie algebra structure (in a sense we will later make precise). In particular, there is a forgetful functor $\mathrm{hLie}(\mathrm{Sp}_{T(n)}) \to \mathrm{hSp}_{T(n)}$. We will begin by describing the composite functor

$$\mathrm{h}\mathcal{S}_* \to \mathrm{h}\mathcal{S}_*^{v_n} \simeq \mathrm{hLie}(\mathrm{Sp}_{T(n)}) \to \mathrm{hSp}_{T(n)}\,.$$

For motivation, we first describe the analogue of this map in the "classical" case $n = 0$. Taking $\mathrm{h}\mathcal{S}_*^{\mathrm{sc}}$ to be the homotopy category of simply connected pointed spaces, we consider the composition

$$\mathrm{h}\mathcal{S}_*^{\mathrm{sc}} \to \mathrm{h}\mathcal{S}_*^{\mathbf{Q}} \xrightarrow{L_*} \mathrm{hLie}_{\mathbf{Q}}^{\mathrm{dg}} \to \mathrm{hSp}_{T(n)}\,.$$

Here the last functor associates to each differential graded Lie algebra its underlying chain complex $\mathfrak{g}_*[1]$ (which, for convenience, we shift by 1), which we will identify with the corresponding generalized Eilenberg-MacLane spectrum. Let us denote this composite functor by $\mathfrak{Q}$. It carries a simply connected space $X$ to a rational spectrum $\mathfrak{Q}(X)$ whose homotopy groups are given by the formula $\pi_*\mathfrak{Q}(X) = \mathbf{Q} \otimes \pi_* X$. To get a feeling for this functor, it will be useful to indulge in a brief algebraic digression.

**Definition 20.** Let $G$ be a group. We will say that $G$ is *rationally nilpotent* if it admits a filtration by normal subgroups $(0) = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n = G$, where each of the quotients $G_n / G_{n-1}$ is isomorphic to $\mathbf{Q}$.

**Example 21.** Let $G$ be the subgroup of $\mathrm{GL}_n(\mathbf{Q})$ consisting of matrices $(M_{ij})$ satisfying $M_{ij} = 0$ for $i < j$ and $M_{ii} = 1$. Then $G$ is rationally nilpotent.

It turns out that every rationally nilpotent group $G$ arises from as a variation on Example 21. More precisely, we have the following:

**Proposition 22.** *The construction $\mathbf{G} \mapsto \mathbf{G}(\mathbf{Q})$ determines an equivalence of categories*

$$\{\textit{Nilpotent algebraic groups over } \mathbf{Q}\} \xrightarrow{\mathbf{G} \mapsto \mathbf{G}(\mathbf{Q})} \{\textit{Rationally nilpotent groups}\}.$$

Note that if $\mathbf{G}$ is any algebraic group over $\mathbf{Q}$ (nilpotent or otherwise), we can associate to $\mathbf{G}$ a rational vector space

$$\mathrm{Lie}(\mathbf{G}) = \ker(\mathbf{G}(\mathbf{Q}[\epsilon]/(\epsilon^2)) \to \mathbf{G}(\mathbf{Q})).$$

Using Proposition 22, one can construct a functor

$$\{\text{Rationally nilpotent groups}\} \to \{\text{Vector spaces over } \mathbf{Q}\}$$

as follows: to every rationally nilpotent group $G$, one assigns the Lie algebra $\mathrm{Lie}(\mathbf{G})$ of the unique nilpotent algebraic group $\mathbf{G}$ satisfying $\mathbf{G}(\mathbf{Q}) \simeq G$. This functor can be regarded as an algebraic analogue of $\mathfrak{L}$ (in fact, there is more than an analogy: the construction $G \mapsto \mathrm{Lie}(\mathbf{G})$ arises naturally when rational homotopy is extended to spaces which are not simply connected).

To carry out a similar construction in homotopy theory, it is convenient to make the following definition:

**Definition 23.** A $\mathbf{Q}$-rational homotopy type is a functor

$$Y : \{\text{Commutative algebras over } \mathbf{Q}\} \to \mathrm{Top}_*^{\mathrm{sc}}$$

with the following property: for each $n \geq 2$, there exists a rational vector space $V_n$ and a natural isomorphism $\pi_n Y(A) \simeq A \otimes_{\mathbf{Q}} V_n$.

We then have the following analogue of Proposition 22:

**Proposition 24.** *The construction $Y \mapsto Y(\mathbf{Q})$ induces an equivalence of homotopy categories*

$$\{\mathbf{Q}\textit{-rational homotopy types}\} \to \mathrm{h}\mathcal{S}_*^{\mathbf{Q}}.$$

If $Y$ is a $\mathbf{Q}$-rational homotopy type, then we can construct another rational space $TY$ by taking the fiber of the natural map $Y(\mathbf{Q}[\epsilon]/(\epsilon^2)) \to Y(\mathbf{Q})$. Note that the homotopy groups of $TY$ are the same as the homotopy groups of $Y(\mathbf{Q})$. However, the space $TY$ has more structure than $X(\mathbf{Q})$: the natural map

$$\mathbf{Q}[\epsilon]/(\epsilon^2) \times_{\mathbf{Q}} \mathbf{Q}[\epsilon']/(\epsilon'^2) \simeq \mathbf{Q}[\epsilon, \epsilon']/(\epsilon^2, \epsilon\epsilon', \epsilon'^2) \xrightarrow{\epsilon, \epsilon' \mapsto \epsilon} \mathbf{Q}[\epsilon]/(\epsilon^2)$$

induces an addition map $TY \times TY \to TY$. This addition is commutative and associative up to coherent homotopy, and therefore endows $TY$ with the structure of an infinite loop space having the same homotopy groups as $Y(\mathbf{Q})$. Combining this construction with Proposition 24, we see that the construction $X(\mathbf{Q}) \mapsto TX$ determines a functor $\mathrm{h}\mathcal{S}_*^{\mathbf{Q}} \to \mathrm{hSp}_{T(0)}$. Composing with the functor $\mathrm{h}\mathcal{S}_*^{\mathrm{sc}} \to \mathrm{h}\mathcal{S}_*^{\mathbf{Q}}$, we obtain the functor $\mathfrak{Q}$ that we are looking for.

**Example 25.** Let $E$ be a rational spectrum whose homotopy groups $\pi_n E$ vanish for $n < 2$. Then the construction $A \mapsto \Omega^\infty(E \wedge HA)$ determines a functor $X$ from the category of commutative algebras over $\mathbf{Q}$ to the category of simply connected pointed spaces. This functor is easily checked to be a $\mathbf{Q}$-rational homotopy type, in the sense of Definition 23. Moreover, we have $X(\mathbf{Q}) = \Omega^\infty(E \wedge H\mathbf{Q}) \simeq \Omega^\infty E$ and $TX \simeq \Omega^\infty(E \wedge H(\epsilon\mathbf{Q})) \simeq \Omega^\infty E$. It follows that the composition

$$\mathrm{hSp}_{T(0)}^{\mathrm{sc}} \xrightarrow{\Omega^\infty} \mathrm{h}\mathcal{S}_*^{\mathrm{sc}} \xrightarrow{\mathfrak{Q}} \mathrm{hSp}_{T(0)}$$

is equivalent to the identity functor (here the superscript on the left hand side indicates that we consider only simply connected rational spectra).

I do not know if it is possible to construct the functor $\mathrm{h}\mathcal{S}_*^{v_n} \to \mathrm{hLie}(\mathrm{Sp}_{T(n)})$ appearing in Theorem 18 using a variation on these ideas. However, Example 25 suggests one property that we might look for in such a functor: after ignoring the Lie algebra structures, it should be left inverse to the functor $\Omega^\infty$. There is a natural candidate for such a functor:

**Theorem 26** (Bousfield, Kuhn)**.** *A spectrum $E \in \mathrm{hSp}_{T(n)}$ is functorially determined by its zeroth space. More precisely, there exists a functor $\Phi : \mathrm{h}\mathcal{S}_* \to \mathrm{hSp}_{T(n)}$ for which the composition*

$$\mathrm{hSp} \xrightarrow{\Omega^\infty} \mathrm{h}\mathcal{S}_* \xrightarrow{\Phi} \mathrm{hSp}_{T(n)}$$

*is isomorphic to the localization functor $L_{T(n)} : \mathrm{hSp} \to \mathrm{hSp}_{T(n)}$.*

The functor $\Phi : \mathrm{h}\mathcal{S}_* \to \mathrm{hSp}_{T(n)}$ is called the *Bousfield-Kuhn functor*. Our first objective in this seminar will be to give a careful definition of $\Phi$ and to establish its basic properties. Once we have done so, the following definition will be precise:

**Definition 27.** Let $n$ be a positive integer. We let $\mathrm{h}\mathcal{S}_*^{v_n}$ denote the category obtained from the homotopy category of pointed spaces $\mathrm{h}\mathcal{S}_*$ by formally inverting all morphisms $f : X \to Y$ for which the induced map $\Phi(f) : \Phi(X) \to \Phi(Y)$ is an isomorphism in $\mathrm{hSp}_{T(n)}$. We will refer to $\mathrm{h}\mathcal{S}_*^{v_n}$ as the *$v_n$-periodic homotopy category*; it can be regarded as a higher chromatic analogue of the rational homotopy category $\mathrm{h}\mathcal{S}_*^{\mathbf{Q}}$.

**Warning 28.** Let $f : X \to Y$ be a map of simply connected topological spaces. One can show that the following conditions are equivalent:

- $(i)$ The induced map of rational homology groups $f_* : \mathrm{H}_*(X;\mathbf{Q}) \to \mathrm{H}_*(Y;\mathbf{Q})$ is an isomorphism.
- $(ii)$ The induced map of rational cohomology groups $f^* : \mathrm{H}^*(Y;\mathbf{Q}) \to \mathrm{H}^*(X;\mathbf{Q})$ is an isomorphism.
- $(iii)$ The induced map of rational homotopy groups $\mathbf{Q}\otimes\pi_* X \to \mathbf{Q}\otimes\pi_* Y$ is an isomorphism.

Moreover, the rational homotopy category $\mathrm{h}\mathcal{S}_*^{\mathbf{Q}}$ can be obtained from the category of simply connected pointed spaces $\mathrm{Top}_*^{\mathrm{sc}}$ by formally inverting the maps satisfying any one of these equivalent conditions.

The higher chromatic analogue of this equivalence fails: for a map of spaces $f : X \to Y$, the condition that $L_{T(n)}\Sigma^\infty(f) : L_{T(n)}\Sigma^\infty(X) \to L_{T(n)}\Sigma^\infty(Y)$ is a homotopy equivalence of $T(n)$-local spectra (which is analogous to $(i)$) is not equivalent to the condition that $\Phi(f) : \Phi(X) \to \Phi(Y)$ is an equivalence of $T(n)$-local spectra (which is analogous to condition $(iii)$).

**Warning 29.** It is not at all obvious from the definitions that the category $\mathrm{h}\mathcal{S}_*^{v_n}$ is a reasonable mathematical object: for example, that the collection of morphisms $\mathrm{Hom}_{\mathrm{h}\mathcal{S}_*^{v_n}}(X,Y)$ forms a set, for each $X, Y \in \mathrm{h}\mathcal{S}_*^{v_n}$. However, Bousfield has shown that, like the rational homotopy category $\mathrm{h}\mathcal{S}_*^{\mathbf{Q}}$, the category $\mathrm{h}\mathcal{S}_*^{v_n}$ can be identified with a full subcategory of the homotopy category $\mathrm{h}\mathcal{S}_*$ of pointed spaces. Our second objective in this seminar will be to understand Bousfield's work on this subject.

It follows immediately from the definition that the Bousfield-Kuhn functor $\Phi : \mathrm{h}\mathcal{S}_* \to \mathrm{hSp}_{T(n)}$ factors (uniquely) through a functor $\mathrm{h}\mathcal{S}_*^{v_n} \to \mathrm{hSp}_{T(n)}$; we will abuse notation by denoting this functor also by $\Phi$. We can now formulate Theorem 18 a bit more precisely: it asserts that the functor $\Phi : \mathrm{h}\mathcal{S}_*^{v_n} \to \mathrm{hSp}_{T(n)}$ is equivalent to the forgetful functor $\mathrm{hLie}(\mathrm{Sp}_{T(n)}) \to \mathrm{hSp}_{T(n)}$. As a first step, we can ask if $\Phi : \mathrm{h}\mathcal{S}_*^{v_n} \to \mathrm{hSp}_{T(n)}$ behaves like a forgetful functor: that is, can we recover an object $X \in \mathrm{h}\mathcal{S}_*^{v_n}$ from the $T(n)$-local spectrum $\Phi(X)$ together with some additional data? To answer this, we will need the following:

**Theorem 30** (Bousfield)**.** *The functor $\Phi : \mathrm{h}\mathcal{S}_*^{v_n} \to \mathrm{hSp}_{T(n)}$ admits a left adjoint $\Theta : \mathrm{hSp}_{T(n)} \to \mathrm{h}\mathcal{S}_*^{v_n}$*

**Warning 31.** In the statement of Theorem 30, it is essential that we regard $\Phi$ as a functor with domain $\mathrm{h}\mathcal{S}_*^{v_n}$. As a functor from $\mathrm{h}\mathcal{S}_*$ to $\mathrm{hSp}_{T(n)}$, $\Phi$ does not have a left adjoint.

Note that, once we have Theorem 18 in hand, we will be able to identify $\Theta$ with some kind of "free Lie algebra" functor.

We now recall a bit of category theory. Given any pair of adjoint functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D},$$

the composition $T = G \circ F$ can be equipped with the structure of a *monad*: that is, an associative algebra in the category of functors from $\mathcal{C}$ to itself. Moreover, the functor $G$ can then be factored as a composition

$$\mathcal{D} \xrightarrow{G'} \mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C},$$

where $\mathrm{LMod}_T(\mathcal{C})$ denotes the category of algebras in $\mathcal{C}$ over the monad $T$. We say that the adjunction $(F, G)$ is *monadic* if the functor $G'$ is an equivalence: in this case, the functor $G'$ identifies $\mathcal{D}$ with the category whose objects are pairs $(C, \alpha)$ where $C$ is an object of $\mathcal{C}$ and $\alpha : TC \to C$ exhibits $C$ as an algebra over $T$. We can now ask if the adjunction

$$\mathrm{hSp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathrm{h}\mathcal{S}_*^{v_n}$$

has this property.

**Warning 32.** Up to this point, all of the results in this talk have been formulated at the level of homotopy categories. However, monadicity is something that essentially *never* holds at the level of homotopy categories. For example, take $\mathcal{C}$ to be the homotopy category of differential graded vector spaces over $\mathbf{Q}$, and $\mathcal{D}$ to be the homotopy category of differential graded algebras over $\mathbf{Q}$. There is a pair of adjoint functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D},$$

where $G$ is the forgetful functor and $F$ is its left adjoint (which carries a differential graded vector space $V_*$ to the tensor algebra $\bigoplus_{n \geq 0} V_*^{\otimes n}$), which determine a monad $T = G \circ F$ on the category $\mathcal{C}$. However, the category $\mathrm{LMod}_T(\mathcal{C})$ of algebras over $T$ is equivalent to the category of associative algebra objects of $\mathcal{C}$ (that is, chain complexes equipped with a homotopy associative multiplication). In general, such an object cannot be rectified to a differential graded algebra.

To address the point raised in Warning 32, we observe that all of the categories of interest in the preceding discussion can be realized as the homotopy categories of $\infty$-categories, and that all of the functors of interest are also defined at the level of $\infty$-categories. We will indicate these $\infty$-categories by omitting the "h" from our notation. For example, we have an adjunction of $\infty$-categories

$$\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathcal{S}_*^{v_n}.$$

We then have the following result:

**Theorem 33** (Eldred-Heuts-Mathew-Meier)**.** *The adjunction of $\infty$-categories $\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathcal{S}_*^{v_n}$ is monadic. That is, the Bousfield-Kuhn functor $\Theta$ identifies the*

*$\infty$-category $\mathcal{S}_*^{v_n}$ of $v_n$-periodic spaces with the $\infty$-category of $T(n)$-local spectra $E$ equipped with an action of the monad $\Phi \circ \Theta$.*

In order to deduce Theorem 18 from Theorem 33, we need to identify the monad $\Phi \circ \Theta$ on $\mathrm{Sp}_{T(n)}$. Note that the $\infty$-category $\mathrm{Sp}_{T(n)}$ admits a symmetric monoidal structure, with tensor product given by the localized smash product $(X, Y) \mapsto L_{T(n)}(X \wedge Y)$.

**Construction 34.** A *symmetric sequence of $T(n)$-local spectra* is a collection $\{E(k)\}_{k \geq 0}$, where each $E(k)$ is a $T(n)$-local spectrum equipped with an action of the symmetric group $\Sigma_k$ (here we mean an action of $\Sigma_k$ in the "naive" sense, rather than the sense of equivariant stable homotopy theory). We let SSeq denote the $\infty$-category of symmetric sequences in $\mathrm{Sp}_{T(n)}$ (that is, the $\infty$-category of functors from the category of finite sets and bijections to the $\infty$-category $\mathrm{Sp}_{T(n)}$).

If $\vec{E} = \{E(k)\}_{k \geq 0}$ is a symmetric sequence of $T(n)$-local spectra, we define a functor $P_{\vec{E}} : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ by the formula

$$P_{\vec{E}}(X) = L_{T(n)}(\bigvee_{k \geq 0} (E(k) \wedge X^{\wedge k})_{h\Sigma k}).$$

We will need the following result:

**Theorem 35** (Kuhn, Mathew)**.** *The construction $\vec{E} \mapsto P_{\vec{E}}$ determines a fully faithful embedding of $\infty$-categories* $\mathrm{SSeq} \to \mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})$.

Theorem 35 articulates a very special feature of $T(n)$-local homotopy: if we were to replace $\mathrm{Sp}_{T(n)}$ by the $\infty$-category Sp of all spectra, then the analogous result would fail. The essential point is that homogeneous functors from the $\infty$-category $\mathrm{Sp}_{T(n)}$ to itself are also cohomogeneous, so that homogeneous functors of different degrees cannot interact (this follows from the $T(n)$-local vanishing of Tate cohomology, which we will discuss later in the seminar).

We will say that a functor $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is *coanalytic* if it belongs to the essential image of the embedding of Theorem 35. (Roughly speaking, one can think of these as functors which admit a "power series" expansion.) It is not difficult to see that the collection of coanalytic functors is stable under composition, so that the $\infty$-category SSeq inherits a monoidal structure from the composition product on $\mathrm{Fun}(\mathrm{Sp}_{T(n)}, \mathrm{Sp}_{T(n)})$. Concretely, this is the usual "composition product" on symmetric sequences (which can be defined without working in the $T(n)$-local setting), whose algebras are given by *operads* in the symmetric monoidal $\infty$-category $\mathrm{Sp}_{T(n)}$.

**Example 36.** Let $\mathcal{O}_{\mathrm{Comm}} = \{\mathcal{O}_{\mathrm{Comm}}(k)\}_{k \geq 0}$ be the symmetric sequence where each $\mathcal{O}_{\mathrm{Comm}}(k)$ is the $T(n)$-local sphere $L_{T(n)}S$ (equipped with the trivial action of $\Sigma_k$). Then $\mathcal{O}_{\mathrm{Comm}}$ is an algebra with respect to the composition product on symmetric sequences, corresponding to the *commutative* operad. Under the embedding of Theorem 35, it corresponds to the monad on $\mathrm{Sp}_{T(n)}$ associated to the (monadic) adjunction

$$\mathrm{Sp}_{T(n)} \underset{\text{Forget}}{\overset{\text{Free}}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Sp}_{T(n)},$$

where $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ is the $\infty$-category of $T(n)$-local $E_\infty$-ring spectra.

**Variant 37.** Let $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}} = \{\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}(k)\}_{k\geq 0}$ be the symmetric sequence with $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}(0) = 0$ and $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}(k) = L_{T(n)}S$ for $k > 0$. Then $\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}$ also has the structure of an operad: the *nonunital* commutative operad. It is associated to the (monadic) adjunction

$$\mathrm{Sp}_{T(n)} \underset{\text{Forget}}{\overset{\text{Free}}{\rightleftarrows}} \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Sp}_{T(n)},$$

where $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Sp}_{T(n)})$ is the $\infty$-category of nonunital $T(n)$-local $E_\infty$-ring spectra.

The main step in the proof of Theorem 18 will be to establish the following:

**Theorem 38** (Heuts)**.** *The endofunctor $\Phi \circ \Theta : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is coanalytic. In other words, the monad $\Phi \circ \Theta$ arises from an operad in $\mathrm{Sp}_{T(n)}$.*

**Remark 39.** The proof of Theorem 38 is not formal: it requires some highly nontrivial results of Arone-Mahowald concerning the $T(n)$-local convergence properties of the Goodwillie derivatives of the identity functor.

One we have Theorem 38 in hand, all that remains to be done is to identify the operad $\Phi \circ \Theta$ with the *Lie operad*. To make this statement precise (and to prove it), we first need to *define* the Lie operad in the setting of stable homotopy theory. We will adopt a definition which depends on the theory of *Koszul duality*, another topic we will meet in the course of this seminar:

**Definition 40.** The Lie operad $\mathcal{O}_{\mathrm{Lie}}$ is defined as the Koszul dual of the nonunital commutative operad of Variant 37.

In order to identify $\Phi \circ \Theta$ with the Lie operad $\vec{\mathrm{Lie}}$, we need to connection the $v_n$-periodic homotopy category $\mathrm{h}\mathcal{S}^{v_n}_*$ to the theory of (nonunital) commutative algebras. In the case $n = 0$, this relation comes from the Sullivan model of rational homotopy theory: that is, we can extract commutative algebras from spaces by passing to their cohomology. We now consider an analogue in the $T(n)$-local world:

**Proposition 41.** *There is a pair of adjoint functors*

$$\mathcal{S}^{v_n}_* \underset{\Omega^\infty_{T(n)}}{\overset{\Sigma^\infty_{T(n)}}{\rightleftarrows}} \mathrm{Sp}_{T(n)}$$

*which exhibits* $\mathrm{Sp}_{T(n)}$ *as the stabilization of the* $\infty$*-category* $\mathcal{S}_*^{v_n}$ *(that is, we can identify objects of* $\mathrm{Sp}_{T(n)}$ *with infinite loop spaces in* $\mathcal{S}_*^{v_n}$*).*

**Warning 42.** The compatibility of the functor $\Sigma^\infty_{T(n)}$ with the usual infinite suspension functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$ is a bit subtle; beware that the diagram

$$\begin{array}{ccc} \mathcal{S}_* & \xrightarrow{\Sigma^\infty} & \mathrm{Sp} \\ \downarrow & & \downarrow L_{T(n)} \\ \mathcal{S}_*^{v_n} & \xrightarrow{\Sigma^\infty_{T(n)}} & \mathrm{Sp}_{T(n)} \end{array}$$

does not commute in general (essentially, the problem is that a map of spaces $f : X \to Y$ which induces a homotopy equivalence $\Phi(f) : \Phi(X) \to \Phi(Y)$ need not induce an equivalence $L_{T(n)}\Sigma^\infty X \to L_{T(n)}\Sigma^\infty Y$; see Warning 28). However, this diagram *does* commute if we restrict our attention to highly connected spaces. We will return to this point later.

Let $\mathbf{D} : \mathrm{Sp}^{\mathrm{op}}_{T(n)} \to \mathrm{Sp}_{T(n)}$ denote the $T(n)$-local Spanier-Whitehead duality functor (given by mapping into the $T(n)$-local sphere $L_{T(n)}S$). The composite functor

$$(\mathcal{S}_*^{v_n})^{\mathrm{op}} \xrightarrow{\Sigma^\infty_{T(n)}} \mathrm{Sp}^{\mathrm{op}}_{T(n)} \xrightarrow{\mathbf{D}} \mathrm{Sp}_{T(n)},$$

which we can think of as an analogue of the reduced cochain functor $X \mapsto C^*_{\mathrm{red}}(X; \mathbf{Q})$. This functor admits a left adjoint, given by the composition

$$\mathrm{Sp}_{T(n)} \xrightarrow{\mathbf{D}} \mathrm{Sp}^{\mathrm{op}}_{T(n)} \xrightarrow{\Omega^\infty_{T(n)}} (\mathcal{S}_*^{v_n})^{\mathrm{op}}.$$

It follows that the composition

$$\mathfrak{D} \circ \Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)} \circ \mathfrak{D} : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$$

can be regarded as a monad.

Beware that the adjunction

$$\mathrm{Sp}_{T(n)} \underset{\Omega^\infty_{T(n)} \circ \mathbf{D}}{\overset{\mathfrak{D} \circ \Sigma^\infty_{T(n)}}{\rightleftarrows}} (\mathcal{S}_*^{v_n})^{\mathrm{op}}$$

is not monadic, and the monad $\mathfrak{D} \circ \Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)} \circ \mathfrak{D}$ is not coanalytic (this is why the "Sullivan model" does not behave as well as the "Quillen model"). However, it is related to a monadic adjunction: the functor $\mathbf{D} \circ \Sigma^\infty_{T(n)}$ factors as a composition

$$(\mathcal{S}_*^{v_n})^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Sp}_{T(n)}) \xrightarrow{\mathrm{Forget}} \mathrm{Sp}_{T(n)}.$$

This factorization induces a map of monads $P_{\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}} \to \mathfrak{D} \circ \Sigma^\infty_{T(n)} \circ \Omega^\infty_{T(n)} \circ \mathfrak{D}$. We will prove the following:

**Theorem 43.** *The preceding map exhibits $P_{\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}}$ as the universal coanalytic approximation to the monad $\mathfrak{D} \circ \Sigma^{\infty}_{T(n)} \circ \Omega^{\infty}_{T(n)} \circ \mathfrak{D}$. That is, if $F : \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is a coanalytic functor (or monad), then any natural transformation $F \to \mathfrak{D} \circ \Sigma^{\infty}_{T(n)} \circ \Omega^{\infty}_{T(n)} \circ \mathfrak{D}$ factors uniquely through $P_{\mathcal{O}^{\mathrm{nu}}_{\mathrm{Comm}}}$.*

To deduce Theorem 18 from Theorem 43, we need to establish some relationship between the adjunctions

$$\mathrm{Sp}_{T(n)} \underset{\Omega^{\infty}_{T(n)} \circ \mathbf{D}}{\overset{\mathfrak{D} \circ \Sigma^{\infty}_{T(n)}}{\rightleftarrows}} (\mathcal{S}^{v_n}_{*})^{\mathrm{op}}$$

$$\mathrm{Sp}_{T(n)} \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathcal{S}^{v_n}_{*} \, ..$$

This is actually quite formal: a defining property of the Bousfield-Kuhn functor is that the composition

$$\mathrm{Sp} \xrightarrow{\Omega^{\infty}} \mathcal{S}_{*} \xrightarrow{\Phi} \mathrm{Sp}_{T(n)}$$

is equivalent to the localization functor $L_{T(n)}$. From this, one can show that the composition

$$\mathrm{Sp}_{T(n)} \xrightarrow{\Omega^{\infty}_{T(n)}} \mathcal{S}^{v_n}_{*} \xrightarrow{\Phi} \mathrm{Sp}_{T(n)}$$

is the identity functor. We will combine this with the theory of Koszul duality to obtain the following:

**Theorem 44.** *The Koszul dual of the operad $\Phi \circ \Theta$ is a universal coanalytic approximation to the monad $\mathfrak{D} \circ \Sigma^{\infty}_{T(n)} \circ \Omega^{\infty}_{T(n)} \circ \mathfrak{D}$. In other words (by virtue of Theorem 43 and Definition 40), $\Phi \circ \Theta$ is the Lie operad.*

Theorem 18 will then follow by combining Theorem 44 with Theorem 33.