Conjugacy Classes and Geodesic Loops (Lecture 35)

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Let $X$ be a path connected topological space and let $f : S^1 \to X$ be a map. Then $f$ determines a conjugacy class $[f]$ in the fundamental group $\pi_1 X$. Our goal in this lecture is to show any nonzero conjugacy class is represented by an essentially canonical map $f$ in the case where $X$ is a hyperbolic surface.

Lemma 1. Assume that $X$ is a compact Riemannian manifold. Then any conjugacy class $\gamma \in \pi_1 X$ can be represented by a closed geodesic $f : S^1 \to X$. 

Proof. Endow the circle $S^1$ with its standard Riemannian metric, normalized so that the circle has total length 1. Define the Lipschitz constant $L(f)$ of a loop $f$ to be the supremum of

$$\frac{d(f(x), f(y))}{d(x,y)}.$$

This supremum may be infinite: however, for a smooth path $f$ it is finite (and coincides with maximum length of the derivative $f'$ on $S^1$). Let $c$ be the infimum of the set $\{L(f)\}$, where $f$ varies over all representatives of $\gamma$. We will show that this infimum is achieved: that is, there exists a loop $f$ with $L(f) = c$. Then $f$ must be a smooth geodesic (of speed $c$) if it fails to be a geodesic near some point $t$, we can obtain a shorter loop representing $\gamma$ by modifying $f$ near $t$ (and then changing our parametrization).

To prove that $c$ is achieved, choose a sequence of loops $\{f_t\}_{t \geq 0}$ such that the real numbers $L(f_t)$ converge to $c$ from above. Passing to a subsequence, we may assume that $L(f_t) < c + 1$. Choose a countable dense subset $\{t_j\} \subseteq S^1$. Since $X$ is compact, we can pass to a subsequence and thereby assume that $f_0(t_0), f_1(t_0), \ldots$ converges to some point $x_0 \in X$. Similarly, we can pass to a subsequence of $\{f_1, f_2, \ldots\}$ and thereby guarantee that the sequence $f_1(t_1), f_2(t_1), \ldots$ converges to a point $x_1 \in X$. Proceeding in this way, we obtain a refinement of the original sequence such that $\{f_i(t_j)\}_{t \geq 0}$ converges to some $x_j \in X$. We define a new map $f : \{t_j\} \to X$ by the formula $f(t_j) = x_j$. We claim that $f$ extends to a continuous map $S^1 \to X$ having $L(f) \leq c$. To prove this, it suffices to show that

$$d(f(t_i), f(t_j)) \leq c d(t_i, t_j)$$

for each pair of integers $i \neq j$. This is clear:

$$d(f(t_i), f(t_j)) \leq d(f(t_i), f_n(t_i)) + d(f(t_j), f_n(t_j)) + d(f_n(t_i), f_n(t_j)) \leq \epsilon + L(f_n) d(t_i, t_j)$$

where $\epsilon$ can be made arbitrarily small (by choosing $n$ large enough) and $L(f_n)$ can be made arbitrarily close to $c$.

Choose $\epsilon > 0$ small enough that every pair of points of $X$ within a distance $\epsilon$ are connected by a unique geodesic. For $n \gg 0$, we have $d(f(t), f_n(t)) < \epsilon$ for all $t$, so that $f$ and $f_n$ can be connected by a geodesic homotopy; it follows that $f$ is homotopic to $f_n$ and therefore represents the free homotopy class $\gamma$. \qed

Let us now suppose that $X$ is a hyperbolic surface, so that $X$ can be represented as $H/\Gamma$ where $H$ is the upper half place $\{x + iy : y > 0\}$ and $\Gamma$ is a group which acts on $H$ by hyperbolic isometries. Then
\( \Gamma \simeq \pi_1 X \), and we can identify \( \Gamma \) with a subgroup of the group \( \text{PSL}_2(\mathbb{R}) \) of linear fractional transformations of the form
\[
    z \mapsto \frac{az + b}{cz + d}.
\]

It is traditional to decompose elements of \( \text{PSL}_2(\mathbb{R}) \) into three types:

(i) An element \( A \in \text{SL}_2(\mathbb{R}) \) is called elliptic if \( |\text{tr}(A)| < 2 \). In this case, the eigenvalues of \( A \) are unit complex numbers (and complex conjugate to one another); the transformation \( A \) itself is given by \( z \mapsto \frac{\cos(\theta)z - \sin(\theta)}{\sin(\theta)z + \cos(\theta)} \) for some real number \( \theta \). Elliptic elements never appear in the discrete groups \( \Gamma \) under consideration, because they always have fixed points in the upper half plane (the above transformation has the complex number \( z = i \) as a fixed point).

(ii) An element \( A \in \text{SL}_2(\mathbb{R}) \) is called parabolic if \( |\text{tr}(A)| = 2 \); in this case, the eigenvalues of \( A \) are both \( \pm 1 \) but \( A \) is generally not semisimple: it is conjugate to a transformation of the form \( z \mapsto z + t \) for some real number \( t \). Nontrivial transformations of this kind cannot appear in \( \Gamma \) when the quotient \( X = H/\Gamma \) is compact. For suppose otherwise: then, by Lemma 1, we would have a geodesic loop \( f : S^1 \to X \) representing the conjugacy class of a parabolic transformation \( z \mapsto z + t \). Then \( f \) lifts to a geodesic path \( \tilde{f} \) with the translation-invariance property \( \tilde{f}(z+1) = \tilde{f}(t) \). There is no geodesic in the upper half plane with this property: the unique geodesic passing through \( \tilde{f}(0) \) and \( \tilde{f}(0) + t \) does not pass through \( \tilde{f}(0) + 2t \).

This argument does not apply if the quotient \( H/\Gamma \) is noncompact. In fact, a finite volume quotient \( H/\Gamma \) is compact if and only if \( \Gamma \) contains no parabolic elements: in fact, there is a bijection between cusps of \( H/\Gamma \) and conjugacy classes of maximal parabolic subgroups of \( \Gamma \).

(iii) An element \( A \in \text{SL}_2(\mathbb{R}) \) is called hyperbolic if \( |\text{tr}(A)| > 2 \) (modifying \( A \) by a sign, we may assume that \( \text{tr}(A) > 2 \)). In this case, \( A \) has distinct real eigenvalues \( \lambda, \frac{1}{\lambda} \) for some \( \lambda > 1 \). Then \( A \) is conjugate to the transformation \( z \mapsto \lambda z \). In this case, there is a unique geodesic path \( \tilde{f} : \mathbb{R} \to H \) satisfying \( \tilde{f}(t+1) = A\tilde{f}(t) \); namely, the path given by the formula \( \tilde{f}(t) = \lambda^t i \). This path descends to a geodesic loop \( f : S^1 \to H/\Gamma \) representing the conjugacy class of \( \pm A \) in \( \Gamma \simeq \pi_1 H/\Gamma \).

The above analysis proves the following result:

**Theorem 2.** Let \( X = H/\Gamma \) be a compact hyperbolic surface. Then every nontrivial element \( \gamma \) of \( \pi_1 X \simeq \Gamma \subseteq \text{PSL}_2(\mathbb{R}) \) is hyperbolic. Moreover, the conjugacy class of \( \gamma \) can be represented by a geodesic loop \( f : S^1 \to X \) which is unique up to reparametrization.

In other words, if \( X \) is a hyperbolic surface, then every conjugacy class in \( \pi_1 X \) has a canonical representative. We now show that these representatives are well-behaved:

**Theorem 3.** Let \( X \) be a hyperbolic surface, and suppose we are given distinct nontrivial conjugacy classes \( \gamma_1, \ldots, \gamma_n \in \pi_1 X \). The following conditions are equivalent:

1. The conjugacy classes \( \gamma_i \) can be represented by simple closed curves \( C_i \subseteq X \) such that \( C_i \cap C_j = \emptyset \) for \( i \neq j \).

2. The canonical geodesic representatives for \( \gamma_1, \ldots, \gamma_n \) are simple closed curves \( C_i \subseteq X \) such that \( C_i \cap C_j = \emptyset \) for \( i \neq j \).

**Proof:** It is clear that (2) \( \Rightarrow \) (1). Suppose that (1) is satisfied. Let \( \{ f_i : S^1 \to X \}_{1 \leq i \leq n} \) be a parametrizations of the curves \( C_i \) which satisfy condition of (1), and let \( \{ g_i : S^1 \to X \}_{1 \leq i \leq n} \) be the geodesic representatives of the conjugacy classes \( \gamma_i \). We wish to prove that each \( g_i \) is a simple curve, and that \( g_i(S^1) \cap g_j(S^1) = \emptyset \) for \( i \neq j \). We will prove the latter; the former follows by the same argument.

Choose a lifting of \( g_i \) to a geodesic path \( \tilde{g}_i : \mathbb{R} \to D \), where \( D \) is the unit disk. If \( g_i(S^1) \cap g_j(S^1) \neq \emptyset \), then we can lift \( g_j \) to a geodesic path \( \tilde{g}_j : \mathbb{R} \to D \) such that \( \tilde{g}_i(\mathbb{R}) \) and \( \tilde{g}_j(\mathbb{R}) \) intersect. Let \( a, b \in \partial D \) be
the endpoints of $\tilde{g}_i$ on the circle at infinity, and let $a', b'$ be the endpoints of $\tilde{g}_j$. Note that $\tilde{g}_i(\mathbb{R})$ and $\tilde{g}_j(\mathbb{R})$ intersect if and only if the sets $\{a, b\}$ and $\{a', b'\}$ are disjoint, and the points $a'$ and $b'$ belong to different components of $\partial D - \{a, b\}$.

Since $f_i$ and $g_i$ represent the same conjugacy class in $\pi_1 X$, there is a homotopy $h$ from $f_i$ to $g_i$. Lifting this homotopy to the universal cover, we get a lift $\tilde{f}_i : \mathbb{R} \to D$ of $f_i$ and a homotopy from $\tilde{f}_i$ to $\tilde{g}_i$. This homotopy moves points by a bounded amount with respect to the hyperbolic metric on $D$. Consequently, it moves points which are close to the boundary $\partial D$ by very small amounts with respect to the Euclidean metric on the closure of $D$. It follows that $\tilde{f}_i$ has the same endpoints $a$ and $b$ as $\tilde{g}_i$.

A similar argument shows that we can lift $f_j$ to a path $\tilde{f}_j : \mathbb{R} \to D$ having endpoints $a', b' \in \partial D$. If $a'$ and $b'$ belong to different components of $\partial D - \{a, b\}$, then $\tilde{f}_i(\mathbb{R})$ and $\tilde{f}_j(\mathbb{R})$ must have a point of intersection $\tilde{x} \in D$. The image of $\tilde{x}$ is a point $x \in f_i(S^1) \cap f_j(S^1) \subseteq X$, contradicting our assumptions. 

\[\square\]