Diffeomorphisms of the 2-Sphere (Lecture 24)

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The goal of this lecture is to compute the homotopy type of the diffeomorphism group of the 2-sphere S^2. The idea is to endow the 2-sphere with some additional structure (a conformal structure). We will show that this structure is essentially unique, and it will follow that the diffeomorphism group $\mathrm{Diff}(S^2)$ is homotopy equivalent to the group of automorphisms which respect this additional structure. The latter group is finite dimensional and easy to describe.

Definition 1. Let M be a smooth manifold. A *(Riemannian) metric* on M consists of a positive definite inner product on each tangent space $T_{M,x}$ which varies smoothly with the chosen point $x \in M$. We will denote the collection of Riemannian metrics on M by $\mathrm{Met}(M)$.

Given a metric g on M and a smooth function $\lambda : M \to \mathbb{R}_{>0}$, the product λg is another metric on M. We will say that two metrics g and g' are *conformally equivalent* if $g = \lambda g'$ for some smooth function $\lambda : M \to \mathbb{R}_{>0}$. The relation of conformal equivalence is an equivalence relation on $\mathrm{Met}(M)$; we will denote the set of equivalence classes by $\mathrm{Conf}(M)$.

There is a natural topology on $\mathrm{Met}(M)$ (we can identify $\mathrm{Met}(M)$ with an open subset of the Frechet space of all smooth sections of the bundle $\mathrm{Sym}^2 T_M^\vee$); we endow $\mathrm{Conf}(M)$ with the quotient topology.

Remark 2. The exact topologies that we place on $\mathrm{Met}(M)$ and $\mathrm{Conf}(M)$ are not really important in what follows: for our purposes it will be enough to work with the singular simplicial sets of $\mathrm{Met}(M)$ and $\mathrm{Conf}(M)$.

Lemma 3. *Let M be a smooth manifold. Then the spaces $\mathrm{Met}(M)$ and $\mathrm{Conf}(M)$ are contractible.*

Proof. The contractibility of $\mathrm{Met}(M)$ follows from the fact that it is a convex subset of a topological vector space. More concretely, choose a metric g_0 on M (such a metric can be constructed by choosing standard metrics on Euclidean charts and averaging them using a partition of unity). Then any other metric g on M can be joined to g_0 by a canonical path of metrics: we simply choose a straight line $g_t = (1-t)g_0 + tg$.

Let G denote the collection of smooth maps from M to $\mathbb{R}_{>0}$. We regard G as a group with respect to pointwise multiplication. The group G is contractible: again, it is a convex subset of the Frechet space of all smooth real-valued functions on M, so every function $f \in G$ is connected to the constant function 1 by a straight-line homotopy $f_t(x) = (1-t)f(x) + t$. The group G acts freely on $\mathrm{Met}(M)$ with quotient $\mathrm{Conf}(M)$. We therefore have a fibration sequence

$$G \to \mathrm{Met}(M) \to \mathrm{Conf}(M)$$

Since G and $\mathrm{Met}(M)$ are contractible (and the map $\mathrm{Met}(M) \to \mathrm{Conf}(M)$ is surjective), we conclude that $\mathrm{Conf}(M)$ is also contractible. $\square$

A conformal structure on an n-manifold M can be thought of as a reduction of the structure group of the tangent bundle of M from $GL_n(\mathbb{R})$ to $\mathbb{R}_{\geq 0} \times O(n)$. If M is an oriented 2-manifold endowed with a conformal structure, then its tangent bundle has structure group reduced to $\mathbb{R}_{\geq 0} \times SO(2)$. If we choose an identification $\mathbb{R}^2 = \mathbf{C}$ (endowing the latter with its standard notion of length), then we can identify $\mathbb{R}_{\geq 0} \times SO(2)$ with the group $\mathbf{C}^*$ of nonzero complex numbers, acting on $\mathbf{C}$ by conjugation. In other words, an orientation of M together with a conformal structure on M give us a reduction of the structure group of M from $GL_2(\mathbb{R})$ to $GL_1(\mathbf{C})$: that is, they give an almost complex structure on M.

Theorem 4 (Existence of Isothermal Coordinates)**.** *Let M be a 2-manifold equipped with an almost complex structure. Then M is a complex manifold: in other words, near each point $x \in M$ we can choose an open neighborhood U and an open embedding $U \hookrightarrow \mathbf{C}$ of almost complex structures.*

Remark 5. In the situation of Theorem 4, suppose that we think of the almost complex structure on M as being given by an orientation together with a conformal structure, where the latter is given by some metric g on M. The assertion of Theorem 4 is equivalent to the assertion that we can choose local coordinate systems on M in which g is *conformally flat*: that is, it has the form λg_0 where g_0 denotes the standard metric on $\mathbb{R}^2 \simeq \mathbf{C}$.

Remark 6. Theorem 4 is a consequence of the Newlander-Nirenberg theorem, which asserts that an almost complex structure on a manifold M is a complex structure if and only if a certain obstruction (called the Nijenhuis tensor) vanishes. When M has dimension 2, the vanishing of this tensor is automatic. However, Theorem 4 is much more elementary. Nevertheless, we will not give a proof here.

Now suppose that M is the 2-sphere S^2, which we regard as an oriented smooth manifold. Every choice of conformal structure $\eta \in \mathrm{Conf}(M)$ endows M with the structure of a complex manifold: that is, a Riemann surface.

Proposition 7. *Up to isomorphism, the 2-sphere S^2 admits a unique complex structure. That is, if X is a Riemann surface which is diffeomorphic to S^2, then X is biholomorphic to the Riemann sphere $\mathbf{C}\mathrm{P}^1$.*

Proof. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. Since X is compact, it has a well-defined holomorphic Euler characteristic

$$\chi(\mathcal{O}_X) = \dim \mathrm{H}^0(X, \mathcal{O}_X) - \dim \mathrm{H}^1(X, \mathcal{O}_X).$$

This Euler characteristic can be computed using the Riemann-Roch theorem: it is given by $1 - g = \frac{\chi(X)}{2} = 1$, since X has genus 0. The space $\mathrm{H}^0(X, \mathcal{O}_X)$ consists of globally defined holomorphic functions on X. By the maximum principle (and the fact that X is compact), every such function must be constant, so that $\mathrm{H}^0(X, \mathcal{O}_X) \simeq \mathbf{C}$. It follows from the Euler characteristic estimate that $\mathrm{H}^1(X, \mathcal{O}_X)$ vanishes.

Now choose a point $x \in X$, and consider the sheaf $\mathcal{O}_X(x)$ of functions on X which are holomorphic except possibly at the point x, and have a pole of order at most 1 at x. We have an exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(x) \to x_* \mathbf{C} \to 0$$

Since the cohomology group $\mathrm{H}^1(X, \mathcal{O}_X)$ vanishes, we get a short exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(X, \mathcal{O}_X(x)) \to \mathrm{H}^0(X, x_* \mathbf{C}) \simeq \mathrm{H}^0(\{x\}, \mathbf{C}) \simeq \mathbf{C} \to 0.$$

This proves that $\mathrm{H}^0(X, \mathcal{O}_X(x))$ is 2-dimensional. In particular, there exists a nonconstant meromorphic function f on X having at most a simple pole at x. Since f cannot be holomorphic (otherwise it would be constant), it must have a pole of exact order 1 at x.

We can regard f as a holomorphic map $X \to \mathbf{C}\mathrm{P}^1$ with $f(x) = \infty$. Since f has unique simple pole at x, this map has degree 1 and is therefore an isomorphism of X with $\mathbf{C}\mathrm{P}^1$. □

Proposition 7 implies that the group $\mathrm{Diff}(S^2)$ acts transitively on the collection $\mathrm{Conf}(S^2)$ of conformal structures on S^2. We have a fiber sequence

$$\mathrm{Diff}_{\mathrm{Conf}}(S^2) \to \mathrm{Diff}(S^2) \to \mathrm{Conf}(S^2),$$

where $\mathrm{Diff}_{\mathrm{Conf}}(S^2)$ denotes the subgroup of $\mathrm{Diff}(S^2)$ consisting of diffeomorphisms which preserve the standard conformal structure on $S^2 = \mathbf{C}\mathrm{P}^1$. Since $\mathrm{Conf}(S^2)$ is contractible, we conclude that the inclusion $\mathrm{Diff}_{\mathrm{Conf}}(S^2) \subseteq \mathrm{Diff}(S^2)$ is a homotopy equivalence.

The group $\mathrm{Diff}_{\mathrm{Conf}}(S^2)$ can be written as a union $\mathrm{Diff}^+_{\mathrm{Conf}}(S^2) \cup \mathrm{Diff}^-_{\mathrm{Conf}}(S^2)$, where $\mathrm{Diff}^+_{\mathrm{Conf}}(S^2)$ denotes the subgroup of orientation preserving conformal diffeomorphisms of S^2 (that is, holomorphic automorphisms of $\mathbf{C}\mathrm{P}^1$), while $\mathrm{Diff}^-_{\mathrm{Conf}}(S^2)$ consists of orientation reversing conformal diffeomorphisms (antiholomorphic automorphisms).

Theorem 8. *The inclusion* $O(3) \hookrightarrow \mathrm{Diff}(S^2)$ *is a homotopy equivalence.*

Proof. It will suffice to show that the inclusion $O(3) \hookrightarrow \mathrm{Diff}_{\mathrm{Conf}}(S^2)$ is a homotopy equivalence. For this, we will show that $SO(3) \hookrightarrow \mathrm{Diff}^+_{\mathrm{Conf}}(S^2)$ is a homotopy equivalence. Both groups act transitively on the sphere S^2, giving rise to a map of fiber sequences

$$\begin{array}{ccccc} \mathrm{SO}(2) & \longrightarrow & \mathrm{SO}(3) & \longrightarrow & S^2 \\ \downarrow{\scriptstyle \theta} & & \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{Diff}^+_{\mathrm{Conf}}(S^2) & \longrightarrow & S^2, \end{array}$$

where G denotes the group of holomorphic automorphisms of $\mathbf{C}\mathrm{P}^1$ that preserve the point ∞. We will prove that θ is a homotopy equivalence.

Elements of G can be identified with biholomorphic maps $f : \mathbf{C}\mathrm{P}^1 \to \mathbf{C}\mathrm{P}^1$ carrying ∞ to itself. Such a map can be viewed as a meromophic function on $\mathbf{C}\mathrm{P}^1$ having a pole of order at most 1 at ∞. The collection of all such meromorphic functions forms a vector space which, by the proof of Proposition 7, has dimension 2. We can write down these meromorphic functions explicitly: they are precisely the maps of the form $z \mapsto az + b$, where $a, b \in \mathbf{C}$. Such a map determines an automorphism of $\mathbf{C}\mathrm{P}^1$ if and only if $a \neq 0$. Consequently, we can identify G with the product $\mathbf{C}^* \times \mathbf{C} = \{(a, b) \in \mathbf{C}^2 : a \neq 0\}$. The map θ has image $S^1 = \{(a, b) \in \mathbf{C}^2 : |a| = 1, b = 0\}$. It is now clear that θ is a homotopy equivalence. □

Remark 9. The automorphism group $\mathrm{Diff}^+_{\mathrm{Conf}}(S^2)$ can be identified with $PGL_2(\mathbf{C})$, which contains $SO(3)$ as a maximal compact subgroup.

We can use the same methods to compute the diffeomorphism group of a surface of genus 1. Such a surface looks like a torus $T = \mathbb{R}^2 / \mathbf{Z}^2$. This description of T as a quotient makes it evident that two different groups act on T:

(*i*) The group T acts on itself by translations.

(*ii*) The group $\mathrm{GL}_2(\mathbf{Z})$ acts on T.

These group actions are in fact compatible with one another, and give a rise to a map $G \to \mathrm{Diff}(T)$, where G denotes the semidirect product of T with $\mathrm{GL}_2(\mathbf{Z})$.

Proposition 10. *The map* $G \to \mathrm{Diff}(T)$ *is a homotopy equivalence.*

The proof proceeds in several steps.

(*a*) The groups G and $\mathrm{Diff}(T)$ both act transitively on T. It will therefore suffice to show that we have a homotopy equivalence $G_0 \to \mathrm{Diff}_0(T)$, where G_0 and $\mathrm{Diff}_0(T)$ denote the subgroups of G and $\mathrm{Diff}(T)$ consisting of maps which fix the origin $0 \in T$. In other words, we must show that the inclusion $\phi : \mathrm{GL}_2(\mathbf{Z}) \to \mathrm{Diff}_0(T)$ is a homotopy equivalence.

(*b*) The map ϕ has an obvious splitting, since $\mathrm{Diff}_0(T)$ maps to $\mathrm{GL}_2(\mathbf{Z})$ via its action on the homology group $\mathrm{H}_1(T; \mathbf{Z})$. It will therefore suffice to show that $\mathrm{Diff}_1(T)$ is contractible, where $\mathrm{Diff}_1(T)$ denotes the group of diffeomorphisms of T which fix the origin 0 and act trivially on the homology of T.

(*c*) The group $\mathrm{Diff}_1(T)$ does not act transitively on $\mathrm{Conf}(T)$. However, it does act freely: suppose that we fix a point of $\mathrm{Conf}(T)$, which endows T with a complex structure. The fixed point $0 \in T$ endows T with the structure of an *elliptic curve*. In particular, it acquires a canonical group structure. If we let $\mathfrak{t}$ denote the (complex) Lie algebra of T at the origin, then we get an exponential map $\mathfrak{t} \to T$ which exhibits T as a quotient $\mathfrak{t}/\Lambda$. Any element f of $\mathrm{Diff}_1(T)$ which preserves the conformal structure must act by a group automorphism of T (since it is complex analytic and fixed the origin), and is therefore determined by its derivative $df : \mathfrak{t} \to \mathfrak{t}$. Since f is required to act trivially on $\mathrm{H}_1(T; \mathbf{Z}) \simeq \Lambda$, we deduce that $df = \mathrm{id}$ so that $f = \mathrm{id}$.

(d) We now have a fiber sequence

$$\mathrm{Diff}_1(T) \to \mathrm{Conf}(T) \to M,$$

where $M = \mathrm{Conf}(T)/\,\mathrm{Diff}_1(T)$ can be thought of as a moduli space for genus 1 Riemann surfaces Σ equipped with a marked point and an oriented trivialization $\mathrm{H}_1(\Sigma, \mathbf{Z}) \simeq \mathbf{Z}^2$. Again, any such Σ must be an elliptic curve and therefore has the form V/Λ, where V is the tangent space to Σ at the origin (a 1-dimensional complex vector space) and $\Lambda \subseteq V$ is a lattice. Our trivialization $\mathbf{Z}^2 \simeq \mathrm{H}_1(\Sigma, \mathbf{Z})$ gives an oriented basis (u, v) for Λ, so a point of M can be identified with an isomorphism class of triples (V, u, v). Any such triple is uniquely isomorphic to $(\mathbf{C}, 1, \tau)$ (namely, the choice of an element u trivializes the vector space V), where τ is an element of the upper half plane $\{x + iy : y > 0\} \subseteq \mathbf{C}$. It follows that M is contractible. Since $\mathrm{Conf}(T)$ is also contractible, we deduce that $\mathrm{Diff}_1(T)$ is contractible, as desired.

We will give a different proof of Proposition 10 shortly.