The Sphere Theorem: Part 2 (Lecture 31)

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In this lecture, we will complete the proof of the sphere theorem.

Let us recall the situation. We are given an oriented, connected 3-manifold \( M \) and a \( \pi_1 M \)-invariant proper subgroup \( N \subset \pi_2 M \). Our goal is to prove that there exists an embedded 2-sphere \( S \subseteq M \) whose homotopy class does not belong to \( N \).

Since \( N \neq \pi_2 M \), there exists a map \( f : S^2 \to M \) whose homotopy class does not belong to \( N \). We may assume that \( f \) is in general position and (as we saw in the last lecture) an immersion. We will suppose that \( f \) has been chosen so as to minimize the number \( t(f) \) of triple points of \( f \).

In the last lecture, we argued as follows:

(1) If the map \( f \) has a simple double curve, then we can modify \( f \) so as to obtain a new map \( f' \) (whose homotopy class again does not belong to \( N \)) which either has fewer triple points \( (t(f') < t(f)) \) or the same number of triple points and fewer double curves. Since \( t(f) \) is minimal, \( f' \) must have fewer double curves. Applying this procedure repeatedly, we can reduce to the case where \( f \) does not have any double curves.

(2) There exists a 3-manifold with boundary \( \tilde{M} \) (namely, the 3-manifold \( V_n \) at the top of the tower that we constructed in the last lecture) and an immersion \( q : \tilde{M} \to M \) with the following properties:

   (i) The map \( f \) lifts to a map \( \tilde{f} : S^2 \to \tilde{M} \).

   (ii) The 3-manifold \( \tilde{M} \) is a regular neighborhood of \( \tilde{f}(S^2) \).

   (iii) The fundamental group \( \pi_1 \tilde{M} \) is finite. As we saw last time, this guarantees that the universal cover of \( \tilde{M} \) is a punctured sphere, so that \( \pi_2 \tilde{M} \) is generated (as a \( \pi_1 \tilde{M} \)-module) by its boundary components.

   (iv) The map \( \tilde{f} \) is not an embedding (otherwise we were able to produce a simple double curve of \( f \).

Let \( \Sigma(\tilde{f}) \) denote the singular locus of the map \( \tilde{f} \). Condition (iv) guarantees that \( \Sigma(\tilde{f}) \) is nonempty. Let \( X \) be a small neighborhood of \( \Sigma(\tilde{f}) \) in \( \tilde{M} \). Since \( f \) is in general position, no point of \( M \) has more than 3 preimages under \( f \). It follows that \( q \) must be injective on \( \Sigma(f) \). Shrinking \( X \), we may assume that \( q \) is injective on \( X \). Let \( T \) denote the closure of \( \tilde{f}(S^2) - X \).

Let \( x \in \Sigma(\tilde{f}) \). Since \( f \) is a general position map, \( q(x) \) has at most 3 preimages under \( f \). At least two of these are preimages of \( x \) under \( \tilde{f} \). There are two possibilities:

(a) The inverse image \( f^{-1}(q(x)) = \tilde{f}^{-1}(x) \). Then \( q(x) \) does not intersect \( q(T) \), so we can choose a neighborhood \( V_x \) of \( x \) such that \( q(V_x) \cap q(T) = \emptyset \).

(b) The inverse image \( f^{-1}(q(x)) \) consists of \( \tilde{f}^{-1}(x) \) together with one additional point \( s \in S^2 \). Let \( y = \tilde{f}(s) \). Since \( q \) is injective on \( X \), we must have \( y \notin X \), so that \( y \in T \). Since \( q \) is an immersion, there exists a neighborhood \( U \) of \( y \) in \( T \) on which \( q \) is injective. Then \( q(x) \) does not intersect \( q(T - U) \), so there is a neighborhood of \( V_x \) of \( x \) such that \( q(V_x) \cap q(T) \subseteq q(U) \).
Let $X_0$ be a regular neighborhood of $\Sigma(f)$ which is contained in the open set $\bigcup V_x$. By construction, if $x \in X_0$ then there is at most one element $y \in \tilde{f}(S^2)$ such that $x \neq y$ but $q(x) = q(y)$.

Let $X_1 \subset X_0$ be a slightly smaller regular neighborhood of $\Sigma(f)$. The map $\tilde{f}$ is an embedding outside of $X_1$; let $S_1, \ldots, S_m$ be the connected components of its image. Then $f(S^2)$ has a regular neighborhood of the form $X_1 \cup (S_1 \times [-1,1]) \cup \ldots \cup (S_m \times [-1,1])$. Shrinking $\tilde{M}$ if necessary, we may assume that it coincides with this regular neighborhood.

Let $\tilde{N}$ denote the inverse image of $N$ in $\pi_2\tilde{M}$. Since $\tilde{N}$ does not contain the homotopy class of $\tilde{f}$, it is a proper subgroup $\pi_1\tilde{M}$-invariant subgroup of $\pi_2\tilde{M}$. Using (iii), we deduce that $\tilde{N}$ does not contain the homotopy class of some boundary component $S$ of $\tilde{M}$. Let $f': S^2 \to \tilde{M}$ be the inclusion of this boundary component. Then the image of $f'$ is contained in

$$X_1 \cup (S_1 \times \{-1,1\}) \cup \ldots \cup (S_m \times \{-1,1\}).$$

Claim 1. For each index $1 \leq i \leq m$, the image of $f'$ cannot intersect both $S_i \times \{-1\}$ and $S_i \times \{1\}$.

Proof. Otherwise, there exists a simple arc $\alpha$ on $f'(S^2)$ joining points $(x, -1)$ and $(y, 1)$, where $x, y \in S_i$. Choose a path joining $y$ to $x$ in $S_i$, which determines a path $\beta$ from $(y, 1)$ to $(x, -1)$ in $S_i \times [-1,1]$. The composition $\alpha \circ \beta$ is a simple loop which meets $\tilde{f}(S^2)$ transversely at exactly one point (belonging to $S_i$). It follows that $\alpha \circ \beta$ represents a nontorsion homology class in $H_1(\tilde{M}, \mathbb{Z})$, which contradicts our assumption that $\pi_1\tilde{M}$ is finite.

Using Claim 1, we can modify the map $f'$ by an isotopy to obtain an embedding $f'' : S^2 \to \tilde{M}$ whose image is contained in $X_0 \cup S_1 \cup S_2 \cup \ldots \cup S_m$. By construction, the homotopy class of $f''$ does not belong to $\tilde{N}$, so the homotopy class of $q \circ f''$ does not belong to $N$. We will obtain a contradiction by showing that $t(q \circ f'')$ has fewer triple points than $f''$.

Let $x \in M$ be a triple point for $q \circ f''$. Since $f''$ is an embedding, we must have three distinct points $x_1, x_2, x_3 \in f''(S^2)$ such that $q(x_1) = q(x_2) = q(x_3) = x$. Note that $f''(S^2) \subseteq T \cup X_0$. Since $q$ is injective on $X_0$, at most one element of $\{x_1, x_2, x_3\}$ belongs to $X_0$. However, if $x_i \in X_0$, then there is at most one element $y \in T$ distinct from $x_i$ such that $q(y) = q(x_i)$. It follows that none of the elements $x_1, x_2$, and $x_3$ belong to $X_0$. Thus $x_1, x_2, x_3 \in T \subseteq \tilde{f}(S^2)$, so that $x$ is also a triple point of $f$. This proves that $t(q \circ f'') \leq t(f)$. To prove that the equality is strict, it suffices to show that $f$ has at least one triple point $x$ such that $q^{-1}(x) \neq \emptyset$. For this, it suffices to show that the map $\tilde{f}$ has a triple point. Assume otherwise. Then the singular locus $\Sigma(\tilde{f})$ is a 1-dimensional submanifold of $M$. This singular locus is nonempty (by (iv)), and therefore contains a circle $C$. This circle is a simple double curve of $\tilde{f}$, so that $q(C)$ is a simple double curve of $f$, which contradicts (1).