Irreducibility and $\pi_2$ (Lecture 27)

April 15, 2009

In the last lecture, we introduced the notion of an irreducible 3-manifold: a 3-manifold $M$ is said to be irreducible if every embedded 2-sphere in $M$ bounds a disk (on exactly one side). Our stated motivation was that embedded 2-spheres were good candidates to represent nontrivial classes in $\pi_2M$. Our first goal in this lecture is to show that this is indeed the case.

**Proposition 1.** Let $M$ be a 3-manifold, and let $S \hookrightarrow M$ be an embedded 2-sphere. The following conditions are equivalent:

1. The sphere $S$ bounds a disk in $M$.
2. The sphere $S$ represents a trivial class in $\pi_2M$.

**Remark 2.** The statement of Proposition 1 is a little sloppy: the homotopy group $\pi_2M$ is really only well-defined after we have chosen a base point on $M$. If $M$ is connected, then the groups $\pi_2(X, x)$ and $\pi_2(X, y)$ can be related by choosing a path from $x$ to $y$, but the identification depends on this choice of path via the action of $\pi_1M$ on $\pi_2M$. This means that the class of $S$ in $\pi_2M$ is only well-defined up to the action of $\pi_1M$; however, the condition that this class vanishes is invariant under the action of $\pi_1M$ (the vanishing is equivalent to the requirement that $S \hookrightarrow M$ is homotopic to a constant map, ignoring the base points).

**Proof.** (In what follows, we do not assume that $M$ is compact.) It is clear that if $S$ bounds a disk, then $S$ is nullhomotopic. Conversely, suppose that $S$ is nullhomotopic. Suppose first that $M$ is simply connected. Since $[S] = 0 \in H_2(M; \mathbb{Z}/2\mathbb{Z})$, the 2-sphere $S$ is separating (though the converse can fail in the noncompact setting); we can therefore write $M = M_0 \coprod_{S^2} M_1$ where $M_0$ and $M_1$ are 3-manifolds with 2-sphere boundary. We have an exact sequence

$$H_2(S) \xrightarrow{j} H_2(M_0) \oplus H_2(M_1) \rightarrow H_2(M)$$

(all homology computed with $\mathbb{Z}/2\mathbb{Z}$ coefficients). Since $[S]$ vanishes in $H_2(M)$, we deduce that the class $([S], 0)$ lies in the image of $j$: in other words, either $([S], 0)$ or $(0, [S])$ vanishes. Assume the former, and let $\tilde{M}_0$ be the 3-manifold obtained from $M_0$ by capping off the boundary sphere. We have an exact sequence

$$H_3(\tilde{M}_0) \rightarrow H_2(S^2) \xrightarrow{i} H_2(D^3) \oplus H_2(M_0).$$

Since the map $i$ is not injective, we deduce that $H_3(\tilde{M}_0)$ is nonzero. By Poincare duality (the simple connectivity of $\tilde{M}_0$ guarantees orientability), we deduce that $H_3(\tilde{M}_0)$ does not vanish, so that $\tilde{M}_0$ is a compact, simply connected 3-manifold. By the Poincare conjecture, $\tilde{M}_0$ is a 3-sphere, so that $M_0$ is a disk bounded by $S$.

Suppose now that $M$ is not simply connected; we still have $M = M_0 \coprod_{S^2} M_1$ as above. Let $\tilde{M}$ be a universal cover of $M$, and $\pi: \tilde{M} \rightarrow M$ the projection map. Since $S$ is simply connected, we can lift $S$ to a 2-sphere $\tilde{S}$ in $\tilde{M}$. Since $\pi_2M \simeq \pi_2\tilde{M}$, the sphere $\tilde{S}$ is nullhomotopic and therefore bounds a disk. This disk might contain other preimages of $S$: however, by adjusting our choice of $\tilde{S}$ we can arrange that $\tilde{S}$ contains a disk $D$ which intersects the inverse image of $\pi^{-1}S$ only in $\tilde{S}$. It follows that $\pi(D) \subseteq M_0$ or $\pi_D \subseteq M_1$;
Remark 5. The hypothesis of orientability in the sphere theorem is essential. If \( P \) denotes the 2-dimensional real projective space, then \( P \times S^1 \) is a nonorientable 3-manifold with \( \pi_2(P \times S^1) \cong \mathbb{Z} \), yet \( P \times S^1 \) does not contain any nontrivial embedded 2-spheres. (It contains many immersed 2-spheres, given by the double covering \( S^2 \to P \)).

We now begin to pave the way for our proofs of the loop and sphere theorems by discussing the notion of a general position map from a surface \( S \) into a 3-manifold \( M \). We will treat this notion informally and not give a precise definition: roughly speaking, a map \( i : S \to M \) is in general position if the behavior of \( i \) satisfies all of the conditions we like that can be guaranteed by moving the map \( i \) by a small amount. In particular, any “singularities” of the map \( i \) can be assumed to appear in the expected codimension, which means they do not appear at all if the expected codimension is \( \geq 3 \) (in \( S \)) or \( \geq 4 \) (in \( M \)).

Assume therefore that we are given a smooth map \( i : S \to M \). How can this map fail to be an embedding? There are essentially two things that can go wrong:

(i) The map \( i \) can fail to be an immersion at a point \( s \in S \). In other words, the derivative \( Di \) can fail to have rank 2 at \( s \). The derivative \( Di_s \) takes values in the 6-dimensional space of linear maps \( T_{S,s} \to T_{M,i(s)} \). A linear map of rank 1 is determined by specifying a 1-dimensional quotient \( Q \) of \( T_{S,s} \) (the set of such choices forms a 1-dimensional space), a 1-dimensional subspace \( Q' \) of \( T_{M,i(s)} \) (where we have a 2-dimensional space of choices), and a linear isomorphism \( Q \cong Q' \) (for which we have 1-dimensional space of choices); in total, we find that the space of maps having rank 1 is a manifold of dimension \( 1+2+1=4 \). Including the zero map does not increase the dimension: we conclude that \( Di_s \) should be expected to have rank \( \leq 2 \) in on a subset of \( S \) having codimension 2. Since \( S \) is a surface, the map \( i \) should fail to be an immersion at a discrete set of points of \( S \). The images of these points in \( M \) are called branch points of the map \( i \).

(ii) The map \( i \) can fail to be injective, so that \( i(x) = i(y) \) for \( x \neq y \). Since \( i(x) \) and \( i(y) \) take values in the 3-manifold \( M \), we should expect the relation \( i(x) = i(y) \) to hold with codimension 3 among \( (x,y) \in S^2 \). We will say that \( x \in M \) is a double point of \( i \) if \( i^{-1}(x) \) has cardinality 2. If \( i \) is in general position, then we expect the set of double points to be a smooth submanifold of codimension 1 in \( M \). We can also
arrange that this set does not intersect the set of branch points (although, as we will see in a moment, every branch point lies in the closure of the set of double points).

(iii) The map $i$ can fail to be injective more drastically: we can have $i(x_1) = i(x_2) = \ldots = i(x_n)$. This behavior is to be expected in codimension $3(n - 1)$ in the space $S^n$ of dimension $2n$. If $n > 3$, then $3(n - 1) > 2n$ so that a generic map $i$ will have not exhibit this behavior. If $n = 3$, then we expect this to happen for a discrete subset of $S^3$: in other words, we expect an isolated set of points $x \in M$ where $i^{-1}\{x\}$ has cardinality 3. We will call such points triple points of the map $i$.

What does the map $i$ look like near a branch point? If we work in the piecewise linear category, then the local structure of a PL map $i : D^2 \to D^3$ is given by taking the cone over some PL map $i_0 : S^1 \to S^2$. If $i_0$ is an embedding, then so is $i$, and we do not have any branching. We may therefore assume that $i_0$ fails to be an embedding and therefore has some double points. It follows that every branch point of $i$ lies at the endpoint of a curve of double points of $i$. (For a generic choice of $i$, the curve $i_0 : S^1 \to S^2$ will have only a single self-intersection so that this double curve is unique. However, we will not need to know this.)